Matrix Subspace

* **Null space** of a matrix A is the subspace $null(A) = \{\vec{x}$ such that $A\vec{x} = \vec{0}\}$
  - Solutions to the homogeneous system
* **Row space** of an $n \times m$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A. $row(A) = span(R_1, R_2, \ldots, R_m)$
* **Column space** of an $n \times m$ matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A: $col(A) = span(C_1, C_2, \ldots, C_n)$
* The **image** or **range** of a linear transformation is the subspace $im(A)$ or $range(A)$ given by the set of vectors $\{A\vec{x}$ where $\vec{x} \in \mathbb{R}^m\}$
  - To find all possible outputs, look at all inputs using $\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n$
    $T(\vec{e}_1) = A\vec{e}_1 \quad T(\vec{e}_2) = A\vec{e}_2 \quad \ldots \quad T(\vec{e}_n) = A\vec{e}_n$
    ↳ The output is spanned by the columns of A
    $\therefore$ $im(A) = range(A) = col(A)$ (Range is column space)
* If A becomes B by elementary row operations, then $row(A) = row(B)$
* If R is a row-echelon matrix, then
  ① The nonzero rows of R are a basis for the rowspace $row(A)$
  ② The columns of R that contain a leading 1 are a basis for the column space $col(A)$
  $\therefore$ The row space and column space always has the same dimensions
* **Rank** of a matrix is the number of leading 1's in row-echelon form
  $rank(A) = dim(row(A)) = dim(col(A))$
  - Can also tell how far a matrix is from being invertible
* Let A be an $m \times n$ matrix with rank r. If A is reduced to row-echelon form R, then
  ① The nonzero rows of R are a basis for $row(A)$
  ② The r columns of A that contain a leading 1 in row echelon form are a basis for the columnspace of A

- Ex: Find a basis for the row and column spaces of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

↳ Basis for $row(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

↳ Basis for $col(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right\}$$

(because first and second columns of REF have leading 1's)

- Ex: $\begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 1 & -4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{row}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\quad \text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$

* Nullity of a matrix is the dimension of the null space $\dim(\text{null}(A))$
* Rank-Nullity Theorem
  - Let A be an $m \times n$ matrix of rank r
    1. $\text{rank}(A) + \text{nullity} = n$ (number of columns)
    2. The $n-r$ basic solutions to $A\vec{x} = \vec{0}$ (as obtained by Gaussian Elimination) are a basis for $\text{null}(A)$, where $n-r$ is the number of free variables
    3. The dimension of the null space of $A = \dim(\text{null}(A)) = n - r$

    $\therefore \dim(\text{range}(A)) = r$

- Ex: $\left[\begin{array}{cccc:c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc:c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$ $\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases}$

  ↳ $\begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$ $\quad \begin{matrix} x_3 = x_3 + 0 \\ x_4 = 0 + x_4 \end{matrix} \rightsquigarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for null space

- Ex: $\begin{cases} x_1 - x_3 - 2x_4 + x_5 = 0 \\ x_2 - 3x_3 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = x_3 + 2x_4 - x_5 \\ x_2 = 3x_3 \end{cases}$ $\quad \begin{matrix} x_3 = x_3 \\ x_4 = 0 + x_4 \\ x_5 = 0 + 0 + x_5 \end{matrix} \rightsquigarrow \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

  ↳ Rank $= 2$; nullity $= 3$; $2 + 3 = 5 =$ # columns