Chapter 5

5.1 - Jointly Distributed Random Variables

Joint Probability Mass Function

Definition

Suppose X and Y are defined on the same sample space of an experiment

$$p(x,y) = P(X = x \, and \, Y = y)$$

$$\sum_x \sum_y p(x,y) = 1$$

where (x, y) is any ordered pair of real numbers

Example

Roll two fair dice green and blue
$X =$ the number of dots on the green die
$Y =$ the number of dots on the blue die
$S = \{(1,1), \ldots, (6,6)\}$
$p(x,y) = \{^{\frac{1}{36},\; x,y=1,2,...,6}_{0,\; otherwise}\}$

Example 75

2. $P(X \leq 15 \, and \, Y \leq 15)$
   - $p(12,12) + p(12,15) + p(15,12) + p(15,15) = .25$

Marginal Probability Mass Function

Definition

$$p_X(x) = \sum p(x,y)$$

$$y : p(x,y) > 0$$

x is a possible value of X

$$p_Y(y) = \sum p(x,y)$$

$$x : p(x,y) > 0$$

y is a possible value of Y

Example 75

The marginal pmf of X:

x	12	15	20
$p_X(x)$	$.05 + .05 + .1 = .2$	$.05 + .1 + .35 = .5$	$0 + .2 + .1 = .3$

The marginal pmf of Y:

y	12	15	20
$p_Y(y)$	$.1$	$.35$	$.55$

Independence of Two Random Variables

Definition

Rv's X and Y are independent if

$$p(x,y) = p_X(x) * p_Y(y)$$

for any (x,y)

5.2 - Expected Values

Definition

Suppose $p(x,y)$ is the joint pmf of X and Y. For $h(X,Y)$,

$$E(h(X,Y)) = \sum_x \sum_y h(x,y)p(x,y)$$

provided the sum exists

Example 24

$(A,B) \Rightarrow$ (seat for A, seat for B)
6 seating options
$P(A = 1, B = 2) = \frac{1}{30}$

$X =$ the A's seat number, $Y =$ the B's seat number
$E(h(X,Y)) = 12(2) * \frac{1}{30} + 12(3) * \frac{1}{30} + 6(4) * \frac{1}{30} = 2.8$

Proposition

Let X and Y be rvs with the joint pmf $p(x,y)$ and expected values $E(X)$ and $E(y)$
For real numbers a and b,

$$E(aX + bY) = aE(X) + bE(Y)$$

Example 75

4. $E(X+Y)$
    - $E(X+Y) = E(X) + E(Y)$
    - $E(X) = 12(.2) + 15(.5) + 20(.3) = 15.9$
    - $E(Y) = 12(.1) + 15(.35) + 20(.55) = 17.45$
    - $E(X+Y) = \$33.35$

5.4 -

$$\sigma_X = \frac{\sigma}{\sqrt{n}}$$

5.5 - The Distribution of a Linear Combination

Definition

Let $X_1, X_2, \ldots, X_n$ be rvs
A linear combination of $X_1, X_2, \ldots, X_n$ is a rv $y = a_1X_1 + a_2X_2 + \ldots + a_nX_n$ where $a_1, a_2, \ldots a_n$ are real numbers

Proposition

Given the RVs $X_1, X_2, \ldots, X_n$ with their means $\mu_1, \mu_2, \ldots, \mu_n$ and the variances $\sigma_1^2, \sigma_2^2, ..\sigma_n^2$

1. $E(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n$
2. If $X_1, X_2, \ldots, X_n$ are independent, then
   $V(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1\sigma_1^2 + a_2\sigma_2^2 + \ldots + a_n\sigma_n^2$

Example

Let $X_1, X_2, \dots, X_n$ be *iid rv*s with mean μ and σ^2

$$E(\overline{X} = \mu)$$

$$V(\overline{X}) = \frac{\sigma^2}{n}$$

Example

Given rvs X_1, X_2 and $\mu_1 = 1, \mu_2 = 2$ and $\sigma_1 = 3, \sigma_2 = 4$
Find $E(X_1 - 2X_2), V(X_1 - 2X_2)$
Assume that X_1 and X_2 are independent
$E(X_1 - 2X_2) = E(X_1) - 2E(X_2) = 1 - 2 * 2 = -3$
$V(X_1 - 2X_2) = 1^2 V(x_1) + (-2)^2 V(X_2)$
$= 1 * 3^2 + 4 * 4^2$

Proposition

Let $X_1, X_2, \dots, X_n$ be independent rvs such that

$$X_i \sim N(\mu_i, \sigma_i^2)$$

where $i = 1, 2, \dots, n$
Then,

$$Y = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

and

$$Y \sim N(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2)$$

Example 60

Given

- Six cylinder cars
  - X_1, X_2
  - $E(X_1) = E(X_2) = 22$
  - $\sigma_{x_1} = \sigma_{x_2} = 1.2$
- Four cylinder cars
  - X_3, X_4, X_5

- $E(X_3) = E(X_4) = E(X_5) = 26$
- $\sigma_{X_3} = \sigma_{X_4} = \sigma_{X_5} = 1.5$

- Consider $Y = \frac{X_1+X_2}{2} - \frac{X_3+X_4+X_5}{3}$
- Find $P(0 \leq Y), P(Y > -2)$, assuming that X_is are normally distributed
  - So $Y \sim N(\mu_Y, \sigma_Y^2)$
  - $E(Y) = \mu_Y = \frac{1}{2}(22+22) + \frac{-1}{3}(26+26+26) = 22 - 26 = -4$
  - $V(Y) = (\frac{1}{2})^2(1.2^2 + 1.2^2) + (\frac{-1}{3})(1.5^2 + 1.5^2 + 1.5^2)$
  - $= \frac{1.2^2}{2} + \frac{1.5^2}{3}$
- $P(Y \geq 0) = 1 - P(Y < 0)$
  - $= 1 - \phi(\frac{0-(-Y)}{\sqrt{1.47}}) = 1 - \phi(3.30)$
  - $1 - .9995 = .0005$
- $P(Y > -2) = 1 - P(Y \leq -2) = 1 - \phi(\frac{-2+4}{\sqrt{1.47}})$
  - =$1 - \phi(1.65) = 1 - .9505 = .0495$