WITH GRAPH PAPER

केन्द्रीय माध्यमिक शिक्षा बोर्ड, दिल्ली

सीनियर स्कूल सर्टिफिकेट परीक्षा (कक्षा बारहवीं)

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अतिरिक्त उत्तर–पुस्तिका (ओं) की संख्या
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विकलांग व्यक्ति : हाँ / नहीं
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Section - A

1. $\bar{r} \cdot \hat{n} = d$

$\hat{n}$ - Unit vector perpendicular to the plane

d - Distance of the plane from origin.

$\bar{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 4$

$\therefore \dfrac{\bar{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{2^2+3^2+6^2}} = \dfrac{4}{\sqrt{2^2+3^2+6^2}} \Rightarrow \bar{r} \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = \frac{4}{7}$ (Distance of plane from origin $= \frac{4}{7}$)

$\bar{r} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) = -30$

$\therefore \dfrac{\bar{r} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k})}{\sqrt{6^2+9^2+18^2}} = \dfrac{-30}{\sqrt{6^2+9^2+18^2}} \Rightarrow \bar{r} \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = \frac{-10}{7}$ (Distance of plane from origin $= \frac{10}{7}$ in the direction opposite to the unit vector)

$\therefore$ Distance between the planes $= \frac{4}{7} - \left(\frac{-10}{7}\right) = \frac{14}{7} = 2$ units

2. $\bar{a} - \sqrt{2}\bar{b}$ is a unit vector

$|\bar{a} - \sqrt{2}\bar{b}| = 1$

$\therefore |\bar{a} - \sqrt{2}\bar{b}|^2 = 1$

$|\bar{a}|^2 + |\sqrt{2}\bar{b}|^2 - 2\sqrt{2}\,\bar{a} \cdot \bar{b} = 1$ $\quad (|\bar{a}| = |\bar{b}| = 1)$ $\quad$ (θ: Angle between vectors $\bar{a}$ & $\bar{b}$)

$1 + 2 - 2\sqrt{2}\,\bar{a} \cdot \bar{b} = 1$ $\quad \bar{a} \cdot \bar{b} = \frac{1}{\sqrt{2}} \Rightarrow |\bar{a}||\bar{b}|\cos\theta = \frac{1}{\sqrt{2}}$ $\quad \therefore \boxed{\theta = 45^\circ}$ (Angle between $\bar{a}$ & $\bar{b}$ is 45°)

3. $|\bar{a}| = \frac{1}{2}$ $\quad |\bar{b}| = \frac{4}{\sqrt{3}}$ $\quad |\bar{a} \times \bar{b}| = \frac{1}{\sqrt{3}}$ $\quad |\bar{a} \cdot \bar{b}| = ?$

$|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin\theta$ (θ: Angle between vectors $\bar{a}$ & $\bar{b}$)

$\frac{1}{\sqrt{3}} = \frac{1}{2} \times \frac{4}{\sqrt{3}} \sin\theta$

$\sin\theta = \frac{1}{2}$ $\quad \theta = 30^\circ$

$|\bar{a} \cdot \bar{b}| = |\bar{a}||\bar{b}| \cos\theta$

$= \frac{1}{2} \times \frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 1$

$\therefore |\bar{a} \cdot \bar{b}| = 1$

4. $A = \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix}$

$kA = \begin{bmatrix} 0 & 3k \\ 2k & -5k \end{bmatrix}$

But given $kA = \begin{bmatrix} 0 & 4a \\ -8 & 5b \end{bmatrix}$

$\begin{bmatrix} 0 & 3k \\ 2k & -5k \end{bmatrix} = \begin{bmatrix} 0 & 4a \\ -8 & 5b \end{bmatrix}$ $\quad$ Equating individual terms,

$2k = -8$ $\quad k = -4$

$3k = 4a$

$3\times(-4) = 4a \qquad a = -3$

$-5k = 5b$

$b = -k = 4$

5. $|AB| = |A||B|$ (Provided A & B are square matrices)

$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1\times(-1) - 2\times 3 = -7$

$|B| = \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} = 1\times(-2) - 3\times(-4) = 10$

$\therefore |AB| = -7\times 10 = -70$

6. $|A| = 5$

$|AA^T| = |A||A^T|$ (As A is a square matrix)

$= |A||A|$ (As $|A| = |A^T|$)

$= |A|^2 = 25$

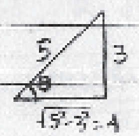
Section-B

7. To prove: $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

Proof: Let $\sin^{-1}\left(\frac{3}{5}\right) = \theta$

$\therefore \sin\theta = \frac{3}{5}$

$\therefore \tan\theta = \frac{3}{4}$ (From triangle ①)



Triangle ①

$\therefore \theta = \tan^{-1}\left(\frac{3}{4}\right)$

Now $2\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)$ $\left\{2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$

$= \tan^{-1}\left(\frac{\frac{3}{2}}{1-\frac{9}{16}}\right) = \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right) = \tan^{-1}\left(\frac{3\times 8}{7}\right) = \tan^{-1}\left(\frac{24}{7}\right)$

$\therefore 2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$ $\left\{\tan^{-1}a - \tan^{-1}b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)\right\}$

$= \tan^{-1}\left(\frac{24\times31-17\times7}{7\times31+24\times17}\right) = \tan^{-1}\left(\frac{625}{625}\right) = \tan^{-1}1 = \frac{\pi}{4}$ Hence Proved

2. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\, dx$ — ①

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\, dx \qquad \left\{ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right\}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x}\, dx \quad — ②$$

Adding ① & ②

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x} \qquad \left\{ \sin^2 x + \cos^2 x = 1 \right\} \qquad \left\{ \cos(A-B) = \cos A \cos B + \sin A \sin B \right\}$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x} = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x \cdot \frac{1}{\sqrt{2}} + \sin x \cdot \frac{1}{\sqrt{2}}} = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

Put $x - \frac{\pi}{4} = t$

$dx = dt$

For $x = 0,\ t = \frac{-\pi}{4}$, for $x = \frac{\pi}{2},\ t = \frac{\pi}{4}$

$$I = \frac{1}{2\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec t \, dt$$

$$= \frac{1}{2\sqrt{2}} \log\left| \sec t + \tan t \right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2\sqrt{2}} \left(\log\left| \sec\frac{\pi}{4} + \tan\frac{\pi}{4} \right| - \log\left| \sec\left(\frac{-\pi}{4}\right) + \tan\left(\frac{-\pi}{4}\right) \right| \right)$$

$$= \frac{1}{2\sqrt{2}} \log\left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right|$$

$$= \frac{1}{2\sqrt{2}} \log\left| \frac{(\sqrt{2}+1)^2}{2-1} \right| = \frac{1}{\sqrt{2}} \log\left|\sqrt{2}+1\right|$$

9. let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$

$$= \int \log(\log x)\, dx + \int \frac{dx}{(\log x)^2}$$

$$= I_1 + I_2 + C \text{ (Say)} \qquad \{C = \text{Arbitrary constant}\}$$

$I_1 = \int \log(\log x)\, dx$ & $I_2 = \int \frac{dx}{(\log x)^2}$

Consider $I_1 = \int \log(\log x) \times 1\, dx$

$= x\log(\log x) - \int x \times \frac{1}{\log x} \times \frac{1}{x}\, dx$ {Applying Integration by parts}

$= x\log(\log x) - \int \frac{dx}{\log x} \times 1$

$= x\log(\log x) - \left[\frac{x}{\log x} - \int x \times \frac{-1}{(\log x)^2} \times \frac{1}{x}\, dx\right]$ {Applying Integration by parts}

$I_1 = x\log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2}$

But $\int \frac{dx}{(\log x)^2} = I_2$

$\therefore \underline{I} = I_1 + I_2 + C = x\log(\log x) - \frac{x}{\log x} + C$

10. Let $I = \int \frac{(1-\sin x)\,dx}{\sin x(1+\sin x)}$

$$= \int \frac{(1-\sin x)(1-\sin x)\,dx}{\sin x(1+\sin x)(1-\sin x)} \quad \left\{ \text{Multiplying numerator \& Denominator by } 1-\sin x \right\}$$

$$= \int \frac{(1-\sin x)^2\,dx}{\sin x(1-\sin^2 x)} = \int \frac{(1+\sin^2 x - 2\sin x)\,dx}{\sin x \cos^2 x}$$

$$= \int \frac{dx}{\sin x \cos^2 x} + \int \frac{\sin x}{\cos^2 x}dx - 2\int \frac{dx}{\cos^2 x}$$

$$= \underbrace{\int \frac{\sin x\,dx}{(1-\cos^2 x)\cos^2 x}}_{= I_1} + \underbrace{\int \frac{\sin x\,dx}{\cos^2 x}}_{= I_2} - 2\tan x + C \quad \left\{ \int \sec^2 x\,d \right.$$

(say)

$$I_2 = \int \frac{\sin x}{\cos^2 x}dx$$

Put $\cos x = t$

$-\sin x\,dx = dt$

$\therefore \frac{I_2}{2} = \int \frac{-dt}{t^2} = \frac{1}{t} + c_1 = \sec x + c_1$

$I_1 = \int \frac{\sin x \, dx}{(1-\cos^2 x)(\cos^2 x)}$

Put $\cos x = u$

$-\sin x \, dx = du$

$I_1 = \int \frac{-du}{(1-u^2)u^2} = \int \frac{du}{(u^2-1)u^2} = \int \frac{(u^2-(u^2-1))\,du}{(u^2-1)u^2}$

$= \int \frac{du}{u^2-1} - \int \frac{du}{u^2}$

$= \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + \frac{1}{u} + c'$

$\left\{ \int \frac{du}{u^2-1} = \int \frac{du}{(u+1)(u-1)} = \frac{1}{2}\int \frac{(u+1)-(u-1)}{(u+1)(u-1)}\,du = \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| \right\}$

$I_1 = \frac{1}{2}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + \sec x + c'$

$\therefore I = \int \frac{(1-\sin x)\,dx}{\sin x(1+\sin x)} = \frac{1}{2}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + 2\sec x - 2\tan x + k$

where k is an arbitrary constant.

ii) $x = am^2$

$ay^2 = (am^2)^3$

$ay^2 = a^3m^6$

$y^2 = a^2m^6$

$\therefore y = \pm am^3$

Considering $(x,y) = (am^2, am^3)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dm}}{\frac{dx}{dm}}$$

$$\frac{dy}{dm} = \frac{d}{dm}(am^3) = 3am^2$$

$$\frac{dx}{dm} = \frac{d}{dm}(am^2) = 2am$$

$$\therefore \left.\frac{dy}{dx}\right]_{(am^2, am^3)} = \frac{3am^2}{2am} = \frac{3m}{2} = \text{Slope of tangent at } (am^2, am^3)$$

$$\text{Slope of normal at } (am^2, am^3) = \frac{-1}{\left.\frac{dy}{dx}\right]_{(am^2, am^3)}} = \frac{-1}{\frac{3m}{2}} = \frac{-2}{3m}$$

$\therefore$ Equation of normal to the curve at (am^2, am^3)

$$y - am^3 = \frac{-2}{3m}\left(x - am^2\right)$$

$$3my - 3am^4 = -2x + 2am^2$$

$$3my + 2x = 3am^4 + 2am^2$$ is the required equation.

12 $f(x) = \begin{cases} k\sin\left(\frac{\pi}{2}(x+1)\right) & x \le 0 \\ \frac{\tan x - \sin x}{x^3} & x > 0 \end{cases}$

$f(x)$ is continuous at $x = 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Now $\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^-} k\sin\left(\frac{\pi}{2}(x+1)\right) = k\sin\frac{\pi}{2} = k$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0^+} \frac{\sin x}{x}\left(\frac{\sec x - 1}{x^2}\right)$$

$$= \lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \frac{1 - \cos x}{\cos x \cdot x^2}$$

$$= \frac{\lim_{x\to 0^+} \frac{\sin x}{x} \times \lim_{x\to 0^+} \frac{2\sin^2\frac{x}{2}}{4\times\left(\frac{x}{2}\right)^2}}{\lim_{x\to 0^+} \cos x}$$

Now $\lim_{x\to 0} \frac{\sin x}{x} = 1$ & $\lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)^2 = 1$

$\therefore \lim_{x\to 0^+} f(x) = \frac{1 \times \frac{1}{2}}{1} = \frac{1}{2}$

$\therefore \boxed{k = \frac{1}{2}}$ $\{$As $f(x)$ is continuous at $x = 0\}$

13. Consider $\tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$

$$= \tan^{-1}\left(\frac{1 - \sqrt{\frac{1-x^2}{1+x^2}}}{1 + \sqrt{\frac{1-x^2}{1+x^2}}}\right)$$

Put ~~But~~ $x^2 = \cos 2\theta$ $\therefore \theta = \frac{1}{2}\cos^{-1} x^2$

$$\therefore \tan^{-1}\left(\frac{1-\sqrt{\frac{1-x^2}{1+x^2}}}{1+\sqrt{\frac{1-x^2}{1+x^2}}}\right) = \tan^{-1}\left(\frac{1-\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}}{1+\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}}\right)$$

$$1-\cos 2\theta = 2\sin^2\theta$$

$$1+\cos 2\theta = 2\cos^2\theta$$

$$\therefore \tan^{-1}\left(\frac{1-\sqrt{\frac{1-x^2}{1+x^2}}}{1+\sqrt{\frac{1-x^2}{1+x^2}}}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

$$= \frac{\pi}{4}-\theta$$

We want: $$\frac{d}{d(\cos^{-1}x^2)}\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)\right) = \frac{d}{d(2\theta)}\left(\frac{\pi}{4}-\theta\right) = \frac{\frac{d}{d\theta}\left(\frac{\pi}{4}-\theta\right)}{\frac{d}{d\theta}(2\theta)}$$

$$= \frac{1}{2}\frac{d}{d\theta}\left(\frac{\pi}{4}-\theta\right) = -\frac{1}{2}$$

14. A plane which passes through $A(3,2,1)$, $B(4,2,-2)$ & $C(6,5,-1)$

is $\begin{vmatrix} x-3 & y-2 & z-1 \\ 4-3 & 2-2 & -2-1 \\ 6-3 & 5-2 & -1-1 \end{vmatrix} = 0$

$0 = \begin{vmatrix} x-3 & y-2 & z-1 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}$

$0 = (x-3)\left(0\times(-2)-(-3)\times3\right) - (y-2)\left(1\times(-2)-(-3)\times3\right) + (z-1)\left(1\times3 - 3\times0\right)$

$9(x-3) - 7(y-2) + 3(z-1) = 0$

$9x - 7y + 3z = 27 - 14 + 3 = 16$

$\therefore$ Plane passing through points A, B, C is $9x - 7y + 3z = 16$

Now A, B, C & $D(\lambda, 5, 5)$ are coplanar

$\therefore$ D lies on the plane of A, B, C

$\therefore$ $9\lambda - 7\times5 + 3\times5 = 16$

$9\lambda = 36$

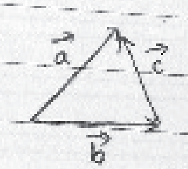
$\lambda = 4$

15. $\vec{a} = \vec{b} + \vec{c}$

Here $\vec{a} = p\hat{i} + q\hat{j} + r\hat{k}$

$\vec{b} = s\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$



$\vec{a} = \vec{b} + \vec{c}$

$\therefore$ $p\hat{i} + q\hat{j} + r\hat{k} = s\hat{i} + 3\hat{j} + 4\hat{k} + 3\hat{i} + \hat{j} - 2\hat{k}$

$p\hat{i} + q\hat{j} + r\hat{k} = (s+3)\hat{i} + 4\hat{j} + 2\hat{k}$

Equating components,

$p = s+3$

$q = 4$

$r = 2$

Area of triangle $= 5\sqrt{6}$

But Area of triangle $= \frac{1}{2}|\vec{a} \times \vec{b}|$

$$5\sqrt{6} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & 4 & 2 \\ p-3 & 2 & 4 \end{vmatrix} = \frac{1}{2}\left| 10\hat{i} - (2p+6)\hat{j} + (12-p)\hat{k} \right|$$

$\therefore 600 = 100 + (2p+6)^2 + (12-p)^2$

$500 = 4p^2 + 36 + \cancel{24p} + 144 + p^2 - \cancel{24p} \Rightarrow 5p^2 = 320$

560
−180
320

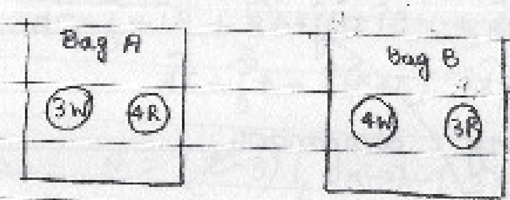
$p^2 = 64$

$p = \pm 8$

If $p = 8$, $S = p - 3 = 5$

If $p = -8$, $S = p - 3 = -11$

16.



Let E_1: Event that the balls are drawn from bag A

E_2: Event that the balls are drawn from Bag B

C: Event that 2 white balls & 1 red ball are drawn

$P(E_2|C) = ?$

Now $P(E_1) = \frac{1}{2} = P(E_2)$

$P(C|E_1) = \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} \times {}^3C_1 = \frac{4}{35} \times 3$ (multiplied by 3 because they can be chosen in any order)

$P(C|E_2) = \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} \times {}^3C_1 = \frac{6}{35} \times 3$

By Bayes theorem,

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2)\,P(C|E_2)}{P(E_1)P(C|E_1) + P(E_2)\times P(C|E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{6}{35}\times 3}{\frac{1}{2}\times\frac{4}{35}\times 3 + \frac{1}{2}\times\frac{6}{35}\times 3} = \frac{6}{10} = \frac{3}{5}$$

17. Let the length of the plot be l.

Let the breadth of the plot be b.

Now $l \times b = A$ (A = Area of the plot)

$(l-50)(b+50) = A$

$(l-10)(b-20) = A - 5300$

$lb - 50b + 50l - 2500 = A$

$\therefore -b + l = 50$ — ①

(A = 20)

$lb - 10b - 20l + 200 = A - 5300$

$10b + 20l = 5500$

$b + 2l = 550$ — ②

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} l \\ b \end{bmatrix} = \begin{bmatrix} 50 \\ 550 \end{bmatrix}$$

$$\quad A \qquad\quad X \qquad\quad B$$

$AX = B$

$\therefore X = A^{-1}B$

When $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ $\quad$ ($ad-bc = 1\times1-(-2)\times1 = 3 \neq 0$ $\therefore A^{-1}$ exists)

$$A^{-1} = \frac{1}{1+2}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 50 \\ 550 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 600 \\ 450 \end{bmatrix} = \begin{bmatrix} 200 \\ 150 \end{bmatrix}$$

$$\therefore \begin{bmatrix} l \\ b \end{bmatrix} = \begin{bmatrix} 200 \\ 150 \end{bmatrix}$$

$l = 200$ m

$b = 150$ m

He wants to donate the plot to the school because he wants rural places to become developed & he is thereby showing his kind heartedness & his intention to help the society to develop by producing more literates. Children should have an opportunity to learn.

18.

$$2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$$

$$\frac{dy}{dx} = \frac{-2ye^{x/y}}{y - 2xe^{x/y}}$$

$$\frac{dx}{dy} = \frac{y - 2xe^{x/y}}{-2ye^{x/y}} = F(x,y) \quad (\text{Say})$$

$$F(\lambda x, \lambda y) = \frac{\lambda y - 2\lambda x e^{\frac{\lambda x}{\lambda y}}}{-2\lambda y e^{\lambda x/\lambda y}} = \lambda^0 F(x,y)$$

$\therefore$ The equation is a homogeneous differential equation

Put $x = vy$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\therefore v + y\frac{dv}{dy} = \frac{y - 2vye^{v}}{-2ye^{v}} = \frac{-1}{2}e^{-v} + v \Rightarrow y\frac{dv}{dy} = \frac{-1}{2}e^{-v}$$

$$\therefore \quad e^{v} dv = \frac{-1}{2} \frac{dy}{y}$$

Integrating

$$e^{v} = \frac{-1}{2} \log y + C$$

$$\therefore \quad e^{x/y} = \frac{-1}{2} \log y + C$$ is the required solution to the differential equation.

19. $$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^{3}$$

$$\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x+1)^{2}$$

This equation is of the form $\frac{dy}{dx} + Py = Q$ {P, Q = functions of x alone}

linear differential equation.

$$\therefore \text{Integrating factor} = e^{\int P dx} = e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = \frac{1}{x+1}$$

Solution to the differential equation:

$$y \times I.F = \int Q \times I.F\, dx + C$$

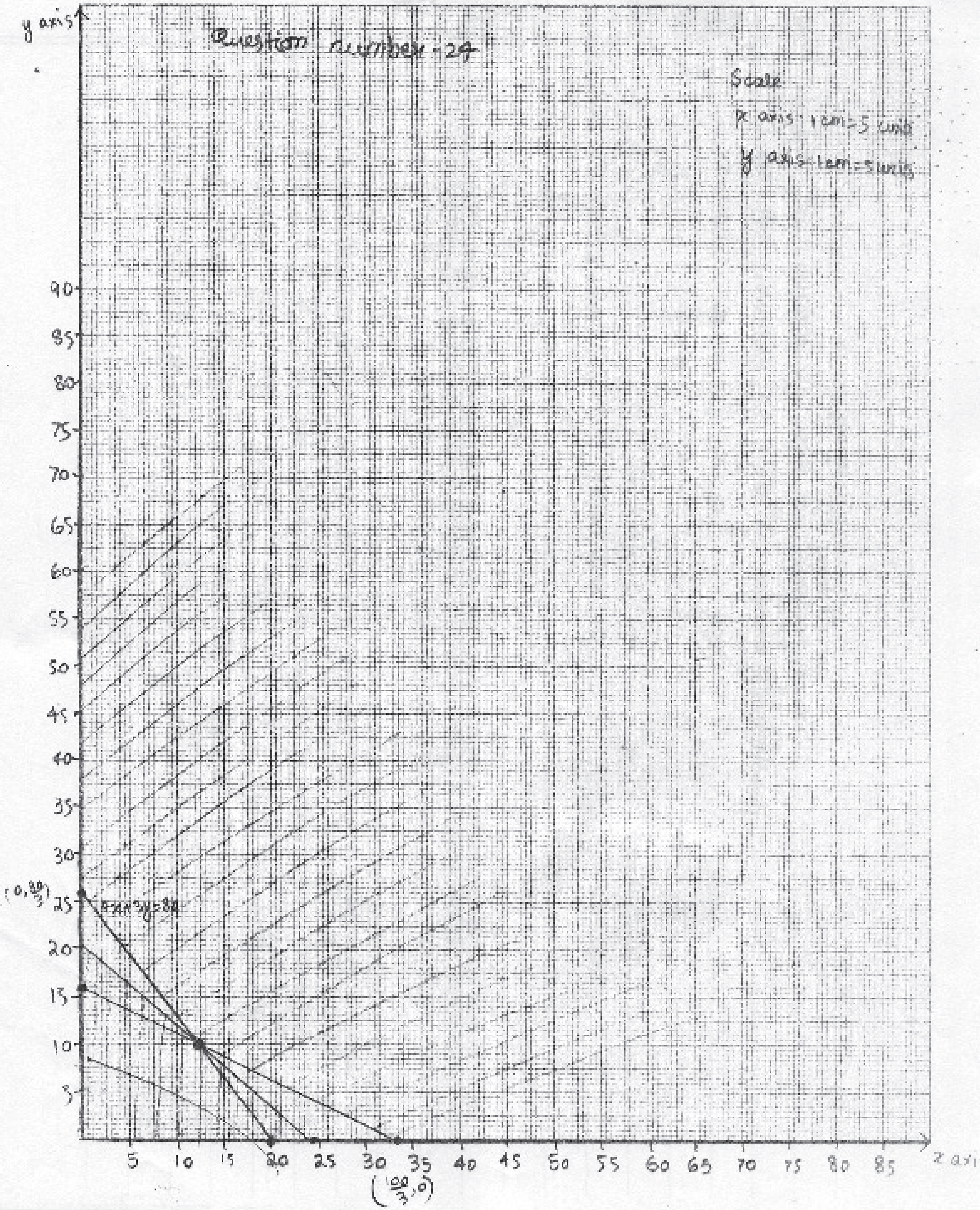
$$\therefore \frac{y}{x+1} = \int e^{3x}(x+1)^{2} \times \frac{1}{(x+1)} dx + C$$

$$= \int x e^{3x} dx + \int e^{3x} dx + C = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x} dx + \int e^{3x} dx + C$$

$$\frac{y}{x+1} = \frac{xe^{3x}}{3} + \frac{2}{9} e^{3x} + C$$ is the solution to the required differential equation

Question number - 29

Scale
x axis : 1 cm = 5 units
y axis : 1 cm = 5 units



Section-C

24. Let the number of units of ~~Vitamin A~~ food F_1 = x

Let the number of units of ~~Vitamin Minerals~~ food F_2 = y

[redacted]

$\cancel{x \geq 80}$ $\cancel{x \geq 0}$ $x \geq 0$ $y \geq 0$

$\cancel{y \geq 100}$ $\cancel{y \geq 0}$ $4x + 3y \geq 80$

$3x + 6y \geq 100$

$Z = 5x + 6y$ Minimize Z

$4x + 3y = 80$

$3x + 6y = 100$

$8x + 6y = 160$

$5x = 60$

$x = 12$

$y = \frac{32}{3}$

(x, y)	Z
$\left(0, \frac{80}{3}\right)$	₹.160
$\left(\frac{100}{3}, 0\right)$	₹. 166.67
$\left(12, \frac{32}{3}\right)$	₹.124

Since the region is unbounded, we should check if any point other than $\left(12, \frac{32}{3}\right)$ is in common to region shaded, & $5x + 6y \leq 124$

Clearly, from the graph, only $\left(12, \frac{32}{3}\right)$ lies on both regions.

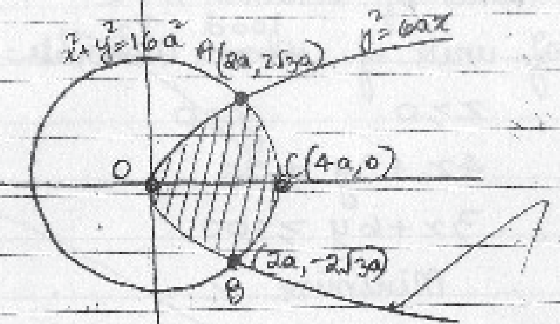
$\therefore$ The Minimum cost of the diet = ₹. 124

with 12 units of food F_1 & $\frac{32}{3}$ units of food F_2

$\frac{60}{+24}$

$\frac{108}{0}$

2a



Solving the two curves

$$x^2+6ax=16a^2$$

$$\left(\frac{x}{a}\right)^2+6\left(\frac{x}{a}\right)-16=0$$

$$\left(\frac{x}{a}+8\right)\left(\frac{x}{a}-2\right)=0$$

$x=-8a$ or $x=2a$

$\therefore$ $x=2a$

$y=\pm 2\sqrt{3}a$ $\quad (y^2=12a^2 \quad y=\pm 2\sqrt{3}a)$

$\therefore$ Required area $= 2\times$ Area OACO

$$= 2\times\int_{0}^{2\sqrt{3}a}(x_1-x_2)\,dy$$

$$= 2\times\int_0^{2\sqrt{3}a}\sqrt{16a^2-y^2}\,dy - 2\times\int_0^{2\sqrt{3}a}\frac{y^2}{6a}\,dy$$

$$= 2\times\left[\frac{y}{2}\sqrt{16a^2-y^2}+\frac{16a^2}{2}\sin^{-1}\frac{y}{4a}\right]_0^{2\sqrt{3}a} - \left[\frac{y^3}{9a}\right]_0^{2\sqrt{3}a}$$

$$= 2\sqrt{3}a\times 2a + 16a^2\sin^{-1}\frac{\sqrt{3}}{2} - 0 - \frac{8\times 3\sqrt{3}a^2}{9}$$

$$= 4\sqrt{3}a^2 - \frac{8\sqrt{3}}{3}a^2 + \frac{16a^2 \times \pi}{3}$$

$$\left\{ \int \sqrt{A^2 - x^2}\,dx = \frac{x}{2}\sqrt{A^2 - x^2} + \frac{A^2}{2}\sin^{-1}\frac{x}{A} \right\}$$

$$= \frac{4}{\sqrt{3}}a^2 + \frac{16a^2\pi}{3} \text{ sq units}$$

21. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$f(x)$ is increasing strictly in an interval if $f'(x) > 0$ in that interval

$f(x)$ is strictly decreasing in an interval if $f'(x) < 0$ in that interval.

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$

$$= 4(x^3 - 6x^2 + 11x - 8)$$

21. Maximum & Minimum values of $f(x) = \sec x + \log \cos^2 x$

$\cancel{= \sec x + 2\log\cos x}$

$f'(x) = 0$

$\therefore \sec x \tan x + \frac{2}{\cos x}(-\sin x) = 0 \qquad x \neq \frac{\pi}{2}$ 2-2log2

$\therefore \tan x (\sec x - 2) = 0$

$x = \pi$ or $x = \frac{\pi}{3}, \frac{5\pi}{3}$

Now $f''(x) < 0$ for maximum

$f''(x) > 0$ for minimum

$f''(x) = \sec^2 x (\sec x - 2) + \tan^2 x \sec x$

$f''(\pi) = 1 \times -3 + 0 = -3 < 0$

$f''(\frac{\pi}{3}) = f''(\frac{5\pi}{3}) = 0 + 3 \times 2 = 6 > 0$

Function attains maximum value at $x = \pi$ & minimum

$\therefore$ value at $x = \frac{\pi}{3}$ & $\frac{5\pi}{3}$

$f(\pi) = -1$

$f(\frac{\pi}{3}) = 2 - 2\log 2$ & $\cancel{2\log 5}$

But when $x = \frac{\pi}{2}$ $\therefore$, function becomes undefined.

~~Minimum & Maximum do not exist as Minimum~~

22. TP: $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ac \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

Proof: $C_1 \to C_1 - 2C_3$

$$\Delta = \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ac \\ a^2+b^2 & c^2 & ab \end{vmatrix}$$

$C_1 \to C_1 + C_2$

Now taking $a^2+b^2+c^2$ common from C_1

$$\Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ac \\ 1 & c^2 & ab \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & ac-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix}$$

Taking $(a-b)$ common from R_2 & $(c-a)$ common from C_3

$$\Delta = (a^2+b^2+c^2)(a-b)(c-a)\begin{vmatrix} 1 & a^2 & bc \\ 0 & -(a+b) & c \\ 0 & a+c & -b \end{vmatrix}$$

Expanding along C_1

$$\Delta = (a^2+b^2+c^2)(a-b)(c-a)\left(-(a+b)\times b + (a+c)c\right)$$

$$= (a^2+b^2+c^2)(a-b)(c-a)\left(b^2 - c^2 + bc - ba\right)$$

$$= (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)$$ H.P.

As $(b-c)(a+b+c) = ab+b^2+bc-ac-bc-c^2$

$b^2-c^2+ab-ac$

25. Equation of plane containing two parallel lines has DRs of perpendicular

$(\vec{a}_2-\vec{a}_1)\times\vec{b}$ $\quad$ $\vec{b}$ = DRs of the line

$\vec{a}_1$ = Point (Position vector on line 1)

$\vec{a}_2$ = Point (Position vector) on line 2

$\vec{a}_1 = \hat{i}-\hat{j}$

$\vec{a}_2 = 2\hat{j}-\hat{k}$

$\vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$

$\vec{a}_2-\vec{a}_1 = -\hat{i}+3\hat{j}-\hat{k}$

$$(\vec{a}_2-\vec{a}_1)\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 8\hat{i}+\hat{j}-5\hat{k}$$

$\therefore$ Equation of plane $= 8x+y-5z = 8\times1+1\times(-1)-5\times0$

$8x+y-5z = 7$

Now consider the line $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{5}$

$8\times3+1\times1+-5\times5=0$

$\therefore$ DRs of line & perpendicular to plane are perpendicular.

& Point on line $=(2,1,2)$ $\quad$ $8\times2+1-5\times2=7$ also satisfies the equation of plane. $\therefore$ The line is contained in the plane.

26) $f(x) = 4x^2 + 12x + 15$

Let $f(x) = y$

$y = 4x^2 + 12x + 15$

$4x^2 + 12x + (15 - y) = 0$

$x = \dfrac{-12 \pm \sqrt{144 - 16(15 - y)}}{8} = \dfrac{-12 + 4\sqrt{9 - 15 + y}}{8}$

$x = \dfrac{\sqrt{y-6} - 3}{2}$

Consider $g(x) = \dfrac{\sqrt{x-6} - 3}{2}$

Now $fog(x) = f(g(x)) = f\left(\dfrac{\sqrt{x-6} - 3}{2}\right) = \dfrac{4}{4}\left(\sqrt{x-6} - 3\right)^2 + 6\left(\sqrt{x-6} - 3\right) + 15$

$= x - 6 + 9 - 6\sqrt{x-6} + 6\sqrt{x-6} - 18 + 15$

$= x$

$gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = \dfrac{\sqrt{4x^2 + 12x + 9} - 3}{2} = \dfrac{2x + 3 - 3}{2} = x$

$\therefore fog(x) = gof(x) = x$

By definition, g is the inverse of f.
As inverse exists, f is invertible &

$\Rightarrow f^{-1}(x) = \dfrac{\sqrt{x-6}-3}{2}$

$f^{-1}(31) = \dfrac{\sqrt{31-6}-3}{2} = \dfrac{5-3}{2} = 1$

$f^{-1}(87) = \dfrac{\sqrt{87-6}-3}{2} = \dfrac{9-3}{2} = 3$

23.

X:	1	2	3	4	5	6
$P(x=)$	$\frac{10}{20} = \frac{1}{2}$	$\frac{6}{20} = \frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0

(as 5 & 6 can't be minimum values)

$P(x=1) = \dfrac{{}^5C_2}{{}^6C_3} = \dfrac{10}{20}$
(Numbers from 2-6 can be chosen)
(Any two)

$P(x=2) = \dfrac{{}^4C_2}{{}^6C_3} = \dfrac{6}{20}$
(Numbers from 3-6 can be chosen)
(Any two)

$P(x=3) = \dfrac{{}^3C_2}{{}^6C_3} = \dfrac{3}{20}$
(Any 2 out of 4,3,6 can be chosen)

$P(x=2) = \dfrac{{}^2C_2}{{}^6C_3} = \dfrac{1}{20}$
only 5,6 can be chosen

Mean $= E(x) = \sum x_i p_i$

$= \frac{10}{20}\times 1 + \frac{6}{20}\times 2 + \frac{3}{20}\times 3 + \frac{1}{20}\times 4 + 0 + 0$

$= \frac{10}{20} + \frac{12}{20} + \frac{9}{20} + \frac{4}{20} = \frac{35}{20} = 1.75$

Variance $= E(x^2) - (E(x))^2$

$= \frac{10}{20}\times 1 + \frac{6}{20}\times 4 + \frac{3}{20}\times 9 + \frac{1}{20}\times 16 - \frac{49}{16}$

$= \frac{10+24+27+16}{20} - \frac{49}{16} = \frac{77}{20} - \frac{49}{16}$

$= \frac{7}{4}\left(\frac{11}{5} - \frac{7}{4}\right)$

$= \frac{7}{4}\times\frac{9}{20}$

$\sigma^2 = \frac{63}{80}$

$\begin{array}{r} 44 \\ -35 \\ \hline 9 \end{array}$

$\begin{array}{r} 39 \\ 27 \\ 16 \\ \hline 77 \end{array}$