Lecture 2: Statistical Learning (Textbook 2.1)

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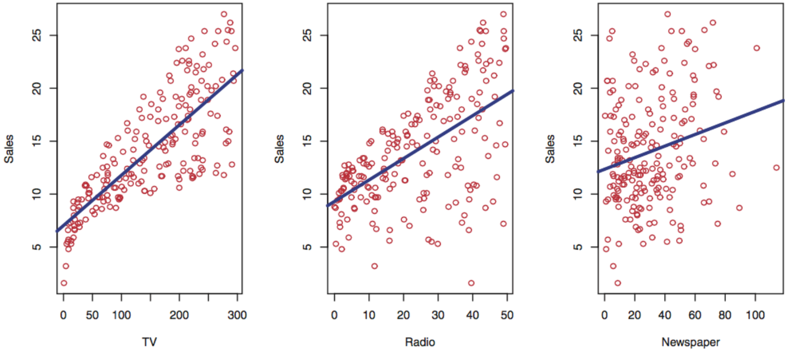
A Simple Example

- **Advertising data set**: sales of a product in 200 different markets, along with advertising budgets for the product in each of those markets for three different media: TV, radio, and newspaper.



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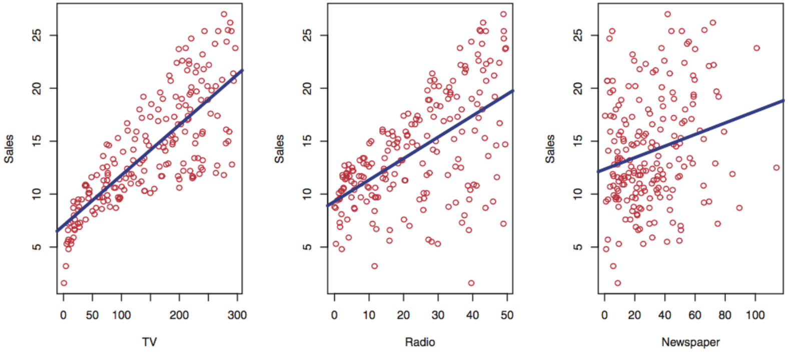
- Suppose that we are statistical consultants hired to investigate the association between advertising and sales of this product.



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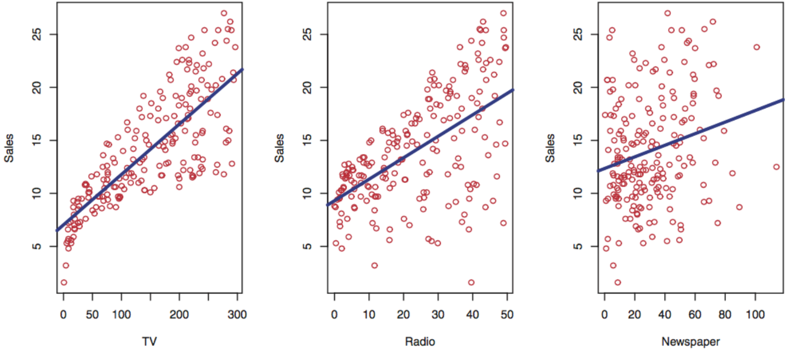

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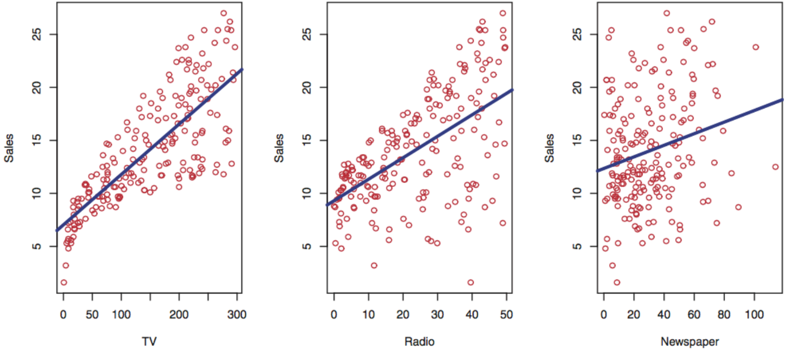

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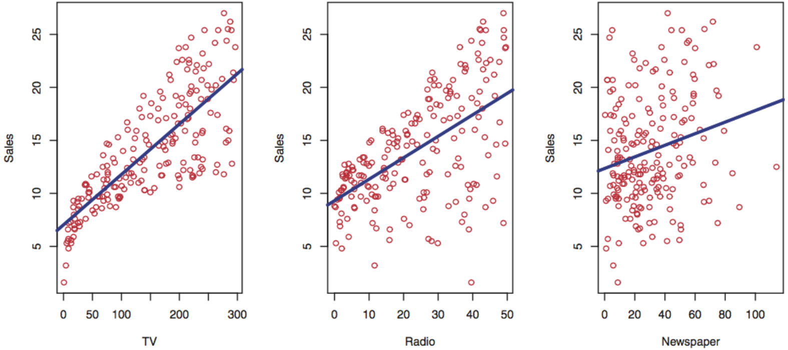

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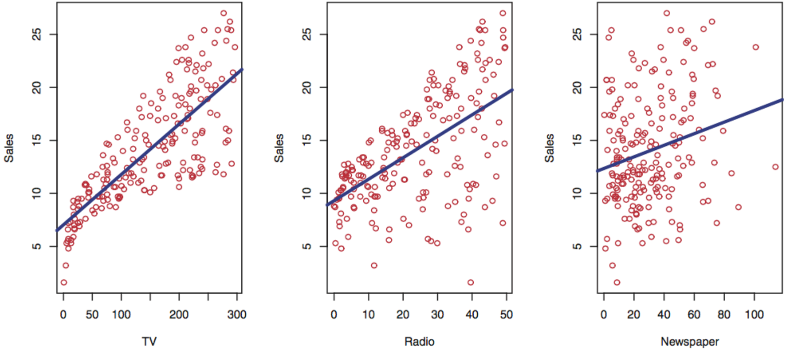
- Goal: Develop a model that can be used to predict sales on the basis of the three media budgets: $Sales \approx f(TV, Radio, Newspaper)$.'



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- Objective: Estimate f based on the observed samples.

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$$\begin{aligned}\mathbb{E}\left[\left(\hat{Y} - Y\right)^2\right] &= \mathbb{E}\left[\left(\hat{f}(X) + \epsilon - f(X)\right)^2\right] \\ &= \underbrace{\left(\hat{f}(X) - f(X)\right)^2}_{\textit{reducible}} + \underbrace{\mathbb{V}(\epsilon)}_{\textit{irreducible}}\end{aligned}$$

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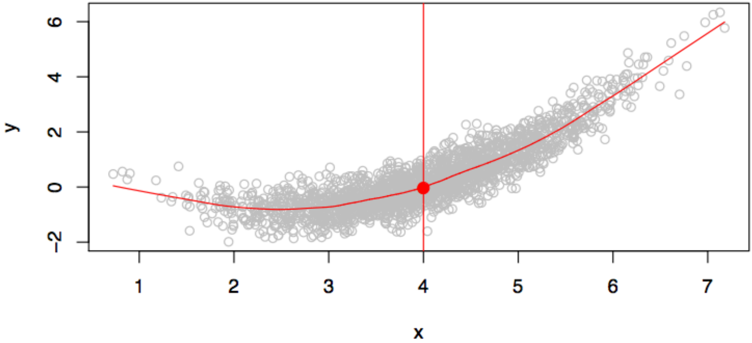

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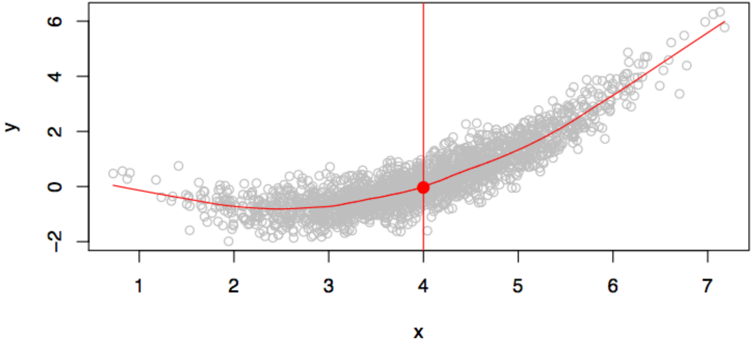

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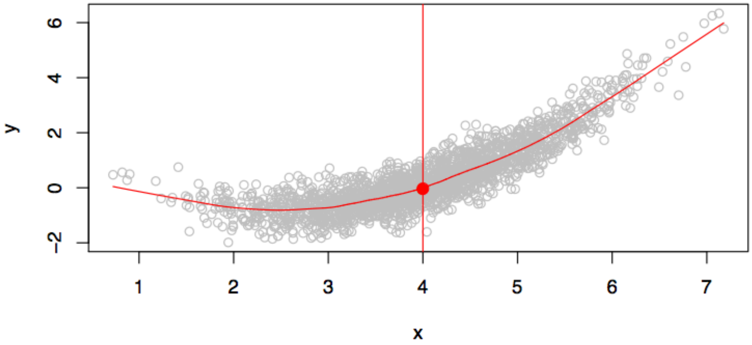

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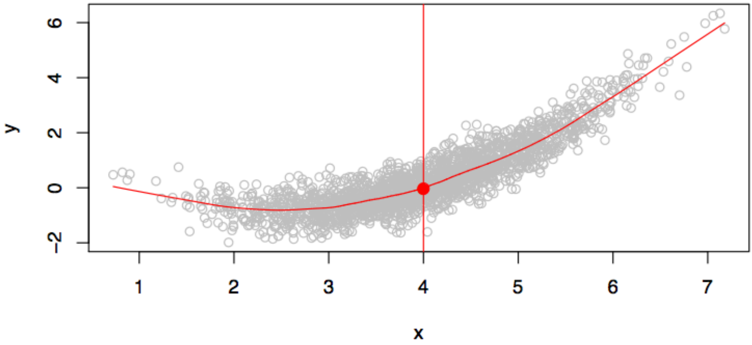

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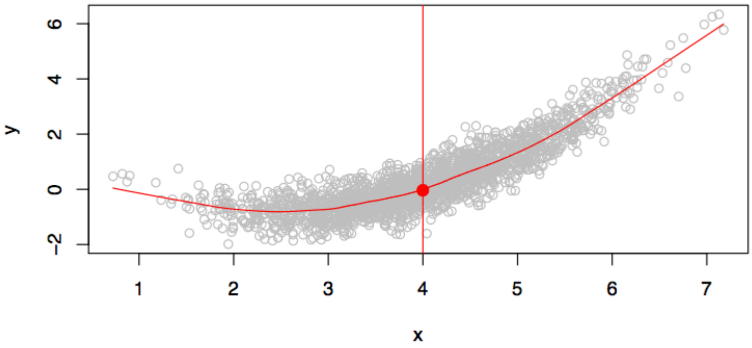

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$\hat{f}(x) = E(Y|X = x)$ is called the *regression function*.

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Two different approaches to estimate f: **Parametric methods** and **Non-parametric methods.**

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The model-based approach just described is referred to as parametric; it reduces the problem of estimating f down to one of estimating a set of parameters.

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  - These more complex models can lead to a phenomenon known as overfitting the data, which essentially means they follow the errors, or noise, too closely.

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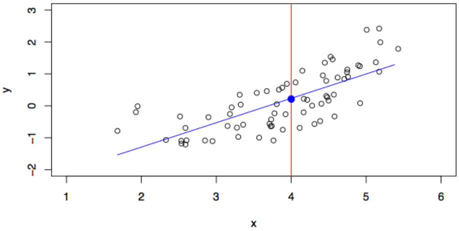

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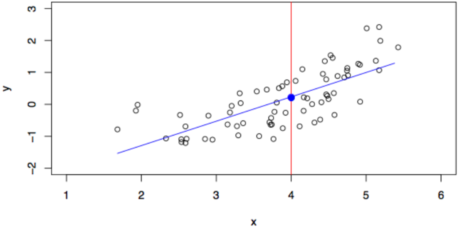
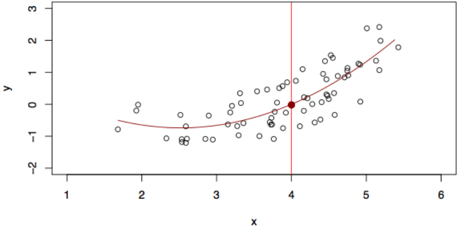


A more flexible model $\hat{f}_Q(X) = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$ gives a slightly better fit



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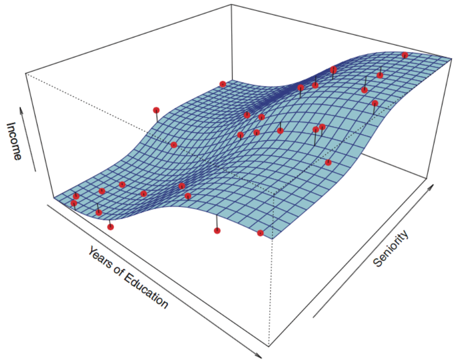
    - Any parametric approach brings with it the possibility that the functional form used to estimate f is very different from the true f, in which case the resulting model will not fit the data well.
- **Cons**: Since they do not reduce the problem of estimating f to a small number of parameters, a very large number of observations is required in order to obtain an accurate estimate for f.

Example: Simulated data points

- Consider the following simulated example.



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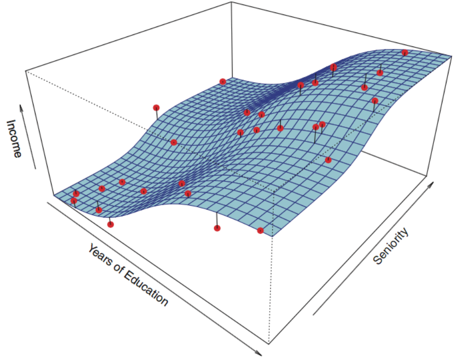

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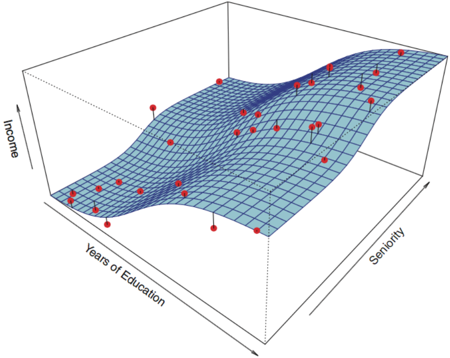
- If we are only given the red points, how can we estimate the blue surface ?



Example 1: Linear regression

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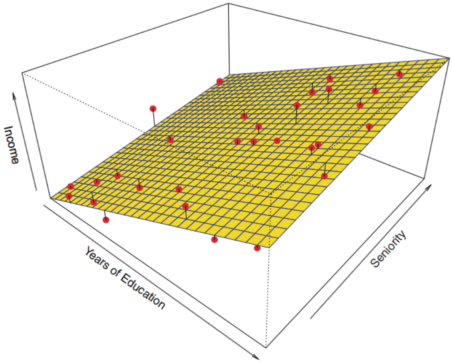

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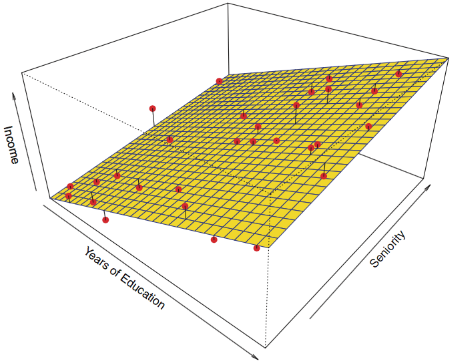
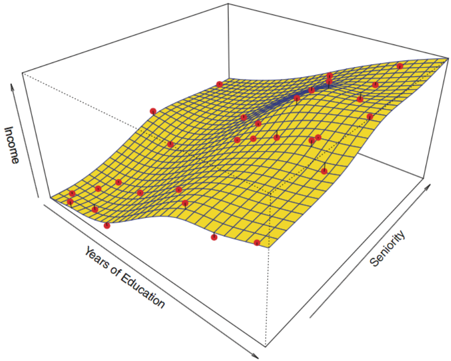
- We estimate the blue surface with linear regression.
- $\hat{f}_L(education, seniority) = \hat{\beta}_0 + \hat{\beta}_1 \times education + \hat{\beta}_2 \times seniority.$



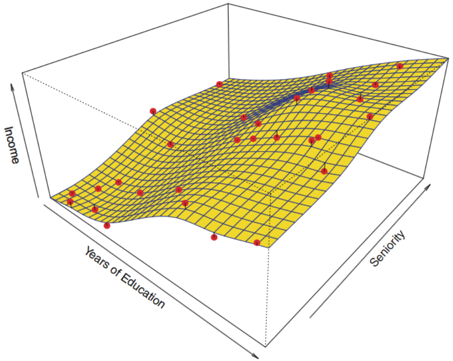
Example2: Non-parametric method

- We estimate the blue surface with a non parametric method.



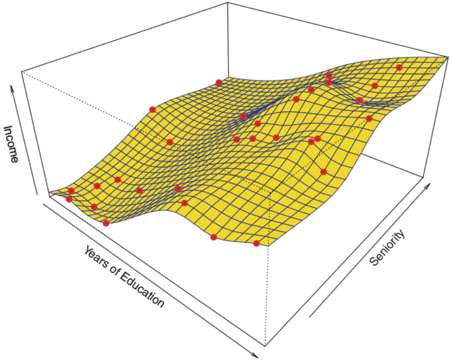
Example2: Non-parametric method

- We estimate the blue surface with a non parametric method.
- Looks closer to the target blue surface !



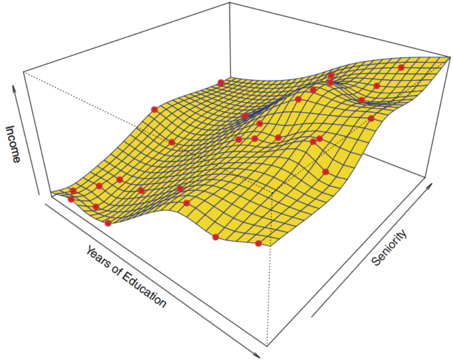
Example3: Overfitting

- We estimate the blue surface with a too flexible non parametric method.



Example3: Overfitting

- We estimate the blue surface with a too flexible non parametric method.
- This fit makes zero errors on the training data!



Some Trade-off

- **Prediction accuracy (flexibility) versus interpretability.**
  Linear models are easy to interpret; thin-plate splines are not.

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Some Trade-off

- **Prediction accuracy (flexibility) versus interpretability.**
  Linear models are easy to interpret; thin-plate splines are not.
- **Good fit versus over-fit or under-fit.**
  How do we know when the fit is just right?
- **Parsimony versus black-box.**
  We often prefer a simpler model involving fewer variables over a black-box predictor involving them all.

Flexibility versus interpretability