# Exam 2 Review

* Eigenvalues and Eigenvectors
  - Eigenvalue: $A\vec{v} = \lambda\vec{v}$ OR $(A-\lambda I)\vec{v} = \vec{0}$
    - Multiplicity of 1 guarantees eigenvector that is linearly independent
  - Characteristic polynomial is found by $\det(A-\lambda I)$, solve for eigenvalue
  - Determinant of a matrix is also product of eigenvalues.

$$\det(P^{-1}AP) = \det(D)$$
$$\det(A) = \det(D)$$
$$= \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

* Diagonalization
  - An $n \times n$ matrix is diagonalizable iff it has $n$ linearly independent eigenvectors
  - $P^{-1}AP = D$ iff (A is diagonalizable)
    - The columns of P are $n$ linearly independent eigenvectors of A
    - The diagonal entries of D are the eigenvalues corresponding to those eigenvectors
    - verify by checking if P is invertible and $AP = PD$
  - Find kth power of diagonalizable matrix:
    - $P^{-1}AP = D$

$$A = PDP^{-1}$$
$$A^2 = (PDP^{-1})(PDP^{-1})$$
$$A^2 = PD^2P^{-1}$$
$$\vdots$$
$$A^k = PD^kP^{-1}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \\ 0 & 0 & \ddots & \\ & \vdots & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots \\ 0 & \lambda_2^k & 0 & \\ 0 & 0 & \ddots & \\ & \vdots & & \lambda_n^k \end{bmatrix}$$

* Vector Spaces
  - Ex: Is the set of vectors $\{\langle s, t, 2t, s^2\rangle$ for $t, s, \mathbb{R}\}$ a vector space?

    No: $s = t = 1$ $\quad \langle 1, 1, 2, 2\rangle \times 2 = \langle 2, 2, 4, 4\rangle$

    $s = t = 2$ $\quad \langle 2, 2, 4, 4\rangle$ not equal $\therefore$ not closed under scalar multiplication
  - All vector spaces are closed under addition and scalar multiplication
  - A subspace is a vector space, etc with the $\vec{0}$ vector

* Linear Independence, Span, and Basis
  - Span: set of all scalar multiples of vector $\vec{v}$
    - Check if $n$ vectors span $\mathbb{R}^n$ if they are linearly independent
    - Check if $\vec{v}$ is in span by checking if $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{v}$ has a solution

- Linear Independence
    - The trivial solution is the only solution to the homogeneous system (is invertible!)
    ↳ $t_1\vec{v}_1 + t_2\vec{v}_2 + \dots t_k\vec{v}_k = \vec{0}$ has only the trivial solution
  ↱ Can find coordinates by solving linear system $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_k]\begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix} = \vec{v}$
- Basis: set of vectors used in linear combinations to form vector space
    - and # of bases determine dimension of a subspace of $\mathbb{R}^n$

* Row, Column, and Nullspace
  - Rowspace: space spanned by rows of A
  - Columnspace: space spanned by columns of A
  - Nullspace is solutions to homogeneous system $A\vec{v} = \vec{0}$
  - Ex: $A = \begin{bmatrix} 2 & 2 & 1 & -44 \\ 1 & 1 & 0 & -1 \\ 3 & 3 & 1 & -45 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & 1 & -44 \\ 3 & 3 & 1 & -45 \end{bmatrix} \begin{matrix} \\ -2R_1 \\ -3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -42 \\ 0 & 0 & 1 & -42 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix}$

    ↳ $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -42 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ RREF!   ∴ row (A) = $\{\langle 1,1,0,-1\rangle, \langle 0,0,1,-42\rangle\}$
    col (A) = $\{\langle 2,1,3\rangle, \langle 1,0,1\rangle\}$
    null (A) = $\{\langle -1,1,0,0\rangle, \langle 1,0,42,1\rangle\}$

    $\begin{cases} x_1 = -x_2 + x_4 \\ x_3 = 42x_4 \end{cases} \sim \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 42 \\ 1 \end{bmatrix}$   rank = 2   nullspace = 2

  - Rank: # of leading 1's
  - Nullity: # of nonleading 1's

  $\Big\}$ rank + nullspace = total # columns
  - If A is $n \times n$, then rank (A) + dim(null(A)) = n
  - Nullspace is orthogonal to rowspace: dot products equal 0