

Ex 4.1

Inverse Trigonometric Functions Ex 4.1 Q1.

Let $\tan^{-1}(-\sqrt{3}) = y$. Then, $\tan y = -\sqrt{3} = -\tan \frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of \tan^{-1} is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}(\sqrt{3})$ is $-\frac{\pi}{3}$.

Concept Insight:

The range for \tan^{-1} is same as \sin^{-1} except that it is an open interval, as $\tan(-\pi/2)$ and $\tan(\pi/2)$ are not defined. So the method of finding principal value is same as \sin^{-1} given in the first problem. Also note that $\tan(-x) = -\tan x$.

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$. Then, $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of \cos^{-1} is

$[0, \pi]$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$. Then, $\operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\left(\frac{\pi}{4}\right) = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of

$\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ and $\operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

We know that for any $x \in [-1, 1]$, $\cos^{-1} x$ represents angle in $[0, \pi]$

$$\begin{aligned}\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) &= \text{an angle in } [0, \pi] \text{ whose cosine is } \left(-\frac{\sqrt{3}}{2}\right) \\ &= \pi - \frac{\pi}{6} = \frac{5\pi}{6} \\ \therefore \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) &= \frac{5\pi}{6}\end{aligned}$$

We know that, for any $x \in \mathbb{R}$, $\tan^{-1} x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x .

So,

$$\begin{aligned}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) &= \text{An angle in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ whose tangent is } \frac{1}{\sqrt{3}} \\ &= \frac{\pi}{6}\end{aligned}$$

$$\therefore \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

We know that, for $x \in \mathbb{R}$, $\sec^{-1} x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$$\begin{aligned}\sec^{-1}(-\sqrt{2}) &= \text{An angle in } [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ whose secant is } (-\sqrt{2}) \\ &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4}\end{aligned}$$

$$\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}.$$

We know that, for any $x \in \mathbb{R}$, $\cot^{-1} x$ represents an angle in $(0, \pi)$

$$\begin{aligned}\cot^{-1}(-\sqrt{3}) &= \text{An angle in } (0, \pi) \text{ whose cotangent is } (-\sqrt{3}) \\ &= \pi - \frac{\pi}{6} \\ &= \frac{5\pi}{6}\end{aligned}$$

$$\therefore \cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}.$$

We know that, for any $x \in \mathbb{R}$, $\sec^{-1} x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$$\begin{aligned}\sec^{-1}(2) &= \text{An angle in } [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ whose secant is } 2 \\ &= \frac{\pi}{3}\end{aligned}$$

$$\therefore \sec^{-1}(2) = \frac{\pi}{3}.$$

We know that, for any $x \in \mathbb{R}$, $\operatorname{cosec}^{-1} x$ is an angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$$\begin{aligned}\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) &= \text{An angle in } \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \text{ whose cosecant is } \left(\frac{2}{\sqrt{3}}\right) \\ &= \frac{\pi}{3}\end{aligned}$$

$$\therefore \operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

Inverse Trigonometric Functions Ex 4.1 Q2.

$$\text{Let } \cos^{-1}\left(\frac{1}{2}\right) = x. \text{ Then, } \cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right).$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\text{Let } \sin^{-1}\left(\frac{1}{2}\right) = y. \text{ Then, } \sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right] = \tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\frac{\pi}{3}\right] = \tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$= \tan^{-1}1 = \frac{\pi}{4}$$

Concept Insight:

Solve the innermost bracket first, so first find the principal value of $\sin^{-1}(1/2)$

$$\text{Let } \tan^{-1}(1) = x. \text{ Then, } \tan x = 1 = \tan\frac{\pi}{4}.$$

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

$$\text{Let } \cos^{-1}\left(-\frac{1}{2}\right) = y. \text{ Then, } \cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right).$$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\text{Let } \sin^{-1}\left(-\frac{1}{2}\right) = z. \text{ Then, } \sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

$$\begin{aligned}\tan^{-1}(\sqrt{3}) &= \text{Angle in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ whose tangent is } \sqrt{3} \\ &= \frac{\pi}{3}\end{aligned}$$

$$\begin{aligned}\sec^{-1}(-2) &= \text{An angle in } [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ whose secant is } (-2) \\ &= \pi - \frac{\pi}{3} \\ &= \frac{2\pi}{3}\end{aligned}$$

$$\begin{aligned}\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) &= \text{An angle in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \text{ whose cosecant is } \left(\frac{2}{\sqrt{3}}\right) \\ &= \frac{\pi}{3}\end{aligned}$$

Hence,

$$\begin{aligned}\tan^{-1} \sqrt{3} - \sec^{-1}(-2) + \operatorname{cosec}^{-1} \frac{2}{\sqrt{3}} \\ = \frac{\pi}{3} - \frac{2\pi}{3} + \frac{\pi}{3} \\ = 0\end{aligned}$$

$$\therefore \tan^{-1} \sqrt{3} - \sec^{-1}(-\sqrt{2}) + \operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = 0$$

$$\begin{aligned}\text{Let } \cos^{-1}\left(\frac{1}{2}\right) &= x. \text{ Then, } \cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \\ \therefore \cos^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{3} \\ \text{Let } \sin^{-1}\left(-\frac{1}{2}\right) &= y. \text{ Then, } \sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right) \\ \therefore \sin^{-1}\left(-\frac{1}{2}\right) &= -\frac{\pi}{6} \\ \therefore \cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right) &= \frac{\pi}{3} - \left(-\frac{2\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}\end{aligned}$$

Inverse Trigonometric Functions Ex 4.1 Q3.

$$\begin{aligned}\text{Let } \sin^{-1}\left(\frac{1}{2}\right) &= x. \text{ Then, } \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) \\ \therefore \sin^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{6} \\ \text{Let } \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) &= y. \text{ Then, } \sin y = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right) \\ \therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) &= \frac{\pi}{4} \\ \therefore \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) &= \frac{\pi}{6} - \frac{2\pi}{4} = \frac{\pi}{6} - \frac{\pi}{2} = \frac{\pi - 3\pi}{6} = -\frac{\pi}{3}\end{aligned}$$

$$\begin{aligned}\text{Let } \sin^{-1}\left(-\frac{1}{2}\right) &= x. \text{ Then, } \sin x = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right) \\ \therefore \sin^{-1}\left(-\frac{1}{2}\right) &= -\frac{\pi}{6} \\ \text{Let } \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) &= y. \text{ Then, } \cos y = \frac{-\sqrt{3}}{2} = \cos\left(\pi - \frac{\pi}{6}\right) \\ \therefore \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) &= \frac{5\pi}{6} \\ \therefore \sin^{-1}\left(-\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) &= -\frac{\pi}{6} + \frac{10\pi}{6} = \frac{-\pi + 10\pi}{6} = \frac{9\pi}{6} = \frac{3\pi}{2}\end{aligned}$$

$$\text{Let } \tan^{-1}(-1) = x. \text{ Then, } \tan x = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$\therefore \tan^{-1}(-1) = \frac{3\pi}{4}$$

$$\text{Let } \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y. \text{ Then, } \cos y = \frac{-1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\therefore \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

$$\therefore \tan^{-1}(-1) + \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

$$\text{Let } \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = x. \text{ Then, } \sin x = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right) = \sin\left(\pi - \frac{\pi}{3}\right)$$

$$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3}$$

$$\text{Let } \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y. \text{ Then, } \cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$$

$$\therefore \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

$$\text{Let } \tan^{-1}(\sqrt{3}) = x. \text{ Then, } \tan x = \sqrt{3} = \tan\left(\frac{\pi}{3}\right)$$

$$\therefore \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\text{Let } \sec^{-1}(-2) = y. \text{ Then, } \sec y = -2 = \sec\left(\pi - \frac{\pi}{3}\right)$$

$$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$$

$$\therefore \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = \frac{\pi - 2\pi}{3} = -\frac{\pi}{3}$$

Ex 4.2

Inverse Trigonometric Functions Ex 4.2 Q1

8.i

$$= \cos^{-1} \left(\frac{a \cos \theta + b}{a + b \cos \theta} \right)$$

$$\left\{ \begin{array}{l} \text{Since } \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \end{array} \right\}$$

= RHS

Hence,

$$2 \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{\theta}{2} \right) = \cos^{-1} \left(\frac{a \cos \theta + b}{a + b \cos \theta} \right)$$