Systems of Equations

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Recap

$4x + 3y = 10$
$x - 2y = -3$

Methods of solving

- Substitution
- Elimination
- Graphing

$$\begin{Bmatrix} x - 2y + 3z = 11 \\ 4x + y - z = 4 \\ 2x - y + 3z = 10 \end{Bmatrix}$$

Method of solving

1. Start by eliminating x from equations 2 and 3
2. Eq 1 stays the same, multiply eq 2 by -4 and eq 3 by -2
3. These two systems are equivalent if they have the same set of solutions
4. Divide eq 3 by 3
5. Switch eq 2 and eq 3
6. Move the 4 in eq 2 to the left side
7. Eq 3 becomes eq 3 - 9 times eq 2
8. Divide eq 3 by -4
9. Solve for y in eq 2

10. Solve for x
11. Write variable values as a system with the left bracket
12. Verify the solution by substituting into the given eqs

Elementary Row Operations (gives equivalent augmented matrices)

- Interchange two rows
- Multiply an row by a nonzero number
- Add a multiple of one row to another

Augmented Matrix (Coefficient Array)

$$\begin{bmatrix} 1 & -2 & 3 & 11 \\ 4 & 1 & -1 & 4 \\ 2 & -1 & 3 & 10 \end{bmatrix}$$

1.2

$x - y + z = 0$
$2x - 3y + 4z = -2$
$-2x - y + z = 7$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{bmatrix}$$

$R_2 - 2 * R_1$
$R_3 + 2 * R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{bmatrix}$$

$R_2 * -1$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 3 & 7 \end{bmatrix}$$

$R_3 + 3R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -3 & 13 \end{bmatrix}$$

$R_3/3$

Row-Echelon Form (The 3 zeroes in the bottom left)

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -13/3 \end{bmatrix}$$

$R_1 - R_3$
$R_2 + 2 * R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 13/3 \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{bmatrix}$$

$R_1 + R_2$

Reduced Row-Echelon Form (The 3 zeroes in the top right)

$$\begin{bmatrix} 1 & 0 & 0 & -7/3 \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{bmatrix}$$

A matrix is in row-echelon form if

1. Any zero rows are at the bottom
2. The first nonzero entry of each row is 1
3. Each leading 1 is to the right of the leading 1s above it

If, in addition

1. Each leading 1 is the only nonzero entry in its column

We say the matrix is in reduced row-echelon form

Theorem
Every matrix can be brought into (R)REF form by a sequence of row operations

Gaussian Algorithm (Gauss-Jacobi, Gaussian Elimination)

1. If a matrix is all zeroes it is in RREF
2. Find the first column containing a nonzero entry and if necessary, swap rows so that the top row has a nonzero entry
3. Multiply the top row so that the leading entry is a 1

4. Subtract multiples of the first row from the rows below it to make each entry below the leading 1 zero
5. Repeat steps 1 - 4 on the matrix formed by disregarding the top row
6. The matrix is now in REF
7. Repeat steps 1 - 5 except working from right-to-left and bottom-to-top (essentially inverse the matrix locations in the steps
8. The matrix is now in RREF

If a bottom row is all zeroes and you end up with 3 variables but 2 equations in REF, there are infinitely many solutions

Free variables are when the variable is equal to itself
Leading variables are ones that depend on the free variable

for free variable ex.
Let $z = t$
t is a parameter

$x = -1 + t$
$y = 1 - t$
$z = t$
for all real numbers t

1.3

A homogeneous system is a system with all zeroes on the right-hand side

All zeroes is always a solution to a homogeneous system and we call it the trivial solution

Any solution with at least one nonzero variable is a nontrivial solution

Any scalar multiple of a solution to a homogeneous system is also a solution to the system

A linear combination of solutions is the sum of scalar multiples of those solutions

If you have the parameter solutions, you can work backwards to find the two basic solutions; ex. $t = 1$, $s = 0$ and $t = 0$, $s = 1$