Ex 15.1

Mean Value Theorems Ex 15.1 Q1(i)

$f(x) = 3 + (x-2)^{\frac{2}{3}}$ on $[1,3]$

Differentiating it with respect to x,

$$f'(x) = \frac{2}{3} \times \frac{1}{(x-2)^{\frac{1}{3}}}$$

Clearly, $\lim_{x\to 2} = \frac{2}{3} \times \frac{1}{(x-2)^{\frac{1}{3}}}$

Thus, $f(x)$ is not differentiable at $x = 2 \in (1,3)$

Hene, Rolle's theorem is not applicable for $f(x)$ in $x \in [1,3]$.

Mean Value Theorems Ex 15.1 Q1(ii)

Here, $f(x) = [x]$ and $x \in [-1,1]$, at $n = 1$

$$\begin{aligned}
\text{LHL} &= \lim_{x\to(1-h)} [x] \\
&= \lim_{h\to 0} [1-h] \\
&= 0 \\
\text{RHL} &= \lim_{x\to(1+h)} [x] \\
&= \lim_{h\to 0} [1+h] \\
&= 1 \\
&\text{LHL} \neq RHL
\end{aligned}$$

So, $f(x)$ is not continuous at $1 \in [-1,1]$

Hence, rolle's theorem is not applicable on $f(x)$ in $[-1,1]$.

Mean Value Theorems Ex 15.1 Q1(iii)

Here, $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in [-1,1]$, at $n = 0$

$$\text{LHS} = \lim_{x \to (0-h)} \sin\left(\frac{1}{x}\right)$$
$$= \lim_{h \to 0} \sin\left(\frac{1}{0-h}\right)$$
$$= \lim_{h \to 0} \sin\left(\frac{-1}{h}\right)$$
$$= -\lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$
$$= -k \qquad \left[\text{Let } \lim_{h \to 0} \sin\left(\frac{1}{h}\right) = k \text{ as } k \in [-1,1]\right]$$

$$\text{RHS} = \lim_{x \to (0+h)} \sin\left(\frac{1}{x}\right)$$
$$= \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$
$$= k$$

$\Rightarrow$ LHS $\neq$ RHS

$\Rightarrow$ $f(x)$ is not continuous at $n = 0$

So, rolle's theorem is not applicable on $f(x)$ in $[-1,1]$

Mean Value Theorems Ex 15.1 Q1(iv)

Here, $f(x) = 2x^2 - 5x + 3$ on $[1,3]$

$f(x)$ is continuous in $[1,3]$ and $f(x)$ is differentiable is $(1,3)$ since it is a polynomial function.

Now,

$$f(x) = 2x^2 - 5x + 3$$
$$f(1) = 3(1)^2 - 5(1) + 3$$
$$= 2 - 5 + 3$$
$$f(1) = 0 \qquad \text{---(i)}$$
$$f(3) = 2(3)^2 - 5(3) + 3$$
$$= 18 - 15 + 3$$
$$f(3) = 6 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(1) \neq f(3)$$

So, rolle's theorem is not applicable on $f(x)$ in $[1,3]$.

Mean Value Theorems Ex 15.1 Q1(v)

Here, $f(x) = x^{\frac{2}{3}}$ on $[-1,1]$

$$f'(x) = \frac{2}{3x^{\frac{1}{3}}}$$
$$f'(0) = \frac{2}{3(0)^{\frac{1}{3}}}$$
$$f'(0) = \infty$$

So, $f'(x)$ does not exist at $x = 0 \in (-1,1)$

$\Rightarrow$ $f(x)$ is not differentiatable in $x \in (-1,1)$

So, rolle's theorem is not applicable on $f(x)$ in $[-1,1]$.

Mean Value Theorems Ex 15.1 Q1(vi)

Here, $f(x) = \begin{cases} -4x+5, & 0 \le x \le 1 \\ 2x-3, & 1 < x \le 2 \end{cases}$

For $n = 1$

LHS $= \lim_{x \to (1-h)} (-4x+5)$

$= \lim_{h \to 0} [-4(1-h)+5]$

$= -4+5$

LHS $= 1$

RHS $= \lim_{x \to (1+h)} (2x-3)$

$= \lim_{h \to 0} [2(1+h)-3]$

$= 2-3$

RHS $= -1$

So, LHS $\ne$ RHS

$\Rightarrow$ $f(x)$ is not contiuuous at $x = 1 \in [0,2]$

$\Rightarrow$ Rolle's theorem is not applicable on $f(x)$ in $[0,2]$.

Mean Value Theorems Ex 15.1 Q2(i)

Here,

$f(x) = x^2 - 8x + 12$ on $[2,6]$

$f(x)$ is continuous is $[2,6]$ and differentiable is $(2,6)$ as it is a polynomial function

And $f(2) = (2)^2 - 8(2) + 12 = 0$

$f(6) = (6)^2 - 8(6) + 12 = 0$

$\Rightarrow$ $f(2) = f(6)$

So, Rolle's theorem is applicable, therefore we show have $f'(c) = 0$ such that $c \in (2,6)$

So, $f(x) = x^2 - 8x + 12$

$\Rightarrow$ $f'(x) = 2x - 8$

So, $f'(c) = 0$

$2c - 8 = 0$

$c = 4 \in (2,6)$

Therefore, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(ii)

The given function is $f(x) = x^2 - 4x + 3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$f(1) = 1^2 - 4 \times 1 + 3 = 0$, $f(4) = 4^2 - 4 \times 4 + 3 = 3$

$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1} = \frac{3-(0)}{3} = \frac{3}{3} = 1$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$f'(c) = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \frac{5}{2}$, where $c = \frac{5}{2} \in (1, 4)$

Hence, Mean Value Theorem is verified for the given function

Mean Value Theorems Ex 15.1 Q2(iii)

Here,

$$f(x) = (x-1)(x-2)^2 \text{ on } (1,2)$$

$f(x)$ is cantinuous is $[1,2]$ and differentiable is $(1,2)$ since it is a polynomial function.

And $f(1) = (1-1)(1-2)^2 = 0$

$f(2) = (2-1)(2-2)^2 = 0$

$\Rightarrow \quad f(1) = f(2)$

So, Rolle's theorem is applicable on $f(x)$ in $[1,2]$, therefore, there exist a $c \in (1,2)$ such that $f'(c) = 0$

Now,

$$f(x) = (x-1)(x-2)^2$$

$$f'(x) = (x-1) \times 2(x-2) + (x-2)^2$$

$$f'(x) = (x-2)(3x-4)$$

So, $f'(c) = 0$

$$(c-2)(3c-4) = 0$$

$\Rightarrow \quad c = 2$ or $c = \frac{4}{3} \in (1,2)$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(iv)

Here,

$$f(x) = x(x-1)^2 \text{ on } [0,1]$$

$f(x)$ is continuous on$[0,1]$ and differentiable on $(0,1)$ as it is a polynomial function.

Now,

$$f(0) = 0(0-1)^2 = 0$$

$$f(1) = 1(1-1)^2 = 0$$

$\Rightarrow \quad f(0) = f(1)$

So, Rolle's theorem is applicable on $f(x)$ in $[0,1]$ therefore, we should show that there exist a $c \in (0,1)$ such that $f'(c) = 0$

Now,

$$f(x) = x(x-1)^2$$

$$f'(x) = (x-1)^2 + x \times 2(x-1)$$

$$= (x-1)(x-1+2x)$$

$$f'(x) = (x-1)(3x-1)$$

So, $f'(c) = 0$

$$(c-1)(3c-1) = 0$$

$\Rightarrow \quad c = 1$ or $c = \frac{1}{3} \in (0,1)$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(v)

Here,

$$f(x) = (x^2 - 1)(x - 2) \text{ on } [-1, 2]$$

$f(x)$ is continuous is $[-1, 2]$ and differentiable in $(-1, 2)$ as it is a polynomial functions.

Now,

$$f(-1) = (1-1)(-1-2) = 0$$

$$f(2) = (4-1)(2-2) = 0$$

$$\Rightarrow \quad f(-1) = f(2)$$

So, Rolle's theorem is applicable on $f(x)$ is $[-1, 2]$ therefore, we have to show that there exist a $c \in (-1, 2)$ such that $f'(c) = 0$

Now,

$$f(x) = (x^2 - 1)(x - 2)$$

$$f'(x) = 2x(x-2) + (x^2 - 1)$$

$$= 2x^2 - 4 + x^2 - 1$$

$$f'(x) = 3x^2 - 5$$

Now,

$$f'(c) = 0$$

$$\Rightarrow \quad 3x^2 - 5 = 0$$

$$\Rightarrow \quad x = -\sqrt{\frac{5}{3}} \text{ or } x = \sqrt{\frac{5}{3}} \in (-1, 2)$$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(vi)

Here, $f(x) = x(x-4)^2$ on $[0, 4]$

$f(x)$ is continuous is $[0, 4]$ and differentiable is $(0, 4)$ since $f(x)$ is a polynomial function.

Now,

$$f(x) = x(x-4)^2$$

$$f(0) = 0(0-4)^2$$

$$f(0) = 0 \qquad \text{---(i)}$$

$$f(4) = 4(4-4)^2$$

$$f(4) = 0 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(0) = f(4)$$

So, Rolle's theorem is applicable, therefore, we have to show that $f'(c) = 0$ for $c \in (0, 4)$

$$f'(x) = x \times 2(x-4) + (x-4)^2$$

$$= 2x^2 - 8x + x^2 + 16 - 8x$$

$$\text{So,} \quad f'(c) = 3c^2 - 16c + 16$$

$$0 = 3c^2 - 12c - 4c + 16$$

$$0 = 3c(c-4) - 4(c-4)$$

$$0 = (c-4)(3c-4)$$

$$\Rightarrow \quad c = 4 \text{ or } c = \frac{4}{3} \in (0, 4)$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(vii)

Here, $f(x) = x(x-2)^2$ on $[0,2]$
$f(x)$ is continuous is $[0,2]$ and differentiable is $(0,2)$
as it is a polynomial function.

And $\quad f(0) = 0(0-2)^2 = 0$

$\quad\quad f(2) = 2(2-2)^2 = 0$

$\Rightarrow \quad f(0) = f(2)$

So, Rolle's theorem is applicable on $f(x)$ is $[0,2]$, therefore,
we have to show that $f'(c) = 0$ as $c \in (0,2)$

$$
\begin{aligned}
& f(x) = x(x-2)^2 \\
& f'(x) = x \times 2(x-2) + (x-2) \\
& f'(x) = 2x(x-2) + (x-2) \\
\Rightarrow \quad & f'(c) = 0 \\
& 2c(c-2) + (c-2) = 0 \\
& (c-2)(2c+1) = 0 \\
& c = 2 \text{ or } c = -\frac{1}{2} \\
\Rightarrow \quad & c = 2 \in (0,2)
\end{aligned}
$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(viii)

Here, $f(x) = x^2 + 5x + 6$ on $[-3,-2]$
$f(x)$ is continuous is $[-3,-2]$ and $f(x)$ is differentiable is $(-3,-2)$
since it is a polynomial function.

Now,

$$
\begin{aligned}
& f(x) = x^2 + 5x + 6 \\
& f(-3) = (-3)^2 + 5(-3) + 6 \\
& \quad\quad = 9 - 15 + 6 \\
& f(-3) = 0 \quad\quad ---\text{(i)} \\
& f(-2) = (-2)^2 + 5(-2) + 6 \\
& \quad\quad = 4 - 10 + 6 \\
& f(-2) = 20 \quad\quad ---\text{(ii)}
\end{aligned}
$$

From equation (i) and (ii),

$$f(-3) = f(-2)$$

So, Rolle's theorem is applicable is $[-3,-2]$, we have to show that
$f'(c) = 0$ as $c \in (-3,-2)$.

Now,

$$
\begin{aligned}
& f(x) = x^2 + 5x + 6 \\
& f'(x) = 2x + 5 \\
\Rightarrow \quad & f'(c) = 0 \\
& 2c + 5 = 0 \\
& c = \frac{-5}{2} \in (-3,-2)
\end{aligned}
$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(i)

Here,

$$f(x) = \cos 2\left(x - \frac{\pi}{4}\right) \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that cosine function is continuous and differentiable every where, so $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left[0, \frac{\pi}{2}\right]$.

Now,

$$f(0) = \cos 2\left(0 - \frac{\pi}{4}\right) = 0$$

$$f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable.

Hence, there must exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f'(x) = -\sin 2\left(x - \frac{\pi}{4}\right) \times 2$$

$$f'(x) = -2\sin\left(2x - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad -2\sin\left(2c - \frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(2c - \frac{\pi}{2}\right) = \sin 0$$

$$\Rightarrow \quad 2c - \frac{\pi}{2} = 0$$

$$c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(ii)

Here,

$$f(x) = \sin 2x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that $\sin x$ is a continuous and differentiable every where. So, $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin 0 = 0$$

$$f\left(\frac{\pi}{2}\right) = \sin \pi = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

Now,

$$f'(x) = 2\cos 2x$$

$$f'(c) = 2\cos 2c = 0$$

$$\Rightarrow \quad \cos 2c = 0$$

$$\Rightarrow \quad 2c = \frac{\pi}{2}$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{4}\right)$$

Thus, Rolle's theorem verified.

Mean Value Theorems Ex 15.1 Q3(iii)

Here,

$$f(x) = \cos 2x \text{ on } \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$$

We know that $\cos x$ is a continuous and differentiable every where. So,

$f(x)$ is continuous in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$ and differentiable is $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$.

Now, $f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

$f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow \quad f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(0, \frac{\pi}{2}\right)$

such that $f'(c) = 0$

Now,

$f'(x) = 2 \sin 2x$

$f'(c) = 2 \sin 2c = 0$

$\Rightarrow \quad \sin 2c = 0$

$\Rightarrow \quad 2c = 0$

$\Rightarrow \quad c = 0 \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

Thus, Rolle's theorem verified.

Mean Value Theorems Ex 15.1 Q3(iv)

Here,

$$f(x) = e^x \times \sin x \quad \text{on } [0, \pi]$$

We know that since and expential function are continuous and differentiable every where so, $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$f(0) = e^0 \sin 0 = 0$

$f(\pi) = e^\pi \sin \pi = 0$

$\Rightarrow \quad f(0) = f(\pi)$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$f(x) = e^x \sin x$

$f'(x) = e^x \cos x + e^x \sin x$

Now, $f'(c) = 0$

$e^c (\cos c + \sin c) = 0$

$\Rightarrow \quad e^c = 0$ or $\cos c = -\sin c$

$\Rightarrow \quad e^c = 0$ gives no value of c or $\tan c = -1$

$\Rightarrow \quad \tan c = \tan\left(\pi - \frac{\pi}{4}\right)$

$c = \frac{3\pi}{4} \in (0, \pi)$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(v)

Here,

$$f(x) = e^x \cos x \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We know that expontial and cosine function are continuous and differentiable every where so, $f(x)$ is continuous is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and differentiable is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now,

$$f\left(-\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right) = 0$$

$$f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^x \cos x$$

$$f'(x) = -e^x \sin x + e^x \cos x$$

So, $\quad f'(c) = 0$

$$e^c(-\sin c + \cos c) = 0$$

$\Rightarrow \quad e^c = 0$ gives no value of c

$$\Rightarrow \quad -\sin c + \cos c = 0$$

$$\Rightarrow \quad \tan c = 1$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(vi)

Here,

$$f(x) = \cos 2x \text{ on } [0, \pi]$$

We know that, cosine function is contiuous and differentiable every where, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = \cos 0 = 1$$

$$f(\pi) = \cos(2\pi) = 1$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = \cos 2x$$

$$f'(x) = -2\sin 2x$$

So, $\quad f'(c) = 0$

$$\Rightarrow \quad -2\sin 2c = 0$$

$$\Rightarrow \quad \sin 2c = 0$$

$$\Rightarrow \quad 2c = 0 \quad \text{or} \quad 2c = \pi$$

$$\Rightarrow \quad c = 0 \quad \text{or} \quad c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(vii)

$f(x) = \frac{\sin x}{e^x}$ *on* $x \in [0, \pi]$

We know that, exponential and sine *both functions are continuous and differentiable every where, so* $f(x)$ *is continuous is* $[0, \pi]$ *and differentiable is* $[0, \pi]$

Now,

$$f(0) = \frac{\sin 0}{e^0} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = 0$$

$\Rightarrow f(0) = f(\pi)$

Since Rolle's theorem applicable, therefore there must exist a point $c \in [0, \pi]$ such that $f'(c) = 0$

Now,

$$f(x) = \frac{\sin x}{e^x}$$

$$\Rightarrow f'(x) = \frac{e^x(\cos x) - e^x(\sin x)}{(e^x)^2}$$

Now,

$$f'(c) = 0$$
$$\Rightarrow e^c(\cos c - \sin c) = 0$$
$$\Rightarrow e^c \neq 0 \text{ and } \cos c - \sin c = 0$$
$$\Rightarrow \tan c = 1$$
$$c = \frac{\pi}{4} \in [0, \pi]$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(viii)

Here,

$$f(x) = \sin 3x \text{ on } [0, \pi]$$

We know that, sine function is continuous and differentiable every where. So, $f(x)$ is continuous is $(0, \pi)$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = \sin 0 = 0$$
$$f(\pi) = \sin 3\pi = 0$$
$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exists a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin 3x$$
$$f'(x) = 3\cos 3x$$

Now,

$$f'(c) = 0$$
$$\Rightarrow \quad 3\cos 3x = 0$$
$$\Rightarrow \quad \cos 3x = 0$$
$$\Rightarrow \quad 3x = \frac{\pi}{2}$$
$$\Rightarrow \quad x = \frac{\pi}{6} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(ix)

Here,

$$f(x) = e^{1-x^2} \text{ on } [-1, 1]$$

We know that, exponential function is continuous and differentiable every where. So, $f(x)$ is continuous is $[-1,1]$ and differentiable is $(-1,1)$.

Now,

$$f(-1) = e^{1-1} = 1$$

$$f(1) = e^{1-1} = 1$$

$$\Rightarrow \quad f(-1) = 1$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1,1)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^{1-x^2}$$

$$f'(x) = e^{1-x^2}(-2x)$$

Now,

$$f'(c) = 0$$

$$-2ce^{1-c^2} = 0$$

$$\Rightarrow \quad c = 0 \quad \text{or} \quad e^{1-c^2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(x)

Here,

$$f(x) = \log\left(x^2 + 2\right) - \log 3 \text{ on } [-1, 1]$$

We know that, logarithmic function is continuous and differentiable is its domain, so $f(x)$ is continuous is $[-1, 1]$ and differentiable is $(-1,1)$.

Now,

$$f(-1) = \log(1 + 2) - \log 3 = 0$$

$$f(1) = \log(1 + 2) - \log 3 = 0$$

$$\Rightarrow \quad f(-1) = f(1)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1, 1)$ such that $f'(c) = 0$.

Now,

$$f(x) = \log\left(x^2 + 2\right) - \log 3$$

$$f'(x) = \frac{(2x)}{x^2 + 2}$$

Now,

$$f'(c) = 0$$

$$\frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xi)

Here,

$$f(x) = \sin x + \cos x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that $\sin x$ and $\cos x$ are continuous and differentiable every where, so $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin 0 + \cos c0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{\sin \pi}{2} + \frac{\cos \pi}{2} = 1$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

Now,

$$f'(c) = 0$$

$$\cos c - \sin c = 0$$

$$\Rightarrow \quad \tan c = 1$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xii)

Here,

$$f(x) = 2\sin x + \sin 2x \text{ on } [0, \pi]$$

We know that sine function is continuous and differentiable every where, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 2\sin 0 + \sin 0 = 0$$

$$f(\pi) = 2\sin \pi + \sin 2\pi = 0$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 2\sin x + \sin 2x$$

$$f'(x) = 2\cos x + 2\cos 2x$$

Now,

$$f'(c) = 0$$

$$2\cos c + 2\cos 2c = 0$$

$$\Rightarrow \quad 2\left(\cos c + 2\cos^2 c - 1\right) = 0$$

$$\Rightarrow \quad \left(2\cos^2 + 2\cos c - \cos c - 1\right) = 0$$

$$\Rightarrow \quad (2\cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \quad \cos c = \frac{1}{2}, \quad \cos c = -1$$

$$\Rightarrow \quad \tan c = 1$$

$$c = \frac{\pi}{3} \in (0, \pi), \; c = \pi$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xiii)

Here,

$$f(x) = \frac{x}{2} - \sin\frac{\pi x}{6} \text{ on } [-1, 0]$$

We know that sine function is continuous and differentiable every where, so $f(x)$ is continuous is $[-1, 0]$ and differentiable is $(-1, 0)$.

Now,

$$f(-1) = \frac{-1}{2} - \sin\left(-\frac{\pi}{6}\right)$$
$$= -\frac{1}{2} + \sin\frac{\pi}{6}$$
$$= -\frac{1}{2} + \frac{1}{2}$$
$$f(-1) = 0 \qquad \text{---(i)}$$

And $\quad f(0) = 0 - \sin 0$

$$f(0) = 0 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(-1) = f\binom{0}{}$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1, 0)$ such that $f'(c) = 0$.

Now,

$$f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)$$
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right)$$

Now,

$$f'(c) = 0$$
$$\frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = 0$$
$$\Rightarrow \quad -\frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = -\frac{1}{2}$$
$$\Rightarrow \quad \cos\left(\frac{\pi c}{6}\right) = 3\pi$$
$$\Rightarrow \quad \frac{\pi c}{6} = \cos^{-1}\left(\frac{66}{7}\right)$$
$$\Rightarrow \quad c = \frac{6}{\pi}\cos^{-1}\left(\frac{66}{7}\right)$$
$$\Rightarrow \quad c = \frac{21}{11}\cos^{-1}\left(\frac{66}{7}\right)$$
$$\Rightarrow \quad c \in \left(-\frac{21}{11}, \frac{21}{11}\right) \qquad \left[\text{since, } \cos^{-1} x \in [-1, 1]\right]$$
$$\Rightarrow \quad c \in (-1.9, 1.9)$$
$$\Rightarrow \quad c \in (-1, 0)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xiv)

Here,

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x \text{ on } \left[0, \frac{\pi}{6}\right]$$

We know that sine and its square function is continuous and differentiable every where, so $f(x)$ is continuous is $\left[0, \frac{\pi}{6}\right]$ and differentiable is $\left(0, \frac{\pi}{6}\right)$.

Now,

$$f(0) = 0 - 0 = 0$$

$$f\left(\frac{\pi}{6}\right) = 1 - 1 = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{6}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{6}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x$$

$$f'(x) = \frac{6}{\pi} - 8\sin x \cos x$$

$$f'(x) = \frac{6}{\pi} - 4\sin 2x$$

Now,

$$f'(c) = 0$$

$$\frac{6}{\pi} - 4\sin 2c = 0$$

$$\Rightarrow \quad 4\sin 2c = \frac{6}{\pi}$$

$$\Rightarrow \quad \sin 2c = \frac{3}{2\pi}$$

$$\Rightarrow \quad 2c = \sin^{-1}\left(\frac{21}{44}\right)$$

$$\Rightarrow \quad c = \frac{1}{2}\sin^{-1}\left(\frac{21}{44}\right)$$

$$\Rightarrow \quad c \in \left(-\frac{1}{2}, \frac{1}{2}\right) \qquad \left[\text{since, } \sin^{-1} x \in [-1, 1]\right]$$

$$\Rightarrow \quad c \in \left(0, \frac{11}{21}\right)$$

$$\Rightarrow \quad c \in \left(0, \frac{\pi}{6}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xv)

Here,

$$f(x) = 4^{\sin x} \text{ on } [0, \pi]$$

We know that exponential and $\sin x$ both are continuous and differentiable, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 4^{\sin 0} = 4^0 = 1$$

$$f(\pi) = 4^{\sin \pi} = 4^0 = 1$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 4^{\sin x}$$

$$f'(x) = 4^{\sin x} \log 4 \times \cos x$$

Now,

$$f'(c) = 0$$

$$4^{\sin c} \times \cos x c \log 4 = 0$$

$$\Rightarrow \quad \cos c = 0$$

$$\Rightarrow \quad c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvi)

Here,

$$f(x) = x^2 - 5x + 4 \text{ on } [1, 4]$$

$f(x)$ is continuous and differentiable as it is a polynomial funciton.

Now,

$$f(1) = (1)^2 - 5(1) + 4 = 0$$

$$f(4) = (4)^2 - 5(4) + 4 = 0$$

$$\Rightarrow \quad f(1) = f(4)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (1, 4)$ such that $f'(c) = 0$.

Now,

$$f(x) = x^2 - 5x + 4$$

$$f'(x) = 2x - 5$$

So,

$$f'(c) = 0$$

$$\Rightarrow \quad 2c - 5 = 0$$

$$\Rightarrow \quad c = \frac{5}{2} \in (1, 4)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Here,

$$f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that *sine* and *cosine* function are differentiable and continuous.

So, $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and it is differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin^4(0) + \cos^4(0) = 1$$

$$f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin^4 x + \cos^4 x$$

$$f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -2(2\sin x \cos x)\left(\cos^2 x - \cos^2 x\right)$$

$$= -2\sin 2x \cos 2x$$

$$f'(x) = -\sin 4x$$

Now,

$$f'(c) = 0$$

$$-\sin 4x = 0$$

$$\sin 4x = 0$$

$$\Rightarrow \quad 4x = 0 \quad \text{or} \quad 4x = \pi$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Since trigonometric functions are differentiable and continuous,
the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

Now $f(0) = \sin 0 - \sin 2 \times 0 = 0$

and

$f(\pi) = \sin\pi - \sin 2 \times \pi = 0$

$\Rightarrow f(0) = f(\pi)$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

Now $f(x) = \sin x - \sin 2x$

$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$

$\Rightarrow \cos x = 2\cos 2x$

$\Rightarrow \cos x = 2\left(2\cos^2 x - 1\right)$

$\Rightarrow \cos x = 4\cos^2 x - 2$

$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$\Rightarrow \cos x = \dfrac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180° - \cos^{-1}(0.5931) \quad \left[\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)\right]$

$\Rightarrow x = 32°32' \text{ or } x = 126°23'$

Both $32°32'$ and $126°23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xviii)

Since trigonometric functions are differentiable and continuous,
the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

Now $f(0) = \sin 0 - \sin 2 \times 0 = 0$

and

$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$

$\Rightarrow f(0) = f(\pi)$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

Now $f(x) = \sin x - \sin 2x$

$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$

$\Rightarrow \cos x = 2\cos 2x$

$\Rightarrow \cos x = 2(2\cos^2 x - 1)$

$\Rightarrow \cos x = 4\cos^2 x - 2$

$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$\Rightarrow \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180° - \cos^{-1}(0.5931) \quad [\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)]$

$\Rightarrow x = 32°32' \text{ or } x = 126°23'$

Both $32°32'$ and $126°23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q7

Let $f(x) = 16 - x^2$, then $f'(x) = -2x$

$f(x)$ is continuous on $[-1, 1]$ because it is a polynomial function.

Also $f(-1) = 16 - (-1)^2 = 15$

$f(1) = 16 - (1)^2 = 15$

$f(-1) = f(1)$

There exists a $c \in [-1, 1]$ such that $f'(c) = 0$

$\Rightarrow -2c = 0$

$\Rightarrow c = 0$

Thus, at $0 \in [-1, 1]$ the tangent is parallel to the x-axis.

Mean Value Theorems Ex 15.1 Q8(i)

Let $f(x) = x^2$, then $f'(x) = 2x$

$f(x)$ is continuous on $[-2, 2]$ because it is a polynomial function.

$f(x)$ is differentiable on $(-2, 2)$ as it is a polynomial function.

Also $f(-2) = (-2)^2 = 4$

$f(2) = 2^2 = 4$

$\Rightarrow \quad f(-2) = f(2)$

$\therefore \quad$ There exists $c \in (-2, 2)$ such that $f'(c) = 0$

$\Rightarrow \quad 2c = 0$

$\Rightarrow \quad c = 0$

Thus, at $0 \in [-2, 2]$ the tangent is parallel to the x-axis.

x = 0, then y = 0

Therefore, the point is (0, 0)

Mean Value Theorems Ex 15.1 Q8(ii)

Let $f(x) = e^{1-x^2}$ on $[-1,1]$

Since, $f(x)$ is a composition of two continuous functions, it is continuous on $[-1,1]$

Also $f(x) = -2xe^{1-x^2}$

$f(2) = 2^2 = 4$

$\therefore \quad f'(x)$ exists for every value of x in $(-1,1)$

$\Rightarrow \quad f(x)$ is differentiable on $(-1,1)$

By rolle's theorem, there exists $c \in (-1,1)$ such that $f'(c) = 0$

$\Rightarrow \quad -2ce^{1-c^2} = 0$

$\Rightarrow \quad c = 0$

Thus, at $c = 0 \in [-1,1]$ the tangent is parallel to the x-axis.

x = 0, then y = e

Therefore, the point is (0, e)

Mean Value Theorems Ex 15.1 Q8(iii)

Let $f(x) = 12(x+1)(x-2)$

Since, $f(x)$ is a polynomial function, it is continuous on $[-1,2]$ and differentiable on $(-1,2)$

Also $f'(x) = 12[(x-2)+(x+1)] = 12[2x-1]$

By rolle's theorem, there exists $c \in (-1,2)$ such that $f'(c) = 0$

$\Rightarrow \quad 12(2c-1) = 0$

$\Rightarrow \quad c = \frac{1}{2}$

Thus, at $c = \frac{1}{2} \in (-1,2)$ the tangent to $y = 12(x+1)(x-2)$ is parallel to x-axis

Mean Value Theorems Ex 15.1 Q9

It is given that $f:[-5,5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists c □ $(-5, 5)$ such that

$$f'(c) = \frac{f(5)-f(-5)}{5-(-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$

$$\Rightarrow 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Mean Value Theorems Ex 15.1 Q10

By Rolle's Theorem, for a function $f:[a, \mathrm{b}] \to \mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

then, there exists some $c \in (a, b)$ such that $f'(c) = 0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$f(x)$ is not continuous in $[5, 9]$.

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x=-2$ and $x=2$

$f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$
$\therefore f(-2)\neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n\in(-2, 2)$.

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x=n$ are not equal, f is not differentiable at $x=n$

f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

Mean Value Theorems Ex 15.1 Q11

It is given that the Rolle's Theorem holds for the function $f(x)=x^3+bx^2+cx,\ x\in[1,2]$ at the point $x=\frac{4}{3}$.

We need to find the values of b and c.

$f(x)=x^3+bx^2+cx$

Since it satisfies the rolle's theorem, we have,

$f(1)=f(2)$

$\Rightarrow 1^3+b\times 1^2+c\times 1=2^3+b\times 2^2+c\times 2$

$\Rightarrow 1+b+c=8+4b+2c$

$\Rightarrow 3b+c=-7\ldots(1)$

Differentiating the given function, we have,

$f'(x)=3x^2+2bx+c$

$f'\left(\frac{4}{3}\right)=3\times\left(\frac{4}{3}\right)^2+2b\times\left(\frac{4}{3}\right)+c$

$\Rightarrow 0=\frac{16}{3}+\frac{8b}{3}+c\ldots(2)$

Solving the equations (1) and (2), we have,

$b=-5$ and $c=8$

Ex 15.2

Mean Value Theorems Ex 15.2 Q1(i)

Here,

$$f(x) = x^2 - 1 \text{ on } [2,3]$$

It is a polynomial function so it is continuous in $[2,3]$ and differentiable in $(2,3)$. So, both conditions of Lagrange's mean value theorem are satisfied.

Therefore, there exist a point $c \in (2,3)$ such that

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$
$$2c = \frac{\left((3)^2 - 1\right) - \left((2)^2 - 1\right)}{1}$$
$$2c = (8 - 3)$$
$$c = \frac{5}{2} \in (2,3)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(ii)

Here,

$$f(x) = x^3 - 2x^2 - x + 3 \text{ on } [0,1]$$

Since, $f(x)$ is a polynomial function. So, $f(x)$ is continuous in $[0,1]$ and differentiable in $(0,1)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{\left[(1)^3 - 2(1)^2 - (1) + 3\right] - 3}{1}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = 1 - 3$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c - 1) - 1(c - 1) = 0$$

$$\Rightarrow \quad (3c - 1)(c - 1) = 0$$

$$\Rightarrow \quad c = \frac{1}{3} \in (0,1)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(iii)

Here,

$$f(x) = x(x - 1)$$

$$f(x) = x^2 - x \quad \text{on } [1,2]$$

We know that, polynomial function is continuous and differentiable. So, $f(x)$ is continuous in $[1,2]$ and $f(x)$ is differentiable in $(1,2)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (1,2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow \quad 2c - 1 = \frac{(4 - 2) - (1 - 1)}{1}$$

$$\Rightarrow \quad 2c - 1 = \frac{2 - 0}{1}$$

$$\Rightarrow \quad 2c = 3$$

$$\Rightarrow \quad c = \frac{3}{2} \in (1,2)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(iv)

Here,

$$f(x) = x^2 - 3x + 2 \text{ on } [-1,2]$$

We know that, polynomial function is continuous and differentiable. So, $f(x)$ is continuous in $[-1,2]$ and differentiable in $(-1,2)$. So, Lagrange's mean value theorem is applicable, so there exist a point $c \in (-1,2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 + 1}$$

$$\Rightarrow \quad 2c - 3 = \frac{(4 - 6 + 2) - (1 + 3 + 2)}{3}$$

$$\Rightarrow \quad 2c - 3 = -\frac{6}{3}$$

$$\Rightarrow \quad 2c = 1$$

$$\Rightarrow \quad c = \frac{1}{2} \in (-1,2)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(v)

Here,

$$f(x) = 2x^2 - 3x + 1 \text{ on } [1,3]$$

We know that, polynomial function is continuous and differentiable. So, $f(x)$ is continuous in $[1,3]$ and $f(x)$ is differentiable in $(1,3)$. So, Lagrange's mean value theorem is applicable, so there exist a point $c \in (1,3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3-1}$$

$$\Rightarrow \quad 4c - 3 = \frac{\left(2(3)^2 - 3(3) + 1\right) - (2 - 3 + 1)}{3-1}$$

$$\Rightarrow \quad 4c - 3 = \frac{10}{2}$$

$$\Rightarrow \quad 4c = 5 + 3$$

$$\Rightarrow \quad 4c = 8$$

$$\Rightarrow \quad c = 2 \in (1,3)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(vi)

Here,

$$f(x) = x^2 - 2x + 4 \text{ on } [1,5]$$

We know that, polynomial is always continuous and differentiable. So, $f(x)$ is continuous in $[1,5]$ and it is differentiable in $(1,5)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (1,5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5-1}$$

$$\Rightarrow \quad 2c - 2 = \frac{\left((5)^2 - 2(5) + 4\right) - (1 - 2 + 4)}{4}$$

$$\Rightarrow \quad 2c - 2 = \frac{19-3}{4}$$

$$\Rightarrow \quad 2c - 2 = 4$$

$$\Rightarrow \quad 2c = 6$$

$$\Rightarrow \quad c = 3 \in (1,5)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(vii)

Here,

$$f(x) = 2x - x^2 \text{ on } [0,1]$$

We know that, polynomial is continuous and differentiable. So, $f(x)$ is continuous in $[0,1]$ and differentiable in $(0,1)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1-0}$$

$$\Rightarrow \quad 2 - 2c = \frac{\left(2(1) - (1)^2\right) - (0)}{1}$$

$$\Rightarrow \quad 2 - 2c = 1$$

$$\Rightarrow \quad 1 = 2c$$

$$\Rightarrow \quad c = \frac{1}{2} \in (0,1)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(viii)

$$f(x) = (x-1)(x-2)(x-3) \text{ on } [0,4]$$

We know that, polynomial is continuous and differentiable every where. So, $f(x)$ is continuous in $[0,4]$ and differentiable in $(0,4)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (0,4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4-0}$$

$$\Rightarrow \quad (c-1)(c-2) + (c-2)(c-3) + (c-1)(c-3) = \frac{(3)(2)(1) - (-1)(-2)(-3)}{4-0}$$

$$\Rightarrow \quad c^2 - 3c + 2 + c^2 + 5c + 6 + c^2 - 4c + 3 = \frac{6+6}{4}$$

$$\Rightarrow \quad 3c^2 - 12c + 11 = 3$$

$$\Rightarrow \quad 3c^2 = 12c + 8 = 0$$

$$\Rightarrow \quad c = \frac{-(-12) \pm \sqrt{144 - 4 \times 3 \times 8}}{6}$$

$$\Rightarrow \quad c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow \quad c = 2 \pm \frac{2\sqrt{3}}{3} \in (0,4)$$

$$\Rightarrow \quad c = 2 \pm \cdot \frac{2}{\sqrt{3}} \in (0,4)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(ix)

Here,

$$f(x) = \sqrt{25 - x^2} \text{ on } [-3,4]$$

Given function is continuous as it has unique value for each $x \in [-3,4]$ and

$$f'(x) = \frac{-2x}{2\sqrt{25 - x^2}}$$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

So, $f'(x)$ exists for all values for $x \in (-3,4)$ so, $f(x)$ is differentiable in $(-3,4)$. So, Lagrange's mean value theorem is applicable. Thus, there exists a point $c \in (-3,4)$ such that

$$f'(c) = \frac{f(4) - f(-3)}{4+3}$$

$$\Rightarrow \quad \frac{-2c}{2\sqrt{25 - c^2}} = \frac{\sqrt{9} - \sqrt{16}}{7}$$

$$\Rightarrow \quad -7c = -\sqrt{25 - c^2}$$

Squaring both the sides,

$$49c^2 = 25 - c^2$$

$$c^2 = \frac{1}{2}$$

$$c = \pm \frac{1}{\sqrt{2}} \in (-3,4)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(x)

Here,

$$f(x) = \tan^{-1} x \text{ on } [0,1]$$

We know that, $\tan^{-1} x$ has unique value in $[0,1]$ so, it is continuous in $[0,1]$

$$f'(x) = \frac{1}{1+x^2}$$

So, $f'(x)$ exists for each $x \in (0,1)$

So, $f'(x)$ is differentiable in $(0,1)$, thus Lagrange's mean value theorem is applicable, so there exist a point $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow \quad \frac{1}{1+c^2} = \frac{\tan^{-1}(1) - \tan^{-1}(0)}{1}$$

$$\Rightarrow \quad \frac{1}{1+c^2} = \frac{\frac{\pi}{4} - 0}{1}$$

$$\Rightarrow \quad \frac{4}{\pi} = 1 + c^2$$

$$\Rightarrow \quad c = \sqrt{\frac{4}{\pi} - 1}$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xi)

Here,

$$f(x) = x + \frac{1}{x} \text{ on } [1,3]$$

$f(x)$ attiams a unique value for each $x \in [1,3]$, so it is continuous

$f'(x) = 1 - \frac{1}{x^2}$ is definded for each $x \in (1,3)$

$\Rightarrow$ $f(x)$ is differentiable in $(1,3)$, so Lagrange's mean value theorem is a applicable, so there exist a point $c \in (1,3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\left(3 + \frac{1}{3} - (1+1)\right)}{2}$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{2}$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{4}{3 \times 2}$$

$$\Rightarrow \quad 1 - \frac{2}{3} = \frac{1}{c^2}$$

$$\Rightarrow \quad c^2 = 3$$

$$\Rightarrow \quad c = \sqrt{3} \in (1,3)$$

So, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xii)

Here,

$$f(x) = x(x+4)^2 \text{ on } [0,4]$$

We know that every polynomial function is continuous and differentiable every wher, so, $f(x)$ is continuous in $[0,4]$ and differentiable in $(0,4)$, so, Lagrange's mean value theorem is applicable, thus there exist a point $c \in (0,4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4-0}$$

$$\Rightarrow \quad 3c^2 + 16c + 16 = \frac{4 \times (8)^2 - 0}{4}$$

$$\Rightarrow \quad 3c^2 + 16c + 16 = 64$$

$$\Rightarrow \quad 3c^2 + 16c - 48 = 0$$

$$\Rightarrow \quad c = \frac{-16 \pm \sqrt{256 + 576}}{6}$$

$$\Rightarrow \quad = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow \quad = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow \quad c = \frac{-8 \pm 4\sqrt{13}}{3}$$

$$c = \frac{-8 + 4\sqrt{13}}{3} \in (0,4)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xiii)

Here,

$$f(x) = x\sqrt{x^2 - 4} \text{ on } [2,4]$$

$f(x)$ is continuous at it attains a unique value for each $x \in [2,4]$ and

$$f'(x) = \frac{2x}{2\sqrt{x^2 - 4}}$$

$$f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

$\Rightarrow \quad f'(x)$ exists for each $x \in (2,4)$

$\Rightarrow \quad f(x)$ is differentiable in $(2,4)$, so

Lagrange's mean value theorem is applicable, so there exist a $c \in (2,4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4-2}$$

$$\Rightarrow \quad \frac{c}{\sqrt{c^2 - 4}} = \frac{\sqrt{12} - 0}{2}$$

Squarintg both the sides,

$$\Rightarrow \quad \frac{c^2}{c^2 - 4} = \frac{12}{4}$$

$$\Rightarrow \quad 4c^2 = 12c^2 - 48$$

$$\Rightarrow \quad 8c^2 = 48$$

$$\Rightarrow \quad c^2 = 6$$

$$\Rightarrow \quad c = \sqrt{6} \in (2,4)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xiv)

Here,

$$f(x) = x^2 + x - 1 \text{ on } [0, 4]$$

$f(x)$ is polynomial, so it is continuous is $[0, 4]$ and differentiable in $(0, 4)$ as every polynomial is continuous and differentiable every where. So, Lagrange's mean value theorem is applicable, so there exists a point $c \in [0, 4]$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow \quad 2c + 1 = \frac{\left((4)^2 + 4 - 1\right) - (0 - 1)}{4}$$

$$\Rightarrow \quad 2c + 1 = \frac{19 + 1}{4}$$

$$\Rightarrow \quad 2c + 1 = 5$$

$$\Rightarrow \quad c = 2 \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xv)

Here,

$$f(x) = \sin x - \sin 2x - x \text{ on } [0, \pi]$$

We know that $\sin x$ and polynomial is continuous and differentiable every where so, $f(x)$ is continuous in $[0, \pi]$ and differentiable in $[0, \pi]$. So, Lagrange's mean value theorem is applicable. So, there exist a point $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow \quad \cos c - 2\cos 2c - 1 = \frac{(\sin \pi - \sin 2\pi - \pi) - (0)}{\pi}$$

$$\Rightarrow \quad \cos c - 2\cos 2c = -1 + 1$$

$$\Rightarrow \quad \cos c - 2\left(2\cos^2 c - 1\right) = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c - \frac{-(-1) \pm \sqrt{1 - 4 \times 4 \times (-2)}}{8}$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q1(xvi)

The given function is $f(x) = x^3 - 5x^2 - 3x$, f being a polynomial function, is continuous in [1,3] and is differentiable in [1,3] whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5(1)^2 - 3(1) = -7$$

$$f(3) = 3^3 - 5(3)^2 - 3(3) = 27 - 45 - 9 = -27$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-27 + 7}{2} = -10$$

Mean value theorem states that there is a point $c(1,3)$ such that $f'(c) = 3c^2 - 10c - 3$

$$f'(c) = -10$$

$$3c^2 - 10c - 3 = -10$$

$$3c^2 - 10c + 7 = 0$$

$$3c^2 - 3c - 7c + 7 = 0$$

$$c = \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3)$$

Hence, Mean value theorem is verified for the given function.

Mean Value Theorems Ex 15.2 Q2

Here,

$$f(x) = |x| \text{ on } [-1,1]$$

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

For differentiability at $x = 0$

$$\text{LHD} = \lim_{x \to 0^-} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$\text{LHD} = -1$$

$$\text{RHD} = \lim_{x \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(0+h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow$ $f(x)$ is not differentiable at $x = 0 \in (-1,1)$

Hence, Lagrange's mean value theorem is verified.

Mean Value Theorems Ex 15.2 Q3

Here,

$$f(x) = \frac{1}{x} \text{ on } [-1,1]$$

$$f'(x) = -\frac{1}{x^2}$$

$\Rightarrow$ $f'(x)$ doesnot exist at $x = 0 \in (-1,1)$

$\Rightarrow$ $f(x)$ is not differentiable in $(-1,1)$

Hence, LMVT is verified

Mean Value Theorems Ex 15.2 Q4

Here,

$$f(x) = \frac{1}{4x-1}, x \in [1,4]$$

f(x) attain unique value for each $x \in [1,4]$, so f(x) is continuous in [1, 4].

$$f'(x) = -\frac{4}{(4x-1)^2}$$

$\Rightarrow f'(x)$ exists for each $x \in (1, 4)$

$\Rightarrow f'(x)$ is differentiable in $(1, 4)$

So, Lagranges mean value theroem is applicable.

So, there exist a point $c \in (1, 4)$ such that,

$$f'(c) = \frac{f(4) - f(1)}{4-1}$$

$$\Rightarrow -\frac{4}{(4x-1)^2} = \frac{\frac{1}{15} - \frac{1}{3}}{3}$$

$$\Rightarrow -\frac{4}{(4x-1)^2} = -\frac{4}{45}$$

$$\Rightarrow (4x-1)^2 = 45$$

$$\Rightarrow 4x - 1 = \pm 3\sqrt{5}$$

$$\Rightarrow x = \frac{3\sqrt{5}+1}{4} \in [1,4]$$

Mean Value Theorems Ex 15.2 Q5

Here,

$$\text{curve is } y = (x-4)^2$$

Since, it a polynomial function so it is differentiable and continuous. So, it Lagrange's mean value theorem is applicable, so, there exist a point c such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2(c-4) = \frac{f(5)-f(4)}{5-4}$$

$$\Rightarrow \quad 2c-8 = \frac{1-0}{1}$$

$$\Rightarrow \quad 2c = 9$$

$$\Rightarrow \quad c = \frac{9}{2}$$

$$\Rightarrow \quad y = \left(\frac{9}{2}-4\right)^2$$

$$y = \frac{1}{4}$$

Thus, $(c,y) = \left(\frac{9}{2},\frac{1}{4}\right)$ is required point.

Mean Value Theorems Ex 15.2 Q6

Here,

$$y = x^2 + x$$

Since, y is a polynomial function, so it continuous differentiable,

$\Rightarrow$ Lagrange's mean value theorem is applicable, so, there exist a point c such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2c+1 = \frac{f(1)-f(0)}{1-0}$$

$$\Rightarrow \quad 2c+1 = 2$$

$$\Rightarrow \quad c = \frac{1}{2}$$

$$\Rightarrow \quad y = \left(\frac{1}{2}\right)^2 + \frac{1}{2}$$

$$\Rightarrow \quad y = \frac{3}{4}$$

So, $(c,y) = \left(\frac{1}{2},\frac{3}{4}\right)$ is the required point.

Mean Value Theorems Ex 15.2 Q7

Here,

$$y = (x-3)^2$$

Since, y is a polynomial function, so it continuous differentiable,

$\Rightarrow$ Lagrange's mean value theorem is applicable

$\Rightarrow$ There exist a point c such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 2(c-3) = \frac{f(4)-f(3)}{4-3}$$

$$\Rightarrow \quad 2c-6 = \frac{1-0}{1}$$

$$\Rightarrow \quad 2c = 7$$

$$\Rightarrow \quad c = \frac{7}{2}$$

$$\Rightarrow \quad y = \left(\frac{7}{2}-3\right)^2$$

$$\Rightarrow \quad y = \frac{1}{4}$$

So, $(c,y) = \left(\frac{7}{2},\frac{1}{4}\right)$ is the required point.

Mean Value Theorems Ex 15.2 Q8

Here,

$$y = x^3 - 3x$$

y is a polynomial function, so it is continuous differentiable, so
Lagrange's mean value theorem is applicable thus there exists a point c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 3c^2 - 3 = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow \quad 3c^2 - 3 = \frac{2 + 2}{1}$$

$$\Rightarrow \quad 3c^2 = 7$$

$$\Rightarrow \quad c = \pm\sqrt{\frac{7}{3}}$$

$$\Rightarrow \quad y = \mp\frac{2}{3}\sqrt{\frac{7}{3}}$$

So, $(c, y) = \left(\pm\sqrt{\frac{7}{3}}, \mp\frac{2}{3}\sqrt{\frac{7}{3}}\right)$ is the required point.

Mean Value Theorems Ex 15.2 Q9

Here,

$$y = x^3 + 1$$

It is a polynomial function, so it is continuous differentiable.

$\Rightarrow$ Lagrange's mean value theorem is applicable, so there exists a point c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 3c^2 = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow \quad 3c^2 = \frac{28 - 2}{2}$$

$$\Rightarrow \quad c^2 = \frac{13}{3}$$

$$\Rightarrow \quad c = \sqrt{\frac{13}{3}}$$

$$\Rightarrow \quad y = \left(\frac{13}{3}\right)^{\frac{3}{2}} + 1$$

So, $(c, y) = \left(\sqrt{\frac{13}{3}}, \left(\frac{13}{3}\right)^{\frac{3}{2}} + 1\right)$ is the required point.

Mean Value Theorems Ex 15.2 Q10

Trigonometric functions are continuous and differentiable.

Thus, the curve C is continuous between the points $(a,0)$ and $(0,a)$ and is differentiable on $[a,a]$

Therefore, by Lagrange's Mean Value Theorem, there exists a real number $c \in (a,a)$ such that

$$f'(c) = \frac{a-0}{0-a} = -1$$

Now consider the parametric functions of the given function

$x = a\cos^3\theta$

and

$y = a\sin^3\theta$

$$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

and

$$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3a\sin^2\theta(\cos\theta)}{3a\cos^2\theta(-\sin\theta)}$$

$$\Rightarrow \frac{dy}{dx} = -\tan\theta$$

Slope of the chord joining the points $(a,0)$ and $(0,a)$

=Slope of the tangent at $(c,f(c))$, where c lies on the curve

$$\Rightarrow \frac{a-0}{0-a} = -\tan\theta$$

$$\Rightarrow -1 = -\tan\theta$$

$$\Rightarrow \tan\theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Now substituting $\theta = \frac{\pi}{4}$, in the parametric representations, we have,

$x = a\cos^3\theta, y = a\sin^3\theta$

$$\Rightarrow x = a\cos^3\left(\frac{\pi}{4}\right), y = a\sin^3\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x = \frac{a}{2\sqrt{2}}, y = \frac{a}{2\sqrt{2}}$$

Thus, $P\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is a point on C, where the tangent is parallel to the chord joining the points $(a,0)$ and $(0,a)$.

Mean Value Theorems Ex 15.2 Q11

Consider the function as

$$f(x) = \tan x, \qquad \left\{x \in [a,b] \text{ such that } 0 < a < b < \frac{\pi}{2}\right\}$$

We know that $\tan x$ is continuous and differentiable in $\left(0, \frac{\pi}{2}\right)$, so, Lagrange's mean value theorem is applicable on (a,b), so there exists a point c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad \sec^2 c = \frac{\tan b - \tan a}{b - a} \qquad \text{---(i)}$$

Now,

$$c \in (a,b)$$

$$\Rightarrow \quad a < c < b$$

$$\Rightarrow \quad \sec^2 a < \sec^2 c < \sec^2 b$$

$$\Rightarrow \quad \sec^2 a < \left(\frac{\tan b - \tan a}{b - a}\right) < \sec^2 b$$

Using equation (i),

$$\Rightarrow \quad (b - a)\sec^2 a < (\tan b - \tan a) < (b - a)\sec^2 b$$