Lecture 4: Simple Linear Regression Assumptions

Module 1: part 3

Spring 2024

Logistics

- Wrap up Module 1 today
- Module assessment due on Feb 11 11:59pm
- Module 2 next week will consider regression with multiple covariates
- Office hour locations: Daniel and Tathagata (Comstock 1187); Nayel in Surge B 159.

Recap

The equation for a line can be put into the following form

$$Y = b_0 + b_1 X \tag{1}$$

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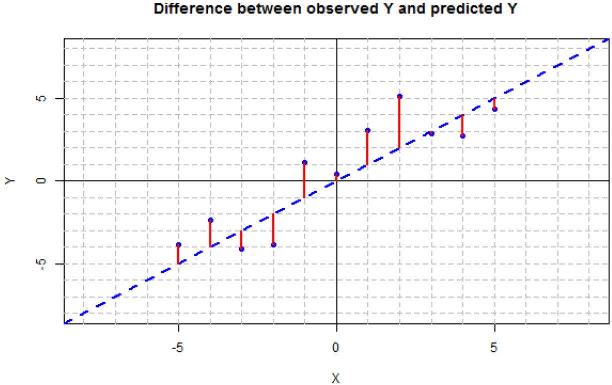
- X and Y are variables
- b_0 is the **Y-intercept**. It is the value of the Y coordinate when $X = 0$
- b_1 is the **slope**. It describes how Y changes as X changes.

Recap

Suppose we observe $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

To select a "best line" we consider the difference between the predicted point and observed value of y_i and choose $\hat{b}_0$ and $\hat{b}_1$ to minimize the RSS:

$$RSS(\hat{b}_0, \hat{b}_1) = \sum_i (y_i - \hat{y}_i)^2 = \sum_i (y_i - (\hat{b}_0 + \hat{b}_1 x_i))^2 \tag{2}$$



Recap

Outliers:

- Points which have x values far from $\bar{x}$ have high leverage
- Points which have high leverage may also have high influence; i.e., change the estimate when included/excluded
- When to include or exclude points with high influence?

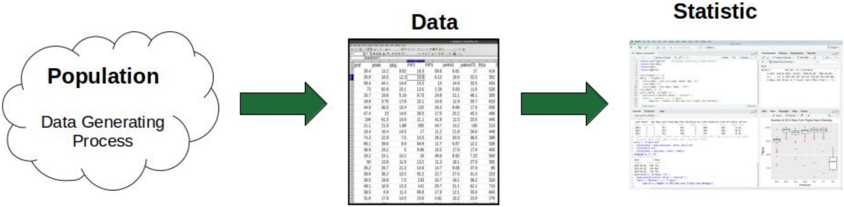
Linear Model

Interpretation

Let's take a step back and consider what we have calculated

- Still have "hat's" on $\hat{b}_0$ and $\hat{b}_1$ because they are calculated from the sample data
- We want to use the sampled data to infer something about the population

Sample data vs population distribution



Linear Models

Much of what we've talked about so far involves calculating coefficients which describe a specific set of data

- Given a sample of data $(x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$, calculate line which minimizes RSS
- Sample is all we have, but most often we are interested in quantities which describe a population
- Given a new sample (potentially repeating the experiment) will give different estimates of $\hat{b}_0$ and $\hat{b}_1$
- What can we say about $\hat{b}_0, \hat{b}_1$ and the "true" population process?

Linear Model Assumptions

Commonly used linear model where ε_i is an error term:

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- Linear function: $E(Y_i \mid X_i = x) = b_0 + b_1 x$
- Independence across observations: ε_i is independent of ε_k where i and k denote different observations
- Independence of errors: ε_i is independent of X_i with mean 0 and variance σ^2

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Less important assumption:

- Normality: sometimes, we assume that $\varepsilon_i \sim N(0, \sigma^2)$

Model Implications

Conditional expectation: $E(Y_i \mid X_i = x) = b_0 + b_1 x$

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Interpretation

- b_0 is the expected value of Y_i when conditioning on $X_i = 0$
- b_1 is the difference of the expected value of Y_i when conditioning on values of X_i which differ by 1 unit.

$$b_1 = E(Y_i \mid X_i = x + 1) - E(Y_i \mid X_i = x)$$

Conditional Expectation

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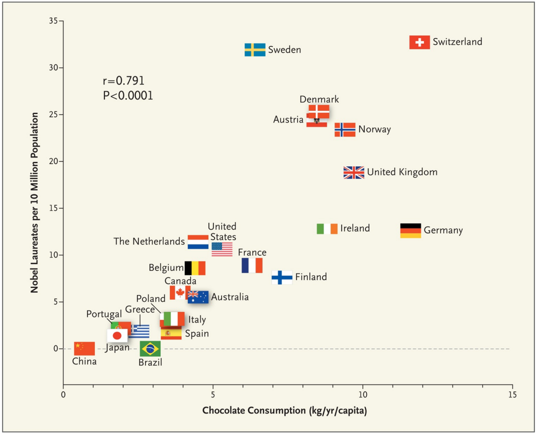

Figure: Messerli 2012, NEJM

Interpretation

Correct Interpretations

- Given two observations whose X values differ by 1 unit, we would **expect** the observation with the larger X value to have a Y value b_1 units larger than the observation with the smaller X value
- Given two observations whose X values differ by 1 unit, **on average** the observation with the larger X value will have a Y value b_1 units larger than the observation with the smaller X value

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Incorrect Interpretations

- Increasing X by 1 unit increases Y by b_1 units
- A 1 unit increase in X causes Y to increase by b_1 units

Statistic is unbiased

Under the assumptions that ε_i is independent of X_i, we have:

$$E(\hat{b}_1) = b_1$$
$$E(\hat{b}_0) = b_0$$

so that the estimated values are "unbiased" estimators of the true values

- If you replicate the experiment many different times, you will get a different estimate, each time, but the average will be the "truth"

Potentially helpful (but not necessary) math

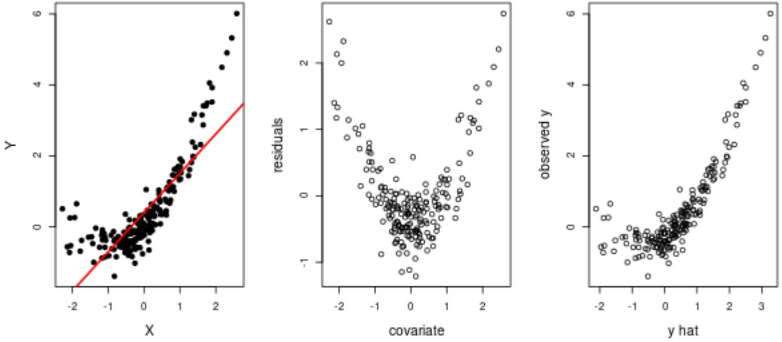
Under the assumptions, we have:

$$\bar{y} = \frac{1}{n}\sum_i (b_0 + b_1 x_i + \varepsilon_i) = b_0 + \frac{1}{n}\sum_i b_1 x_i + \frac{1}{n}\sum_i \varepsilon_i = b_0 + b_1\bar{x} + \bar{\varepsilon}$$

$$\begin{aligned}
E(\hat{b}_1 \mid X) &= E\left(\frac{\sum_i (y_i - \bar{y})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} \mid X\right) \\
&= E\left(\frac{\sum_i (b_0 + b_1 x_i + \varepsilon_i - b_0 + b_1\bar{x} + \bar{\varepsilon})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} \mid X\right) \\
&= E\left(\frac{b_1 \sum_i (x_i - \bar{x})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} \mid X\right) + \underbrace{E\left(\frac{\sum_i (\varepsilon_i - \bar{\varepsilon})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} \mid X\right)}_{\mathrm{cov}(\varepsilon_i, X_i) = 0} \\
&= b_1 + 0
\end{aligned}$$

Model Assumptions: Linearity

Look for patterns in residuals if the linearity assumption is violated



Model Assumptions: Linearity

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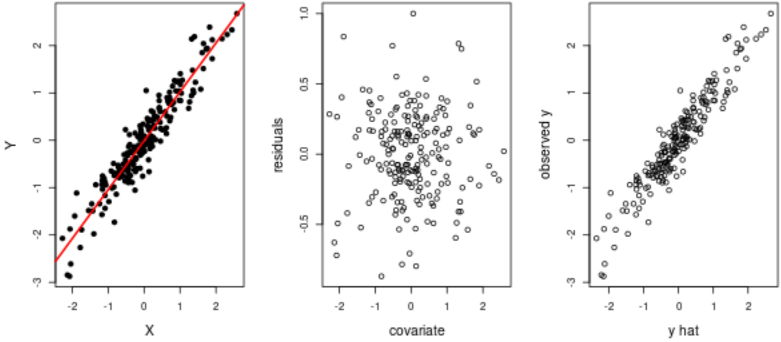


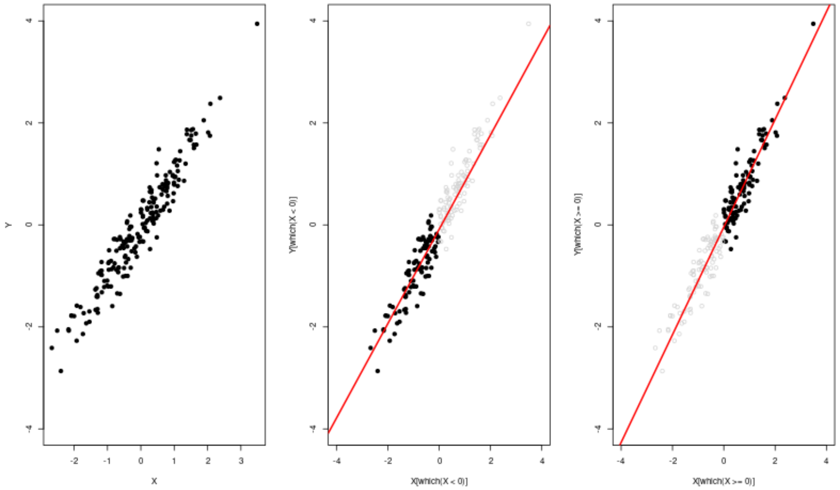
Model Assumptions: Linearity

What happens if the linearity assumption is violated?

- Consider transforming your data with a non-linear transformation
- Adding other covariates can be "helpful"
- b_1 no longer corresponds to change in conditional expectation, but the sign of coefficient can still be useful for interpretability
- Parameters are the best "linear approximation"
- Best linear approximation depends on the range of the X values

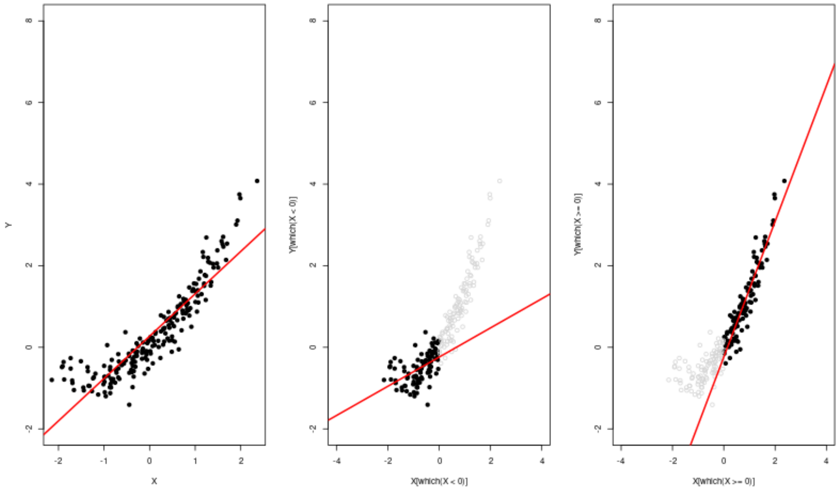
Model Assumptions: Linearity

Best linear approximation depends on the range of the X values



Model Assumptions: Linearity

Best linear approximation depends on the range of the X values

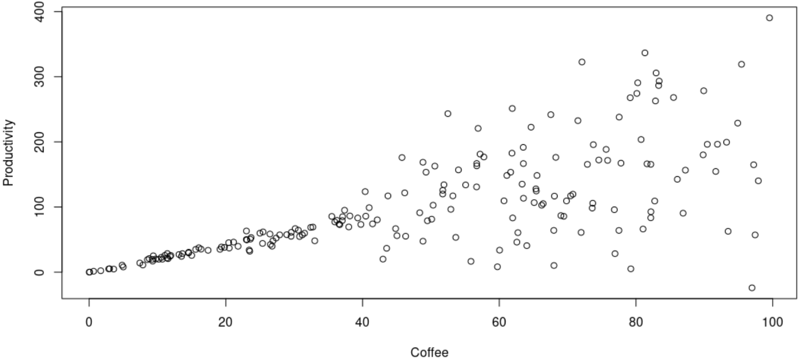


Model Assumptions: independence across observations

Model Assumptions: independence of error and covariate

We made a strong assumption that ε_i is mean 0 and independent of X_i

- What if the variance of ε_i depends on X_i? i.e., model is heteroscedastic
- As long as $E(\varepsilon_i \mid X_i) = 0$, estimates are still unbiased $E(\hat{b}_1) = b_1$
- Will effect testing procedures!



Discussion

- What is a scientific question that you are interested in?
- Are you trying to do prediction or modeling?
- Are the assumptions we discussed today reasonable for your setting?
  - Linearity
  - Independence across observations
  - Independence of errors and covariates

Assessing explanatory power

Components of the squared error

How can we assess how useful the explanatory variable is for predicting the response variable?

$$
\begin{aligned}
(y_i - \bar{y}) &= (y_i - \hat{y}_i + \hat{y}_i - \bar{y}) \\
&= (y_i - \hat{y}_i) + (\hat{y}_i - \bar{y}) \\
&= \text{residual} + \text{predicted deviation from mean}
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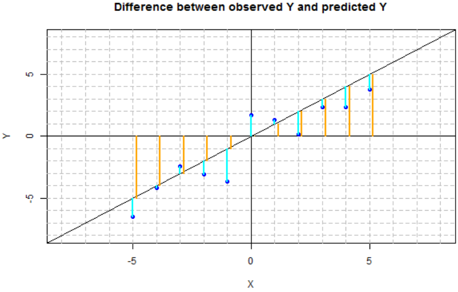
Using a bit of algebra, we can decompose the total sum of squares for Y into

$$
SS_{total} = \sum_i (y_i - \bar{y})^2 = \underbrace{\sum_i (\hat{y}_i - \bar{y})^2}_{SS_{\text{regression}}} + \underbrace{\sum_i (y_i - \hat{y}_i)^2}_{SS_{\text{error}}} \tag{4}
$$

Components of the squared error

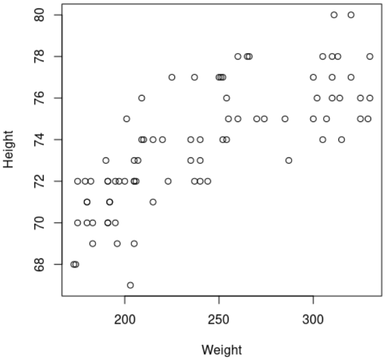
If $SS_{regression}$ is large compared to SS_{error}, then the explanatory variable is a good predictor of the response variable

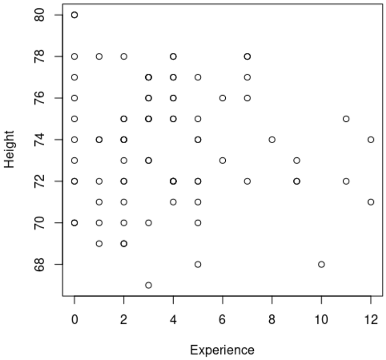
$$\underbrace{1 - \frac{SS_{error}}{SS_{total}} = \frac{SS_{regression}}{SS_{total}} = \frac{\sum_i(\hat{y}_i - \bar{y})}{\sum_i(y_i - \bar{y})} = r^2_{XY}}_{\text{Referred to as } R^2} \tag{5}$$



Example: Components of the squared error

The R^2 for height and weight is .59 while the R^2 for height and experience is .01.





Wrap-up

- If we assume the true population process is a linear model, we can describe properties of the estimated regression coefficients
- Estimated slope is estimated difference in conditional expectation associated with difference in X
- If assumptions are violated, interpretation is not as straightforward
- Explanatory power of regression can be summarized by R^2 value
- Next module will consider setting with more than 1 covariate