- $\det(A - \lambda I) = 0$ gives quadratic equation (up to 2 solutions) for 2×2 matrices $\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)$
  gives cubic equations (up to 3 solutions) for 3×3 matrices
  gives polynomials (up to n solutions) for n×n matrices
  ↳ n^{th} degree polynomial
  because the entries of the main diagonal are multiplied by each other
- For quadratic equations. $(ax^2 + bx + c = 0)$
  - 2 solutions: $b^2 - 4ac > 0$
  - 1 solution: $b^2 - 4ac = 0$
  - 0 solutions: $b^2 - 4ac < 0$ (complex conjugate solutions)

* If A is an $n \times n$ matrix, then the characteristic polynomial of A is
$C_A(x) = \det(xI - A)$

* Let A be a square matrix. The eigenvalues of A are the roots of the characteristic polynomial
  - The roots of a polynomial are the values of x such that the polynomial evaluates to $\varnothing$

* The eigenvectors are the nonzero solutions to $(\lambda I - A) = (A - \lambda I)\vec{v} = \vec{0}$

- Ex: find the eigenvalue(s) and eigenvector(s) of $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$\det(A - \lambda I) = |A - \lambda I| = \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0 \rightsquigarrow (3-\lambda)(1-\lambda) + 1 = 0$

$\lambda^2 - 4\lambda + 4 = 0$ ← eigenvalue

$(\lambda - 2)^2 = 0 \rightsquigarrow \boxed{\lambda = 2}$

(multiplicity of 2)

↳ $\lambda = 2$

$\begin{bmatrix} 3-2 & -1 \\ 1 & 1-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x - y = 0$

↳ zero row! (proves there is an eigenvector)

↳ y is free variable, $x = 1 \rightsquigarrow y = 1$

↳ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ← eigenvector corresponding with $\lambda = 2$

* λ has multiplicity m if it occurs m times as a root of the characteristic polynomial

- Ex: find the eigenvalue(s) and eigenvector(s) of $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 2 \\ 1 & 0 & -2-\lambda \end{bmatrix}$

$|A - \lambda I| = -\lambda \begin{vmatrix} 2-\lambda & 2 \\ 0 & -2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)(-2-\lambda) = 0$

↳ $-(1+\lambda)(\lambda-2)(\lambda+2) = 0 \rightsquigarrow \lambda = -1, 2, -2$

↳ $\lambda = -1$

$\left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{matrix} \\ -R_3 \\ \\ \end{matrix} \rightsquigarrow \left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{cases} x - z = 0 \\ y + z = 0 \\ z \text{ is free, pick } z = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ corresponds with $\lambda = -1$

zero row!

↳ $\lambda = 2$

$\left[\begin{array}{ccc:c} -3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & -4 & 0 \end{array}\right] \begin{matrix} \\ \\ +2R_2 \end{matrix} \rightsquigarrow \left[\begin{array}{ccc:c} -3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ +\frac{1}{3}R_1 \\ +R_1 \end{matrix} \rightsquigarrow \left[\begin{array}{ccc:c} -3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{cases} x = 0 \\ 2z = 0 \\ y \text{ is free, } y = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

↳ zero row!

↳ $\lambda = -2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x = 0 \\ 2y + z = 0 \\ y \text{ is free, pick } y = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \text{ corresponds with } \lambda = -2$$

↳ zero row!

↳ can pick any variable and still get scalar multiple of eigenvector

- Ex: Find the eigen value(s) and eigenvector(s) of $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & -\lambda & -2 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\left[(-\lambda)(3-\lambda) + 2\right] = 0$$

↳ $(2-\lambda)\left[\lambda^2 - 3\lambda + 2\right] = 0$

↳ $(2-\lambda)(\lambda - 1)(\lambda - 2) = 0$

↳ $\lambda = 1, 2$ (multiplicity of 2!)

↳ $\lambda = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} +2R_2 \\ \rightsquigarrow \\ +R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x - 2z = 0 \\ y + 2z = 0 \\ \text{pick } z = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \text{ for } \lambda = 1$$

↳ $\lambda = 2$

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} -2R_3 \\ +2R_3 \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{cases} y + z = 0 \rightsquigarrow y = -z \\ x \text{ is free} \rightsquigarrow x = t \\ \text{pick } z \text{ is free} \rightsquigarrow z = s \end{cases}$$

↳ $\vec{v} = \begin{bmatrix} t \\ -s \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ ∴ 2 basic solutions $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

* Multiplicity of 2 produces 2 eigenvectors for $\lambda = 2$ (<u>not</u> scalar multiples of each other)