Ex 18.1

Maxima and Minima 18.1 Q1

$$f(x) = 4x^2 - 4x + 4 \quad \text{on } R$$
$$= 4x^2 - 4x + 1 + 3$$
$$= (2x-1)^2 + 3$$
$$\because \quad (2x-1)^2 \geq 0$$
$$\Rightarrow \quad (2x-1)^2 + 3 \geq 3$$
$$\Rightarrow \quad f(x) \geq f\left(\frac{1}{2}\right)$$

Thus, the minimum value of $f(x)$ is 3 at $x = \frac{1}{2}$

Since, $f(x)$ can be made as large as we please. Therefore maximum value does not exist

Maxima and Minima 18.1 Q2

The given function is $f(x) = -(x-1)^2 + 2$

It can be observed that $(x-1)^2 \geq 0$ for every $x \in \mathbf{R}$.

Therefore, $f(x) = -(x-1)^2 + 2 \leq 2$ for every $x \in \mathbf{R}$.

The maximum value of f is attained when $(x-1) = 0$.

$(x-1) = 0 \Rightarrow x = 1$

$\therefore$Maximum value of $f = f(1) = -(1-1)^2 + 2 = 2$

Hence, function f does not have a minimum value.

Maxima and Minima 18.1 Q3

$f(x) = |x+2|$ on R

$\because \quad |x+2| \geq 0$ for $x \in R$

$\Rightarrow \quad f(x) \geq 0$ for all $x \in R$

So, the minimum value of $f(x)$ is 0, which attains at $x = -2$

Clearly, $f(x) = |x+2|$ does not have the maximum value.

Maxima and Minima 18.1 Q4

$h(x) = \sin 2x + 5$

We know that $-1 \leq \sin 2x \leq 1$.

$\Rightarrow -1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$

Hence, the maximum and minimum values of h are 6 and 4 respectively.

Maxima and Minima 18.1 Q5

$f(x) = |\sin 4x + 3|$

We know that $-1 \leq \sin 4x \leq 1$.

$\Rightarrow 2 \leq \sin 4x + 3 \leq 4$

$\Rightarrow 2 \leq |\sin 4x + 3| \leq 4$

Hence, the maximum and minimum values of f are 4 and 2 respectively.

Maxima and Minima 18.1 Q6

$f(x) = 2x^3 + 5$ on R

Here, we observe that the values of $f(x)$ increase when the values of x are increased and $f(x)$ can be made as large as possible, we please.

So, $f(x)$ does not have the maximum value.

Similarly $f(x)$ can be made as small as we please by giving smaller values to x.

So, $f(x)$ does not have the minimum value.

Maxima and Minima 18.1 Q7

$g(x) = -|x+1| + 3$

We know that $-|x+1| \leq 0$ for every $x \in \mathbf{R}$.

Therefore, $g(x) = -|x+1| + 3 \leq 3$ for every $x \in \mathbf{R}$.

The maximum value of g is attained when $|x+1| = 0$

$|x+1| = 0$

$\Rightarrow x = -1$

$\therefore$Maximum value of $g = g(-1) = -|-1+1| + 3 = 3$

Hence, function g does not have a minimum value.

Maxima and Minima 18.1 Q8

$$f(x) = 16x^2 - 16x + 28 \text{ on } R$$
$$= 16x^2 - 16x + 4 + 24$$
$$= (4x - 2)^2 + 24$$

Now,

$$(4x - 2)^2 \geq 0 \text{ for all } x \in R$$
$$\Rightarrow \quad (4x - 2)^2 + 24 \geq 24 \text{ for all } x \in R$$
$$\Rightarrow \quad f(x) \geq f\left(\frac{1}{2}\right)$$

Thus, the minimum value of $f(x)$ is 24 at $x = \frac{1}{2}$

Since $f(x)$ can be made as large as possible by giving difference values to x.

Thus, maximum values does not exist.

Maxima and Minima 18.1 Q9

$f(x) = x^3 - 1$ on R

Here, we observe that the values of $f(x)$ increases when the values of x are increased and $f(x)$ can be made as large as we please by giving large values to x.

So, $f(x)$ does not have the maximum value.

Similarly, $f(x)$ can be made as small as we please by giving smaller values to x.

So, $f(x)$ does not have the minimum value.

Ex 18.2

Maxima and Minima Ex 18.2 Q1

$f(x) = (x-5)^4$

$\therefore \quad f'(x) = 4(x-5)^3$

For local maxima and minima

$f'(x) = 0$

$\Rightarrow \quad 4(x-5)^3 = 0$

$\Rightarrow \quad x - 5 = 0$

$\Rightarrow \quad x = 5$

$f'(x)$ changes from $-$ve to $+$ve as passes through 5.

So, $x = 5$ is the point of local minima

Thus, local minimum value is $f(5) = 0$

Maxima and Minima Ex 18.2 Q2

$$g(x) = x^3 - 3x$$

$$\therefore g'(x) = 3x^2 - 3$$

Now,

$$g'(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$$

$$g''(x) = 6x$$

$$g''(1) = 6 > 0$$

$$g''(-1) = -6 < 0$$

By second derivative test, $x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = 1^3 - 3 = 1 - 3 = -2$. However,

$x = -1$ is a point of local maxima and local maximum value of g at

$x = -1$ is $g(1) = (-1)^3 - 3(-1) = -1 + 3 = 2$.

Maxima and Minima Ex 18.2 Q3

$$f(x) = x^3 (x-1)^2$$

$$\begin{aligned}
\therefore \quad f'(x) &= 3x^2 (x-1)^2 + 2x^3 (x-1) \\
&= (x-1)\left(3x^2 (x-1) + 2x^3\right) \\
&= (x-1)\left(3x^3 - 3x^2 + 2x^3\right) \\
&= (x-1)\left(5x^3 - 3x^2\right) \\
&= x^2 (x-1)(5x-3)
\end{aligned}$$

For all maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad x^2 (x-1)(5x-3) = 0$$

$$\Rightarrow \quad x = 0, 1, \frac{3}{5}$$

At $x = \frac{3}{5}$ $f'(x)$ changes from $+ve$ to $-ve$

$\therefore \quad x = \frac{3}{5}$ is point of minima.

At $x = 1$ $f'(x)$ changes from $-ve$ to $+ve$

$\therefore \quad x = 1$ is point of maxima

Maxima and Minima Ex 18.2 Q4

$$f(x) = (x-1)(x+2)^2$$

$$\begin{aligned}
\therefore \quad f'(x) &= (x+2)^2 + 2(x-1)(x+2) \\
&= (x+2)(x+2+2x-2) \\
&= (x+2)(3x)
\end{aligned}$$

For point of maxima and minima

$$f'(x) = 0$$

$$\Rightarrow \quad (x+2) \times 3x = 0$$

$$\Rightarrow \quad x = 0, -2$$

At $x = -2$ $f'(x)$ changes from $+ve$ to $-ve$

$\therefore \quad x = -2$ is point of local maxima

At $x = 0$ $f'(x)$ changes from $-ve$ to $+ve$

$\therefore \quad x = 0$ is point of local minima

Thus, local min value $= f(0) = -4$

local max value $= f(-2) = 0$.

Maxima and Minima Ex 18.2 Q5

$f(x) = (x-1)^3 (x+1)^2$

$\therefore \quad f'(x) = 3(x-1)^2 (x+1)^2 + 2(x-1)^3 (x+1)$

$= (x-1)^2 (x+1)\left(3(x+1) + 2(x-1)\right)$

$= (x-1)^2 (x+1)(5x+1)$

For the point of local maxima and minima,

$f'(x) = 0$

$\Rightarrow \quad (x-1)^2 (x+1)(5x+1) = 0$

$\Rightarrow \quad x = 1, -1, -\frac{1}{5}$

Here,

At $x = -1$ $f'(x)$ changes from $+$ve to $-$ve so $x = -1$ is point of maxima.

At $x = -\frac{1}{5}$, $f'(x)$ changes from $-$ve to $+$ve so $x = -\frac{1}{5}$ is point of minima

Hence, local max value = 0

local min value = $-\frac{3456}{3125}$.

Maxima and Minima Ex 18.2 Q6

$f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore \quad f'(x) = 3x^2 - 12x + 9$

$= 3\left(x^2 - 4x + 3\right)$

$= 3(x-3)(x-1)$

For, the point of local maxima and minima,

$f'(x) = 0$

$\Rightarrow \quad 3(x-3)(x-1) = 0$

$\Rightarrow \quad x = 3, 1$

At $\quad x = -1$, $f'(x)$ changes from $+$ve to $-$ve

$\therefore \quad x = 1$ is point of local maxima

At $\quad x = 3$, $f'(x)$ changes from $-$ve to $+$ve

$\therefore \quad x = 3$ is point of local manima

Hence, local max value = $f(1) = 19$

local min value = $f(3) = 15$.

Maxima and Minima Ex 18.2 Q7

$f(x) = \sin 2x$, $0 < x, \pi$

$\therefore \quad f'(x) = 2\cos 2x$

For, the point of local maxima and minima,

$f'(x) = 0$

$\Rightarrow \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$

At $\quad x = \frac{\pi}{4}$, $f'(x)$ changes from $+$ve to $-$ve

$\therefore \quad x = \frac{\pi}{4}$ is point of local maxima

At $\quad x = \frac{3\pi}{4}$, $f'(x)$ changes from $-$ve to $+$ve

$\therefore \quad x = \frac{3\pi}{4}$ is point of local minima,

Hence, local max value = $f\left(\frac{\pi}{4}\right) = 1$

local min value = $f\left(\frac{3\pi}{4}\right) = -1$.

Maxima and Minima Ex 18.2 Q8

$f(x) = \sin x - \cos x, 0 < x < 2\pi$

$\therefore f'(x) = \cos x + \sin x$

$f'(x) = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$

$f''(x) = -\sin x + \cos x$

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

Therefore, by second derivative test, $x = \frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is

$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$. However, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x = \frac{7\pi}{4}$ is $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

Maxima and Minima Ex 18.2 Q9

$f(x) = \cos x, \ 0 < x < \pi$

$\therefore \quad f'(x) = -\sin x$

For, the point of local maxima and minima,

$f'(x) = 0$

$\Rightarrow \quad -\sin x = 0$

$\Rightarrow \quad x = 0$, and π

But, these two points lies outside the interval $(0, \pi)$

So, no local maxima and minima will exist in the interval $(0, \pi)$.

Maxima and Minima Ex 18.2 Q10

$\therefore \quad f'(x) = 2\cos 2x - 1$

For, the point of local maxima and minima,

$f'(x) = 0$

$\Rightarrow \quad 2\cos 2x - 1 = 0$

$\Rightarrow \quad \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \quad 2x = \frac{\pi}{3}, -\frac{\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{6}, -\frac{\pi}{6}$

At $\quad x = -\frac{\pi}{6}$, $f'(x)$ changes from $-$ve to $+$ ve

$\therefore \quad x = -\frac{\pi}{6}$ is point of local manima

At $\quad x = \frac{\pi}{6}$, $f'(x)$ changes from $+$ve to $-$ve

$\therefore \quad x = \frac{\pi}{6}$ is point of local maxima

Hence, local max value $= f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

local min value $= f\left(-\frac{\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{\pi}{6}$.

Maxima and Minima Ex 18.2 Q11

$f(x) = 2\sin x - x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

For checking the minima and maxima, we have

$f'(x) = 2\cos x - 1 = 0$

$\Rightarrow \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{3}$

At $x = -\frac{\pi}{3}$, *f(x) changes from* $-ve$ *to* $+ve$

$\Rightarrow x = -\frac{\pi}{3}$ *is point of local minima with value* $= -\sqrt{3} - \frac{\pi}{3}$

At $x = \frac{\pi}{3}$, *f(x) changes from* $+ve$ *to* $+ve$

$\Rightarrow x = \frac{\pi}{3}$ *is point of local maxima with value* $= \sqrt{3} - \frac{\pi}{3}$

Maxima and Minima Ex 18.2 Q12

$$\therefore f'(x) = \sqrt{1-x} + x \cdot \frac{1}{2\sqrt{1-x}}(-1) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x) = 0 \Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2-3x = 0 \Rightarrow x = \frac{2}{3}$$

$$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$= \frac{\sqrt{1-x}(-3)+(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$= \frac{-6(1-x)+(2-3x)}{4(1-x)^{\frac{3}{2}}}$$

$$= \frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f''\left(\frac{2}{3}\right) = \frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}} = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$$

Therefore, by second derivative test, $x = \frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x = \frac{2}{3}$ is

$$f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

Maxima and Minima Ex 18.2 Q13

We have,

$$f(x) = x^3(2x-1)^3$$

$$\therefore \quad f'(x) = 3x^2(2x-1)^3 + 3x^3(2x-1)^2 \times 2$$

$$= 3x^2(2x-1)^2(2x-1+2x)$$

$$= 3x^2(4x-1)$$

For, the point of local maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2(4x-1) = 0$$

$$\Rightarrow \quad x = 0, \frac{1}{4}$$

At $\quad x = \frac{1}{4}$, $f'(x)$ changes from $-$ve to $+$ ve

$\therefore \quad x = \frac{1}{4}$ is the point of local minima,

$$\therefore \quad \text{local min value} = f\left(\frac{1}{4}\right) = \frac{-1}{512}.$$

Maxima and Minima Ex 18.2 Q14

We have,

$$f(x) = \frac{x}{2} + \frac{2}{x}, \ x > 0$$

$$\therefore \quad f'(x) = \frac{1}{2} - \frac{2}{x^2}$$

For the point of local maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{1}{2} - \frac{2}{x^2} = 0$$

$$\Rightarrow \quad x^2 - 4 = 0$$

$$\Rightarrow \quad x = \sqrt{4}, -\sqrt{4}$$

$$\Rightarrow \quad x = 2, -2$$

At $\quad x = 2$, $f'(x)$ changes from $-$ve to $+$ ve

$\therefore \quad x = 2$ is point of local minima.

$$\therefore \quad \text{local min value} = f(2) = 2.$$

Maxima and Minima Ex 18.2 Q15

$$g(x) = \frac{1}{x^2+2}$$

$$\therefore g'(x) = \frac{-(2x)}{(x^2+2)^2}$$

$$g'(x) = 0 \Rightarrow \frac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$$

Now, for values close to $x = 0$ and to the left of 0, $g'(x) > 0$. Also, for values close to $x = 0$ and to the right of 0, $g'(x) < 0$.

Therefore, by first derivative test, $x = 0$ is a point of local maxima and the local maximum value of $g(0)$ is $\frac{1}{0+2} = \frac{1}{2}$..

Ex 18.3

Maxima and Minima 18.3 Q1(i)

$f(x) = x^4 - 62x^2 + 120x + 9$

$\therefore \quad f'(x) = 4x^3 - 124x + 120 = 4(x^3 - 31x + 30)$

$\quad f''(x) = 12x^2 - 124 = 4(3x^2 - 31)$

For maxima and minima,

$\quad f'(x) = 0$

$\Rightarrow \quad 4(x^3 - 31x + 30) = 0$

$\Rightarrow \quad 4(x^3 - 31x + 30) = 0$

$\Rightarrow \quad x = 5, 1, -6$

Now,

$\quad f''(5) = 176 > 0$

$\Rightarrow \quad x = 5$ is point of local minima

$\quad f''(1) = -112 < 0$

$\Rightarrow \quad x = 1$ is point of local maxima

$\quad f''(-6) = 308 > 0$

$\Rightarrow \quad x = -6$ is point of local minima

$\therefore \quad$ local max value $= f(1) = 68$

local min value $= f(5) = -316$

and $= f(-6) = -1647$.

Maxima and Minima 18.3 Q1(ii)

We have,

$$f(x) = x^3 - 6x^2 + 9x + 15$$

$$\therefore \quad f'(x) = 3x^2 - 12x + 9$$

$$= 3\left(x^2 - 4x + 3\right)$$

$$f''(x) = 6x - 12$$

$$= 6(x-2)$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad 3\left(x^2 - 4x + 3\right) = 0$$

$$\Rightarrow \quad 3(x-3)(x-1) = 0$$

$$\Rightarrow \quad x = 3, 1$$

Now,

$$f''(3) = 6 > 0$$

$\therefore$ $x = 3$ is point of local minima

$$f''(1) = -6 < 0$$

$\therefore$ $x = 1$ is point of local maxima

$\therefore$ local max value $= f(1) = 19$

local min value $= f(3) = 15.$

Maxima and Minima 18.3 Q1(iii)

We have,

$$f(x) = (x-1)(x+2)^2$$

$$\therefore \quad f'(x) = (x+2)^2 + 2(x-1)(x+2)$$

$$= (x+2)(x+2+2x-2)$$

$$= (x+2)(3x)$$

and, $f''(x) = 3(x+2) + 3x$

$$= 6x + 6$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad 3x(x+2) = 0$$

$$\Rightarrow \quad x = 0, -2$$

Now,

$$f''(0) = 6 > 0$$

$\therefore$ $x = 0$ is point of local minima

$$f''(-2) = -6 < 0$$

$\therefore$ $x = -2$ is point of local maxima

$\therefore$ local max value $= f(-2) = 0$

local min value $= f(0) = -4.$

Maxima and Minima 18.3 Q1(iv)

We have,

$$f(x) = \frac{2}{x} - \frac{2}{x^2},\ x > 0$$

$\therefore \quad f'(x) = \frac{-2}{x^2} + \frac{4}{x^3}$

and, $\quad f''(x) = \frac{+4}{x^3} - \frac{12}{x^4}$

For maxima and minima,

$$f'(x) = 0$$

$\Rightarrow \quad \frac{-2}{x^2} + \frac{4}{x^3} = 0$

$\Rightarrow \quad \frac{-2(x-2)}{x^3} = 0$

$\Rightarrow \quad x = 2$

Now,

$$f''(2) = \frac{4}{8} - \frac{12}{6} = \frac{1}{2} - \frac{3}{4} = \frac{-1}{4} < 0$$

$\therefore \quad x = 2$ is point of local maxima

local max value $= f(2) = \frac{1}{2}$.

Maxima and Minima 18.3 Q1(v)

We have,

$$f(x) = xe^x$$

$\therefore \quad f'(x) = e^x + xe^x = e^x(x+1)$

$$f''(x) = e^x(x+1) + e^x$$
$$= e^x(x+2)$$

For maxima and minima,

$$f'(x) = 0$$

$\Rightarrow \quad e^x(x+1) = 0$

$\Rightarrow \quad x = -1$

Now,

$$f''(-1) = e^{-1} = \frac{1}{e} > 0$$

$\therefore \quad x = -1$ is point of local minima

Hence,

local min value $= f(-1) = \frac{-1}{e}$.

Maxima and Minima 18.3 Q1(vi)

We have,

$$f(x) = \frac{x}{2} + \frac{2}{x},\ x > 0$$

$\therefore \quad f'(x) = \frac{1}{2} - \frac{2}{x^2}$

and, $\quad f''(x) = \frac{4}{x^3}$

For maxima and minima,

$$f'(x) = 0$$

$\Rightarrow \quad \frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow \quad \frac{x^2 - 4}{2x^2} = 0$

$\Rightarrow \quad x = 2, -2$

Now,

$$f''(2) = \frac{1}{2} > 0$$

$\therefore \quad x = 2$ is point of minima

We will not consider $x = -2$ as $x > 0$

$\therefore \quad$ local min value $= f(2) = 2$.

Maxima and Minima 18.3 Q1(vii)

We have,

$$f(x) = (x+1)(x+2)^{\frac{1}{3}},\ x \geq -2$$

$$\therefore \quad f'(x) = (x+2)^{\frac{1}{3}} + \frac{1}{3}(x+1)(x+2)^{\frac{-2}{3}}$$

$$= (x+2)^{\frac{-2}{3}}\left(x+2+\frac{1}{3}(x+1)\right)$$

$$= \frac{1}{3}(x+2)^{\frac{-2}{3}}(4x+7)$$

and, $$f''(x) = -\frac{2}{9}(x+2)^{\frac{-5}{3}}(4x+7) + \frac{1}{3}(x+2)^{\frac{-2}{3}} \times 4$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{1}{3}(x+2)^{\frac{-2}{3}}(4x+7) = 0$$

$$\Rightarrow \quad x = -\frac{7}{4}$$

Now,

$$f''\left(-\frac{7}{4}\right) = \frac{4}{3}\left(-\frac{7}{4}+2\right)^{\frac{-2}{3}}$$

$\therefore \quad x = \frac{-7}{4}$ is point of minima

$\therefore \quad$ local min value $= f\left(\frac{-7}{4}\right) = \frac{-3}{4^{\frac{4}{3}}}$.

Maxima and Minima 18.3 Q1(viii)

We have,

$$f(x) = x\sqrt{32-x^2}, -5 \leq x \leq 5$$

$$\therefore \quad f'(x) = \sqrt{32-x^2} + \frac{x}{2\sqrt{32-x^2}} \times (-2x)$$

$$= \frac{2\left(32-x^2\right) - 2x^2}{2\sqrt{32-x^2}}$$

$$= \frac{64-4x^2}{2\sqrt{32-x^2}}$$

and, $$f''(x) = \frac{2\sqrt{32-x^2} \times (-8x) \frac{-2\left(64-4x^2\right)}{2\sqrt{32-x^2}} \times (-2x)}{4\left(32-x^2\right)}$$

$$= \frac{-4\left(32-x^2\right) \times 8x + 4x\left(64-x^2\right)}{8\left(32-x^2\right)^{\frac{3}{2}}}$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{4\left(16-x^2\right)}{2\sqrt{32-x^2}} = 0$$

$$\Rightarrow \quad x = \pm 4$$

Now,

$$f''(4) = \frac{4 \times 4(64-16-8 \times 32+8 \times 16)}{8(32-16)^{\frac{3}{2}}} < 0$$

$\therefore \quad x = 4$ is point of maxima

Maxima and Minima 18.3 Q1(ix)

$Local\ Maximum\ value\ = f(4)$

$$= 4\sqrt{32-4^2}$$
$$= 4\sqrt{32-16}$$
$$= 4\sqrt{16}$$
$$= 16$$

$Local\ minimum\ at\ x = -4;$

$Local\ Minimum\ value\ = f(-4)$

$$= -4\sqrt{32-(-4)^2}$$
$$= -4\sqrt{32-16}$$
$$= -4\sqrt{16}$$
$$= -16$$

Maxima and Minima 18.3 Q1(x)

$$f(x) = x + \frac{a^2}{x}$$

$\therefore \quad f'(x) = 1 - \frac{a^2}{x^2}$

$$f''(x) = \frac{2a^2}{x^3}$$

For maxima and minima,

$$f'(x) = 0$$

$\Rightarrow \quad 1 - \frac{a^2}{x^2} = 0$

$\Rightarrow \quad x^2 - a^2 = 0$

$\Rightarrow \quad x = \pm a$

Now,

$$f''(a) = \frac{2}{a} > 0 \text{ as } a > 0$$

$\therefore \quad x = a$ is point of minima

$$f''(-a) = \frac{-2}{a} < 0 \text{ as } a > 0$$

$\therefore \quad x = -a$ is point of maxima

Hence,

local max value $= f(-a) = -2a$

local min value $= f(a) = 2a$.

Maxima and Minima 18.3 Q1(xi)

$$f(x) = x\sqrt{2-x^2}$$

$$\therefore \quad f'(x) = \sqrt{2-x^2} - \frac{2x^2}{2\sqrt{2-x^2}}$$

$$= \frac{2\left(2-x^2\right) - 2x^2}{2\sqrt{2-x^2}}$$

$$= \frac{2-2x^2}{\sqrt{2-x^2}}$$

$$f''(x) = \frac{\sqrt{2-x^2}\,(-4x) + \frac{\left(2-2x^2\right)2x}{\sqrt{2-x^2}}}{\left(\sqrt{2-x^2}\right)^2}$$

$$= \frac{-\left(2-x^2\right)4x + 4x - 4x^3}{\left(2-x^2\right)^{\frac{3}{2}}}$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{2\left(1-x^2\right)}{\sqrt{2-x^2}} = 0$$

$$\Rightarrow \quad x = \pm 1$$

Now,

$$f''(1) < 0$$

$\Rightarrow$ $x = 1$ is point of local maxima

$$f''(-1) > 0$$

$\Rightarrow$ $x = -1$ is point of local minima

Hence,

local max value $= f(1) = 1$

local min value $= f(-1) = -1$.

Maxima and Minima 18.3 Q1(xii)

$$f(x) = x + \sqrt{1-x}$$

$$\therefore \quad f'(x) = 1 - \frac{1}{2\sqrt{1-x}} = \frac{2\sqrt{1-x} - 1}{2\sqrt{1-x}}$$

$$\therefore \quad f'(x) = \frac{2\sqrt{1-x}\left(\frac{-1}{\sqrt{1-x}}\right) + \frac{\left(2\sqrt{1-x} - 1\right)}{\sqrt{1-x}}}{4(1-x)}$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{2\sqrt{1-x} - 1}{2\sqrt{1-x}} = 0$$

$$\Rightarrow \quad \sqrt{1-x} = \frac{1}{2}$$

$$\Rightarrow \quad x = 1 - \frac{1}{4} = \frac{3}{4}$$

Now,

$$f''\left(\frac{3}{4}\right) < 0$$

$\Rightarrow$ $x = \frac{3}{4}$ is point of local maxima

Hence,

local max value $= f\left(\frac{3}{4}\right) = \frac{5}{4}$.

Maxima and Minima 18.3 Q2(i)

$$f(x) = (x-1)(x-2)^2$$

$$\therefore \quad f'(x) = (x-2)^2 + 2(x-1)(x-2)$$

$$= (x-2)(x-2+2x-2)$$

$$= (x-2)(3x-4)$$

$$f''(x) = (3x-4) + 3(x-2)$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow \quad (x-2)(3x-4) = 0$$

$$\Rightarrow \quad x = 2, \frac{4}{3}$$

Now,

$$f''(2) > 0$$

$$\therefore \quad x = 2 \text{ is local minima}$$

$$f''\left(\frac{4}{3}\right) = -2 < 0$$

$$\therefore \quad x = \frac{4}{3} \text{ is point of local maxima}$$

$$\therefore \quad \text{local max value} = f\left(\frac{4}{3}\right) = \frac{4}{27}$$

$$\text{local min value} = f(2) = 0.$$

Maxima and Minima 18.3 Q2(ii)

$$f(x) = x\sqrt{1-x}$$

$$\therefore \quad f'(x) = \sqrt{1-x} + \frac{x}{2\sqrt{1-x}}(-1)$$

$$= \frac{2(1-x) - x}{2\sqrt{1-x}}$$

$$= \frac{2-3x}{2\sqrt{1-x}}$$

$$f''(x) = \frac{2\sqrt{1-x}(-3) + \frac{(2-3x)}{\sqrt{1-x}}}{4(1-x)}$$

For maximum and minimum,

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{2-3x}{2\sqrt{1-x}} = 0$$

$$\Rightarrow \quad x = \frac{2}{3}$$

Now,

$$f''\left(\frac{2}{3}\right) < 0$$

$$\therefore \quad x = \frac{2}{3} \text{ is point ofmaxima}$$

$$\therefore \quad \text{local max value} = f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}}.$$

Maxima and Minima 18.3 Q2(iii)

$$f(x) = -(x-1)^3(x+1)^2$$

$\therefore \quad f'(x) = -3(x-1)^2(x+1)^2 - 2(x-1)^3(x+1)$

$$= -(x-1)^2(x+1)(3x+3+2x-2)$$

$$= -(x-1)^2(x+1)(5x+1)$$

$\therefore \quad f''(x) = -2(x-1)(x+1)(5x+1) - (x-1)^2(5x+1) - 5(x-1)^2(x+1)$

For maximum and minimum value,

$$f'(x) = 0$$

$\Rightarrow \quad -(x-1)^2(x+1)(5x+1) = 0$

$\Rightarrow \quad x = 1, -1, -\frac{1}{5}$

Now,

$$f''(1) = 0$$

$\therefore \quad x = 1$ is inflection point

$$f''(-1) = -4 \times -4 = 16 > 0$$

$\therefore \quad x = -1$ is point of minima

$$f''\left(\frac{-1}{5}\right) = -5\left(\frac{36}{25}\right) \times \frac{4}{5} = \frac{-144}{25} < 0$$

$\therefore \quad x = \frac{-1}{5}$ is point of maxima

Hence,

local max value $= f\left(-\frac{1}{5}\right) = \frac{3456}{3125}$

local min value $= f(-1) = 0.$

Maxima and Minima 18.3 Q3

We have,

$$y = a\log x + bx^2 + x$$

$\therefore \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

and $\quad \frac{d^2y}{dx^2} = \frac{-a}{x^2} + 2b$

For maximum and minimum value,

$$\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{a}{x} + 2bx + 1 = 0$

Given that extreme value exist at $x = 1, 2$

$\Rightarrow \quad a + 2b = -1 \qquad$ ---(i)

$$\frac{a}{2} + 4b = -1$$

$\Rightarrow \quad a + 8b = -2 \qquad$ ---(ii)

Solving (i) and (ii), we get

$$a = \frac{-2}{3},\ b = \frac{-1}{6}.$$

Maxima and Minima 18.3 Q4

The given function is $f(x)=\frac{\log x}{x}$.

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

Now, $f'(x)=0$

$\Rightarrow 1-\log x=0$

$\Rightarrow \log x=1$
$\Rightarrow \log x=\log e$
$\Rightarrow x=e$

$$\text{Now, } f''(x)=\frac{x^2\left(-\frac{1}{x}\right)-(1-\log x)(2x)}{x^4}$$
$$=\frac{-x-2x(1-\log x)}{x^4}$$
$$=\frac{-3+2\log x}{x^3}$$
$$\text{Now, } f''(e)=\frac{-3+2\log e}{e^3}=\frac{-3+2}{e^3}=\frac{-1}{e^3}<0$$

Therefore, by second derivative test, f is the maximum at $x=e$.

Maxima and Minima 18.3 Q5

$$f(x)=\frac{4}{x+2}+x$$

$\therefore \quad f'(x)=\frac{-4}{(x+2)^2}+1$

$$f''(x)=\frac{8}{(x+2)^3}$$

For maximum and minimum value,

$f'(x)=0$

$\Rightarrow \quad \frac{-4}{(x+2)^2}+1=0$

$\Rightarrow \quad (x+2)^2=4$

$\Rightarrow \quad x^2+4x=0$

$\Rightarrow \quad x(x+4)=0$

$x=0,-4$

Now,

$f''(0)=1>0$

$\therefore \quad x=0$ is point of minima

$f''(-4)=-1<0$

$\therefore \quad x=-4$ is point of maxima

$\therefore \quad$ local max value $=f(-4)=-6$

local min value $=f(0)=2$.

Maxima and Minima 18.3 Q6

We have,

$$y = \tan x - 2x$$

$\therefore \quad y' = \sec^2 x - 2$

$$y'' = 2\sec^2 x \tan x$$

For maximum and minimum value,

$$y' = 0$$

$\Rightarrow \quad \sec^2 x = 2$

$\Rightarrow \quad \sec x = \pm\sqrt{2}$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore \quad y''\left(\frac{\pi}{4}\right) = 4 > 0$

$\therefore \quad x = \frac{\pi}{4}$ is point of minima

$$y''\left(\frac{3\pi}{4}\right) = -4 < 0$$

$\therefore \quad x = \frac{3\pi}{4}$ is point of maxima

Hence,

$$\text{max value} = f\left(\frac{3\pi}{4}\right) = -1 - \frac{3\pi}{2}$$

$$\text{min value} = f\left(\frac{\pi}{4}\right) = 1 - \frac{\pi}{2}.$$

Maxima and Minima 18.3 Q7

Consider the function

$f(x) = x^3 + ax^2 + bx + c$

Then $f'(x) = 3x^2 + 2ax + b$

It is given that $f(x)$ is maximum at $x = -1$.

$\therefore f'(-1) = 3(-1)^2 + 2a(-1) + b = 0$

$\Rightarrow f'(-1) = 3 - 2a + b = 0 \ldots(1)$

It is given that $f(x)$ is minimum at $x = 3$.

$\therefore f'(3) = 3(3)^2 + 2a(3) + b = 0$

$\Rightarrow f'(3) = 27 + 6a + b = 0 \ldots(2)$

Solving equations (1) and (2), we have,

$a = -3$ and $b = -9$

Since $f'(x)$ is independent of constant c, it can be any real number.

Ex 18.4

Maxima and Minima 18.4 Q1(i)

The given function is $f(x) = 4x - \frac{1}{2}x^2$.

$$\therefore f'(x) = 4 - \frac{1}{2}(2x) = 4 - x$$

Now,

$$f'(x) = 0 \Rightarrow x = 4$$

Then, we evaluate the value of f at critical point $x = 4$ and at the end points of the interval $\left[-2, \frac{9}{2}\right]$.

$$f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$$

$$f(-2) = -8 - \frac{1}{2}(4) = -8 - 2 = -10$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = 18 - 10.125 = 7.875$$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$ and the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is -10 occurring at $x = -2$.

Maxima and Minima 18.4 Q1(ii)

The given function is $f(x)=(x-1)^2+3$.

$\therefore f'(x)=2(x-1)$

Now,

$f'(x)=0 \Rightarrow 2(x-1)=0 \Rightarrow x=1$

Then, we evaluate the value of f at critical point $x = 1$ and at the end points of the interval [−3, 1].

$$f(1)=(1-1)^2+3=0+3=3$$
$$f(-3)=(-3-1)^2+3=16+3=19$$

Hence, we can conclude that the absolute maximum value of f on [−3, 1] is 19 occurring at $x = -3$ and the minimum value of f on [−3, 1] is 3 occurring at $x = 1$.

Maxima and Minima 18.4 Q1(iii)

Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

$$\begin{aligned}\therefore f'(x)&=12x^3-24x^2+24x-48\\&=12(x^3-2x^2+2x-4)\\&=12[x^2(x-2)+2(x-2)]\\&=12(x-2)(x^2+2)\end{aligned}$$

Now, $f'(x)=0$ gives $x = 2$ or $x^2 + 2 = 0$ for which there are no real roots.

Therefore, we consider only $x = 2 \in [0, 3]$.

Now, we evaluate the value of f at critical point $x = 2$ and at the end points of the interval [0, 3].

$$\begin{aligned}f(2)&=3(16)-8(8)+12(4)-48(2)+25\\&=48-64+48-96+25\\&=-39\\f(0)&=3(0)-8(0)+12(0)-48(0)+25\\&=25\\f(3)&=3(81)-8(27)+12(9)-48(3)+25\\&=243-216+108-144+25=16\end{aligned}$$

Hence, we can conclude that the absolute maximum value of f on [0, 3] is 25 occurring at $x = 0$ and the absolute minimum value of f at [0, 3] is − 39 occurring at $x = 2$.

Maxima and Minima 18.4 Q1(iv)

$f(x) = (x-2)\sqrt{x-1}$

$\Rightarrow \quad f'(x) = \sqrt{x-1} + (x-2)\dfrac{1}{2\sqrt{x-1}}$

Put $f'(x) = 0$

$\Rightarrow \quad \sqrt{x-1} + \dfrac{x-2}{2\sqrt{x-1}} = 0$

$\Rightarrow \quad \dfrac{2(x-1)+(x-2)}{2\sqrt{x-1}} = 0$

$\Rightarrow \quad \dfrac{3x-4}{2\sqrt{x-1}} = 0$

$\Rightarrow \quad x = \dfrac{4}{3}$

Now,

$f(1) = 0$

$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}-2\right)\sqrt{\frac{4}{3}-1} = \frac{4-6}{3\sqrt{3}} = \frac{-2}{3\sqrt{3}} = \frac{-2\sqrt{3}}{9}$

$f(9) = (9-2)\sqrt{9-1} = 7\sqrt{8} = 14\sqrt{2}$

$\therefore$ The absolute maximum value of $f(x)$ is $14\sqrt{2}$ at $x = 9$ and the absolute minimum value is $\dfrac{-2\sqrt{3}}{9}$ at $x = \dfrac{4}{3}$.

Maxima and Minima 18.4 Q2

Let $f(x) = 2x^3 - 24x + 107$.

$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

Now,

$f'(x) = 0 \Rightarrow 6(x^2-4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

We first consider the interval [1, 3].

Then, we evaluate the value of f at the critical point $x = 2 \in [1, 3]$ and at the end points of the interval [1, 3].

$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$

$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$

$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$

Hence, the absolute maximum value of $f(x)$ in the interval [1, 3] is 89 occurring at $x = 3$.

Next, we consider the interval $[-3, -1]$.

Evaluate the value of f at the critical point $x = -2 \in [-3, -1]$ and at the end points of the interval [1, 3].

$f(-3) = 2(-27) - 24(-3) + 107 = -54 + 72 + 107 = 125$

Maxima and Minima 18.4 Q3

$$f(x)=\cos^2 x+\sin x$$
$$f'(x)=2\cos x(-\sin x)+\cos x$$
$$=-2\sin x\cos x+\cos x$$
Now, $f'(x)=0$
$$\Rightarrow 2\sin x\cos x=\cos x \Rightarrow \cos x(2\sin x-1)=0$$
$$\Rightarrow \sin x=\frac{1}{2} \text{ or } \cos x=0$$
$$\Rightarrow x=\frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x\in[0,\pi]$$

Now, evaluating the value of f at critical points $x=\frac{\pi}{2}$ and $x=\frac{\pi}{6}$ and at the end points of the interval $[0,\pi]$ (i.e., at $x=0$ and $x=\pi$), we have:

$$f\left(\frac{\pi}{6}\right)=\cos^2\frac{\pi}{6}+\sin\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{5}{4}$$
$$f(0)=\cos^2 0+\sin 0=1+0=1$$
$$f(\pi)=\cos^2\pi+\sin\pi=(-1)^2+0=1$$
$$f\left(\frac{\pi}{2}\right)=\cos^2\frac{\pi}{2}+\sin\frac{\pi}{2}=0+1=1$$

Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at $x=\frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x=0,\frac{\pi}{2},$ and π.

Maxima and Minima 18.4 Q4

We have

$$f(x)=12x^{\frac{4}{3}}-6x^{\frac{1}{3}}$$
$$\therefore \qquad f'(x)=16x^{\frac{1}{3}}-\frac{2}{x^{\frac{2}{3}}}=\frac{2(8x-1)}{x^{\frac{2}{3}}}$$

Thus, $f'(x)=0$

$$\Rightarrow \qquad x=\frac{1}{8}$$

Further note that $f'(x)$ is not defined at $x=0$.

So, the critical points are $x=0$ and $x=\frac{1}{8}$.

Evaluating the value of f at critical points $x=0,\frac{1}{8}$ and at end points of the interval $x=-1$ and $x=1$

$$f(-1)=12(-1)^{4/3}-6(-1)^{1/3}=18$$
$$f(0)=12(0)-6(0)=0$$
$$f\left(\frac{1}{8}\right)=12\left(\frac{1}{8}\right)^{4/3}-6\left(\frac{1}{8}\right)^{1/3}=\frac{-9}{4}$$
$$f(1)=12(1)^{4/3}-6(1)^{1/3}=6$$

Hence we conclude that absolute maximum value of f is 18 at $x=-1$ and absolute minimum value of f is $\frac{-9}{4}$ at $x=\frac{1}{8}$.

Maxima and Minima 18.4 Q5

Given,

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$\therefore \quad f'(x) = 6x^2 - 30x + 36 = 6\left(x^2 - 5x + 6\right) = 6(x-2)(x-3)$$

Note that $f'(x) = 0$ gives $x = 2$ and $x = 3$

We shall now evaluate the value of f at these points

and at the end points of the interval [1,5],

i.e at x= 1, 2, 3 and 5

At $x = 1$, $f(1) = 2\left(1^3\right) - 15\left(1^2\right) + 36(1) + 1 = 24$

At $x = 2$, $f(2) = 2\left(2^3\right) - 15\left(2^2\right) + 36(2) + 1 = 29$

At $x = 3$, $f(3) = 2\left(3^3\right) - 15\left(3^2\right) + 36(3) + 1 = 28$

At $x = 5$, $f(5) = 2\left(5^3\right) - 15\left(5^2\right) + 36(5) + 1 = 56$

Thus we conclude that the absolute maximum value of f on [1,5] is 56, occurring at x=5, and absolute minimum value of f on [1,5] is 24 which occurs at x=1.

Ex 18.5

Maxima and Minima 18.5 Q1

Let x and y be the two numbers.

Given that $x + y = 16$ ---(i)

$Let \quad s = x^2 + y^2$ ---(ii)

From (i) and (ii)

$$S = x^2 + (15 - x)^2$$

$$\therefore \quad \frac{ds}{dx} = 2x + 2(15 - x)(-1)$$
$$= 2x - 30 + 2x$$
$$= 4x - 30$$

Now, $\frac{ds}{dx} = 0$

$\Rightarrow \quad 4x - 30 = 0$

$\Rightarrow \quad x = \frac{15}{2}$

Since,

$$\frac{d^2s}{dx^2} = 4 > 0$$

$\therefore \quad x = \frac{15}{2}$ is the point of local minima.

So, from (i)

$$y = 15 - \frac{15}{2} = \frac{15}{2}$$

Hence, the required numbers are $\frac{15}{2}, \frac{15}{2}$.

Maxima and Minima 18.5 Q2

Let x and y be the two parts of 64.

$\therefore \quad x + y = 64 \qquad$ ---(i)

Let $\quad S = x^3 + y^3 \qquad$ ---(ii)

From (i) and (ii), we get

$$S = x^3 + (64 - x)^3$$

$$\therefore \quad \frac{dS}{dx} = 3x^2 + 3(64 - x)^2 \times (-1)$$

$$= 3x^2 - 3(4096 - 128x + x^2)$$

$$= -3(4096 - 128x)$$

For maxima and minima,

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad -3(4096 - 128x) = 0$$

$$\Rightarrow \quad x = 32$$

Now,

$$\frac{d^2s}{dx^2} = 384 > 0$$

$\therefore \quad x = 32$ is the point of local minima.

Thus, the two parts of 64 are $(32, 32)$.

Maxima and Minima 18.5 Q3

Let x and y be the two numbers, such that, $x, y \geq -2$ and

$$x + y = \frac{1}{2} \qquad \text{---(i)}$$

Let $S = x + y^3$ ---(ii)

From (i) and (ii), weget

$$S = x + \left(\frac{1}{2} - x\right)^3$$

$$\therefore \quad \frac{dS}{dx} = 1 + 3\left(\frac{1}{2} - x\right)^2 \times (-1)$$

$$= 1 - 3\left(\frac{1}{4} - x + x^2\right)$$

$$= \frac{1}{4} + 3x - 3x^2$$

For maximum and minimum,

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{4} + 3x - 3x^2 = 0$$

$$\Rightarrow \quad 1 + 12x - 12x^2 = 0$$

$$\Rightarrow \quad 12x^2 - 12x - 1 = 0$$

$$\Rightarrow \quad x = \frac{12 \pm \sqrt{144 + 48}}{24}$$

$$\Rightarrow \quad x = \frac{1}{2} \pm \frac{8\sqrt{3}}{24}$$

$$\Rightarrow \quad x = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad x = \frac{1}{2} - \frac{1}{\sqrt{3}}, \frac{1}{2} + \frac{1}{\sqrt{3}}$$

Now,

$$\frac{d^2S}{dx^2} = 3 - 6x$$

At $x = \frac{1}{2} - \frac{1}{\sqrt{3}}$, $\frac{d^2S}{dx^2} = 3\left(1 - 2\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\right)$

$$= 3\left(+\frac{2}{\sqrt{3}}\right) = 2\sqrt{3} > 0$$

$\therefore$ $x = \frac{1}{2} - \frac{1}{\sqrt{3}}$ is point of local minima

$\therefore$ from (i)

$$y = \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

Hence, the required numbers are $\frac{1}{2} - \frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$.

Maxima and Minima 18.5 Q4

Let x and y be the two parts of 15, such that

$\therefore \quad x + y = 15 \qquad$ ---(i)

Also, $S = x^2y^3 \qquad$ ---(ii)

From (i) and (ii), weget

$$S = x^2(15 - x)^3$$

$$\therefore \frac{dS}{dx} = 2x(15 - x)^3 - 3x^2(15 - x)^2$$

$$= (15 - x)^2\left[30x - 2x^2 - 3x^2\right]$$

$$= 5x(15 - x)^2(6 - x)$$

For maxima and minima,

$$\frac{dS}{dx} = 0$$

$$\Rightarrow 5x(15 - x)^2(6 - x) = 0$$

$$\Rightarrow x = 0,\ 15,\ 6$$

Now,

$$\frac{d^2S}{dx^2} = 5(15 - x)^2(6 - x) - 5x \times 2(15 - x)(6 - x) - 5x(15 - x)^2$$

$\therefore$ At $x = 0$, $\frac{dS^2}{dx^2} = 1125 > 0$

$\therefore$ $x = 0$ is point of local minima

At $x = 15$, $\frac{d^2s}{dx^2} = 0$

$\therefore$ $x = 15$ is an inflection point.

At $x = 6$, $\frac{ds^2}{dx^2} = -2430 < 0$

$\therefore$ $x = 6$ is the point of local maxima

Thus the numbers are 6 and 9.

Maxima and Minima 18.5 Q5

Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 100 \qquad \text{(given)}$$

$$\therefore h = \frac{100}{\pi r^2}$$

Surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2},\ \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0$.

$\therefore$By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

$$\text{When } r=\left(\frac{50}{\pi}\right)^{\frac{1}{3}},\ h=\frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}}=\frac{2\times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}}=2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm.}$$

Hence, the required dimensions of the can which has the minimum surface area is given by radius $=\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $=2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Maxima and Minima 18.5 Q6

We are given that the bending moment M at a distance x from one end of the beam is given by

(i) $M=\frac{WL}{2}x-\frac{W}{2}x^2$

$\therefore \quad \frac{dM}{dx}=\frac{WL}{2}-Wx$

For maxima and minima,

$$\frac{dM}{dx}=0 \Rightarrow \quad \frac{WL}{2}-Wx=0 \Rightarrow \quad x=\frac{L}{2}$$

Now,

$$\frac{d^2M}{dx^2}=-W<0$$

$\therefore \quad x=\frac{L}{2}$ is point of local maxima.

(ii) $M=\frac{Wx}{3}-\frac{Wx^3}{3L^2}$

$\therefore \quad \frac{dM}{dx}=\frac{W}{3}-\frac{Wx^2}{L^2}$

For maxima and minima,

$$\frac{dM}{dx}=0 \Rightarrow \quad \frac{W}{3}-\frac{Wx^2}{L^2}=0 \Rightarrow \quad x=\frac{L}{\sqrt{3}}$$

Now,

$$\frac{d^2M}{dx^2}=-\frac{2xW}{L^2}$$

At $\quad x=\frac{L}{\sqrt{3}},\ \frac{d^2M}{dx^2}=-\frac{2W}{\sqrt{3}L}<0$

$\therefore \quad x=\frac{L}{\sqrt{3}}$ is point of local maxima

$$\Rightarrow \quad \frac{d^2s}{dx^2}=-\frac{\sqrt{2}r}{\frac{r^2}{2}}$$

$$=\frac{2\sqrt{2}}{r}<0$$

$\therefore \quad x=\frac{r}{\sqrt{2}}$ is the point of local maxima

From (i)

$$y=\frac{r}{\sqrt{2}}$$

Hence, $x=\frac{r}{\sqrt{2}}, y=\frac{r}{\sqrt{2}}$ is the required number.

Maxima and Minima 18.5 Q7

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28 - l)$ m.

Now, side of square $= \frac{l}{4}$.

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28-l)$.

The combined areas of the square and the circle (A) is given by,

$$A = (\text{side of the square})^2 + r^2$$
$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$
$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$
$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$$
$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$
$$\text{Now, } \frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28-l) = 0$$

$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$
$$\Rightarrow (\pi+4)l - 112 = 0$$
$$\Rightarrow l = \frac{112}{\pi+4}$$

Thus, when $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0$.

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm.

Maxima and Minima 18.5 Q8

Let the wire of length 20 m be cut into x cm and y cm and bent into a square and equilateral triangle, so that the sum of area of square and triangle is minimum.

Now,

$$x + y = 20 \quad \text{---(i)}$$
$$x = 4l \text{ and } y = 3a$$

Let s = sum of area of square and triangle

$$s = l^2 + \frac{\sqrt{3}}{4}a^2 \quad \text{---(ii)}$$

$\left[\because \text{area of equilateral } \Delta = \frac{\sqrt{3}}{4}(\text{one side})^2\right]$

We have, $4l + 3a = 20$

$\Rightarrow 4l = 20 - 3a$

$\Rightarrow l = \dfrac{20-3a}{4}$

From (*i*), we have,

$$s = \left(\frac{20-3a}{4}\right)^2 + \frac{\sqrt{3}}{4}a^2$$

$$\frac{ds}{da} = 2\left(\frac{20-3a}{4}\right)\left(\frac{-3}{4}\right) + 2a \times \frac{\sqrt{3}}{4}$$

To find the maximum or minimum, $\dfrac{ds}{da} = 0$

$$\Rightarrow 2\left(\frac{20-3a}{4}\right)\left(\frac{-3}{4}\right) + 2a \times \frac{\sqrt{3}}{4} = 0$$

$\Rightarrow -3(20-3a) + 4a\sqrt{3} = 0$

$\Rightarrow -60 + 9a + 4a\sqrt{3} = 0$

$\Rightarrow 9a + 4a\sqrt{3} = 60$

$\Rightarrow a\left(9 + 4\sqrt{3}\right) = 60$

$\Rightarrow a = \dfrac{60}{9+4\sqrt{3}}$

Differentiating once again, we have,

$$\frac{d^2s}{da^2} = \frac{9+4\sqrt{3}}{8} > 0$$

Thus, the sum of the areas of the square and triangle is minimum when $a = \dfrac{60}{9+4\sqrt{3}}$

We know that, $l = \dfrac{20-3a}{4}$

$$\Rightarrow l = \frac{20 - 3\left(\frac{60}{9+4\sqrt{3}}\right)}{4}$$

$$\Rightarrow l = \frac{180 + 80\sqrt{3} - 180}{4\left(9+4\sqrt{3}\right)}$$

$$\Rightarrow l = \frac{20\sqrt{3}}{9+4\sqrt{3}}$$

Maxima and Minima 18.5 Q9

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$2\pi r + 4a = k$ (where k is constant)

$\Rightarrow a = \dfrac{k - 2\pi r}{4}$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k-2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k-2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k-2\pi r)}{4}$$

Now, $\dfrac{dA}{dr} = 0$

$\Rightarrow 2\pi r = \dfrac{\pi(k-2\pi r)}{4}$

$8r = k - 2\pi r$

$$\Rightarrow (8+2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8+2\pi} = \frac{k}{2(4+\pi)}$$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

$\therefore$ When $r = \frac{k}{2(4+\pi)}$, $\frac{d^2A}{dr^2} > 0.$

$\therefore$ The sum of the areas is least when $r = \frac{k}{2(4+\pi)}$.

When $r = \frac{k}{2(4+\pi)}$, $a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{k(4+\pi)-\pi k}{4\ 4(4+\pi)} = \frac{4k}{4\ 4(4+\pi)} = \frac{k}{4+\pi} = 2r.$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

Maxima and Minima 18.5 Q10

ABC is a right angled triangle. Hypotenuse $h = AC = 5$ cm.

Let x and y one the other two side of the triangle.

$\therefore \quad x^2 + y^2 = 25 \qquad$ ---(i)

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}BC \times AB$

$\Rightarrow \quad S = \frac{1}{2}xy \qquad$ ---(ii)

From (i) and (ii)

$$S = \frac{1}{2}x\sqrt{25-x}$$

$$\therefore \quad \frac{ds}{dx} = \frac{1}{2}\left[\sqrt{25-x^2} - \frac{2x^2}{2\sqrt{25-x^2}}\right]$$

$$= \frac{1}{2}\frac{\left[25-x^2-x^2\right]}{\sqrt{25-x^2}}$$

$$= \frac{1}{2}\left[\frac{25-2x^2}{\sqrt{25-x^2}}\right]$$

For maxima and minima,

$$\frac{ds}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{25-2x^2}{\sqrt{25-x^2}}\right] = 0$$

$$\Rightarrow \quad x = 5\sqrt{2}$$

Now,

$$\frac{d^2s}{dx^2} = \frac{1}{2}\frac{\sqrt{25-x^2}\times(-4x) + \frac{(25-2x^2)2x}{2\sqrt{25-x^2}}}{(25-x^2)}$$

At $\quad x = \frac{5}{\sqrt{2}}$, $\frac{d^2s}{dx^2} = \frac{1}{2}\frac{\left[-\frac{25}{\sqrt{2}}\times\frac{5}{\sqrt{2}}+0\right]}{\frac{25}{2}}$

$$= -\frac{5}{2} < 0$$

$\therefore \quad x = \frac{5}{\sqrt{2}}$ is a point local maxima,

Maxima and Minima 18.5 Q11

ABC is a given triangle with $AB = a, BC = b$ and $\angle ABC = \theta$.
AD in perpendicular to BC.

$\therefore \quad BD = a\sin\theta$

Now,

$$\text{Area of } \Delta ABC = \frac{1}{2} \times BC \times AD$$

$$\Rightarrow \quad A = \frac{1}{2} b \times a\sin\theta \qquad \text{---(i)}$$

$$\therefore \quad \frac{dA}{d\theta} = \frac{1}{2} ab\cos\theta$$

For maxima and minima,

$$\frac{dA}{d\theta} = 0$$

$$\Rightarrow \quad \frac{1}{2} ab\cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

Now,

$$\frac{d^2A}{d\theta^2} = -\frac{1}{2} ab\sin\theta$$

$$\text{At} \quad \theta = \frac{\pi}{2}, \quad \frac{d^2A}{d\theta^2} = -\frac{1}{2} ab < 0$$

$$\therefore \quad \theta = \frac{\pi}{2} \text{ is point of local maxima}$$

$$\therefore \quad \text{Maximum area of } \Delta = \frac{1}{2} ab\sin\frac{\pi}{2} = \frac{1}{2} ab.$$

Maxima and Minima 18.5 Q12

Let the side of the square to be cut off be x cm. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Therefore, the volume $V(x)$ of the box is given by,

$$V(x) = x(18 - 2x)^2$$

$$\begin{aligned}
\therefore V'(x) &= (18-2x)^2 - 4x(18-2x) \\
&= (18-2x)[18-2x-4x] \\
&= (18-2x)(18-6x) \\
&= 6\times 2(9-x)(3-x) \\
&= 12(9-x)(3-x)
\end{aligned}$$

$$\begin{aligned}
\text{And, } V''(x) &= 12[-(9-x)-(3-x)] \\
&= -12(9-x+3-x) \\
&= -12(12-2x) \\
&= -24(6-x)
\end{aligned}$$

$$\textit{Maximum volume is } V_{x=3} = 3 \times (18 - 2 \times 3)^2$$

$$\Rightarrow V = 3 \times 12^2$$

$$\Rightarrow V = 3 \times 144$$

$$\Rightarrow V = 432\ cm^3$$

Maxima and Minima 18.5 Q13

Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is $45 - 2x$, and the breadth is $24 - 2x$.

Therefore, the volume $V(x)$ of the box is given by,

$$\begin{aligned} V(x) &= x(45-2x)(24-2x) \\ &= x\left(1080-90x-48x+4x^2\right) \\ &= 4x^3-138x^2+1080x \end{aligned}$$
$$\begin{aligned} \therefore V'(x) &= 12x^2-276x+1080 \\ &= 12\left(x^2-23x+90\right) \\ &= 12(x-18)(x-5) \end{aligned}$$
$$V''(x) = 24x-276 = 12(2x-23)$$

Now, $V'(x) = 0 \Rightarrow x = 18$ and $x = 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

$\therefore x = 5$

Now, $V''(5) = 12(10-23) = 12(-13) = -156 < 0$

$\therefore$ By second derivative test, $x = 5$ is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

Maxima and Minima 18.5 Q14

Let l, b, and h represent the length, breadth, and height of the tank respectively.

Then, we have height $(h) = 2$ m

Volume of the tank $= 8\text{m}^3$

Volume of the tank $= l \times b \times h$

$\therefore 8 = l \times b \times 2$

$\Rightarrow lb = 4 \Rightarrow b = \dfrac{4}{l}$

Now, area of the base $= lb = 4$

Area of the 4 walls $(A) = 2h\,(l + b)$

$$\therefore A = 4\left(l+\frac{4}{l}\right)$$
$$\Rightarrow \frac{dA}{dl} = 4\left(1-\frac{4}{l^2}\right)$$

Now, $\dfrac{dA}{dl} = 0$

$$\Rightarrow 1-\frac{4}{l^2} = 0$$
$$\Rightarrow l^2 = 4$$

$\Rightarrow l = \pm 2$

However, the length cannot be negative.

Therefore, we have $l = 4$.

$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$

Now, $\frac{d^2A}{dl^2} = \frac{32}{l^3}$

When $l = 2$, $\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$.

Thus, by second derivative test, the area is the minimum when $l = 2$.

We have $l = b = h = 2$.

$\therefore$Cost of building the base = Rs 70 × (lb) = Rs 70 (4) = Rs 280

Cost of building the walls = Rs $2h$ ($l + b$) × 45 = Rs 90 (2) (2 + 2)

= Rs 8 (90) = Rs 720

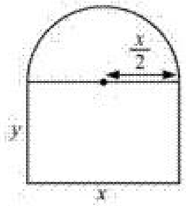
Required total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

Maxima and Minima 18.5 Q15

Maxima and Minima 18.5 Q15

Radius of the semicircular opening $= \dfrac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1+\frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1+\frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

$\therefore$Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx}=0$

$$\Rightarrow 5-x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x=0$$

$$\Rightarrow 5-x-\frac{\pi}{4}x=0$$

$$\Rightarrow x\left(1+\frac{\pi}{4}\right)=5$$

$$\Rightarrow x=\frac{5}{\left(1+\frac{\pi}{4}\right)}=\frac{20}{\pi+4}$$

Thus, when $x=\frac{20}{\pi+4}$ then $\frac{d^2A}{dx^2}<0$.

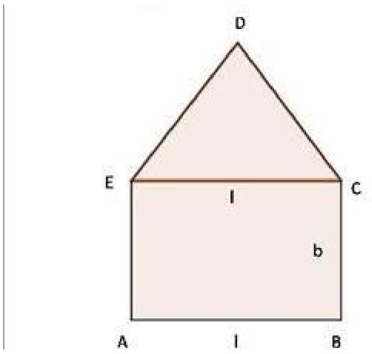
Therefore, by second derivative test, the area is the maximum when length $x=\frac{20}{\pi+4}$ m.

Now,

$$y=5-\frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right)=5-\frac{5(2+\pi)}{\pi+4}=\frac{10}{\pi+4}\text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by length $=\frac{20}{\pi+4}$ m and breadth $=\frac{10}{\pi+4}$ m.

Maxima and Minima 18.5 Q16



The perimeter of the window = 12 m

$\Rightarrow$ (l + 2b) + (l + l) = 12

$\Rightarrow$ 3l + 2b = 12 ----- (i)

Let S = Area of the rectangle + Area of the equilateral Δ

From (i),

$$S = l\left(\frac{12-3l}{2}\right) + \frac{\sqrt{3}}{4}l^2$$

$$\therefore \quad \frac{dS}{dl} = 6 - 3l + \frac{\sqrt{3}}{2}l = 6 - \sqrt{3}\left(\sqrt{3} - \frac{1}{2}\right)l$$

For maxima and minima,

$$\frac{dS}{dl} = 0$$

$$\Rightarrow \quad 6 - \sqrt{3}\left(\sqrt{3} - \frac{1}{2}\right)l = 0$$

$$\Rightarrow \quad l = \frac{6}{\sqrt{3}\left(\sqrt{3} - \frac{1}{2}\right)} = \frac{12}{6 - \sqrt{3}}$$

Now, $\frac{d^2S}{dl^2} = -\sqrt{3}\left(\sqrt{3} - \frac{1}{2}\right) = -3 + \frac{\sqrt{3}}{2} < 0$

$\therefore$ $l = \frac{12}{6 - \sqrt{3}}$ is the point of local maxima

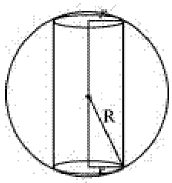
From (i),

$$b = \frac{12 - 3l}{2} = \frac{12 - 3\left(\frac{12}{6-\sqrt{3}}\right)}{2} = \frac{24 - 6\sqrt{3}}{6 - \sqrt{3}}$$

Maxima and Minima 18.5 Q17

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2\sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r\left(R^2 - r^2\right) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\text{Now, } \frac{d^2V}{dr^2} = \frac{\sqrt{R^2-r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2-r^2}}}{\left(R^2-r^2\right)}$$

$$= \frac{\left(R^2-r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2-r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2-r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$.

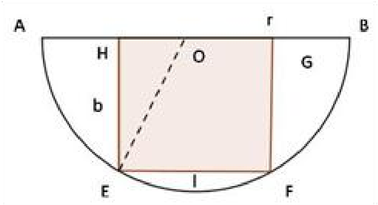
$\therefore$The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Maxima and Minima 18.5 Q18

Let $EFGH$ be a rectangle inscribed in a semi-circle with radius r.



Let l and b are the length and width of rectangle.
In ΔOHE

$$HE^2 = OE^2 - OH^2$$

$$\Rightarrow \quad HE = b = \sqrt{r^2 - \left(\frac{r}{2}\right)^2} \qquad \text{---(i)}$$

Let S = Area of rectangle

$$= lb = l \times \sqrt{r^2 - \left(\frac{l}{2}\right)^2}$$

$$\therefore \quad S = \frac{1}{2} l \sqrt{4r^2 - l^2}$$

$$\therefore \quad \frac{ds}{dl} = \frac{1}{2}\left[\sqrt{4r^2 - l^2} - \frac{l^2}{\sqrt{4r^2 - l^2}}\right]$$

$$= \frac{1}{2}\left[\frac{4r^2 - l^2 - l^2}{\sqrt{4r^2 - l^2}}\right]$$

$$= \frac{2r^2 - l^2}{\sqrt{4r^2 - l^2}}$$

For maxima and minima,

$$\frac{ds}{dl = 0}$$

$$\Rightarrow \quad \frac{2r^2 - l^2}{\sqrt{4r^2 - l^2}} = 0$$

$$\Rightarrow \quad l = \pm\sqrt{2}r$$

Also,

$$\frac{d^2s}{dl^2} = 0 \text{ at } l = \sqrt{2}r$$

So, the dimension of the rectangle

$$l = \sqrt{2}r,\ b = \sqrt{r^2 - \left(\frac{l}{2}\right)^2} = \frac{r}{\sqrt{2}}$$

Area of rectangle = $lb = \sqrt{2}r \times \dfrac{r}{\sqrt{2}}$

$$= r^2.$$

Maxima and Minima 18.5 Q19

Let r and h be the radius and the height (altitude) of the cone respectively.

Then, the volume (V) of the cone is given as:

$$V = \frac{1}{3\pi}\pi r^2 h \Rightarrow h = \frac{3V}{r^2}$$

The surface area (S) of the cone is given by,

$S = \pi rl$ (where l is the slant height)

$$= \pi r\sqrt{r^2 + h^2}$$

$$= \pi r\sqrt{r^2 + \frac{9\cancel{\pi}^2}{\pi^2 r^4}} \overset{\pi}{=} \frac{r\sqrt{9r^6 + V^2}}{\pi r^2}$$

$$= \frac{1}{r}\sqrt{\pi^2 r^6 + 9V^2}$$

$$\therefore \frac{dS}{dr} = \frac{r \cdot \frac{6\pi^2 r^5}{2\cancel{\pi}\sqrt{\ {}^2 r^6\ 9\ \ V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2}$$

$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} > 0.$

$\therefore$ By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}$.

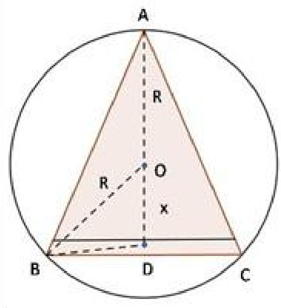
When $r^6 = \frac{9V^2}{2\pi^2},\ h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2}\cdot\frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r.$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

Maxima and Minima 18.5 Q20

We have a cone, which is inscribed in a sphere.

Let v be the volume of greatest cone ABC. If is obvious that, for maximum volume the axis of the cone must be along the diameter of sphere.



Let $OD = x$ and $AO = OB = R$

$\Rightarrow \quad BD = \sqrt{R^2 - x^2}$ and $AD = R + x$

Now,

$$v = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi BD^2 \times AD$$
$$= \frac{1}{3}\pi\left(R^2 - x^2\right) \times (R + x)$$

$$\therefore \quad \frac{dv}{dx} = \frac{\pi}{3}\left[-2x(R + x) + R^2 - x^2\right]$$
$$= \frac{\pi}{3}\left[R^2 - 2xR - 3x^2\right]$$

For maximum and minimum

$$\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{\pi}{3}\left[R^2 - 2xR - 3x^2\right] = 0$$

$$\Rightarrow \quad \frac{\pi}{3}\left[(R - 3x)(R + x)\right] = 0$$

$$\Rightarrow \quad R - 3x = 0 \text{ or } x = -R$$

$$\Rightarrow \quad x = \frac{R}{3} \qquad \left[\because x = -R \text{ is not possible as, } x = -R \text{ will make the altitude } 0\right]$$

Now,

$$\frac{d^2v}{dx^2} = \frac{\pi}{3}\left[-2R - 6x\right]$$

At $\quad x = \frac{R}{3}, \quad \frac{d^2v}{dx^2} = \frac{\pi}{3}\left[-2R - 2R\right]$

$$= \frac{-4\pi R}{3} < 0$$

$\therefore \quad x = \frac{R}{3}$ is the point of local maxima.

Maxima and Minima 18.5 Q21

Volume of the cone$=\frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi r^2 h$

Squaring both the sides, we have,

$V^2 = \left(\frac{1}{3}\pi r^2 h\right)^2$

$= \frac{1}{9}\pi^2 r^4 h^2 \ldots(1)$

$\Rightarrow \pi^2 r^2 h^2 = \frac{9V^2}{r^2} \ldots(2)$

Consider the curved surface area of the cone.

Thus,

$C=\pi rl$

Squaring both the sides, we have,

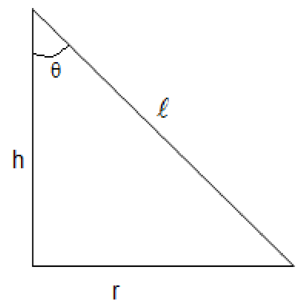
$C^2=\pi^2 r^2 l^2$

We know that $l^2 = r^2 + h^2$

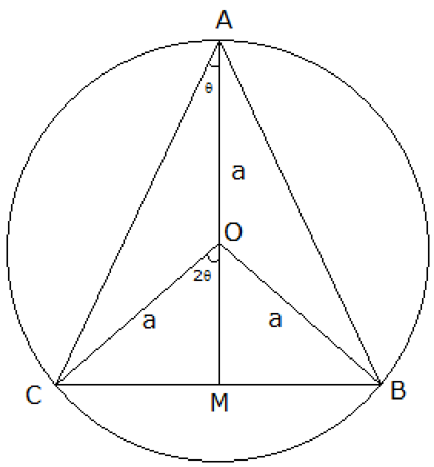
$\Rightarrow C^2=\pi^2 r^2 \left(r^2 + h^2\right)$

$\Rightarrow C^2=\pi^2 r^4 + \pi^2 r^2 h^2$

$\Rightarrow C^2=\pi^2 r^4 + \frac{9V^2}{r^2} \ldots$(From equation (2))



Maxima and Minima 18.5 Q22



ABC is an isosceles triangle such that AB = AC.
The vertical angle $\angle BAC = 2\theta$.
Triangle is inscribed in the circle with center O and radius a.

Draw AM perpendicular to BC.
$\because \Delta ABC$ is an isoscales triangle the circumcentre of the circle will lie on the perpendicular from A to BC.

Let O be the circumcentre.

$\angle BOC = 2 \times 2\theta = 4\theta$ ....[Using central angle theorem]

$\angle COM = 2\theta$ ....[$\because \Delta OMB$ and ΔOMC are congruent triangles]

$OA = OB = OC = a$.......[Radius of the circle]

In ΔOMC,

$CM = a\sin 2\theta$ and $OM = a\cos 2\theta$

$BC = 2CM$...[Perpendicular from the center bisects the chord]

$BC = 2a\sin 2\theta$..........(1)

Height of $\Delta ABC = AM = AO + OM$

$AM = a + a\cos 2\theta$........(2)

Area of ΔABC is,

$A = \frac{1}{2} \times BC \times AM$

Differentiating equation (3) with respect to θ

$$\frac{dA}{d\theta} = a^2\left(2\cos 2\theta + \frac{1}{2} \times 4\cos 4\theta\right)$$

$$\frac{dA}{d\theta} = 2a^2(\cos 2\theta + \cos 4\theta)$$

Differentiating agin with respect to θ

$$\frac{d^2A}{d\theta^2} = 2a^2(-2\sin 2\theta - 4\sin 4\theta)$$

For maximum value of area equating $\frac{dA}{d\theta} = 0$

$2a^2(\cos 2\theta + \cos 4\theta) = 0$

$\cos 2\theta + \cos 4\theta = 0$

$\cos 2\theta + 2\cos^2 2\theta - 1 = 0$

$(2\cos 2\theta - 1)(2\cos 2\theta + 1) = 0$

$\cos 2\theta = \frac{1}{2}$ or $\cos 2\theta = -1$

$2\theta = \frac{\pi}{3}$ or $2\theta = \pi$

$\theta = \frac{\pi}{6}$ or $\theta = \frac{\pi}{2}$

If $2\theta = \pi$ it will not form a triangle.

$\therefore \theta = \frac{\pi}{6}$

Also $\frac{d^2A}{d\theta^2}$ is negative for $\theta = \frac{\pi}{6}$.

Thus the area of the triangle is maximum when $\theta = \frac{\pi}{6}$.

Maxima and Minima 18.5 Q23

Here, $ABCD$ is a rectangle with width $AB = x$ cm and length $AD = y$ cm.

The rectangle is rotated about AD. Let v be the volume of the cylinder so formed.

$\therefore \quad v = \pi r^2 y \qquad$ ---(i)

Again,

$$\text{Perimeter of } ABCD = 2(l+b) = 2(x+y) \qquad \text{---(ii)}$$

$\Rightarrow \quad 36 = 2(x+y)$

$\Rightarrow \quad y = 18 - x \qquad$ ---(iii)

From (i) and (ii), we get

$$v - \pi r^2 (18 - x) = \pi \left(18x^2 - x^3\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = \pi \left(36x - 3x^2\right)$$

For maxima or minima, we have,

$$\frac{dv}{dx} = 0$$

$\Rightarrow \quad \pi \left(36x - 3x^2\right) = 0$

$\Rightarrow \quad 3\pi \left(12x - x^2\right) = 0$

$\Rightarrow \quad x(12 - x) = 0$

$\Rightarrow \quad x = 0$ (Not possible) or 12

$\therefore \quad x = 12$ cm

From (iii)

$$y = 18 - 12 = 6 \text{ cm}$$

Now,

$$\frac{d^2v}{dx^2} = \pi(36 - 6x)$$

At $\quad (x = 12, y = 6) \dfrac{d^2v}{dx^2} = \pi(36 - 72) = -36\pi < 0$

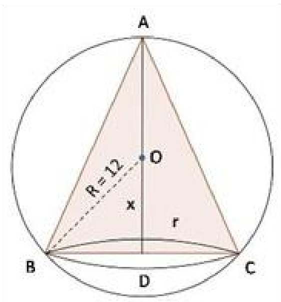
$\therefore \quad (x = 12, y = 6)$ is the point of local maxima,

Hence,

The dimension of rectangle, which wiout maximum value, when revolved about one of its side is width = 12 cm and length = 6 cm.

Maxima and Minima 18.5 Q24

Let r and h be the radius of the base of cone and height of the cone respectively.



Let $OD = x$

It is abvious that the axis of cone must be along the diameter of shpere for maximum volume of cone.

Now,

$$\text{In } \Delta BOD,\ BD = \sqrt{R^2 - x^2}$$
$$= \sqrt{144 - x^2}$$
$$AD = AO + OD = R + x = 12 + x$$
$$v = \text{volume of cone} = \frac{1}{3}\pi r^2 h$$
$$\Rightarrow \quad v = \frac{1}{3}\pi BD^2 \times AD$$
$$= \frac{1}{3}\pi\left(144 - x^2\right)(2 + x)$$
$$= \frac{1}{3}\pi\left(1728 + 144x - 12x^2 - x^3\right)$$
$$\therefore \quad \frac{dv}{dx} = \frac{1}{3}\pi\left(144 - 24x - 3x^2\right)$$

For maximum and minimum of v.

$$\frac{dv}{dx} = 0$$
$$\Rightarrow \quad \frac{1}{3}\pi\left(144 - 24x - 3x^2\right) = 0$$
$$\Rightarrow \quad x = -12, 4$$
$$x = -12 \text{ is not possible}$$
$$\therefore \quad x = 4$$

Now,

$$\frac{d^2v}{dx^2} = \frac{\pi}{3}\left(-24 - 6x\right)$$
$$\text{At} \quad x = 4, \frac{d^2v}{dx^2} = -2\pi\left(4 + x\right)$$
$$= -2\pi \times 8 = -16\pi < 0$$
$$\therefore \quad x = 4 \text{ is point of local maxima.}$$

Hence,

$$\text{Height of cone of maximum volume} = R + x$$
$$= 12 + 4$$
$$= 16 \text{ cm.}$$

Maxima and Minima 18.5 Q25

We have, a closed cylinder whose volume $v = 2156 \text{ cm}^3$

Let r and h be the radius and the height of the cylinder. Then,

$$\therefore \quad v = \pi r^2 h = 2156 \quad \text{---(i)}$$

Total surface area = $S = 2\pi rh + 2\pi r^2$

$$\Rightarrow \quad S = 2\pi r(h + r) \quad \text{---(ii)}$$

From (i) and (ii)

$$S = \frac{2156 \times 2}{r} + 2\pi r^2$$

$$\therefore \quad \frac{ds}{dr} = -\frac{4312}{r^2} + 4\pi r$$

For maximum and minimum

$$\frac{ds}{dr} = 0$$

$$\Rightarrow \quad \frac{-4312 + 4\pi r^3}{r^2} = 0$$

$$\Rightarrow \quad r^3 = \frac{4312}{4\pi}$$

$$\Rightarrow \quad r = 7$$

Now,

$$\frac{d^2s}{dr^2} = \frac{8624}{r^3} + 4\pi > 0 \text{ for } r = 7.$$

$\therefore$ $r = 7$ is the point of local minima

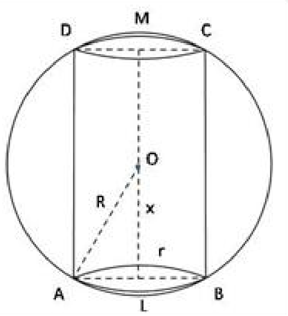
Hence,

The total surface area of closed cylinder will be munimum at $r = 7$ cm.

Maxima and Minima 18.5 Q26

Let r be the radius of the base of the cylinder and h be the height of the cylinder.

$\therefore$ $LM = h$.



Let $\quad R = 5\sqrt{3}$ cm be the radius of the sphere.

It is obvious, that for maximum volume of cylinder $ABCD$, the axis of cylinder must be along the diameter of sphere.

Let $\quad OL = x$

$\therefore \quad h = 2x$

Now,

$$\text{In } \Delta AOL,\ AL = \sqrt{AO^2 - OL^2}$$
$$= \sqrt{75 - x^2}$$

Now,

$$v = \text{volume of cylinder} = \pi r^2 h$$
$$\Rightarrow \quad v = \pi AL^2 \times ML$$
$$= \pi\left(75 - x^2\right) \times 2x$$

For maxima and minima of v, we must have,

$$\frac{dv}{dx} = \pi\left[150 - 6x^2\right] = 0$$
$$\Rightarrow \quad x = 5 \text{ cm}$$

Also, $\quad \frac{d^2v}{dx^2} = -12\pi x$

At $\quad x = 5,\ \frac{d^2v}{dx^2} = -60\pi x < 0$

$\therefore \quad x = 5$ is point of local maxima.

Hence,

The maximum volume of cylinder is $= \pi\left(75 - 25\right) \times 10 = 500\pi$ cm^3.

Maxima and Minima 18.5 Q27

Let x and y be two positive numbers with

$$x^2 + y^2 = r^2 \qquad \text{---(i)}$$

Let $\quad S = x + y \qquad \text{---(ii)}$

$\therefore \quad S = x + \sqrt{r^2 - x^2} \qquad$ from (ii)

$\therefore \quad \frac{dS}{dx} = 1 - \frac{x}{\sqrt{r^2 - x^2}}$

For maxima and minima,

$\frac{dS}{dx} = 0$

$\Rightarrow \quad 1 - \frac{x}{\sqrt{r^2 - x^2}} = 0$

$\Rightarrow \quad x = \sqrt{r^2 - x^2}$

$\Rightarrow \quad 2x^2 = r^2$

$\Rightarrow \quad x = \frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}$

$\therefore \quad x$ & y are positive numbers

$\therefore \quad x = \frac{r}{\sqrt{2}}$

Also, $\quad \frac{d^2S}{dx^2} = \frac{-\left(\sqrt{r^2 - x^2} + \frac{x^2}{\sqrt{r^2 - x^2}}\right)}{r^2 - x^2}$

At, $\quad x = \frac{r}{\sqrt{2}}, \frac{d^2S}{dx^2} = -\left[\frac{\frac{r}{\sqrt{2}} + \frac{\frac{r^2}{2}}{\frac{r}{\sqrt{2}}}}{\frac{r^2}{2}}\right] < 0$

Since $\frac{d^2S}{dx^2} < 0$, *the sum is largest when* $x = y = \frac{r}{\sqrt{2}}$

Maxima and Minima 18.5 Q28

The given equation of parabola is

$$x^2 = 4y \qquad \text{---(i)}$$

Let $P(x, y)$ be the nearest point on (i) from the point $A(0,5)$

Let S be the square of the distance of P from A.

$\therefore \quad S = x^2 + (y-5)^2 \qquad \text{---(ii)}$

From (i),

$$S = 4y + (y-5)^2$$

$\Rightarrow \quad \dfrac{dS}{dy} = 4 + 2(y-5)$

For maxima or minima, we have

$$\frac{dS}{dy} = 0$$

$\Rightarrow \quad 4 + 2(y-5) = 0$

$\Rightarrow \quad 2y = 6$

$\Rightarrow \quad y = 3$

From (i)

$$x^2 = 12$$

$\therefore \quad x = \pm 2\sqrt{3}$

$\Rightarrow \quad P = \left(2\sqrt{3}, 3\right)$ and $P' = \left(-2\sqrt{3}, 3\right)$

Now,

$$\frac{d^2S}{dy^2} = 2 > 0$$

$\therefore \quad P$ and P' are the point of local minima

Hence, the nearest points are $P\left(2\sqrt{3}, 3\right)$ and $P'\left(-2\sqrt{3}, 3\right)$.

Maxima and Minima 18.5 Q29

Let $\quad P(x,y)$ be a point on

$$y^2 = 4x \qquad \text{---(i)}$$

Let $\quad S$ be the square of the distance between $A(2,-8)$ and P.

$$\therefore \quad S = (x-2)^2 + (y+8)^2 \qquad \text{---(ii)}$$

Using (i),

$$S = \left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2$$

$$\therefore \quad \frac{dS}{dy} = 2\left(\frac{y^2}{4} - 2\right) \times \frac{y}{2} + 2(y+8)$$

$$= \frac{y^3 - 8y}{4} + 2y + 16$$

$$= \frac{y^3}{4} + 16$$

For maxima and minima,

$$\frac{dS}{dy} = 0$$

$$\Rightarrow \quad \frac{y^3}{4} + 16 = 0$$

$$\Rightarrow \quad y = -4$$

Now,

$$\frac{d^2S}{dy^2} = \frac{3y^2}{4}$$

At $\quad y = -4, \ \frac{d^2S}{dy^2} = 12 > 0$

$\therefore \quad y = -4$ is the point of local minima

From (i)

$$x = \frac{y^2}{4} = 4$$

Thus, the required point is $(4,-4)$ nearest to $(2,-8)$.

Maxima and Minima 18.5 Q30

Let $P(x, y)$ be a point on the curve,

$$x^2 = 8y \qquad \text{---(i)}$$

Let $A = (2, 4)$ be a point and

let $S =$ square of the distance between P and A

$$\therefore \quad S = (x-2)^2 + (y-4)^2 \qquad \text{---(ii)}$$

Using (i), we get

$$S = (x-2)^2 + \left(\frac{x^2}{8} - 4\right)^2$$

$$\therefore \quad \frac{dS}{dy} = 2(x-2) + 2\left(\frac{x^2}{8} - 4\right) \times \frac{2x}{8}$$

$$= 2(x-2) + \frac{(x^2 - 32)x}{16}$$

Also, $$\frac{d^2S}{dx^2} = 2 + \frac{1}{16}\left[x^2 - 32 + 2x^2\right]$$

$$= 2 + \frac{1}{16}\left[3x^2 - 32\right]$$

For maxima and minima,

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 2(x-2) + \frac{x(x^2 - 32)}{16} = 0$$

$$\Rightarrow \quad 32x - 64 + x^3 - 32x = 0$$

$$\Rightarrow \quad x^3 - 64 = 0$$

$$\Rightarrow \quad x = 4$$

Now,

At $x = 4$, $\frac{d^2S}{dx^2} = 2 + \frac{1}{16}\left[16 \times 3 - 32\right] = 2 + 1 = 3 > 0$

$\therefore$ $x = 4$ is point of local minima

From (i)

$$y = \frac{x^2}{8} = 2$$

Thus, $P(4, 2)$ is the nearest point.

Maxima and Minima 18.5 Q31

Let $P(x, y)$ be a point on the curve $x^2 = 2y$ which is closest to $A(0,5)$

Let S = square of the length of AP

$\Rightarrow \quad S = x^2 + (y-5)^2 \qquad$ ---(ii)

Using (i),

$S = 2y + (y-5)^2$

$\therefore \quad \frac{dS}{dy} = 2 + 2(y-5)$

For maxima and minima,

$\frac{dS}{dy} = 0$

$\Rightarrow \quad 2 + 2y - 10 = 0$

$\Rightarrow \quad y = 4$

Now,

$\frac{d^2S}{dy^2} = 2 > 0$

$\therefore \quad y = 4$ is the point of local minima

From (i)

$r = \pm 2\sqrt{2}$

Hence, $\left(\pm 2\sqrt{2}, 4\right)$ is the closest point on the curve to $A(0,5)$.

Maxima and Minima 18.5 Q32

The given equations are

$y = x^2 + 7x + 2 \qquad$ ---(i)

and $\quad y = 3x - 3 \qquad$ ---(ii)

Let $P(x, y)$ be the point on parabola (i) which is closest to the line (ii)

Let S be the perpendicular distance from P to the line (ii).

$\therefore \quad S = \frac{|y - 3x + 3|}{\sqrt{1^2 + (-3)^2}}$

$\Rightarrow \quad S = \frac{|x^2 + 7x + 2 - 3x + 3|}{\sqrt{10}} \qquad$ ---(iii)

$\Rightarrow \quad \frac{dS}{dx} = \frac{2x + 4}{\sqrt{10}}$

For maxima or minima, we have

$\frac{dS}{dx} = 0$

$\Rightarrow \quad \frac{2x + y}{\sqrt{10}} = 0$

$\Rightarrow \quad x = -2$

From (i)

$y = 4 - 14 + 2 = -8$

Now,

$\frac{d^2S}{dx^2} = \frac{2}{\sqrt{10}} > 0$

$\therefore .(x = -2, y = -8)$ is the point of local minima,

Hence,

The closest point on the parabola to the line $y = 3x - 3$ is $(-2, -8)$.

Maxima and Minima 18.5 Q33

Let $P(x, y)$ be a point on the curve $y^2 = 2x$ which is minimum distance from the point $A(1, 4)$.
Let

$$S = \text{square of the length of } AP$$

$$S = (x-1)^2 + (y-4)^2$$

Using this equation, we have

$$S = x^2 + 1 - 2x + y^2 + 16 - 8y$$

$$S = x^2 - \cancel{2x} + \cancel{2x} + 17 - 8y$$

$$S = \frac{y^4}{4} - 8y + 17 \qquad \left[\text{Since } x = \frac{y^2}{2}\right]$$

$$\frac{dS}{dy} = y^3 - 8$$

For maxima and minima, we have

$$\frac{dS}{dy} = 0$$

$$y^3 - 8 = 0$$

$$y^3 = 2^3$$

$$y = 2$$

Now,

$$\frac{d^2S}{dy^2} = 3y^2$$

$$\frac{d^2S}{dy^2} = 12 > 0$$

$\therefore y = 2$ is minimum point

We have

$$x = \frac{y^2}{2}$$

$$= \frac{4}{2}$$

$$= 2$$

Hence, $\boxed{(2,2)}$ is at a minimum distance from the point $(1, 4)$.

Maxima and Minima 18.5 Q34

The given equation of curve is

$$y = x^3 + 3x^2 + 2x - 27 \qquad \text{---(i)}$$

Slope of (i)

$$m = \frac{dy}{dx} = -3x^2 + 6x + 2 \qquad \text{---(ii)}$$

Now,

$$\frac{dm}{dx} = -6x + 6$$

and $\dfrac{d^2m}{dx^2} = -6 < 0$

For maxima and minima,

$$\frac{dm}{dx} = 0$$

$\Rightarrow \quad -6x + 6 = 0$

$\Rightarrow \quad x = 1$

$\therefore \quad \dfrac{d^2m}{dx^2} = -6 < 0$

$\therefore \quad x = 1$ is point of local maxima

Hence, maximum slope $= -3 + 6 + 2 = 5$

Maxima and Minima 18.5 Q35

We have,

Cost of producing x radio sets is Rs. $\frac{x^2}{4} + 35x + 25$

Selling price of x radio is Rs. $x\left(50 - \frac{x}{2}\right)$

So,

Profit on x radio sets is

$$P = \text{Rs}\ \left(50x - \frac{x^2}{2} - \frac{x^2}{4} - 35x - 25\right)$$

$$\therefore \quad \frac{dP}{dx} = 50 - x - \frac{x}{2} - 35$$

$$= 15 - \frac{3}{2}x$$

For maxima and minima,

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \quad 15 - \frac{3}{2}x = 0$$

$$\Rightarrow \quad x = 10$$

Also,

$$\frac{d^2P}{dx^2} = \frac{-3}{2} < 0$$

$\therefore \quad x = 10$ is the point of local maxima

Hence, the daily output should be 10 radio sets.

Maxima and Minima 18.5 Q35

We have,

Cost of producing x radio sets is Rs. $\frac{x^2}{4} + 35x + 25$

Selling price of x radio is Rs. $x\left(50 - \frac{x}{2}\right)$

So,

Profit on x radio sets is

$$P = \text{Rs}\ \left(50x - \frac{x^2}{2} - \frac{x^2}{4} - 35x - 25\right)$$

$$\therefore \quad \frac{dP}{dx} = 50 - x - \frac{x}{2} - 35$$

$$= 15 - \frac{3}{2}x$$

For maxima and minima,

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \quad 15 - \frac{3}{2}x = 0$$

$$\Rightarrow \quad x = 10$$

Also,

$$\frac{d^2P}{dx^2} = \frac{-3}{2} < 0$$

$\therefore \quad x = 10$ is the point of local maxima

Hence, the daily output should be 10 radio sets.

Maxima and Minima 18.5 Q36

Let $S(x)$ be the selling price of x items and let $C(x)$ be the cost price of x items.

Then, we have $S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$

and $C(x) = \frac{x}{5} + 500$

Thus, the profit function $P(x)$ is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500 = \frac{24}{5}x - \frac{x^2}{100} - 500$$

$\therefore \quad P'(x) = \frac{24}{5} - \frac{x}{50}$

Now, $P'(x) = 0$

$\Rightarrow \frac{24}{5} - \frac{x}{50} = 0$

$\Rightarrow \quad x = \frac{24}{5} \times 50 = 240$

Also $P''(x) = -\frac{1}{50}$

So, $P''(240) = -\frac{1}{50} < 0$

Thus, $x = 240$ is a point of maxima.
Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Maxima and Minima 18.5 Q37

Let l be the length of side of square base of the tank and h be the height of tank.
Then,

Volume of tank $(v) = l^2h$
Total surface area $(s) = l^2 + 4lh$

Since the tank holds a given quantity of water the volume (v) is constant.

$\therefore \quad v = l^2h \quad$ ---(i)

Also, cost of lining with lead will be least if the total surface area is least.
So we need to minimise the surface area.

$\therefore \quad S = l^2 + 4lh \quad$ ---(ii)

Now,

From (i) and (ii)

$S = l^2 + \frac{4v}{l}$

$\therefore \quad \frac{ds}{dl} = 2l - \frac{4v}{l^2}$

For maximum and minimum

$\frac{ds}{dl} = 0$

$\Rightarrow \quad 2l - \frac{4v}{l^2} = 0$

$\Rightarrow \quad 2l^3 - 4v = 0$

$\Rightarrow \quad l^3 = 2v = 2t^2h$

$\Rightarrow \quad l^2[l - 2h] = 0$

$\Rightarrow \quad l = 0$ or $2h$

$l = 0$ is not possible.

$\therefore \quad l = 2h$

Now,

$\frac{d^2s}{dl^2} = 2 + \frac{8v}{l^3}$

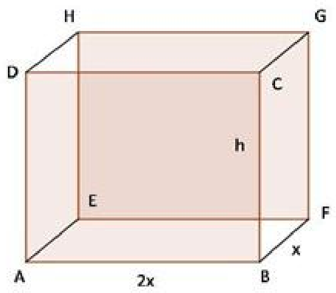
At $\quad l = 2h, \ \frac{d^2s}{dl^2} > 0 \quad$ for all h.

$\therefore \quad l = 2h$ is point of local minima

$\therefore \quad S$ is minimum when $l = 2h$

Maxima and Minima 18.5 Q38

Let $ABCDEFGH$ be a box of constant volume c. We are given that the box is twice as long as its width.



$\therefore$ Let $BF = x$

$\Rightarrow$ $AB = 2x$

Cost of material of top and front side = 3 × cost of material of the bottom of the box.

$\Rightarrow$ $2x \times x + xh + xh + 2xh + 2xh = 3 \times 2x^2$

$\Rightarrow$ $2x^2 + 2xh + 4xh = 6x^2$

$\Rightarrow$ $4x^2 - 6xh = 0$

$\Rightarrow$ $2x(2x - 3h) = 0$

$\Rightarrow$ $x = \frac{3h}{2}$ or $h = \frac{2x}{3}$

Volume of box = $2x \times x \times h$

$\Rightarrow$ $c = 2x^2h$

$\Rightarrow$ $h = \frac{c}{2x^2}$ ---(ii)

Now,

S = Surface area of box = $2\left(2x^2 + 2xh + xh\right)$

$\Rightarrow$ $S = 2\left(2x^2 + 3xh\right)$

From (i)

$S = 2\left(2x^2 + \frac{3xc}{2x^2}\right)$

$\Rightarrow$ $S = 2\left(2x^2 + \frac{3}{2}\frac{c}{x}\right)$

For maxima and minima,

$\frac{dS}{dx} = 2\left(4x - \frac{3}{2}\frac{c}{x^2}\right) = 0$

$\Rightarrow$ $8x^3 - 3c = 0$

$\Rightarrow$ $x = \left(\frac{3c}{8}\right)^{\frac{1}{3}}$

Now,

$\frac{d^2s}{dx^2} = 2\left(4 + 3\frac{c}{x^3}\right) > 0$ as $x = \left(\frac{3c}{8}\right)^{\frac{1}{3}}$

$x = \left(\frac{3c}{8}\right)^{\frac{1}{3}}$ is point of local minima

$\therefore$ Most economic dimension will be

x = width = $\left(\frac{3c}{8}\right)^{\frac{1}{3}}$

$2x$ = length = $2\left(\frac{3c}{8}\right)^{\frac{1}{3}}$

h = height = $\frac{2x}{3} = \frac{2}{3}\left(\frac{3c}{8}\right)^{\frac{1}{3}}$.

Maxima and Minima 18.5 Q39

Let s be the sum of the surface areas of a sphere and a cube.

$$\therefore \quad s = 4\pi r^2 + 6l^2 \qquad \text{---(i)}$$

Let v = volume of sphere + volume of cube

$$\Rightarrow \quad v = \frac{4}{3}\pi r^3 + l^3 \qquad \text{---(ii)}$$

From (i)

$$l = \sqrt{\frac{s - 4\pi r^2}{6}}$$

$$\therefore \quad v = \frac{4}{3}\pi r^2 + \left(\frac{s - 4\pi r^2}{6}\right)^{\frac{3}{2}}$$

$$\therefore \quad \frac{dV}{dr} = 4\pi r^2 + \frac{3}{2}\left(\frac{s - 4\pi r^2}{6}\right)^{\frac{1}{2}} \times \left(\frac{-4\pi}{6}\right)^{2r}$$

For maxima and minima,

$$\frac{dv}{dr} = 0$$

$$\Rightarrow \quad 4\pi r^2 = \frac{\pi}{6}\left(s - 4\pi r^2\right)^{\frac{1}{2}} \times 2r = 0$$

$$\Rightarrow \quad 2r\pi\left[2r - l\right] = 0$$

$$\therefore \quad r = 0, \ \frac{l}{2}$$

Now,

$$\frac{d^2v}{dr^2} = 8\pi r - \frac{2\pi}{\sqrt{6}}\left[\left(s - 4\pi r^2\right)\right]^{\frac{1}{2}} - \frac{8\pi r^2}{2\left(s - 4\pi r^2\right)^{\frac{1}{2}}}$$

At $\quad r = \frac{l}{2}$

$$\frac{d^2v}{dr^2} = \pi\frac{l}{2} - \frac{2\pi}{\sqrt{6}}\left[\sqrt{6}l - \frac{8\pi\frac{l^2}{4}}{2\sqrt{6}l}\right] = 4\pi l - \frac{2\pi}{\sqrt{6}}\left[\frac{12l^2 - 2\pi l^2}{2\sqrt{6}l}\right]$$

Maxima and Minima 18.5 Q40

Let ABCDEF be a half cylinder with rectangular base and semi-circular ends.
Here AB = height of the cylinder
AB = h
Let r be the radius of the cylinder.

Volume of the half cylinder is $V = \frac{1}{2}\pi r^2 h$

$\Rightarrow \frac{2v}{\pi r^2} = h$

$\therefore$ TSA of the half cylinder is
S = LSA of the half cylinder + area of two semi-circular ends + area of the rectangle (base)

$S = \pi rh + \frac{\pi r^2}{2} + \frac{\pi r^2}{2} + h \times 2r$

$S = (\pi r + 2r)h + \pi r^2$

$S = (\pi r + 2r)\frac{2v}{\pi r^2} + \pi r^2$

$S = (\pi + 2)\frac{2v}{\pi r} + \pi r^2$

Differentiate S wrt r we get,

$\frac{ds}{dr} = \left[(\pi + 2) \times \frac{2v}{\pi}\left(\frac{-1}{r^2}\right) + 2\pi r\right]$

For maximum and minimum values of S, we have $\frac{ds}{dr} = 0$

$\Rightarrow (\pi + 2) \times \frac{2v}{\pi}\left(\frac{-1}{r^2}\right) + 2\pi r = 0$

$\Rightarrow (\pi + 2) \times \frac{2v}{\pi r^2} = 2\pi r$

But 2r = D

$\therefore h : D = \pi : \pi + 2$

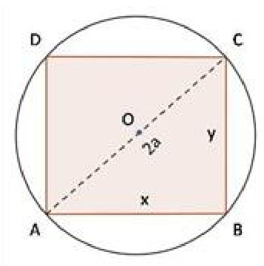
Differentiate $\frac{ds}{dr}$ wrt r we get,

$\frac{d^2s}{dr^2} = (\pi + 2)\frac{V}{\pi} \times \frac{2}{r^3} + 2\pi > 0$

Thus S will be minimum when h : 2r is $\pi : \pi - 12$.
Height of the cylinder : Diameter of the circular end
$\pi : \pi + 2$

Maxima and Minima 18.5 Q41

Let *ABCD* be the cross-sectional area of the beam which is cut from *a* circular log of radius *a*.



$\therefore \qquad AO = a \Rightarrow AC = 2a$

Let x be the width of log and y be the depth of log $ABCD$

Let S be the strength of the beam according to the question,

$$S = xy^2 \qquad \text{---(i)}$$

In ΔABC

$$x^2 + y^2 = (2a)^2$$

$$\Rightarrow \qquad y = (2a)^2 - x^2 \qquad \text{---(ii)}$$

From (i) and (ii), we get

$$S = x\left((2a)^2 - x^2\right)$$

$$\Rightarrow \qquad \frac{dS}{dx} = \left(4a^2 - x^2\right) - 2x^2$$

$$\Rightarrow \qquad \frac{dS}{dx} = 4a^2 - 3x^2$$

For maxima or minima

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \qquad 4a^2 - 3x^2 = 0$$

$$\Rightarrow \qquad x^2 = \frac{4a^2}{3}$$

$$\therefore \qquad x = \frac{2a}{\sqrt{3}}$$

From (ii),

$$y^2 = 4a^2 - \frac{4a^2}{3} = \frac{8a^2}{3}$$

$$\therefore \qquad y = 2a \times \sqrt{\frac{2}{3}}$$

Now,

$$\frac{d^2S}{dx^2} = -6x$$

$$\text{At} \qquad x = \frac{2a}{\sqrt{3}},\ y = \sqrt{\frac{2}{3}}2a,\ \ \frac{d^2S}{dx^2} = -\frac{12a}{\sqrt{3}} < 0$$

$$\therefore \qquad \left(x = \frac{2a}{\sqrt{3}}, y = \sqrt{\frac{2}{3}}2a\right) \text{ is the point of local maxima.}$$

Hence,

The dimension of strongest beam is width $= x = \frac{2a}{\sqrt{3}}$ and depth $= y = \sqrt{\frac{2}{3}}2a$.

Maxima and Minima 18.5 Q42

Let l be a line through the point $P(1,4)$ that cuts the x-axis and y-axis.

Now, equation of l is

$$y - 4 = m(x - 1)$$

$\therefore$ x – Intercept is $\frac{m-4}{m}$ and y – Intercept is $4 - m$

Let $S = \frac{m-4}{m} + 4 - m$

$\therefore$ $\frac{dS}{dm} = +\frac{4}{m^2} - 1$

For maxima and minima,

$$\frac{dS}{dm} = 0$$

$\Rightarrow$ $\frac{4}{m^2} - 1 = 0$

$\Rightarrow$ $m = \pm 2$

Now,

$$\frac{d^2S}{dm^2} = -\frac{8}{m^3}$$

At $m = 2$, $\frac{d^2S}{dm^2} = -1 < 0$

$m = -2$ $\frac{d^2S}{dm^2} = 1 > 0$

$\therefore$ $m = -2$ is point of local minima.

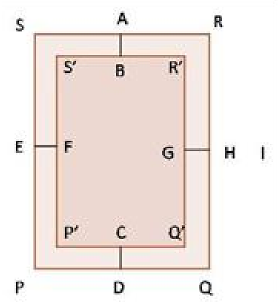
$\therefore$ least value of sum of intercept is

$$\frac{m-4}{m} + 4 - m$$
$$= 3 + 6 = 9$$

Maxima and Minima 18.5 Q43

The area of the page $PQRS$ in 150 cm^2

Also, $AB + CD = 3$ cm
$EF + GH = 2$ Cm



Let x and y be the combined width of margin at the top and bottom and the sides respectively.

$\therefore \quad x = 3$ cm and $y = 2$ cm.

Now, area of printed matter = area of $P'Q'R'S'$

$\Rightarrow \quad A = P'Q' \times Q'R'$

$\Rightarrow \quad A = (b-y)(l-x)$

$\Rightarrow \quad A = (b-2)(l-3) \qquad$ ---(i)

Also,

Area of $PQRS = 150 \text{ cm}^2$

$\Rightarrow \quad lb = 150 \qquad$ ---(ii)

From (i) and (ii)

$$A = (b-2)\left(\frac{150}{b} - 3\right)$$

$\therefore$ For maximum and minimum,

$$\frac{dA}{db} = \left(\frac{150}{b} - 3\right) + (b-2)\left(-\frac{150}{b^2}\right) = 0$$

$$\Rightarrow \quad \frac{(150 - 3b)}{b} + (-150)\frac{(b-2)}{b^2} = 0$$

$$\Rightarrow \quad 150b - 3b^2 - 150b + 300 = 0$$

$$\Rightarrow \quad -3b^2 + 300 = 0$$

$$\Rightarrow \quad b = 10$$

From (ii)

$$l = 15$$

Now,

$$\frac{d^2A}{db^2} = \frac{-150}{b^2} - 150\left[-\frac{1}{b^2} + \frac{4}{b^3}\right]$$

At $b = 10$

$$\frac{d^2A}{db^2} = -\frac{15}{10} - 150\left[-\frac{1}{100} + \frac{4}{1000}\right]$$

$$= -1.5 - .15[-10 + 4]$$

$$= -1.5 + .9$$

$$= -0.6 < 0$$

$\therefore \quad b = 10$ is point of local maxima.

Hence,

The required dimension will be $l = 15$ cm, $b = 10$ cm.

Maxima and Minima 18.5 Q44

The space s described in time t by a moving particle is given by

$$s = t^5 - 40t^3 + 30t^2 + 80t - 250$$

$\therefore$ velocity $= \frac{ds}{dt} = 5t^4 - 120t^2 + 60t + 80$

Acceleration $= a = \frac{d^2s}{dt^2} = 20t^3 - 240t + 60t$ ---(i)

Now,

$$\frac{da}{dt} = 60t^2 - 240$$

For maxima and minima,

$$\frac{da}{dt} = 0$$

$\Rightarrow$ $60t^2 - 240 = 0$

$\Rightarrow$ $60\left(t^2 - 4\right) = 0$

$\Rightarrow$ $t = 2$

Now,

$$\frac{d^2a}{dt^2} = 120t$$

At $t = 2$, $\frac{d^2a}{dt^2} = 240 > 0$

$\therefore$ $t = 2$ is point of local minima

Hence, minimum acceleration is $160 - 480 + 60 = -260$.

Maxima and Minima 18.5 Q45

We have,

$$\text{Distance, } s = \frac{t^4}{4} - 2t^3 + 4t^2 - 7$$

$$\text{Velocity,} v = \frac{ds}{dt} = t^3 - 6t^2 + 8t$$

$$\text{Acceleration,} a = \frac{d^2s}{dt^2} = 3t^2 - 12t + 8$$

For velocity to be maximum and minimum,

$$\frac{dv}{dt} = 0$$

$$\Rightarrow \quad 3t^2 - 12t + 8 = 0$$

$$\Rightarrow \quad t = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$= 2 \pm \frac{4\sqrt{3}}{6}$$

$$\therefore \quad t = 2 + \frac{2}{\sqrt{3}},\ 2 - \frac{2}{\sqrt{3}}$$

Now,

$$\frac{d^2v}{dt^2} = 6t - 12$$

At $\quad t = 2 - \frac{2}{\sqrt{3}},\ \frac{d^2v}{dt^2} = 6\left(2 - \frac{2}{\sqrt{3}}\right) - 12 = \frac{-12}{\sqrt{3}} < 0$

$$t = 2 + \frac{2}{\sqrt{3}},\ \frac{d^2r}{dt^2} = 6\left(2 + \frac{2}{\sqrt{3}}\right) - 12 = \frac{12}{\sqrt{3}} > 0$$

$\therefore \quad$ At $t = 2 - \frac{2}{\sqrt{3}}$, velocity is maximum

For acceleration to be maximum and minimum

$$\frac{da}{dt} = 0$$

$$\Rightarrow \quad 6t - 12 = 0$$

$$\Rightarrow \quad t = 2$$

Now,

$$\frac{d^2a}{dt^2} = 6 > 0$$

$\therefore$ At, $t = 2$ Acceleration is minimum.