Chapter 2

cx

2.1 - Sample Spaces and Events

Sample Space

Definition

The collection of all outcomes of an experiment, represented by S

Example

- Roll a die twice, $S = \{(1,1),(1,2),\dots,(6,6)\}$
- Flip a coin until you get heads, $S = \{H, TH, TTH, TTTH, \dots\}$

Event

Definition

An event E of an experiment is a subset of the sample space S of the experiment

Example

- Experiment: toss a die twice
    - $S = \{(1,1),(1,2),\dots,(6,6)\}$
    - Event: getting a double
        - $E = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$
- Experiment: flip a coin until two heads are obtained (not necessarily consecutively)
    - $S = \{HH, THH, TTHH, \dots\}$
    - Event A: the coin is tossed 4 times
        - $A = \{HTTH, THTH, TTHH\}$

Tip

Event E has occurred if the resulting outcome is in E

Simple

Only one outcome

Compound

More than one outcome

Null

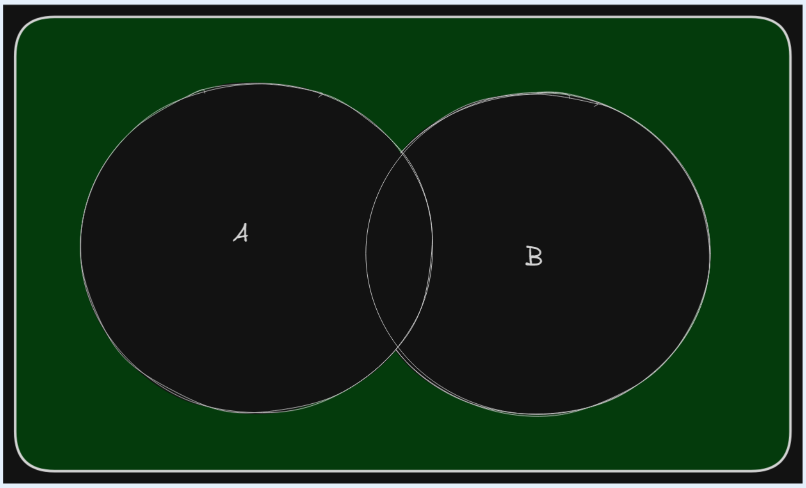
Event with no outcomes

Set Operations

Complement

Definition

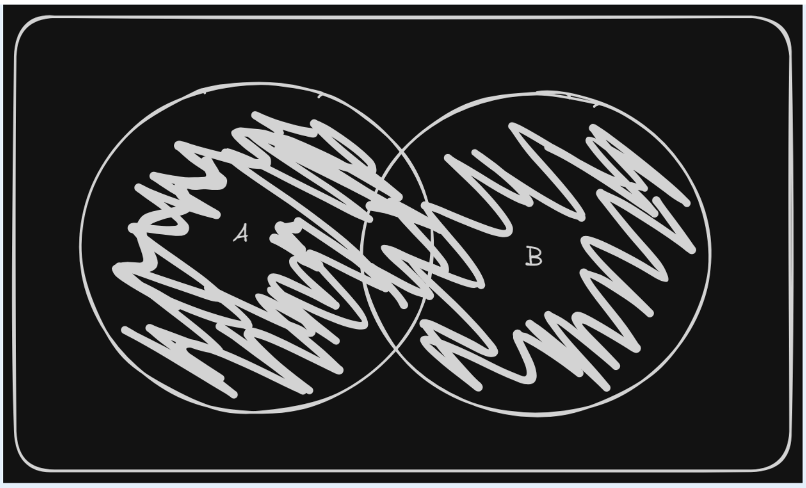
The complement of event A, A^1, is the set of all outcomes in the sample space S that are not in A



Union

Definition

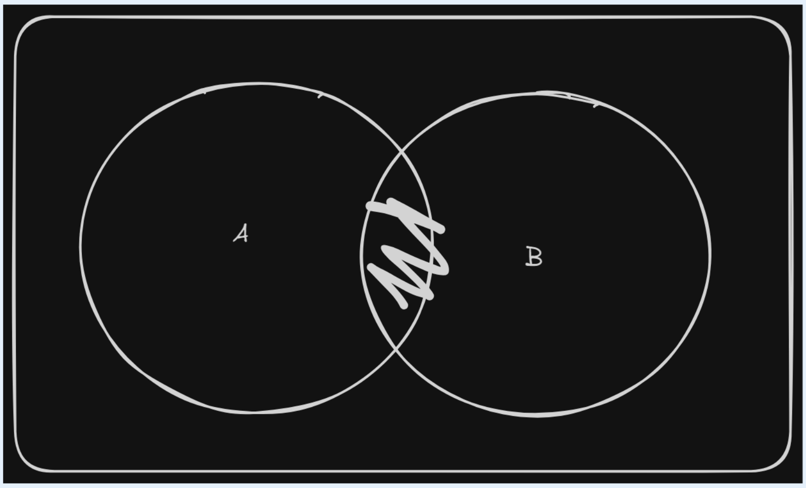
The union of events A and B, $A \cup B$, read as "A or B", is the set of all outcomes S that are in A or B or in both A and B



Intersection

Definition

The intersection of events A and B, represented by $A \cap B$ read as "A and B", is the set of all outcomes that are in both A and B



Mutual Exclusivity

Definition

A and B are disjoint or mutually exclusive if $A \cup B = \emptyset$

Pairwise Disjoint

Definition

$A_1, A_2, A_3, \ldots$ are mutually exclusive or pairwise disjoint if for any $i \neq j$, $i, j = 1, 2, 3, \ldots$, $A_i \cap A_j = \emptyset$

Example

- Experiment: roll the die twice
  - Events
    - A_1: getting a sum of 7 dots
      - $A_1 = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}$

Textbook Exercise 8

- Let A_i denote the event that the plant at site i is completed by the contract date
- Events: A_1, A_2, A_3
- At least one plant is completed by the contract date
  - $A_1 \cup A_2 \cup A_3$
- All plants are completed by the contract date
  - $A_1 \cap A_2 \cap A_3$
- Only the plant at site 1 is completed by the contract date
  - $A_1 \cap A_2^1 \cap A_3^1$
- Exactly one plant is completed by the contract date
  - $(A_1 \cap A_2^1 \cap A_3^1) \cup (A_1^1 \cap A_2 \cap A_3^1) \cup (A_1^1 \cap A_2^1 \cap A_3)$
- Exactly the plant at site 1 or both of the other two plants are completed by the contract date
  - $A_1 \cup (A_2 \cap A_3)$

2.2 - The Axioms and Properties of Probability

The probability of event A, $P(A)$ is a measure of chance that event A will occur

The Three Axioms

1. For any event $A, P(A) \geq 0$
2. $P(S) = 1$
3. Let $A_1, A_2, A_3, \ldots$ be an infinite collection of pairwise disjoint events
   - Then, $P(A_1, A_2, A_3, \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

Properties

1. $P(\emptyset) = 0$

2. Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint events
   - Then $P(A_1 \cup \ldots \cup A_n) = P(A_1) + P(A_2)$
3. For any event $A, P(A) + P(A^1) = 1$ or $P(A) = 1 - P(A^1)$

Example

- Roll a fair die twice. What is the chance that the sum of the dots will be at least 4?
  - $A = \{(1,3),(3,1),(2,2),\ldots\}$
  - $A^1 = \{(1,1),(1,2),(2,1)\}$
  - $P(A) = 1 - \frac{3}{36} = \frac{11}{12}$

1. For any $A, P(A) \leq 1$
2. For any event A and $B, P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. For any events A, B, C,
   $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Textbook Example 18

5 $10 bills, 4 $5 bills, 6 $1 bills

If the bills are selected one by one in random order, what is the probability that at least two bills must be selected to obtain a first $10 bill?

$P(\text{the first bill is not } 10) = \$\frac{10}{15} = \frac{2}{3}$

Textbook Example 26

Let $A_i (i = 1, 2, 3)$

- What is the probability that the system does not have a type 1 defect?
  - $P(A_1^1) = 1 - P(A_1) = 0.88$
- What is the probability that the system has both type 1 and type 2 defects?
  - $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = 0.06$
- What is the probability that the system has both type 1 an type 2 defects but not a type 3 defect?
  - $P(A_1 \cap A_2 \cap A_3^1) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) = 0.05$
- What is the probability that the system has at most two of these defects?
  - P(at most 2 defects) = $1 - P$(all three defects)
  - $1 - .01 = 0.99$
- Probability of at least one defect?
  - $P(\text{at least one defect}) = P(A_1 \cup A_2 \cup A_3) =$ *use property 6*

Relative Frequency Approach to Probability

Consider an experiment that can be repeated n times in and identical and independent fashion.

Definition

The relative frequency of event A, is$f_n(A) =$ # of time A occurs / n
$f_n(A) =$ a limit value as $n \to \infty$
$P(A)$ is the limiting value

2.3 - Counting Techniques

The Product Rule

Definition

Suppose an experiment or procedure consists of k steps, and there are
n_1 ways to complete step 1,
n_2 ways to complete step 2
.
.
.
n_k ways to complete step k
Then, there are $n_1 * n_2 * \ldots * n_k$ ways to perform the experiment or complete the procedure

Example

Flip a coin and then roll a 6 faced die

1. Flip a coin: H or T (two branches)
2. Roll a die $1, 2, 3, 4, 5, 6$ (each branch from step 1 gets 6 branches)
3. Outcomes
   - $H1, H2, H3, \ldots, H6$
   - $T1, T2, T3, \ldots T6$
   - $2 * 6 = 12$

Textbook Example 32

Receiver: Kenwood, Onkyo, Pioneer, Sony, Sherwood (5)
Compact disk player: Onkyo, Pioneer, Sony, Techniques (4)

Speakers: Boston, Infinity, Polk (3)
Turntable: Onkyo, Sony, Teac, Techniques (4)

- In how many ways can one component of each type be selected?
  - $5 * 4 * 3 * 4 = 240$
- In how many ways can components be selected if both the receiver and compact disk player are to be Sony?
  - $1 * 1 * 3 * 4 = 12$
- In how many ways can components be selected if none is to be Sony?
  - $4 * 3 * 3 * 3 = 108$
- In how many ways can a selection be made if at least one Sony component is to be included?
  - All selections - selections without Sony $= 240 - 108 = 132$
- If someone flips switches on the selection in a completely random fashion, what is the probability that the system selected contains at least one Sony component?
  - $P(\text{at least one Sony}) = \frac{132}{240}$
  - Exactly one Sony component?
    - $P(\text{exactly one Sony}) = \frac{1*3*3*3+4*1*3*3+4*3*3*1}{240}$

Permutations

Definition

Let $0 < k \leq n$, k, n are integers
A k-permutation of a set of n distinct objects (elements) is an **ordered** selection of k objects of the set
$P_{k,n} =$ the number of k-permutations of n elements
It can be shown that $P_{k,n} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$
$P_{n,n} = n!$

Example

$n = 7,\ k = 3 \Rightarrow \{a, b, c, d, e, f, g\}$
Some 3-permutations: abc, cab, bca, bfe
$P_{3,7} = 7 * 6 * 5 = 7 * 6 * 5 * \frac{4!}{4!} = \frac{7!}{(7-3)!}$

Textbook Example 30

A combination lock with 4 digits 0-9
Find the # of non-repetitive combinations $\Rightarrow \frac{10!}{6!} = 10 * 9 * 8 * 7$

Combinations

> ⓘ **Definition**
>
> A k-combination of a set of n distinct elements is subset of k elements of the set
> $C_{k,n} =$ The # of k-combinations of n elements
> $\binom{n}{k}$
> $C_{k,n} = \frac{P_{k,n}}{k!} = \frac{n!}{(n-k)!*k!}$

> **Example**
>
> $\{a, b, c, d, e, f, g\}, n = 7, k = 3$
> Some 3-combinations: $\{a, b, c\} = \{b, c, a\}$
> Permutations: $abc, bca, \ldots, cba \Rightarrow 3! = 6$
> $C_{3,7} = \frac{P_{3,7}}{3!} = \frac{7!}{(7-3)!*3!}$

Textbook Example 34

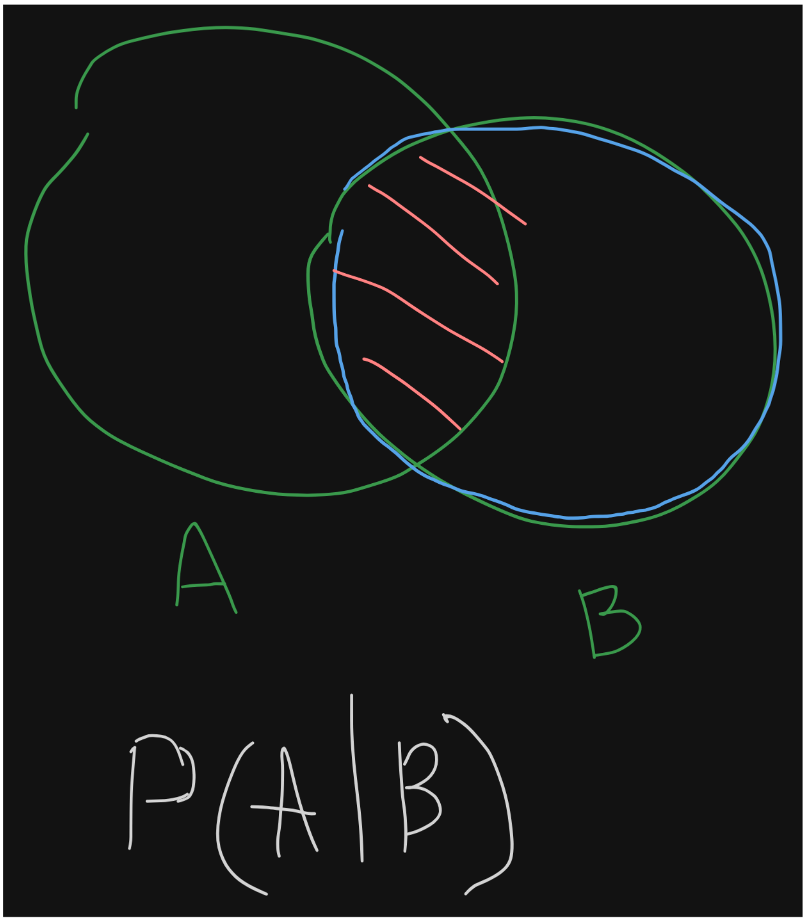
- 25 failed keyboards
  - 6 with electrical defects
  - 19 with mechanical defects
- How many ways are there to randomly select 5 of these keyboards for a thorough inspection (without regard to order)?
  - $\binom{25}{5} = 53130$
- In how many ways can a sample of 5 keyboards be selected so that exactly two have an electrical defect?
  - $\binom{6}{2} * \binom{19}{3} = 15 * 969 = 14535$
- If a sample of 5 keyboards is randomly selected, what is the probability that at least 4 of these will have a mechanical defect?
  - $P(\text{exactly } 4) + P(\text{all } 5)$
  - $\frac{\binom{19}{4}*\binom{6}{1}+\binom{19}{5}}{\binom{25}{5}}$

2.4 - Conditional Probability

- Suppose A and B are events
- $P(A|B)$ is read as "the probability of A given B"
- $P(A|B) =$ the probability of A given that B has occurred
- B is the conditioning event

> ⓘ **Definition**

For events A and B with $P(B) > 0$, $P(A|B) = \frac{P(A \cap B)}{P(B)}$



Textbook Example 50

- Given that the shirt that just sold was a short-sleeved plaid, what is the probability that its size was medium?
  - $P(M$ | short sleeve and plaid)
  - $= P(M$ & short sleeve & plaid) / P(short sleeve & plaid)
  - $= \frac{0.08}{0.04+0.08+0.03} = \frac{8}{15}$
- Given that the shirt that just sold was a medium plaid, what is the probability that it was short-sleeved?
  - P(short sleeve | M & plaid)
  - $= P$(short sleeve & M & plaid) / $P(M$ & plaid)
  - $= \frac{0.08}{0.08+0.10} = \frac{8}{18}$

Multiplication Rule

ⓘ **Definition**

For any event A and B, the $P(A \cap B) = P(A|B) * P(B)$ if $P(B) > 0$

Example

A box contains five blue balls and eight red ones. Two balls are removed, one at a time, at random without replacement. What is the probability that both balls are red?

- P(1st red & 2nd red)
- $=$ P(1st red)$*$$P$(2nd red | 1st red)
- $= \frac{8}{13} * \frac{7}{12}$

For events A, B, C, $P(A \cap B \cap C) = P(A) * P(B|A) * P(C|A \cap B)$

Example 22

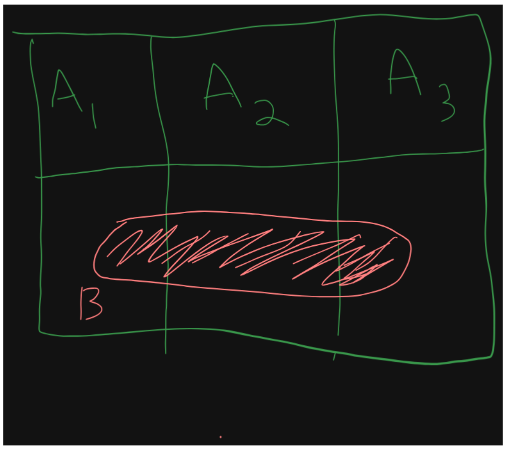
400 backup power supply units, 8 are defective

- If 3 of the units are randomly selected for testing, what is the probability that the entire batch will be accepted
  - P(1st ok & 2nd ok & 3rd ok)
  - $= \frac{392}{400} * \frac{391}{399} * \frac{390}{398} = 0.94\ldots$

The Law of Total Probability

Definition

Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint events such that $S = A_1 \cup A_2 \cup \ldots \cup A_n$. Then for any event B,
$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n)$

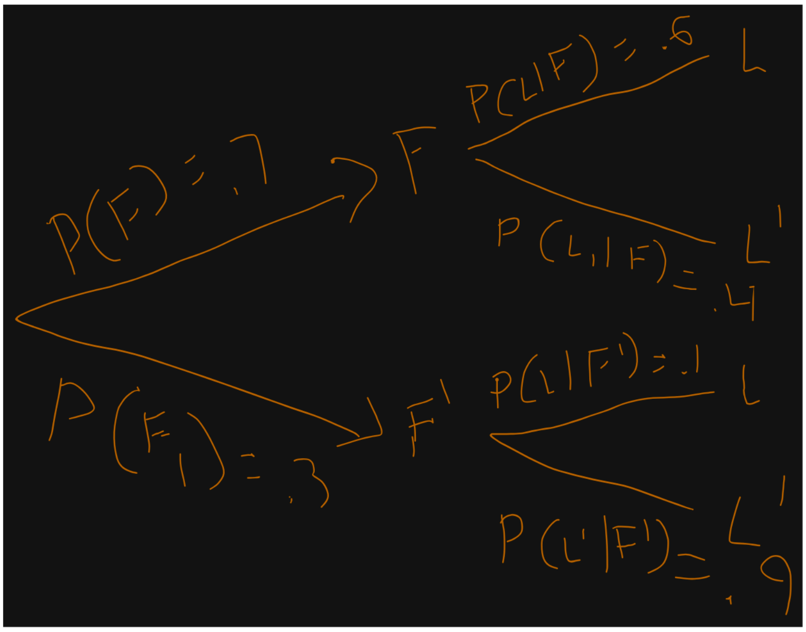


$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$
$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$

Example 60

70% discovered after disappearing
Of discovered - 60% with locator
Of not discovered - 90% without locator

F = aircraft found
F^1 = aircraft not found
L = aircraft has locator
L^1 = aircraft has no locator



$\Rightarrow P(F \cap L) = .42$
$\Rightarrow P(F \cap L^1) = .28$
$\Rightarrow P(F^1 \cap L) = .03$
$\Rightarrow P(F^1 \cap L^1) = .27$

- If is has a locator, what is the probability that it will not be discovered?
  - $P(F^1|L) = \frac{P(F^1 \cap L)}{P(L)} = \frac{.03}{P(F \cap L)+P(F^1 \cap L)} = \frac{.03}{.42+.03} = \frac{3}{45} = \frac{1}{15} \approx 0.07$
- If it does not have a locator, what is the probability that it will be discovered?
  - $P(F|L^1) = \frac{P(F \cap L^1)}{P(L^1)} = \frac{.28}{1-.45} = \frac{.28}{.55} \approx 0.51$

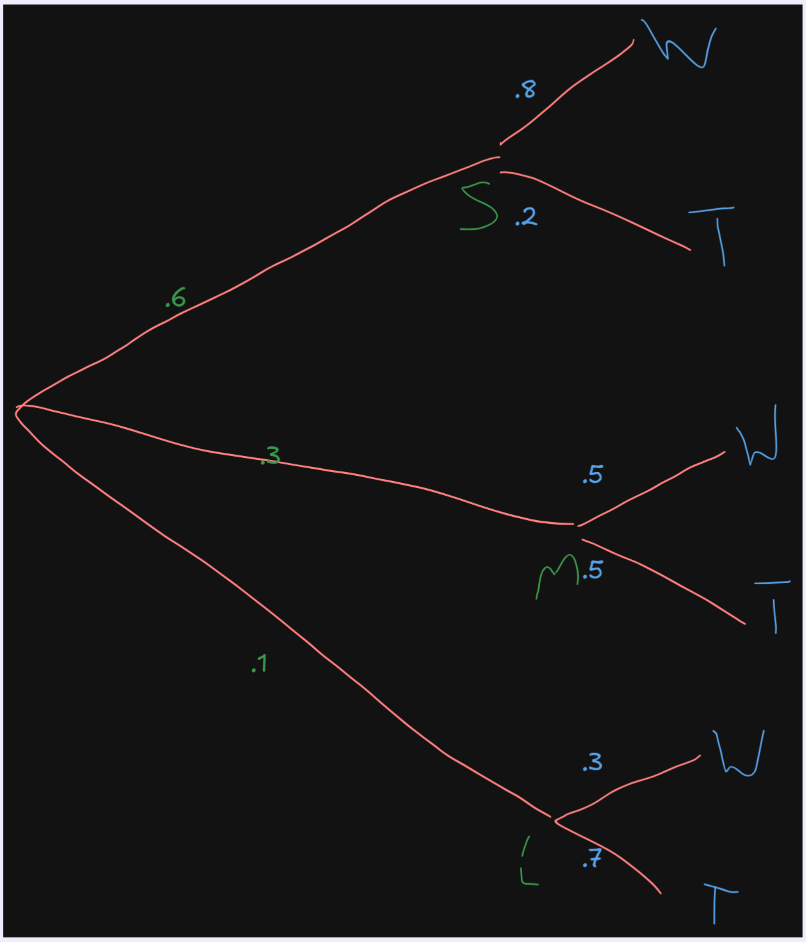
Bayes Theorem

Definition

$A_1, A_2, \ldots, A_n$ are pairwise disjoint and $S = A_1 \cup A_2 \cup \ldots \cup A_n$ with $P(A) > 0$ $i = 1, \ldots, n$ ($P(A_i)$ is prior probability of A_i). Then for with $P(B) > 0$, Posterior probability of $A_i = P(A_i|B) = \frac{P(B|A_i)*P(A_i)}{P(B|A_1)*P(A_1)+P(B|A_2)*P(A_2)+\ldots+P(B|A_n)*P(A_n)}$

Example 64

60% short, 30% medium, 10% long
short - 80% word, medium - 50% word, long, 30% word
randomly selected



S = short
M = medium
L = long
W = word
T = LaTeX

- probability that selected review was submitted in word
  - $P(W) = P(S \cap W) + P(M \cap W) + P(L \cap W) = .48 + .15 + .03 = .66$
- if in word, posterior probabilities of it being short, medium, or long
  - $P(S|W) = \frac{P(S \cap W)}{P(W)} = \frac{.48}{.66} \approx .73$
  - $P(M|W) = \frac{P(M \cap W)}{P(W)} = \frac{.15}{.66} \approx .23$
  - $P(L|W) = \frac{P(L \cap W)}{P(W)} = \frac{.03}{.66} \approx .04$

2.5 - Independence of Events

Definition

Events A and B are independent if $P(A \cap B) = P(A) * P(B)$
Otherwise, A and B are dependent

Proposition

Let A and B be events with $P(A \cap B) > 0, P(B) > 0$
A and B are independent if and only if $P(A|B) = P(A)$
$(P(B|A) = P(B))$

Example

Roll a fair die
$A =$ "the # of dots is even" $= \{2, 4, 6\}$
$P(A) = \frac{1}{2}$
$B =$ "the # of dots is divisible by 3" $= \{3, 6\}$
$P(B) = \frac{1}{3}$C=$ "at least 5 dots" $= \{5, 6\}$
$P(C) = \frac{1}{3}$
$P(A|B) = \frac{1}{2}$
$P(A) = \frac{1}{2}$, so A and B are independent
$P(B|C) = \frac{1}{2}$
$P(B) = \frac{1}{3}$, so B, C are dependent

???

Definition

Events $A_1, A_2, \ldots, A_n$ are mutually independent for any subset of indices$i_1, i_2, \ldots, i_n$, where $m = 1, 2, \ldots, n$
$P(A_{i_1}, A_{i_2}, \ldots, A_{i_m}) = P(A_{i_1}) * P(A_{i_2}) * \ldots * P(A_{i_m})$

Example

A, B, C are events $\Rightarrow$ mutually independent
$P(A \cap B) = P(A)P(B)$
$P(A \cap C) = P(A)P(C)$
$P(B \cap C) = P(B)P(C)$
$P(A \cap B \cap C) = P(A)P(B)P(C)$

Example 82

red, green dice
Let A be red 3, B be green 4, C be total on both dice is 7
Are these pairwise independent?
$A = \{(3,1),(3,2),\ldots,(3,6)\}$
$B = \{(1,4),(2,4),\ldots,(6,4)\}$
$C = \{(1,6),(2,5),(3,4),(6,1),(5,2),(4,3)\}$
$P(A) = P(B) = P(C) = \frac{6}{36} = \frac{1}{6}$
$P(B|A) = \frac{|\{(3,4)\}|}{6} = \frac{1}{6} = P(B)$
$P(B \cap C) = \frac{1}{36} = P(B) * P(C)$
$P(A|C) = \frac{1}{6}$
Yes
Are these mutually independent?
$P(A \cap B \cap C) = \frac{|\{3,4\}|}{36} = \frac{1}{36} \neq P(A)P(B)P(C) \Rightarrow No$

Multiplication Rule for Independent Events

Definition

If events $A_1, A_2, \ldots, A_n$ are independent, then
$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) * P(A_2) * \ldots * P(A_n)$

Tip

Mutually independent = independent

Example 80

System of components: 1 and 2 connected in parallel (subsystem works iff either 1 or 2 work); since 3 and 4 connected in series, that subsystem works. iff both 3 and 4 work.
Components work independently of one another
P(component i works)$= 0.9$ for $i = 1,2$ and $= 0.8$ for $i = 3,4$; calculate P(system works)

- $sys \Rightarrow sub1, sub2$
- $P(sys) = P(sub1 \cup sub2) = 1 - P(sub1^1 \cap sub2^1)$
- $= 1 - P(sub1^1) * P(sub2^1)$
- $P(sub1^1) = P(com1^1 \cap com2^1) = (0.1)^2 = 0.01$
- $P(sub2^1) = 1 - P(sub2) = 1 - (0.8)^2 = 0.36$

- $P(sys) = 1 - 0.01 * 0.36 = 0.9964$