Vectors

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Vector Geometry in 2D

$\overrightarrow{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} =< 1, 3 >= (1, 3)$
$\overrightarrow{y} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} =< 2, -1 >= (2, -1)$
$\overrightarrow{x} + \overrightarrow{y} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Consider two vectors $\overrightarrow{x}, \overrightarrow{y}$ in R^2
If the arrows for $\overrightarrow{x}$ and $\overrightarrow{y}$ are drawn then the arrow for $\overrightarrow{x} + \overrightarrow{y}$ corresponds to the fourth vertex of the parallelogram determined by $\overrightarrow{o}, \overrightarrow{x}, \overrightarrow{y}$

If $\overrightarrow{x}$ in R^2 then the arrow for $k\overrightarrow{x}$ is $|k|$ times as long as the arrow for $\overrightarrow{x}$. It is in the same direction as $\overrightarrow{x}$ for $k > 0$ and opposite direction if $k < 0$

Transformations in 2D

Given an mxn matrix A we can define a transformation $T_A : R^n \to R^n$ by $T_A(\overrightarrow{x}) = A\overrightarrow{x}$ for every $\overrightarrow{x}$ in R^n
T_A is called the matrix transformation induced by A

Elementary Matrices

Type I: Swap Rows

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Reflection over $y = x$

Type II: Multiply by a scalar

$E_2 = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ $E_2^1 = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

$a = 2$

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

x extension/compression, y extension/compression
If the x has a negative scalar, it reflects over the y axis, and vice versa for the x axis

Type III: Add a multiple of one row to another

$E_3 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

x shear

There is also y shear

Matrix Multiplication

We can take compositions of transformations
$T_A(T_B(\overrightarrow{x})) = T_A(B\overrightarrow{x}) = AB\overrightarrow{x} = T_{AB}(\overrightarrow{x})$

Inverse Transformations

A is invertible if and only if the transformation T_A has an inverse
In this case, the inverse is unique and the inverse of $(T_A)^{-1} = T_{A^{-1}}$

Identity Matrix

Identity transformation doesn't do any transformation

Zero Matrix

Zero transformation makes everything 0

Projection

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

This is an x projection, it makes a flat line along x axis

For all 3 of these they have 0 area, so they are not invertible

2.6

A transformation $T : R^n \to R^n$ is called a Linear Transformation if it satisfies the following two conditions for all $\overrightarrow{x}, \overrightarrow{y}$ in R^n and all numbers a

1. $T(\overrightarrow{x} + \overrightarrow{y}) = T(\overrightarrow{x}) + T(\overrightarrow{y})$
2. $T(a\overrightarrow{x}) = aT(\overrightarrow{x})$

A Linear Transformation preserves Linear Combinations
$T(a_1, \overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_k\overrightarrow{x_k}) = a_1T(\overrightarrow{x_1}) + ...$

If A s a mxn matrix then T_A is a Linear Transformation from R^n to R^n

Linear

T is linear if

1. $T(\overrightarrow{x} + \overrightarrow{y}) = T(\overrightarrow{x}) + T(\overrightarrow{y})$
2. $T(a\overrightarrow{x}) = aT(\overrightarrow{x})$

If T is linear
$A = [T(e_1)\ T(e_2)...T(e_3)]$

LU Factorization

The main diagonal of a matrix is the entries a_{11}, a_{22}, a_{33}

A is called upper triangular if all of the entries below/left of the main diagonal are 0, and vice versa for all entries above/right of the main diagonal
All REF matrices are upper triangular

If A and B are both upper/lower triangular then AB is triangular of the same type
If A is invertible and triangular then A_{-1} is triangular of the same type

$[A : I] \to [U : E_kE_{k-1}...E_2, E_1I]$
$E_kE_{k-1}...E_2E_1A = U$

$A = (E_k E_{k-1} ... E_2 E_1)^{-1} U = LU$

We can solve a matrix in REF quickly and efficiently using back substitution

Forward substitution

Find an LU factorization for $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$

REF form is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = U$

$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

The first matrix in this A has each column before it gets reduced

L must be invertible therefore L must be square
If any columns don't exist or don't have row operations, use the corresponding column of the identity matrix

How to change LU without changing A if we want the one row diagonal
$LIU = A$
$LEE^{-1}U = A$
Use the scaling matrix
Row scale U with E^{-1}
Column scale L with E (gets to 1 diagonal)