Dot Product in $\mathbb{R}^n$

- Ex. Find the coordinates of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to both the elementary basis and the basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightsquigarrow \begin{cases} c_1 = 1 \\ c_2 = 2 \\ c_3 = 3 \end{cases}$$

$\vec{e}_1 = \hat{x}$ $\quad \vec{e}_2 = \hat{y}$ $\quad \vec{e}_3 = \hat{z}$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -1 & -1 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\hookrightarrow \begin{cases} c_1 = 1 \\ c_2 = 2 \\ c_3 = 6 \end{cases}$$

* Dot Product of 2 vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ is given by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

* $(\vec{x})^T \vec{y} = \vec{x} \cdot \vec{y}$
  - Proof: $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ Becomes matrix multiplication, and results in 1×1 matrix with same answer

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} \vec{x} \cdot \vec{y} \end{bmatrix}$$

* $\vec{x} \cdot \vec{x} = x_1 x_1 + x_2 x_2 + \ldots + x_n x_n = x_1^2 + x_2^2 + \ldots + x_n^2 = \|\vec{x}\|^2$

  length / magnitude of $\vec{x}$ (squared)

* Properties of the Dot Product

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$
$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$
$$(a\vec{x})\vec{y} = a\vec{x}\vec{y} = \vec{x}(a\vec{y})$$
$$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$$
$$\|\vec{x}\| \geq 0$$
$$\|\vec{x}\| = 0 \text{ iff } \vec{x} = \vec{0}$$
$$\|a\vec{x}\| = |a| \, \|\vec{x}\|$$

* Assuming Euclidean Geometry, we can define the angle between two vectors by

$$\vec{x} \, \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

* 2 vectors in $\mathbb{R}^n$ are orthogonal if $\vec{x} \cdot \vec{y} = 0$
* Distance between 2 vectors in $\mathbb{R}^n$ is $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$
* Cauchy Inequality

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$$

* Triangle Inequality

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

* A set of nonzero vectors is orthogonal if $\vec{x}_i \cdot \vec{x}_j = 0$ for all $i \neq j$
* An orthogonal set of vectors is orthonormal if $\vec{x}_i \cdot \vec{x}_i = 1$ for each $\vec{x}_i$

  $\|\vec{x}_i\| = 1$

* Every set of orthogonal vectors in $\mathbb{R}^n$ is linearly independent

$$t_1 \vec{x}_1 + t_2 \vec{x}_2 + \ldots + t_k \vec{x}_k = \vec{0}$$
$$t_1 \vec{x}_1 \cdot \vec{x}_1 + t_2 \vec{x}_2 \cdot \vec{x}_1 + \ldots + t_k \vec{x}_k \cdot \vec{x}_1 = \vec{0} \cdot \vec{x}_1$$

$t_1 \|x_1\|^2 + 0 + \ldots + 0 = 0$ / $t_2 \|x_2\|^2 = 0$ ... $\therefore$ only trivial solution

$t_1 = 0$ $\qquad$ $t_2 = 0$

* If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ are an orthonormal set, then we have the Pythagorean Theorem

$$\|x_1 + x_2 + \ldots + x_k\|^2 = \|x_1\|^2 + \|x_2\|^2 + \ldots + \|x_k\|^2$$

- Ex. Show that the vectors $\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ are orthogonal, the scale them to form an orthonormal set

$$\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0+0+0=0 \checkmark$$

$\therefore$ This is an orthogonal set

$$\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = 12+0-12=0 \checkmark$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = 0+0+0=0 \checkmark$$

For an orthonormal set, all lengths of the vectors must be 1

$$\|\langle 3, 0, -4 \rangle\| = \sqrt{3^2+0^2+(-4)^2} = \sqrt{25} = 5$$

$$\|\langle 0, 1, 0 \rangle\| = \sqrt{0^2+1^2+0^2} = \sqrt{1} = 1$$

$$\|\langle 4, 0, 3 \rangle\| = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$$

$$V_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \frac{1}{5}\vec{x}_1 \qquad V_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \vec{x}_2 \qquad V_3 = \frac{\vec{x}_3}{\|\vec{x}_3\|} = \frac{1}{5}\vec{x}_3$$

$$\underbrace{V_1 = \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad V_3 = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}}_{\text{orthonormal set}}$$

- Ex: Find a basis for the space of vectors orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

$$1x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$$
$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right]$$

* Solving the homogeneous solution $A\vec{x} = 0$, where A is the matrix whose rows are the given set

* The set of all solutions to the homogeneous solution $A\vec{x} = 0$ is called the null space of A : null (A)

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -4 & -8 & -12 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

2 free variables!

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases} \qquad \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

* Can check if the dot products of the 4 vectors are 0!