Best Approximation & Least Squares

- Let A be an $m\times n$ matrix. Let $\vec{b}$ be a vector in $\mathbb{R}^m$. Consider the system $A\vec{x}=\vec{b}$.

① Any solution $\vec{z}$ to the normal equations $(A^TA)\vec{z} = A^T\vec{b}$ is a best approximation to $A\vec{x}=\vec{b}$ in the sense that $\|A\vec{z}-\vec{b}\| \le \|A\vec{x}-\vec{b}\|$ for all $\vec{x}$ in $\mathbb{R}^n$

② If the columns of A are linearly independent, then A^TA is invertible and $\vec{z}$ is the unique solution $\vec{z} = (A^TA)^{-1}A^T\vec{b}$

- Ex:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 3 & 0 & 4 & -1 \\ 5 & 3 & 6 & 5 \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -5R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 9 & -2 & -7 \\ 0 & 18 & -4 & -5 \end{array}\right] \begin{array}{l} \\ \\ -2R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 9 & -2 & -7 \\ 0 & 0 & 0 & 9 \end{array}\right] \nearrow 0=9 \text{ X}$$

inconsistent

$$A^TA = \begin{bmatrix} 1 & 3 & 5 \\ -3 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 2 \\ 3 & 0 & 4 \\ 5 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 35 & 12 & 44 \\ 12 & 18 & 12 \\ 44 & 12 & 56 \end{bmatrix}$$

← transpose is the same (symmetric)

$$A^T\vec{b} = \begin{bmatrix} 1 & 3 & 5 \\ -3 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 24 \\ 9 \\ 30 \end{bmatrix}$$

$$A^TA\vec{z} = A^T\vec{b}: \left[\begin{array}{ccc|c} 35 & 12 & 44 & 24 \\ 12 & 18 & 12 & 9 \\ 44 & 12 & 56 & 30 \end{array}\right] \times\tfrac{1}{12} \sim \left[\begin{array}{ccc|c} 35 & 12 & 44 & 24 \\ 1 & \frac{3}{2} & 1 & \frac{3}{4} \\ 44 & 12 & 56 & 30 \end{array}\right]$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 1 & \frac{3}{4} \\ 35 & 12 & 44 & 24 \\ 44 & 12 & 56 & 30 \end{array}\right] \begin{array}{l} \\ -35R_1 \\ -44R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 1 & \frac{3}{4} \\ 0 & \frac{-81}{2} & 9 & -\frac{9}{4} \\ 0 & -54 & 12 & -3 \end{array}\right] \begin{array}{l} \\ \times\frac{1}{9} \\ \times\frac{1}{12} \end{array} \sim \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 1 & \frac{3}{4} \\ 0 & -\frac{9}{2} & 1 & -\frac{1}{4} \\ 0 & -\frac{9}{2} & 1 & -\frac{1}{4} \end{array}\right] \begin{array}{l} -R_2 \\ \\ -R_2 \end{array}$$

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 6 & 0 & 1 \\ 0 & -\frac{9}{2} & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 = 1-6x_2 \\ x_3 = -\frac{1}{4}+\frac{9}{2}x_2 \\ x_2 \text{ is free, pick } 0 \end{cases} \sim \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{4} \end{bmatrix}$$

System is consistent!

← A minimizer

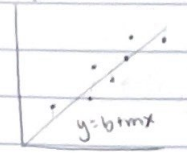
$$\hookrightarrow \begin{bmatrix} 1-6t \\ t \\ -\frac{1}{4}+\frac{9}{2}t \end{bmatrix}$$ is minimizer for all $t \in \mathbb{R}$

↳ can choose 0 because it's a non-homogeneous system and no data is lost

Check:

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & 0 & 4 \\ 5 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1-6t \\ t \\ -\frac{1}{4}+\frac{9}{2}t \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1-6t-3t-\frac{1}{2}+9t \\ 3-18t-1+18t \\ 5-30t+3t-\frac{3}{2}+27t \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 3 \\ -\frac{3}{2} \end{bmatrix}$$

*Best Fit Lines



→ Want to minimize the sum of the squares of errors

$$\left.\begin{aligned} y_1 &= b+mx_1 \\ y_2 &= b+mx_2 \\ &\vdots \\ y_k &= b+mx_k \end{aligned}\right\} \text{system of equations!} : \underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_k \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b \\ m \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}}_{\vec{b}}$$

← very small chance of having a solution due to many zero rows
↳ may be inconsistent

↳ $$A\vec{x} - \vec{b} = \begin{bmatrix} mx_1 + b - y_1 \\ mx_2 + b - y_2 \\ \vdots \\ mx_k + b - y_k \end{bmatrix}$$

$$\|A\vec{x} - \vec{b}\| = \sqrt{(mx_1+b-y_1)^2 + (mx_2+b-y_2)^2 + \dots + (mx_k+b-y_k)^2}$$

- Ex: 2020 Olympic Table Tennis, Womens Singles, Round 3 (first no-bye)

Point Difference	Final Place
26	1st
25	2nd
15	3rd
10	9th
-4	17th

$$A = \begin{bmatrix} 1 & 26 \\ 1 & 25 \\ 1 & 15 \\ 1 & 10 \\ 1 & -4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 9 \\ 17 \end{bmatrix}$$

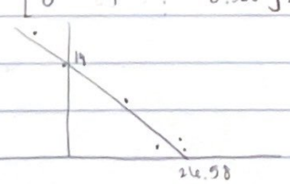
$A\vec{x} = \vec{b}$ will be inconsistent :(

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 26 & 25 & 15 & 10 & -4 \end{bmatrix} \begin{bmatrix} 1 & 26 \\ 1 & 25 \\ 1 & 15 \\ 1 & 10 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 72 \\ 72 & 1642 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 26 & 25 & 15 & 10 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 32 \\ 143 \end{bmatrix}$$

Solve. $$\left[\begin{array}{cc|c} 5 & 72 & 32 \\ 72 & 1642 & 143 \end{array}\right] \times \tfrac{1}{5} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 14.4 & 6.4 \\ 72 & 1642 & 143 \end{array}\right] -72R_1 \rightsquigarrow \left[\begin{array}{cc|c} 1 & 14.4 & 6.4 \\ 0 & 605.2 & -317.8 \end{array}\right] -14.4R_2$$

↳ $$\left[\begin{array}{cc|c} 1 & 0 & 13.96 \\ 0 & 1 & -0.525 \end{array}\right] \begin{array}{l} = b \\ = m \end{array} \Bigg\} \; y = 13.96 - 0.525x$$



4th seed: -15 point difference
↳ $y = 13.96 - 0.525(15) = 21.84$ (actual 17th)

5th seed: 14 point difference
↳ $y = 13.96 - 0.525(14) = 6.61$ (actual 5th)

pretty close!