Diagonalization (Continued)

Ex. Each month, people without Netflix sign up with a probability of $1/6$. Each month, Netflix subscribers quit with a probability of $1/3$. If nobody starts with Netflix, which proportion of people will be signed up after 12 months?

* Markov Chain Model: only considers current state, not past history or how long the state has lasted (AKA Discrete Dynamical Systems)

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → State of nobody having Netflix
→ State of everybody not having Netflix

↓

$\begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$ → After 1 month, $1/6$ of all people have signed up for Netflix
→ After 1 month, $5/6$ of all people have not signed up for Netflix

↳ This is a linear transformation!

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$ $\qquad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ ← if everyone starts off with netflix

$A = [T(e_1) \quad T(e_2)] = \begin{bmatrix} 2/3 & 1/6 \\ 1/3 & 5/6 \end{bmatrix}$ ← Stochastic Matrix: All columns of A sum to 1 (in this case, because probability always sums to 1)

* Transition matrix (encapsulates all the information of how we transition from one state to another state

↳ Initial State $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (AKA state vector)

State after m months is S_m

Goal: Find S_{12}

↳ $S_1 = AS_0 = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$

$S_2 = AS_1 = AAS_0 = A^2 S_0$

$S_3 = AS_2 = AA^2S_0 = A^3 S_0$

$S_m = A^m S_0 \quad \therefore S_{12} = A^{12} S_0 = PD^{12}P^{-1}S_0$

↳ Use diagonalization to take matrix to high power!

$P^{-1}AP = D$

$\underbrace{PP^{-1}}AP\underbrace{P^{-1}} = PDP^{-1}$

$A = PDP^{-1}$

$A^2 = PDP^{-1}PDP^{-1}$

$A^2 = PDDP^{-1}$

$A^2 = PD^2P^{-1}$

$A^3 = PD^2P^{-1}PDP^{-1}$

$A^3 = PD^2DP^{-1}$

$A^3 = PD^3P^{-1}$

* $A^m = PD^mP^{-1}$

$\therefore A^{12} = PD^{12}P^{-1}$

$\det(A-\lambda I) = \begin{vmatrix} 2/3-\lambda & 1/6 \\ 1/3 & 5/6-\lambda \end{vmatrix} = \begin{vmatrix} 2-3\lambda & 1 \\ 1 & 5-6\lambda \end{vmatrix} = 0$ (scaling rows / columns multiplies determinant by factors, so $0 \times 3 \times 6 = 0$)

$\times 3 \quad \times 6$

$(2-3\lambda)(5-6\lambda) - 1 = 0$
$18\lambda^2 - 27\lambda + 9 = 0$
$9(2\lambda^2 - 3\lambda + 1) = 0$
$9(2\lambda-1)(\lambda-1) = 0$
$\lambda = \frac{1}{2}, 1$

or

$\frac{3 \pm \sqrt{9-4(2)(1)}}{2(2)} = \frac{3 \pm \sqrt{1}}{4} = \frac{3 \pm 1}{4}$

$\lambda = \frac{3+1}{4} = \frac{4}{4} = 1$

$\lambda = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

* Every regular stochastic matrix has an eigenvalue of 1 and a unique positive eigenvector that sums to 1
↳ all entries are positive

↳ $\lambda = 1$; $A - \lambda I = \vec{0}$

$\begin{bmatrix} 2/3-1 & 1/6 \\ 1/3 & 5/6-1 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/6 \\ 1/3 & -1/6 \end{bmatrix} \begin{matrix} \times -3 \\ +R_1 \end{matrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{cases} x - \frac{1}{2}y = 0 \\ y \text{ is free, choose } y = 2 \end{cases}$

basic eigenvector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, scale by $\frac{1}{3}$ to get stochastic matrix $\rightsquigarrow \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

↳ $\lambda = \frac{1}{2}$; $A - \lambda I = \vec{0}$

$\begin{bmatrix} 2/3-1/2 & 1/6 \\ 1/3 & 5/6-1/2 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 \\ 1/3 & 1/3 \end{bmatrix} \begin{matrix} \times 6 \\ -2R_1 \end{matrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{cases} x + y = 0 \\ x \text{ is free, pick } x = 1 \end{cases}$

basic eigenvector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

↳ $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$S_{12} = PD^{12}\underbrace{P^{-1}S_0}_{X_0} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4096} \end{bmatrix} \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1/3 \\ -\frac{1}{3 \times 4096} \end{bmatrix} = \begin{bmatrix} 1/3 - \frac{1}{3 \times 4096} \\ 2/3 + \frac{1}{3 \times 4096} \end{bmatrix}$

← ppl with netflix
← ppl w/o netflix

$X_0 = P^{-1}S_0$

$PX_0 = S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ← system of equations

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 1 \end{array}\right] \underset{-2R_1}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 1 \end{array}\right] \underset{\div -3}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1/3 \end{array}\right] \overset{-R_2}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -1/3 \end{array}\right] \begin{cases} x = 1/3 \\ y = -1/3 \end{cases}$

∴ The proportion of people signed up after 12 months is $\frac{1}{3} - \frac{1}{3 \times 4096} = 33.325\%$

BUT $S_{12} \approx \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$, the unique positive eigenvector, is the steady state or the longterm probabilities over an extended period of time.