# Unit 6

## 6.1
## Basics of Counting

### Product Rule

Definition
There are $n_1$ ways to do task 1 and $n_2$ ways to do task 2.
Also, for each way you do task 1, there $n_2$ ways to do task 2, then there are $n_1 * n_2$ ways to do task 1 and 2.

Examples
There are 5 fruits and 6 vegetables for sale at a grocery store
If you just want to buy exactly 1 fruit and exactly 1 vegetable, how many ways can you shop?
$5 * 6 = 30$

How many bit strings of length 5 are there?
$2^5 = 32$

How many subsets does a set with 5 elements have?
$2^5 = 32$

3 friends go to a movie. How many ways can they sit in a row?
$3! = 6$

How many functions are there from A to B?
$|A| = 3$
$|B| = 4$
$4^3 = 64$

How many injective (1-to-1) functions are there from A to B?
$|A| = 3$
$|B| = 5$
$\frac{5!}{2!} = 60$

How many items in a cartesian product between A and B?
$|A| * |B| = ...$

### Inclusion-Exclusion (Subtraction) Rule

Definition
$|A \cup B| = |A| + |B| - |A \cap B|$
When counting, make sure you take care of double counting

### Division Rule

Example
4 people sit around a round table. 2 arrangements are considered the same if the same person sits t your left or to your right.
How many different arrangements are there?
$3! = 6$

## 6.2
## Pigeonhole Principle

Pigeonhole Principle

Definition
Let $k$ be an integer
If $k+1$ or more objects are placed in $k$ boxes, then at least one box would contain two or more objects

Example
If 7 balls are put into 6 boxes, at least one box would have at least 2 balls

Corollary of the Pigeonhole Principle

Definition
A function from a set with $k+1$ elements to a set with $k$ elements is not an injection.
Some 2 elements in the domain must be mapped to the same element in the codomain

Example
Among 367 people, 2 must have the same birthday
Proof
There are 366 days in a year
Since you have 367 people, by the pigeonhole principle, at least 2 people must have the same birthday

Generalized Pigeonhole Principle

Definition
If $n$ objects are placed in $k$ boxes, then at least 1 box has at least $[\frac{n}{k}]$ (ceiling brackets) objects

Example
If there are 13 balls and 6 boxes
At least one box has $[\frac{13}{6}]$
$[\frac{13}{6}] = [2...] = 3$

## 6.3
## Permutations/Combinations

Permutation

Definition
An ordered arrangement of a set of objects
An r-permutation is an ordered arrangement of objects chosen from a set of objects

Example
$S = \{1, 2, 3, 4\}$

## Theorem

Definition
If $n$ is a positive integer and $0 \leq r \leq n$, then there are $P(n,r) = \frac{n!}{(n-r)!}$ r-permutations

Example
$\{A, B, C, D, E, F\}$
Find number of 4 permutations
$\frac{6!}{(6-4)!} = 360$

## Combination

Definition
An r-combination is an unordered arrangement of $r$ objects chosen from a set
The number of r-combinations chosen from $n$ objects is denoted $C(n, r)$

## Theorem

Definition
$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!}$ r-combinations

Example
$\{A, B, C, D, E\}$
Find number of 3 combinations
$\frac{5!}{3!(5-3)!} = 10$

## Theorem

Definition
$C(n,r) = C(n, n-r)$, $r \leq n$

Example
How many bit strings of length 7 has exactly 3 ones
$= \frac{7!}{3!(7-3)!} = 35$

## Practice

1. Bit strings of length 6

a) How many bit strings total
$2^6 = 64$

b) How many with exactly 2 1's
$\frac{6!}{2!(6-2)!} = 15$

c) How many with at least 2 1's

$2^6 - C(6,1) + C(6,2) = 57$

2. $\{A, B, C, D, E, F\}$

a) How many permutations
$6! = 720$

b) How many 4-permutations
$\frac{6!}{(6-4)!} = 360$

c) How many 4-combinations
$\frac{360}{4!} = 15$

d) How many 4-permutations containing A or F
$360 - P(4,4) = 336$

e) How many 4-combinations with either A or F
$15 - C(4,4) = 1$

## 6.4
## Binomial Coefficients and Identities

Binomial Theorem

Definition
Let $x$ and $y$ be variables and integer $n \leq 0$. Then,
$(x+y)^h = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + ... + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$

Example
Find the coefficient of $x^5y^7$ in the expansion of $(x+y)^{12}$
$(x+y)^12 = \sum_{i=0}^{12} \binom{12}{i} x^{12-i} y^i$
$\binom{12}{7} x^5 y^7$
$\binom{12}{7} = \frac{12!}{7!5!}$

Theorem

Definition
$\sum_{k=0}^{n} = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n-1} + \binom{n}{n}$

Theorem

Definition
$\sum_{k=0}^{n}(-1)^k\binom{n}{k}$

Theorem

Definition
$\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$

Pascal's Theorem

Definition
$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$