Independence & Bases

* The set of all linear combinations is called the <u>span</u> of the $\vec{x_i}$ and is written
  span $\{\vec{x_1}, \vec{x_2}, \ldots \vec{x_k}\} = V = \{t_1\vec{x_1} + t_2\vec{x_2} + \ldots + t_k\vec{x_k}$ for $t_i \in \mathbb{R}\}$,
  V is spanned by the vectors / the vectors span V
* V is a subspace that each of the $\vec{x_i}$
* Every subspace that contains each $\vec{x_i}$ contains all of V
* A set $\{x_1, x_2, \ldots x_k\}$ of vectors is <u>linearly independent</u> if the only solution to
  $t_1x_1 + t_2x_2 + \ldots t_kx_k = 0$ is the trivial solution, $t_1 + t_2 + \ldots + t_k = 0$
* If $\{\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k}\}$ is an independent set in $\mathbb{R}^n$, then every vector in the span of $\{\vec{x_1}, \vec{x_2}, \ldots \vec{x_k}\}$ has a <u>unique</u> representation as a linear combination of $\vec{x_i}$
* The $t_1, t_2, \ldots t_k$ are coordinates with respect to the $\vec{x_i}$
  $\vec{x} = t_1\vec{x_1} + t_2\vec{x_2} + \ldots + t_k\vec{x_k} = s_1\vec{x_1} + s_2\vec{x_2} + \ldots s_k\vec{x_k}$ (subtract)
  $0 = (t_1 - s_1)x_1 + (t_2 - s_2)x_2 + \ldots (t_k - s_k)x_k$ ∴ The only solution is the only solution
  $t_1 = s_1,\ t_2 = s_2,\ t_k = s_k$
  ↳ The representations are identical

- Ex: Are the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}3\\2\\1\end{bmatrix} \begin{bmatrix}-1\\2\\5\end{bmatrix}$ linearly independent?

$$t_1\begin{bmatrix}1\\2\\3\end{bmatrix} + t_2\begin{bmatrix}3\\2\\1\end{bmatrix} + t_3\begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$ ⤳ Check if the trivial solution is the only solution (or can check if invertible!) determinant $\neq 0$

$$\left[\begin{array}{ccc|c}1&3&-1&0\\2&2&2&0\\3&1&5&0\end{array}\right]\begin{array}{c}\\-2R_1\\-3R_1\end{array} \rightsquigarrow \left[\begin{array}{ccc|c}1&3&-1&0\\0&-4&4&0\\0&-8&8&0\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1&3&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right] \leftarrow \text{zero row}$$
∴ not linearly independent (are dependent)

↳ Find t_1, t_2, t_3 such that $t_1v_1 + t_2v_2 + t_3v_3 = 0$ for not all $t_i = 0$

$$\left[\begin{array}{ccc|c}1&3&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1&0&2&0\\0&1&-1&0\\0&0&0&0\end{array}\right] \begin{cases} t_1 + 2t_3 = 0 \\ t_2 - t_3 = 0 \\ \text{pick } t_3 = 1 \end{cases} \rightsquigarrow \begin{bmatrix}-2\\1\\1\end{bmatrix}$$

- Ex: Are the vectors $\begin{bmatrix}4\\7\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}8\\6\end{bmatrix}$ linearly independent?
  No, because there are too many vectors
  ↳ Too many columns will always result in a free variable!
- Ex: Do the vectors $\begin{bmatrix}12\\2\\-3\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix}$ span $\mathbb{R}^3$
  No, because only 2 vectors will give a plane (not enough vectors)
  ↳ Too many rows will always result in a zero row, and there is a right hand side in $\mathbb{R}^3$ that gives an inconsistent solution

* If V is a subspace of $\mathbb{R}^n$, then a set $\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_m\}$ is a basis of V if
① $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ are linearly independent
② $V = \text{span}\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_m\}$
* If $\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_m\}$ and $\{\vec{y}_1, \vec{y}_2, \dots \vec{y}_k\}$ are both bases for V, then $m = k$
* If V is a subspace of $\mathbb{R}^n$ and $\{x_1, x_2, \dots \vec{x}_m\}$ is any basis of V the dimension of V is the number of basis vectors. ($\dim V = m$)

Ex: Are the vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ a basis for $\mathbb{R}^3$?

① Check for linear independence (no free variable $\therefore$ the matrix is invertible)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \det = 1 \neq 0 \quad \therefore \text{ Linearly independent!} \checkmark$$

↳ Lower triangular $\therefore$
det = product of diagonal

② Check that the vectors span the space of $\mathbb{R}^3$
↳ True as long as there's no zero row
↳ True if the matrix is invertible (already proved :))
$\therefore$ The vectors span $\mathbb{R}$ and they are a basis

Invertible Matrix Theorem

The following is equivalent for an $n \times n$ matrix A:
① A is invertible
② The homogenous system $A\vec{x} = \vec{0}$ has only the trivial solution
③ A is row-equivalent to the identity
④ $A\vec{x} = \vec{b}$ has at least 1 solution for all $\vec{b}$
⑤ There exists a matrix C such that $AC = I$
⑥ A is a product of elementary matrices
⑦ Columns of A are linearly independent
⑧ Columns of A span $\mathbb{R}^n$
⑨ Columns of A are a basis for $\mathbb{R}^n$
⑩ Rows of A are linearly independent
⑪ Rows of A span $\mathbb{R}^n$ *
⑫ Rows of A are a basis for $\mathbb{R}^n$

} because the matrix transpose has the same determinant $\therefore$ A is invertible...

* Let $V \subseteq W$ be subspaces of $\mathbb{R}^n$
    ↳ $\dim V \leq \dim W$
* If $\dim V = \dim W$, then $V = W$
* $\{\vec{0}\}$ is a subspace with no basis vectors
    ↳ $\dim \{\vec{0}\} = 0$ (no dimensions)
* $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$
    ↳ $\dim(\mathbb{R}^n) = n$
* Every other subspace V of $\mathbb{R}^n$ satisfies $0 < \dim(V) < n$
    ↳ These are proper subspaces