1.4 Predicates and Quantifier

Let $P(x)$ be the statement "x is greater than 3"

x is the variable and "is greater than 3" is the predicate

$P(x)$ is "x likes math"

Domain: Everyone in this room

P(Albert) is "Albert likes math"

the proposition

A predicate $P(x)$ is also called a propositional function

$P(x)$ only possible outputs are T or F

for any x in Domain

Let $Q(x,y)$ be the statement "$x = y+1$"

$Q(1,1)$ is "$1 = 1+1$"

so $Q(1,1)$ is false

$Q(1,0)$ is "$1+0=1$" so $Q(0,1)$ is true

Definition: The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain" we write it as $\forall x\, P(x)$. $\forall$ is the universal quantifier. $\forall x\, P(x)$ reads as "for all x $P(x)$, or "For every x $P(x)$"

An x for which $P(x)$ is false is called a counterexample

ex. Let $P(x)$ be "$x+1 > x$"

Let the domain be $\mathbb{R}$

$\forall x\, P(x)$ is "For every x in $\mathbb{R}$ $x+1 > x$"

true

Proof

We know $1>0$ is true

$\Rightarrow 1+x > 0+x$

$\Rightarrow x+1 > x$

let $P(x)$ be "$2x > x$"

The domain is $\mathbb{R}$

$\forall x\, P(x)$ is "for every x in $\mathbb{R}$, $2x > x$."

$2(0) > 0$

$0 > 0$ false

$x=0$ is a counter example

Definition

The existential quantification of $P(x)$ is the statement "There exists an element x in the domain such that $P(x)$."

We write it as $\exists x P(x)$. $\exists$ is called the existential quantifier

$\exists x P(x)$ is true when $P(x)$ is true for some x. It is false when $P(x)$ is true for some x.

It is false when $P(x)$ is false for every x

ex. let $P(x)$ be "$x = x+1$"

Domain: $\mathbb{R}$

$\underbrace{\exists x P(x)}_{\text{false}}$ is "There exists an x in $\mathbb{R}$ s. t. $x = x+1$

$x \neq x+1$

$\forall$ and $\exists$ have higher precedence than all other logical operator

$P(x)$ is called the scope of $\forall x$

In $\forall x P(x)$, the x is bound

suppose 1 is the domain. If you set $x=1$, then $P(x)$ is bound

$P(x)$ is called free

$\exists x (x+y=1)$ x is bound and y is free

$\forall x (P(x) \wedge Q(x))$ $\quad \forall x P(x) \wedge \forall x Q(x)$

negation

$\neg \exists x P(x) \equiv \neg(\exists x P(x)) \equiv \forall x \neg P(x)$

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

$\exists x \neg P(x)$ $\quad \forall x (\neg P(x))$

ex. show that $\neg \forall x (P(x) \rightarrow Q(x)) \equiv \exists x (P(x) \wedge \neg Q(x))$

Proof

$\neg \forall x (P(x) \rightarrow Q(x)) \equiv \exists x \neg (P(x) \rightarrow Q(x)) \equiv \exists x \neg(\neg P(x) \vee Q(x))$

$\equiv \exists x (\neg(\neg P(x)) \vee \neg Q(x)) \equiv \exists x (P(x) \vee \neg Q(x))$

Worksheet

① For what truth value assignments to A and B is $((A \wedge B) \to A) \to (A \vee B)$ true?

truth table can be used for these questions.

② negate the statement "$p \to (q \vee (r \wedge s))$ (answer can only have $\neg, \vee, \wedge$

switch these columns

A	B	$A \wedge B$	$(A \wedge B) \to A$	$A \vee B$	$((A \wedge B) \to A) \to (A \vee B)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	F

$\neg(p \to (q \vee (r \wedge s)))$

$p \wedge \neg(q \vee (r \wedge s))$

$p \wedge (\neg q \wedge \neg(r \wedge s))$

$p \wedge (\neg q \wedge (\neg r \vee \neg s))$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$

$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

☆ $(A \wedge B) \vee (C \wedge D) \equiv (A \vee C) \wedge (A \vee D) \wedge (B \vee C) \wedge (B \vee D)$

$(A + B) \cdot (C + D)$

② negate the statement $(\neg p \to q) \wedge (p \to q)$ (can only have $\neg, \wedge, \vee$)

$\neg((\neg p \to q) \wedge (p \to q))$

$(\neg(\neg p \to q) \vee \neg(p \to q))$

$(\neg p \wedge \neg q) \vee (p \wedge \neg q)$

$\neg(p \vee q) \vee \neg(\neg p \vee q)$

$\neg((p \vee q) \wedge (\neg p \vee q))$

$\neg(p \to q)$

$\equiv p \wedge \neg q$

$(p \to q)$

$\equiv \neg p \vee q$

$\equiv \neg(\neg p \to q) \vee \neg(p \to q)$

$\equiv [\neg(\neg(\neg p) \vee q)] \vee [\neg(\neg p \vee q)]$

$\equiv [\neg(p \vee q)] \vee [\neg(\neg p \vee q)]$

$\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg q)$

③ show that $(p \wedge q) \vee \neg p \equiv p \to q$ (no truth table)

$(p \vee q) \vee \neg p$

$(\neg p \to q) \vee \neg p$

$\neg(p \vee q) \to \neg p$

$(\neg p \wedge \neg q) \to \neg p$

$(p \wedge q) \vee \neg p$

$\equiv (p \vee \neg p) \wedge (q \vee \neg p)$ distribute

$\equiv T \wedge (q \vee \neg p)$

$\equiv q \vee \neg p \equiv \neg p \vee q \equiv p \to q$ $\square$

④ show $\neg((p \wedge q) \to r) \equiv p \wedge q \wedge \neg r$

$\neg((p \wedge q) \to r)$

$\equiv \neg(\neg(p \wedge q) \vee r)$

$\equiv \neg((\neg p \vee \neg q) \vee r)$

$\equiv (\neg(\neg p \vee \neg q) \wedge \neg r)$

$\equiv p \wedge q \wedge \neg r$ $\square$

Assignment 2

1) show that $(p \to r) \wedge (q \to r)$

$\equiv (p \vee q) \to r$

by using $p \to q \equiv \neg p \vee q$

$r \to s \equiv \neg r \vee s$

Pf:

$(p \vee q) \to r$

$\equiv \neg(p \vee q) \vee r$

$\equiv (\neg p \wedge \neg q) \vee r$

distribute!

1.6

ex. If I study (p), then I will do well on an exam (q). If I do well on an exam then I will get an A in the course (r). Therefore, if I study, then I will get an A in the course.

Valid argument (Has a valid argument form

$p \to q$

$\underline{q \to r}$ hypo.

$\therefore p \to r$ syllogism

Fallacies (Invalid argument forms)

① $((p \to q) \wedge q) \to p$ is not tautology

when p is false and q is true, the implication is false

$$\begin{array}{l} p \to q \\ \underline{q} \\ \therefore p \end{array} \quad X$$

② $\overbrace{((p \to q) \wedge \neg p)}^{T} \to \overbrace{\neg q}^{F}$ is false when p false q true

$\neg p$ true $\neg q$ false

$$\begin{array}{l} p \to q \\ \underline{\neg p} \\ \therefore \neg q \end{array} \quad X$$

denying the hypothesis

Rules of Inference for quantified statement

Rule of Inference	name
$\underline{\forall x P(x)}$ $\therefore P(c)$ for some c in the domain	Universal Instantiation ↳ general to specific
$\underline{P(c) \text{ for any arbitrary } c}$ $\therefore \forall x P(x)$	Universal Generalization ↳ specific to general
$\underline{\exists x P(c)}$ $\therefore P(c)$ for some element x in the domain	Existential Instantiation
$\underline{P(c) \text{ for some } c \text{ in the domain}}$ $\therefore \exists x P(x)$	Existential Generalization

$\overbrace{P(a) \to \neg Q(a)}^{T}$

Universal Modus Ponens ← universal instantiation + modus ponens

$\forall x\,(P(x) \to Q(x))$

P(a), where a is a particular element in the domain

$\therefore Q(a)$

Universal Modus Tollens ← univ. instant. + modus tollens

$\forall x\,(P(x) \to Q(x))$ $\quad P(a) \to Q(a)$

$\neg Q(a)$

$\therefore \neg P(a)$

p: you send me an email

q: I will finish writing the program

r: I will go to bed early

s: I will wake up refreshed

Valid argument

1. $p \to \neg q$ (premise)
3. $\neg p \to r$ (premise)
5. $r \to s$ (premise)

$\therefore \neg q \to s$

2. $\neg q \to \neg p$ (modus tollens of 1.)
4. $\neg q \to r$ (hyp. syllogism of 2.)
6. $\neg q \to s$ (hyp. syllogism of 4. and 5.)

↳ says "If I do not finish writing the program, then I will wake up refreshed

Show that the premises "A student in this class has not read the textbook,"
"Everyone in this class passed the course."
lead to the conclusion "Someone in this class did not read the book and passed the course."

$C(x)$ is "x is in this class"

$B(x)$ is "x has read the textbook"

$P(x)$ is "x passed the course"

premise	1. $\exists x\,(C(x) \wedge \neg B(x))$
exist. inst. of 1	2. $C(a) \wedge \neg B(a)$
simplification	3. $C(a)$
premise	4. $\forall x\,(C(x) \to P(x))$
univ. inst. of 4	5. $C(a) \to P(a)$
modus ponens from 3&5	6. $P(a)$
simplification from 2	7. $\neg B(a)$

8. $P(a) \wedge \neg B(a)$ conjunction of 6 & 7

9. $E_x\,(P(x) \wedge \neg B(x))$ exist. gen. from 8

1.7 Introduction to proofs

A theorem is a statement that can be shown to be true

The truth of a theorem is established with a proof

True statements are used in a proof are: axioms, premises of the theorem, and previously established theorems

A lemma is a preliminary theorem proven before a main theorem

A corollary is a theorem that follows from a theorem

A conjecture is a statement that is proposed to be true

Once a conjecture is proven, it becomes a theorem

$\boxed{\text{Def}}$ An even integer n is an integer with the form $n=2k$, where k is an integer

An odd integer n is an integer with the form $n=2k+1$, where k is an integer

$\boxed{\text{Direct Proof}}$ $(p \to q)$

ex. Prove the theorem "If n is an odd integer, then n^2 is also an odd integer

Pf

Since n is an odd integer, $n=2k+1$, where k is an integer, now, $n^2=(2k+1)^2=4k^2+4k+1=2\underbrace{(2k^2+2k)}_{\text{integer}}+1$

ex. Prove the statement "If m and n are perfect squares, then $n \cdot m$ is also a perfect square"

Pf

Since m and n are perfect squares, we have $m=s^2$ and $n=t^2$, where s and t are integers. And so, $(m)\cdot(n) = (s^2)(t^2) = s \cdot s \cdot t \cdot t = s \cdot t \cdot s \cdot t = (s \cdot t)^2$. Since $s \cdot t$ is an integer, we conclude that $m \cdot n$ is a perfect square

Proof by contraposition $(\neg q \to \neg p)$

ex. Prove the statement "Suppose n is an integer. If $3n+2$ is odd, then n is odd

Pf

Suppose n is not odd. Which means n is even, then $n=2k$, where k is an integer.

And so, $3n+2 = 3(2k)+2) = 2(3k+2) = 2(3k+1)$

And so, $3n+2$ is not odd

Proof by contradiction

We want to show $p \to q$

If you can show that p is true and $\neg q$

leads to a contradiction, then it must be that $p \to q$ is true

ex. Prove the statement "suppose there is an integer. If $3n+2$ is odd, then n is odd"

Pf Suppose $3n+2$ is odd

Also, Suppose n is not odd. Then n is even

$\Rightarrow n = 2k$

$\Rightarrow 3n+2 = 3(2k)+2 = 2(3k)+2 = 2(3k+1)$

$\Rightarrow 3n+2$ is even

But this is a contradiction

$3n+2$ is odd and

$3n+2$ is even

Therefore, the statement is true