Lecture 4: Simple Linear Regression (3.1.1)

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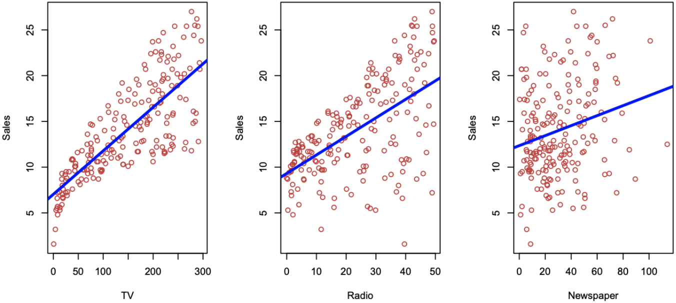
- Understanding linear regression is crucial for studying more complex learning methods.
- Many advanced statistical learning approaches can be seen as generalizations or extensions of linear regression.
- **True regression functions are never linear!**

Example: Advertising dataset (Lecture 2)

- The Figure displays sales (in thousands of units) for a particular product as a function of advertising budgets (in thousands of dollars) for TV, radio, and newspaper media.



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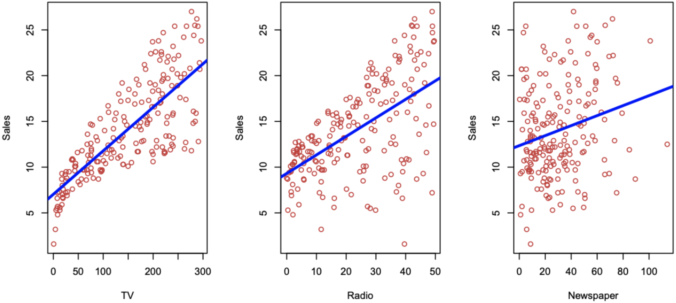

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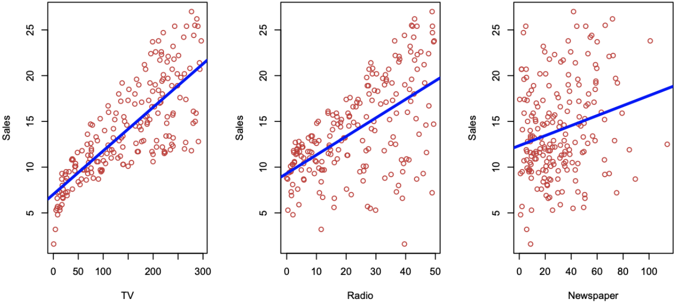
- What information would be useful in order to provide such a recommendation?.



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- How large is the association between each medium and sales?
  - *For every dollar spent on advertising in a particular medium, by what amount will sales increase? How accurately can we predict this amount of increase?*
- How accurately can we predict future sales?
  - *For any given level of television, radio, or newspaper advertising, what is our prediction for sales, and what is the accuracy of this prediction?*

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- Is there synergy among the advertising media?
  - *Perhaps spending $50,000 on television advertising and $50,000 on radio advertising is associated with higher sales than allocating $100,000 to either television or radio individually. In marketing, this is known as a synergy effect, while in statistics it is called an interaction effect.*

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- The hat symbol denotes the estimated value for an unknown parameter or coefficient, or the predicted value of the response.

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  - $\bar{y}$ and $\bar{x}$ are the sample means.

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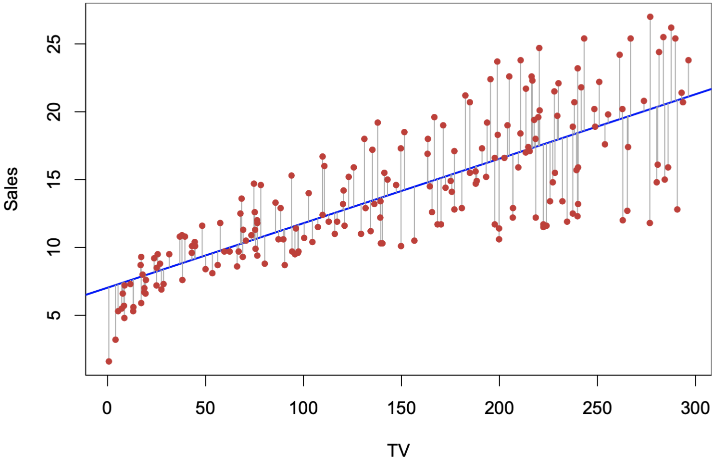
For the Advertising data, the fit is found by minimizing the residual sum of squares. Each grey line segment represents a residual. In this case a linear fit captures the essence of the relationship, although it overestimates the trend in the left of the plot.



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  - The true relationship is probably not linear,
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  - There may be measurement errors.
  - We typically assume that the error term is independent of X.

Question

If the true relationship between X and Y is actually linear, can we have $\hat{\beta}_0 = \beta_0$ and $\hat{\beta}_1 = \beta_1$?