Span & Linear Independence

- A subset V of $\mathbb{R}^n$ is a subspace if
  1. The zero vector is in V
  2. If $\vec{x} \in V$ and $\vec{y} \in V$, then $\vec{x}+\vec{y} \in V$ ("closed under addition")
  3. If $\vec{x} \in V$ and $a \in \mathbb{R}$, then $a \times \vec{v} \in V$ ("closed under multiplication")
- Ex: The set of all scalar multiples of the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$

  $a_1\begin{bmatrix}1\\2\\3\end{bmatrix} + a_2\begin{bmatrix}1\\2\\3\end{bmatrix} = (a_1+a_2)\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the set (closed under addition ✓)

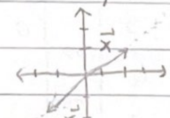
  By definition, closed under multiplication ✓

  $\therefore$ Is a subspace

* The set of all scalar multiples of a vector $\vec{v}$ is a subspace called span $(\vec{v})$ or span $\{\vec{v}\}$ or span $(\{\vec{v}\})$

- Ex: span $\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ is the set of all vectors with a 0 as the second component
  $a\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix}$ for any real a

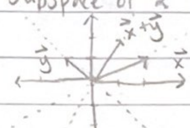
- Ex: $\vec{x} = \langle 2, 1\rangle$



Span $(\vec{x}) =$ span$(\langle 2,1\rangle)$

* The span of a nonzero vector gives a line through the origin

- Ex: Subspace of 2 vectors $\vec{x} = \langle 2, 1\rangle$ and $\vec{y} = \langle -1, 1\rangle$



(all possible combinations of $a_1\vec{x} + a_2\vec{y}$ forms a plane)

Proof: $a_1\vec{x} + a_2\vec{y} = \vec{b}$

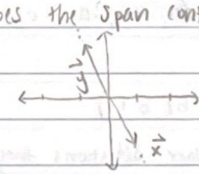
$$a_1\begin{bmatrix}2\\1\end{bmatrix} + a_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \quad \begin{cases}2a_1 - a_2 = b_1\\ a_1 + a_2 = b_2\end{cases}$$

$$\left[\begin{array}{cc:c}2 & -1 & b_1\\ 1 & 1 & b_2\end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c}1 & 1 & b_2\\ 2 & -1 & b_1\end{array}\right]_{-2R_1} \rightsquigarrow \left[\begin{array}{cc:c}1 & 1 & b_2\\ 0 & -3 & b_1 - 2b_2\end{array}\right]$$

No zero row; so there is a solution and the result is the whole plane

- Ex: $\vec{x} = \langle 1, -2\rangle$ and $\vec{y} = \langle -1, 2\rangle$ Does the span containing the two vectors include the whole plane?

$$\det\left(\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}\right) = 2-2 = 0$$



Since $\vec{x}$ and $\vec{y}$ are scalar multiples, the result is only a line, not the whole plane

* The span of 2 vectors (which are not scalar multiples of each other) gives a plane, no matter the dimension
  - Ex: 2 3D vectors will still give a plane

* A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is a sum of scalar multiples of the vectors
  $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k$ for real numbers $a_1, a_2, \ldots a_k$

* The span of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$
  span $(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k) =$ span $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} =$ span $(\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\})$

- Ex: Span$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ is the set of all vectors in $\mathbb{R}^3$ with the third component as 0
  $a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}a_1\\a_2\\0\end{bmatrix}$

- Ex: Span$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$ does not give the entirety of $\mathbb{R}^3$ because the third vector was already in the span of the first two! $a_1 = -1$, $a_2 = 1$ $\rightsquigarrow \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$

- Ex: Is $\begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}$ in the span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$?

↳ $a_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}$ ← system of equations!

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ -1 & -2 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & 3 & 10 \end{array}\right] \begin{array}{l} \\ +R_1 \\ -2R_1 \\ -3R_1 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & -3 & 3 & -3 \\ 0 & -1 & 6 & 4 \end{array}\right] \begin{array}{l} \\ \times -1 \\ -3R_2 \\ -R_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 7 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↳ Since there are 3 columns, 4 rows, the last row will always be a zero row

If found inconsistent, the vector is not in the span

Consistent! ∴ There is a solution and it is in the span

Find a_1, a_2, and a_3

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_3 = 1 \end{cases}$$

Check:

$$1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \checkmark$$

* A set of vectors $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ is linearly independent if the equation $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k = \vec{0}$ has only the trivial solution $(a_1 = 0, a_2 = 0, \dots, a_k = 0)$

- Ex: $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0}$

$a_1\vec{v}_1 = -a_2\vec{v}_2$ (must be 0!)

↳ If there are any other solutions, they are scalar multiples of each other

- Ex: $$\begin{bmatrix} 1 & 1 & -1 & 2 \\ -1 & -2 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & 3 & 10 \end{bmatrix} \begin{array}{l} \\ \\ +R_1 \\ +3R_1 \end{array} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ -1 & -2 & 0 & -5 \\ 3 & 0 & 0 & 3 \\ 6 & 5 & 0 & 16 \end{bmatrix}$$

$$\det = -\begin{vmatrix} -1 & -2 & -5 \\ 3 & 0 & 3 \\ 6 & 5 & 16 \end{vmatrix} = 3\begin{vmatrix} -2 & -5 \\ 5 & 16 \end{vmatrix} + 3\begin{vmatrix} -1 & -2 \\ 6 & 5 \end{vmatrix}$$

$$= 3[32 + 25 - 5 + 12] = 3[-37 + 37] = 0 \checkmark$$