Ex-10.1

Q1

$\angle A = 45^\circ, \angle B = 60^\circ \; and \angle C = 75^\circ$

Using sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin 45} = \frac{b}{\sin 60} = \frac{c}{\sin 75} = k$$

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = k$$

$$a : b : c = 2 : \sqrt{6} : (\sqrt{3}+1)$$

Q2

$\angle C = 105^\circ, \angle B = 45^\circ, a = 2$

From here we can calculate that

$\angle A = 30°$

$$a \sin B = b \sin A$$

$$\Rightarrow 2 \sin 45 = b \sin 30$$

$$\Rightarrow 2 \times \frac{1}{\sqrt{2}} = b \times \frac{1}{2}$$

$$\Rightarrow \sqrt{2} = \frac{b}{2}$$

$$\Rightarrow b = 2\sqrt{2}$$

Q3

$a = 18, b = 24, c = 30, \angle C = 90°$

let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\frac{\sin A}{18} = \frac{\sin B}{24} = \frac{\sin 90}{30}$$

$$\frac{\sin A}{18} = \frac{\sin B}{24} = \frac{1}{30}$$

$$\frac{\sin A}{18} = \frac{1}{30} \Rightarrow \sin A = \frac{18}{30} = \frac{3}{5}$$

$$\frac{\sin B}{24} = \frac{1}{30} \Rightarrow \sin B = \frac{24}{30} = \frac{4}{5}$$

$$\therefore \sin A = \frac{3}{5}, \sin B = \frac{4}{5}, \sin C = 1$$

Q4

$$\frac{a-b}{a+b}=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

Let $a=k\sin A, b=k\sin B$ (Using sine rule)

LHS

$$=\frac{a-b}{a+b}$$

$$=\frac{k\sin A-k\sin B}{k\sin A+k\sin B}$$

$$=\frac{\sin A-\sin B}{\sin A+\sin B}$$

$$=\frac{2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})}{2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})}$$

$$=\frac{\tan(\frac{A-B}{2})}{\tan(\frac{A+B}{2})}=RHS$$

Q5

$$(a-b)\cos\frac{C}{2}=c\sin\left(\frac{A-B}{2}\right)$$

Let $a=k\ \sin\ A, b=k\sin\ B, c=k\sin C$

LHS

$$(a-b)\cos\frac{C}{2}$$

$$=k(\sin A-\sin B).\cos\frac{C}{2}$$

$$=2k\cos(\frac{A+B}{2})\sin(\frac{A-B}{2}).\cos\frac{C}{2}$$

$$=2k\cos(\frac{\pi-C}{2})\sin(\frac{A-B}{2}).\cos\frac{C}{2}$$

$$=2k\sin(\frac{C}{2}).\cos\frac{C}{2}.\sin(\frac{A-B}{2}) \qquad [\cos(\frac{\pi}{2}-\theta)=\sin\theta]$$

$$=k\sin C.\sin(\frac{A-B}{2})$$

$$=c.\sin(\frac{A-B}{2})=RHS$$

Q6

$$\frac{c}{a-b}=\frac{\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right)-\tan\left(\frac{B}{2}\right)}$$

LHS

$$\frac{c}{a-b}$$

$$=\frac{k\sin C}{k\sin A-k\sin B}$$

$$=\frac{\sin C}{\sin A-\sin B}$$

$$=\frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{\sin A-\sin B}$$

$$=\frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})}$$

$$=\frac{\sin\frac{C}{2}\cos\frac{(\pi-(A+B))}{2}}{\cos(\frac{\pi-C}{2})\sin(\frac{A-B}{2})}$$

$$=\frac{\sin\frac{C}{2}\sin\frac{(A+B)}{2}}{\sin\frac{C}{2}\sin(\frac{A-B}{2})}$$

$$=\frac{\sin\frac{(A+B)}{2}}{\sin(\frac{A-B}{2})}$$

$$=\frac{\sin(\frac{A}{2}).\cos(\frac{B}{2})+\sin(\frac{B}{2}).\cos(\frac{A}{2})}{\sin(\frac{A}{2}).\cos(\frac{B}{2})-\sin(\frac{B}{2}).\cos(\frac{A}{2})}$$

$$=\frac{\tan(\frac{A}{2})+\tan(\frac{B}{2})}{\tan(\frac{A}{2}).-\tan(\frac{B}{2}).}\text{ [Dividing both Numerator and Denominator by }\cos(\frac{A}{2}).\cos(\frac{B}{2})\text{]}$$

$$=RHS$$

Q7

$$\frac{c}{a+b}=\frac{1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$$

LHS

$$=\frac{c}{a+b}$$

$$=\frac{k\sin C}{k\sin A+k\sin B}$$

$$=\frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin(\frac{A+B}{2}).\cos(\frac{A-B}{2})}$$

$$=\frac{\sin\frac{C}{2}\cos\frac{C}{2}}{\sin(\frac{\pi-C}{2}).\cos(\frac{A-B}{2})}$$

$$=\frac{\sin(\frac{\pi-(A+B)}{2})\cos\frac{C}{2}}{\cos(\frac{C}{2}).\cos(\frac{A-B}{2})}$$

$$=\frac{\cos(\frac{A+B}{2})}{\cos(\frac{A-B}{2})}$$

$$=\frac{\cos\frac{A}{2}.\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}.\cos\frac{B}{2}+\sin\frac{A}{2}.\sin\frac{B}{2}}$$

$$=\frac{1-\tan\frac{A}{2}\tan\frac{B}{2}}{1+\tan\frac{A}{2}.\tan\frac{B}{2}}\text{[Dividing both Numerator and Denominator by }\cos(\frac{A}{2}).\cos(\frac{B}{2})\text{]}$$

$=RHS$

Q8

$$\frac{a+b}{c}=\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

Let $a=k\sin\ A, b=k\sin\ B, c=k\sin C$

LHS

$$\frac{k\sin A+k\sin B}{k\sin C}$$

$$=\frac{\sin A+\sin B}{\sin C}$$

$$=\frac{2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}}{2\sin\frac{C}{2}.\cos\frac{C}{2}}$$

$$=\frac{\sin(\frac{\pi-C}{2}).\cos\frac{A-B}{2}}{\sin\frac{C}{2}.\cos\frac{C}{2}}$$

$$=\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}=RHS$$

Q9

$$\sin\left(\frac{B-C}{2}\right)=\frac{b-c}{a}\cos\frac{A}{2}$$

Let $a=k\sin A, b=k\sin B, c=k\sin C$

RHS

$$\frac{b-c}{a}\cos\frac{A}{2}$$

$$=\frac{k\sin B-k\sin C}{k\sin A}.\cos\frac{A}{2}$$

$$\frac{\sin B-\sin C}{\sin A}.\cos\frac{A}{2}$$

$$=\frac{2\cos\frac{B+C}{2}.\sin\frac{B-C}{2}}{2\sin\frac{A}{2}.\cos\frac{A}{2}}\cos\frac{A}{2}$$

$$=\frac{\cos\frac{\pi-A}{2}\sin\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$=\frac{\sin\frac{A}{2}\sin\frac{B-C}{2}}{\sin\frac{A}{2}}=\sin\frac{B-C}{2}=RHS$$

Q10

$$\text{let } \frac{\text{a}}{\sin \text{A}}=\frac{\text{b}}{\sin \text{B}}=\frac{\text{c}}{\sin \text{C}}=k$$

LHS,

$$\frac{a^2-c^2}{b^2}$$

$$=\frac{k^2\sin^2 A-k^2\sin^2 C}{k^2\sin^2 B}$$

$$=\frac{k^2(\sin^2 A-\sin^2 C)}{k^2\sin^2 B}$$

$$=\frac{(\sin^2 A-\sin^2 C)}{\sin^2(\pi-(A+C))}$$

$$=\frac{\sin(A+C)\sin(A-C)}{\sin^2(A+C)}$$

$$=\frac{\sin(A-C)}{\sin(A+C)}=RHS$$

Q11

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$$

RHS,

$a \ \sin(B-C)$

$=a\sin B.\cos C-a\sin C.\cos B$

$$=a(bk).\left(\frac{a^2+b^2-c^2}{2ab}\right)-a(ck).\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$=k.\frac{(a^2+b^2-c^2)}{2}-k\frac{(a^2+c^2-b^2)}{2}$$

$$=2k.\frac{(b^2-c^2)}{2}$$

$=b.(kb)-c(kc)$

$=b(\sin B)-c(\sin C)$

LHS

Q12

$a^2\sin(B-C)= \ (b^2-c^2)\sin A$

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$$

$LHS,$

$a^2\sin(B-C)$

$=a^2\{\sin B.\cos C-\sin C.\cos B\}$

$$=a^2kb.\frac{a^2+b^2-c^2}{2ab}-a^2ck.\frac{a^2+c^2-b^2}{2ac}\text{[Using cos rule and sine rule]}$$

$$=a^2k.\frac{a^2+b^2-c^2}{2a}-a^2k.\frac{a^2+c^2-b^2}{2a}$$

$$=a^2k.\left(\frac{a^2+b^2-c^2-a^2-c^2+b^2}{2a}\right)$$

$$=a^2k.\left(\frac{2b^2-2c^2}{2a}\right)$$

$=ak.(b^2-c^2)$

$=\sin A(b^2-c^2)=RHS$

Hence Proved

Q13

$$\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}}=\frac{a+b-2\sqrt{ab}}{a-b}$$

RHS

$$\frac{a+b-2\sqrt{ab}}{a-b}$$

$$=\frac{\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2-2\sqrt{ab}}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2}$$

$$=\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2}$$

$$=\frac{\left(\sqrt{a}-\sqrt{b}\right)}{(\sqrt{a}+\sqrt{b})}$$

$$=\frac{\left(\sqrt{k\sin A}-\sqrt{k\sin B}\right)}{(\sqrt{k\sin A}+\sqrt{k\sin B})}$$

$$=\frac{\left(\sqrt{\sin A}-\sqrt{\sin B}\right)}{(\sqrt{\sin A}+\sqrt{\sin B})}[\text{taking } k \text{ common and cancelling them}]$$

$= LHS$

Hence Proved

Q14

$LHS,$

$a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)$

$=a\sin B-a\sin C+b\sin C-b\sin A+c\sin A-c\sin B$

$=b\sin A-c\sin A+c\sin B-b\sin A+c\sin A-c\sin B[\because b\sin A=a\sin B, b\sin C=c\sin B, c\sin A=a\sin C]$

$=0=RHS$

Hence Proved

Q15

$$\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0$$

$$\frac{a}{\sin A}=\frac{b}{b\sin C}=\frac{c}{\sin C}=k$$

LHS

$$\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}$$

$=ak\sin(B-C)+bk\sin(C-A)+ck\sin(A-B)$

$=\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)$

$=\sin(\pi-(B+C))\sin(B-C)+\sin(\pi-(C+A))\sin(C-A)$

$+\sin(\pi-(A+B))\sin(A-B)$

$=\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A)$

$+\sin(A+B)\sin(A-B)$

$=\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B=0=RHS$

Q16

$a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

LHS

$=a^2(1-\sin^2 B-1+\sin^2 C) + b^2(1-\sin^2 C-1+\sin^2 A)$

$+ c^2(1-\sin^2 A-1+\sin^2 B)$

$=a^2(\sin^2 C-\sin^2 B) + b^2(\sin^2 A-\sin^2 C)+ c^2(\sin^2 B-\sin^2 A)$

$=a^2(k^2c^2-k^2b^2) + b^2(k^2a^2-k^2c^2)+ c^2(k^2b^2-k^2a^2)$

$=k^2(a^2c^2-a^2b^2+b^2a^2-b^2c^2+b^2c^2-a^2c^2)$

$=k^2\times 0=0=RHS$

Q17

Let $a=k\sin A, b=k\sin B, c=k\sin C$

LHS

$b\cos B+c\cos C$

$=k\sin B\cos B+k\sin C\cos C$

$=\frac{k}{2}(2\sin B\cos B+2\sin C\cos C)$

$=\frac{k}{2}(\sin 2B+\sin 2C)$

$=\frac{k}{2}2\sin(B+C)\cos(B-C)$

$=k\sin(\pi-A)\cos(B-C)$

$=k\sin A\cos(B-C)$

$=a\cos(B-C)=RHS$

Q18

$$\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$$

LHS

$$=\frac{1-2\sin^2 A}{a^2}-\frac{1-2\sin^2 B}{b^2}$$

$$=\frac{1}{a^2}-\frac{1}{b^2}-2\left(\frac{\sin^2 A}{a^2}-\frac{\sin^2 B}{b^2}\right)$$

$$=\frac{1}{a^2}-\frac{1}{b^2}-2\left(k^2-k^2\right)[\text{Using sine rule}]$$

$$=\frac{1}{a^2}-\frac{1}{b^2}=RHS$$

hence Proved

Q19

$$\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b}=0$$

LHS

$$\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b}$$

$$=\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b}$$

$$=\frac{1-\sin^2 B-1+\sin^2 C}{b+c}+\frac{1-\sin^2 C-1+\sin^2 A}{c+a}+\frac{1-\sin^2 A-1+\sin^2 B}{a+b}$$

$$=\frac{\sin^2 C-\sin^2 B}{b+c}+\frac{\sin^2 A-\sin^2 C}{c+a}+\frac{\sin^2 B-\sin^2 A}{a+b}$$

$$=\frac{k^2c^2-k^2b^2}{b+c}+\frac{k^2a^2-k^2c^2}{c+a}+\frac{k^2b^2-k^2a^2}{a+b}$$

$$=k^2\left(\frac{c^2-b^2}{b+c}+\frac{a^2-c^2}{c+a}+\frac{b^2-a^2}{a+b}\right)$$

$$=k^2(c-b+a-c+b-a)[\text{Using } b^2-a^2=(b-a)(b+a)]$$

$$=0=RHS$$

Hence Proved

Q20

We know $a\sin B = b\sin A, c\sin B = b\sin C, a\sin C = c\sin B$

$$a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right)+b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right)+c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right)=0$$

LHS

$$=a\sin\left(\frac{\pi-(B+C)}{2}\right)\sin\left(\frac{B-C}{2}\right)+b\sin\left(\frac{\pi-(C+A)}{2}\right)\sin\left(\frac{C-A}{2}\right)$$

$$+c\sin\left(\frac{\pi-(A+B)}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$=a\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)+b\cos\left(\frac{C+A}{2}\right)\sin\left(\frac{C-A}{2}\right)$$

$$+c\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$=a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)$$

$$=a\sin B-a\sin C+b\sin C-b\sin A+c\sin A-c\sin B$$

$$=b\sin A-a\sin C+b\sin C-b\sin A+a\sin C-b\sin C$$

$$=0=RHS$$

Q21

$$\frac{b\sec B+c\sec C}{\tan B+\tan C}=\frac{c\sec C+a\sec A}{\tan C+\tan A}=\frac{a\sec A+b\sec B}{\tan A+\tan B}$$

$$\frac{b\sec B+c\sec C}{\tan B+\tan C}$$

$$=\frac{k\sin B\sec B+k\sin C\sec C}{\tan B+\tan C}$$

$$=\frac{k\sin B\dfrac{1}{\cos B}+k\sin C\dfrac{1}{\cos C}}{\tan B+\tan C}$$

$$=\frac{k\tan B+k\tan C}{\tan B+\tan C}=\frac{k(\tan B+\tan C)}{\tan B+\tan C}=k$$

$$Similarly,\ \frac{c\sec C+a\sec A}{\tan C+\tan A}=k$$

$$Similarly,\ \frac{a\sec A+b\sec B}{\tan A+\tan B}=k$$

Q22

$$a\cos A + b\cos B + c\cos C = 2b\sin A \sin C = 2c\sin A \sin B$$

LHS

$$a\cos A + b\cos B + c\cos C$$

$$= k \sin A\cos A + k\sin B\cos B + k\sin C\cos C$$

$$= \frac{k}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{k}{2}(2\sin(A+B).\cos(A-B) + 2\sin C.\cos C)$$

$$= \frac{2k}{2}(\sin(\pi - C).\cos(A-B) + \sin C.\cos C)$$

$$= k(\sin C.\cos(A-B) + \sin C.\cos C)$$

$$= k\sin C(\cos(A-B) + \cos C)$$

$$= k\sin C.2\cos(\frac{A-B+C}{2}).\cos(\frac{A-B-C}{2})$$

$$= k\sin C.2\cos(\frac{\pi - 2B}{2}).\cos(\frac{A-\pi+A}{2})$$

$$= k\sin C.2\sin B.\cos(\frac{2A-\pi}{2})$$

$$= k\sin C.2\sin B.\cos(\frac{\pi-2A}{2})$$

$$= k\sin C.2\sin B.\sin A$$

$$= 2\sin B\sin C(k\sin A) = 2a\sin B\sin C$$

$$= RHS$$

Similarly, $a\cos A + b\cos B + c\cos C = 2c\sin A \sin B$

Q23

$$a(\cos B\cos C + \cos A) = b(\cos A\cos C + \cos B) = c(\cos A\cos B + \cos C)$$

$$a(\cos B\cos C - \cos(\pi - (B+C)))$$

$$= a(\cos B\cos C - \cos(B+C))$$

$$= a(\cos B\cos C - \cos B.\cos C + \sin B\sin C)$$

$$= a\sin B\sin C$$

$$= k\sin A\sin B\sin C$$

Similarly, $b(\cos A\cos C + \cos B) = k\sin A\sin B\sin C$

Similarly, $c(\cos A\cos B + \cos C) = k\sin A\sin B\sin C$

Q24

Let $a = k\sin A$

$a(\cos C - \cos B) = \ 2(b-c)\ \cos^2\frac{A}{2}$

LHS

$= a(\cos C - \cos B)$

$= a2.\sin\frac{C+B}{2}.\sin\frac{B-C}{2}$

$= 2k\sin A\sin\frac{\pi-A}{2}.\sin\frac{B-C}{2}$

$= 2k2\sin\frac{A}{2}.\cos\frac{A}{2}.\cos\frac{A}{2}.\sin\frac{B-C}{2}$

$= 2k\cos^2\frac{A}{2}\left(2\sin\frac{B-C}{2}.\sin\frac{A}{2}\right)$

$= 2k\cos^2\frac{A}{2}\left(2\sin\frac{B-C}{2}.\sin\frac{\pi-(B+C)}{2}\right)$

$= 2k\cos^2\frac{A}{2}\left(2\sin\frac{B-C}{2}.\cos\frac{B+C}{2}\right)$

$= 2k\cos^2\frac{A}{2}(\sin B - \sin C)$

$= 2\cos^2\frac{A}{2}(k\sin B - k\sin C)$

$= 2\cos^2\frac{A}{2}(b-c) = RHS$

Q25

$b\cos\theta = c\cos(A-\theta)\ +a\cos(C+\theta)$

Let $a\sin C = c\sin A$[Using sine rule]

RHS

$= c\cos(A-\theta) + a\cos(C+\theta)$

$= c\cos A\cos\theta + c\sin A\cos\theta + a\cos C.\cos\theta - a\sin C\sin\theta$

$= k\sin C\cos A.\cos\theta + k\sin C\sin A\cos\theta + k\sin A\cos C.\cos\theta$

$-k\sin A\sin C\sin\theta$

$= k\sin C\cos A.\cos\theta + k\sin A\cos C.\cos\theta$

$= k\cos\theta(\sin C\cos A + \sin A\cos C)$

$= k\cos\theta\sin(C+A)$

$= k\cos\theta\sin(\pi - B)$

$= k\cos\theta\sin B$

$= k\sin B.\cos\theta = b\cos\theta = \ LHS$

Q26

Let $\sin A = ak, \sin B = bk, \sin C = ck$

$\sin^2 A + \sin^2 B = \sin^2 C$

$\Rightarrow k^2a^2 + k^2b^2 = k^2c^2$[Using sine rule]

$\Rightarrow a^2 + b^2 = c^2$

Since the triangle satisfies the Pythagoras theorem, therefore it is right angled.

Q27

a^2, b^2, c^2 are in A.P.

$\Rightarrow -2a^2, -2b^2, -2c^2$ are in A.P.

$\Rightarrow (a^2+b^2+c^2)-2a^2, (a^2+b^2+c^2)-2b^2, (a^2+b^2+c^2)-2c^2$ are in A.P.

$\Rightarrow (b^2+c^2-a^2), (c^2+a^2-b^2), (b^2+a^2-c^2)$ are in A.P.

$$\Rightarrow \frac{(b^2+c^2-a^2)}{2abc}, \frac{(c^2+a^2-b^2)}{2abc}, \frac{(b^2+a^2-c^2)}{2abc} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{a}\frac{(b^2+c^2-a^2)}{2bc}, \frac{1}{b}\frac{(c^2+a^2-b^2)}{2ac}, \frac{1}{c}\frac{(b^2+a^2-c^2)}{2ab} \text{ are in A.P.}$$

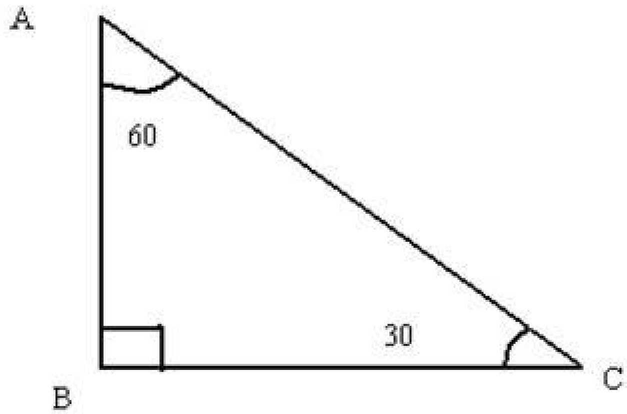
$$\Rightarrow \frac{1}{a}\cos A, \frac{1}{b}\cos B, \frac{1}{c}\cos C \text{ are in A.P.}$$

$$\Rightarrow \frac{k}{a}\cos A, \frac{k}{b}\cos B, \frac{k}{c}\cos C \text{ are in A.P.}$$

$$\Rightarrow \frac{\cos A}{\sin A}, \frac{\cos B}{\sin B}, \frac{\cos C}{\sin C} \text{ are in A.P.}$$

$\Rightarrow \cot A, \cot B, \cot C$ are in A.P.

Q28



BC=15m,AB=h

From the diagram we can calculate, $\angle A = 60°$

Using sine rule,

$$\frac{\sin A}{15} = \frac{\sin C}{h}$$

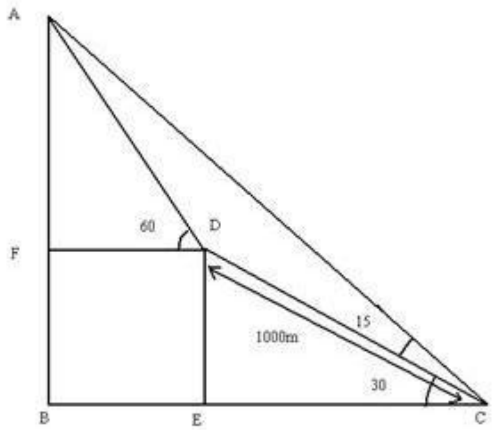
$$\Rightarrow \frac{\sin 60}{15} = \frac{\sin 30}{h}$$

$$\Rightarrow \frac{\sqrt{3}}{2\times 15} = \frac{1}{2\times h}$$

$$\Rightarrow \frac{\sqrt{3}}{15} = \frac{1}{h}$$

$$\Rightarrow h = \frac{15}{\sqrt{3}} \Rightarrow h = 5\sqrt{3}$$

Q29



$DE = 1000\sin 30 = 1000 \times \frac{1}{2} = 500m = FB$

$EC = 1000\cos 30 = 1000 \times \frac{\sqrt{3}}{2} = 500\sqrt{3}m$

Let $AF = x\ m$

$DF = \frac{x}{\sqrt{3}}m = BE$

We know,

From ΔABC,

$\tan 45 = \frac{AB}{BC}$

$\Rightarrow 1 = \frac{AF + FB}{BE + EC}$

$\Rightarrow 1 = \frac{x + 500}{\frac{x}{\sqrt{3}} + 500\sqrt{3}}$

$\Rightarrow \frac{x}{\sqrt{3}} + 500\sqrt{3} = x + 500$

$\Rightarrow x + 1500 = x\sqrt{3} + 500\sqrt{3}$

$\Rightarrow 1500 - 500\sqrt{3} = x\sqrt{3} - x$

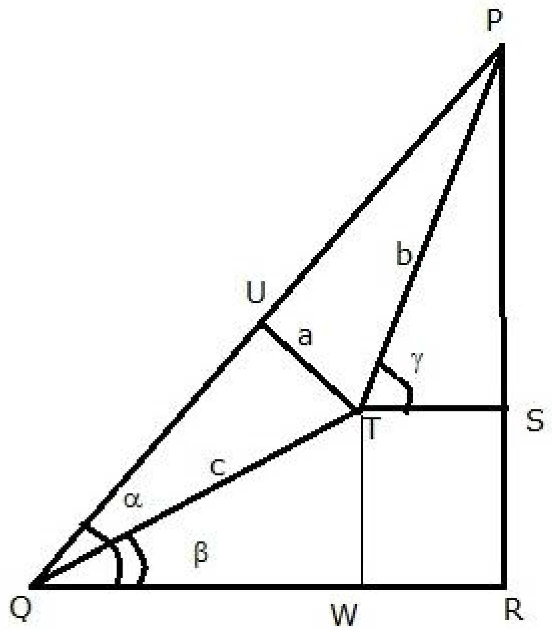
$\Rightarrow 500\sqrt{3}(\sqrt{3} - 1) = x(\sqrt{3} - 1)$

$\therefore x = 500\sqrt{3}m$

The height of the triangle is $AB = AF + FB = 500(\sqrt{3} + 1)m$

Q30

Consider the following figure.



The person is observing the peak P from the point Q.
The distance he travelled is $QT = c$ metres and the angle of inclination of QT is β.
He is observing the peak from the point and the angle of inclination is γ.
Now consider the triangle $\triangle QUT$.

$\angle TQU = \beta - \alpha$

Thus, $\sin(\alpha - \beta) = \dfrac{a}{c}$

$\Rightarrow a = c \times \sin(\alpha - \beta) \ldots.(1)$

Now consider the triangle $\triangle PQR$.
We know that $\angle QPR = 90° - \alpha$

In triangle $\triangle PTS$, $\angle TPS = 90° - \gamma$

Thus, $\angle TPU = \angle QPR - \angle TPS$

$\Rightarrow \angle TPU = (90° - \alpha) - (90° - \gamma)$

$\Rightarrow \angle TPU = \gamma - \alpha$

Now consider the $\triangle TPU$,

Thus, $\sin(\gamma - \alpha) = \dfrac{a}{b}$

$$\Rightarrow b = \frac{a}{\sin(\gamma - \alpha)}$$

Substituting the value of a from equation (1), *we have*,

$$b = \frac{c \times \sin(\alpha - \beta)}{\sin(\gamma - \alpha)} \ldots (2)$$

We need to find the total height of the peak PR.

Here, $PR = PS + SR \ldots . (3)$

From the triangle PST,

$$\sin\gamma = \frac{PS}{PT} = \frac{PS}{b}$$

$\Rightarrow PS = b\sin\gamma \ldots . (4)$

From the triangle QTW,

$$\sin\beta = \frac{TW}{QT} = \frac{TW}{c}$$

$\Rightarrow TW = SR = c\sin\beta \ldots . (5)$

Substituting the values of PS and SR from equations (4) *and* (5)

in equation (3), *we have*

$PR = PS + SR$

$\Rightarrow PR = b\sin\gamma + c\sin\beta$

$$\Rightarrow PR = \frac{c \times \sin(\alpha - \beta)}{\sin(\gamma - \alpha)}\sin\gamma + c\sin\beta \qquad [\textit{from equation } (2)]$$

$$\Rightarrow PR = \frac{c \times \sin(\alpha - \beta) \times \sin\gamma + c\sin\beta \times \sin(\gamma - \alpha)}{\sin(\gamma - \alpha)}$$

$$\Rightarrow PR = c\left[\frac{\sin\alpha \times \cos\beta \times \sin\gamma - \cos\alpha \times \sin\beta \times \sin\gamma + \sin\beta \times \sin\gamma \times \cos\alpha - \sin\beta \times \sin\alpha \times \cos\gamma}{\sin(\gamma - \alpha)}\right]$$

$$\Rightarrow PR = c\left[\frac{\sin\alpha \times \cos\beta \times \sin\gamma - \sin\beta \times \sin\alpha \times \cos\gamma}{\sin(\gamma - \alpha)}\right]$$

$$\Rightarrow PR = \frac{c\sin\alpha \times (\cos\beta \times \sin\gamma - \sin\beta \times \cos\gamma)}{\sin(\gamma - \alpha)}$$

$$\Rightarrow PR = \frac{c\sin\alpha \times \sin(\gamma - \beta)}{\sin(\gamma - \alpha)}$$

Q31

If the sides a, b, c of a Δ ABC are in H.P.

$\therefore \frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in AP

$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

$\Rightarrow \frac{a-b}{ba} = \frac{b-c}{ca}$

$\Rightarrow \frac{\sin A - \sin B}{\sin B \sin A} = \frac{\sin B - \sin C}{\sin C \sin B}$.......[Using sine rule]

$\Rightarrow \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin A} = \frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{\sin C}$

But $A + B + C = \pi$

$A + B = \pi - C$

$\cos\frac{A+B}{2} = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\frac{C}{2}$

$\sin^2\frac{C}{2}\cos\frac{C}{2}\sin\frac{A-B}{2} = \sin\frac{B-C}{2}\cos\frac{A}{2}\sin^2\frac{A}{2}$

$\sin^2\frac{C}{2}\sin\frac{A+B}{2}\sin\frac{A-B}{2} = \sin\frac{B-C}{2}\cos\frac{B+C}{2}\sin^2\frac{A}{2}$

$\sin^2\frac{C}{2}\left[\sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right] = \sin^2\frac{A}{2}\left[\sin^2\frac{B}{2} - \sin^2\frac{C}{2}\right]$

$\sin^2\frac{C}{2}\sin^2\frac{A}{2} - \sin^2\frac{C}{2}\sin^2\frac{B}{2} = \sin^2\frac{A}{2}\sin^2\frac{B}{2} - \sin^2\frac{A}{2}\sin^2\frac{C}{2}$

$\frac{1}{\sin^2\frac{B}{2}} - \frac{1}{\sin^2\frac{A}{2}} = \frac{1}{\sin^2\frac{C}{2}} - \frac{1}{\sin^2\frac{B}{2}}$

Hence $\frac{1}{\sin^2\frac{A}{2}}, \frac{1}{\sin^2\frac{B}{2}}, \frac{1}{\sin^2\frac{C}{2}}$ are in AP.

$\therefore \sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are in HP.

Ex-10.2

Q1

The area of a triangle ABC is given by

$$\Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 5 \times 6 \sin 60^\circ$$

$$= \frac{15\sqrt{3}}{2} \text{ sq.unit}$$

Q2

The area of a triangle ABC is given by

$$\Delta = \frac{1}{2} ab \sin C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{2 + 3 - 5}{2\sqrt{6}}$$

$$= 0$$

$$\sin C = \sqrt{1 - \cos^2 C}$$

$$= 1$$

Therefore,

$$\Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2}\sqrt{6}$$

Q3

We have, $a = 4, b = 6$ and $c = 8$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7}{8}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{11}{16}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = -\frac{1}{4}$$

$$8\cos A + 16\cos B + 4\cos C = 8 \times \frac{7}{8} + 16 \times \frac{11}{16} + 4 \times \left(-\frac{1}{4}\right)$$

$$= 17$$

Q4

In any $\triangle ABC$, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

we have,

$a = 18, b = 24, c = 30$

Therefore,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1152}{1440} = \frac{4}{5}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{648}{1080} = \frac{3}{5}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{0}{864} = 0$$

Q5

$b(c\cos A - a\cos C) = c^2 - a^2$

RHS

$= c^2 - a^2$

$= k^2\sin^2 C - k^2\sin^2 A$

$= k^2(\sin^2 C - \sin^2 A)$

$= k^2\sin(C+A).\sin(C-A)$

$= k^2\sin(\pi - B).\sin(C-A)$

$= k^2\sin B.\sin(C-A)$

$= k\sin B.k\sin(C-A)$

$= bk\sin(C-A)$

$= bk(\sin C.\cos A - \sin A.\cos C)$

$= b(k\sin C.\cos A - k\sin A.\cos C)$

$= b(c\cos A - a\cos C) = LHS$

Q6

$$c(a\cos B - b\cos A)$$
$$= ac.\cos B - bc\cos A$$
$$= ac.\frac{a^2+c^2-b^2}{2ac} - bc.\frac{b^2+c^2-a^2}{2bc}$$
$$= \frac{a^2+c^2-b^2}{2} - \frac{b^2+c^2-a^2}{2}$$
$$= \frac{a^2+c^2-b^2-b^2-c^2+a^2}{2}$$
$$= \frac{2a^2-2b^2}{2} = \left(a^2-b^2\right) = RHS$$

Q7

$$2(bc\cos A + ca\cos B + ab\cos C) = a^2 + b^2 + c^2$$
$$LHS$$
$$= 2bc\cos A + 2ca\cos B + 2ab\cos C$$
$$= 2bc\frac{b^2+c^2-a^2}{2bc} + 2ca\frac{a^2+c^2-b^2}{2ca} + 2ab\frac{a^2+b^2-c^2}{2ab}$$
$$= b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2$$
$$= a^2+b^2+c^2 = RHS$$

Q8

For any $\triangle ABC$, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

therefore,

$$\left(c^2+b^2-a^2\right)\tan A = \left(c^2+b^2-a^2\right)\frac{\sin A}{\cos A}$$

$$= \left(c^2+b^2-a^2\right)\frac{ka}{\frac{b^2+c^2-a^2}{2bc}}$$

$$= 2kabc$$

Also,

$$\left(a^2+c^2-b^2\right)\tan B = \left(a^2+c^2-b^2\right)\frac{\sin B}{\cos B}$$

$$= \left(a^2+c^2-b^2\right)\frac{kb}{\frac{a^2+c^2-b^2}{2ac}}$$

$$= 2kabc$$

Now,

$$\left(a^2+b^2-c^2\right)\tan C = \left(a^2+b^2-c^2\right)\frac{\sin C}{\cos C}$$

$$= \left(a^2+b^2-c^2\right)\frac{kc}{\frac{a^2+b^2-c^2}{2ab}}$$

$$= 2kabc$$

Hence proved.

Q9

$$\frac{c-b\cos A}{b-c\cos A} = \frac{\cos B}{\cos C}$$

LHS

$$= \frac{c-b\cos A}{b-c\cos A}$$

$$= \frac{k\sin C - k\sin B\cos A}{k\sin B - k\sin C\cos A}$$

$$= \frac{\sin(\pi-(A+B)) - \sin B\cos A}{\sin(\pi-(A+C)) - \sin C\cos A}$$

$$= \frac{\sin(A+B) - \sin B\cos A}{\sin(A+C) - \sin C\cos A}$$

$$= \frac{\sin A\cos B + \cos A\sin B - \sin B\cos A}{\sin A\cos C + \cos A\sin C - \sin C\cos A}$$

$$= \frac{\sin A\cos B}{\sin A\cos C}$$

$$= \frac{\cos B}{\cos C} = RHS$$

Q10

In any $\triangle ABC$, we have

$a = b\cos C + c\cos B$

$b = c\cos A + a\cos C$

$c = a\cos B + b\cos A$

Therefore,

$$\begin{aligned}
L.H.S &= a(\cos B + \cos C - 1) + b(\cos C + \cos A - 1) + c(\cos A + \cos B - 1) \\
&= a\cos B + a\cos C - a + b\cos C + b\cos A - b + c\cos A + c\cos B - c \\
&= c - b\cos A + a\cos C - a + a - c\cos B + b\cos A - b + b - a\cos C + c\cos B - c \\
&= 0 \\
&= R.H.S
\end{aligned}$$

Hence proved.

Q11

By sine rule we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$k\sin A = a, k\sin B = b, k\sin C = c$

$$\begin{aligned}
&a\cos A + b\cos B + c\cos C = k\sin A\cos A + k\sin B\cos B + k\sin C\cos C \\
&= \left(\frac{1}{2}\right)k\left[2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C\right] \\
&= \left(\frac{1}{2}\right)k\left[\sin 2A + \sin 2B + \sin 2C\right] \\
&= k\left[\sin(A+B)\cos(A-B) + \sin C\cos C\right] \\
&= k\left[\sin(\pi - C)\cos(A-B) + \sin C\cos\left(\pi - (A+B)\right)\right] \\
&= k\left[\sin C\cos(A-B) - \sin C\cos(A+B)\right] \\
&= k\left[\sin C\left(\cos(A-B) - \cos(A+B)\right)\right] \\
&= k\sin C[2\sin A\sin B] \\
&= 2\sin C(k\sin A)\sin B \\
&= 2a\sin B\sin C
\end{aligned}$$

Q12

We know that by cosine rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow 2bc \cos A = b^2 + c^2 - a^2$$

$$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Rightarrow a^2 = b^2 + c^2 - 2bc\left(2\cos^2 \frac{A}{2} - 1\right)$$

$$\Rightarrow a^2 = b^2 + c^2 + 2bc - 4bc\cos^2 \frac{A}{2}$$

$$\Rightarrow a^2 = (b+c)^2 - 4bc\cos^2 \frac{A}{2}$$

Q13

$$4\left(bc\cos^2 \frac{A}{2} + ca\cos^2 \frac{B}{2} + ab\cos^2 \frac{C}{2}\right) = (a+b+c)^2$$

LHS,

$$4\left(bc\cos^2 \frac{A}{2} + ca\cos^2 \frac{B}{2} + ab\cos^2 \frac{C}{2}\right)$$

$$= 2\left(bc.2\cos^2 \frac{A}{2} + ca.2\cos^2 \frac{B}{2} + ab.2\cos^2 \frac{C}{2}\right)$$

$$= 2\left(bc.(1 - \cos A) + ca.(1 - \cos B) + ab.(1 - \cos C)\right)$$

$$= 2bc - 2bc\cos A + 2ca - 2ca\cos B + 2ab - 2ab\cos C$$

$$= 2bc - 2bc\frac{b^2 + c^2 - a^2}{2bc} + 2ca - 2ca\frac{a^2 + c^2 - b^2}{2ca} + 2ab$$

$$-2ab\frac{b^2 + a^2 - c^2}{2ab}[\text{cos rule}]$$

$$= 2bc - b^2 - c^2 + a^2 + 2ca - a^2 - c^2 + b^2 + 2ab - b^2 - a^2 + c^2$$

$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$= (a+b+c)^2 = RHS$$

Q14

$\sin^3 A\cos(B-C)+\sin^3 B.\cos(C-A)+\sin^3 C.\cos(A-B)$

$=\sin^2 A\sin A\cos(B-C)+\sin^2 B.\sin B\cos(C-A)+\sin^2 C.\sin C.\cos(A-B)$

$=\sin^2 A\sin\left(\pi-(B+C)\right)\cos(B-C)+\sin^2 B.\sin\left(\pi-(A+C)\right).\cos(C-A)$

$+\sin^2 C.\sin\left(\pi-(A+B)\right).\cos(A-B)$

$=\sin^2 A\sin(B+C)\cos(B-C)+\sin^2 B.\sin(C+A).\cos(C-A)$

$+\sin^2 C.\sin(A+B).\cos(A-B)$

$=\sin^2 A.(\sin 2B+\sin 2C)+\sin^2 B.(\sin 2C+\sin 2A)+\sin^2 C.(\sin 2A+\sin 2B)$

$=\sin^2 A.(2\sin B.\cos B+2\sin C\cos C)+\sin^2 B.(2\sin C\cos C+2\sin A\cos A)$

$+\sin^2 C.(2\sin A\cos A+2\sin B\cos B)$

$=\sin^2 A.(2\sin B\cos B+2\sin C\cos C)+\sin^2 B.(2\sin C\cos C+2\sin A\cos A)$

$+\sin^2 C.(2\sin A\cos A+2\sin B\cos B)$

$=\sin^2 A.2\sin B\cos B+\sin^2 A.2\sin C\cos C+\sin^2 B.2\sin C\cos C$

$+\sin^2 B.2\sin A\cos A+\sin^2 C.2\sin A\cos A+\sin^2 C.2\sin B\cos B$

$=k^2a^2\,2kb\cos B+k^2a^2.2kc\cos C+k^2b^2.2ka\cos C$

$+k^2b^2.2ka\cos A+k^2c^2.2ka\cos A+k^2c^2.2kb\cos B$

$=k^3ab(a\cos B+b\cos A)+k^3ac(a\cos C+c\cos A)+k^3bc(c\cos B+b\cos C)$

$=k^3abc+k^3acb+k^3bca$

$=k^3 3abc$

$=3(k\sin A.k\sin B.k\sin C)$

$=3abc=RHS$

Q15

$$\text{Let } \frac{b+c}{12} = \frac{c+a}{13} = \frac{a+b}{15} = \lambda \text{ (say)}$$

$$b+c = 12\lambda, c+a = 13\lambda, a+b = 15\lambda$$

$$(b+c+c+a+a+b) = 12\lambda + 13\lambda + 15\lambda$$

$$2(a+b+c) = 40\lambda$$

$$a+b+c = 20\lambda$$

$$b+c = 12\lambda \text{ and } a+b+c = 20\lambda \Rightarrow a = 8\lambda$$

$$c+a = 13\lambda \text{ and } a+b+c = 20\lambda \Rightarrow b = 7\lambda$$

$$a+b = 15\lambda \text{ and } a+b+c = 20\lambda \Rightarrow c = 5\lambda$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{49\lambda^2+25\lambda^2-64\lambda^2}{70\lambda^2} = \frac{1}{7}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{64\lambda^2+25\lambda^2-49\lambda^2}{80\lambda^2} = \frac{1}{2}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{64\lambda^2+49\lambda^2-25\lambda^2}{112\lambda^2} = \frac{11}{14}$$

$$\cos A : \cos B : \cos C = \frac{1}{7} : \frac{1}{2} : \frac{11}{14} = 2:7:11$$

Q16

We have, $\angle B = 60^\circ$

$$\cos B = \frac{1}{2} \Rightarrow \frac{a^2+c^2-b^2}{2ac} = \frac{1}{2}$$

$$\Rightarrow a^2+c^2-b^2 = ac$$

$$\Rightarrow a^2+c^2-ac = b^2 \qquad \ldots\ldots(i)$$

$$(a+b+c)(a-b+c) = 3ca$$

$$a^2-ab+ac+ab-b^2+bc+ac-bc+c^2 = 3ac$$

$$a^2+c^2-b^2+2ac-3ac = 0$$

$$a^2+c^2-ac = b^2$$

which is given.

Q17

Consider the given equation:

$$\cos^2 A + \cos^2 B + \cos^2 C = 1$$

$$\Rightarrow 1 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 C = 1$$

$$\Rightarrow 3 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 C = 1$$

Q18

Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$. Then, $\sin A = ka, \sin B = kb, \sin C = kc$

Now, $\cos C = \frac{\sin A}{2\sin B}$

$2\sin B \cos C = \sin A$

$2\left(\frac{a^2 + b^2 - c^2}{2ab}\right)kb = ka$

$a^2 + b^2 - c^2 = a^2$

$b^2 = c^2$

$b = c$

$\triangle ABC$ is isosceles.

Q19

Let P and Q be the position of two ships at the end of 3 hours.

Then,

$OP = 3 \times 24 = 72\,km$ and $OQ = 3 \times 32 = 96\,km$

Using cosine formula in ΔOPQ, we get

$PQ^2 = OP^2 + OQ^2 - 2OP \times OQ \cos 90^\circ$

$PQ^2 = 72^2 + 96^2 - 2 \times 72 \times 96 \cos 90^\circ$

$PQ^2 = 14400$

$PQ = 120\,km$