Unit 7

7.2
Probability Theory

Probability Distribution
(assigning probabilities to various outcomes)

Definition
Let S be a finite set and suppose
1) $0 \leq p(a) \leq 1$ for each $a \in S$
2) $\sum_{a \in S} p(a) = 1$
The function p is called a probability distribution

Example
Roll a die
$\{1, 2, 3, 4, 5, 6\}$
p(1) $= \frac{1}{6}$
p(2) $= \frac{1}{6}$
p(3) $= \frac{1}{6}$
p(4) $= \frac{1}{6}$
p(5) $= \frac{1}{6}$
p(6) $= \frac{1}{6}$
add up all 6 to get 1
this example has a uniform distribution

Probability of an Event

Definition
$p(E) = \sum_{a \in E} p(a)$

Example
$E = \{2, 4, 6\}$
$p(E) = p(2) + p(4) + p(6)$

Complements

Definition
$p(\bar{E}) + p(E) = 1 \Rightarrow p(\bar{E}) = 1 - p(E)$

Union of events

Definition
$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
Suppose $E_1, E_2, ..., E_k$ are positive disjoint events in S, then $P(\cup_{i=1}^{k} E_i) = \sum_{i=1}^{k} p(E_i)$
$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} p(E_i)$ can also work

Example

Roll a die
$p(E_1)$: roll a 1
$p(E_2)$: roll a 2
$p(E_1 \cup E_2) = p(E_1) + p(E_2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Theorem

Definition
Let E and F be events in a sample space and $p(F) > 0$. The conditional probability of E given F denotes $p(E|F)$, is defined as $p(E|F) = \frac{p(E \cap F)}{p(F)}$

Examples
What is the probability the sum of two rolls of a die is 7, given that the 1st roll is a 3?
Let F be the event "1st roll is a 3"
Let E be the event "the sum of the 2 rolls is 7"
$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$

A,B,C,D,E
(no replacement permutation)
Find the probability of having A,D,E as the letters in a 3 letter word given that the first letter of the word is a
Let F be "the first letter is A
Let E be the 3 letters in the word are A,D,E
$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{\frac{2}{60}}{\frac{1}{5}} = \frac{1}{6}$

Theorem

Definition
Events E and F are independent if and only if $p(E \cap F) = p(E) * p(F)$

Example
Flip coin: head - H, tail - T
$S = \{HH, HT, TH, TT\}$
E: getting an H on roll 1
F: getting an H on roll 2
$p(E \cap F) = \frac{|E \cap F|}{|S|} = \frac{1}{4}$
$p(E) = \frac{1}{2}$
$p(E) = \frac{1}{2}$
$\Rightarrow p(E \cap F) = p(E) * p(F)$ since $\frac{1}{4} = \frac{1}{2} * \frac{1}{2}$

Theorem

Definition
$E_1, E_2, ..., E_n$ are pairwise independent if $p(E_i \cap E_j) = p(E_i) * p(E_j)$ for $1 \leq i, j \leq n, i/neqj$ (can only be two events)
$E_1, E_2, ..., E_n$ are mutually independent if $p(E_i \cap E_j \cap E_k) = p(E_i) * p(E_j) * p(E_k)$ (can be any amount of events)

Bernoulli Trials

Definition
Bernoulli trials are mutual independent events that have probability of success p and probability of failure q. Also, $p + q = 1$, which means $q = 1 - p$

Example
Flip coin
success: H failure: T
$p = \frac{1}{2}$ $q = \frac{1}{2}$

Theorem

Definition
The probability of having exactly k successes in n independent Bernoulli Trials is $\binom{n}{k} p^k q^{n-k}$

7.3
Bayes Theorem

Bayes Theorem

Definition
Let A and B be events in a sample space S
Then, $P(A|B) = \frac{P(B|A)*P(A)}{P(B)} = \frac{P(B|A)*P(A)}{P(B|A)*P(A)+P(B|\overline{A})*P(\overline{A})}$

Example
A drug test is 90 percent accurate on drug users
The test is 10 percent false positive for non users
Also, 5 percent of population use drugs
Find the probability that a person is a user given that the test is positive
Let A be event "the person is a drug user"
Let B be event "the test is positive"
$P(A|B) = \frac{P(B|A)*P(A)}{P(B)} = \frac{P(B|A)*P(A)}{P(B|A)*P(A)+P(B|\overline{A})*P(\overline{A})}$
$\frac{(0.90)(0.05)}{(0.90)(0.05)+(0.10)(0.95)} =$ 0.32