Gram Schmidt Orthogonalization

- Let $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$ be an orthogonal basis of a subspace U in $\mathbb{R}^n$
- If $\vec{x}$ is any vector in U,

$$\vec{x} = \left(\frac{\vec{x}\cdot\vec{f}_1}{\|\vec{f}_1\|^2}\right)\vec{f}_1 + \left(\frac{\vec{x}_2\cdot\vec{f}_2}{\|\vec{f}_2\|^2}\right)\vec{f}_2 + \dots + \left(\frac{\vec{x}_k\cdot\vec{f}_k}{\|\vec{f}_k\|^2}\right)\vec{f}_k$$

Expansion Theorem

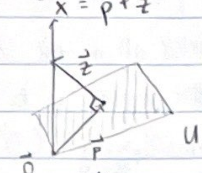
$\vec{x} = \vec{p} + \vec{z}$

- If $\vec{x}$ is in $\mathbb{R}^n$, the orthogonal projection of $\vec{x}$ onto U is

$$\text{proj}_U\,\vec{x} = \left(\frac{\vec{x}\cdot\vec{f}_1}{\|\vec{f}_1\|^2}\right)\vec{f}_1 + \left(\frac{\vec{x}_2\cdot\vec{f}_2}{\|\vec{f}_2\|^2}\right)\vec{f}_2 + \dots + \left(\frac{\vec{x}_k\cdot\vec{f}_k}{\|\vec{f}_k\|^2}\right)\vec{f}_k$$

↳ If $U = \{\vec{0}\}$, $\text{proj}_U\,\vec{x} = \vec{0}$

- $\vec{p}$ is the vector in U closest to $\vec{x}$ in the sense $\|\vec{x}-\vec{p}\| < \|\vec{x}-\vec{y}\|$ for all $\vec{y}$ in U, $\vec{y} \neq \vec{p}$



↳ $\vec{p} = \text{proj}_U\,\vec{x}$

$\vec{p} \in U$

$\vec{z} \in U^\perp$

- Ex: $U = \text{span}\{\langle 2,0,1\rangle, \langle -1,0,2\rangle\}$ Show that this is an orthogonal basis for U and find the projection of $\vec{x} = \langle 1,1,1\rangle$ onto U and $U^\perp$

$\langle 2,0,1\rangle \cdot \langle -1,0,2\rangle = -2+0+2 = 0$ ✓

↳ The set is orthogonal (thus linearly independent by definition)

By definition, they span U, so they are an orthogonal basis

$$\text{proj}_U \vec{x} = \frac{\langle 1,1,1\rangle \cdot \langle 2,0,1\rangle}{\langle 2,0,1\rangle \cdot \langle 2,0,1\rangle} \times \langle 2,0,1\rangle + \frac{\langle 1,1,1\rangle \cdot \langle -1,0,2\rangle}{\langle -1,0,2\rangle \cdot \langle -1,0,2\rangle} \times \langle -1,0,2\rangle$$

$$= \frac{2+0+1}{4+0+1} \langle 2,0,1\rangle + \frac{-1+0+2}{1+0+4} \langle -1,0,2\rangle = \frac{3}{5}\langle 2,0,1\rangle + \frac{1}{5}\langle -1,0,2\rangle$$

$$= \frac{1}{5}\langle 5,0,5\rangle = \boxed{\langle 1,0,1\rangle} \quad \vec{p} \in U$$

$$z = \langle 1,1,1\rangle - \langle 1,0,1\rangle = \boxed{\langle 0,1,0\rangle} \quad \vec{z} \in U^\perp$$

* Let U be a subspace of $\mathbb{R}^n$
  - Every orthogonal set in U is a subset of an orthogonal basis for U
  - U has an orthogonal basis

* Let $\{f_1, f_2, \ldots, f_m\}$ be an orthogonal set in $\mathbb{R}^n$

$$\vec{f}_{m+1} = \vec{x} - \frac{\vec{x}_1 \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \frac{\vec{x}_2 f_2}{\|\vec{f}_2\|^2}\vec{f}_2 - \cdots - \frac{\vec{x}_m \vec{f}_m}{\|\vec{f}_m\|^2}\vec{f}_m$$

- $\vec{f}_{m+1} \cdot \vec{k} = 0$ for $k = 1, 2, \ldots, m$
- If $\vec{x}$ is <u>not</u> in $\text{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$, then $\vec{f}_{m+1} \neq \vec{0}$ and $\{f_1, f_2, \ldots, f_m, f_{m+1}\}$ is an orthogonal set

- Ex: (from above)

$\vec{f}_{m+1} = \langle 0,1,0\rangle$ and $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$

- Ex: Find an orthogonal basis for the space $U = \text{span}\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

$\langle 0,1,-1\rangle \cdot \langle 1,2,3\rangle = 0+2-3 = -1$

↳ <u>not</u> orthogonal

$$\vec{f}_2 = \langle 1,2,3\rangle - \frac{\langle 1,2,3\rangle \cdot \langle 0,1,-1\rangle}{\langle 0,1,-1\rangle \cdot \langle 0,1,-1\rangle}\langle 0,1,-1\rangle = \langle 1,2,3\rangle - \frac{0+2-3}{0+1+1}\langle 0,1,-1\rangle$$

$$= \langle 1,2,3\rangle - \frac{-1}{2}\langle 0,1,-1\rangle = \langle 1, \tfrac{5}{2}, \tfrac{5}{2}\rangle$$

∴ The set $\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\5/2\\5/2\end{bmatrix}\right\}$ is an orthogonal basis for U

* Gram-Schmidt orthogonalization Algorithm

- If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is any basis of U in $\mathbb{R}^n$:
  - ↳ $\vec{f}_1 = \vec{x}_1$
  - ↳ $\vec{f}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1$
  - ↳ $\vec{f}_3 = \vec{x}_3 - \dfrac{\vec{x}_3 \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 - \dfrac{\vec{x}_3 \vec{f}_2}{\|\vec{f}_2\|} \vec{f}_2$
  - $\vdots$
  - ↳ $\vec{f}_k = \vec{x}_k - \dfrac{\vec{x}_k \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 - \dfrac{\vec{x}_k \vec{f}_2}{\|\vec{f}_1\|^2} \vec{f}_2 - \ldots - \dfrac{\vec{x}_k \vec{f}_{k-1}}{\|\vec{f}_{k-1}\|^2} \vec{f}_{k-1}$

    for $k = 2, 3, \ldots, m$
  - * $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ is an orthogonal basis for $\vec{u}$
  - * span $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\}$ = span $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ for $k = 1, 2, 3, \ldots, m$

- Ex: Find an orthogonal basis for the space $U = \text{span}\{\underset{\vec{x}_1}{\langle 1,1,1,1\rangle}, \underset{\vec{x}_2}{\langle 0,1,1,1\rangle}, \underset{\vec{x}_3}{\langle 0,0,1,1\rangle}\}$

$\vec{f}_1 = \vec{x}_1 = \langle 1,1,1,1\rangle$

$\vec{f}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 = \langle 0,1,1,1\rangle - \dfrac{\langle 0,1,1,1\rangle \cdot \langle 1,1,1,1\rangle}{\langle 1,1,1,1\rangle \cdot \langle 1,1,1,1\rangle} \cdot \langle 1,1,1,1\rangle$

$= \langle 0,1,1,1\rangle - \langle \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\rangle = \langle -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\rangle \times 4$

↳ $\langle -3, 1, 1, 1\rangle$

$\vec{f}_3 = \vec{x}_3 - \dfrac{\vec{x}_3 \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \dfrac{\vec{x}_3 \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2$

$= \langle 0,0,1,1\rangle - \dfrac{\langle 0,0,1,1\rangle \langle 1,1,1,1\rangle}{\langle 1,1,1,1\rangle \cdot \langle 1,1,1,1\rangle}\langle 1,1,1,1\rangle - \dfrac{\langle 0,0,1,1\rangle \langle -3,1,1,1\rangle}{\langle -3,1,1,1\rangle \cdot \langle -3,1,1,1\rangle}\langle -3,1,1,1\rangle$

$= \langle 0,0,1,1\rangle - \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle - \langle \frac{-1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\rangle$

$= \langle 0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\rangle \times 3$

↳ $\langle 0, -2, 1, 1\rangle$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for U

$\therefore \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -3/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for U

* QR - Factorization
  - $A = QR$, Q has orthonormal columns that span col (A)
    R is upper triangular and invertible with positive diagonal entries $(R = Q^T A)$