Ex 13.1

Q1(i)

We know that $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$= i^{4p} \times i^q$

$= \left(i^4\right)^p \times i^q$

$= 1^p \times i^q$

$= i^q \qquad \left[\because 1^{p=1}\right]$

Hence $i^n = i^q$, *where* $o \le q < 4$

$\therefore i^{457} = i^{4\times 114} \times i^1$

$= i^1$

$= i$

Q1(ii)

We know that $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$= i^{4p} \times i^q$

$= \left(i^4\right)^p \times i^q$

$= 1^p \times i^q$

$= i^q \qquad \left[\because 1^{p=1}\right]$

Hence $i^n = i^q$, *where* $o \le q < 4$

$\therefore i^{528} = i^{4\times 132}$

$= \left(i^4\right)^{132}$

$= 1^{132}$

$= 1$

$\therefore \left(i^{528}\right) = 1$

Q1(iii)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$$= i^{4p} \times i^q$$
$$= \left(i^4\right)^p \times i^q$$
$$= 1^p \times i^q$$
$$= i^q \qquad \left[\because 1^{p=1}\right]$$

Hence $i^n = i^q$, *where* $o \le q < 4$

$$\therefore \frac{1}{i^{58}} = \frac{1}{i^{4\times 14} \times i^2}$$
$$= \frac{1}{1 \times i^2}$$
$$= \frac{1}{-1} \qquad \left[\because i^2 = -1\right]$$
$$= -1$$

Q1(iv)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$$= i^{4p} \times i^q$$
$$= \left(i^4\right)^p \times i^q$$
$$= 1^p \times i^q$$
$$= i^q \qquad \left[\because 1^{p=1}\right]$$

Hence $i^n = i^q$, *where* $o \le q < 4$

$$\therefore i^{37} + \frac{1}{i^{67}} = i^{4\times 9} \times i^1 + \frac{1}{i^{4\times 16} \times i^3}$$
$$= 1 \times i^1 + \frac{1}{1 \times i^3}$$
$$= i + \frac{1}{i^3 \times i} \times i$$
$$= i + \frac{i}{i^4}$$
$$= i + \frac{i}{1} \qquad \left[\because i^4 = 1\right]$$
$$= 2i$$

$$\therefore i^{37} + \frac{1}{i^{67}} = 2i$$

Q1(v)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$$= i^{4p} \times i^q$$
$$= \left(i^4\right)^p \times i^q$$
$$= 1^p \times i^q$$
$$= i^q \qquad \left[\because 1^{p=1}\right]$$

Hence $i^n = i^q$, *where* $o \le q < 4$

$$\left(i^{41} + \frac{1}{i^{257}}\right)^9 = \left(i^{4\times10} \times i^1 + \frac{1}{i^{4\times64} \times i^1}\right)^9$$
$$= \left(1 \times i + \frac{1}{1 \times i}\right)^9$$
$$= \left(i + \frac{1}{i}\right)^9$$
$$= \left(i + \frac{1}{i \times i} \times i\right)^9$$
$$= \left(i + \frac{i}{-1}\right)^9$$
$$= (i - i)^9$$
$$= 0$$

Q1(vi)

We know that $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$= i^{4p} \times i^q$

$= \left(i^4\right)^p \times i^q$

$= 1^p \times i^q$

$= i^q \qquad \left[\because 1^{p-1}\right]$

Hence $i^n = i^q$, *where* $o \le q < 4$

$$\left(i^{77} + i^{70} + i^{87} + i^{414}\right)^3 = \left(i^{4\times19} \times i^1 + i^{4\times17} \times i^2 + i^{4\times21} \times i^3 + i^{4\times103} \times i^2\right)^3$$

$$= \left(1 \times i + 1 \times i^2 + 1 \times i^3 + 1 \times i^2\right)^3$$

$$= \left(i - 1 - i - 1\right)^3$$

$$= \left(-2\right)^3$$

$$= -8$$

$\therefore \left(i^{77} + i^{70} + i^{87} + i^{414}\right)^3 = -8$

Q1(vii)

We know that $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

Then $i^n = i^{4p+q}$

$= i^{4p} \times i^q$

$= \left(i^4\right)^p \times i^q$

$= 1^p \times i^q$

$= i^q \qquad \left[\because 1^{p-1}\right]$

Hence $i^n = i^q$, *where* $o \le q < 4$

$$\therefore i^{30} + i^{40} + i^{60} = i^{4\times7} \times i^2 + i^{4\times10} + i^{4\times15}$$

$$= 1 \times i^2 + 1 + 1$$

$$= -1 + 1 + 1$$

$$= 1$$

$\therefore i^{30} + i^{40} + i^{60} = 1$

Q1(viii)

We know that $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q, so that

$$n = 4p + q, o \le q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= \left(i^4\right)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \qquad \left[\because 1^{p-1}\right] \end{aligned}$$

Hence $i^n = i^q$, *where* $o \le q < 4$

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{4\times12} \times i^1 + i^{4\times17} + i^{4\times22} \times i^1 + i^{4\times27} \times i^2 \\ &= 1 \times i + 1 + 1 \times i + 1 \times i^2 \\ &= i + 1 + i - 1 \\ &= 2i \end{aligned}$$

$\therefore i^{49} + i^{68} + i^{89} + i^{110} = 2i$

Q2

$$\begin{aligned} 1 + i^{10} + i^{20} + i^{30} &= 1 + i^{4\times2} \times i^2 + i^{4\times5} + i^{4\times7} \times i^2 \\ &= 1 + 1 \times i^2 + 1 + 1 \times i^2 \\ &= 1 - 1 + 1 - 1 \\ &= 0, \text{ which is real number} \end{aligned}$$

Q3(i)

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{4\times12} \times +i^1 + i^{4\times17} + i^{4\times22} \times i^1 + i^{4\times27} \times i^2 \\ &= 1 \times i + 1 + 1 \times i + 1 \times i^2 \\ &= i + 1 + i - 1 \\ &= 2i \end{aligned}$$

$\therefore i^{49} + i^{68} + i^{89} + i^{110} = 2i$

Q3(ii)

$$i^{30}+i^{80}+i^{120} = i^{4\times 7}\times i^{2}+i^{4\times 20}+i^{4\times 30}$$
$$= 1\times i^{2}+1+1$$
$$= -1+1+1$$
$$= 1$$

$$\therefore\ i^{30}+i^{80}+i^{120} = 1$$

Q3(iii)

$$i+i^{2}+i^{3}+i^{4} = 1+(-1)+(-i)+1$$
$$= 0$$

$$\therefore\ i+i^{2}+i^{3}+i^{4} = 0$$

Q3(iv)

$$i^{5}+i^{10}+i^{15} = i^{4\times 1}\times i^{1}+i^{4\times 2}\times i^{2}+i^{4\times 3}\times i^{3}$$
$$= 1\times i+1\times i^{2}+1\times i^{3}$$
$$= i-1-i$$
$$= -1$$

$$\therefore\ i^{5}+i^{10}+i^{15} = -1$$

Q3(v)

$$\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}} = \frac{i^{4\times 148}+i^{147}\times i^{2}+i^{4\times 147}+i^{4\times 146}\times i^{2}+i^{4\times 146}}{i^{4\times 145}\times i^{2}+i^{4\times 145}+i^{4\times 144}\times i^{2}+i^{4\times 144}+i^{4\times 143}\times i^{2}}$$
$$= \frac{1+1\times i^{2}+1+1\times i^{2}+1}{1\times i^{2}+1+1\times i^{2}+1+1\times i^{2}}$$
$$= \frac{1-1+1-1+1}{-1+1-1+1-1}$$
$$= \frac{1}{-1}$$
$$= -1$$

$$\therefore\ \frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}} = -1$$

Q3(vi)

$$1+i^2+i^4+i^6+i^8+\ldots+i^{20}$$
$$=1+i^2+i^4+i^{4\times1}\times i^2+i^{4\times2}+i^{4\times2}\times i^2+i^{4\times3}+i^{4\times3}\times i^2+i^{4\times4}+i^{4\times4}\times i^2+i^{4\times5}$$
$$=1-1+1+1\times i^2+1+1\times i^2+1+1\times i^2+1+1\times i^2+1$$
$$=1-1+1-1+1-1+1-1+1-1+1$$
$$=1$$

Q3(vii)

$$(1+i)^6+(1-i)^3=\left[(1+i)^2\right]^3+(1-i)^3$$
$$=\left(1+i^2+2i\right)^3+\left(1-3i+3i^2-i^3\right)$$
$$=(1-1+2i)^3+(1-3i-3+i)$$
$$=8i^3-2-2i$$
$$=-8i-2-2i$$
$$=-2-10i$$

Ex 13.2

Q1(i)

$$\begin{aligned}(1+i)(1+2i) &= 1\times(1+2i)+i(1+2i)\\ &= 1+2i+i+2i^2\\ &= 1+3i-2\\ &= -1+3i\end{aligned}$$

$$\therefore\ (1+i)(1+2i) = -1+3i$$

Q1(ii)

$$\begin{aligned}\frac{3+2i}{-2+i} &= \frac{3+2i}{(-2+i)}\times\frac{(-2-i)}{-2-i} && \text{[Rationalising the denominator]}\\ &= \frac{3(-2-i)+2i(-2-i)}{(-2)^2-(i)^2} && [\because (a+ib)(a-ib)=a^2+b^2]\\ &= \frac{-6-3i-4i+2}{4+1} && [\because -i^2=1]\\ &= \frac{-4-7i}{5}\\ &= \frac{-4}{5}-\frac{7}{5}i\end{aligned}$$

$$\therefore\ \frac{3+2i}{-2+i} = \frac{-4}{5}-\frac{7}{5}i$$

Q1(iii)

$$\begin{aligned}\frac{1}{(2+i)^2} &= \frac{1}{2^2+(i)^2+2\times2\times i}\\ &= \frac{1}{4-1+4i}\\ &= \frac{1}{3+4i}\\ &= \frac{1}{(3+4i)}\times\frac{(3-4i)}{(3-4i)} && \text{[on rationalising the denominator]}\\ &= \frac{3-4i}{3^2+4^2} && [\because (a+ib)(a-ib)=a^2+b^2]\\ &= \frac{3-4i}{25}\\ &= \frac{3}{25}-\frac{4}{25}i\end{aligned}$$

$$\therefore\ \frac{1}{(2+i)^2} = \frac{3}{25}-\frac{4}{25}i$$

Q1(iv)

$$\frac{1-i}{1+i}=\frac{(1-i)}{(1+i)}\times\frac{(1-i)}{(1-i)} \qquad \text{(Rationalising the denominator)}$$
$$=\frac{(1-i)^2}{1^2+1^2} \qquad \left[\because (a+ib)(a-ib)=a^2+b^2\right]$$
$$=\frac{1^2+i^2-2\times i\times 1}{2}$$
$$=\frac{-2i}{2}$$
$$=-i$$
$$=0-i$$

$$\therefore \frac{1-i}{1+i}=0-i$$

Q1(v)

$$\frac{(2+i)^3}{2+3i}=\frac{2^3+i^3+3\times 2\times i\,(2+i)}{2+3i} \qquad \left[\because (a+b)^3=a^3+b^3+3ab\,(a+b)\right]$$
$$=\frac{\left(8-i+6i\,(2+i)\right)}{2+3i}\times\frac{(2-3i)}{2-3i} \qquad \text{(On rationalising the denominator)}$$
$$=\frac{\left(8-i+12i+6i^2\right)(2-3i)}{2^2+3^2}$$
$$=\frac{(8-6+11i)(2-3i)}{4+9} \qquad \left(\because i^2=-1\right)$$
$$=\frac{(2+11i)(2-3i)}{13}$$
$$=\frac{4-6i+22i+33}{13}$$
$$=\frac{37+16i}{13}$$
$$=\frac{37}{13}+\frac{16}{13}i$$

$$\therefore \frac{(2+i)^3}{2+3i}=\frac{37}{13}+\frac{16}{13}i$$

Q1(vi)

$$\frac{\left(1+i\right)\left(1+\sqrt{3}i\right)}{1-i}=\frac{1\left(1+\sqrt{3}i\right)+i\left(1+\sqrt{3}i\right)}{1-i}$$
$$=\frac{\left(1+\sqrt{3}i+i-\sqrt{3}\right)}{1-i}\qquad\left(\because i^2=-1\right)$$
$$=\frac{\left(1-\sqrt{3}\right)+i\left(1+\sqrt{3}\right)}{1-i}\times\frac{\left(1+i\right)}{1+i}\qquad\text{(Rationalising the denominator)}$$
$$=\frac{\left(1-\sqrt{3}\right)\left(1+i\right)+i\left(1+\sqrt{3}\right)\left(1+i\right)}{1^2+1^2}$$
$$=\frac{1+i-\sqrt{3}\left(1+i\right)+i\left(1+i+\sqrt{3}\left(1+i\right)\right)}{2}$$
$$=\frac{1+i-\sqrt{3}-\sqrt{3}i+i\left(1+i+\sqrt{3}+\sqrt{3}i\right)}{2}$$
$$=\frac{1-\sqrt{3}+i-\sqrt{3}i+i-1+\sqrt{3}i-\sqrt{3}}{2}$$
$$=\frac{-2\sqrt{3}+2i}{2}$$
$$=-\sqrt{3}+i$$

$$\therefore\ \frac{\left(1+i\right)\left(1+\sqrt{3}i\right)}{1-i}=-\sqrt{3}+i$$

Q1(vii)

$$\frac{2+3i}{4+5i}=\frac{2+3i}{4+5i}\times\frac{\left(4-5i\right)}{\left(4-5i\right)}\qquad\text{(rationalising the denominator)}$$
$$=\frac{2\left(4-5i\right)+3i\left(4-5i\right)}{4^2+5^2}$$
$$=\frac{8-10i+12i+15}{16+25}\qquad\left(\because i^2=-1\right)$$
$$=\frac{23+2i}{41}$$
$$=\frac{23}{41}+\frac{2}{41}i$$

$$\therefore\ \frac{2+3i}{4+5i}=\frac{23}{41}+\frac{2}{41}i$$

Q1(viii)

$$\frac{(1-i)^3}{1-i^3} = \frac{1^3 - i^3 - 3\times1\times i\,(1-i)}{1-(-i)} \qquad \left[\begin{array}{l}\because (a-b)^3 = a^3 - b^3 - 3ab\,(a-b)\\ \text{and}\, i^3 = -i\end{array}\right]$$

$$= \frac{1-(-i)-3i\,(1-i)}{1+i}$$

$$= \frac{1+i-3i-3}{1+i}$$

$$= \frac{-2-2i}{1+i}$$

$$= \frac{-2\,(1+i)}{1+i}$$

$$= -2$$

$$= -2+0i$$

$$\therefore\ \frac{(1-i)^3}{1-i^3} = -2+0i$$

Q1(ix)

$$(1+2i)^{-3} = \frac{1}{(1+2i)^3} \qquad \left(\because z^{-3} = \frac{1}{z^3}\right)$$

$$= \frac{1}{1^3+(2i)^3+3\times1\times2i\,(1+2i)}$$

$$= \frac{1}{1^3+2^3\times i^3+6i\,(1+2i)}$$

$$= \frac{1}{1-8i+6i-12} \qquad \left(\because i^3 = -i\ \text{and}\ i^2 = -1\right)$$

$$= \frac{1}{-11-2i}$$

$$= \frac{1}{-11-2i}\times\frac{(-11+2i)}{(-11+2i)}$$

$$= \frac{-11+2i}{(-11)^2+2^2}$$

$$= \frac{-11+2i}{121+4}$$

$$= \frac{-11}{125}+\frac{2}{125}i$$

$$\therefore\ (1+2i)^{-3} = \frac{-11}{125}+\frac{2}{125}i$$

Q1(x)

$$\frac{3-4i}{(4-2i)(1+i)} = \frac{3-4i}{4(1+i)-2i(1+i)}$$
$$= \frac{3-4i}{4+4i-2i+2}$$
$$= \frac{3-4i}{6+2i}$$
$$= \frac{3-4i}{6+2i}\times\frac{6-2i}{6-2i}$$
$$= \frac{3(6-2i)-4i(6-2i)}{6^2+2^2}$$
$$= \frac{18-6i-24i-8}{36+4}$$
$$= \frac{10-30i}{40}$$
$$= \frac{10(1-3i)}{40}$$
$$= \frac{1-3i}{4}$$
$$= \frac{1}{4}-\frac{3}{4}i$$

$$\therefore \frac{3-4i}{(4-2i)(1+i)} = \frac{1}{4}-\frac{3}{4}i$$

Q1(xi)

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \frac{(1+i-2(1-4i))}{(1-4i)(1+i)}\times\frac{3-4i}{5+i}$$
$$= \frac{(1+i-2+8i)}{1(1+i)-4i(1+i)}\times\frac{3-4i}{5+i}$$
$$= \frac{(1+i-2+8i)}{1(1+i)-4i(1+i)}\times\frac{3-4i}{5+i}$$
$$= \frac{(-1+9i)}{(1+i-4i+4)}\times\frac{3-4i}{5+i}$$
$$= \frac{-1(3-4i)+9i(3-4i)}{(5-3i)(5+i)}$$
$$= \frac{-3+4i+27i+36}{5(5+i)-3i(5+i)}$$
$$= \frac{33+31i}{25+5i-15i+3}$$
$$= \frac{33+31i}{28-10i}$$
$$= \frac{(33+31i)}{28-10i}\times\frac{(28+10i)}{28+10i}$$
$$= \frac{33\times 28+33\times 10i+31i\times 28+31i\times 10i}{28^2+10^2}$$
$$= \frac{924+330i+868i-310}{784+100}$$
$$= \frac{614+1198i}{884}$$
$$= \frac{614}{884}+\frac{1198}{884}i$$
$$= \frac{307}{442}+\frac{599}{442}i$$

$$\therefore \left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \frac{307}{442}+\frac{599}{442}i$$

Q1(xii)

We have

$$\frac{5+\sqrt{2}i}{1-\sqrt{2}i}=\frac{5+\sqrt{2}i}{1-\sqrt{2}i}\times\frac{1+\sqrt{2}i}{1+\sqrt{2}i}$$

$$=\frac{5\left(1+\sqrt{2}i\right)+\sqrt{2}i\left(1+\sqrt{2}i\right)}{1+2}$$

$$=\frac{5+5\sqrt{2}i+\sqrt{2}i-2}{3}$$

$$=\frac{3+6\sqrt{2}i}{3}$$

$$=1+2\sqrt{2}i$$

Therefore, $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}=1+2\sqrt{2}i$

Q2(i)

We have $(x+iy)(2-3i)=4+i$

$\Rightarrow x(2-3i)+iy(2-3i)=4+i$

$\Rightarrow 2x-3xi+2yi+3y=4+i$

$\Rightarrow 2x+3y+i(-3x+2y)=4+i$

Equating the real and imaginary parts we get

$2x+3y=4\ldots\ldots\ldots\ldots(i)$

$-3x+2y=1\ldots\ldots\ldots\ldots(ii)$

Multiplying (i) by 3 and (ii) by 2 and adding

$6x-6x-9y+4y=12+2$

$\Rightarrow \quad 13y=14$

$\Rightarrow \quad y=\frac{14}{13}$

Substituting the value of y in (i), we get

$2x+3\times\frac{14}{13}=4$

$\Rightarrow 2x+\frac{42}{13}=4$

$\Rightarrow \quad 2x=4-\frac{42}{13}$

$\Rightarrow \quad 2x=\frac{52-42}{13}$

$\Rightarrow \quad 2x=\frac{10}{13}$

$\Rightarrow \quad x=\frac{5}{13}$

Hence

$x=\frac{5}{13}$ and $y=\frac{14}{13}$

Q2(ii)

$\left(3x - 2iy\right)\left(2 + i\right)^2 = 10\left(1 + i\right)$

$\Rightarrow \left(3x - 2iy\right)\left(2^2 + i^2 + 2 \times 2 \times i\right) = 10 + 10i$
$\Rightarrow \left(3x - 2iy\right)\left(4 - 1 + 4i\right) = 10 + 10i$
$\Rightarrow 3x\left(3 + 4i\right) - 2iy\left(3 + 4i\right) = 10 + 10i$
$\Rightarrow 9x + 12xi - 6yi + 8y = 10 + 10i$
$\Rightarrow 9x + 8y + i\left(12x - 6y\right) = 10 + 10i$

Equating the real and imaginary parts we get
$9x + 8y = 10 \ldots\ldots\ldots\ldots(i)$
$12x - 6y = 10 \ldots\ldots\ldots\ldots(ii)$

Multiplying (i) by 6 and (ii) by 8 and adding
$54x + 96x + 48y - 48y = 60 + 80$
$\Rightarrow 150x = 140$
$\Rightarrow \quad x = \frac{140}{150}$
$\Rightarrow \quad x = \frac{14}{15}$

Substituting value of x in (i) we get
$9 \times \frac{14}{15} + 8y = 10$
$\Rightarrow \frac{42}{5} + 8y = 10$
$\Rightarrow \quad 8y = 10 - \frac{42}{5}$
$\Rightarrow \quad 8y = \frac{50 - 42}{5}$
$\Rightarrow \quad 8y = \frac{8}{5}$
$\Rightarrow \quad y = \frac{1}{5}$

Q2(iii)

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$

$$\Rightarrow \frac{(3-i)((1+i)x-2i)+(3+i)((2-3i)y+i)}{(3+i)(3-i)}=i$$

$$\Rightarrow \frac{(3-i)(1+i)x-2i(3-i)+(3+i)(2-3i)y+i(3+i)}{3^2+1^2}=i$$

$$\Rightarrow \frac{(3+3i-i+1)x-6i-2+(6-9i+2i+3)y+3i-1}{9+1}=i$$

$$\Rightarrow \frac{(4+2i)x-6i-2+(9-7i)y+3i-1}{10}=1$$

$$\Rightarrow 4x+2ix-6i-2+9y-7iy+3i-1=10i$$

$$\Rightarrow 4x+9y-3+i(2x-7y-3)=10i$$

Equatingrealandimaginaryparts we get

$4x+9y-3=0 \ldots\ldots\ldots\ldots (i)$

$and\ 2x-7y-3=10$

$i.e\ 2x-7y=13 \ldots\ldots\ldots\ldots (ii)$

Multiplying (i) by 7, (ii) by 9 and adding we get

$28x+18x+63y-63y=117+21$

$\Rightarrow \quad 46x=117+21$

$\Rightarrow \quad 46x=138$

$\Rightarrow \quad x=\frac{138}{46}$

$= 3$

Substituting the value of $x=3$ in (i), we get

$4\times 3+9y=3$

$\Rightarrow \quad 9y=-9$

$\Rightarrow \quad y=\frac{-9}{9}$

$\Rightarrow \quad y=-1$

Hence

$x=3, y=-1$

Q2(iv)

$(1+i)(x+iy) = 2-5i$

$\Rightarrow 1(x+iy)+i(x+iy) = 2-5i$
$\Rightarrow \quad x+iy+ix-y = 2-5i$
$\Rightarrow \quad x-y+i(x+y) = 2-5i$

Equating real and imaginary parts we get
$x-y = 2 \ldots\ldots\ldots\ldots(i)$
$x+y = -5 \ldots\ldots\ldots\ldots(ii)$

Adding (i) and (ii) we get
$2x = 2-5$
$\Rightarrow 2x = -3$
$\Rightarrow \quad x = \frac{-3}{2}$

Substituting the value of x in (i), we get
$\frac{-3}{2} - y = 2$
$\Rightarrow \frac{-3}{2} - 2 = y$
$\Rightarrow \quad y = \frac{-3-4}{2}$
$\Rightarrow \quad y = \frac{-7}{2}$

Hence
$x = \frac{-3}{2}, y = \frac{-7}{2}$

Q3(i)

If $z = x+iy$ is a complex number, then the conjugate of z denoted by $\overline{z}$ is defined as $\overline{z} = x - iy$

$let\ z = 4-5i$
$\Rightarrow \overline{z} = 4+5i$

Q3(ii)

$$\text{let } z = \frac{1}{3+5i}$$
$$= \frac{1}{3+5i} \times \frac{(3-5i)}{3-5i} \quad \text{(Onrationalising the denominator)}$$
$$= \frac{3-5i}{3^2+5^2}$$
$$\Rightarrow z = \frac{3-5i}{9+25}$$

$$So\ \bar{z} = \frac{3+5i}{34}$$
$$= \frac{3}{34} + \frac{5}{34}i$$

Q3(iii)

$$\text{let } z = \frac{1}{1+i}$$
$$= \frac{1}{1+i} \times \frac{(1-i)}{(1-i)}$$
$$= \frac{1-i}{1^2+1^2}$$
$$= \frac{1-i}{2}$$

$$\therefore \bar{z} = \frac{1+i}{2}$$
$$= \frac{1}{2} + \frac{1}{2}i$$

Q3(iv)

$$\text{let } z = \frac{(3-i)^2}{2+i}$$
$$= \frac{3^2 + i^2 - 2 \times 3 \times i}{2+i}$$
$$= \frac{9 - 1 - 6i}{2+i}$$
$$= \frac{8-6i}{2+i}$$
$$= \frac{8-6i}{2+i} \times \frac{2-i}{2-i}$$
$$= \frac{8(2-i) - 6i(2-i)}{2^2 + 1^2}$$
$$= \frac{16 - 8i - 12i - 6}{4+1}$$
$$= \frac{10-20i}{5}$$
$$\Rightarrow z = 2 - 4i$$

Hence
$$\bar{z} = 2 + 4i$$

Q3(v)

$$\text{let } z = \frac{(1+i)(2+i)}{3+i}$$
$$= \frac{2 + i + i(2+i)}{3+i}$$
$$= \frac{2 + i + 2i - 1}{3+i}$$
$$= \frac{1+3i}{3+i}$$
$$= \frac{(1+3i)}{(3+i)} \times \frac{(3-i)}{(3-i)}$$
$$= \frac{3 - i + 3i(3-i)}{3^2 + 1^2}$$
$$= \frac{3 - i + 9i + 3}{9+1}$$
$$= \frac{6+8i}{10}$$
$$= \frac{2(3+4i)}{10}$$
$$\Rightarrow z = \frac{3+4i}{5}$$

Hence
$$\bar{z} = \frac{3-4i}{5}$$
$$= \frac{3}{5} - \frac{4}{5}i$$

Q3(vi)

$$\text{let } z = \frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$$
$$= \frac{3(2+3i)-2i(2+3i)}{2-i+2i(2-i)}$$
$$= \frac{6+9i-4i+6}{2-i+4i+2}$$
$$= \frac{12+5i}{4+3i}$$
$$= \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$
$$= \frac{12(4-3i)+5i(4-3i)}{4(4-3i)+3i(4-3i)}$$
$$= \frac{48-36i+20i+15}{16-12i+12i+9}$$
$$= \frac{63-16i}{16+9}$$
$$\Rightarrow z = \frac{63-16i}{25}$$

$$\therefore \bar{z} = \frac{63+16i}{25}$$
$$= \frac{63}{25} + \frac{16}{25}i$$

Q4(i)

If $z = x + iy$ is a complex number, then the multiplicative inverse of z, denoted by z^{-1} or $\frac{1}{z}$ is defined as $z^{-1} = \frac{1}{z}$

$$= \frac{1}{x+iy}$$
$$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$
$$= \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

Given

$z = 1 - i$

$$\therefore z^{-1} = \frac{1}{1^2+1^2} - \frac{(-1)}{1^2+1^2} \times i$$
$$= \frac{1}{2} + \frac{1}{2}i$$

Q4(ii)

$$\text{let } z = \left(1 + i\sqrt{3}\right)^2$$
$$= 1^2 + \left(i\sqrt{3}\right)^2 + 2 \times 1 \times i\sqrt{3}$$
$$= 1 - 3 + 2\sqrt{3}i$$
$$= -2 + 2\sqrt{3}i$$

$$\therefore\ z^{-1} = \frac{-2}{(-2)^2 + \left(2\sqrt{3}\right)^2} - \frac{2\sqrt{3}i}{(-2)^2 + \left(2\sqrt{3}\right)^2}$$
$$= \frac{-2}{4+12} - \frac{2\sqrt{3}i}{4+12}$$
$$= \frac{-2}{16} - \frac{2\sqrt{3}i}{16}$$
$$= \frac{-1}{8} - \frac{\sqrt{3}i}{8}$$

Q4(iii)

$$\text{let } z = 4 - 3i$$
$$\text{Then } z^{-1} = \frac{4}{4^2 + (-3)^2} - \frac{(-3)}{4^2 + (-3)^2}$$
$$= \frac{4}{16+9} + \frac{3}{16+9}i$$
$$= \frac{4}{25} + \frac{3}{25}i$$

Q4(iv)

$$\text{let } z = \sqrt{5} + 3i$$
$$\text{Then } z^{-1} = \frac{\sqrt{5}}{\left(\sqrt{5}\right)^2 + (3)^2} - \frac{3}{\left(\sqrt{5}\right)^2 + (3)^2}i$$
$$= \frac{\sqrt{5}}{5+9} - \frac{3}{5+9}i$$
$$= \frac{\sqrt{5}}{14} - \frac{3}{14}i$$

Q5

If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$

We have,

$$z_1 = 2 - i, z_2 = 1 + i$$

$$z_1 + z_2 = 2 - i + 1 + i$$
$$= 3$$

And $\quad z_1 - z_2 = 2 - i - 1 - i$
$$= 1 - 2i$$

$$\frac{z_1 + z_2 + 1}{z_1 - z_2 + i} = \frac{3 + 1}{1 - 2i + i}$$
$$= \frac{4}{1 - i}$$
$$= \frac{4}{1 - i} \times \frac{1 + i}{1 + i}$$
$$= \frac{4(1 + i)}{1^2 + 1^2}$$
$$= \frac{4(1 + i)}{2}$$
$$= 2(1 + i)$$

$$\therefore \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right| = |2(1 + i)|$$
$$= |2||1 + i| \qquad (\because |z_1 z_2| = |z_1| \times |z_2|)$$
$$= 2 \times \sqrt{1^2 + 1^2}$$
$$= 2 \times \sqrt{2}$$
$$= 2\sqrt{2}$$

Q6

(i)

$$\frac{z_1 z_2}{z_1} = \frac{z_1 z_2}{z_1} \times \frac{z_1}{z_1} \qquad \text{(rationalising the denominator)}$$

$$= \frac{(z_1)^2 z_2}{z_1 z_1}$$

$$= \frac{(2-i)^2(-2+i)}{|z_1|^2} \qquad \left(\because z\bar{z} = |z|^2\right)$$

$$= \frac{\left(2^2 + i^2 - 2 \times 2 \times i\right)(-2+i)}{|2-i|^2}$$

$$= \frac{(4 - 1 - 4i)(-2+i)}{2^2 + (-1)^2}$$

$$= \frac{(3-4i)(-2+i)}{4+i}$$

$$= 3(-2+i) - 4i(-2+i)$$

$$= \frac{-6 + 3i + 8i + 4}{5}$$

$$= \frac{-2 + 11i}{5}$$

$$\therefore \mathrm{Re}\left(\frac{z_1 z_2}{z_1}\right) = \mathrm{Re}\left(\frac{-2}{5} + \frac{11}{5} i\right)$$

$$= \frac{-2}{5}$$

(ii)

$$\frac{1}{z_1 \overline{z_1}} = \frac{1}{|z_1|^2}$$

$$= \frac{1}{|2-i|^2}$$

$$= \frac{1}{2^2 + (-1)^2}$$

$$= \frac{1}{4+1}$$

$$= \frac{1}{5}, \text{ which is purely real}$$

$$\therefore \mathrm{Im}\left(\frac{1}{z_1 z_1}\right) = 0$$

Q7

$$\text{let } z = \frac{1+i}{1-i} - \frac{1-i}{1+i}$$

$$= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$

$$= \frac{1^2 + i^2 + 2 \times 1 \times i - (1^2 + i^2 - 2 \times 1 \times i)}{1^2 + 1^2}$$

$$= \frac{1 - 1 + 2i - (1 - 1 - 2i)}{2}$$

$$= \frac{2i + 2i}{2}$$

$$= \frac{4i}{2}$$

$$\Rightarrow z = 2i$$

$$\therefore |z| = |2i|$$

$$= 2|i| \qquad (\because |z_1 z_2| = |z_1| \times |z_2|)$$

$$= 2 \times i \qquad (\because |i| = 1)$$

$$= 2$$

Q8

$$x + iy = \frac{a+ib}{a-ib}$$

$$\Rightarrow \left(\overline{x+iy}\right) = \overline{\left(\frac{a+ib}{a-ib}\right)} \qquad \text{(on taking conjugate both sides)}$$

$$\Rightarrow \quad x - iy = \frac{\left(\overline{a+ib}\right)}{\left(\overline{a-ib}\right)} \qquad \left(\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}\right)$$

$$= \frac{a-ib}{a+ib}$$

$$\because (x+iy)(x-iy) = \frac{a+ib}{a-ib} \times \frac{a-ib}{a+ib}$$

$$\Rightarrow x^2 + y^2 = 1$$

proved

Q9

For $n = 1$, we have,

$$\left(\frac{1+i}{1-i}\right)^1 = \frac{1+i}{1-i}$$
$$= \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$
$$= \frac{(1+i)^2}{1^2+1^2}$$
$$= \frac{1^2+i^2+2\times1\times i}{2}$$
$$= \frac{2i}{2} \qquad (\because i^2 = -1)$$
$$= i, \text{ which is not real}$$

For $n = 2$, we have

$$\left(\frac{1+i}{1-i}\right)^2 = i^2 \qquad \left(\because \frac{1+i}{1-i} = 1 \text{ form above}\right)$$
$$= -1, \text{ which is real}$$

Hence the least positive integral value of n is 2.

Q10

$$\text{let } z = \frac{1+i\cos\theta}{1-2i\cos\theta}$$
$$= \frac{1+i\cos\theta}{1-2i\cos\theta} \times \frac{1+2i\cos\theta}{1+2i\cos\theta}$$
$$= \frac{1+2i\cos\theta+i\cos\theta(1+2i\cos\theta)}{1^2+(2\cos\theta)^2}$$
$$= \frac{1+2i\cos\theta+i\cos\theta-2\cos^2\theta}{1+4\cos^2\theta}$$
$$= \frac{1-2\cos^2\theta+3i\cos\theta}{1+4\cos^2\theta}$$
$$= \frac{1-2\cos^2\theta}{1+4\cos^2\theta} + \frac{3\cos\theta}{1+4\cos^2\theta}i$$

we know that z is purely real if and only if $\operatorname{Im} z = 0$

$$\therefore \frac{3\cos\theta}{1+4\cos^2\theta} = 0 \qquad (\because \text{z is given to be purely real})$$
$$\Rightarrow 3\cos\theta = 0$$
$$\Rightarrow \cos\theta = 0$$
$$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$$

$\therefore$ The general solution is given by

$$\theta = 2n\pi \pm \frac{\pi}{2}, n \in Z$$

Q11

$$\text{let } z = \frac{(1+i)^n}{(1-i)^{n-2}}$$

$$= \frac{(1+i)^n}{(1-i)^n}(1-i)^2$$

$$= \left(\frac{1+i}{1-i}\right)^n \times (1-i)^2$$

$$= i^n\left(1+i^2-2\times 1\times i\right) \qquad \left(\because \frac{1+i}{1-i} = i, \text{using problem 10}\right)$$

$$= i^n(1-1-2i)$$

$$= -2i \times i^n$$

$$= -2i^{n+1}$$

$\therefore$ *For* $n = 1$

$z = -2i^{1+1}$

$= -2i^2$

$= 2$, which is a real number

$\therefore$ The smallest positive integer value of n is 1.

Q12

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$$

$$\Rightarrow \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^3 - \left(\frac{(1-i)(1-i)}{(1+i)(1-i)}\right)^3 = x+iy \text{[Rationalizing the denomiantor]}$$

$$\Rightarrow \left(\frac{1+2i-1}{1+1}\right)^3 - \left(\frac{1-2i-1}{1+1}\right)^3 = x+iy$$

$$\Rightarrow \left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3 = x+iy$$

$$\Rightarrow i^3 - (-i)^3 = x+iy$$

$$\Rightarrow -i-i = x+iy$$

$$\Rightarrow -2i = x+iy$$

Comparing the real and imaginary parts,

$(x, y) = (0, 2)$

Q13

$$\frac{(1+i)^2}{2-i} = x+iy$$

$$\Rightarrow \frac{(1+2i-1)}{2-i} = x+iy$$

$$\Rightarrow \frac{2i}{2-i} = x+iy$$

$$\Rightarrow \frac{2i(2+i)}{(2-i)(2+i)} = x+iy\text{[Rationalizing the denominator]}$$

$$\Rightarrow \frac{2(2i-1)}{4+1} = x+iy$$

$$\Rightarrow \frac{4i-2}{5} = x+iy$$

$$\Rightarrow -\frac{2}{5}+i\frac{4}{5} = x+iy$$

Comparing the real and imaginary parts, we get

$$x = -\frac{2}{5}, y = \frac{4}{5}$$

$$x+y = \frac{2}{5}$$

Q14

$$\left(\frac{1-i}{1+i}\right)^{100} = a+ib$$

$$\Rightarrow\left(\frac{(1-i)(1-i)}{(1+i)(1-i)}\right)^{100} = a+ib\text{[Rationalizing the denominator]}$$

$$\Rightarrow\left(\frac{(1-2i-1)}{(1+1)}\right)^{100} = a+ib$$

$$\Rightarrow\left(\frac{-2i}{2}\right)^{100} = a+ib$$

$$\Rightarrow(-i)^{100} = a+ib$$

$$\Rightarrow 1 = a+ib$$

Comparing, we get (a,b)=(1,0)

Q15

a $=\cos\theta + i\sin\theta$

$\frac{1+a}{1-a}$

$= \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$

$= \frac{(1+\cos\theta+i\sin\theta)(1-\cos\theta+i\sin\theta)}{(1-\cos\theta-i\sin\theta)(1-\cos\theta+i\sin\theta)}$[Rationalizing the denominator]

$= \frac{(1+\cos\theta+i\sin\theta)(1-\cos\theta+i\sin\theta)}{(1-\cos\theta)^2-(i\sin\theta)^2}$

$= \frac{(1+i\sin\theta)^2-\cos^2\theta}{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$

$= \frac{1+2i\sin\theta-\sin^2\theta-\cos^2\theta}{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$

$= \frac{1+2i\sin\theta-1}{1-2\cos\theta+\cos^2\theta+\sin^2\theta}[\because \cos^2\theta+\sin^2\theta=1]$

$= \frac{2i\sin\theta}{1-2\cos\theta+1}$

$= \frac{2i\sin\theta}{2-2\cos\theta}$

$= \frac{i\sin\theta}{1-\cos\theta}$

$= \frac{i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$= \frac{i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = i\cot\frac{\theta}{2}$

Q16(i)

We have,

$x = \dfrac{3-5i}{2}$

$\Rightarrow \quad 2x = 3 - 5i$

$\Rightarrow \quad 2x - 3 = -5i$

$\Rightarrow (2x-3)^2 = (-5i)^2$

$\Rightarrow \quad 4x^2 + 9 - 12x = -25$

$\Rightarrow \quad 4x^2 - 12x + 34 = 0$

$\Rightarrow 2(2x^2 - 6x + 17) = 0$

$\Rightarrow \quad 2x^2 - 6x + 17 = 0 \qquad \ldots\ldots\ldots\ldots(i)$

$\therefore 2x^3 + 2x^2 - 7x + 72$

$= x(2x^2 - 6x + 17) + 6x^2 - 17x + 2x^2 - 7x + 72$ (adding and subtracting $6x^2$ and $17x$)

$= x \times 0 + 8x^2 - 24x + 72$ (using (i))

$= 4(2x^2 - 6x + 17) + 4$

$= 4 \times 0 + 4$ (using (i))

$= 4$

Q16(ii)

We have,

$x = 3 + 2i$

$\Rightarrow x - 3 = 2i$

$\Rightarrow (x-3)^2 = (2i)^2$

$\Rightarrow x^2 + 3^2 - 2 \times 3 \times x = -4$

$\Rightarrow x^2 + 9 - 6x + 4 = 0$

$\Rightarrow x^2 - 6x + 13 = 0 \qquad \ldots\ldots\ldots\ldots(i)$

Now,

$x^4 - 4x^3 + 4x^2 + 8x + 44$

$= x^2(x^2 - 6x + 13) + 6x^2 - 13x^2 - 4x^3 + 4x^2 + 8x + 44$ (adding and subtracting $6x^3$ and $13x^2$)

$= x^2 \times 0 + 2x^3 - 9x^2 + 8x + 44$ (using (i))

$= 2x(x^2 - 6x + 13) + 12x^2 - 26x - 9x^2 + 8x + 44$ (adding and subtracting $12x^2$ and $26x$)

$= 2x \times 0 + 3x^2 - 18x + 44$ (using (i))

$= 3(x^2 - 6x + 13) + 5$

$= 3 \times 0 + 5$ (using (i))

$= 5$

Q16(iii)

We have,

$x = -1 + i\sqrt{2}$

$\Rightarrow x + 1 = i\sqrt{2}$

$\Rightarrow (x+1)^2 = \left(i\sqrt{2}\right)^2$ (squaring both sides)

$\Rightarrow x^2 + 1 + 2x = -2$

$\Rightarrow x^2 + 2x + 3 = 0 \ldots\ldots\ldots\ldots (i)$

Now,

$x^4 + 4x^3 + 6x^2 + 4x + 9$

$= x^2\left(x^2 + 2x + 3\right) + 2x^3 + 3x^2 + 4x + 9$

$= x^2 \times 0 + 2x\left(x^2 + 2x + 3\right) - x^2 - 2x + 9$ (using (i))

$= 2x \times 0 - \left(x^2 + 2x + 3\right) + 3 + 9$ (using (i) and adding and subtracting 3)

$= -0 + 3 + 9$ (using (i))

$= 12$

Q16(iv)

We have,

$x = \frac{1+i}{\sqrt{2}}$

$\Rightarrow \sqrt{2}x = 1 + i$

$\Rightarrow \left(\sqrt{2}x\right)^2 = (1+i)^2$ (squaring both sides)

$\Rightarrow 2x^2 = 1^2 + (i)^2 + 2 \times 1 \times i$

$= 1 - 1 + 2i$

$\Rightarrow 2x^2 = 2i$

$\Rightarrow x^2 = i$

$\Rightarrow \left(x^2\right)^2 = (i)^2$ (squaring both sides)

$\Rightarrow x^4 = -1$

$\Rightarrow x^4 + 1 = 0 \quad \ldots\ldots\ldots\ldots (i)$

Now

$x^6 + x^4 + x^2 + 1$

$= x^6 + x^2 + x^4 + 1$

$= x^2\left(x^4 + 1\right) + 1\left(x^4 + 1\right)$

$= x^2 \times 0 + 1 \times 0$ (using (i))

$= 0$

Q16(v)

$$x = \left(-2-\sqrt{3}i\right)$$

$$x^2 = \left(-2-\sqrt{3}i\right)^2 = 4+4\sqrt{3}i+3i^2 = 1+4\sqrt{3}i$$

$$x^3 = \left(1+4\sqrt{3}i\right)\left(-2-\sqrt{3}i\right) = -2-8\sqrt{3}i-\sqrt{3}i-12i^2 = 10-9\sqrt{3}i$$

$$x^4 = \left(1+4\sqrt{3}i\right)^2 = 1+8\sqrt{3}i+48i^2 = -47+8\sqrt{3}i$$

$$\begin{aligned}2x^4+5x^3+7x^2-x+41 &= 2\left(-47+8\sqrt{3}i\right)+5\left(10-9\sqrt{3}i\right)+7\left(1+4\sqrt{3}i\right)-\left(-2-\sqrt{3}i\right)+41\\ &= -94+16\sqrt{3}i+50-45\sqrt{3}i+7+28\sqrt{3}i+2+\sqrt{3}i+41\\ &= \left(-94+50+7+2+41\right)+\left(16\sqrt{3}i-45\sqrt{3}i+28\sqrt{3}i+\sqrt{3}i\right)\\ &= 6+0\\ &= 6\end{aligned}$$

Q17

$$(1-i)^n\left(1-\frac{1}{i}\right)^n$$

$$= (1-i)^n\left(\frac{i-1}{i}\right)^n$$

$$= \left\{\frac{(1-i)(i-1)}{i}\right\}^n$$

$$= \left\{\frac{(1-i)(1-i)}{-i}\right\}^n$$

$$= \left\{\frac{(1-i)^2}{-i}\right\}^n$$

$$= \left\{\frac{1-2i-1}{-i}\right\}^n$$

$$= \left\{\frac{-2i}{-i}\right\}^n = 2^n$$

Q18

$$(1+i)z = (1-i)\bar{z}$$

$$\Rightarrow z = \frac{(1-i)}{(1+i)}\bar{z}$$

$$\Rightarrow z = \frac{(1-i)(1-i)}{(1+i)(1-i)}\bar{z} \quad \text{[Rationalizing the denominator]}$$

$$\Rightarrow z = \frac{(1-2i-1)}{(1+1)}\bar{z}$$

$$\Rightarrow z = \frac{-2i}{2}\bar{z}$$

$$\Rightarrow z = -i\bar{z}$$

Q19

$\text{Re}(z^2) = 0, |z|=2$

Let $z=x+iy$

$z^2 = 0$

$\Rightarrow (x+iy)^2 = 0$

$\Rightarrow x^2 + 2ixy - y^2 = 0$

$\Rightarrow x^2 - y^2 = 0 \ldots\ldots(i)$, which is the real part of $(x+iy)^2$.

$|z|=2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2 + y^2 = 4 \ldots\ldots(ii)$

Adding (i) and (ii), we get

$2x^2 = 4$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}, y = \pm\sqrt{2}$

$x+iy = \sqrt{2}+i\sqrt{2}$

$\quad = -\sqrt{2}-i\sqrt{2}$

$\quad = \sqrt{2}-i\sqrt{2}$

$\quad = -\sqrt{2}+i\sqrt{2}$

Q20

let $z = x + iy,$

$$\frac{z-1}{z+1}$$

$$=\frac{x+iy-1}{x+iy+1}$$

$$=\frac{x-1+iy}{x+1+iy}$$

$$=\frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}\text{[Rationalizing the denominator]}$$

$$=\frac{(x-1+iy)(x+1-iy)}{(x+1)^2-(iy)^2}$$

$$=\frac{x^2+x-ixy-x-1+iy+ixy+iy+y^2}{x^2+2x+1+y^2}$$

$$=\frac{x^2-1+2iy+y^2}{x^2+2x+1+y^2}$$

$$=\frac{x^2+y^2-1}{x^2+2x+1+y^2}+i\frac{2y}{x^2+2x+1+y^2}$$

$\because$ It is a purely imaginary no. therefore real part $=0$

$$\frac{x^2+y^2-1}{x^2+2x+1+y^2}=0$$

$$\Rightarrow x^2+y^2-1=0$$

$$\Rightarrow x^2+y^2=1$$

$$\Rightarrow \sqrt{x^2+y^2}=1$$

$$\Rightarrow |z| = 1$$

Q21

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$|z_1| = 1 \Rightarrow x_1^2 + y_1^2 = 1$

$$z_2 = \frac{z_1 - 1}{z_1 + 1}$$

$$x_2 + iy_2 = \frac{x_1 + iy_1 - 1}{x_1 + iy_1 + 1}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1 - 1 + iy_1}{x_1 + 1 + iy_1}$$

$$\Rightarrow x_2 + iy_2 = \frac{(x_1 - 1 + iy_1)(x_1 + 1 - iy_1)}{(x_1 + 1 + iy_1)(x_1 + 1 - iy_1)} \text{[Rationalizing the denominator]}$$

$$\Rightarrow x_2 + iy_2 = \frac{(x_1 - 1)(x_1 + 1) - iy_1(x_1 - 1) + iy_1(x_1 + 1) + y_1^2}{(x_1 + 1)^2 - (iy_1)^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1^2 - 1 + y_1^2 - iy_1x_1 + iy_1 + iy_1x_1 + iy_1}{(x_1 + 1)^2 - (iy_1)^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1^2 + y_1^2 - 1 + 2iy_1}{(x_1 + 1)^2 - (iy_1)^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{1 - 1 + 2iy_1}{(x_1 + 1)^2 - (iy_1)^2} [\because x_1^2 + y_1^2 = 1]$$

$$\Rightarrow x_2 + iy_2 = \frac{2iy_1}{(x_1 + 1)^2 - (iy_1)^2} [\because x_1^2 + y_1^2 = 1]$$

Since there is no real part in the RHS, therefore $x_2 = 0$.

The real part of the $z_2 = 0$.

Q22

Let $z = x + iy$

$|z+1| = z + 2(1+i)$

$\Rightarrow |x+iy+1| = x+iy+2+2i$

$\Rightarrow \sqrt{(x+1)^2 + y^2} = (x+2)+i(y+2)$

Comparing, real and imaginary parts, we get

$x+2 = \sqrt{x^2 + 2x+1+y^2}$ and $y+2=0$

$y+2=0$

$\Rightarrow y=-2$

& $(x+2)^2 = x^2+2x+1+y^2$

$\Rightarrow x^2+4x+4 = x^2+2x+1+y^2$

$\Rightarrow 2x+3 = y^2$

$\Rightarrow 2x+3 = (-2)^2$

$\Rightarrow 2x+3=4$

$\Rightarrow 2x=1$

$\Rightarrow x=\frac{1}{2}$

$\therefore z = x+iy = \frac{1}{2} - i2$

Q23

Let $z = x + iy$

$|z| = z + 1 + 2i$

$\Rightarrow |x+iy| = x+iy+1+2i$

$\Rightarrow \sqrt{x^2+y^2} = (x+1)+i(y+2)$

$\Rightarrow x^2+y^2 = (x+1)^2+2i(x+1)(y+2)-(y+2)^2$ [Squaring both sides]

$\Rightarrow x^2+y^2 = x^2+2x+1+2i(xy+2x+y+2)-(y^2+4y+4)$

$\Rightarrow 2y^2-2x+4y+4 = 2i(xy+2x+y+2)$

$\Rightarrow y^2-x+2y+2 = i(xy+2x+y+2)$

$\Rightarrow (y^2-x+2y+2)-i(xy+2x+y+2) = 0$

Comparing we get,

$(xy+2x+y+2) = 0$

$\Rightarrow (x+1)(y+2) = 0$

$\Rightarrow x = -1 \,\&\, y = -2$

Also, $(y^2-x+2y+2) = 0$

Taking $x = -1$, $(y^2-(-1)+2y+2) = 0$

$\Rightarrow (y^2+2y+3) = 0$

Doesnot have a solution since roots will be imaginary

Taking $y = -2$, $(4-x-4+2) = 0$

$\Rightarrow x = 2$

$\therefore z = x+iy = 2-2i$

Q24

$$(1+i)^{2n} = (1-i)^{2n}$$
$$\Rightarrow \left(\frac{1+i}{1-i}\right)^{2n} = 1$$
$$\Rightarrow \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^{2n} = 1 \quad \text{[Rationalizing the denominator]}$$
$$\Rightarrow \left(\frac{1+2i-1}{1+1}\right)^{2n} = 1$$
$$\Rightarrow \left(\frac{2i}{2}\right)^{2n} = 1$$
$$\Rightarrow i^{2n} = 1$$
$$\therefore n = 2$$

Q25

$$|z_1 + z_2 + z_3| = \left|\frac{z_1\overline{z_1}}{\overline{z_1}} + \frac{z_2\overline{z_2}}{\overline{z_2}} + \frac{z_3\overline{z_3}}{\overline{z_3}}\right|$$
$$= \left|\frac{|z_1|^2}{\overline{z_1}} + \frac{|z_2|^2}{\overline{z_2}} + \frac{|z_3|^2}{\overline{z_3}}\right|$$
$$= \frac{\left|\frac{1}{\overline{z_1}} + \frac{1}{\overline{z_2}} + \frac{1}{\overline{z_3}}\right|}{\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|} \cdots\cdots\cdots\cdots [\because |z_1| = |z_2| = |z_3| = 1]$$
$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$$
$$= 1$$

Q26

Let $z = x + iy$

$z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$|z|^2 = z\overline{z} = (x+iy)(x-iy) = x^2 + y^2$

$z^2 + |z|^2 = 0$

$x^2 - y^2 + 2xyi + x^2 + y^2 = 0$

$2x^2 + 2xyi = 0$

$\Rightarrow 2x^2 = 0$ and $2xy=0$

$\Rightarrow x = 0$ and $y \in R$

$\therefore z = 0 + iy$ where $y \in R$

Ex 13.3

Q1(i)

Let $z = -5 + 12i$

$$\Rightarrow |z| = \sqrt{(-5)^2 + 12^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13$$

$$\therefore \sqrt{-5 + 12i} = \pm\left\{\sqrt{\frac{13 + (-5)}{2}} + i\sqrt{\frac{13 - (-5)}{2}}\right\} \qquad (\because y > 0)$$
$$= \pm\left\{\sqrt{\frac{8}{2}} + i\sqrt{\frac{18}{2}}\right\}$$
$$= \pm\{2 + 3i\}$$

Q1(ii)

let $z = -7\ \ -24i$

$$\text{then } |z| = \sqrt{(-7)^2 + (-24)^2}$$
$$= \sqrt{49 + 576}$$
$$= \sqrt{625}$$
$$= 25$$

$$\therefore \sqrt{-7 - 24i} = \pm\left\{\sqrt{\frac{25 - 7}{2}} - i\sqrt{\frac{25 + 7}{2}}\right\} \quad (\because y < 0)$$
$$= \pm\left\{\sqrt{\frac{18}{2}} - i\sqrt{\frac{32}{2}}\right\}$$
$$= \pm\left\{\sqrt{9} - i\sqrt{16}\right\}$$
$$= \pm\{3 - 4i\}$$

Q1(iii)

$$\text{let } z = 1 - i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{1+1} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{1-i} &= \pm\left(\sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \qquad (\because y < 0) \\ &= \pm\left(\sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}}\right)\end{aligned}$$

Q1(iv)

$$\text{let } z = -8 - 6i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{(-8)^2 + (-6)^2} \\ &= \sqrt{64+36} \\ &= \sqrt{100} \\ &= 10\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{-8-6i} &= \pm\left\{\sqrt{\frac{10-8}{2}} - i\sqrt{\frac{10+8}{2}}\right\} \quad (\because y < 0) \\ &= \pm\left\{\sqrt{\frac{2}{2}} - i\sqrt{\frac{18}{2}}\right\} \\ &= \pm\left\{\sqrt{1} - i\sqrt{9}\right\} \\ &= \pm\{1 - 3i\}\end{aligned}$$

Q1(v)

$$\text{let } z = 8 - 15i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{(8)^2 + (-15)^2} \\ &= \sqrt{64+225} \\ &= \sqrt{289} \\ &= 17\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{8-5i} &= \pm\left\{\sqrt{\frac{17+8}{2}} - i\sqrt{\frac{17-8}{2}}\right\} \qquad (\because y < 0) \\ &= \pm\left\{\sqrt{\frac{25}{2}} - i\sqrt{\frac{9}{2}}\right\} \\ &= \pm\left\{\frac{5}{\sqrt{2}} - i\frac{3}{\sqrt{2}}\right\} \\ &= \pm\frac{1}{\sqrt{2}}\{5 - 3i\}\end{aligned}$$

Q1(vi)

Let $z = -11 - 60\sqrt{-1}$

$\Rightarrow z = -11 - 60i \quad \left(\because \sqrt{-1} = i\right)$

Then $|z| = \sqrt{(-11)^2 + (-60)^2}$

$= \sqrt{121 + 3600}$

$= \sqrt{3721}$

$= 61$

$\therefore \sqrt{-11 - 60i} = \pm\left\{\sqrt{\frac{61-11}{2}} - i\sqrt{\frac{61+11}{2}}\right\} \quad (\because y < 0)$

$= \pm\left\{\sqrt{\frac{50}{2}} - i\sqrt{\frac{72}{2}}\right\}$

$= \pm\left\{\sqrt{25} - i\sqrt{36}\right\}$

$= \pm\{5 - 6i\}$

Q1(vii)

let $z = 1 + 4\sqrt{-3}$

$= 1 + 4\sqrt{3} \times \sqrt{-1} \quad \left(\because \sqrt{-3} = \sqrt{3} \times \sqrt{-1}\right)$

$\Rightarrow z = 1 + 4\sqrt{3i}$

$\therefore |z| = \sqrt{(1)^2 + \left(4\sqrt{3}\right)^2}$

$= \sqrt{1 + 48}$

$= \sqrt{49}$

$= 7$

Hence $\sqrt{1 + 4\sqrt{-3}} = \pm\left\{\sqrt{\frac{7+1}{2}} + i\sqrt{\frac{7-1}{2}}\right\} \quad (\because y > 0)$

$= \pm\left\{\sqrt{\frac{8}{2}} + i\sqrt{\frac{6}{2}}\right\}$

$= \pm\left\{\sqrt{4} + i\sqrt{3}\right\}$

$= \pm\left\{2 + \sqrt{3}i\right\}$

Q1(viii)

$\text{let } z = 4i$

$$\begin{aligned}
\text{then } |z| &= |4i| \\
&= 4\,|i| && \left(\because |z_1 z_2| = |z_1| \times |z_2|\right) \\
&= 4 && \left(\because |i| = 1\right)
\end{aligned}$$

$$\begin{aligned}
\therefore \sqrt{4i} &= \pm\left\{\sqrt{\frac{4+0}{2}} + i\sqrt{\frac{4-0}{2}}\right\} && \left(\because y > 0\right) \\
&= \pm\left\{\sqrt{2} + i\sqrt{2}\right\} \\
&= \pm\sqrt{2}\,(1+i)
\end{aligned}$$

Q1(ix)

$\text{let } z = -i$

$$\begin{aligned}
\text{then } |z| &= |-i| \\
&= |-1| \times |i| && \left(\because |z_1 z_2| = |z_1| \times |z_2|\right) \\
&= 1 \times i && \left(\because |i| = 1\right) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\therefore \sqrt{-i} &= \pm\left\{\sqrt{\frac{1+0}{2}} - i\sqrt{\frac{1-0}{2}}\right\} && \left(\because y < 0\right) \\
&= \pm\left\{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right\} \\
&= \pm\frac{1}{\sqrt{2}}\,(1-i)
\end{aligned}$$

Ex 13.4

Q1(i)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$
where,
$|z| = \sqrt{x^2 + y^2}$ and
$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = 1 + i$

$$|z| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$

$\because x, y > 0$, so θ lies in first quadrant

Now,

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$
$$= \tan^{-1}\left(\frac{1}{1}\right) \qquad [\because a = 1 \text{ and } b = 1]$$
$$= \tan^{-1}(1)$$
$$= \tan^{-1}\left(\frac{\tan\pi}{4}\right) \qquad \left(\because \frac{\tan\pi}{4} = 1\right)$$
$$= \frac{\pi}{4} \qquad \left(\because \tan^{-1}(\tan x) = x\right)$$

$$\Rightarrow \arg(z) = \frac{\pi}{4}$$

Polar form of $1 + i$ is given by $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Q1(ii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$

where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = \sqrt{3} + i$

$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$$
$$= \sqrt{3+1}$$
$$= \sqrt{4}$$
$$= 2$$

$\because x = \sqrt{3} > 0$ & $y = 1 > 0$,

$\therefore$ θ lies in first quadrant

Hence

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \tan^{-1}\left(\frac{\tan\pi}{6}\right)$$
$$= \tan^{-1}\left(\because \tan^{-1}(\tan x) = x\right)$$

polar form is given by $z = |z|(\cos\theta + i\sin\theta)$

i.e $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Q1(iii)

Modulus, $|1-i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Argument, $\arg(1-i) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

Polar form, $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

Q1(iv)

$$\frac{1-i}{1+i}=\frac{(1-i)(1-i)}{(1+i)(1-i)}=\frac{(1-i)^2}{1^2-i^2}=\frac{1-2i-1}{1+1}=\frac{-2i}{2}=-i$$

$$\text{Modulus, } \left|\frac{1-i}{1+i}\right|=|-i|=1$$

$$\text{Argument, } \tan^{-1}\left(\frac{-1}{0}\right)=-\frac{\pi}{2}$$

$$\text{Polar Form, } z=r\left(\cos\theta+i\sin\theta\right)$$

$$z=\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$$

Q1(v)

$$\text{Modulus, } \left|\frac{1}{1+\mathrm{i}}\right|$$

$$=\left|\frac{1(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}\right|\text{[Rationalizing the denominator]}$$

$$=\left|\frac{1-i}{1^2-i^2}\right|=\left|\frac{1-i}{2}\right|=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

$$\text{Argument, } \tan^{-1}(-1)=-\frac{\pi}{4}$$

$$\text{Polar Form}=\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)$$

Q1(vi)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$
where,

$$|z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{let } z = \frac{1+2i}{1-3i}$$

$$= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{1(1+3i) + 2i(1+3i)}{1^2 + 3^2}$$

$$= \frac{1+3i+2i-6}{1+9}$$

$$= \frac{-5+5i}{10}$$

$$= \frac{-5}{10} + \frac{5}{10}i$$

$$= \frac{-1}{2} + \frac{1}{2}i$$

$$\therefore |z| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{2}{4}}$$

$$= \frac{1}{\sqrt{2}}$$

Here $x = \frac{-1}{2} < 0$ & $y = \frac{1}{2} > 0, \therefore \theta$ lies in quadrant II

$$\theta = \arg(z) = \tan^{-1}\frac{1/2}{-1/2}$$

$$= \tan^{-1}(-1)$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) \qquad (\because \tan(\pi - \theta) = -\tan\theta)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

The polar form is given by $z = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Q1(vii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$

where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = \sin 120° - i\cos 120°$

$= \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) - i\cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) \qquad \left(\because 120° = \frac{\pi}{2} + \frac{\pi}{6}\right)$

$\Rightarrow z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \qquad \left(\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \;\&\; \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right)$

Here z is already in polar form

with $|z| = 1$ & $\theta = \arg(z) = \frac{\pi}{6}$

Q1(viii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$

where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\begin{aligned}
\text{let } z &= \frac{-16}{1+i\sqrt{3}} \\
&= \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} \\
&= \frac{-16\left(1-i\sqrt{3}\right)}{(1)^2+\left(\sqrt{3}\right)^2} \\
&= \frac{-16\left(1-i\sqrt{3}\right)}{1+3} \\
&= \frac{-16}{4}\left(1-i\sqrt{3}\right) \\
&= -4\left(1-i\sqrt{3}\right) \\
&= -4+4\sqrt{3}i
\end{aligned}$$

$$\begin{aligned}
\therefore |z| &= \sqrt{(-4)^2+\left(4\sqrt{3}\right)^2} \\
&= \sqrt{16+48} \\
&= \sqrt{64} \\
&= 8
\end{aligned}$$

Here $x = -4 < 0$ & $y = 4R3 > 0$, $\therefore$ θ lies in quadrant II

$$\begin{aligned}
\theta = \arg(z) &= \tan^{-1}\left(\frac{4\sqrt{3}}{-4}\right) \\
&= \tan^{-1}\left(-\sqrt{3}\right) \\
&= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \\
&= \tan^{-1}\left(\tan\left(\pi-\frac{\pi}{3}\right)\right) \qquad \left(\because \tan(\pi-\theta) = -\tan\theta\right) \\
&= \pi - \frac{\pi}{3} \\
&= \frac{2\pi}{3}
\end{aligned}$$

The polar form is given by $z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Q2

$$z = \left(i^{25}\right)^3 = (i)^3 = -i$$

$$|z| = 1,$$

$$\arg(z) = \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$$

$$\text{Polar Form: } \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

Q3(i)

Let $z = 1 + i\tan\alpha$

$\tan\alpha$ is periodic function with period π.

So, let us take α lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case – I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$|z| = \sqrt{1 + \tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$\tan\beta = |\tan\alpha| = \tan\alpha$

$\Rightarrow \beta = \alpha$

As z is represented by a point in first quadrant.

$\therefore \arg(z) = \beta = \alpha$.

So polar form of z is $\sec\alpha(\cos\alpha + i\sin\alpha)$

Case – II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$|z| = \sqrt{1 + \tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha| = -\sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$\tan\beta = |\tan\alpha| = -\tan\alpha = \tan(\pi - \alpha)$

$\Rightarrow \beta = \pi - \alpha$

As z is represented by a point in fourth quadrant.

$\therefore \arg(z) = -\beta = \alpha - \pi$.

So polar form of z is $-\sec\alpha\left(\cos(\alpha - \pi) + i\sin(\alpha - \pi)\right)$.

Q3(ii)

Let $z = \tan\alpha - i$

$\tan\alpha$ is periodic function with period π.

So, let us take α lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case – I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$|z| = \sqrt{\tan^2\alpha + 1} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = \cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$

$\Rightarrow \beta = \frac{\pi}{2} - \alpha$

As z is represented by a point in fourth quadrant.

$\therefore \arg(z) = -\beta = \alpha - \frac{\pi}{2}$.

So polar form of z is $\sec\alpha\left(\cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right)\right)$

Case – II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$|z| = \sqrt{\tan^2\alpha + 1} = \sqrt{\sec^2\alpha} = |\sec\alpha| = -\sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = -\cot\alpha = \tan\left(\alpha - \frac{\pi}{2}\right)$

$\Rightarrow \beta = \alpha - \frac{\pi}{2}$

As z is represented by a point in third quadrant.

$\therefore \arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha$.

So polar form of z is $-\sec\alpha\left(\cos\left(\frac{\pi}{2} + \alpha\right) + i\sin\left(\frac{\pi}{2} + \alpha\right)\right)$.

Q3(iii)

Let $z = (1 - \sin\alpha) + i\cos\alpha$

Since sine and cosine are periodic functions with period 2π

So, let us take α lying in the interval $[0, 2\pi]$.

Now, $z = (1 - \sin\alpha) + i\cos\alpha$

$\Rightarrow |z| = \sqrt{(1 - \sin\alpha)^2 + \cos^2\alpha} = \sqrt{2 - 2\sin\alpha} = \sqrt{2}\sqrt{1 - \sin\alpha}$

$\Rightarrow |z| = \sqrt{2}\sqrt{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$$\tan\beta = \frac{|\cos\alpha|}{|1\text{-}\sin\alpha|} = \left|\frac{\cos\alpha}{1\text{-}\sin\alpha}\right| = \left|\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}\right| = \left|\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right|$$

$$\Rightarrow \tan\beta = \left|\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right| = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right|$$

Following cases arise:

Case I: When $0 \le \alpha < \frac{\pi}{2}$

$\cos\frac{\alpha}{2} > \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\therefore\ |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \Rightarrow \beta = \frac{\pi}{4} + \frac{\alpha}{2}$

Clearly, z lies in the first quadrant.

$\therefore\ \arg(z) = \frac{\pi}{4} + \frac{\alpha}{2}$

So polar form of z is $\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)$

Case II: When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore\ |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = -\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = \tan\left\{\pi - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} = \tan\left(\frac{3\pi}{4} - \frac{\alpha}{2}\right)$

$\Rightarrow \beta = \frac{3\pi}{4} - \frac{\alpha}{2}$

Since $1 - \sin\alpha > 0$ and $\cos\alpha < 0$.

Clearly, z lies in the fourth quadrant.

$\therefore\ \arg(z) = -\beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

So polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right)$

Case III: When $\frac{3\pi}{2} < \alpha < 2\pi$

$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\pi, \frac{5\pi}{4}\right)$

$\therefore\ |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = -\tan\left\{\pi - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} = \tan\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)$

$\Rightarrow \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

Clearly, $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$.

So, z lies in the first quadrant.

$\therefore\ \arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

So polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right)$.

Q3(iv)

Let $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\ \sin\frac{\pi}{3}} = \dfrac{1-i}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \dfrac{2-2i}{1+i\sqrt{3}} = \dfrac{(2-2i)(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} =$

$\dfrac{(2-2\sqrt{3}) - i(2\sqrt{3}+2)}{4} = \dfrac{(1-\sqrt{3})}{2} - i\dfrac{(\sqrt{3}+1)}{2}$

$$|z| = \sqrt{\frac{(1-\sqrt{3})^2}{4} + \frac{(\sqrt{3}+1)^2}{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}$$

Let β be acute angle given by $\tan\beta = \dfrac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$$\tan\beta = \frac{\left|-\frac{(\sqrt{3}+1)}{2}\right|}{\left|\frac{(1-\sqrt{3})}{2}\right|} = \left|\frac{-(\sqrt{3}+1)}{(1-\sqrt{3})}\right| = |2+\sqrt{3}| = \tan\left(\frac{7\pi}{12}\right)$$

$\Rightarrow \beta = \dfrac{7\pi}{12}$

Z is represented by a point in second quadrant.

So polar form of z is $\sqrt{2}\left(\cos\dfrac{7\pi}{12} - i\sin\dfrac{7\pi}{12}\right)$.

$|z_1| = |z_2|$

Let $\arg(z_1)=\theta$

$\therefore \arg(z_2)=\pi-\theta$

In polar form, $z_1=|z_1|(\cos\theta + i\sin\theta)$......$(i)$

$z_2 = |z_2|\{\cos(\pi-\theta) + i\sin(\pi-\theta)\}$

$= |z_2|(-\cos\theta + i\sin\theta)$

$= -|z_2|(\cos\theta - i\sin\theta)$

Finding conjugate of

$\overline{z_2} = -|z_2|(\cos\theta + i\sin\theta)$......$(ii)$

(i)/(ii) is equal to

$\dfrac{z_1}{\overline{z_2}} = -\dfrac{|z_1|(\cos\theta + i\sin\theta)}{|z_2|(\cos\theta + i\sin\theta)}$

$\dfrac{z_1}{\overline{z_2}} = -\dfrac{|z_1|}{|z_1|} \qquad [\because |z_1| = |z_2|]$

$\dfrac{z_1}{\overline{z_2}} = -1$

$z_1 = -\overline{z_2}$

Hence Proved.

Q4

z_1, z_2 are conjugates implies $z_2 = \overline{z_1}$

z_3, z_4 are conjugates implies $z_4 = \overline{z_3}$

Also we know that $\arg(z_1) + \arg(\overline{z_1}) = 0$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$$

$$= \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$$

$$= \arg(z_1) - \arg(\overline{z_3}) + \arg(\overline{z_1}) - \arg(z_3)$$

$$= \arg(z_1) + \arg(\overline{z_1}) - \arg(\overline{z_3}) - \arg(z_3)$$

$$= \arg(z_1) + \arg(\overline{z_1}) - \left[\arg(\overline{z_3}) + \arg(z_3)\right] [\because \arg(z_1) + \arg(\overline{z_1}) = 0]$$

$$= 0 + 0 = 0$$

Q5

$$\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$$

$$= 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i\,2\sin^2\frac{\pi}{10} [\text{Using } \sin 2\theta = 2\sin\theta\cos\theta \ \& \ 1 - \cos 2\theta = 2\sin^2\theta]$$

$$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$