Eigenvalues & Eigenvectors

* Eigenvector: Nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ (eigenvalue)
  ↳ $A\vec{v} = \lambda\vec{v}$
  $A\vec{v} - \lambda\vec{v} = \vec{0}$
  $A\vec{v} - \lambda I\vec{v} = \vec{0}$
  $(A-\lambda I)\vec{v} = \vec{0}$ (homogeneous system for $\vec{v}$)
  - $A-\lambda I$ must have a nontrivial solution
  - $A-\lambda I$ must have at least one free variable
  - $A-\lambda I$ cannot be invertible

  ↳ Need nontrivial solution, where $A-\lambda I$ must be singular/noninvertible
  $\therefore$ The determinant of $A-\lambda I = 0$
* λ is an eigenvalue of A iff the determinant of $A-\lambda I = 0$
* The eigenvectors associated with λ are the basic solutions of $(A-\lambda I)\vec{v} = 0$
  ↳ Recap: Basic solution is the parameterized solution to homogeneous system
- Ex: Find the eigenvalues of $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)(-\lambda) - (-2)(1) = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda-1)(\lambda-2) = 0$$
$$\lambda = 1, 2$$

$$\frac{3 \pm \sqrt{9-8}}{2} = \frac{3}{2} \pm \frac{1}{2}$$

↳ λ is just subtracted from entries on main diagonal

Find the eigenvectors associated with $\lambda = 1$

$$\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right] -2R_2 \rightsquigarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightsquigarrow x - y = 0 \text{ OR } x = y, \text{ where } y \text{ is free } (y = a$$

$x = y = a$

$\vec{v} = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector

Check:

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

Find the eigenvectors associated with $\lambda = 2$

$$\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow x - 2y = 0 \rightsquigarrow x = 2y$$ where y is free variable

Pick $y = 1 \rightsquigarrow x = 2$ $\therefore$ Eigenvector $= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ * Scalar multiple of result is 2 because $\lambda = 2$

- Ex: $A = \begin{bmatrix} 7 & 0 \\ 39 & 4 \end{bmatrix}$ Find the eigenvalues and eigenvectors of A

$$\begin{vmatrix} 7-\lambda & 0 \\ 39 & 4-\lambda \end{vmatrix} = (7-\lambda)(4-\lambda) - 0 = 0$$

$$(\lambda-7)(\lambda-4) = 0 \rightsquigarrow \boxed{\lambda = 4, 7}$$

* The eigenvalues of a square triangular matrix are the entries of the main diagonal

↳ $\lambda = 4$

$$\left[\begin{array}{cc|c} 3 & 0 & 0 \\ 39 & 0 & 0 \end{array}\right] \xrightarrow{-13R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left.\begin{array}{l} x = 0 \\ \text{Pick } y = 1 \end{array}\right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{v}$$

↳ y is free due to 0 column

↳ $\lambda = 7$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 39 & -3 & 0 \end{array}\right] \quad \left.\begin{array}{l} 39x - 3y = 0 \\ 13x - y = 0 \end{array}\right\} \begin{array}{l} \text{Pick } x = 1 \\ \hookrightarrow y = 13 \end{array} \quad \begin{bmatrix} 1 \\ 13 \end{bmatrix} = \vec{v}$$