Ex 12.1

Q1

$P(n) : n(n+1)$ is even

$P(3) : 3.(3+1)$ is even

Q2

$P(n) : n^3 + n$ is divisible by 3

$P(3) : 3^3 + 3$ is divisible by 3

$\Rightarrow P(3) : 30$ is divisible by 3.

$\therefore \quad P(3)$ is true

Now,

$P(4) : 4^3 + 3 = 67$ is divisible by 3

Since, 67 is not divisible by 3

So, $P(4)$ is not true

Q3

$P(n) : 2^n \geq 3n$

Given that $P(r)$ is true

$\Rightarrow \quad 2^r \geq 3r$

Multiplying both the sides by 2,

$2 . 2^r \geq 2 . 3r$

$2^{r+1} \geq 6r$

$\quad \geq 3r + 3r$

$\quad \geq 3 + 3r, \qquad\qquad\qquad\qquad$ [Since $3r \geq 3 \Rightarrow \qquad 3r + 3r \geq 3 + 3r$]

$2^{r+1} \geq 3(r+1)$

$\Rightarrow \quad P(r+1)$ is true

Q4

Here, $P(n) : n^2 + n$ is even

Given, $P(r)$ is true

$\Rightarrow \qquad r^2 + r$ is even

$\Rightarrow \qquad r^2 + r = 2\lambda \qquad\qquad\qquad\qquad ---(1)$

Now,

$(r + 1)^2 + (r + 1)$

$= r^2 + 2r + 1 + r + 1$

$= (r^2 + r) + 2r + 2$

$= 2\lambda + 2r + 2 \qquad\qquad\qquad [\text{Using equation } (1)]$

$= 2(\lambda + r + 1)$

$= 2\mu$

$\Rightarrow \qquad (r + 1)^2 + (r + 1)$ is even

$\Rightarrow \qquad P(r + 1)$ is true

Q5

$P(n) : 1 + 2 + 3 + ---- + n = \dfrac{n(n + 1)}{2}$ is true for all $n \in N$

Q6

$P(n) : n^2 - n + 41$ is prime

$P(1) : 1 - 1 + 41$ is prime
$\Rightarrow P(1) : 41$ is prime
$\therefore P(1)$ is true.

$P(2) : 2^2 - 2 + 41$ is prime
$\Rightarrow P(2) : 43$ is prime
$\therefore P(2)$ is true.

$P(3) : 3^2 - 3 + 41$ is prime
$\Rightarrow P(3) : 47$ is prime
$\therefore P(3)$ is true.

$P(41) : (41)^2 - 41 + 41$ is prime
$P(41) : (41)^2$ is prime
$\Rightarrow \quad P(41)$ is not true

Ex 12.2

Q1

Let $P(n) : 1 + 2 + 3 + ---- + n = \dfrac{n(n+1)}{2}$

For $n = 1$,

LHS of P(n) = 1

RHS of P(n) $= \dfrac{1(1+1)}{2} 1 = 1$

Since, LHS = RHS

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ be true for $n = k$, so

$1 + 2 + 3 + --- + k = \dfrac{k(k+1)}{2}$ $\qquad\qquad ---(1)$

Now

$(1 + 2 + 3 + --- + k) + (k+1)$

$= \dfrac{k(k+1)}{2} + (k+1)$

$= (k+1)\left(\dfrac{k}{2} + 1\right)$

$= \dfrac{(k+1)(k+2)}{2}$

$= \dfrac{(k+1)\big[(k+1)+1\big]}{2}$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$

So, by the principle of mathematical induction

$P(n) : 1 + 2 + 3 + --- + n = \dfrac{n(n+1)}{2}$ is true for all $n \in N$

Q2

Let $P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

For $n = 1$

$P(1) : 1 = \dfrac{1(1+1)(2+1)}{6}$

$\qquad 1 = 1$

$\Rightarrow \qquad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$P(k) : 1^2 + 2^2 + 3^2 + \ldots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ $\qquad\qquad ---(1)$

We have to show that $P(n)$ is true for $n = k+1$

$\Rightarrow \qquad 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$

So, $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ $\qquad\qquad$ [Using equation (1)]

$= (k+1)\left[\dfrac{2k^2 + k}{6} + \dfrac{(k+1)}{1}\right]$

$= (k+1)\left[\dfrac{2k^2 + k + 6k + 6}{6}\right]$

$= (k+1)\left[\dfrac{2k^2 + 7k + 6}{6}\right]$

$= (k+1)\left[\dfrac{2k^2 + 4k + 3k + 6}{6}\right]$

$= (k+1)\left[\dfrac{2k(k+2) + 3(k+2)}{6}\right]$

$= \dfrac{(k+1)(2k+3)(k+2)}{6}$

$\Rightarrow \qquad P(n)$ is true for $n = k+1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q3

Let $P(n) : 1 + 3 + 3^2 + \ldots + 3^{n-1} = \dfrac{3^n - 1}{2}$

For $n = 1$

$P(1) : 1 = \dfrac{3^1 - 1}{2}$

$\quad\quad 1 = 1$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$1 + 3 + 3^2 + \ldots + 3^{k-1} = \dfrac{3^k - 1}{2}$ $\quad\quad\quad\quad\quad - - - (1)$

We have to show $P(n)$ is true for $n = k + 1$

i.e. $1 + 3 + 3^2 + \ldots + 3^k = \dfrac{3^{k+1} - 1}{2}$

Now,

$\left\{ 1 + 3 + 3^2 + \ldots + 3^{k-1} \right\} + 3^{k+1-1}$

$= \dfrac{3k - 1}{2} + 3^k$ $\quad\quad\quad\quad\quad\quad\quad$ $\left[\text{Using equation } (1) \right]$

$= \dfrac{3^k - 1 + 2.3^k}{2}$

$= \dfrac{3.3^k - 1}{2}$

$= \dfrac{3^{k+1} - 1}{2}$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q4

Let $P(n): \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

For $n = 1$

$P(1): \dfrac{1}{1.2} = \dfrac{1}{1+1}$

$\quad \dfrac{1}{2} = \dfrac{1}{2}$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{k(k+1)} = \dfrac{k}{k+1}$ $\qquad\qquad --- (1)$

We have to show that

$\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{k(k+1)} + \dfrac{k}{(k+1)(k+2)} = \dfrac{k+1}{(k+2)}$

Now,

$\left\{ \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{k(k+1)} \right\} + \dfrac{1}{(k+1)(k+2)}$

$= \dfrac{k}{k+1} + \dfrac{1}{(k+1)(k+2)}$ $\qquad\qquad$ [Using equation (1)]

$= \dfrac{1}{k+1}\left[\dfrac{k(k+2)+1}{(k+2)} \right]$

$= \dfrac{1}{k+1}\left[\dfrac{k^2+2k+1}{(k+2)} \right]$

$= \dfrac{1}{k+1}\left[\dfrac{(k+1)(k+1)}{(k+2)} \right]$

$= \dfrac{(k+1)}{(k+2)}$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q5

Let $P(n) : 1 + 3 + 5 + \ldots + (2n - 1) = n^2$

For $n = 1$

$P(1) : 1 = 1^2$

$\qquad 1 = 1$

$\Rightarrow \qquad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$P(k) : 1 + 3 + 5 + \ldots + (2k - 1) = k^2 \qquad\qquad ---(1)$

We have to show that

$1 + 3 + 5 + \ldots + (2k - 1) + 2(k + 1) - 1 = (k + 1)^2$

Now,

$\{1 + 3 + 5 + \ldots + (2k - 1)\} + (2k + 1)$

$= k^2 + (2k + 1) \qquad\qquad\qquad\qquad \left[\text{Using equation } (1)\right]$

$= k^2 + 2k + 1$

$= (k + 1)^2$

$\Rightarrow \qquad P(n)$ is true for $n = k + 1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q6

Let $P(n) : \dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + \ldots + \dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{(6n+4)}$

Put $n = 1$

$P(1) : \dfrac{1}{2.5} = \dfrac{1}{6+4}$

$\dfrac{1}{10} = \dfrac{1}{10}$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + \ldots + \dfrac{1}{(3k-1)(3k+2)} = \dfrac{k}{(6k+4)} \qquad \text{---(1)}$$

We have to show that,

$$\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + \ldots + \dfrac{1}{(3k-1)(3k+2)} + \dfrac{1}{(3k+2)(3k+5)} = \dfrac{(k+1)}{(6k+10)}$$

Now,

$$\left\{\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + \dfrac{1}{(3k-1)(3k+2)}\right\} + \dfrac{1}{(3k+2)(3k+5)}$$

$$= \dfrac{k}{6k+4} + \dfrac{1}{(3k+2)(3k+5)}$$

$$= \dfrac{k}{2(3k+2)} + \dfrac{1}{(3k+2)(3k+5)}$$

$$= \dfrac{k(3k+5)+2}{2(3k+2)(3k+5)}$$

$$= \dfrac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$= \dfrac{3k^2+3k+2k+2}{2(3k+2)(3k+5)}$$

$$= \dfrac{3k(k+1)+2(k+1)}{2(3k+2)(3k+5)}$$

$$= \dfrac{(k+1)\,\cancel{(3k+2)}}{2\,\cancel{(3k+2)}(3k+5)}$$

$$= \dfrac{(k+1)}{2(3k+5)}$$

$P(n)$ is true for $n = k+1$

$P(n)$ is true for all $n \in N$ by PMI

Q7

Let $P(n) : \dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$

Put $n = 1$

$P(1) : \dfrac{1}{1.4} = \dfrac{1}{4}$

$\qquad \dfrac{1}{4} = \dfrac{1}{4}$

$\Rightarrow \qquad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3k-2)(3k+1)} = \dfrac{k}{3k+1} \qquad \qquad ---(1)$

We have to show that,

$\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3k-2)(3k+1)} + \dfrac{1}{(3k+1)(3k+4)} = \dfrac{(k+1)}{(3k+4)}$

Now,

$\left\{ \dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3k-2)(3k+1)} \right\} + \dfrac{1}{(3k+1)(3k+4)}$

Now,

$\left\{ \dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \ldots + \dfrac{1}{(3k-2)(3k+1)} \right\} + \dfrac{1}{(3k+1)(3k+4)}$

$= \dfrac{k}{(3k+1)} + \dfrac{1}{(3k+1)(3k+4)}$

$= \dfrac{1}{(3k+1)} \left[\dfrac{k}{1} + \dfrac{1}{(3k+4)} \right]$

$= \dfrac{1}{(3k+1)} \left[\dfrac{k(3k+4)+1}{(3k+4)} \right]$

$= \dfrac{1}{(3k+1)} \left[\dfrac{3k^2 + 4k + 1}{(3k+4)} \right]$

$= \dfrac{1}{(3k+1)} \dfrac{(3k^2 + 3k + k + 1)}{(3k+4)}$

$= \dfrac{3k(k+1)+(k+1)}{(3k+1)(3k+4)}$

$= \dfrac{(k+1)(3k+1)}{(3k+1)(3k+4)}$

$= \dfrac{(k+1)}{(3k+4)}$

$\Rightarrow \qquad P(n)$ is true for $n = k+1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q8

Let $P(n): \dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + \dots + \dfrac{1}{(2n+1)(2n+3)} = \dfrac{n}{3(2n+3)}$

Put $n = 1$

$\dfrac{1}{3.5} = \dfrac{1}{3(5)}$

$\dfrac{1}{15} = \dfrac{1}{15}$

$\Rightarrow \qquad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + \dots + \dfrac{1}{(2k+1)(2k+3)} = \dfrac{k}{3(2k+3)} \qquad ---(1)$

We have to show that,

$\dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + \dots + \dfrac{1}{(2k+1)(2k+3)} + \dfrac{1}{(2k+3)(2k+5)} = \dfrac{(k+1)}{3(2k+5)}$

Now,

$\left\{ \dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + \dots + \dfrac{1}{(2k+1)(2k+3)} \right\} + \dfrac{1}{(2k+3)(2k+5)}$

$= \dfrac{k}{3(2k+3)} + \dfrac{1}{(2k+3)(2k+5)}$ \qquad $\left[\text{Using equation } (1) \right]$

$= \dfrac{1}{(2k+3)} \left[\dfrac{k}{3} + \dfrac{1}{(2k+5)} \right]$

$= \dfrac{1}{(2k+3)} \left[\dfrac{k(2k+5)+3}{(2k+5)} \right]$

$= \dfrac{1}{(2k+3)} \left[\dfrac{2k^2 + 5k + 3}{(2k+5)} \right]$

$= \dfrac{1}{(2k+3)} \left[\dfrac{2k^2 + 2k + 3k + 3}{(2k+5)} \right]$

$= \dfrac{1}{(2k+3)} \left[\dfrac{2k(k+1) + 3(k+1)}{(2k+5)} \right]$

$= \dfrac{1}{(2k+3)} \left[\dfrac{(k+1)(2k+3)}{(2k+5)} \right]$

$= \dfrac{(k+1)}{2k+5}$

$\Rightarrow \qquad P(n)$ is true for $n = k+1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q9

Let $P(n): \dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \ldots + \dfrac{1}{(4n-1)(4n+3)} = \dfrac{n}{3(4n+3)}$

For $n = 1$

$\dfrac{1}{3.7} = \dfrac{1}{3(7)}$

$\dfrac{1}{21} = \dfrac{1}{21}$

$\Rightarrow \qquad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \ldots + \dfrac{1}{(4n-1)(4n+3)} = \dfrac{k}{3(4k+3)} \qquad ---(1)$

We have to show that,

$\dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \ldots + \dfrac{1}{(4k-1)(4k+3)} + \dfrac{1}{(4k+3)(4k+7)} = \dfrac{(k+1)}{3} \dfrac{}{(4k+7)}$

Now,

$\left\{ \dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \ldots + \dfrac{1}{(4k-1)(4k+3)} \right\} + \dfrac{1}{(4k+3)(4k+7)}$

Now,

$\left\{ \dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \ldots + \dfrac{1}{(4k-1)(4k+3)} \right\} + \dfrac{1}{(4k+3)(4k+7)}$

$= \dfrac{k}{3(4k+3)} + \dfrac{1}{(4k+3)(4k+7)}$

$= \dfrac{1}{(4k+3)} \left[\dfrac{k}{3} + \dfrac{1}{4k+7} \right]$

$= \dfrac{1}{(4k+3)} \left[\dfrac{k(4k+7)+3}{3(4k+7)} \right]$

$= \dfrac{1}{(4k+3)} \left[\dfrac{4k^2+7k+3}{3(4k+7)} \right]$

$= \dfrac{1}{(4k+3)} \left[\dfrac{4k^2+4k+3k+3}{3(4k+7)} \right]$

$= \dfrac{1}{(4k+3)} \left[\dfrac{4k(k+1)+3(k+1)}{3(4k+7)} \right]$

$= \dfrac{1}{(4k+3)} \left[\dfrac{(4k+3)(k+1)}{3(4k+7)} \right]$

$= \dfrac{(k+1)}{3(4k+7)}$

$\Rightarrow \qquad P(n)$ is true for $n = k+1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q10

Let $P(n) : 1.2 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n-1)2^{n+1} + 2$

For $n = 1$

$\quad 1.2 = 0.2^0 + 2$

$\quad\quad 2 = 2$

$\Rightarrow \quad\quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k = (k-1)2^{k+1} + 2 \quad\quad ---(1)$

We have to show that,

$\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\} + (k+1)2^{k+1} = k2^{k+2} + 2$

Now,

$\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\} + (k+1)2^{k+1}$

$= \left[(k-1)2^{k+1} + 2\right] + (k+1)2^{k+1} \quad\quad\quad\quad\quad \left[\text{Using equation } (1)\right]$

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}(k - 1 + k + 1) + 2$

$= 2^{k+1}.2k + 2$

$= k2^{k+2} + 2$

$\Rightarrow \quad\quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad\quad P(n)$ is true for all $n \in N$ by PMI

Q11

Let $P(n) : 2 + 5 + 8 + 11 + \ldots + (3n - 1) = \frac{1}{2}n(3n + 1)$

For $n = 1$

$P(1)\ 2 = \frac{1}{2} \cdot 1 \cdot (4)$

$\quad 2 = 2$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$2 + 5 + 8 + 11 + \ldots + (3k - 1) = \frac{1}{2}k(3k + 1) \qquad \qquad --- (1)$

We have to show that,

$2 + 5 + 8 + 11 + \ldots + (3k - 1) + (3k + 2) = \frac{1}{2}(k + 1)(3k + 4)$

Now,

$\{2 + 5 + 8 + 11 + \ldots + (3k - 1)\} + (3k + 2)$

$= \frac{1}{2}k(3k + 1) + (3k + 2)$

$= \dfrac{3k^2 + k + 2(3k + 2)}{2}$

$= \dfrac{3k^2 + k + 6k + 4}{2}$

$= \dfrac{3k^2 + 7k + 4}{2}$

$= \dfrac{3k^2 + 3k + 4k + k}{2}$

$= \dfrac{3k(k + 1) + 4(k + 1)}{2}$

$= \dfrac{(k + 1)(3k + 4)}{2}$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q12

Let $1.3 + 2.4 + 3.5 + \ldots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

For $n = 1$

$1.3 = \frac{1}{6} \cdot 1 \cdot (2)(9)$

$3 = 3$

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$1.3 + 2.4 + 3.5 + \ldots + k(k+2) = \frac{1}{6}k(k+1)(2k+7)$ $\quad - - -(1)$

We have to show that,

$1.3 + 2.4 + 3.5 + \ldots + k(k+2) + (k+1)(k+3) = \frac{(k+1)}{6}(k+2)(2k+9)$

Now,

$\{1.3 + 2.4 + 3.5 + \ldots + k(k+2)\} + (k+1)(k+3)$

$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$ $\qquad\qquad$ [Using equation (1)]

$= (k+1)\left[\frac{k(2k+7)}{6} + \frac{k+3}{1}\right]$

$= (k+1)\left[\frac{2k^2 + 7k + 6k + 18}{6}\right]$

$= (k+1)\left(\frac{2k^2 + 13k + 18}{6}\right)$

$= (k+1)\left[\frac{2k^2 + 4k + 9k + 18}{6}\right]$

$= (k+1)\left[\frac{2k(k+2) + 9(k+2)}{6}\right]$

$= (k+1)\left[\frac{(2k+9)(k+2)}{6}\right]$

$= \frac{1}{6}(k+1)(k+2)(2k+9)$

$\Rightarrow$ $P(n)$ is true for $n = k+1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q13

Let $P(n) : 1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \dfrac{n\left(4n^2 + 6n - 1\right)}{3}$

For $n = 1$

$1.3 = \dfrac{1\left(4 + 6 - 1\right)}{3}$

$3 = 3$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) = \dfrac{k\left(4k^2 + 6k - 1\right)}{3} \qquad \text{---}(1)$

We have to show that,

$1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) + (2k + 1)(2k + 3) = \dfrac{(k + 1)\left[4(k + 1)^2 + 6(k + 1) - 1\right]}{3}$

Now,

$\left\{1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1)\right\} + (2k + 1)(2k + 3)$

$= \dfrac{k\left(4k^2 + 6k - 1\right)}{3} + (2k + 1)(2k + 3) \qquad\qquad \left[\text{Using equation } (1)\right]$

$= \dfrac{k\left(4k^2 + 6k - 1\right) + 3\left(4k^2 + 6k + 2k + 3\right)}{3}$

$= \dfrac{4k^3 + 6k^2 - k + 12k^2 + 18k + 6k + 9}{3}$

$= \dfrac{4k^3 + 18k^2 + 23k + 9}{3}$

$= \dfrac{4k^3 + 4k^2 + 14k^2 + 14k + 9k + 9}{3}$

$= \dfrac{(k + 1)\left(4k^2 + 8k + 4 + 6k + 6 - 1\right)}{3}$

$= \dfrac{(k + 1)\left[4(k + 1)^2 + 6(k + 1) - 1\right]}{3}$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q14

Let $P(n): 1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

For $n = 1$

$$1.2 = \frac{1(1+1)(1+2)}{3}$$

$$2 = 2$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$\Rightarrow \quad 1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \dfrac{k(k+1)(k+2)}{3} \qquad \text{---}(1)$

We have to show that,

$$1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Now,

$$\{1.2 + 2.3 + 3.4 + \ldots + k(k+1)\} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{(k+1)(k+2)}{1}$$

$$= (k+1)(k+2)\left[\frac{k}{3} + 1\right]$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q15

Let $P(n): \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

For $n = 1$

$$\dfrac{1}{2} = 1 - \dfrac{1}{2^1}$$

$$\dfrac{1}{2} = \dfrac{1}{2}$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^k} = 1 - \dfrac{1}{2^k} \qquad\qquad ---(1)$$

We have to show that,

$$\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^k} + \dfrac{1}{2^{k+1}} = 1 - \dfrac{1}{2^{k+1}}$$

Now,

$$\left\{ \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^k} \right\} + \dfrac{1}{2^{k+1}}$$

$$= 1 - \dfrac{1}{2^k} + \dfrac{1}{2^{k+1}} \qquad\qquad \left[\text{Using equation } (1)\right]$$

$$= 1 - \left(\dfrac{2-1}{2^{k+1}} \right)$$

$$= 1 - \dfrac{1}{2^{k+1}}$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q16

Let $P(n) : 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$

For $n = 1$

$\quad 1 = \frac{1}{3}.1.(4-1)$

$\quad 1 = 1$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{1}{3}k(4k^2 - 1)$ $\qquad$ ---(1)

We have to show that,

$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}(k+1)\left[4(k+1)^2 - 1\right]$

Now,

$\left\{1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2\right\} + (2k+1)^2$

$= \frac{1}{3}k(4k^2 - 1) + (2k+1)^2$ $\qquad$ $\left[\text{Using equation } (1)\right]$

$= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2$

$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right]$

$= (2k+1)\left[\frac{2k^2 - k + 3(2k+1)}{3}\right]$

$= (2k+1)\left[\frac{2k^2 - k + 6k + 3}{3}\right]$

$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$

$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$

$= \frac{(2k+1)(2k(k+1) + 3(k+1))}{3}$

$= \frac{(2k+1)(2k+3)(k+1)}{3}$

$= \frac{(k+1)}{2}\left[4k^2 + 6k + 2k + 3\right]$

$= \frac{(k+1)}{2}\left[4k^2 + 8k + 4 - 1\right]$

$= \frac{(k+1)}{2}\left[4(k+1)^2 - 1\right]$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q17

Let $P(n): a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\dfrac{r^n - 1}{r - 1}\right), r \neq 1$

For $n = 1$

$$a = a\left(\dfrac{r^1 - 1}{r - 1}\right)$$

$$a = a$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$a + ar + ar^2 + \ldots + ar^{k-1} = a\left(\dfrac{r^k - 1}{r - 1}\right), r \neq 1 \qquad\qquad ---(1)$$

We have to show that,

$$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = a\left(\dfrac{r^{k+1} - 1}{r - 1}\right)$$

Now,

$$\left\{a + ar + ar^2 + \ldots + ar^{k-1}\right\} + ar^k$$

$$= a\left(\dfrac{r^k - 1}{r - 1}\right) + ar^k \qquad\qquad\qquad \left[\text{Using equation } (1)\right]$$

$$= \dfrac{a\left[r^k - 1 + r^k(r - 1)\right]}{r - 1}$$

$$= \dfrac{a\left[r^k - 1 + r^{k+1} - r^k\right]}{r - 1}$$

$$= \dfrac{a\left(r^{k+1} - 1\right)}{r + 1}$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q18

Let $P(n) : a + (a + d) + (a + 2d) + ... + (a + (n - 1)d) = \dfrac{n}{2}[2a + (n - 1)d]$

For $n = 1$

$\qquad a = \dfrac{1}{2}[2a + (1 - 1)d]$

$\qquad a = a$

$\Rightarrow \qquad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$a + (a + d) + (a + 2d) + ... + (a + (k - 1)d) = \dfrac{k}{2}[2a + (k - 1)d] \qquad\qquad ---(1)$

We have to show that,

$a + (a + d) + (a + 2d) + ... + (a + (k - 1)d) + (a + (k)d) = \dfrac{(k + 1)}{2}[2a + kd]$

Now,

$\{a + (a + d) + (a + 2d) + ... + (a + (k - 1)d)\} + (a + kd)$

$= \dfrac{k}{2}[2a + (k - 1)d] + (a + kd) \qquad\qquad\qquad$ [Using equation (1)]

$= \dfrac{2ka + k(k - 1)d + 2(a + kd)}{2}$

$= \dfrac{2ka + k^2 d - kd + 2a + 2kd}{2}$

$= \dfrac{2ka + 2a + k^2 d + kd}{2}$

$= \dfrac{2a(k + 1) + d(k^2 + k)}{2}$

$= \dfrac{(k + 1)}{2}[2a + kd]$

$\Rightarrow \qquad P(n)$ is true for $n = k + 1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by *PMI*

Q19

Let $P(n) : \left(5^{2n} - 1\right)$ is divisible by 24

For $n = 1$

$5^2 - 1 = 24$

Which is divisible by 24

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$\Rightarrow \quad \left(5^{2k} - 1\right)$ is divisible by 24

$\Rightarrow \quad 5^{2k} - 1 = 24\lambda \qquad\qquad\qquad ---(1)$

We have to show that,

$\left(5^{2k} - 1\right)$ is divisible by 24

$5^{2(k+1)} - 1 = 24\mu$

Now,

$5^{2(k+1)} - 1$

$= 5^{2k} . 5^2 - 1$

$= 25 . 5^{2k} - 1$

$= 25\left(24\lambda + 1\right) - 1 \qquad\qquad \left[\text{Using equation } (1)\right]$

$= 25 . 24\lambda + 24$

$= 24\left(25\lambda + 1\right)$

$= 24\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q20

Let $P(n): 3^{2n} + 7$ is divisible by 8

For $n = 1$

$3^2 + 7 = 16$

Which is divisible by 8

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$3^{2k} + 7$ is divisible by 8

$\Rightarrow \quad 3^{2k} + 7 = 8\lambda \qquad\qquad ---(1)$

We have to show that,

$3^{2(k+1)} + 7$ is divisible by 8

$3^{2(k+1)} + 7 = 8\mu$

Now,

$3^{2(k+1)} + 7$

$= 3^{2k}.3^2 + 7$

$= 9.3^{2k} + 7$

$= 9.(8\lambda - 7) + 7$

$= 72\lambda - 56$

$= 8(9\lambda - 7)$

$= 8\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q21

Let $P(n) : 5^{2n+2} - 24n - 25$ is divisible by 576

For $n = 1$

$5^4 - 24 - 25$

$= 625 - 49$

$= 576$

Which is divisible by 576

Let $P(n)$ is true for $n = k$, so

$5^{2k+2} - 24k - 25$ is divisible by 576

$5^{2k+2} - 24k - 25 = 576\lambda$ $---(1)$

We have to show that,

$5^{2k+4} - 24(k+1) - 25$ is divisible by 576

$5^{(2k+2)+2} - 24(k+1) - 25 = 576\mu$

Now,

$5^{(2k+2)+2} - 24(k+1) - 25$

$= 5^{(2k+2)}.5^2 - 24k - 24 - 25$

$= (576\lambda + 24k + 25)25 - 24k - 49$ $\left[\text{Using equation } (1)\right]$

$= 25.576\lambda + 600k + 625 - 24k - 49$

$= 25.576\lambda + 576k + 576$

$= 576(25\lambda + k + 1)$

$= 576\mu$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q22

Let $P(n): 3^{2n+2} - 8n - 9$ is divisible by 8

For $n = 1$

$3^{2+2} - 8 - 9$

$= 81 - 17$

$= 64$

It is divisible by 8

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\left(3^{2k+2} - 8k - 9\right)$ is divisible by 8

$\Rightarrow$ $3^{2k+2} - 8k - 9 = 8\lambda$ $\qquad\qquad --- (1)$

We have to show that,

$3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8

$3^{2(k+1)}.3^2 - 8(k+1) - 9 = 8\mu$

Now,

$3^{2(k+1)}.9 - 8k - 8 - 9$

$= (8\lambda + 8k + 9)9 - 8k - 8 - 9$

$= 72\lambda + 72k + 81 - 8k - 17$

$= 72\lambda + 64k + 64$

$= 8(9\lambda + 8k + 8)$

$= 8\mu$

$\Rightarrow$ $P(n)$ is true for $n = 1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q23

Let $P(n) : (ab)^n = a^n b^n$

For $n = 1$

$(ab)^1 = a^1 b^1$

$ab = ab$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$$(ab)^k = a^k b^k \qquad\qquad ---(1)$$

We have to show that,

$$(ab)^{k+1} = a^{k+1} b^{k+1}$$

Now,

$(ab)^{k+1}$

$= (ab)^k (ab)$

$= (a^k b^k)(ab) \qquad\qquad [\text{Using equation } (1)]$

$= (a^{k+1})(b^{k+1})$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q24

Let $P(n) : n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$

For $n = 1$
$1 \cdot (1+1)(1+5)$
$= (2)(6)$

$= 12$

it is a multiple of 3

Let $P(n)$ is true for $n = k$
$k(k+1)(k+5)$ is a multiple of 3

$$k(k+1)(k+5) = 3\lambda \qquad\qquad ---(1)$$

We have to show that,

$(k+1)\big[(k+1)+1\big]\big[(k+1)+5\big]$ is a multiple of 3
$(k+1)\big[(k+1)+1\big]\big[(k+1)+5\big] = 3\mu$

Now,

$(k+1)(k+2)\big[(k+1)+5\big]$
$= \big[k(k+1)+2(k+1)\big]\big[(k+5)+1\big]$
$= k(k+1)(k+5)+k(k+1)+2(k+1)(k+5)+2(k+1)$
$= 3\lambda + k^2 + k + 2\big(k^2+6k+5\big)+2k+2 \qquad\qquad \big[\text{Using equation }(1)\big]$
$= 3\lambda + k^2 + k + 2k^2 + 12k + 10 + 2k + 2$
$= 3\lambda + 3k^2 + 15k + 12$
$= 3\big(\lambda + k^2 + 5k + 4\big)$

$= 3\mu$

$\Rightarrow \qquad P(n)$ is true for $n = k+1$
$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q25

Let $P(n) : 7^{2n} + 2^{3n-3}.3^{n-1}$ is divisible by 25

For $n = 1$

$7^2 + 2^0.3^0$

$= 49 + 1$

$= 50$

it is divisible of 25

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$7^{2k} + 2^{3k-3}.3^{k-1}$ is divisible by 25

$\Rightarrow \quad 7^{2k} + 2^{3k-3}.3^{k-1} = 25\lambda$ $\qquad\qquad ---(1)$

We have to show that,

$7^{2(k+1)} + 2^{3k}3^k$ is divisible by 25

$7^{2(k+1)} + 2^{3k}.3^k = 25\mu$

Now,

$7^{2(k+1)} + 2^{3k}.3^k$

$= 7^{2k}.7^2 + 2^{3k}.3^k$

$= \left(25\lambda - 2^{3k-3}.3^{k-1}\right)49 + 2^{3k}.3k$ $\qquad$ $\left[\text{Using equation }(1)\right]$

$= 25\lambda.49 - \dfrac{2^{3k}}{8}.\dfrac{3^k}{3}.49 + 2^{3k}.3^k$

$= 24.25.49\lambda - 2^{3k}.3^k.49 + 24.2^{3k}.3^k$

$= 24.25.49\lambda - 25.2^{3k}.3^k$

$= 25\left(24.49\lambda - 2^{3k}.3^k\right)$

$= 25\mu$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q26

Let $P(n): 2.7^n + 3.5^n - 5$ is divisible by 24

For $n = 1$

$2.7 + 3.5 - 5$

$= 24$

it is divisible of 24

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$2.7^k + 3.5^k - 5$ is divisible by 24

$2.7^k + 3.5^k - 5 = 24\lambda$ $\qquad\qquad$ $---(1)$

We have to show that,

$2.7^{(k+1)} + 3.5^{(k+1)} - 5$

$= 2.7^k.7 + 3.5^k.5 - 5$

$= \left(24\lambda - 3.5^k + 5\right)7 + 15.5^k - 5$

$= 24.7\lambda - 21.5^k + 35 + 15.5^k - 5$

$= 24.7\lambda - 6.5^k + 30$

$= 24.7\lambda - 6\left(5^k - 5\right)$

$= 24.7\lambda - 6.(20\nu)$ $\qquad\qquad$ $\left[\text{Since } 5^k - 5 \text{ is multiple of } 20\right]$

$= 24\left(7\lambda - 5\nu\right)$

$= 24\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q27

Let $P(n) : 11^{n+2} + 12^{2n+1}$ is divisible by 133

For $n = 1$

$11^3 + 12^3$

$= 1331 + 1728$

$= 3059$

it is divisible of 133

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$11^{k+2} + 12^{2k+1}$ is divisible by 133

$11^{k+2} + 12^{2k+1} = 133\lambda$ $---(1)$

We have to show that,

$11^{k+3} + 12^{2k+3}$ is divisible by 133

Now,

$11^{k+2}.11 + 12^{2k+1}.12^2$

$= \left(133\lambda - 12^{2k+1}\right)11 + 12^{2k+1}.144$

$= 11.133\lambda - 11.12^{2k+1} + 144.12^{2k+1}$

$= 11.133\lambda + 133.12^{2k+1}$

$= 133\left(11\lambda + 12^{2k+1}\right)$

$= 133\mu$

$\Rightarrow$ $P(n)$ is true for $n = k+1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q28

Consider equation

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots n \times n!$

Lets take $(n+1)! - n! = n! (n+1 - 1) = n \times n!$

Now substitue n=1,2,3,4,...n in above equation we get

$2! - 1! = 1 \times 1!$

$3! - 2! = 2 \times 2!$

$4! - 3! = 3 \times 3!$

....

$(n+1)! - n! = n \times n!$

Adding all the above terms gives

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots n \times n! = 2! - 1! + 3! - 2! + 4! - 3! \ldots + (n+1)! - n!$

$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots n \times n! = (n+1)! - 1$

Q29

Let P(n) be the statement given by

$P(n) : n^3 - 7n + 3$ is divisible by 3.

Step I:

$P(1) : 1^3 - 7(1) + 3$ is divisible by 3

$\because 1 - 7 + 3 = -3$ is divisible by 3

$\therefore P(1)$ is true.

Step II:

Let P(m) is true. Then,

$m^3 - 7m + 3$ is divisible by 3

$\Rightarrow m^3 - 7m + 3 = 3\lambda$ for some $\lambda \in N$ ....(i)

We have to prove that $P(m+1)$ is true.

$$(m+1)^3 - 7(m+1) + 3 = m^3 + 3m^2 + 3m + 1 - 7m - 7 + 3$$
$$= m^3 - 7m + 3 + 3m^2 + 3m + 1 - 7$$
$$= (m^3 - 7m + 3) + 3(m^2 + m - 2)$$
$$= 3\lambda + 3(m^2 + m - 2) \ldots \ldots \ldots \ldots [\text{Using (i)}]$$
$$= 3[\lambda + (m^2 + m - 2)] \text{ which is divisible by 3}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q30

Let $P(n)$ be the statement given by

$P(n): 1 + 2 + 2^2 + \cdots\cdots + 2^n = 2^{n+1} - 1$ for all $n \in N$.

Step I:

$P(1): 1 + 2^1 = 2^{1+1} - 1$

$\Rightarrow 1 + 2 = 4 - 1$

$\Rightarrow 3 = 3$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$1 + 2 + 2^2 + \cdots\cdots + 2^m = 2^{m+1} - 1 \ \ldots\ldots\ldots(i)$

We have to prove that $P(m+1)$ is true.

$$1 + 2 + 2^2 + \cdots\cdots + 2^{m+1} = 1 + 2 + 2^2 + \cdots\cdots + 2^m + 2^{m+1}$$

$$= \left(2^{m+1} - 1\right) + 2^{m+1} \ldots\ldots\ldots\ldots\ldots[\text{Using (i)}]$$

$$= \left(2^{m+1} + 2^{m+1}\right) - 1$$

$$= 2 \times 2^{m+1} - 1$$

$$= 2^{m+2} - 1$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q31

Let P(n) be the statement given by

$$P(n): 7 + 77 + 777 + \cdots + \underset{n-digits}{777\ldots\ldots7} = \frac{7}{81}\left[10^{n+1} - 9n - 10\right] \text{ for all } n \in N.$$

Step I:

$$P(1): 7 = \frac{7}{81}\left[10^{1+1} - 9(1) - 10\right]$$

$$\Rightarrow 7 = \frac{7}{81} \times (100 - 9 - 10)$$

$$\Rightarrow 7 = \frac{7}{81} \times 81$$

$$\Rightarrow 7 = 7 \times (1)$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$7 + 77 + 777 + \cdots + \underset{m-digits}{777\ldots\ldots7} = \frac{7}{81}\left[10^{m+1} - 9m - 10\right]\ldots\ldots(i)$$

We have to prove that $P(m+1)$ is true.

$$7 + 77 + 777 + \cdots + \underset{m+1-digits}{777\ldots\ldots7} = 7 + 77 + 777 + \cdots + \underset{m-digits}{777\ldots\ldots7} + \underset{m+1-digits}{777\ldots\ldots7}$$

$$= \frac{7}{81}\left[10^{m+1} - 9m - 10\right] + 7[\underset{m+1-digits}{1111\ldots\ldots1}] \qquad \left[\text{Using (i)}\right]$$

$$= \frac{7}{81}\left[10^{m+1} - 9m - 10\right] + \frac{7}{9}[\underset{m+1-digits}{9999\ldots\ldots9}]$$

$$= \frac{7}{81}\left[10^{m+1} - 9m - 10\right] + \frac{7}{9}\left[10^{m+1} - 1\right]$$

$$= \frac{7}{81}\left[(1+9)10^{m+1} - 9m - 19\right]$$

$$= \frac{7}{81}\left[10 \times 10^{m+1} - 9(m+1) - 10\right]$$

$$= \frac{7}{81}\left[10^{m+2} - 9(m+1) - 10\right]$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q32

Let $P(n): \dfrac{n^7}{7} + \dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{n^2}{2} - \dfrac{37}{210}$ n is a positive integer

For $n = 1$

$$\dfrac{1}{7} + \dfrac{1}{5} + \dfrac{1}{3} + \dfrac{1}{2} - \dfrac{37}{210}$$

$$= \dfrac{30 + 42 + 70 + 105 - 37}{210}$$

$$= \dfrac{247 - 37}{210}$$

It is a positive integer

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$$\dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{k^3}{3} + \dfrac{k^2}{2} - \dfrac{37}{210} k \text{ is positive integer}$$

$$\dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{k^3}{3} + \dfrac{k^2}{2} - \dfrac{37}{210} k = \lambda$$

For $n = k + 1$,

$$\dfrac{(k+1)^7}{7} + \dfrac{(k+1)^5}{5} + \dfrac{(k+1)^3}{3} + \dfrac{(k+1)^2}{2} - \dfrac{37}{210}(k+1)$$

$$= \dfrac{1}{7}\left[k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1\right] + \dfrac{1}{5}\left[k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\right]$$

$$+ \dfrac{1}{3}\left[k^3 + 3k^2 + 3k + 1\right] + \dfrac{1}{2}\left[k^2 + 2k + 1\right] - \dfrac{37k}{210} - \dfrac{37}{210}$$

$$= \left[\dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{k^3}{3} + \dfrac{k^2}{2} - \dfrac{37k}{210}\right] + \left[\begin{array}{l} k^6 + 3k^5 + 5k^4 + 5k^3 + 3k^2 + k + \dfrac{1}{7} + k^4 + 2k^3 + 2k^2 + \dfrac{1}{5} + k^2 \\ \\ \qquad\qquad +k + \dfrac{1}{3} + k + \dfrac{1}{2} - \dfrac{37}{210} \end{array}\right]$$

$$= \lambda + k^6 + 3k^5 + 6k^4 + 7k^3 + 6k^2 + 3k + \dfrac{1}{7} + \dfrac{1}{5} + \dfrac{1}{3} + \dfrac{1}{2} - \dfrac{37}{210}$$

$$= \lambda + k^6 + 3k^5 + 6k^4 + 7k^3 + 6k^2 + 3k + 1$$

$$= \text{Positive integer}$$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q33

$P(n): \quad \dfrac{n^{11}}{11} + \dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{62}{165}n \quad$ is a positive integer

For $n = 1$

$$\dfrac{1}{11} + \dfrac{1}{5} + \dfrac{1}{3} + \dfrac{62}{165}$$

$$= \dfrac{15 + 33 + 55 + 62}{165}$$

$$= \dfrac{165}{165}$$

Which is a positive integer

Let $P(n)$ is true for $n = k$, so

$$\dfrac{k^{11}}{11} + \dfrac{k^5}{5} + \dfrac{k^3}{3} + \dfrac{62k}{165} \quad \text{is a positive integer}$$

$$\dfrac{k^{11}}{11} + \dfrac{k^5}{5} + \dfrac{k^3}{3} + \dfrac{62k}{165} = \lambda \qquad\qquad \text{--- (i)}$$

For $n = k + 1$

$$\dfrac{(k+1)^{11}}{11} + \dfrac{(k+1)^5}{5} + \dfrac{(k+1)^3}{3} + \dfrac{62}{165}(k+1)$$

$$= \dfrac{1}{11}\left[k^{11} + 11k^{10} + 55k^9 + 165k^8 + 330k^7 + 462k^6 + 462k^5 + 330k^4 + 165k^3 + 55k^2 + 11k + 1 \right]$$

$$+ \dfrac{1}{5}\left[k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 \right] + \dfrac{1}{3}\left[k^3 + 3k^2 + 3k + 1 \right] + \dfrac{62}{165}[k+1]$$

$$= \left[\dfrac{k^{11}}{11} + \dfrac{k^5}{5} + \dfrac{k^3}{3} + \dfrac{62k}{165} \right] + k^{10} + 5k^9 + 15k^8 + 30k^7 + 42k^6 + 42k^5 + 30k^4 + 15k^3 + 5k^2 + 1 + \dfrac{1}{11}$$

$$+ k^4 + 2k^3 + 2k^2 + k + \dfrac{1}{5} + k^2 + k + \dfrac{1}{3} + \dfrac{62}{165}$$

$$= \lambda + k^{10} + 5k^9 + 15k^8 + 30k^7 + 42k^6 + 42k^5 + 31k^4 + 17k^3 + 8k^2 + 2k + 1$$

$$= \text{An integer}$$

$\Rightarrow \qquad P(n)$ is true for $n = k + 1$

$\Rightarrow \qquad P(n)$ is true for all $n \in N$ by PMI

Q34

Let $P(n): \dfrac{1}{2}\tan\left(\dfrac{x}{2}\right) + \dfrac{1}{4}\tan\left(\dfrac{x}{4}\right) + \ldots + \dfrac{1}{2^n}\tan\left(\dfrac{x}{2^n}\right) = \dfrac{1}{2^n}\cot\left(\dfrac{x}{2^n}\right) - \cot x$

For $n = 1$

$\dfrac{1}{2}\tan\dfrac{x}{2} = \dfrac{1}{2}\cot\left(\dfrac{x}{2}\right) - \cot x$

$= \dfrac{1}{2}\dfrac{1}{\tan\dfrac{x}{2}} - \dfrac{1}{\tan x}$

$= \dfrac{1}{2\tan\dfrac{x}{2}} - \dfrac{1}{\left(\dfrac{2\tan\dfrac{x}{2}}{1 - \tan^2\dfrac{x}{2}}\right)}$

$= \dfrac{1}{2\tan\dfrac{x}{2}} - \dfrac{1 - \tan^2\dfrac{x}{2}}{2\tan\dfrac{x}{2}}$

$= \dfrac{1 - 1 + \tan^2\dfrac{x}{2}}{2\tan\dfrac{x}{2}}$

$= \dfrac{\tan^2\dfrac{x}{2}}{2\tan\dfrac{x}{2}}$

$= \dfrac{1}{2}\tan\dfrac{x}{2}$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\dfrac{1}{2}\tan\left(\dfrac{x}{2}\right) + \dfrac{1}{4}\tan\left(\dfrac{x}{4}\right) + \ldots + \dfrac{1}{2^k}\tan\left(\dfrac{x}{2^k}\right) = \dfrac{1}{2^k}\cot\left(\dfrac{x}{2^k}\right) - \cot x \qquad\qquad ---(1)$

We have to show that,

$$\frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^k}\tan\left(\frac{x}{2^k}\right) + \frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right) = \frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right) - \cot x$$

Now,

$$\left\{\frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)\right\} + \frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

$$= \frac{1}{2^k}\cot\left(\frac{x}{2^k}\right) - \cot x + \frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

$$= \frac{1}{2^k}\cot\left(\frac{x}{2^k}\right) - \cot x + \frac{1}{2 \cdot 2^k}\frac{1}{\cot\left(\frac{x}{2^k} \cdot \frac{1}{2}\right)}$$

$$= \frac{1}{2^k}\left[\frac{1}{\tan\left(\frac{x}{2^k}\right)} + \frac{1}{2} \cdot \tan\left\{\left(\frac{x}{2^k}\right) \cdot \frac{1}{2}\right\}\right] - \cot x$$

$$= \frac{1}{2^k}\left[\frac{1 - \tan^2\left(\frac{x}{2^{k+1}}\right)}{2\tan\left(\frac{x}{2^{k+1}}\right)} + \frac{1}{2}\tan\left(\frac{x}{2 \cdot 2^k}\right)\right] - \cot x$$

$$= \frac{1}{2^k}\left[\frac{1 - \tan^2\left(\frac{x}{2^{k+1}}\right) + \tan^2\left(\frac{x}{2^{k+1}}\right)}{2\tan\left(\frac{x}{2^{k+1}}\right)}\right] - \cot x$$

$$= \frac{1}{2^{k+1}}\left[\frac{1}{\tan\left(\frac{x}{2^{k+1}}\right)}\right] - \cot x$$

$$= \frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right) - \cot x$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q35

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

Above can be written as

$$=\left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\left(\frac{4^2-1}{4^2}\right)\cdots\left(\frac{n^2-1}{n^2}\right)$$

$$=\left(\frac{(2+1)(2-1)}{2^2}\right)\left(\frac{(3+1)(3-1)}{3^2}\right)$$

$$\left(\frac{(4+1)(4-1)}{4^2}\right)\cdots\left(\frac{(n+1)(n-1)}{n^2}\right)$$

$$=\left(\frac{3\cdot1}{2^2}\right)\left(\frac{4\cdot2}{3^2}\right)\left(\frac{5\cdot3}{4^2}\right)\cdots\left(\frac{(n+1)\cdot(n-1)}{n^2}\right)$$

In the above product, there are two series in numerator

$3\cdot4\cdot5\ldots\ldots(n+1)$ and $1\cdot2\cdot3\ldots\ldots(n-1)$

All numbers from 3 to (n-1) are repeated twice

and 1, 2, n are appeared once in numerator

So after cancelling like terms we get

$$=\frac{(n+1)}{2n}$$

Q36

$$P(n) : \frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{\sqrt{3n+1}}$$

For $n = 1$

$$\frac{2!}{2^2 \cdot 1} \le \frac{1}{\sqrt{4}}$$

$$= \frac{1}{2} \le \frac{1}{2}$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{(2k)!}{2^{2k}(k!)^2} \le \frac{1}{\sqrt{3k+1}} \qquad\qquad ---(1)$$

We have to show that,

$$\frac{2(k+1)!}{2^{2(k+1)}\left[(k+1)!\right]^2} \le \frac{1}{\sqrt{3k+4}}$$

Now,

$$\frac{2(k+1)!}{2^{2(k+1)}\left[(k+1)!\right]^2}$$

$$= \frac{(2k+2)!}{2^{2k} \cdot 2^2 (k+1)!(k+1)!}$$

$$= \frac{(2k+2)(2k+1)(2k)!}{4 \cdot 2^2 (k+1)(k!)(k+1)(k!)}$$

$$= \frac{2(k+1)(2k+1)(2k)!}{4 \cdot (k+1)^2 \cdot 2^{2k} \cdot (k!)^2}$$

$$\le \frac{2(2k+1)}{4(k+1)} \cdot \frac{1}{\sqrt{3k+1}} \qquad\qquad \left[\text{Using equation } (1)\right]$$

$$\le \frac{(2k+1)}{2(k+1)} \cdot \frac{1}{\sqrt{3k+1}}$$

$$\le \frac{(2k+2)}{2(k+1)} \cdot \frac{1}{\sqrt{3k+3+1}}$$

$$\le \frac{1}{\sqrt{3k+4}} \qquad\qquad \left[\begin{array}{l}\text{Since, } 2k+1 < 2k+2 \\ 3k+1 \le 3k+4\end{array}\right]$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q37

Let $P(n): 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \geq 2$

For $n = 2$

$1 + \frac{1}{4} < 2 - \frac{1}{4}$

$= \frac{5}{4} < \frac{7}{4}$

$\Rightarrow \quad P(n)$ is true for $n = 2$

Let $P(n)$ is true for $n = k$,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... + \frac{1}{k^2} < 2 - \frac{1}{k} \qquad\qquad ---(1)$$

Now, we have to show that,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{(k+1)}$$

Now,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2}$$

$$< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \qquad\qquad [\text{Using (1)}]$$

$$< 2 - \frac{k^2 + 2k + 1 - k}{k(k+1)^2}$$

$$< 2 - \frac{k^2 + k + 1}{k(k+1)^2}$$

$$< 2 - \frac{k^2 + k}{k(k+1)^2}$$

$$< 2 - \frac{k(k+1)}{k(k+1)^2}$$

$$< 2 - \frac{1}{k+1}$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q38

Let $P(n): x^{2n-1} + y^{2n-1}$ is divisible by $(x+y)$

For $n = 1$

$x^{2(1)-1} + y^{2(1)-1}$

$= x + y$

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$x^{2k-1} + y^{2k-1}$ is divisible by $(x+y)$

$x^{2k-1} + y^{2k-1} = (x+y)\lambda$ $\qquad\qquad$ $---(1)$

We have to show that,

$x^{2k+1} + y^{2k+1} = \qquad (x+y)\mu$

Now,

$x^{2k+1} + y^{2k+1}$

$= x^{2k-1}x^2 + y^{2k-1}y^2$

$= \left[(x+y)\lambda - y^{2k-1} \right]x^2 + y^{2k-1}.y^2$

$= (x+y)\lambda x^2 - y^{2k-1}.x^2 + y^{2k-1}.y^2$

$= (x+y)\lambda x^2 - y^{2k-1}\left(x^2 - y^2\right)$

$= (x+y)\lambda x^2 - y^{2k-1}(x+y)(x-y)$

$= (x+y)\left[\lambda x^2 - y^{2k-1}(x-y)\right]$

$= (x+y)\mu$

$\Rightarrow$ $P(n)$ is true for $n = k+1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q39

Let $P(n) : \sin x + \sin 3x + \ldots + \sin(2n-1)x = \dfrac{\sin^2 nx}{\sin x}$

For $n = 1$

$$\sin x = \frac{\sin^2 x}{\sin x}$$

$$\sin x = \sin x$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\sin x + \sin 3x + \ldots + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x} \qquad \text{---(i)}$$

We have to show that

$$\sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x = \frac{\sin^2 (k+1)x}{\sin x}$$

Now,

$$\{\sin x + \sin 3x + \ldots + \sin(2k-1)x\} + \sin(2k+1)x$$

$$= \frac{\sin^2 kx}{\sin x} + \frac{\sin(2k+1)x}{1}$$

Using equation (i),

$$= \frac{\sin^2 kx + \sin(2k+1)x \sin x}{\sin x}$$

$$= \frac{2\sin^2 kx + \cos[(2k+1)x - x] - \cos[2kx + x + x]}{2\sin x}$$

$$= \frac{2\sin^2 kx + \cos 2kx - \cos(2kx + 2x)}{2\sin x}$$

$$= \frac{1 - \cos 2kx + \cos 2kx - \cos 2x(k+1)}{2\sin x}$$

$$= \frac{1 - \cos 2x(k+1)}{2\sin x}$$

$$= \frac{2\sin^2 x(k+1)}{2\sin x}$$

$$= \frac{\sin^2 x(k+1)}{\sin x}$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI.

Q40

Let $P(n)$ be the statement given by

$P(n) : \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (n-1)\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}} \text{ for all } n \in N.$$

Step I:

$$P(1) : \cos \alpha = \frac{\cos\left\{\alpha + \left(\frac{1-1}{2}\right)\beta\right\}\sin\left(\frac{1\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$\Rightarrow \cos \alpha = \frac{\cos\left\{\alpha + 0\right\}\sin\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$\Rightarrow \cos \alpha = \cos \alpha$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (m-1)\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

We have to prove that $P(m+1)$ is true.

$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (m)\beta)$

$= \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (m-1)\beta) + \cos(\alpha + (m)\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}} + \cos(\alpha + (m)\beta)\ldots\ldots\ldots\ldots\ldots[\text{Using (i)}]$$

$$= \frac{\sin\frac{\beta}{2}\cos\left(\alpha+(m)\beta\right)+\cos\left\{\alpha+\left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m+1}{2}\right)\beta\right)-\sin\left(\alpha+\left(\frac{2m-1}{2}\right)\beta\right)\right]+\cos\left\{\alpha+\left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m+1}{2}\right)\beta\right)-\sin\left(\alpha+\left(\frac{2m-1}{2}\right)\beta\right)\right]+\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m-1}{2}\right)\beta\right)+\sin\left(-\alpha+\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m+1}{2}\right)\beta\right)\right]+\frac{1}{2}\left[\sin\left(-\alpha+\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin\alpha\cos\left(\frac{2m+1}{2}\right)\beta+\cos\alpha\sin\left(\frac{2m+1}{2}\right)\beta+\sin\frac{\beta}{2}\cos\alpha-\cos\frac{\beta}{2}\sin\alpha\right]}{\sin\frac{\beta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin\alpha\left(\cos\left(\frac{2m+1}{2}\right)\beta-\cos\frac{\beta}{2}\right)+\cos\alpha\left(\sin\left(\frac{2m+1}{2}\right)\beta+\sin\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$= \frac{\frac{1}{2}\left[-2\sin\alpha\left(\left(\sin\left(\frac{m+1}{2}\right)\beta\right)\sin\frac{m\beta}{2}\right)+2\cos\alpha\left(\left(\sin\left(\frac{m+1}{2}\right)\beta\right)\cos\frac{m\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\frac{(m+1)\beta}{2}\right)\left[\cos\alpha\cos\frac{m\beta}{2}-\sin\alpha\sin\left(\frac{(m+1)\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\frac{(m+1)\beta}{2}\right)\cos\left(\alpha+\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q41

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24},$$

Using induction we first show this is true for n=2:

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12} = \frac{14}{24} > \frac{13}{24} \; (True)$$

Now lets assume it is true for some n=k,

$$S_k = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} > \frac{13}{24}$$

Finally we need to prove that this implies

it is also true for n=k+1:

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+2}$$

$$= \frac{-1}{k+1} + \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$= \frac{-1}{k+1} + S_k + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$= S_k + \frac{1}{2(2k+1)(k+1)}$$

$$> S_k$$

$$\therefore S_{k+1} > \frac{13}{24}$$

Q42

$$a_1 = \frac{1}{2}\left(a_0 + \frac{A}{a_0}\right), A_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right) \text{ and } a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$$

Let $P(n): \dfrac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\dfrac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$

For $n = 1$

$$\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$$

$$\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)$$

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right) \qquad\qquad ---(i)$$

We have to show that

$$\frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^k}$$

$$\left(\frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}}\right)^{2^0}$$

$$= \left[\frac{\dfrac{1}{2}\left(a_k + \dfrac{A}{a_k}\right) - \sqrt{A}}{\dfrac{1}{2\left(a_k + \dfrac{A}{a_k}\right) + \sqrt{A}}}\right]^{2^0}$$

$$= \left[\frac{\left(a_k\right)^2 + A - 2ak\sqrt{A}}{\left(ak\right)^2 + A + 2a_k\sqrt{A}}\right]^{2^0}$$

$$= \frac{\left(a_k - \sqrt{A}\right)^2}{\left(a_k + \sqrt{A}\right)^2}$$

$$= \left[\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}}\right]^{2^1}$$

$$= \left[\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right]^{2^k}$$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$

$\Rightarrow$ $P(n)$ is true for all $n \in NE$ by PMI

Q43

$P(n) : 2^n \geq 3n$

It is given that $P(r)$ is true, so

$2^r \geq 3r$ --- (1)

Multiplying both the sides by 2,

$2^r.2 \geq 3r.2$
$2^{r+1} \geq 6r$
$2^{r+1} \geq 3r + 3r$
$2^{r+1} \geq 3 + 3r$ $\left[\text{Since } 3r \geq 3, \ 6r \geq 3 + 3r\right]$

$2^{r+1} \geq 3(r+1)$

So, $P(r+1)$ is true

But for $r = 1$
$2 \geq 3$

It is true, so

$P(n)$ is not true for all $n \in N$ by PMI

Q44

$S_n = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \ldots$

Using induction we first show this is true for n=2,

we get $S_2 = 1^2 + 2 \times 2^2 = 1 + 8 = 9$

From RHS, we have if n is even $S_n = \dfrac{n(n+1)^2}{2}$

$S_2 = \dfrac{2 \times 9}{2} = 9$

Now using induction we first show this is true also

for n=3, we get $S_3 = 1 + 8 + 9 = 18$

From RHS, we have if n is odd $S_n = \dfrac{n^2(n+1)}{2}$

$S_3 = \dfrac{9 \times 4}{2} = 18$

Lets assume above is true for n=k, we get

k is even, $S_k = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \ldots + 2 \times k^2 \quad --1$

k is odd, $S_k = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \ldots + k^2 \quad ----2$

Now lets prove for n=k+1

If k is even, k+1 is odd we get

$S_{k+1} = 1^2 + 2 \times 2^2 + 3^2 + \ldots + 2 \times k^2 + (k+1)^2 \quad ---3$

From above relation, we get

$S_k = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \ldots + 2 \times k^2 = \dfrac{k(k+1)^2}{2}$

Substitute this in 3, we get

$S_{k+1} = \dfrac{k(k+1)^2}{2} + (k+1)^2 = \dfrac{(k+1)^2(k+2)}{2}$

$= RHS$ (when '$k+1$'is odd)

Hence Proved

Q45

Let P(n) be the statement given by

P(n) : The number of subsets of a set containing n distinct elements is 2^n

for all $n \in N$.

Step I:

$P(1) : 2^1 = 2$

For any set A contaiining 1 element, empty set and set A are two sets always subsets of A.

∴ P(1) is true.

Step II:

Let P(m) is true. Then,

A set containing m distinct elements has 2^m subsets..........(i)

We have to prove that $P(m + 1)$ is true.

Let the set A has (m+1) elements.

$A = \{1,2,.......,m,m+1\}$

$A = \{1,2,.......,m\} \cup \{m+1\}$

Now using (i) we can say that $\{1,2,.......,m\}$ being m elemets has 2^m subsets.

For $\{m+1\}$, empty set and set itself $\{m+1\}$ are subsets.

So, $\{m+1\}$ has 2 subsets.

⇒ Set A has $2^m + 2$ subsets

⇒ Set A has 2^{m+1} subsets

⇒ P(m+1) is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q46

Let $P(n)$ be the statement given by

$P(n): a_n = 3 \times 7^{n-1}$ for all $n \in N$.

Step I:

$P(2): a_2 = 3 \times 7^{2-1} = 21$

Given that $a_k = 7\, a_{k-1}$ for all natural numbers $k \geq 2$

$a_2 = 7a_1 = 7 \times 3 = 21$

$\therefore P(2)$ is true.

Step II:

Let $P(m)$ is true. Then,

$a_m = 3 \times 7^{m-1}$ ....(i)

We have to prove that $P(m+1)$ is true.

$a_{m+1} = 7a_m$

$a_{m+1} = 7 \times a_m$

$a_{m+1} = 7^1 \times 3 \times 7^{m-1}$ ....$\left[\text{from(i)} \right]$

$a_{m+1} = 3 \times 7^{m-1+1}$

$a_{m+1} = 3 \times 7^m$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q47

Let $P(n)$ be the statement given by

$P(n) : x_n = \dfrac{2}{n!}$ for all $n \in N$.

Step I:

$P(2) : x_2 = \dfrac{2}{2!} = 1$

Given that $x_k = \dfrac{x_{k-1}}{n}$ for all natural numbers $k \geq 2$

$x_2 = \dfrac{x_1}{2} = \dfrac{2}{2} = 1$

$\therefore P(2)$ is true.

Step II:
Let $P(m)$ is true. Then,

$x_m = \dfrac{2}{m!} \ldots \ldots \ldots (i)$

We have to prove that $P(m+1)$ is true.

$x_{m+1} = \dfrac{x_{m+1-1}}{m+1}$

$x_{m+1} = \dfrac{x_m}{m+1}$

$x_{m+1} = \dfrac{\dfrac{2}{m!}}{m+1} \ldots \ldots \ldots \big[\text{from}\,(i)\big]$

$x_{m+1} = \dfrac{2}{m!(m+1)}$

$x_{m+1} = \dfrac{2}{(m+1)!}$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q48

Let $P(n)$ be the statement given by

$P(n): x_n = 5 + 4n$ for all $n \in N$.

Step I:

$P(1): x_1 = 5 + 4(1) = 5 + 4 = 9$

Given that $x_k = 4 + x_{k-1}$ for all natural numbers k

$x_1 = 4 + x_0 = 4 + 5 = 9$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$x_m = 5 + 4m \ldots \ldots (i)$

We have to prove that $P(m+1)$ is true.

$x_{m+1} = 4 + x_{m+1-1}$

$x_{m+1} = 4 + x_m$

$x_{m+1} = 4 + 5 + 4m \ldots \ldots \ldots [\text{from(i)}]$

$x_{m+1} = 5 + 4(m + 1)$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q49

Let P(n) be the statement given by

$P(n): \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$.

Step I:

$P(2): \sqrt{2} = 1.4142$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{1.4142} = 1 + 0.7071 = 1.7071$

$\therefore \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$

$\therefore P(2)$ is true.

Step II:

Let P(m) is true. Then,

$\sqrt{m} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} \ldots\ldots\ldots(i)$

We have to prove that P(m+1) is true.

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} > \sqrt{m} \ldots\ldots\ldots[\text{from}(i)]$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m} + \frac{1}{\sqrt{m+1}}$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{\sqrt{m^2 + m} + 1}{\sqrt{m+1}}$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{\sqrt{m^2} + 1}{\sqrt{m+1}}$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{m+1}{\sqrt{m+1}}$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q50

The distributive law from algebra states that for all real numbers c, a1 and a2, we have c (a1 + a2) = ca1 + ca2
Use this law and mathematical induction to prove that, for all natural numbers, n ≥ 2, if c (a1 + a2 + &. + an)

$$= ca1 + ca2 + ... + can$$

Let P(n) be the statement given by

$P(n): c(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ca_3 + ... + ca_n$ for all natural numbers n ≥ 2.

Step I:
$P(2): c(a_1 + a_2) = ca_1 + ca_2$

∴ P(2) is true.

Step II:
Let P(m) is true. Then,
$c(a_1 + a_2 + ... + a_m) = ca_1 + ca_2 + ca_3 + ... + ca_m$...........(i)

We have to prove that P(m + 1) is true.

$c(a_1 + a_2 + ... + a_m + a_{m+1}) = c[(a_1 + a_2 + ... + a_m) + a_{m+1}]$

$c(a_1 + a_2 + ... + a_m + a_{m+1}) = c(a_1 + a_2 + ... + a_m) + ca_{m+1}$

$c(a_1 + a_2 + ... + a_m + a_{m+1}) = ca_1 + ca_2 + ca_3 + ... + ca_m + ca_{m+1}$

$$........... [from(i)]$$

⇒ P(m+1) is true.

Hence by the principle of mathematical induction, the given result is true for all n ∈ N.