10.16

5.1
Mathemtical Induction

Principle of Mathematical Induction
To prove P(n) is true for all positive integers n, we prove the following 2 steps
Basis step: verify that P(1) is true
Inductive step: show that P(k) is true implies P(k+1) is true
P(k) is true is called the inductive hypothesis
k = 1 P(1) true
P(k) true $\Rightarrow$ P(k+1) true
Let, k = 1, then P(1), true $\Rightarrow$ P(2) true
Let, k = 2, then P(2), true $\Rightarrow$ P(3) true
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Example
Show that 1 + 2 + 3 + ... + n = $\frac{n(n+1)}{2}$, for integers n $\geq$ 1
Proof by induction
Let P(n) be the statement 1 + 2 + 3 + ... + n = $\frac{n(n+1)}{2}$
Basis step: P(1) is true, since 1 = $\frac{1(1+1)}{2}$
Inductive step: suppose P(k) is true. And so 1 + 2 + ... + k = $\frac{k(k+1)}{2}$
Now, 1 + 2 + ... + k + (k+1) = $\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k)+(k+1)(2)}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$
And so, P(k+1) is true
Therefore, P(n) is true for all integers $n \geq 1$

Example
Show that 1 + 3 + 5 + ... + (2n - 1) = n^2 for integers $n \geq 1$ (n terms)
Proof by induction
Let P(n) be "1 + 3 + 5 + ... + (2n - 1) = n^2"
Basis step: P(1) is true since 1 = 1^2
Inductive step: suppose P(k) is true. And so 1 + 3 + 5 + ... + (2k - 1) = k^2
Now, 1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = $k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2$
And so, P(k+1) is true
Therefore, P(n) is true for all integers $n \geq 1$

Example
Show that 1 + 2 + 2^2 + ... + 2^n = $2^{n+1} - 1$ for all integers $n \geq 1$
Proof by induction
Let P(n) be "1 + 2 + 2^2 + ... + 2^n = $2^{n+1} - 1$"
Basis step: P(2) is true, since 1 = $2^{1+1} - 1$
Inductive step: suppose P(k) is true. And so, 1 + 2 + 2^2 + ... + 2^k = $2^{k+1} - 1$
Now, 1 + 2 + 2^2 + ... + $2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2 * 2^{k+1} - 1 = 2^{(k+1)+1} - 1$
And so, P(k+1) is true
Therefore, P(n) is true for all integers $n \geq 1$

Example
Show that $n < 2^n$ for all integers $n \geq 1$
Proof by induction
Let P(n) be "$n < 2^n$"
Basis step: P(1) is true since $1 < 2^1$
Inductive step: Suppose P(k) is true. And so, $k < 2^k$
Now, $k + 1 < 2^k + 1 < 2^k + 2^k = 2 * 2^k = 2^{k+1}$
And so, P(k+1) is true
Therefore, P(n) is true for all integrs $n \geq 1$

Example
Show that $2^n < n!$ for all $n \geq 4$
Proof by induction
Let P(n) be "$2^n < n!$"
Basis step: P(4) is true since $2^4 < 4!$
Inductive step: Suppose P(k) is true. And so, $2^k < k!$
$\Rightarrow 2 * 2^k < 2 * k! < (k + 1)k!$
$\Rightarrow 2^{k+1} < (k + 1)!$
And so, P(k+1) is true
Therefore, P(n) is true for all integers $n \geq 4$

Example
Let $F_0 = 0$, $F_n = 5 * F_{n-1} + 2$ for $n \geq 1$
Show that $2|F_n$ for all $n \geq 0$
F_0, F_1, F_2, F_3
0, 2, 12, 62
Proof by induction
Let P(n) be the statement "$2|F(n)$"
Basis step: P(0) is true since $2|0$
Inductive step: Suppose $2|F(k)$. Now, $F_{k+1} = 5F_{k+2}$
Since $2|5F_k$ and $2|2$, we have $2|(5F_{k+2})$.
And so P(k+1) is true
Therefore, P(n) is true for all integers $n \geq 0$

Suppose $f : A \to B$ $g : B \to C$
f and g injective $\Rightarrow$ g o f injective
f and g surjective $\Rightarrow$ g o f surjective
f and g bijective $\Rightarrow$ g o f bijective

5.2
Strong Induction and Well-ordering

Strong Induction
To prove P(n) is true for all positive integers, we prove the following 2 steps
Basis step: we verify that P(1) is true
Inductive step: We show that P(1), P(2), ..., P(k) are true imply that P(k+1) is true
P(1), P(2), P(k) are true is called the inductive hypothesis
P(1) true
P(1) true $\Rightarrow$ P(2) true
P(1), P(2) true $\Rightarrow$ P(3) k + 1 = 3

P(1), P(2), P(3) $\Rightarrow$ P(4) true

Example
Show that every integer n $\geq$ 2 can be written as a product of primes
Proof by strong induction
Let P(n) be "n can be written as a product of primes"
Basis step: P(2) is true, since 2 = 2 $\leftarrow$ prime factor form
Inductive step: Suppose P(2), P(3), P(4), ..., P(k) are true, meaning 2, 3, 4, ..., k can be written as products of primes
case 1: if k+1 is prime, then k + 1 = k + 1 $\leftarrow$ already in prime factor form
case 2: if k + 1 is composite, then k + 1 = $a * b$, where $2 \leq a < k+1$ and $2 \leq b < k+1$. But we know P(a) and P(b) are true, meaning a and b can be written as produts of primes by the inductive hypothesis
And so
$a = P_1 P_2 ... P_r$
$b = q_1 q_2 ... q_s$
This means $k + 1 = a * b = (P_1 P_2 ... P_2)(q_1 q_2 ... q_s)$
And so P(k+1) is true
Therefore P(n) is true for all integers $n \geq 2$

Example
Show that every postage of 12 cents or more can be formed using 4-cent and 5-cent stamps
12 = 4 + 4 + 4 14 = 4 + 5 + 5
13 = 4 + 4 + 5 15 = 5 + 5 + 5
Proof by strong induction
Let P(n) be "n cents can be formed using 4-cent and 5-cent stamps"
Basis step: P(12), P(13), P(14), and P(15) are true
Inductive step: suppose P(12), P(13), P(14), P(15), ..., P(k) are true
For P(k+1), we know P(k-3) is true by the inductive hypothesis. And so just add one 4-cent stamp to the combination of stamps used to make (k-3) cents, we have the mix needed to make (k+1) cents. And so P(k+1) is also true.
Therefore, the statement P(n) is true for all integers $n \geq 12$

Practice

1. Show that 1 + 4 + 7 + ... + (3n-2) = $\frac{(3n-1)(n)}{2}$ by induction
Proof
P(n) is "1 + 4 + 7 + ... + (3n-2) = $\frac{(3n-1)(n)}{2}$ by induction
Basis step: P(1) is true, since 1 = $\frac{(3*1-1)(1)}{2}$
Induction step: Suppose P(k) is true.
1 + 4 + 7 + ... + (3k-2) + (3(k+1)-2) = $\frac{(3k-1)(k)}{2}$ + (3(k+1)-2) = $\frac{(3k-1)(k)+(2)(3k+1)}{2}$
$= \frac{(3k^2-k)+(6k+2)}{2} = \frac{3k^2+5k+2}{2} = \frac{(3k+2)(k+1)}{2} = \frac{(3(k+1)-1)(k+1)}{2}$
3k + 3 - 1 = 3k + 2 (show on side to algebraically prove the last part
And so $1 + 4 + 7 + ... + (3k-2) + (3(k+1)-2) = \frac{(3(k+1)-1)(k+1)}{2}$
And so P(k+1) is true
Therefore P(n) is true for all integers n $\geq$ 1

2. Show that $n < 3^n$ for all integers $n \geq 1$ by induction
Proof

P(n) is $n < 3^n$
Basis step: P(1) is true, since $1 < 3^1$
Induction step: Suppose P(k) is true, And so, $k < 3^k$
$\Rightarrow k + 1 < 3^k + 1 < 3^k + 3^k + 3^k = 3 * 3^k = 3^{k+1}$
$\Rightarrow k + 1 < 3^{k+1}$
And so P(k+1) is true
Therefore P(n) is true for all integers n ≥ 1