Ex 2.1

Functions Ex 2.1 Q1(i)

Example of a function which is one-one but not only.

let $f : N \to N$ given by $f(x) = x^2$

Check for injectivity:

let $x, y \in N$ such that

$$f(x) = f(y)$$

$\Rightarrow \quad x^2 = y^2$

$\Rightarrow \quad (x - y)(x + y) = 0 \qquad [\because x, y \in N \Rightarrow x + y > 0]$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

$\therefore \quad f$ is one-one

Surjectivity: let $y \in N$ be arbitrary, then

$$f(x) = y$$

$\Rightarrow \quad x^2 = y$

$\Rightarrow \quad x = \sqrt{y} \notin N$ for non-perfect square value of y.

$\therefore$ No non-perfect square value of y has a pre image in domain N.

$\therefore$ $f : N \to N$ given by $f(x) = x^2$ is one-one but not onto.

Functions Ex 2.1 Q1(ii)

Example of a function which is onto but not one-one.

let $f : R \to R$ defined by $f(x) = x^3 - x$

Check for injectivity:

let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow \quad x^3 - x = y^3 - y$

$\Rightarrow \quad x^3 - y^3 - (x - y) = 0$

$\Rightarrow \quad (x - y)(x^2 + xy + y^2 - 1) = 0$

$\because \quad x^2 + xy + y^2 \geq 0 \quad \Rightarrow \quad x^2 + xy + y^2 - 1 \geq -1$

$\therefore \quad x \neq y$ for some $x, y \in R$

$\therefore \quad f$ is not one-one.

Surjectivity: let $y \in R$ be arbitrary

then, $f(x) = y$

$\Rightarrow \quad x^3 - x = y$

$\Rightarrow \quad x^3 - x - y = 0$

we know that a degree 3 equation has a real root.

let $x = \alpha$ be that root

$\therefore \quad \alpha^3 - \alpha = y$

$\Rightarrow \quad f(\alpha) = y$

Thus for clearly $y \in R$. there exist $\alpha \in R$ such that $f(x) = y$

$\therefore \quad f$ is onto

$\therefore$ Hence $f : R \to R$ defined by $f(x) = x^3 - x$ is not one-one but onto.

Functions Ex 2.1 Q1(iii)

Example of a function which is neither one-one nor onto.

let $f : R \to R$ defined by $f(x) = 2$

We know that a constant function in neither one-one nor onto

Here $f(x) = 2$ is a constant function

$\therefore \quad f : R \to R$ defined by $f(x) = 2$ is neither one-one nor onto.

Functions Ex 2.1 Q2

i) $f_1 = \{(1,3),(2,5),(3,7)\}$

$A = \{1,2,3\}$, $B = \{3,5,7\}$

We can earily observe that in f_1 every element of A has different image from B.

$\therefore$ f_1 in one-one

also, each element of B is the image of some element of A.

$\therefore$ f_1 in onto.

ii)

$f_2 = \{(2,a),(3,b),(4,c)\}$

$A = \{2,3,4\}$ $B = \{a,b,c\}$

It in clear that different elements of A have different images in B

$\therefore$ f_2 in one-one

Again, each element of B is the image of some element of A.

$\therefore$ f_2 in onto

iii) $f_3 = \{(a,x),(b,x),(c,z)(d,z)\}$

$A = \{a,b,c,d\}$ $B = \{x,y,z\}$

Since, $f_3(a) = x = f_3(b) \text{ and } f_3(c) = z = f_3(d)$

$\therefore$ f_3 in not one-one

Again, $y \in B$ in not the image of any of the element of A

$\therefore$ f_3 in not onto

Functions Ex 2.1 Q3

We have, $f : N \to N$ defined by $f(x) = x^2 + x + 1$

Check for injectivity:

Let $x, y \in N$ such that

$$f(x) = f(y)$$

$$\Rightarrow \quad x^2 + x + 1 = y^2 + y + 1$$

$$\Rightarrow \quad x^2 - y^2 + x - y = 0$$

$$\Rightarrow \quad (x - y)(x + y + 1) = 0$$

$$\Rightarrow \quad x - y = 0 \qquad [\because x, y \in N \Rightarrow x + y + 1 > 0]$$

$$\Rightarrow \quad x = y$$

$\therefore$ f is one-one.

Surjectivity:

Let $y \in N$, then

$$f(x) = y$$

$$\Rightarrow \quad x^2 + x + 1 - y = 0$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1 - 4(1 - y)}}{2} \notin N \quad \text{for } y > 1$$

$\therefore$ for $y > 1$, we do not have any pre-image in domain N.

$\therefore$ f is not onto.

Functions Ex 2.1 Q4.

We have, $A = \{-1, 0, 1\}$ and $f : A \to A$

defined by $f = \{(x, x^2) : x \in A\}$

clearly $f(1) = 1$ and $f(-1) = 1$

$\therefore \quad f(1) = f(-1)$

$\therefore \quad f$ is not one-one

Again $y = -1 \in A$ in the co-domain does not have any pre image in domain A.

$\therefore \quad f$ is not onto.

Functions Ex 2.1 Q5(i)

$f : N \to N$ given by $f(x) = x^2$

let $\quad x_1 = x_2 \quad$ for $\quad x_1, x_2 \in N$

$\Rightarrow \quad x_1^2 = x_2^2 \quad \Rightarrow \quad f(x_1) = f(x_2)$

$\therefore \quad f$ in one-one.

Surjectivity: Since f takes only square value like 1,4,9,16......

so, non-perfect square values in N (co-domain) do not have pre image in domain N.

Thus, f is not onto.

Functions Ex 2.1 Q5(ii)

$f : Z \to Z$ given by $f(x) = x^2$

Injectivity: let $x_1 \,\&\, -x_1 \in Z$

$\Rightarrow \quad x_1 \neq -x_1$

$\Rightarrow \quad x_1^2 = (-x_1)^2 \quad \Rightarrow \quad f(x_1) = f(-x_1)$

$\Rightarrow \quad f$ is not one-one.

Surjective: Again, f takes only square values 1,4,9,16,...

So, no non-perfect square values in Z have a pre image in domain Z.

$\therefore \quad f$ is not onto.

Functions Ex 2.1 Q5(iii)

$f : N \to N$, given by $f(x) = x^3$

Injectivity: let $y, x \in N$ such that

$x = y$

$\Rightarrow \quad x^3 = y^3$

$\Rightarrow \quad f(x) = f(y)$

$\therefore \quad f$ is one-one

Surjective:

$\because \ f$ attain only cubic number like 1,8,27,64,...

So, no non-cubic values of N (co-domain) have pre image in N (Domain)

$\therefore \quad f$ is not onto.

Functions Ex 2.1 Q5(iv)

$f : Z \to Z \quad$ given by $f(x) = x^3$

Injectivity: let $x, y \in Z$ such that

$x = y$

$\Rightarrow \quad x^3 = y^3$

$\Rightarrow \quad f(x) = f(y)$

$\Rightarrow \quad f(x) = f(y)$

$\Rightarrow \quad f$ is one-one.

Surjective: Since f attains only cubic values like $\pm 1, \pm 8, \pm 27, \ldots\ldots\ldots\ldots$

so, no non-cubic values of Z (co-domain) have pre image in Z (domain)

$\therefore \quad f$ is not onto.

Functions Ex 2.1 Q5(v)

$f : R \to R$ given by $f(x) = |x|$

Injectivity: let $x, y \in R$ such that

$x = y$ but if $y = -x$

$\Rightarrow$ $|x| = |y|$ $\Rightarrow$ $|y| = |-x| = x$

$\therefore$ f is not one-one.

Surjective: Since f attains only positive values, for negative real numbers in R, there is no pre-image in domain R.

$\therefore$ f is not onto.

Functions Ex 2.1 Q5(vi)

$f : Z \to Z$ given by $f(x) = x^2 + x$

Injective: let $x, y \in Z$ such that

$f(x) = f(y)$

$\Rightarrow$ $x^2 + x = y^2 + y$

$\Rightarrow$ $x^2 - y^2 + x - y = 0$

$\Rightarrow$ $(x - y)(x + y + 1) = 0$

$\Rightarrow$ either $x - y = 0$ or $x + y + 1 = 0$

Case I: if $x - y = 0$

$\Rightarrow$ $x = y$

$\therefore$ f is injective

Case II if $x + y + 1 = 0$

$\Rightarrow$ $x + y = -1$

$\Rightarrow$ $x \neq y$

$\therefore$ f is not one to one

Thus, in general, f is not one-one

Surjective:

Since $1 \in Z$ (co-domain)

Now, we wish to find if there is any pre-image in domain Z.

let $x \in Z$ such that $f(x) = 1$

$\Rightarrow$ $x^2 + x = 1$ $\Rightarrow$ $x^2 + x - 1 = 0$

$\Rightarrow$ $x = \dfrac{-1 \pm \sqrt{1 + 4}}{2} \notin Z.$

So, f is not onto.

Functions Ex 2.1 Q5(vil)

$f : Z \to Z$ given by $f(x) = x - 5$

Injective: let $x, y \in Z$ such that

$f(x) = f(y)$

$\Rightarrow$ $x - 5 = y - 5$

$\Rightarrow$ $x = y$

$\therefore$ f is one-one.

Surjective: let $y \in Z$ be an arbitrary element

then $f(x) = y$

$\Rightarrow$ $x - 5 = y$

$\Rightarrow$ $x = y + 5 \in Z$ (domain)

Thus, for each element in co-domain Z there exists an element in domain Z such that $f(x) = y$

$\therefore$ f in onto.

Since, f in one-one and onto,

$\therefore$ f in bijective.

Functions Ex 2.1 Q5(viii)

$f : R \to R$ given by $f(x) = \sin x$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $\sin x = \sin y$

$\Rightarrow$ $x = n\pi + (-1)^n y$

$\Rightarrow$ $x \neq y$

$\therefore$ f is not one-one.

Surjective: let $y \in R$ be arbitrary such that

$f(x) = y$

$\Rightarrow$ $\sin x = y$

$\Rightarrow$ $x = \sin^{-1} y$

Now, for $y > 1$ $x \notin R$ (domain)

$\therefore$ f is not onto.

Functions Ex 2.1 Q5(ix)

$f : R \to R$ difined by $f(x) : x^3 + 1$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $x^3 + 1 = y^3 + 1$

$\Rightarrow$ $x^3 = y^3$

$\Rightarrow$ $x = y$

$\therefore$ f is one-one.

Surjective:

let $y \in R$, then

$f(x) = y$

$\Rightarrow$ $x^3 + 1 = y$ $\Rightarrow$ $x^3 + 1 - y = 0$

We know that degree 3 equation has atleast one real root.

$\therefore$ let $x = \alpha$ be the real root.

$\therefore$ $\alpha^3 + 1 = y$

$\Rightarrow$ $f(\alpha) = y$

Thus, for each $y \in R$, there exist $\alpha \in R$ such that $f(\alpha) = y$

$\therefore$ f is onto.

Since f is one-one and onto, f is bijective.

Functions Ex 2.1 Q5(x)

$f: R \to R$ defined by $f(x) = x^3 - x$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $x^3 - x = y^3 - y$

$\Rightarrow$ $x^3 - y^3 - (x - y) = 0$

$\Rightarrow$ $(x - y)(x^2 + xy + y^2 - 1) = 0$

$\because$ $x^2 + xy + y^2 \geq 0 \Rightarrow x^2 + xy + y^2 - 1 \geq -1$

$\therefore$ $x^2 + xy + y^2 - 1 \neq 0$

$\Rightarrow$ $x - y = 0 \Rightarrow x = y$

$\therefore$ f is one-one.

Surjective:

let $y \in R$, then

$f(x) = y$

$\Rightarrow$ $x^3 - x - y = 0$

We know that a degree 3 equation has atleast one real solution.

let $x = \alpha$ be that real solution

$\therefore$ $\alpha^3 - \alpha = y$

$\Rightarrow$ $f(\alpha) = y$

$\therefore$ For each $y \in R$, there exist $x = \alpha \in R$

such that $f(\alpha) = y$

$\therefore$ f is onto.

Functions Ex 2.1 Q5(xi)

$f: R \to R$ defined by $f(x) = sin^2 x + cos^2 x$.

Injective: since $f(x) = sin^2 x + cos^2 x = 1$

$\Rightarrow$ $f(x) = 1$ which is a constant function we know that a constant function in neither injective nor surjective

$\therefore$ f is not one-one and not onto.

Functions Ex 2.1 Q5(xii)

$f : Q - [3] \to Q$ defined by $f(x) = \frac{2x+3}{x-3}$

Injective: let $x, y \in Q - [3]$ such that

$f(x) = f(y)$

$\Rightarrow \quad \frac{2x+3}{x-3} = \frac{2y+3}{y-3}$

$\Rightarrow \quad 2xy - 6x + 3y - 9 = 2xy + 3x - 6y - 9$

$\Rightarrow \quad -6x + 3y - 3x + 6y = 0$

$\Rightarrow \quad -9(x - y) = 0$

$\Rightarrow \quad x = y$

$\Rightarrow \quad f$ is one-one.

Surjective:

let $y \in Q$ be arbitrary. then

$f(x) = y$

$\Rightarrow \quad \frac{2x+3}{x-3} = y$

$\Rightarrow \quad 2x + 3 = xy - 3y$

$\Rightarrow \quad x(2 - y) = -3(y + 1)$

$\therefore \quad x = \frac{-3(y+1)}{2-y} \notin Q - [3]$ for $y = 2$

$\therefore \quad f$ is not onto

Functions Ex 2.1 Q5(xiii)

$f : Q \to Q$ defined by $f(x) = x^3 + 1$

Injective: let $x, y \in Q$ such that

$f(x) = f(y)$

$\Rightarrow \quad x^3 + 1 = y^3 + 1$

$\Rightarrow \quad (x^3 - y^3) = 0$

$\Rightarrow \quad (x - y)(x^2 + xy + y^2) = 0$

but $x^2 + xy + y^2 \geq 0$

$\therefore \quad x - y = 0$

$\Rightarrow \quad x = y$

$\therefore \quad f$ is injective.

Surjective: let $y \in Q$ be arbitrary, then

$f(x) = y$

$\Rightarrow \quad x^3 + 1 - y = 0$

we know that a degree 3 equation has alteast one real solution.

let $x = \alpha$ be that solution

$\therefore \quad \alpha^3 + 1 = y$

$\therefore \quad f(\alpha) = y$

$\therefore \quad f$ is onto.

Functions Ex 2.1 Q5(xiv)

$f: R \to R$ defined by $f(x) = 5x^3 + 4$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $5x^3 + 4 = 5y^3 + 4$

$\Rightarrow$ $5\left(x^3 - y^3\right) = 0$

$\Rightarrow$ $5(x - y)\left(x^2 + xy + y^2\right) = 0$

but $5\left(x^2 + xy + y^2\right) \geq 0$

$\Rightarrow$ $x - y = 0$ $\Rightarrow$ $x = y$

$\therefore$ f is one-one

Surjective: let $y \in R$ be arbitrary, then

$f(x) = y$

$\Rightarrow$ $5x^3 + 4 = y$

$\Rightarrow$ $5x^3 + 4 - y = 0$

we know that a degree 3 equation has alteast one real solution.

let $x = \alpha$ be that real solution

$\therefore$ $5\alpha^3 + 4 = y$

$\therefore$ $f(\alpha) = y$

$\therefore$ For each $y \in Q$, there $\alpha \in R$ such that $f(\alpha) = y$

$\therefore$ f is onto

Since f in one-one and onto

$\therefore$ f in bijective.

Functions Ex 2.1 Q5(xv)

$f: R \to R$ defined by $f(x) = 3 - 4x$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $3 - 4x = 3 - 4y$

$\Rightarrow$ $-4(x - y) = 0$

$\Rightarrow$ $x = y$

$\therefore$ f is one-one.

Surjective: let $y \in R$ be arbitrary, such that

$f(x) = y$

$\Rightarrow$ $3 - 4x = y$

$\Rightarrow$ $x = \frac{3 - y}{4} \in R$

Thus for each $y \in R$, there exist $x \in R$ such that

$f(x) = y$

$\therefore$ f is onto.

Hence, f is one-one and onto and therefore bijective.

Functions Ex 2.1 Q5(xvi)

$f : R \to R$ defined by $f(x) = 1 + x^2$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow \quad 1 + x^2 = 1 + y^2$

$\Rightarrow \quad x^2 - y^2 = 0$

$\Rightarrow \quad (x - y)(x + y) = 0$

either $x = y$ or $x = -y$ or $x \neq y$

$\therefore \quad f$ is not one-one.

Surjective: let $y \in R$ be arbitrary, then

$f(x) = y$

$\Rightarrow \quad 1 + x^2 = y$

$\Rightarrow \quad x^2 + 1 - y = 0$

$\therefore \quad x = \pm\sqrt{y - 1} \notin R$ for $y < 1$

$\therefore \quad f$ is not onto.

Functions Ex 2.1 Q6

Given, $f : A \to B$ is injective such that range$(f) = \{a\}$

We know that in injective map different elements have different images.

$\therefore$ A has only one element.

Functions Ex 2.1 Q7

A = **R** − {3}, B = **R** − {1}

f: A → B is defined as $f(x) = \left(\frac{x-2}{x-3}\right)$.

Let x, $y \in$ A such that $f(x) = f(y)$.

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$$

$$\Rightarrow -3x - 2y = -3y - 2x$$

$$\Rightarrow 3x - 2x = 3y - 2y$$

$$\Rightarrow x = y$$

Therefore, f is one-one.

Let $y \in$B = **R** − {1}.

Then, $y \neq 1$.

The function f is onto if there exists $x \in$A such that $f(x) = y$.

Now,

$$f(x) = y$$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x(1-y) = -3y + 2$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in \text{A} \qquad [y \neq 1]$$

Thus, for any $y \in$ B, there exists $\frac{2-3y}{1-y} \in \text{A}$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right) - 2}{\left(\frac{2-3y}{1-y}\right) - 3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y.$$

$\therefore f$ is onto.

Hence, function f is one-one and onto.

Functions Ex 2.1 Q8

We have $f : R \to R$ given by $f(x) = x - [x]$

Now,

check for injectivity:

$\because\ f(x) = x - [x] \Rightarrow f(x) = 0$ for $x \in Z$

$\therefore$ Range of $f = [0, 1] \neq R$

$\therefore$ f is not one-one, where as many-one

Again, Range of $f = [0, 1] \neq R$

$\therefore$ f is an into function

Functions Ex 2.1 Q9

Suppose $f(n_1) = f(n_2)$

If n_1 is odd and n_2 is even, then we have

$n_1 + 1 = n_2 - 1 \Rightarrow n_2 - n_1 = 2$, not possible

If n_1 is even and n_2 is odd, then we have

$n_1 - 1 = n_2 + 1 \Rightarrow n_1 - n_2 = 2$, not possible

Therefore, both n_1 and n_2 must be either odd or even.

Suppose both n_1 and n_2 are odd.

Then, $f(n_1) = f(n_2) \Rightarrow n_1 + 1 = n_2 + 1 \Rightarrow n_1 = n_2$

Suppose both n_1 and n_2 are even.

Then, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1 \Rightarrow n_1 = n_2$

Thus, f is one – one.

Also, any odd number $2r + 1$ in the co – domain $\mathbf{N}$ will have an even number as image in domain $\mathbf{N}$ which is

$f(n) = 2r + 1 \Rightarrow n - 1 = 2r + 1 \Rightarrow n = 2r + 2$

any even number $2r$ in the co – domain $\mathbf{N}$ will have an odd number as image in domain $\mathbf{N}$ which is

$f(n) = 2r \Rightarrow n + 1 = 2r \Rightarrow n = 2r - 1$

Thus, f is onto.

Functions Ex 2.1 Q10

We have $A = \{1, 2, 3\}$

All one-one functions from $A = \{1, 2, 3\}$ to itself are obtained by re-arranging elements of A.

Thus all possible one-one functions are:

i) $f(1) = 1,\quad f(2) = 2,\quad f(3) = 3$

ii) $f(1) = 2,\quad f(2) = 3,\quad f(3) = 1$

iii) $f(1) = 3,\quad f(2) = 1,\quad f(3) = 2$

iv) $f(1) = 1,\quad f(2) = 3,\quad f(3) = 2$

v) $f(1) = 3,\quad f(2) = 2,\quad f(3) = 1$

vi) $f(1) = 2,\quad f(2) = 1,\quad f(3) = 3$

Functions Ex 2.1 Q11

We have $f : R \to R$ given by $f(x) = 4x^3 + 7$

Let $x, y \in R$ such that

$$f(a) = f(b)$$

$$4a^3 + 7 = 4b^3 + 7$$

$$a = b$$

f is one-one.

Now let $y \in R$ be arbitrary, then

$$f(x) = y$$

$$4x^3 + 7 = y$$

$$x = (y - 7)^{\frac{1}{3}} \in R$$

f is onto.

Hence the function is a bijection

Functions Ex 2.1 Q12

We have $f : R \to R$ given by $f(x) = e^x$

let $x, y \in R$, such that

$f(x) = f(y)$

$\Rightarrow \quad e^x = e^y$

$\Rightarrow \quad e^{x-y} = 1 = e^0$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

$\therefore$ f is one-one

clearly range of $f = (0, \infty) \neq R$

$\therefore$ f is not onto

When co-domain in replaced by R_0^+ i.e, $(0, \infty)$ then f becomes an onto function.

Functions Ex 2.1 Q13

We have $f : R_0^+ \to R$ given by $f(x) = log_a x : a > 0$

let $x, y \in R_0^+$, such that

$f(x) = f(y)$

$\Rightarrow \quad log_a x = log_a y$

$\Rightarrow \quad log_a^x \left(x/y\right) = 0$

$\Rightarrow \quad x/y = 1$

$\Rightarrow \quad x = y$

$\therefore$ f is one-one

Now, let $y \in R$ be arbitrany, then

$f(x) = y$

$\Rightarrow \quad log_a x = y \quad \Rightarrow \quad x = a^y \in R_0^+ \quad \left[\because \; a > 0 \Rightarrow a^y > 0\right]$

Thus, for all $y \in R$, there exist $x = a^y$ such that $f(x) = y$

$\therefore$ f is onto

$\because$ f is one-one and onto $\quad \therefore$ f is bijective

Functions Ex 2.1 Q14

Since f is one-one, three elements of {1, 2, 3} must be taken to 3 different elements of the co-domain {1, 2, 3} under f.

Hence, f has to be onto.

Functions Ex 2.1 Q15

Suppose f is not one-one.

Then, there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same.

Also, the image of 3 under f can be only one element.

Therefore, the range set can have at most two elements of the co-domain {1, 2, 3}

i.e f is not an onto function, a contradiction.

Hence, f must be one-one.

Functions Ex 2.1 Q16

Onto functions from the set $\{1, 2, 3, \ldots, n\}$ to itself is simply a permutation on n symbols $1, 2, \ldots, n$.
Thus, the total number of onto maps from $\{1, 2, \ldots, n\}$ to itself is the same as the total number of permutations on n symbols $1, 2, \ldots, n$, which is $n!$.

Functions Ex 2.1 Q17

Let $f_1 : R \to R$ and $f_2 : R \to R$ be two functions given by:

$$f_1(x) = x$$
$$f_2(x) = -x$$

We can earily verify that f_1 and f_2 are one-one functions.

Now,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x - x = 0$$

$\therefore$ $f_1 + f_2 : R \to R$ is a function given by

$$(f_1 + f_2)(x) = 0$$

Since $f_1 + f_2$ is a constant function, it is not one-one.

Functions Ex 2.1 Q18

Let $f_1 : Z \to Z$ defined by $f_1(x) = x$ and
$f_2 : Z \to Z$ defined by $f_2(x) = -x$

Then f_1 *and* f_2 are surjective functions.

Now,

$f_1 + f_2 : Z \to Z$ is given by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x - x = 0$$

Since $f_1 + f_2$ is a constant function, it is not surjective.

Functions Ex 2.1 Q19

Let $f_1 : R \to R$ be defined by $f_1(x) = x$
and $f_2 : R \to R$ be defined by $f_2(x) = x$

clearly f_1 *and* f_2 are one-one functions.

Now,

$F = f_1 \times f_2 : R \to R$ *is* defined by

$$F(X) = (f_1 \times f_2)(x) = f_1(x) \times f_2(x) = x^2 \quad \ldots\ldots\ldots \text{(i)}$$

Clearly, $F(-1) = 1 = F(1)$

$\therefore$ F is not one-one

Hence, $f_1 \times f_2 : R \to R$ is not one-one.

Functions Ex 2.1 Q20

Let $f_1 : R \to R$ and $f_2 : R \to R$ are two functions defined by $f_1(x) = x^3$ and

$$f_2(x) = x$$

clearly f_1 & f_2 are one-one functions.

Now,

$$\frac{f_1}{f_2} : R \to R \text{ given by}$$

$$\left(\frac{f_1}{f_2}\right)(x) = \frac{f_1(x)}{f_2(x)} = x^2 \text{ for all } x \in R.$$

let $\frac{f_1}{f_2} = f$

$\therefore$ $F = R \to R$ defined by $f(x) = x^2$

now, $F(1) = 1 = F(-1)$

$\therefore$ F is not one-one

$\therefore$ $\frac{f_1}{f_2} = R \to R$ is not one-one.

Functions Ex 2.1 Q22

We have $f : R \to R$ given by $f(x) = x - [x]$

Now,

check for injectivity:

$\because$ $f(x) = x - [x] \Rightarrow f(x) = 0$ for $x \in Z$

$\therefore$ Range of $f = [0, 1] \neq R$

$\therefore$ f is not one-one, where as many-one

Again, Range of $f = [0, 1] \neq R$

$\therefore$ f is an into function

Functions Ex 2.1 23

Suppose $f(n_1) = f(n_2)$

If n_1 is odd and n_2 is even, then we have

$n_1 + 1 = n_2 - 1 \Rightarrow n_2 - n_1 = 2$, not possible

If n_1 is even and n_2 is odd, then we have

$n_1 - 1 = n_2 + 1 \Rightarrow n_1 - n_2 = 2$, not possible

Therefore, both n_1 and n_2 must be either odd or even.

Suppose both n_1 and n_2 are odd.

Then, $f(n_1) = f(n_2) \Rightarrow n_1 + 1 = n_2 + 1 \Rightarrow n_1 = n_2$

Suppose both n_1 and n_2 are even.

Then, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1 \Rightarrow n_1 = n_2$

Thus, f is one – one.

Also, any odd number $2r + 1$ in the co – domain **N** will have an even number as image in domain **N** which is

$f(n) = 2r + 1 \Rightarrow n - 1 = 2r + 1 \Rightarrow n = 2r + 2$

any even number $2r$ in the co – domain **N** will have an odd number as image in domain **N** which is

$f(n) = 2r \Rightarrow n + 1 = 2r \Rightarrow n = 2r - 1$

Thus, f is onto.

Ex 2.2

Functions Ex2.2 Q1(i)

Since, $f : R \to R$ and $g : R \to R$

$\therefore \quad f \circ g : R \to R$ and $gof : R \to R$

$Now,\ f(x) = 2x + 3$ and $g(x) = x^2 + 5$

$g \circ f(x) = g(2x+3) = (2x+3)^2 + 5$

$\Rightarrow g \circ f(x) = 4x^2 + 12x + 14$

$f \circ g(x) = f(g(x)) = f(x^2+5) = 2(x^2+5) + 3$

$\Rightarrow f \circ g(x) = 2x^2 + 13$

Functions Ex2.2 Q1(ii)

$f(x) = 2x + x^2$ and $g(x) = x^3$

$g \circ f(x) = g(f(x)) = g(2x + x^2)$

$g \circ f(x) = (2x + x^2)^3$

$f \circ g(x) = f(g(x)) = f(x^3)$

$\therefore \quad f \circ g(x) = 2x^3 + x^6$

Functions Ex2.2 Q1(iii)

$f(x) = x^2 + 8$ *and* $g(x) = 3x^3 + 1$

Thus, $g \circ f(x) = g[f(x)]$

$\Rightarrow\ g \circ f(x) = g\left[x^2 + 8\right]$

$\Rightarrow\ g \circ f(x) = 3\left[x^2 + 8\right]^3 + 1$

Similarly, $f \circ g(x) = f[g(x)]$

$\Rightarrow\ f \circ g(x) = f\left[3x^3 + 1\right]$

$\Rightarrow\ f \circ g(x) = \left[3x^3 + 1\right]^2 + 8$

$\Rightarrow\ f \circ g(x) = \left[9x^6 + 1 + 6x^3\right] + 8$

$\Rightarrow\ f \circ g(x) = 9x^6 + 6x^3 + 9$

Functions Ex2.2 Q1(iv)

$f(x) = x$ and $g(x) = |x|$

Now, $g \circ f(x) = g(f(x)) = g(x)$

$\therefore\ g \circ f(x) = |x|$

and, $f \circ g(x) = f(g(x)) = f(|x|)$

$\therefore\ f \circ g(x) = |x|$

Functions Ex2.2 Q1(v)

$f(x) = x^2 + 2x - 3$ and $g(x) = 3x - 4$

Now, $g \circ f(x) = g(f(x)) = g(x^2 + 2x - 3)$

$\therefore\ g \circ f(x) = 3(x^2 + 2x - 3) - 4$

$\Rightarrow\ g \circ f(x) = 3x^2 + 6x - 13$

and, $f \circ g(x) = f(g(x)) = f(3x - 4)$

$\therefore\ f \circ g(x) = (3x - 4)^2 + 2(3x - 4) - 3$

$= gx^2 + 16 - 24x + 6x - 8 - 3$

$\therefore\ f \circ g(x) = 9x^2 - 18x + 5$

Functions Ex2.2 Q1(vi)

$f(x) = 8x^3$ and $g(x) = x^{1/3}$

Now, $g \circ f(x) = g(f(x)) = g(8x^3)$

$= (8x^3)^{1/3}$

$\therefore\ g \circ f(x) = 2x$

and, $f \circ g(x) = f(g(x)) = f\left(x^{1/3}\right)$

$= 8\left(x^{1/3}\right)^3$

$\therefore\ f \circ g(x) = 8x$

Functions Ex2.2 Q2

Let $f = \{(3,1), (9,3), (12,4)\}$ and
$g = \{(1,3), (3,3), (4,9), (5,9)\}$

Now,
range of $f = \{1, 3, 4\}$
domain of $f = \{3, 9, 12\}$
range of $g = \{3, 9\}$
domain of $g = \{1, 3, 4, 5\}$

since, range of $f \subset$ domain of g
$\therefore$ $g \circ f$ in well defined.

Again, range of $g \subseteq$ domain of f
$\therefore$ $f \circ g$ in well defined.

Now $g \circ f = \{(3,3), (9,3), (12,9)\}$
$f \circ g = \{(1,1), (3,1)(4,3), (5,3)\}$

Functions Ex2.2 Q3

We have,
$f = \{(1,-1), (4,-2), (9,-3), (16,4)\}$ and
$g = \{(-1,-2), (-2,-4), (-3,-6), (4,8)\}$

Now,
Domain of $f = \{1, 4, 9, 16\}$
Range of $f = \{-1, -2, -3, 4\}$
Domain of $g = \{-1, -2, -3, 4\}$
Range of $g = \{-2, -4, -6, 8\}$

Clearly range of f = domain of g
$\therefore$ $g \circ f$ is defined.

but, range of $g \neq$ domain of f
$\therefore$ $f \circ g$ in not defined.

Now,
$g \circ f(1) = g(-1) = -2$
$g \circ f(4) = g(-2) = -4$
$g \circ f(9) = g(-3) = -6$
$g \circ f(16) = g(4) = 8$

$\therefore$ $g \circ f = \{(1,-2), (4,-4), (9,-6), (16,8)\}$

Functions Ex2.2 Q4

$A = \{a, b, c\}$, $B = \{u, v, w\}$ and
$f = A \rightarrow B$ and $g : B \rightarrow A$ defined by
$f = \{(a, v), (b, u), (c, w)\}$ and
$g = \{(u, b), (v, a), (w, c)\}$

For both f and g, different elements of domain have different images
$\therefore$ f and g are one-one

Again for each element in co-domain of f and g, there in a pre image in domain
$\therefore$ f and g are onto

Thus, f and g are bijectives.

Now,
$$g \circ f = \{(a, a), (b, b), (c, c)\} \text{ and}$$
$$f \circ g = \{(u, u), (v, v), (w, w)\}$$

Functions Ex2.2 Q5

We have, $f : R \rightarrow R$ given by $f(x) = x^2 + 8$ and
$g : R \rightarrow R$ given by $g(x) = 3x^3 + 1$

$$\therefore\ f \circ g(x) = f(g(x)) = f(3x^3 + 1)$$
$$= (3x^3 + 1)^2 + 8$$

$$\therefore\ f \circ g(2) = (3 \times 8 + 1)^2 + 8 = 625 + 8 = 633$$

Again
$$g \circ f(x) = g(f(x)) = g(x^2 + 8)$$
$$= 3(x^2 + 8)^3 + 1$$

$$\therefore\ g \circ f(1) = 3(1 + 8)^3 + 1 = 2188$$

Functions Ex2.2 Q6

We have, $f : R^+ \rightarrow R^+$ given by
$f(x) = x^2$
$g : R^+ \rightarrow R^+$ given by
$g(x) = \sqrt{x}$

$$\therefore\ f \circ g(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$
Also,
$$g \circ f(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = x$$

Thus,
$$f \circ g(x) = g \circ f(x)$$

Functions Ex2.2 Q7

We have, $f : R \rightarrow R$ and $g : R \rightarrow R$ are two functions defined by
$f(x) = x^2$ and $g(x) = x + 1$

Now,
$$f \circ g(x) = f(g(x)) = f(x + 1) = (x + 1)^2$$
$$\therefore f \circ g(x) = x^2 + 2x + 1 \ \ldots\ldots\ldots\ldots\ldots(i)$$
$$g \circ f(x) = g(f(x)) = g(x^2) = x^2 + 1 \ \ldots\ldots\ldots\ldots(ii)$$
from (i) & (ii)
$$f \circ g \neq g \circ f$$

Functions Ex2.2 Q8

Let $f: R \to R$ and $g: R \to R$ are defined as
$f(x) = x+1$ and $g(x) = x-1$

Now,
$$f \circ g(x) = f(g(x)) = f(x-1) = x-1+1$$
$$= x = I_R \;\ldots\ldots\ldots \text{(i)}$$

Again,
$$f \circ g(x) = f(g(x)) = g(x+1) = x+1-1$$
$$= x = I_R \;\ldots\ldots\ldots \text{(ii)}$$
from (i) & (ii)
$$f \circ g = g \circ f = I_R$$

Functions Ex2.2 Q9

We have, $f: N \to Z_0$, $g: Z_0 \to Q$ and
$h: Q \to R$

Also, $f(x) = 2x$, $g(x) = \dfrac{1}{x}$ and $h(x) = e^x$

Now, $f: N \to Z_0$ and $h \circ g: Z_0 \to R$
$\therefore$ $(h \circ g) \circ f: N \to R$

also, $g \circ f: N \to Q$ and $h: Q \to R$
$\therefore$ $h \circ (g \circ f): N \to R$

Thus, $(h \circ g) \circ f$ and $h \circ (g \circ f)$ exist and are function from N to *set* R.

Finally. $(h \circ g) \circ f\ (x) = (h \circ g)(f(x)) = (h \circ g)(2x)$
$$= h\left(\tfrac{1}{2x}\right)$$
$$= e^{1/2x}$$

now, $h \circ (g \circ f)(x) = h \circ (g(2x)) = h\left(\tfrac{1}{2x}\right)$
$$= e^{1/2x}$$
Hence, associativity verified.

Functions Ex2.2 Q10

We have,
$$h \circ (g \circ f)(x) = h(g \circ f(x)) = h(g(f(x)))$$
$$= h(g(2x)) = h(3(2x)+4)$$
$$= h(6x+4) = \sin(6x+4) \quad \forall x \in \mathbf{N}$$
$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = (h \circ g)(2x)$$
$$= h(g(2x)) = = h(3(2x)+4)$$
$$= h(6x+4) = \sin(6x+4) \quad \forall x \in \mathbf{N}$$
This shows, $h \circ (g \circ f) = (h \circ g) \circ f$

Functions Ex2.2 Q11

Define f: $\mathbf{N} \to \mathbf{N}$ by, $f(x) = x + 1$

And, g: $\mathbf{N} \to \mathbf{N}$ by,
$$g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We first show that f *is* not onto.

For this, consider element 1 in co-domain **N**. It is clear that this element is not an image of any of the elements in domain **N**.

Therefore, f is not onto.

Now, gof: $\mathbf{N} \to \mathbf{N}$ is defined by,

Functions Ex2.2 Q12

Define f: $\mathbf{N} \to \mathbf{Z}$ as $f(x) = x$ and g: $\mathbf{Z} \to \mathbf{Z}$ as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that:

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

Therefore, $g(-1) = g(1)$, but $-1 \neq 1$.

Therefore, g is not injective.

Now, gof: $\mathbf{N} \to \mathbf{Z}$ is defined as $gof(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in \mathbf{N}$ such that $gof(x) = gof(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in \mathbf{N}$, both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, gof is injective

Functions Ex2.2 Q13

We have, $f : A \to B$ and $g : B \to C$ are one-one functions

Now we have to prove : $g \circ f : A \to C$ in one-one

let $x, y \in A$ such that

$$g \circ f(x) = g \circ f(y)$$

$\Rightarrow \quad g(f(x)) = g(f(y))$

$\Rightarrow \quad f(x) = f(y) \qquad [\because g \text{ in one-one}]$

$\Rightarrow \quad x = y \qquad [\because f \text{ in one-one}]$

$\therefore \quad g \circ f$ is one-one function

Functions Ex2.2 Q14

We have, $f : A \to B$ and $g : B \to C$ are onto functions.

Now, we need to prove: $g \circ f : A \to C$ in onto.

let $y \in C$, then

$$g \circ f(x) = y$$

$\Rightarrow \quad g(f(x)) = y \ \ldots\ldots\ldots\ldots(i)$

Since g is onto, for each element in C, then exists a preimage in B.

$\therefore \quad g(x) = y \ \ldots\ldots\ldots\ldots(ii)$

From (i) & (ii)

$$f(x) = \alpha.$$

Since f is onto, for each element in B there exists a preimage in A

$\therefore f(x) = \alpha \ \ldots\ldots\ldots\ldots(iii)$

From (ii) and (iii) we can conclude that for each $y \in C$, there exists a pre image in A such that $g \circ f(x) = y$

$\therefore g \circ f$ is onto

Ex 2.3

Functions Ex 2.3 Q 1(i)

$f(x) = e^x$ and $g(x) = \log_e x$

Now, $f \circ g(x) = f(g(x)) = f(\log_e x) = e^{\log_e x} = x$

$f \circ g(x) = x$

$g \circ f(x) = g(f(x)) = g(e^x) = \log_e e^x = x$

$\Rightarrow \quad g \circ f(x) = x$

Functions Ex 2.3 Q 1(ii)

$f(x) = x^2, \quad g(x) = \cos x$

Domain of f and Domain of $g = R$

Range of $f = (0, \infty)$

Range of $g = (-1, 1)$

$\therefore$ Range of $f \subset$ domain of $g \Rightarrow g \circ f$ exist

Range of $g \subset$ domain of $f \Rightarrow f \circ g$ exist

Now,

$g \circ f(x) = g(f(x)) = g(x^2) = \cos x^2$

And

$f \circ g(x) = f(f(x)) = f(\cos x) = \cos^2 x$

Functions Ex 2.3 Q1(iii)

$f(x) = |x|$ and $g(x) = \sin x$

$$\text{Range of } f = (0, \infty) \subset \text{ Domain } g(R) \Rightarrow g \circ f \text{ exist}$$
$$\text{Range of } g = [-1, 1] \subset \text{ Domain of } (R) \Rightarrow f \circ g \text{ exist}$$

Now,

$$f \circ g(x) = f(g(x)) = f(\sin x) = |\sin x|$$

And

$$g \circ f(x) = g(f(x)) = g(|x|) = \sin|x|$$

Functions Ex 2.3 Q1(iv)

$f(x) = x + 1$ and $g(x) = e^x$

$$\text{Range of } f = R \subset \text{ Domain of } g = R \Rightarrow g \circ f \text{ exist}$$
$$\text{Range of } g = (0, \infty) \subset \text{ Domain of } f = R \Rightarrow f \circ g \text{ exist}$$

Now,

$$g \circ f(x) = g(f(x)) = g(x+1) = e^{x+1}$$

And

$$f \circ g(x) = f(g(x)) = f(e^x) = e^x + 1$$

Functions Ex 2.3 Q1(v)

$f(x) = \sin^{-1} x$ and $g(x) = x^2$

$$\text{Range of } f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset \text{ Domain of } g = R \Rightarrow g \circ f \text{ exist}$$
$$\text{Range of } g = (0, \infty) \subseteq \text{ Domain of } f = R \Rightarrow f \circ g \text{ exist}$$

Now,

$$f \circ g(x) = f(g(x)) = f(x^2) = \sin^{-1} x^2$$

And

$$g \circ f(x) = g(f(x)) = g(\sin^{-1} x) = (\sin^{-1} x)^2$$

Functions Ex 2.3 Q 1(vi)

$f(x) = x + 1$ and $g(x) = \sin x$

$$\text{Range of } f = R \subset \text{Domain of } g = R \Rightarrow g \circ f \text{ exists}$$
$$\text{Range of } g = [-1, 1] \subset \text{ Domain of } f = R \Rightarrow f \circ g \text{ exists}$$

Now,

$$f \circ g(x) = f(g(x)) = f(\sin x) = \sin x + 1$$

And

$$g \circ f(x) = g(f(x)) = g(x+1) = \sin(x+1)$$

Functions Ex 2.3 Q1(vii)

$f(x) = x + 1$ and $g(x) = 2x + 3$

$$\text{Range of } f = R \subseteq \text{Domain of } g = R \Rightarrow g \circ f \text{ exist}$$
$$\text{Range of } g = R \subseteq \text{Domain of } R = R \Rightarrow f \circ g \text{ exist}$$

Now,

$$f \circ g(x) = f(g(x)) = f(2x+3) = (2x+3) + 1 = 2x + 4$$

And

$$g \circ f(x) = g(f(x)) = g(x+1) = 2(x+1) + 3$$

$$\Rightarrow \quad g \circ f(x) = 2x + 5$$

Functions Ex 2.3 Q1(viii)

$f(x) = c, \quad c \in R$ and
$g(x) = \sin x^2$

$\text{Range of } f = R \subset \text{Domain of } g = R \Rightarrow g \circ f \text{ exist}$
$\text{Range of } g = [-1, 1] \subset \text{Domain of } f = R \Rightarrow f \circ g \text{ exist}$

Now,

$$g \circ f(x) = g(f(x)) = g(c) = \sin c^2$$

And

$$f \circ g(x) = f(g(x)) = f(\sin x^2) = c$$

Functions Ex 2.3 Q1(ix)

$f(x) = x^2 + 2$ and $g(x) = 1 - \frac{1}{1-x}$

$\text{Range of } f = (2, \infty) \subset \text{Domain of } g = R \Rightarrow g \circ f \text{ exist}$
$\text{Range of } g = R - [1] \subset \text{Domain of } f = R \Rightarrow f \circ g \text{ exist}$

Now,

$$f \circ g(x) = f(g(x)) = f\left(\frac{-x}{1-x}\right) = \frac{x^2}{(1-x)^2} + 2$$

And

$$g \circ f(x) = g(f(x)) = g(x^2 + 2) = \frac{-(x^2+2)}{1-(x^2+2)}$$

$$\Rightarrow \quad g \circ f(x) = \frac{x^2+2}{x^2+1}$$

Functions Ex 2.3 Q2

We have, $f(x) = x^2 + x + 1$ and $g(x) = \sin x$

Now,

$$f \circ g(x) = f(g(x)) = f(\sin x)$$

$$\Rightarrow \quad f \circ g(x) = \sin^2 x + \sin x + 1$$

Again, $g \circ f(x) = g(f(x)) = g(x^2 + x + 1)$

$$\Rightarrow \quad g \circ f(x) = \sin(x^2 + x + 1)$$

Clearly

$$f \circ g \neq g \circ f$$

Functions Ex 2.3 Q3

We have $\quad f(x) = |x|$

We assume the domain of $f = R$
Range of $f = (0, \infty)$
$\therefore$ Range of f $\subset$ domain of f
$\therefore$ $f \circ f$ exists.

Now,

$$f \circ f(x) = f(f(x)) = f(|x|) = ||x|| = f(x)$$

$\therefore \; f \circ f = f$

Functions Ex 2.3 Q4

$f(x) = 2x + 5$ and $g(x) = x^2 + 1$

$\therefore$ Range of $f = R$ and range of $g = [1, \infty]$

$\therefore$ Range of $f \subseteq$ Domain of $g(R)$ and range of $g \subseteq$ domain of $f(R)$

$\therefore$ both fog and gof exist.

i) $f \circ g(x) = f(g(x)) = f(x^2 + 1)$

$= 2(x^2 + 1) + 5$

$\Rightarrow$ $f \circ g(x) = 2x^2 + 7$

ii) $g \circ f(x) = g(f(x)) = g(2x + 5)$

$= (2x + 5)^2 + 1$

$\Rightarrow$ $g \circ f(x) = 4x^2 + 20x + 26$

iii) $f \circ f(x) = f(f(x)) = f(2x + 5)$

$= 2(2x + 5) + 5$

$f \circ f(x) = 4x + 15$

iv) $f^2(x) = [f(x)]^2 = (2x + 5)^2$

$= 4x^2 + 20x + 25$

$\therefore$ from (iii) & (iv)

$f \circ f \neq f^2$

Functions Ex 2.3 Q5

We have, $f(x) = \sin x$ and $g(x) = 2x$.

Domain of f and g is R

Range of $f = [-1, 1]$

Range of $g = R$

$\therefore$ Range of $f \subset$ Domain g and

Range of $g \subseteq$ Domain f

$\therefore$ fog and gof both exist.

i) $g \circ f(x) = g(f(x)) = g(\sin x) = g \circ f(x) = 2 \sin x$

ii) $f \circ g(x) = f(g(x)) = f(2x) = \sin 2x$

$\therefore$ $g \circ f \neq f \circ g$

Functions Ex 2.3 Q6

$f, g,$ and h are real functions given by $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$

To prove: $f \circ g = g \circ (fh)$

L.H.S

$f \circ g(x) = f(g(x))$

$= f(2x) = \sin 2x$

$\Rightarrow f \circ g(x) = 2 \sin x \cos x$ ....(A)

R.H.S

$g \circ (fh)(x) = go(f(x).h(x))$

$= g(\sin x \cos x)$

$g \circ (fh)(x) = 2 \sin x \cos x$ ....(B)

from A & B

$f \circ g(x) = g \circ (fh)(x)$

Functions Ex 2.3 Q7

We are given that f is a real function and g is a function given by $g(x) = 2x$

To prove; $g \circ f = f + f$.

L.H.S

$$g \circ f(x) = g(f(x)) = 2f(x)$$

$$= f(x) + f(x) = \text{R.H.S}$$

$\Rightarrow \quad g \circ f = f + f$

Functions Ex 2.3 Q8

$$f(x) = \sqrt{1-x}\ , \qquad g(x) = log_e^x$$

Domain of f and g are R.

Range of $f = (-\infty, 1)$

Range of $g = (0, e)$

Clearly Range $f \subset$ Domain $g \Rightarrow g \circ f$ exists

Range $g \subset$ Domain $f \Rightarrow f \circ g$ exists

$\therefore\ g \circ f(x) = g(f(x)) = g\left(\sqrt{1-x}\right)$

$$g \circ f(x) = log_e^{\sqrt{1-x}}$$

Again

$$f \circ g(x) = f(g(x)) = f\left(log_e^x\right)$$

$$f \circ g(x) = \sqrt{1 - log_e^x}$$

Functions Ex 2.3 Q9

$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ and $g : [-1, 1] \to R$ defined as $f(x) = tan\,x$ and $g(x) = \sqrt{1-x^2}$

Range of f : let $y = f(x) \Rightarrow y = tan\,x$

$$\Rightarrow x = tan^{-1} y$$

Since $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), y \in (-\infty, \infty)$

$\therefore$ Range of f $\subset$ domain of $g = [-1, 1]$

$\therefore\ g \circ f$ exists.

By similar argument $f \circ g$ exists.

Now,

$$f \circ g(x) = f(g(x)) = f\left(\sqrt{1-x^2}\right)$$

$$f \circ g(x) = tan\sqrt{1-x^2}$$

Again

$$g \circ f(x) = g(f(x))$$

$$= g(tan\,x)$$

$$g \circ f(x) = \sqrt{1 - tan^2 x}$$

Functions Ex 2.3 Q10

$f(x) = \sqrt{x+3}$ and $g(x) = x^2 + 1$

Now,

Range of $f = [-3, \infty]$ and

Range of $g = (1, \infty)$

Then, Range of $f \subset$ Domain g and

Range of $g \subset$ Domain f

$\therefore$ $f \circ g$ and $g \circ f$ exist.

Now,

$$f \circ g(x) = f(g(x)) = f(x^2 + 1)$$

$$f \circ g(x) = \sqrt{x^2 + 4}$$

Again,

$$g \circ f(x) = g(f(x)) = g(\sqrt{x+3})$$

$$= (\sqrt{x+3})^2 + 1$$

$$g \circ f(x) = x + 4$$

Functions Ex 2.3 Q11(i)

We have, $f(x) = \sqrt{x-2}$

Clearly, Domain$(f) = [2, \infty)$ and Range$(f) = [0, \infty)$.

We observe that range(f) is not a subset of domain of f.

$$\therefore \text{Domain of } (fof) = \{x : x \in \text{Domain } (f) \text{ and } f(x) \in \text{Domain } (f)\}$$

$$= \{x : x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x : x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x : x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x : x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Now,

$$(fof)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2} - 2}$$

$\therefore$ $fof : [6, \infty) \to R$ defined as

$$(fof)(x) = \sqrt{\sqrt{x-2} - 2}$$

Functions Ex 2.3 Q11(ii)

We have, $f(x) = \sqrt{x-2}$

Clearly, Domain$(f) = [2, \infty)$ and Range$(f) = [0, \infty)$.

We observe that range(f) is not a subset of domain of f.

$\therefore$ Domain of $(fof) = \{x : x \in \text{Domain } (f) \text{ and } f(x) \in \text{Domain } (f)\}$

$$= \{x : x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$
$$= \{x : x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$
$$= \{x : x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$
$$= \{x : x \in [2, \infty) \text{ and } x \geq 6\}$$
$$= [6, \infty)$$

Clearly, range of $f = [0, \infty) \not\subset$ Domain of (fof).

$\therefore$ Domain of $((fof)of) = \{x : x \in \text{Domain } (f) \text{ and } f(x) \in \text{Domain } (fof)\}$

$$= \{x : x \in [2, \infty) \text{ and } \sqrt{x-2} \in [6, \infty)\}$$
$$= \{x : x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 6\}$$
$$= \{x : x \in [2, \infty) \text{ and } x - 2 \geq 36\}$$
$$= \{x : x \in [2, \infty) \text{ and } x \geq 38\}$$
$$= [38, \infty)$$

Now,

$(fof)(x) = f(f(x)) = f\left(\sqrt{x-2}\right) = \sqrt{\sqrt{x-2} - 2}$

$(fofof)(x) = (fof)(f(x)) = (fof)\left(\sqrt{x-2}\right) = \sqrt{\sqrt{\sqrt{x-2} - 2} - 2}$

$\therefore$ fofof : $[38, \infty) \to R$ defined as

$(fofof)(x) = \sqrt{\sqrt{\sqrt{x-2} - 2} - 2}$

Functions Ex 2.3 Q11(iii)

We have, $f(x)=\sqrt{x-2}$

Clearly, $\text{Domain}(f)=[2,\infty)$ and $\text{Range}(f)=[0,\infty)$.

We observe that range(f) is not a subset of domain of f.

$\therefore$ Domain of $(fof)=\{x : x\in \text{Domain }(f) \text{ and } f(x)\in \text{Domain }(f)\}$

$=\{x : x\in[2,\infty) \text{ and } \sqrt{x-2}\in[2,\infty)\}$

$=\{x : x\in[2,\infty) \text{ and } \sqrt{x-2}\geq 2\}$

$=\{x : x\in[2,\infty) \text{ and } x-2\geq 4\}$

$=\{x : x\in[2,\infty) \text{ and } x\geq 6\}$

$=[6,\infty)$

Clearly, range of $f=[0,\infty)\not\subset$ Domain of (fof).

$\therefore$ Domain of $((fof)of)=\{x : x\in \text{Domain }(f) \text{ and } f(x)\in \text{Domain }(fof)\}$

$=\{x : x\in[2,\infty) \text{ and } \sqrt{x-2}\in[6,\infty)\}$

$=\{x : x\in[2,\infty) \text{ and } \sqrt{x-2}\geq 6\}$

$=\{x : x\in[2,\infty) \text{ and } x-2\geq 36\}$

$=\{x : x\in[2,\infty) \text{ and } x\geq 38\}$

$=[38,\infty)$

Now,

$(fof)(x)=f(f(x))=f\left(\sqrt{x-2}\right)=\sqrt{\sqrt{x-2}-2}$

$(fofof)(x)=(fof)(f(x))=(fof)\left(\sqrt{x-2}\right)=\sqrt{\sqrt{\sqrt{x-2}-2}-2}$

$\therefore$ $fofof:[38,\infty)\to R$ defined as

$(fofof)(x)=\sqrt{\sqrt{\sqrt{x-2}-2}-2}$

$(fofof)(38)=\sqrt{\sqrt{\sqrt{38-2}-2}-2}=\sqrt{\sqrt{\sqrt{36}-2}-2}=\sqrt{\sqrt{6-2}-2}=\sqrt{\sqrt{4}-2}=\sqrt{2-2}=0$

Functions Ex 2.3 Q11(iv)

We have, $f(x)=\sqrt{x-2}$

Clearly, $\text{Domain}(f)=[2,\infty)$ and $\text{Range}(f)=[0,\infty)$.

We observe that range(f) is not a subset of domain of f.

$\therefore$ Domain of $(fof)=\{x : x\in \text{Domain }(f) \text{ and } f(x)\in \text{Domain }(f)\}$

$=\{x : x\in[2,\infty) \text{ and } \sqrt{x-2}\in[2,\infty)\}$

$=\{x : x\in[2,\infty) \text{ and } \sqrt{x-2}\geq 2\}$

$=\{x : x\in[2,\infty) \text{ and } x-2\geq 4\}$

$=\{x : x\in[2,\infty) \text{ and } x\geq 6\}$

$=[6,\infty)$

Now,

$(fof)(x)=f(f(x))=f\left(\sqrt{x-2}\right)=\sqrt{\sqrt{x-2}-2}$

$\therefore$ $fof:[6,\infty)\to R$ defined as

$(fof)(x)=\sqrt{\sqrt{x-2}-2}$

$f^2(x)=[f(x)]^2=\left[\sqrt{x-2}\right]^2=x-2$

$\therefore$ $f^2:[2,\infty)\to R$ defined as

$f^2(x)=x-2$

$\therefore$ $fof\neq f^2$

Functions Ex 2.3 Q12

$$f(x)\begin{cases}1+x & 0\le x\le 2\\ 3-x & 2\le x\le 3\end{cases}$$

$\therefore$ Range of $f=[0,3]\subseteq$ Domain of f.

$$\therefore\ f\circ f(x)=f(f(x))=f\begin{cases}1+x & 0\le x\le 2\\ 3-x & 2<x\le 3\end{cases}$$

$$f\circ f(x)=\begin{cases}2+x & 0\le x\le 1\\ 2-x & 1<x\le 2\\ 4-x & 2<x\le 3\end{cases}$$

Ex 2.5

Functions Ex 2.5 Q 1.

i) $f : \{1,2,3,4\} \rightarrow \{10\}$ given by

$f\{(1,10),(2,10),(3,10),(4,10)\}$

clearly f is many-one function

$\Rightarrow$ f is not bijective

$\Rightarrow$ f is not invertible

ii) $g : \{5,6,7,8\} \rightarrow \{1,2,3,4\}$ given by

$g\{(5,4),(6,3),(7,4),(8,2)\}$

Since, 5 and 7 have same image 4

$\therefore$ g is not bijectible

$\Rightarrow$ g is not bijective

$\Rightarrow$ g is not invertible

iii) $h : \{2,3,4,5\} \rightarrow \{7,9,11,13\}$ given by

$h\{(2,7),(3,9),(4,11),(5,13)\}$

We can observe that different element of domain have defferent image is co-domain.

Functions Ex 2.5 Q2

$A = \{0, -1, -3, 2\}$, $B = \{-9, -3, 0, 6\}$
$f : A \to B$ is defined by $f(x) = 3x$

Since different elements of A have different images in B.
$\therefore$ f is one-one

Again, each element in B has a preimage in A.
$\therefore$ f is onto

$\because$ f in one-one bijective
$\Rightarrow$ $f^{-1} : B \to A$ exists and is given by
$$f^{-1}(x) = \frac{x}{3}$$

$A = \{1, 3, 5, 7, 9\}$, $B = \{0, 1, 9, 25, 49, 81\}$

$f : A \to B$ be a function defined by $f(x) = x^2$

Since different elements of A have different images in B.
$\therefore$ f is one-one

Again, $0 \in B$ does not have a preimage in A.
$\therefore$ f is not onto

Hence, f^{-1} does not exist.

Functions Ex 2.5 Q3

Given that $f:\{1,2,3\} \to \{a,b,c\}$ *and* $g:\{a,b,c\} \to \{apple, ball, cat\}$ *such that*
$f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a) = apple$, $g(b) = ball$ *and* $g(c) = cat$
We need to prove that f, g *and* $g \circ f$ *are invertible.*
In order to prove that f *is invertiblem is is sufficient to show that*
$f:\{1,2,3\} \to \{a,b,c\}$ *is a bijection.*
f *is one – one:*
Each and every element of the set $\{1,2,3\}$ *is having an image in the set* $\{a,b,c\}$
Thus, f *is one – one.*
Obviously, the number of element of the sets $\{1,2,3\}$ *and* $\{a,b,c\}$ *are equal and hence*
f *is onto.*
Thus, the function f *is invertible.*
Similarly, let us observe for the function g:
g *is one – one:*
Each and every element of the set $\{a,b,c\}$ *is having an image in the set* $\{apple, ball, cat\}$
Thus, g *is one – one.*
Obviously, the number of element of the sets $\{a,b,c\}$ *and* $\{apple, ball, cat\}$ *are equal and hence*
g *is onto.*
Thus, the function g *is invertible.*

$Now\ let\ us\ consider\ the\ function\ g\circ f = g[f(x)]$
$Each\ and\ every\ element\ of\ of\ the\ set\ \{1,2,3\}\ is\ having\ an\ image\ in\ the\ set$
$\{apple,\ ball,\ cat\}.$
$Therefore,\ g\circ f = \{(1,apple),\ (2,ball),\ (3,cat)\ \}$
$Thus,\ g\circ f\ is\ one-one.$
$Since\ the\ number\ of\ elemenets\ in\ the\ sets\ \{1,2,3\}\ and\ \{apple,\ ball,\ cat\}\ are\ equal.$
$Hence\ \ g\circ f\ is\ onto.$
$Therefore, function\ \ g\circ f\ is\ invertible.$
$Let\ us\ now\ find\ f^{-1}:$
$We\ have\ f:\{1,2,3\}\to\{a,b,c\}$
$Thus,\ f^{-1}:\{a,b,c\}\to\{1,2,3\}$
$\Rightarrow f^{-1} = \{(a,1),(b,2),(c,3)\}$
$Let\ us\ now\ find\ g^{-1}:$
$We\ have\ g:\{a,b,c\}\to\{apple,ball,cat\}$
$Thus,\ g^{-1}:\{apple,ball,cat\}\to\{a,b,c\}$
$\Rightarrow g^{-1} = \{(apple,a),(ball,b),(cat,c)\}$
$Let\ us\ now\ find\ f^{-1}\circ g^{-1}:$
$\Rightarrow\ f^{-1}\circ g^{-1} = \{(apple,1),\ (ball,\ 2),\ (cat,3)\}.....(1)$
$Also,\ let\ us\ find,\ (g\circ f)^{-1}:$
$\Rightarrow\ (g\circ f)^{-1} = \{(apple,\ 1),\ (ball,2),\ (cat,3)\ \}....(2)$
$From\ (1)\ and\ (2),\ we\ have,$
$(g\circ f)^{-1} = f^{-1}\circ g^{-1}$

Functions Ex 2.5 Q4

Given that

$$A = \{1, 2, 3, 4\},\quad B = \{3, 5, 7, 9\},\quad C = \{7, 23, 47, 79\}$$

$f : A \to B$ and $g : B \to C$ are two functions defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$

Now,

$$g\circ f(x) = g(f(x)) = g(2x+1) = (2x+1)^2 - 2$$

$$\Rightarrow \qquad g\circ f(x) = 4x^2 + 4x - 1$$

Now,

$f : A \to B$ given by $f(x) = 2x + 1$

Clearly f in one-one and onto, $\therefore$ f in bijective

$\Rightarrow \qquad f^{-1}$ exist

$\therefore \quad f^{-1} = \{(3,1), (5,2), (7,3), (9,7_1)\}$

Again, $g : B \to C$ given by $g(x) = x^2 - 2$

Clearly g in one-one and onto $\Rightarrow g^{-1}$ exists

$g^{-1} = \{(7,3), (23,5), (47,7), (79,9)\}$

$f\circ^{-1} g^{-1} = \{(7,1), (23,2), (47,3), (79,4)\}$ .... (A)

Now, $g\circ f(x) = 4x^2 + 4x - 1$

Clearly gof in one-one and onto $\Rightarrow (g\circ f)^{-1}$ exists.

Hence,

$(g\circ f)^{-1} = \{(7,1), (23,2), (47,3), (79,4)\}$ .... (B)

From (A) & (B) we have $\qquad g\circ f^{-1} = f\circ^{-1} g^{-1}$

Functions Ex 2.5 Q5

Given that $f: Q \to Q$ *defined by* $f(x) = 3x + 5$.

To prove that f *is invertible, we need to prove that* f *is one – one and onto.*

Let $(x, y) \in Q$ *be such that,* $f(x) = f(y)$

$\Rightarrow 3x + 5 = 3y + 5$

$\Rightarrow x = y$

So, f *is an injection.*

Let y *be an arbitrary element of* Q *such that* $f(x) = y$.

$\Rightarrow 3x + 5 = y$

$\Rightarrow 3x = y - 5$

$\Rightarrow x = \dfrac{y-5}{3}$

Thus, for any $y \in Q$ *there exists* $x = \dfrac{y-5}{3} \in Q$ *such that*

$f(x) = f\left(\dfrac{y-5}{3}\right) = 3\dfrac{y-5}{3} + 5 = y$

Thus, $f: Q \to Q$ *is a bijection and hence invertible.*

Let f^{-1} *denotes the inverse of* f.

Thus, $f \circ f^{-1}(x) = x$ *for all* $x \in Q$

$\Rightarrow f\left[f^{-1}(x)\right] = x$ *for all* $x \in Q$.

$\Rightarrow 3f^{-1}(x) + 5 = x$ *for all* $x \in Q$.

$\Rightarrow f^{-1}(x) = \dfrac{x-5}{3}$ *for all* $x \in Q$

Functions Ex 2.5 Q6

f: **R** → **R** is given by, $f(x) = 4x + 3$

One-one:

Let $f(x) = f(y)$.

$\Rightarrow 4x + 3 = 4y + 3$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

Therefore f is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$\Rightarrow x = \dfrac{y-3}{4} \in \mathbf{R}$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \dfrac{y-3}{4} \in \mathbf{R}$ such that

$f(x) = f\left(\dfrac{y-3}{4}\right) = 4\left(\dfrac{y-3}{4}\right) + 3 = y.$

Therefore, f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: **R**→ **R** by $g(x) = \dfrac{x-3}{4}$

Now, $(gof)(x) = g(f(x)) = g(4x+3) = \dfrac{(4x+3)-3}{4} = x$

$(fog)(y) = f(g(y)) = f\left(\dfrac{y-3}{4}\right) = 4\left(\dfrac{y-3}{4}\right) + 3 = y - 3 + 3 = y$

Therefore, gof = fog = I_R

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \dfrac{y-3}{4}.$

Functions Ex 2.5 Q7

f: $\mathbf{R}_+ \to [4, \infty)$ is given as $f(x) = x^2 + 4$.
One-one:
Let $f(x) = f(y)$.

$$\Rightarrow x^2+4=y^2+4$$

$$\Rightarrow x^2=y^2$$

$$\Rightarrow x=y \qquad [\text{as } x=y\in\mathbf{R}_+]$$

Therefore, f is a one-one function.
Onto:
For $y \in [4, \infty)$, let $y = x^2 + 4$.

$$\Rightarrow x^2=y-4\geq 0 \qquad [\text{as } y\geq 4]$$

$$\Rightarrow x=\sqrt{y-4}\geq 0$$

Therefore, for any $y \in \mathbf{R}$, there exists $x=\sqrt{y-4}\in\mathbf{R}$ such that

$$f(x)=f\left(\sqrt{y-4}\right)=\left(\sqrt{y-4}\right)^2+4=y-4+4=y.$$

Therefore, f is onto.
Thus, f is one-one and onto and therefore, f^{-1} exists.
Let us define g: $[4, \infty) \to \mathbf{R}_+$ by,

$$g(y)=\sqrt{y-4}$$

$$\text{Now, } gof(x)=g(f(x))=g(x^2+4)=\sqrt{(x^2+4)-4}=\sqrt{x^2}=x$$

$$\text{And, } fog(y)=f(g(y))=f\left(\sqrt{y-4}\right)=\left(\sqrt{y-4}\right)^2+4=(y-4)+4=y$$

Therefore, gof = fog = I_R
Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y)=g(y)=\sqrt{y-4}.$$

Functions Ex 2.5 Q8

It is given that $f(x)=\dfrac{(4x+3)}{(6x-4)},\ x\neq\dfrac{2}{3}$.

$$(fof)(x)=f(f(x))=f\left(\frac{4x+3}{6x-4}\right)$$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x$$

$$\text{Therefore, } fof(x)=x,\ \text{for all } x\neq\frac{2}{3}.$$

$$\Rightarrow fof=\mathrm{I}$$

Hence, the given function f is invertible and the inverse of f is f itself.

Functions Ex 2.5 Q9

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.
Let y be an arbitrary element of $[-5, \infty)$.
Let $y = 9x^2 + 6x - 5$.

$$\Rightarrow y=(3x+1)^2-1-5=(3x+1)^2-6$$

$$\Rightarrow (3x+1)^2=y+6$$

$$\Rightarrow 3x+1=\sqrt{y+6} \qquad [\text{as } y\geq -5\Rightarrow y+6>0]$$

$$\Rightarrow x=\frac{\sqrt{y+6}-1}{3}$$

Therefore, f is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \to \mathbf{R}_+$ as $g(y)=\dfrac{\sqrt{y+6}-1}{3}$.

We now have:

$$\begin{aligned}(gof)(x)&=g(f(x))=g(9x^2+6x-5)\\&=g\left((3x+1)^2-6\right)\\&=\frac{\sqrt{(3x+1)^2-6+6}-1}{3}\\&=\frac{3x+1-1}{3}=x\end{aligned}$$

$$\text{And}, (fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right)$$

$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$$

$$= \left(\sqrt{y+6}\right)^2 - 6 = y + 6 - 6 = y$$

Therefore, gof = I_R and fog = $I_{(-5, \infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Functions Ex 2.5 Q10

$f : R \to R$ be a function defined by

$f(x) = x^3 - 3$

Injectivity:

let $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^3 - 3 = x_2^3 - 3$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad f$ in one-one

Surjectivity: let $y \in R$ be arbitrary such that

$f(x) = y$

$\Rightarrow \quad x^3 - 3 - y = 0$

We know that an equation of odd degree must have atleast one real solution.

let $x = \alpha$ be that solution

$\therefore \quad \alpha^3 - 3 = y$

$\Rightarrow \quad f(\alpha) = y$

so, for each $y \in R$ in co-domain there exist $\alpha \in R$ in domain

$\Rightarrow \quad f$ in onto

Thus, f in one-one and onto, so

f^{-1} exists

Now,

$\because \quad f(x) = x^3 - 3 = y$

$\Rightarrow \quad x^3 = 3 + y$

$\Rightarrow \quad x = 3\sqrt{3+y}$

$\Rightarrow \quad f^{-1}(x) = 3\sqrt{3+x}$

Thus, $f^{-1} : R \to R$ be the inverse function defined by $f^{-1}(x) = (x+3)^{1/3}$

finally,

$f^{-1}(24) = (24+3)^{1/3} = 3$

$f^{-1}(5) = (5+3)^{1/3} = 2$

Functions Ex 2.5 Q11

We have,

$f : R \to R$ in a function defined by

$f(x) = x^3 + 4$

Injectivity: let $f(x_1) = f(x_2)$ for $x_1 x_2 \in R$

$\Rightarrow \quad x_1^3 + 4 = x_2^3 + 4$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad f$ in one-one

Surjectivity: let $y \in R$ be artritrary such that

$f(x) = y$

$\Rightarrow \quad x^3 + 4 = y$

$\Rightarrow \quad x^3 + 4 - y = 0$

We know that an odd degree equation must have a real root.

$\Rightarrow \quad \alpha^3 + 4 = y \Rightarrow f(\alpha) = y$

$\Rightarrow \quad f$ in onto

Since f in one-one and onto

$\Rightarrow \quad f$ in bijective

finally.

$f(x) = y$

$\Rightarrow \quad x^3 + 4 = y$

$\Rightarrow \quad x^3 = y - 4$

$\Rightarrow \quad x = (y - 4)^{1/3}$

$\therefore \quad f^{-1}(x) = (x - 4)^{1/3}$

$\therefore \quad f^{-1}(3) = (3 - 4)^{1/3} = -1$

Functions Ex 2.5 Q12

Given that $f(x)=2x$ and $g(x)=x+2$.
We need to prove that f and g are bijective maps.
Let $x,y \in Q$.
Consider $f(x)=f(y)$
$\Rightarrow 2x=2y$
$\Rightarrow x=y$
$\Rightarrow f$ is one $-$ one.
Let y be an arbitrary element of Q such that $f(x)=y$
Then $f(x)=y=2x \Rightarrow x=\frac{y}{2}$
Thus, for any $y \in Q$, there exists $x=\frac{y}{2} \in Q$ such that,
$f(x)=f\left(\frac{y}{2}\right)=2\frac{y}{2}=y$
So $f: Q \to Q$ is a bijection and hence invertible.
Let f^{-1} denote the inverse of f.
Thus, $f^{-1}(x)=\frac{x}{2}$ ...(1)
Let $x,y \in Q$.
Consider $g(x)=g(y)$
$\Rightarrow x+2=y+2$
$\Rightarrow x=y$
$\Rightarrow g$ is one $-$ one.
Let y be an arbitrary element of Q such that $g(x)=y$
Then $g(x)=y=x+2 \Rightarrow x=y-2$
Thus, for any $y \in Q$, there exists $x=y-2$, $y \in Q$ such that,
$g(x)=g(y-2)=y-2+2=y$
So $g: Q \to Q$ is a bijection and hence invertible.
Let g^{-1} denote the inverse of g.
Thus, $g^{-1}(x)=x-2$...(2)
Now consider $g \circ f=g[f(x)]=g(2x)=2x+2$
Thus, $(g \circ f)^{-1}=\frac{x-2}{2}$ ...(3)
From (1) and (2), we have
$f^{-1} \circ g^{-1}=f^{-1}\left[g^{-1}(x)\right]=f^{-1}[x-2]=\frac{x-2}{2}$ ...(4)
From (3) and (4), it is clear that
$(g \circ f)^{-1}=f^{-1} \circ g^{-1}$

Functions Ex 2.5 Q13

Given that $f(x) = \dfrac{x-2}{x-3}$;

Let $f(x) = y$;

$\Rightarrow y = \dfrac{x-2}{x-3}$

Interchange x *and* y;

$\Rightarrow x = \dfrac{y-2}{y-3}$

$\Rightarrow (y-3)x = y-2$

$\Rightarrow xy - 3x = y - 2$

$\Rightarrow xy - y = 3x - 2$

$\Rightarrow y(x-1) = 3x - 2$

$\Rightarrow y = \dfrac{3x-2}{x-1}$

$\Rightarrow f^{-1}(x) = \dfrac{3x-2}{x-1}$

Functions Ex 2.5 Q14

$f : R^+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$

For any $x,\ y \in R^+$

$f(x) = f(y)$

$\Rightarrow 5x^2 + 6x - 9 = 5y^2 + 6y - 9$

$\Rightarrow 5(x^2 - y^2) + 6(x - y) = 0$

$\Rightarrow (x - y)[5(x + y) + 6] = 0$

$\Rightarrow x - y = 0 \qquad [\because 5(x + y) + 6 \neq 0 \text{ as } x, y \in R^+]$

$\Rightarrow x = y$

So, f is an injection.

Let y be an arbitrary element of $[-9, \infty)$.

$f(x) = y$

$\Rightarrow 5x^2 + 6x - 9 = y$

$\Rightarrow 25x^2 + 30x - 45 = 5y$

$\Rightarrow 25x^2 + 30x + 9 - 54 = 5y$

$\Rightarrow (5x + 3)^2 = 5y + 54$

$\Rightarrow (5x + 3) = \sqrt{5y + 54}$

$\Rightarrow x = \dfrac{\sqrt{5y + 54} - 3}{5}$

Now, $y \in [-9, \infty)$

$\Rightarrow y \geq -9$

$\Rightarrow 5y + 54 \geq 9$

$\Rightarrow \sqrt{5y+54} \geq 3$

$\Rightarrow \sqrt{5y+54} - 3 \geq 0$

$\Rightarrow \dfrac{\sqrt{5y+54}-3}{5} \geq 0$

$\Rightarrow x \geq 0 \Rightarrow x \in R^{+}$

Thus, for every $y \in [-9, \infty)$ there exist $x = \dfrac{\sqrt{5y+54}-3}{5} \in R^{+}$ such that $f(x) = y$.

So, $f : R^{+} \to [-9, \infty)$ is onto.

Thus, $f : R^{+} \to [-9, \infty)$ is a bijection and hence invertible.

Let f^{-1} denote the inverse of f.

Then,

$(fof^{-1})(y) = y$ for all $y \in [-9, \infty)$

$f(f^{-1}(y)) = y$ for all $y \in [-9, \infty)$

$\Rightarrow 5\{f^{-1}(y)\}^2 + 6\{f^{-1}(y)\} - 9 = y$ for all $y \in [-9, \infty)$

$\Rightarrow 25\{f^{-1}(y)\}^2 + 30\{f^{-1}(y)\} - 45 = 5y$ for all $y \in [-9, \infty)$

$\Rightarrow 25\{f^{-1}(y)\}^2 + 30\{f^{-1}(y)\} + 9 = 5y + 54$ for all $y \in [-9, \infty)$

$\Rightarrow \{5f^{-1}(y) + 3\}^2 = 5y + 54$ for all $y \in [-9, \infty)$

$\Rightarrow 5f^{-1}(y) + 3 = \sqrt{5y+54}$ for all $y \in [-9, \infty)$

$\Rightarrow f^{-1}(y) \dfrac{\sqrt{5y+54}-3}{5}$

Functions Ex 2.5 Q15

We have given that

$f : R \to (-1, 1)$ defined by

$f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is invertible

let $f(x) = y$

$$\Rightarrow \quad \frac{10^x - 10^{-x}}{10^x + 10^{-x}} = y$$

$$\Rightarrow \quad \frac{10^{2x} - 1}{10^{2x} - 1} = y$$

$$\Rightarrow \quad 10^{2x} - 1 = y\left(10^{2x} + 1\right)$$

$$\Rightarrow \quad 10^{2x} - 10^{2x}y = y + 1$$

$$\Rightarrow \quad 10^{2x}(1 - y) = y + 1$$

$$\Rightarrow \quad 10^{2x} = \frac{y+1}{1-y}$$

$$\Rightarrow \quad 2x = log_{10}\left(\frac{1+y}{1-y}\right)$$

$$x = \frac{1}{2} log_{10}\left(\frac{1+y}{1-y}\right)$$

$$\therefore \quad f^{-1}(x) = \frac{1}{2} log_{10}\left(\frac{1+x}{1-x}\right)$$

Functions Ex 2.5 Q16

We have given that

$f : R \to (0, 2)$ defined by

$f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 1$ is invertible.

let $f(x) = y$

$$\Rightarrow \quad \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1 = y$$

$$\Rightarrow \quad \frac{2e^x}{e^x + e^{-x}} = y$$

$$\Rightarrow \quad \frac{2e^{2x}}{e^{2x} + 1} = y$$

$$\Rightarrow \quad 2e^{2x} = y\left(e^{2x} + 1\right)$$

$$\Rightarrow \quad e^{2x}\left(2 - y\right) = y$$

$$\Rightarrow \quad e^{2x} = \frac{y}{2 - y} \Rightarrow x = \frac{1}{2} log_e\left(\frac{y}{2 - y}\right)$$

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2} log_e\left(\frac{x}{2 - x}\right)$$

Functions Ex 2.5 Q17

Given: that

$f : [-1, \infty] \to [-1, \infty]$ is a function

given by $f(x) = (x+1)^2 - 1$

In order to show that f in invertible, we need to prove that f in bijective.

Injective: let $x, y \in [-1, \infty]$, Such that

$f(x) = f(y)$

$\Rightarrow \quad (x+1)^2 - 1 = (y+1)^2 - 1$

$\Rightarrow \quad (x+1)^2 = (y+1)^2$

$\Rightarrow \quad x+1 = y+1 \qquad [x, y \in [-1, \infty]]$

$\Rightarrow \quad x = y$

$\Rightarrow \quad f$ is one-one

Surjectivity: let $y \in [-1, \infty]$ be arbitrary

such that $f(x) = y$

$\Rightarrow \quad (x+1)^2 - 1 = y$

$= \quad (x+1)^2 = y+1$

$\Rightarrow \quad x+1 = \sqrt{y+1}$

$\Rightarrow \quad x = \sqrt{y+1} - 1 \in [-1, \infty]$

So, for each $y \in [-1, \infty]$ (co-domain) there exist $x = \sqrt{y+1} - 1 \in [-1, \infty]$ (domain)

$\therefore \quad f$ is onto

Thus, f is bijective $\Rightarrow \qquad f$ is invertible.

Now,

$f(x) = f^{-1}(x)$

$\Rightarrow \quad (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow \quad (x+1)^2 - \sqrt{x+1} = 0$

$\Rightarrow \quad \sqrt{x+1}\left((x+1)^{3/2} - 1\right) = 0$

$\Rightarrow \quad \sqrt{x+1} = 0$ or $(x+1)^{3/2} - 1 = 0$

$\Rightarrow \quad x = -1 \quad$ or $x = 0$

$\therefore \quad x = 0, -1$

Hence, $S = \{0, -1\}$

Functions Ex 2.5 Q18

$A = \{x \in R : -1 \le x \le 1\}$ and $f : A \to A$, $g : A \to A$ are two functions defined by $f(x) = x^2$ and $g(x) = sin\left(\frac{\pi x}{2}\right)$

Here, $f : A \to A$ is defined by

$$f(x) = x^2$$

Clearly f in not injective, $\because f(1) = f(-1) = 1$

$So,$ f is not bijective and hence not invertible.
Hence, f^{-1} does not exist

Now, $g : A \to A$ defined by

$$g(x) = sin\left(\frac{\pi x}{2}\right)$$

Injectivity: Let $x_1 = x_2$

$\Rightarrow \quad \frac{\pi x_1}{2} = \frac{\pi x_2}{2}$

$\Rightarrow \quad sin\left(\frac{\pi x_1}{2}\right) = sin\left(\frac{\pi x_2}{2}\right) \qquad [\because -1 \le x \le 1]$

$\Rightarrow \quad g(x_1) = g(x_2)$

$\Rightarrow \quad g$ is one-one ....(i)

Surjectivity: let y be aribitrary such that

$g(x) = y$

$\Rightarrow \quad sin\left(\frac{\pi x}{2}\right) = y$

$\Rightarrow \quad \frac{\pi x}{2} = sin^{-1} y$

$\Rightarrow \quad x = \frac{2}{\pi} sin^{-1} y = [-1, 1]$

Thus, for each y in codomain, there exists x in domain, such that

$g(x) = y$

$\Rightarrow \quad g$ is surjective ....(ii)

From (i) & (ii)

Functions Ex 2.5 Q19

Given: $f : R \to R$ is a function defined by

$$f(x) = cos(x + 2)$$

Injectivity: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow \quad cos(x + 2) = cos(y + 2)$

$\Rightarrow \quad x + 2 = 2n\pi \pm y + 2$

$\Rightarrow \quad x = 2n\pi \pm y$

$\Rightarrow \quad x \ne y$

$\Rightarrow \quad f$ is not one-one

Hence, f is not bijective

$\Rightarrow \quad f$ is not invertible

Functions Ex 2.5 Q20

We have, $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$

We know that a function from A to B is said to be bijection if it is one-one and onto. This means different elements of A has different image in B. Also each element of B has preimage in A.

Let f_1, f_2, f_3 and f_4 are the functions from A to B.

$$f_1 = \{(1, a), (2, b), (3, c), (4, d)\}$$
$$f_2 = \{(1, b), (2, c), (3, d), (4, a)\}$$
$$f_3 = \{(1, c), (2, d), (3, a), (4, b)\}$$
$$f_4 = \{(1, d), (2, a), (3, b), (4, c)\}$$

we can verify that f_1, f_2, f_3 and f_4 are bijective from A to B.

Now,

$$f_1^{-1} = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$$
$$f_2^{-1} = \{(b, 1), (c, 2), (d, 3), (a, 4)\}$$
$$f_3^{-1} = \{(c, 1), (d, 2), (a, 3), (b, 4)\}$$
$$f_4^{-1} = \{(d, 1), (a, 2), (b, 3), (c, 4)\}$$

Functions Ex 2.5 Q21

Given: A and B are two sets with finite elements.

$f : A \to B$ and $g : B \to A$ are injective map.

To prove: f in bijective

Proof: *Since*, $f : A \to B$ in injective we need to show f in surjective only.

Now,

$g : B \to A$ in injective

$\Rightarrow$ each element of B has image in A.

Functions Ex 2.5 Q22

We have,

$f : Q \to Q$ and $g : Q \to Q$ are two function defined by

$f(x) = 2x$ and $g(x) = x + 2$

Now, $f : Q \to Q$ defined by $f(x) = 2x$

Injectivity: let $x, y \in Q$ such that

$f(x) = f(y) \Rightarrow 2x = 2y \qquad \Rightarrow x = y$

$\Rightarrow$ f in one-one

Surjectivity: let $y \in Q$ such that

$f(x) = y \qquad \Rightarrow \qquad 2x = y \qquad \Rightarrow x = y/2 \in Q$

$\therefore$ For each $y \in Q$ (co-domain) there exist $x = y/2 \in Q$ (domain) such that $f(x) = y$

$\Rightarrow$ f is onto

$\therefore$ f in bijective

Again for $g : Q \to Q$ defined by

$g(x) = x + 2$

Injectivity: let $x, y \in Q$ such that

$g(y) = g(x) \Rightarrow \qquad y + 2 = x + 2 \Rightarrow \qquad y = x$

$\Rightarrow$ g is one-one

Surjectivity: let $y \in Q$ be arbitrary such that

$g(x) = y \Rightarrow \quad x + 2 = y \Rightarrow \quad x = y - 2 \in Q$

Thus, for each $y \in Q$ (co-domain), there exist $x = y - 2 \in Q$ such that $g(x) = y$

$\therefore$ g in onto

Hence, g is bijective.

$g \circ f(x) = g(f(x)) = g(2x) = 2x + 2$

$\Rightarrow$ $gof(x) = 2x + 2$

f and g are bijective $\Rightarrow g \circ f$ is bijective

$\Rightarrow$ $(g \circ f)^{-1}$ exist

Now, $(g \circ f)(x) = 2x + 2$

$\Rightarrow (g \circ f)^{-1}(2x + 2) = x$

$\Rightarrow (g \circ f)^{-1}(2x) = x - 2$

$(g \circ f)^{-1}(x) = \frac{1}{2}(x - 2) \qquad \ldots A$

Again,

f is bijective $\Rightarrow f^{-1}$ exist

$\therefore$ $f^{-1} : Q \to Q$ defined by

$f^{-1}(x) = x/2$

Also, g is bijective $\Rightarrow g^{-1}$ exist.

$\therefore$ $g^{-1} : Q \to Q$ defined by

$g^{-1}(x) = x - 2$

$\therefore$ $f^{-1} \circ g^{-1}(x) = f^{-1}(g^{-1}(x))$

$= f^{-1}(x - 2)$

$(f^{-1} \circ g^{-1})(x) = \frac{1}{2}(x - 2) \ldots\ldots\ldots\ldots\ldots (B)$

From (A) & (B)

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$