Chapter 7

7.1 - Basic Properties of Confidence Intervals

Point Estimate

ⓘ **Definition**

A point estimate of a population parameter is a single number used to approximate a population parameter

Interval Estimate

ⓘ **Definition**

An interval estimate (or a confidence interval) of a population parameter is an interval that with a certain degree of confidence contains the value of the population parameter

Confidence Intervals

ⓘ **Definition**

Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution that has a parameter θ
Suppose $h_(X_1, X_2, \dots X_n, \theta)$ is a function whose distribution is known and it (the distribution) doesn't depend on θ
Consider the probability

$$P(a < h(X_1, X_2, \dots X_n, \theta) < b) = 1 - \alpha$$

where $1 - \alpha$ is called the **confidence interval**, usually expressed as a percentage $100(1 - \alpha)$%
Typical confidence intervals are

- 90% $(\alpha = 10 \dots$

Interpretation of a Confidence Interval

ⓘ **Definition**

A confidence level is the proportion of times that the confidence interval actually does contain the population parameter, assuming that the estimation process is repeated a large number of times

Procedure for Constructing a CI for μ (a normal distribution with known α)

1. Verify that a random sample is drawn from a normally distributed population
2. Refer to Table $A3$ and find the critical value …
   3.….

Example 6

Given

- $X_i =$ the yield point of the i^{th} bar
- $i = 1, \ldots, 25;\ n = 25$
- $X_i \sim N(\mu, \sigma^2 = 100^2)$
  Compute a 90% CI for μ
- $\mu =$ the true average yield point = mean yield

1. $1 - \alpha = .90 \Rightarrow \alpha = .1 \Rightarrow \frac{\alpha}{2} = .05$
2. Critical value $Z_{\frac{\alpha}{2}} = Z_{.05}$
   - Table $A3 \Rightarrow Z_{.05} = 1.645$
3. The margin of error
   - $E = Z_{\frac{\alpha}{2}} \frac{\alpha}{\sqrt{n}} = 1.645 * \frac{100}{\sqrt{25}} = 329$
4. Confidence limits
   - Lower: $\overline{X} - E = 8439 - 32.9 = 8406.1$
   - Upper: $\overline{X} + E = 8439 + 32.9 = 8471.9$
   - A 90% confidence interval for μ is (8406.1, 8471.9)
   - Interpretation
     - We are 90% confident that the true average yield point of all modified bars is between 8406.1 lb and 8471.9 lb

Relationship between Confidence Level, Sample Size, and Width

- If a sample size is fixed, then any increase in the confidence level makes the corresponding confidence interval wider
- For a given confidence level, the width of the confidence interval can be reduced if we increase the sample size

Sample Size Determination

Given: $1-\alpha$ and width $w=2E$
Find: n, sample size

- $w=2E=2Z_{\frac{\alpha}{2}}*\frac{\sigma}{\sqrt{n}} \Rightarrow n=(\frac{2Z_{\frac{\alpha}{2}}\sigma}{w})^2$

Example 4

Given

- $\sigma=3$
- $1-\alpha=.99$
- $w=1$
  Find n

1. $\alpha=0.01,\ \frac{\alpha}{2}=.005$
2. $Z_{.005}=2.575$
3. $n=(\frac{2(2.575)*3}{1})^2=238.1$
   - $n=239$

7.2 - Confidence Intervals for Large Samples

Central Limit Theorem

Definition

If $n>30$

$$\overline{X}\sim approx\ N(\mu,\frac{\sigma^2}{n})$$

where $X_1,X_2,\ldots,X_n$ are iid with mean μ and standard deviation σ

Let $X_1,X_2,\ldots,X_n$ be a random sample from some distribution and let n be sufficiently large. Then,

$$Z=\frac{\overline{X}-\mu}{S}*\sqrt{n}\sim appox\ N(0,1)$$

where μ is the mean of the distribution of X_i's. Then,

$$P(-Z_{\alpha/2}<\frac{\overline{X}-\mu}{S}<Z_{\alpha/2})\approx 1-\alpha$$

An approximate $100(1-\alpha)\%$ CI for μ:

$$\overline{x} - z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}} < \mu < \overline{x} + z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}}$$

Example 16

Given

- $X_i =$ the breakdown voltage of the i^{th} circuit for $i = 1, \ldots, 48$
- $n = 48$ (n is large)
- $\overline{x} = 54.7$
- $S = 5.2$

Calculate A 95% CI for μ, the average breakdown voltage

1. $\alpha = .05,\ \frac{\alpha}{2} = .025$
2. Critical values $Z_{.025} = 1.96$
3. $Z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}} = 1.96 \frac{52}{\sqrt{48}} = 1.47$
4. - Lower limit: $54.7 - 1.47 = 52.23 \approx 53.2$
   - Upper limit: $54.7 + 1.47 = 56.17 \approx 56.2$
   - An approximate 95% CI for μ is (53.2,56.2)
5. We are approximately 95% confident that μ is between $53.2\ kV$ and $56.2\ kV$

Find the sample size

- Given: $w = 2, 1 - \alpha = .95, range = 70 - 40 = 30$
- $S \approx \frac{30}{4} = 1.5$
- $n = (2 * 1.96 * \frac{7.5}{2})^2 = 216.09$
- $n = 217$

Estimating Sample Size

Let w be the width of a CI

$$w = 2 * Z_{\frac{\alpha}{2}} * \frac{S}{\sqrt{n}}$$

Solve for $n \Rightarrow n = (2 * Z_{\frac{\alpha}{2}} * \frac{S}{w})^2$
If the distribution of X_i's is not too skewed, then a possible estimate of S is

$$S \approx \frac{range}{4} = \frac{max - min}{4}$$

Bounds for Unknown μ

An approximate 100(1-α)% lower confidence bound for μ is

$$\overline{x} - Z_\alpha * \frac{S}{\sqrt{n}} < \mu$$

and an upper confidence bound for μ is

$$\overline{x} + z_\alpha * \frac{S}{\sqrt{n}} > \mu$$

Population Proportions

- Population proportion of successes: $p = \frac{M}{N}$
- Sample proportion: $\hat{p} = \frac{X}{n}$

Example 20

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Traditional Confidence Interval

$$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Equation

$$n \approx \frac{4z\hat{p}\hat{q}}{w^2}$$

7.3 - Intervals Based on a Normal Distribution

Assumption

- $X_1, X_2, \ldots, X_n$ form a random sample from $N(\mu, \sigma^2)$ where both μ and σ are unknown

Compute

- A $100(1-\alpha)\%$ CI for μ

Function

- $T = \frac{\overline{X}-\mu}{S} * \sqrt{n}$ where S is a sample standard deviation

Student t-Distribution

Definition

Suppose that $X_1, X_2, \ldots, X_n$ are iid rvs and $X_i \sim N(\mu, \sigma^2)$. Then

$$T = \frac{\overline{X} - \mu}{\ldots}$$

…

Properties

1. The Student t-Distribution is different for different sample sizes
2. The Student t-Distribution has the same general symmetric bell shape as the standard normal distribution but it reflects the greater variability
3. $E(T) = 0$ if $\nu > 1$
4. $V(T) = \frac{\nu}{\nu - 2}$ if $\nu > 2$
   * Note: $V(T) \to 1$ as $\nu \to \infty$
   5.…
   $T \sim t\text{-}dist(\nu = df)$
   $\nu = df =$degrees of freedom $= n - 1$

Example 39

Given

- $X_i = \text{work of adhesion for the } i^{th} \text{ specimen}$
- $i = 1, \ldots, 5$

Assumption

- $X_I \sim N(\mu, \sigma^2)$
- $\mu = \text{the true average work of adhesion for all such specimens}$

Calculate a 95% CI for μ

- $\overline{x} = 107.78, \ s = 1.076$

1. $\alpha = .05 \Rightarrow \frac{\alpha}{2} = .025$
2. Critical value: $t_{.025, 5-1} = 2.776$
   - Use Table $A5$
3. Margin of Error
   - $E = 2.776 * \frac{1.076}{\sqrt{5}} = 1.34$
4. Confidence Limits
   - Lower: 106.44
   - Upper: 109.12

5. A 95% CI for μ is $(106.44, 109.12)$
   - With 95% confidence we conclude that the true average work μ is between 106.44 and 109.12

One-Sided CI

Let $X_1, X_2, \ldots, X_n$ be iid rvs such that $X_i \sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$
μ and σ are unknown
$T = \frac{\overline{X}-\mu}{\frac{S}{\sqrt{n}}} \sim \text{t-dist}(\nu = n-1)$
Consider ...
Collect the data $x_1, x_2, \ldots, x_n$
Calculate $\overline{x}$ and S
$\frac{\overline{x}-\mu}{S}\sqrt{n} < t_{\alpha,n-1}$
Solve for μ
$\overline{x} - t_{\alpha,n-1} * \frac{S}{\sqrt{n}} < \mu \Rightarrow 100(1-\alpha)$% lower confidence bound for μ
$\mu < \overline{x} + t_{\alpha,n-1} * \frac{S}{\sqrt{n}} \Rightarrow 100(1-\alpha)$% upper confidence bound for μ

Example 34

Given

- $X_i =$ the proportional limit stress of the i^{th} joint for $i = 1, \ldots, 14$ and $n = 14$
- Assume that $X_i \sim N(\mu, \sigma^2)$
- $\mu =$ the true average proportional limit stress of all joints
- $\overline{x} = 8.48$, $s = .79$

Find a 95% lower confidence bound for μ

1. $1 - \alpha = .95 \Rightarrow \alpha = .05$
2. Critical value $t_{\alpha,n-1} = t_{.05,13} = 1.771$
3. A 95% lower confidence bound for μ
   - $\overline{x} - t_{\alpha,n-1} * \frac{S}{\sqrt{n}} = 8.49 - 1.771 * \frac{.79}{\sqrt{14}}$
   - We are 95% confident that $\mu > 8.11\ MPa$

If $n > 30$

$$T = \frac{\overline{X}-\mu}{S} * \sqrt{n} \sim approx\ N(0,1)$$

7.4 - Confidence Interval for the Variance and Standard Deviation

Chi Squared Distribution

$\chi^2(\nu = n-1)$

The pdf

- Positively skewed

Assumption

Let $X_1, X_2, \ldots, X_n$ be iid, where $X_i \sim N(\mu, \sigma^2)$

Proposition

Let S be a sample variance. Then,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(\nu = n-1)$$

$$P(\chi^2_{1-\frac{\alpha}{2},n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\frac{\alpha}{2},n-1}) = 1-\alpha$$

A $100(1-\alpha)\%$ CI for σ^2 is

$$\frac{(n-1)S}{\chi^2_{\frac{\alpha}{2},n-1}} < \sigma^2 < \frac{(n-1)S}{\chi^2_{1-\frac{\alpha}{2},n-1}}$$

Example 4

Given

- $X_i =$ the amount of lateral expansion of the i^{th} arc weld
- $i = 1, \ldots, 9$
- $n = 9$
- $S = 2.81$
- Assume $X_i \sim N(\mu, \sigma^2)$

Find a 95% CI for σ^2 and σ

1. $\alpha = .05 \Rightarrow \frac{\alpha}{2} = .025$
2. Critical Values
   - Table $A7$
   - $\chi^2_{.975,8} = 2.180$
   - $\chi^2_{.025,8} = 17.534$
3. Confidence Interval

- Lower limit: $\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},n-1}} = \frac{8*2.81^2}{17.534} = 3.60$
- Upper limit: $\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},n-1}} = \frac{8*2.81^2}{2.18} = 28.98$
- A 95% CI for σ is $(1.90,\ 5.38)$