Complex Eigenvalues

If $P^{-1}AP = D$

$\det(P^{-1}AP) = \det(D)$

$\det(P^{-1})\det(A)\det(P) = \det(D)$

$\cancel{\frac{1}{\det(P)}}\det(A)\cancel{\det(P)} = \det(D)$

$\det(A) = \det(D)$

↳ $\det(D)$ is the product of all diagonal entries, which are the eigenvalues of A

∴ $\det(A)$ is equal to the product of the eigenvalues of A (taken with multiplicity)

↳ True for all matrices!

∴ A is invertible if and only if none of the eigenvalues are zero

* Even if the eigenvalues are complex, their product will always be a real number, so the determinant will always exist

* Introduction to Complex Numbers

Q: Is it possible to take the square root of a negative number?

A: There is no real number whose square is negative

BUT if we really want to...

$\sqrt{-1} = i$

- Ex: $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$

$3i - i = 2i$

$4 \times 2i = 8i$

* Can do normal operations like with real numbers, just with an "i", acting as a variable / vector

* These are "purely imaginary" numbers

↳ $i\mathbb{R}$

* A complex number is written as $z = a + bi$ ($\mathbb{R} + i\mathbb{R}$), where a and b are real numbers

* Can also interpret complex numbers as a 2-vector: $z = \langle a, b \rangle$

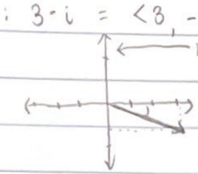
* Addition and scalar multiplication work as expected

* BUT multiplication is different $\quad \rightarrow i^2 = (\sqrt{-1})^2 = -1$

- Ex. $(a+bi)(c+di) = ac + \underline{adi + cbi} + bdi^2 = ac - bd + (ad+bc)i$

* <u>Fundamental Theorem of Algebra</u>
  - Every nonconstant polynomial has at least one (possibly complex) root
  - Every order n polynomial has exactly n (possibly complex) roots (taken with multiplicity)
- Visualizing complex numbers:
  - Ex: $3-i = \langle 3, -1 \rangle$ $\quad$ $(z = a+bi = \langle a, b \rangle) \rightsquigarrow$ Real part of z is $Re[z] = a$

Imaginary part of z is $Im[z] = b$



Complex plane
(<u>not</u> Cartesian!)

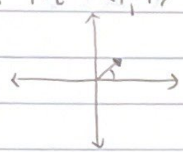
$\therefore Re[3-i] = 3$

$Im[3-i] = -1$

$r = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

$\theta = \arctan(\frac{-1}{3})$

- Ex: $1+i = \langle 1, 1 \rangle$



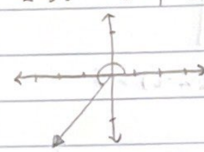
$Re[1+i] = 1$

$Im[1+i] = 1$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \arctan(\frac{1}{1}) = \frac{\pi}{4} / 45°$

$-2-3i = \langle -2, -3 \rangle$



$Re[-2-3i] = -2$

$Im[-2-3i] = -3$

$r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$

$\theta = \arctan(\frac{-3}{-2})$

* Polar coordinates are very useful for complex numbers

$r = \sqrt{a^2+b^2} = |z|$ $\quad$ $\theta = \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right) = \tan^{-1}\left(\frac{b}{a}\right) = \arctan\left(\frac{b}{a}\right)$

↳ indicates magnitude of complex number

- $i^6 = (-1)(-1)(-1) = -1$ $\quad$ * i and $-i$ should somehow be "interchangeable"

$(i^3)(i^3) = -1$

$(-i)(-i) = -1$

$(-i)^2 = -1$

$i^2 = -1$

* If $z = a+bi$, then the complex conjugate of z is $\bar{z} = z^* = a - bi$, which represents reflection over the real axis on the complex plane

$\therefore z\bar{z} = (a+bi)(a-bi) = a^2 \cancel{-abi} \cancel{+abi} - b^2i^2 = a^2 - b^2i^2 = a^2+b^2$

- Ex: $A = \begin{bmatrix} 7 & 3 \\ -3 & 7 \end{bmatrix}$ $\quad$ $\det(A - \lambda I) = 0 \rightsquigarrow (7-\lambda)^2 + 9 = 0$

$\lambda^2 - 14\lambda + 49 + 9 = 0$

$\lambda^2 - 14\lambda + 58 = 0$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{14 \pm \sqrt{196 - 232}}{2}$

$\lambda = 7 \pm \sqrt{-36}/2 = 7 \pm 3i$

$A - \lambda I = \begin{bmatrix} 7-\lambda & 3 \\ -3 & 7-\lambda \end{bmatrix}$

↳ This constant is <u>always</u> the determinant!

$\det(A) =$ product of eigenvalues
$= (7+3i)(7-3i) = 49 + 9 = 58$

↳ The trace is the sum of the real parts of the eigenvalues / the sum of the diagonal entries of A

↳ $\lambda = 7+3i$

$$\left[\begin{array}{cc|c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right] \times i \rightsquigarrow \left[\begin{array}{cc|c} -3i & 3 & 0 \\ -3i & 3 & 0 \end{array}\right] -R \rightsquigarrow \left[\begin{array}{cc|c} -3i & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \div 3 \rightsquigarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} -ix+y=0 \\ x \text{ is free, pick } x=i \end{cases} \rightsquigarrow \begin{matrix} -i^2+y=0 \\ y=-1 \end{matrix} \quad \therefore \begin{bmatrix} i \\ -1 \end{bmatrix}$$

↳ For $\lambda = 7-3i$, the basic solution would be $\begin{bmatrix} -i \\ -1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 7+3i & 0 \\ 0 & 7-3i \end{bmatrix}$

$P^{-1}AP = D$

- Ex: $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ $\quad \det(A-\lambda I) = (3-\lambda)(1-\lambda)+2=0 \quad \dfrac{4 \pm \sqrt{16-20}}{2}$

$\lambda^2 - 4\lambda + 3 + 2 = 0 \quad = 2 \pm \sqrt{-4}/2$

$A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}$ $\quad \lambda^2 - 4\lambda + 5 = 0 \quad \lambda = 2 \pm i$

↳ det(A)!

$\det(A) = (2+i)(2-i) = 5$

↳ $\lambda = 2+i$

$$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right] \times (1-i) \rightsquigarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1-i & -2 & 0 \end{array}\right] -R_1 \rightsquigarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

↳ $\begin{cases} (1-i)x - 2y = 0 \\ \text{pick } x=2 \text{ and } y=1-i \text{ to balance equation} \end{cases}$ $\quad \therefore \begin{bmatrix} 2 \\ 1-i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\vec{v}_{RE}$ $\quad$ ↳ $i \times \vec{v}_{IM}$

↳ $\lambda = 2-i$ is conjugate $\therefore \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$

$\therefore P = \begin{bmatrix} 2 & 2 \\ 1-i & 1+i \end{bmatrix}$ and $D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$

* Standard form $\begin{bmatrix} Re[\lambda] & Im[\lambda] \\ -Im[\lambda] & Re[\lambda] \end{bmatrix}$

↳ $P = [\vec{v}_{RE} \ \ \vec{v}_{IM}] = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \rightsquigarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$ $\quad$ ↙ Jordan Block

$$D = P^{-1}AP = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ \frac{1}{2} & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$