Vector Spaces

- Let V be some set of stuff and the elements of V be vectors ($\vec{v}, \vec{u}, \vec{w}$ are elements of V AKA $\vec{v}, \vec{u}, \vec{w} \in V$)

* Addition
  - $(\vec{u} + \vec{v}) \in V$ (closed under addition)
  - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative property)
  - $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (associative property)
  - $\vec{0} \in V$, $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ (zero property)
  - $-\vec{v} \in V$, $\vec{v} + (-\vec{v}) = \vec{v} - \vec{v} = \vec{0}$ (inverse property)

  * If any of the above is not true, it is not a vector space

* Scalar Multiplication
  - $a\vec{v} \in V$ for any real number a
  - $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ for any real number a (distributive property)
  - $(a+b)\vec{v} = a\vec{v} + b\vec{v}$ for any real number a (distributive property
  - $(ab)\vec{v} = a(b\vec{v})$, $a, b \in \mathbb{R}$ (associative property)
  - $1\vec{v} = \vec{v}$ (identity property / unit)

  * If any of the above is not true, it is not a vector space

* Examples of Vector Spaces
  - Columns with n components $(\mathbb{R}^n)$
  - $m \times n$ matrices $M_{m,n}(\mathbb{R})$
  - Complex numbers are a real vector space
  - Complex vectors and matrices (over the complex numbers $\mathbb{C}$)
  - Polynomials
  - Continuous functions on (a,b)

- Ex: Is the set of all scalar multiples of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a vector space?
  - Check closed under addition

$$a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ 3b \end{bmatrix} = \begin{bmatrix} a+b \\ 2a+2b \\ 3a+3b \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

  - Check commutative: true, because the vector is $\in \mathbb{R}^3$
  - Check associative: true, because the vector is $\in \mathbb{R}^3$
  - Check zero:

$$0 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is in the space } (\vec{0} \in V) \checkmark$$

- Check inverse: true, by definition, Scalar multiples are in the set (closed under scalar multiplication) ✓
- Check $a\vec{v} \in V$ ✓
- Check distributive properties: true, because the vectors are $\in \mathbb{R}^3$
  - associative ...
  - identity ...

$\therefore$ The set is a vector space! (and a subspace of $\mathbb{R}^3$)

* A subset U of a vector space is a subspace if
  1. The zero vector $\vec{0} \in U$
  2. it is closed under addition (If $\vec{x} \in U$ and $\vec{y} \in U$, then $\vec{x}+\vec{y} \in U$)
  3. it is closed under scalar multiplication (If $\vec{x} \in U$, then $a\vec{x} \in U$ for all $a \in \mathbb{R}$)
* Every subspace is a vector space
* Examples of Subspaces
  - All scalar multiples of a vector
  - $\vec{0}$ is a trivial subspace
  - Solutions to a homogeneous system
  - Lines through the origin
  - Planes through the origin
  - The set of all eigenvectors for a specific λ (plus $\vec{0}$)
  - The set of all outputs of any linear transformation (AKA the range of a linear transformation)