Diagonalization

- Eigenvalue and eigenvector review
  - Eigenvalues
    - $\det(A - \lambda I) = 0$ gave an n^{th} order polynomial with up to n solutions
    - Multiplicity of an eigenvalue λ is the number of times that λ appears as a root of the characteristic polynomial
  - Eigenvectors
    - $(A - \lambda I)\vec{v} = \vec{0}$ and solve for the basic solutions / basic eigenvectors
  - Ex: Find the eigenvalue(s) and eigenvector(s) of:

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & -\lambda & -2 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$\hookrightarrow (2-\lambda)[(-\lambda)(3-\lambda)+2] = (2-\lambda)[\lambda^2 - 3\lambda + 2] = (2-\lambda)(\lambda-2)(\lambda-1) = -(\lambda-1)(\lambda-2)^2$

$\hookrightarrow \lambda = 1, 2$ ← multiplicity of 2!

$\hookrightarrow \lambda = 1$

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2-1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & 3-1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{matrix} \\ \times -1 \\ +R_2 \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} -2R_2 \\ \\ \\ \end{matrix} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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↳ $\begin{cases} x - 2z = 0 \\ y + 2z = 0 \\ z \text{ is free, pick } 1 \end{cases} \rightsquigarrow \begin{cases} x = 2z = 2(1) = 2 \\ y = -2z = -2(1) = -2 \\ z = 1 \end{cases} \rightsquigarrow \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \vec{v}$

↳ $\lambda = 2$

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} -2R_3 \\ +2R_3 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{cases} y + z = 0 \rightsquigarrow z = -y \\ x \text{ is free} \\ z \text{ is free} \end{cases}$$

↳ Pick $x = 1, z = 0$ / Pick $x = 0, z = 1$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

* An $n \times n$ matrix A is <u>diagonalizable</u> if $P^{-1}AP$ is diagonal for some invertible $n \times n$ matrix P (diagonalizing matrix)

- $P^{-1}AP = D$ (some diagonal matrix

$PP^{-1}AP = PD$

$AP = PD$

$A[\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n] = [\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n] \begin{bmatrix} d_1 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & \cdots \\ 0 & 0 & \ddots & d_n \\ \vdots & \vdots & & \end{bmatrix}$

$A\vec{x}_1 = d_1\vec{x}_1 \qquad (A - d_1 I)\vec{x}_1 = \vec{0}$

$A\vec{x}_2 = d_2\vec{x}_2 \rightsquigarrow (A - d_2 I)\vec{x}_2 = \vec{0}$

$\vdots \qquad \vdots$

$A\vec{x}_n = d_n\vec{x}_n \qquad (A - d_2 I)\vec{x}_n = \vec{0}$

* P is the matrix of all eigenvectors

* D is the matrix of all eigenvalues along the diagonal

* Let A be an $n \times n$ matrix where

- A is diagonalizable iff it has eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ such that $P = [\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n]$ is invertible
- when this is the case, $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$, where λ_k corresponds to $\vec{x}_k$ for $1 \le k \le n$
- Ex. $P = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (see previous example)

$\therefore P^{-1}AP = D \rightsquigarrow$ Check $AP = PD$

$AP = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 0 & -2 \\ 1 & 0 & 2 \end{bmatrix}$

column-scaled version of P

$PD = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 0 & -2 \\ 1 & 0 & 2 \end{bmatrix}$ ✓

* A square matrix is diagonalizable iff every eigenvalue of multiplicity m (yields m basic eigenvectors / has m-dimensional solution / has m basis vectors / has geometric multiplicity of 2)
  - AKA the algebraic and geometric multiplicities are the same for each individual eigenvalue

- Ex: Determine if A is diagonalizable. If so, find P and D.

① $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $\lambda = 1$ with $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = 2$ with $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Yes, it is diagonalizable

$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

② $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ $\lambda = 2$ with $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

No, it is not diagonalizable

P would be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, but not invertible

③ $A = \begin{bmatrix} 4 & 0 \\ 39 & 3 \end{bmatrix}$ $\lambda = 4$ with $\vec{v} = \begin{bmatrix} 1 \\ 39 \end{bmatrix}$ and $\lambda = 3$ with $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Yes, it is diagonalitable

$P = \begin{bmatrix} 1 & 0 \\ 39 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ OR $P = \begin{bmatrix} 0 & 1 \\ 1 & 39 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$