# Lecture 2: Correlation

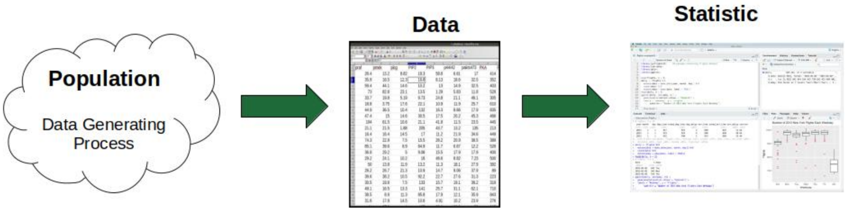
Module 1, part 1

Spring 2025

# Logistics

- Please take a look at the syllabus if you haven't already
- Population, data, and statistics
- Start Module 1 (3 lectures total)
- Correlation

# Sample data vs Population distribution



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- Mean:
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right)$$
- Median: "middle value"
- Mode: most frequent value

# Alternative way

We can think about the mean through a different lens...

- Let $\hat{b}_0$ be a "candidate"
- The residual for the *i*th observation is $e_i = x_i - \hat{b}_0$

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If you need a refresher on notation:
https://www.youtube.com/watch?v=bPvtv780h3k

# Measure of centrality

The **mean** is the value $\hat{b}_0$ which minimizes

$$RSS(\hat{b}_0) = \sum_i^n (x_i - \hat{b}_0)^2 = \sum_i |e_i|^2$$

We often also use $\bar{y}$ to denote the mean of the $x_1, x_2, \ldots x_n$.
The **median** is a value $\hat{b}_0$ which minimizes

$$\sum_i^n |x_i - \hat{b}_0| = \sum_i |e_i|$$

The **mode** is a value $\hat{b}_0$ which minimizes

$$\sum_i^n |x_i - \hat{b}_0|^0 = \sum_i |e_i|^0,$$

with here the (unusual) convention $0^0 = 0$.

# Measuring spread of data

The **variance** of a data set is defined as:

$$\hat{\sigma}_X^2 = \mathrm{var} = \frac{1}{n}\sum_i (x_i - \bar{x})^2 = \frac{RSS(\bar{x})}{n}$$

The **standard deviation** of a data set is defined as:

$$\mathrm{sd} = \sqrt{\hat{\sigma}_X^2}$$

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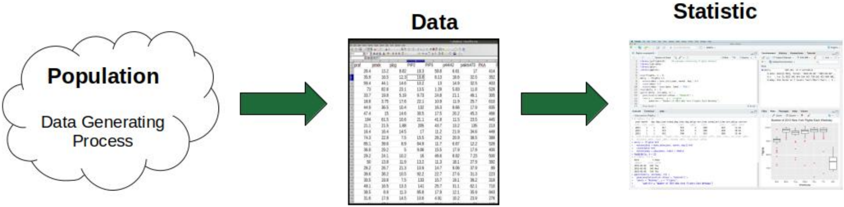

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Let $X_i$ denote a random variable (sometimes we will drop the subscript).

- Roughly speaking, random variables take a "process" and output a number
- $E(\cdot)$ will denote the "expectation" which roughly speaking means the average in the population or what we would get if we could take an infinite number of samples
- $E(X)$ denotes the (population) mean of $X$, also sometimes will use $\mu_X$
- We will denote the (population) variance of $X$ as

$$\sigma_X^2 = E\left[(X - \mu_X)^2\right]$$

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We will generally use lower case letters to denote numbers

- Typically, $x_i$ will denote the realization of random variable $X_i$
- $\bar{x}$ denotes the mean of the observations $x_1, x_2, \ldots, x_n$
- $\hat{\sigma}_x^2$ denotes the variance of the observations

# Estimating the variance

Suppose we have some observations $x_1, x_2, \ldots, x_n$ which are sampled from a population with a true mean of $\mu_X$ and true variance of $\sigma_x^2$. How would we estimate the true variance if it is unknown?

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If we knew $\mu_X$, we could use

$$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i}^{n}(x_i - \mu_X)^2 = \frac{1}{n}RSS(\mu_X)$$

and

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When we don't know $\mu_X$, we can plug in $\bar{x}$, and use

$$s_x^2 = \frac{1}{n}\sum_i^n (x_i - \bar{x})^2 = \frac{1}{n} RSS(\bar{x})$$

# Estimating the variance

Unfortunately, $\bar{x}$ minimizes RSS, so

$$\frac{1}{n}RSS(\bar{x}) \leq \frac{1}{n}RSS(\mu_x)$$

and

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Instead of dividing by $n$, we divide by $n-1$ and redefine

$$s_x^2 = \frac{1}{n-1}\sum_i^n (x_i - \bar{x})^2 = \frac{1}{n-1}RSS(\bar{x})$$

and we now have

$$E(s_x^2) = \sigma_x^2$$

# Group Discussion

- What is a scientific problem you are interested in?
- Describe the population process, the data you might gather, and the statistic you might be interested in

# Correlation

# Wine data

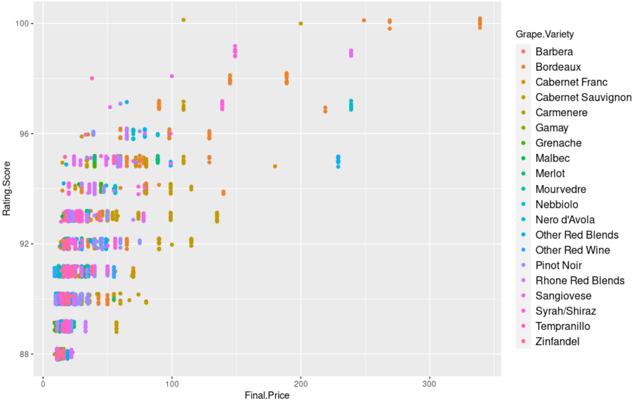


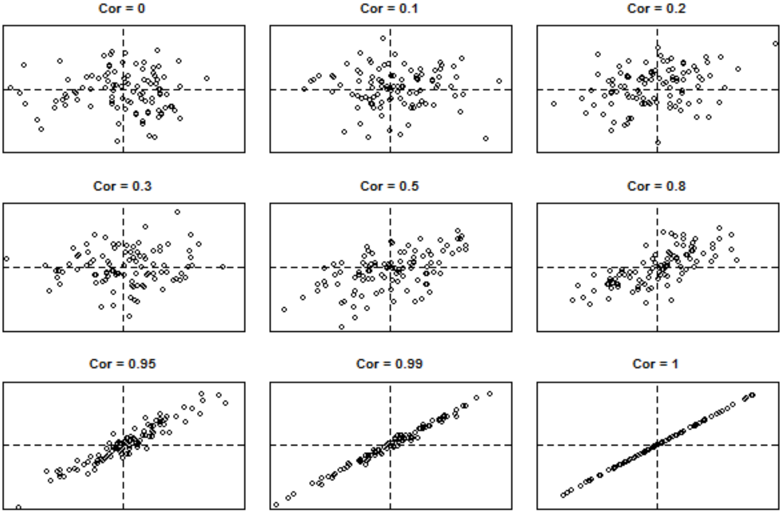

Figure: Wine Price vs Wine Rating from wine.com

# Correlation

Correlation measures the linear dependence between two variables.

- For two variables, $X$ and $Y$, correlation is denoted by $r_{XY}$
- Correlation is between -1 and 1
- $r_{XY} = 0$ indicates no **linear** relationship
- $r_{XY} > 0$ indicates positive **linear** relationship
- $r_{XY} < 0$ indicates negative **linear** relationship
- $r_{XY} = \pm 1$ indicates perfect **linear** relationship

# Correlation



# Correlation

For two variables, $X$ and $Y$, the sample correlation is

$$r_{XY} = \frac{s_{XY}}{s_X s_Y}$$

where

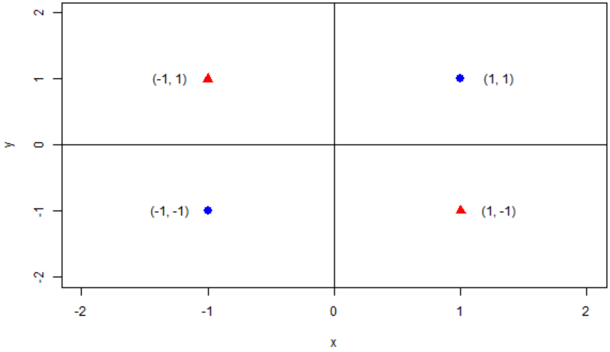
$$\text{Sample SD of X} = \mathrm{s}_X = \sqrt{\frac{1}{n-1}\sum_i (x_i - \bar{x})^2}$$

$$\text{Sample SD of Y} = \mathrm{s}_Y = \sqrt{\frac{1}{n-1}\sum_i (x_i - \bar{y})^2}$$

$$\text{Sample Covariance} = \mathrm{s}_{XY} = \frac{1}{n-1}\sum_i (x_i - \bar{x})(x_i - \bar{y})$$

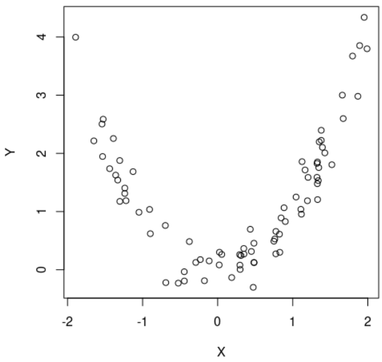
# Sample Covariance

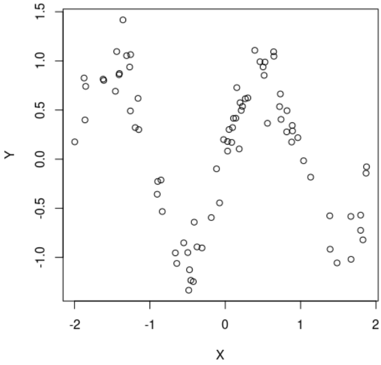
$$s_{XY} = \frac{1}{n-1} \sum_i (x_i - \bar{x})(x_i - \bar{y})$$



# Non-linear association

Correlation only measure **linear** association



# Wrap-up

- Population: process of interest
- Data: measurements gathered
- Statistic: calculation based on data
- Describe linear relationship between two variables using correlation