EE403: Digital Communications

Lecture 9: Digital Communications with Noise

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April 6, 2022

Announcement

- Midterm
  - 4/20 Wed, 10:30-11:59, E3-112
  - Closed book

- In person class next week
  - The univ recommends to offer in person lectures for small-size classes
  - Mon, Wed, 10:30-11:59, E3-112

- Office hour will be also in person by default
  - Every Monday, 10:00-10:30, E3-311 or walk-ins/emails are welcome
  - Email me if you want to discuss online

- Homework 2 was posted
  - No late submissions are allowed, due 4/13 wed
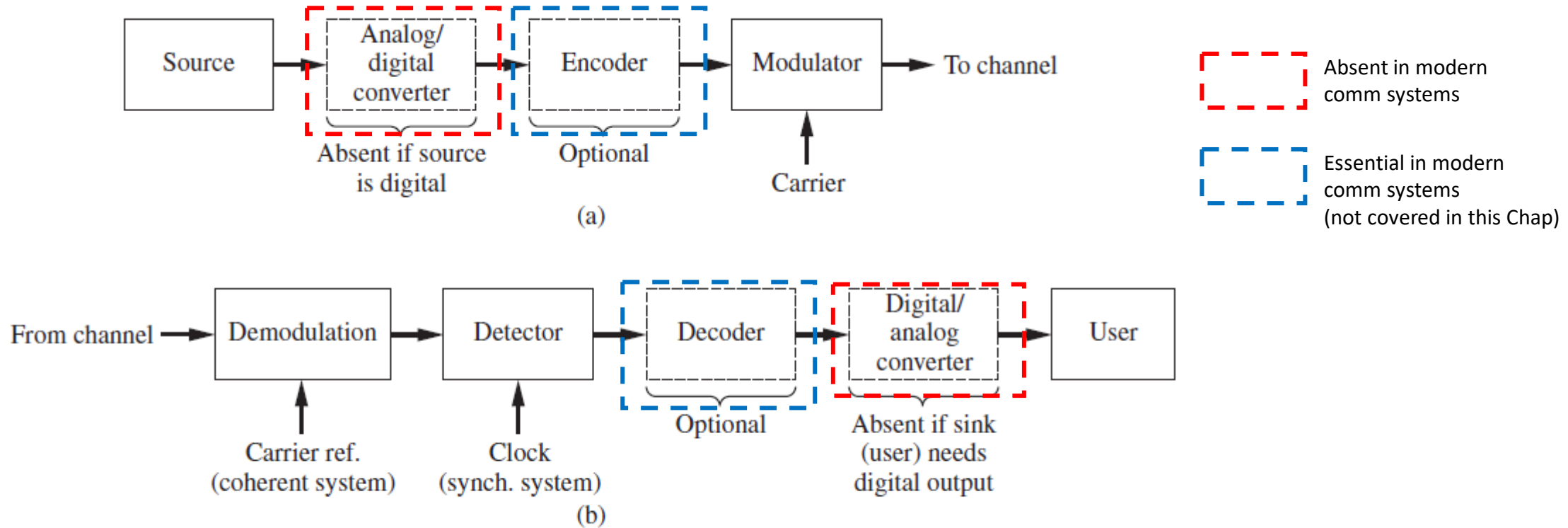
Digital Communications Systems


Figure 9.1
Block diagram of a digital data transmission system. (a) Transmitter. (b) Receiver.

Digital Communications Systems

- (assumption) bits are equally likely
  - If not, bits can be mapped to equally likely bits, which is called source coding, e.g., Huffman code
  - Encoder also maps to another bit streams to be robust against noise, called channel coding, e.g., Turbo code, LDPC code

- Coherent systems: Rx knows Tx frequency and phase exactly
  - Otherwise, called noncoherent

- Synchronous: Rx knows start/end time of each symbol exactly
  - Otherwise, called asynchronous

- Unlike Chap. 5, our focus is a system in the presence of **noise**
  - Wireless communication, the signal power is often close to the noise power
 - How to minimize error probability (P_E)?

Baseband Transmission with Noise

- Given continuous signal with (white Gaussian) noise, how to distinguish "0" and "1"?

- What we learned in Chap. 5 is: 1) sample at mT and 2) say "0" if $y(mT) < 0$, say "1" otherwise $\Rightarrow$ it does not work well with noisy channel
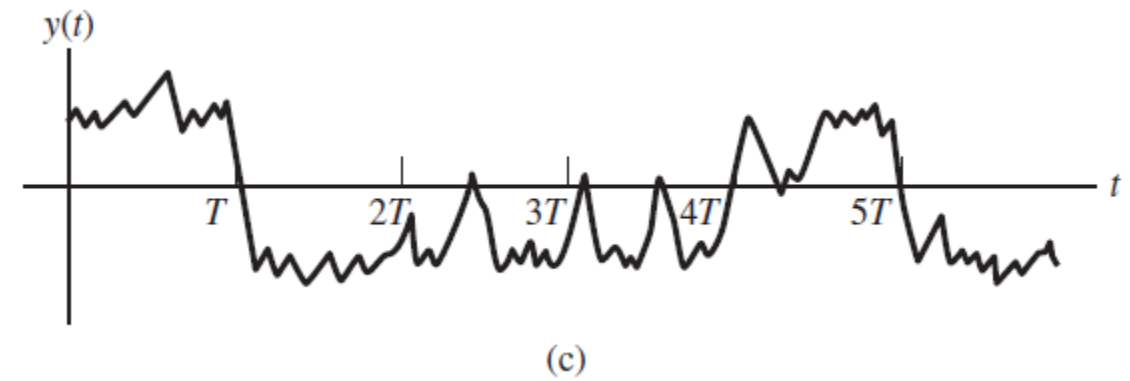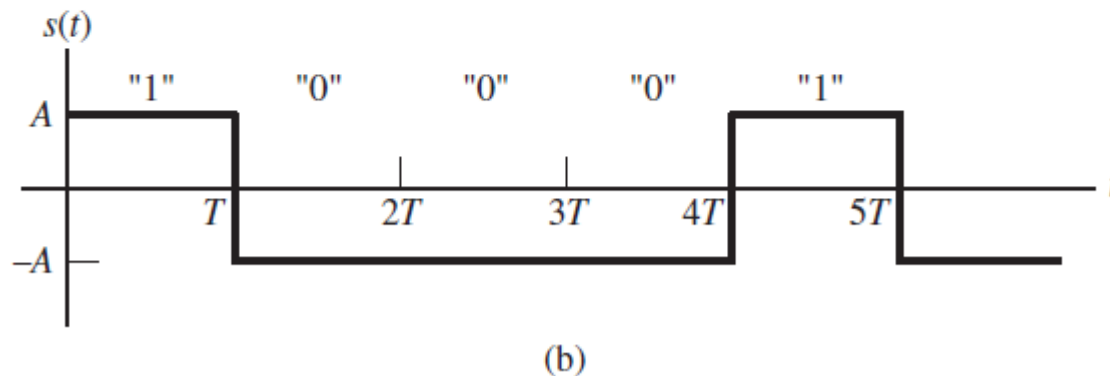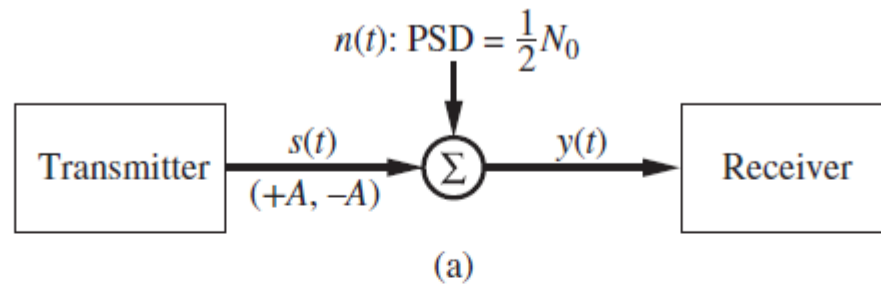
Baseband Transmission with Noise

- A better way is to use "integrate-and-dump" detector

- Calculate the area under each symbol and say "0" if the area > 0. Say "1" otherwise

- After each symbol, reset (=dump) the calculation

(c)

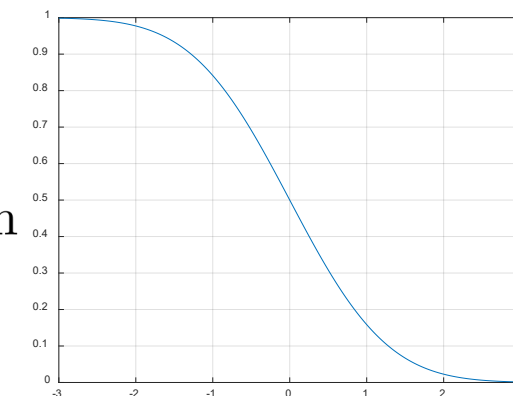(a)

(b)
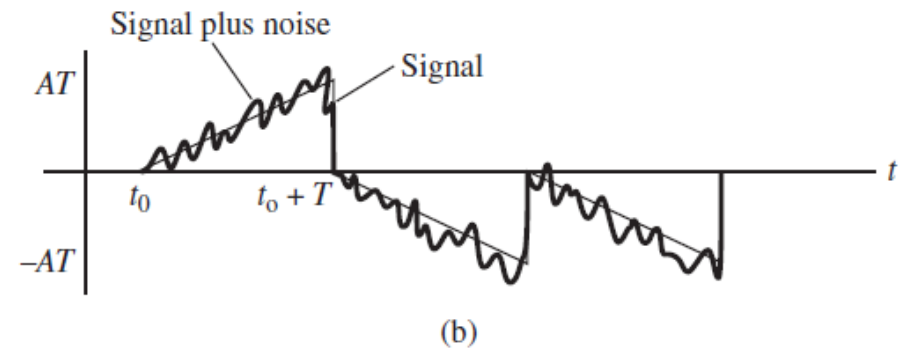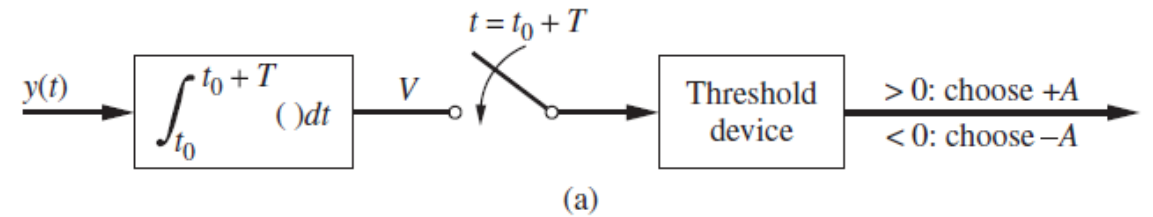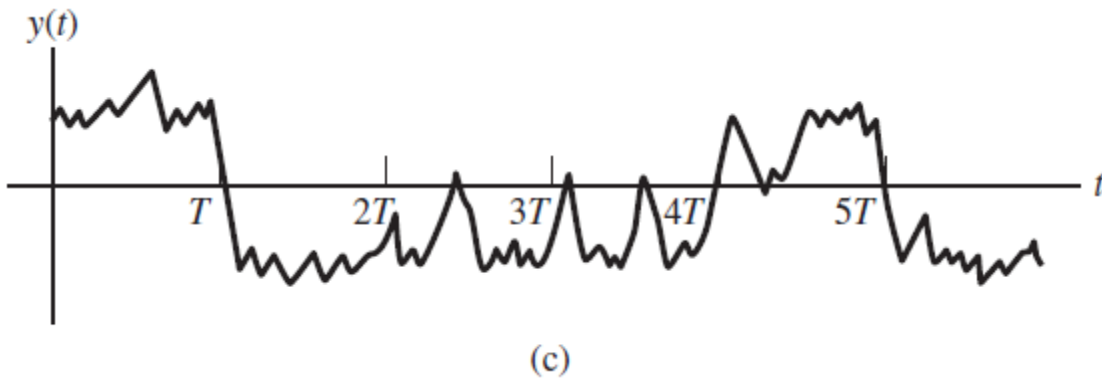
Q-function

- A Gaussian RV has a pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

  - Often denoted by $\mathcal{N}(\mu, \sigma^2)$
  - When $\mu = 0$, $\sigma^2 = 1$, it is called **standard** Gaussian or standard normal.

- The Q-function is the tail distribution of the standard Gaussian distribution. Formally,

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) dt$$

- The Q-function is non-analytic, i.e., does not have a closed-form expression

- A good approximation for large x is $Q(x) \approx \frac{e^{-x^2/2}}{x\sqrt{2\pi}}$
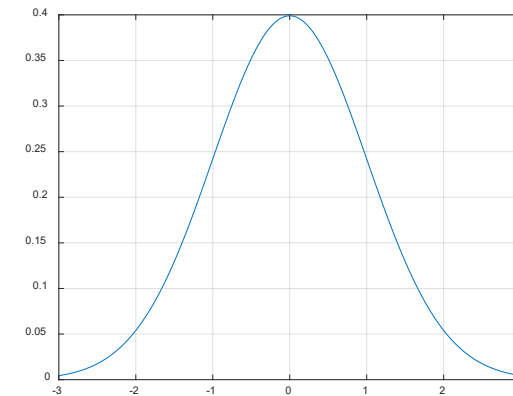
Baseband Transmission with Noise

- Performance analysis: The output at the end of each symbol is

$$V = \int_{t_0}^{t_0+T} s(t) + n(t)dt = \begin{cases} +AT + N & \text{if } +A \text{ is sent} \\ -AT + N & \text{if } -A \text{ is sent,} \end{cases}$$

where $N = \int_{t_0}^{t_0+T} n(t)dt$ is a random variable

- White Gaussian noise: the most basic/essential noise in communication

  - Stationary and ergodic
  - $n(t)$ is Gaussian
  - $\mathbb{E}[n(t)] = 0$ for all t
  - $\mathbb{E}[n(t)n(\tau)] = \frac{N_0}{2}\delta(t-\tau)$
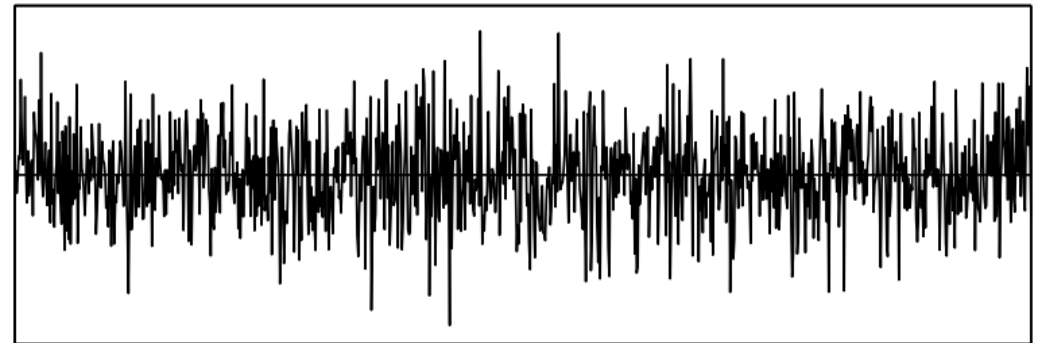
Baseband Transmission with Noise

- As $n(t)$ is a Gaussian process, its time average is zero $\Rightarrow \mathbb{E}[N] = 0$

- Variance

$$Var(N) = \mathbb{E}[N^2] = \mathbb{E}\left[\left(\int_{t_0}^{t_0+T} n(t)dt\right)^2\right] = \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} \mathbb{E}[n(t)n(\tau)]dtd\tau$$

$$= \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} \frac{N_0}{2}\delta(t-\tau)dtd\tau = \int_{t_0}^{t_0+T} \frac{N_0}{2}d\tau = \frac{N_0 T}{2}$$

- $\int n(t)dt$ is a sum of (infinitesimal) independent Gaussians $\Rightarrow N$ is Gaussian too

$$f_N(z) = \frac{1}{\sqrt{\pi N_0 T}} e^{-\frac{z^2}{N_0 T}}$$

Baseband Transmission with Noise

- If $+A$ was sent, $AT + N$ is Gaussian with mean AT and variance $\frac{N_0 T}{2}$

- Decision threshold is 0

- Error probability

$$P(\text{say "-A"} \mid \text{"A" sent}) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0 T}} e^{-\frac{(z - AT)^2}{N_0 T}} \, dz$$

$$= \int_{\sqrt{2A^2 T/N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du \triangleq Q\left(\sqrt{\frac{2A^2 T}{N_0}}\right)$$

- $Q\left(x\right) \triangleq \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du$ is the Q-function: It is the tail CDF of the standard Gaussian RV

Baseband Transmission with Noise

- $P(\text{say ``-A''} \mid \text{``A'' sent})$ is the same by symmetry

- $P_E = P(+A)P(-A|+A) + P(-A)P(+A|-A) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$

- Energy in each pulse (=each bit) is $A^2T \triangleq E_b$

- Noise PSD $= N_0$

    - Noise PSD is N_0 or $N_0/2$?
    - N_0 for single sideband
    - $N_0/2$ for double sideband $\Rightarrow$ PSD$=N_0$ considering positive/negative band both

- Signal-to-noise-ratio (SNR) $= \frac{E_b}{N_0} \Rightarrow P_E = Q(\sqrt{2SNR}) = Q(\sqrt{2E_b/N_0})$

Baseband Transmission with Noise

- As $Q(x) \approx \dfrac{e^{-x^2/2}}{x\sqrt{2\pi}}$ when x is large

$$P_E \approx \frac{e^{-E_b/N_0}}{2\sqrt{\pi E_b/N_0}}$$
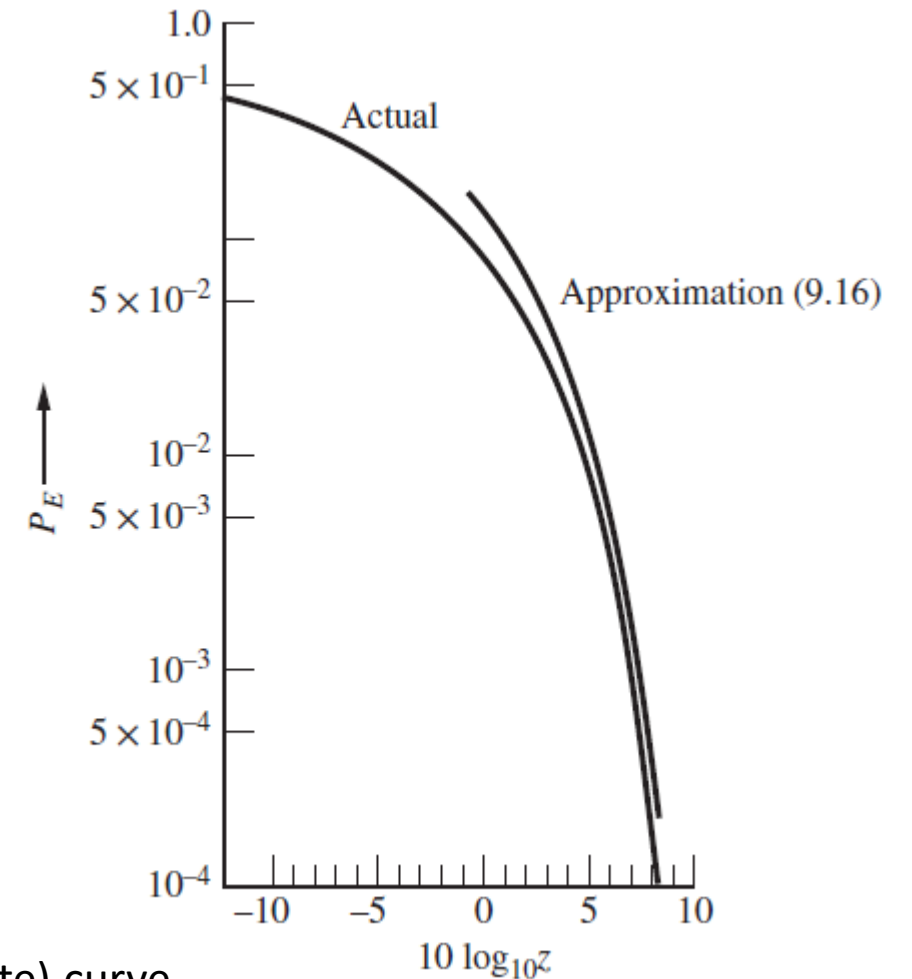
It is called **BER** (bit error rate) curve

Inner Product and CS Inequality

- For Euclidean space, the Cauchy-Schwartz inequality says

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

  - The equality holds when $x = cy$ for some scalar c
  - This is the nature for all general vector spaces, e.g., functional space (L_2)

- It means, if x is fixed, y that maximizes the inner product must be **aligned** with x

$$|\langle x, y \rangle| = \|x\| \cdot \|y\| = c\|x\|^2 \quad \text{if } x, y \text{ are aligned}$$

- Filtering (=convolution) is indeed an inner product between two functions

$$x(t) * h(t)\big|_{t=T} = \int x(t)h(T-t)dt = \langle x(t), h(T-t) \rangle$$

- That is, for the output of filter to be maximized, our filter must be aligned with input

Transmission with Arbitrary Shapes

- Instead of square pulse, let us consider arbitrary pulse shapes

- Let $s_1(t)$ be the shape of bit "1"

- Let $s_2(t)$ be the shape of bit "0"

- Energies $E_1 = \int_0^T s_1^2(t)dt$ and $E_2 = \int_0^T s_2^2(t)dt$

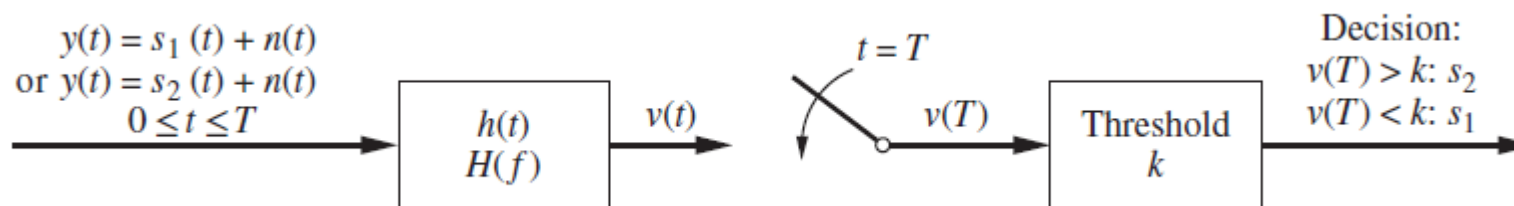- Consider a filter-based detector with impulse response $h(t), H(f)$

$$y(t) = s_1(t) + n(t) \quad \text{or} \quad y(t) = s_2(t) + n(t)$$

- After filtering, $v(t) = y(t) * h(t)$

- Then, the threshold detector makes a decision

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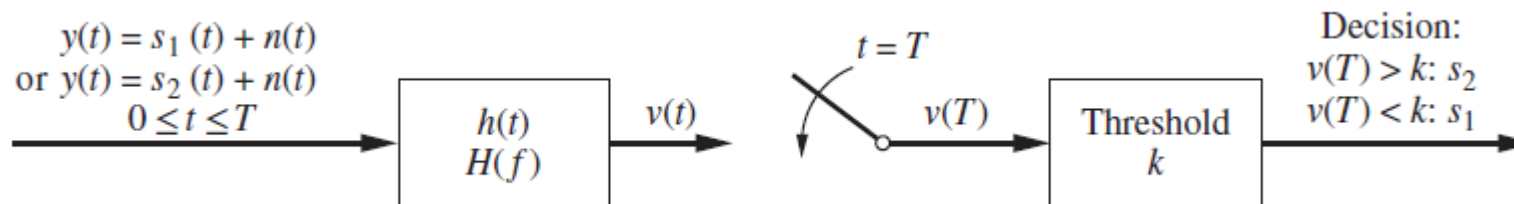
Transmission with Arbitrary Shapes

- Let $s_{01}(t) = s_1(t) * h(t)$ be the filter output when the input is clean $s_1(t)$

- Let $s_{02}(t) = s_2(t) * h(t)$ is for the clean $s_2(t)$. Assume $s_{01}(T) < s_{02}(T)$

- Let $n_0(t) = n(t) * h(t)$ be the noise at filter output

- Noise PSD at output $= S_{n_0}(f) = \frac{1}{2}N_0|H(f)|^2$

- $\Rightarrow$ noise power $= \sigma_0^2 = \int_{-\infty}^{\infty} \frac{1}{2}N_0|H(f)|^2 df$

- $V = s_{01}(T) + N$ or $s_{02}(T) + N$

- It means, V is Gaussian with mean $s_{01}(T)$ or $s_{02}(T)$, variance σ_0^2

$y(t) = s_1(t) + n(t)$
or $y(t) = s_2(t) + n(t)$
$0 \le t \le T$

$h(t)$
$H(f)$

$v(t)$

$t = T$

$v(T)$

Threshold
k

Decision:
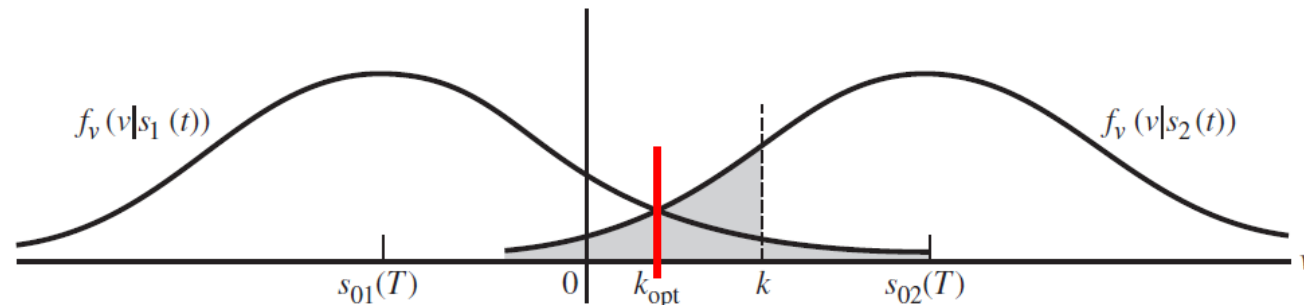$v(T) > k$: s_2
$v(T) < k$: s_1

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Transmission with Arbitrary Shapes

- Error analysis

$$P(s_2|s_1) = \int_k^\infty f_V(v|s_1(t))dv = \int_k^\infty \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(v-s_{01}(T))^2}{2\sigma_0^2}} dv$$

$$P(s_1|s_2) = \int_{-\infty}^k f_V(v|s_2(t))dv = \int_{-\infty}^k \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(v-s_{02}(T))^2}{2\sigma_0^2}} dv$$

- Optimal threshold is $k_{opt} = \frac{1}{2}[s_{01}(T) + s_{02}(T)]$

- $\Rightarrow P(s_2|s_1) = P(s_1|s_2) = Q\left(\frac{s_{02}(T) - s_{01}(T)}{2\sigma_0}\right)$

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Matched Filter

- Let $\zeta = \frac{s_{02}(T) - s_{01}(T)}{\sigma_0}$, which we have to maximize to minimize P_e

$$\zeta^2 = \frac{(s_{02}(T) - s_{01}(T))^2}{\sigma_0^2} = \left. \frac{(s_{02}(t) - s_{01}(t))^2}{\sigma_0^2} \right|_{t=T}$$

- Note that

$$s_{02}(t) - s_{01}(t) = s_2(t) * h(t) - s_1(t) * h(t)$$

$$= (s_2(t) - s_1(t)) * h(t) = \int_{-\infty}^{\infty} (S_2(f) - S_1(f)) H(f) e^{j2\pi ft} df$$

- Using $\sigma_0^2 = \frac{N_0}{2} \int |H(f)|^2 df$,

$$\zeta^2 = \frac{\left| \int_{-\infty}^{\infty} (S_2(f) - S_1(f)) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int |H(f)|^2 df}$$

Matched Filter

- Cauchy-Schwartz inequality for general vector spaces

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

  - The equality holds when $x = cy$ for some scalar c

  - The Euclidean space is an instance of vector spaces

  - For functions in L_2, it is $\left| \int x(t) y^*(t) dt \right| \leq \left(\int |x(t)|^2 dt \right)^{1/2} \left(\int |y(t)|^2 dt \right)^{1/2}$

- Then, $x \leftarrow H(f), y^* \leftarrow red$

$$\zeta^2 = \frac{\left| \int_{-\infty}^{\infty} (S_2(f) - S_1(f)) H(f) e^{j 2 \pi f T} df \right|^2}{\frac{N_0}{2} \int |H(f)|^2 df}$$

$$\leq \frac{\int |S_2(f) - S_1(f)|^2 df \int |H(f)|^2 df}{\frac{N_0}{2} \int |H(f)|^2 df} = \frac{2 \int |S_2(f) - S_1(f)|^2 df}{N_0}$$

- The equality holds when $H(f) = c(S_2^*(f) - S_1^*(f)) e^{-j 2 \pi f T}$ for some (complex) constant c

Matched Filter

- $H(f) = c(S_2^*(f) - S_1^*(f))e^{-j2\pi fT}$ for some (complex) constant c

- Its time response is (letting $c = 1$)

$$h(t) = \int (S_2^*(f) - S_1^*(f))e^{-j2\pi fT}e^{j2\pi ft}df$$

$$= \int (S_2(-f) - S_1(-f))e^{-j2\pi fT}e^{j2\pi ft}df$$

Time reversal   Time shift

- Therefore, $h(t) = s_2(T - t) - s_1(T - t)$

- That is, the optimal receiver must have two parallel filters whose impulses responses are time reversal of pulse shapes (**"matched"**)
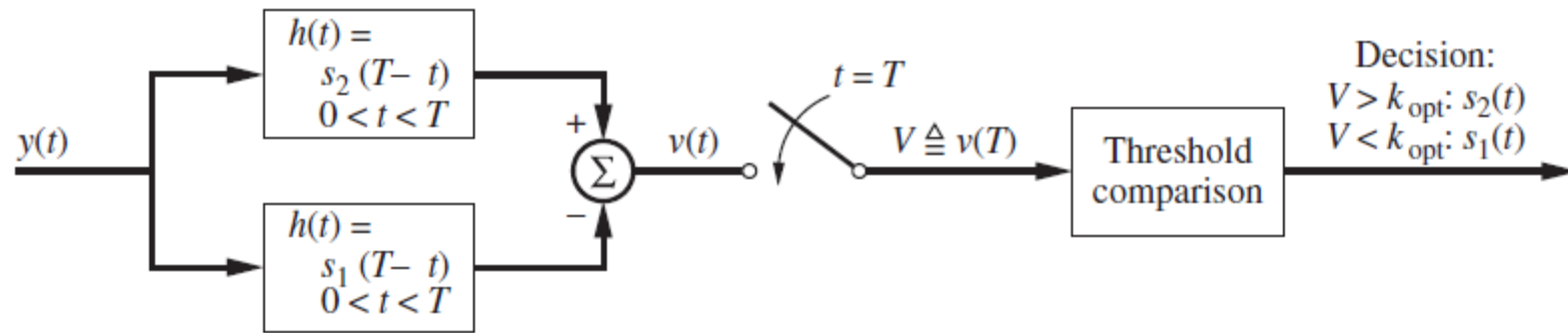
Matched Filter

Figure 9.9
Matched-filter receiver for binary signaling in white Gaussian noise.

Matched Filter

- Error analysis: We know that $P_E = Q(\zeta/2)$, where

$$\zeta_{max} = \sqrt{\frac{2 \int |S_2(f) - S_1(f)|^2 df}{N_0}}$$

$$= \sqrt{\frac{2}{N_0}} \sqrt{\int |s_2(t) - s_1(t)|^2 dt} = \sqrt{\frac{2}{N_0}} \sqrt{\int s_2^2(t) - 2s_2(t)s_1(t) + s_1^2(t) dt}$$

$$\triangleq \sqrt{\frac{2}{N_0}} \sqrt{E_1 + E_2 - 2\sqrt{E_1 E_2} \rho_{12}} \quad \text{by defining } \rho_{12} = \frac{1}{\sqrt{E_1 E_2}} \int s_1(t) s_2(t) dt$$

$$\frac{\zeta_{max}}{2} = \sqrt{\frac{1}{N_0}} \sqrt{\frac{E_1 + E_2 - 2\sqrt{E_1 E_2} \rho_{12}}{2}} = \sqrt{\frac{1}{N_0}} \sqrt{E_b - \sqrt{E_1 E_2} \rho_{12}}$$

$$= \sqrt{\frac{E_b}{N_0}} \sqrt{1 - \frac{\sqrt{E_1 E_2} \rho_{12}}{E_b}}$$

1. When $\rho_{12} = -1$ (when s_1, s_2 are the most dissimilar, called antipodal), P_E is minimized

2. When $\rho_{12} = -1$, $\frac{\zeta_{max}}{2} = \sqrt{\frac{2E_b}{N_0}}$

Matched Filter

- The smallest error probability is at ζ_{max} and the error is

$$Q(\zeta_{max}/2) = Q\left(\sqrt{\frac{E_b}{N_0}}\sqrt{1 - \frac{\sqrt{E_1 E_2}\rho_{12}}{E_b}}\right)$$

where

- $E_b = \frac{E_1 + E_2}{2}$

- $\rho_{12} = \frac{1}{\sqrt{E_1 E_2}}\int_{-\infty}^{\infty} s_1(t)s_2(t)dt$

Matched Filter

- If the matched filter is used,

$$s_{01}(T) = \int_{-\infty}^{\infty} h(\lambda) s_1(T - \lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} [s_2(T - \lambda) - s_1(T - \lambda)] s_1(T - \lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} s_2(u) s_1(u) du - \int_{-\infty}^{\infty} s_1^2(u) du$$

$$= \sqrt{E_1 E_2} \rho_{12} - E_1,$$

$$s_{02}(T) = E_2 - \sqrt{E_1 E_2} \rho_{12}$$

- Since the threshold is $k_{opt} = \frac{1}{2}[s_{01}(T) + s_{02}(T)] = \frac{1}{2}(E_2 - E_1)$

- Signal shape does not matter, only signal energy determines the threshold
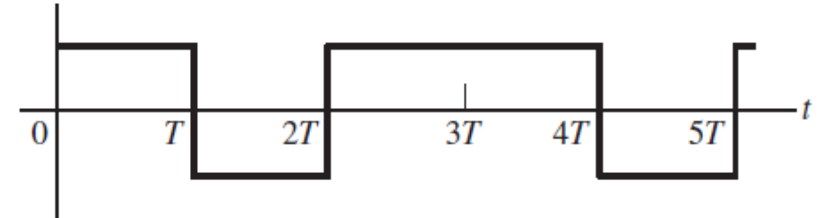
Binary X-shift keying

Table 9.1   Possible Signal Choices for Binary Digital Signaling

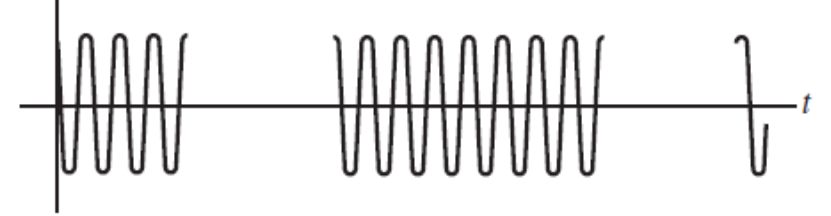| Case	$s_1(t)$	$s_2(t)$	Type of signaling
1	0	$A\cos(\omega_c t)\,\Pi\left(\frac{t-T/2}{T}\right)$	Amplitude-shift keying
2	$A\sin(\omega_c t + \cos^{-1} m)\Pi\left(\frac{t-T/2}{T}\right)$	$A\sin(\omega_c t - \cos^{-1} m)\Pi\left(\frac{t-T/2}{T}\right)$	Phase-shift keying with carrier ($\cos^{-1} m \triangleq$ modulation index)
3	$A\cos(\omega_c t)\,\Pi\left(\frac{t-T/2}{T}\right)$	$A\cos\left[(\omega_c + \Delta\omega)\,t\right]\Pi\left(\frac{t-T/2}{T}\right)$	Frequency-shift keying

Phase difference $= 2\cos^{-1} m$
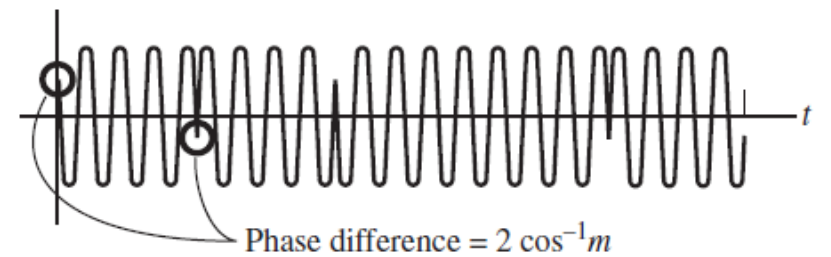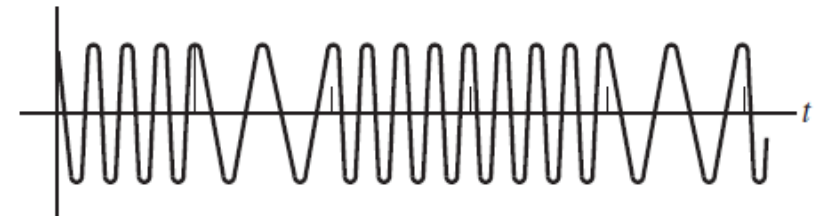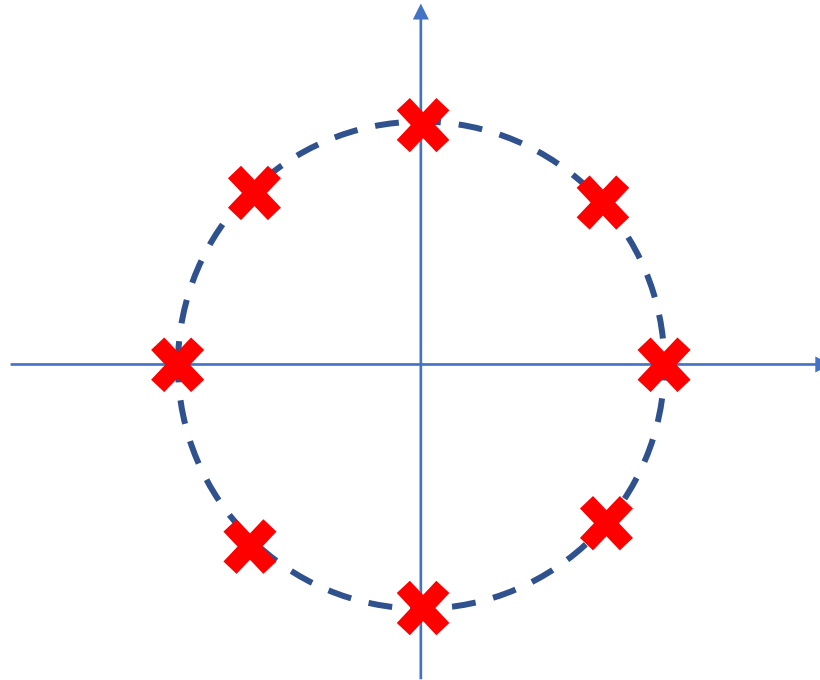
M-ary PSK

- Transmitting pulses with $M = 2^i$ different carrier phases

$$s_i(t) = A\cos(w_c t + i\frac{2\pi}{M})$$

Error Analysis: ASK

- ASK: $s_1(t) = 0$, $s_2(t) = A\cos(w_c t)$

This is also called "On-off keying" (OOK)

$$Q(\zeta_{max}/2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where $E_b = \frac{E_1 + E_2}{2} = \frac{E_2}{2}$

- s_1, s_2 are not opposite signals; not antipodal. It leads to the absence of $\sqrt{2}$ factor

- $k_{opt} = \frac{E_1 + E_2}{2} = \frac{0 + \frac{A^2 T}{2}}{2} = \frac{A^2 T}{4}$

Error Analysis: FSK

- PSK: $s_1(t) = A\cos(w_c t)$, $s_2(t) = A\cos((w_c + \Delta w)t)$

- For simplicity, let $w_c = \frac{2\pi n}{T}$ and $\Delta w = \frac{2\pi m}{T}$ with $m \neq n \Rightarrow$ Then both will go through an integer number of cycles during T

$$\sqrt{E_1 E_2}\rho_{12} = \int s_1(t)s_2(t)dt = \int_0^T A^2 \cos(w_c t)\cos((w_c + \Delta w)t)dt$$

$$= \frac{A^2}{2}\int_0^T \cos(\Delta w t) + \cos((2w + \Delta w)t)dt$$

$$= 0$$

- It leads to

$$Q(\zeta_{max}/2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The error prob is the same as ASK, but ASK needs higher peak power

- $k_{opt} = 0$

M-ary PAM

- Baseband M-ary PAM (Pulse-Amplitude Modulation): A signal set is

$$s_i(t) = A_i p(t), i = 1, 2, \ldots, M$$

- Assume $p(t)$ has unit energy

- Receiver of interest: given $y(t) = s_i(t) + n(t)$, using matched filtering,

$$Y = \int_0^T (s_i(t) + n(t))p(t)dt = A_i + N$$

where $N = \int_0^T n(t)p(t)dt$ is a Gaussian RV with mean 0 and variance $N_0/2$ (Why?)
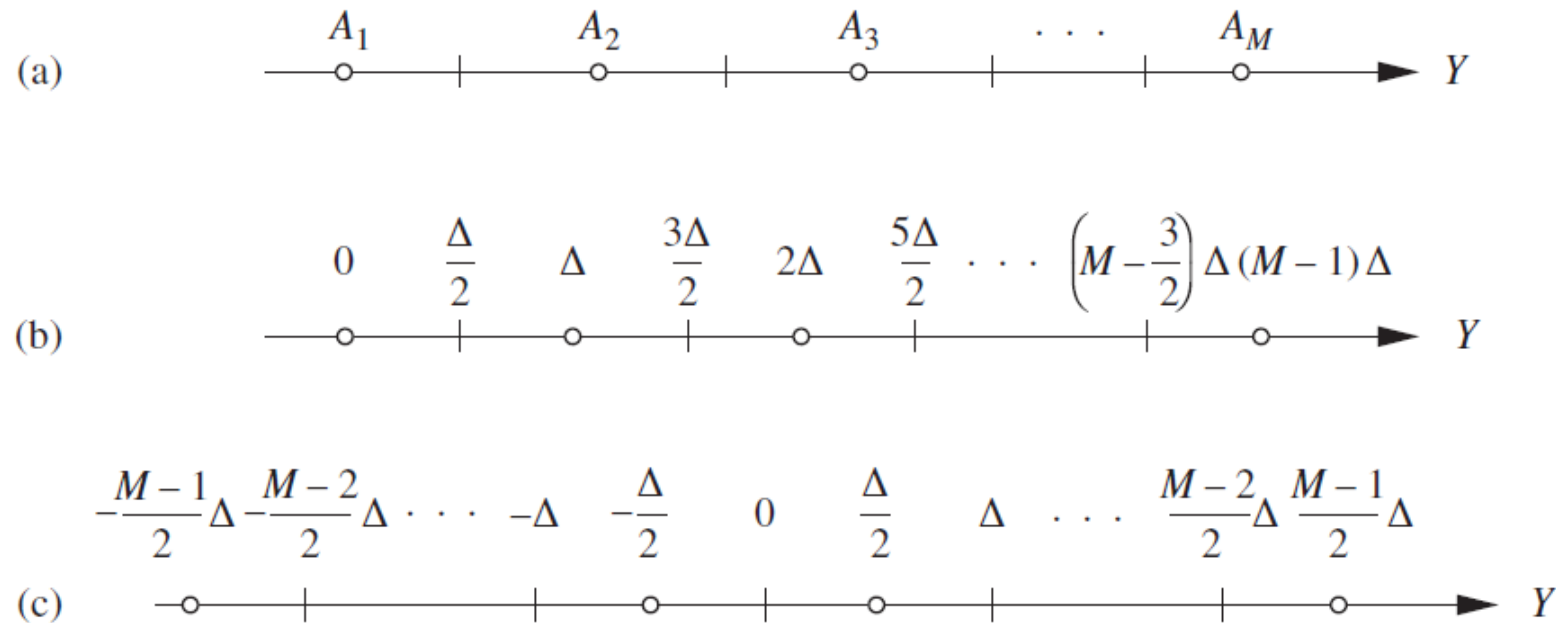
M-ary PAM

Figure 9.21

(a) Amplitudes and thresholds for PAM. (b) Nonnegative-amplitude equally spaced case. (c) Antipodal equally spaced case.

M-ary PAM

- For beginning/end symbols, $j = 1, M$, the probability of error is

$$P(\text{error}|A_j \text{ sent}) = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

- For middle symbols, $2 \leq j \leq M-1$ the probability of error is

$$P(\text{error}|A_j \text{ sent}) = 1 - \int_{\Delta/2}^{\Delta/2} \frac{1}{\sqrt{\pi N_0}} e^{-t^2/N_0} dt = 2\int_{\Delta/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-t^2/N_0} dt = 2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

- As symbols are equally likely, total error is

$$P_E = \frac{1}{M}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{2}{M}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \cdots + \frac{2}{M}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + \frac{1}{M}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

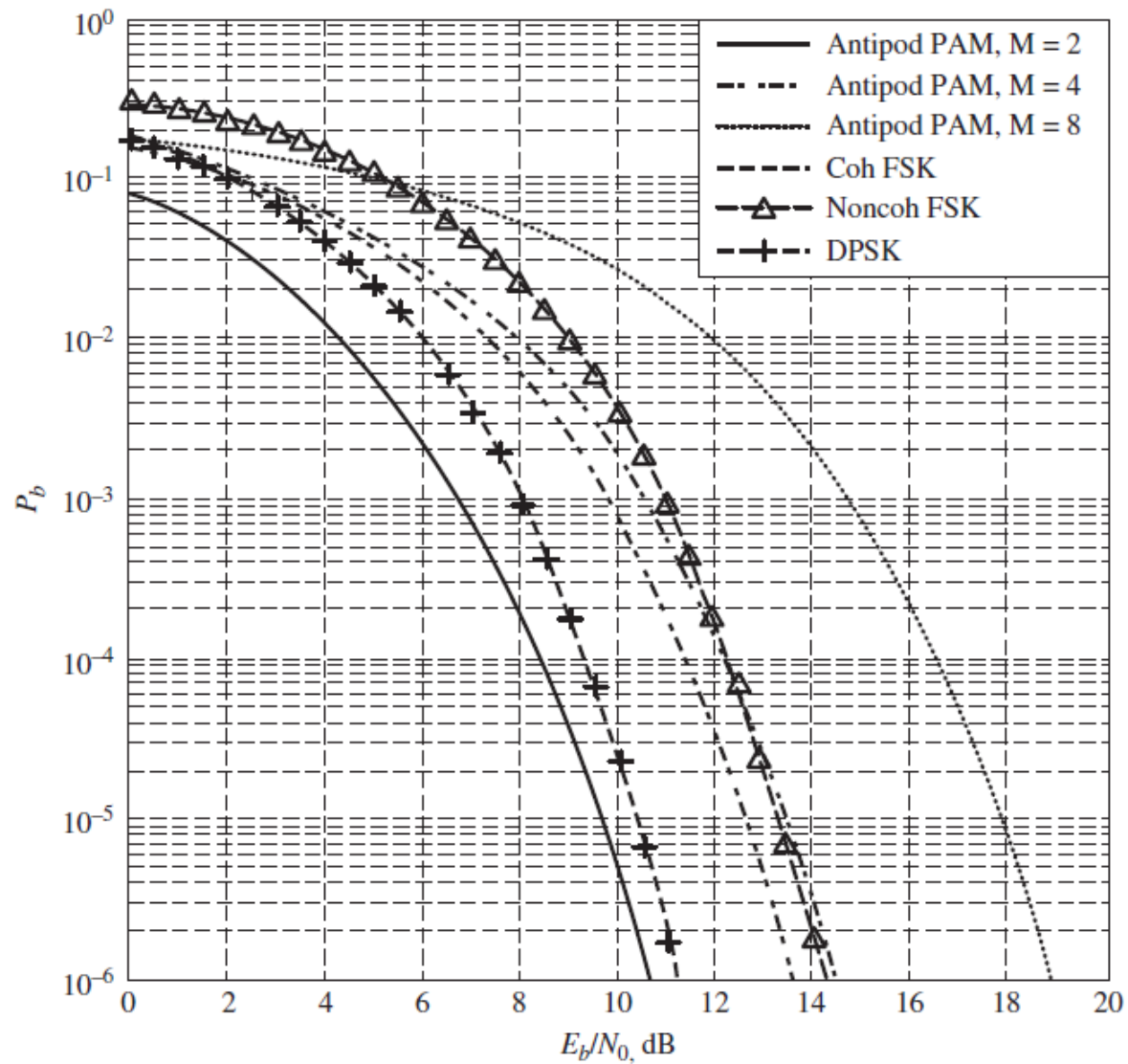$$= \frac{2M-2}{M}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

M-ary PAM

- Energy calculation: Non-negative equally spaced case

$$E_{ave} = \frac{1}{M}\sum_{j=1}^{M}E_j = \frac{1}{M}\sum_{j=1}^{M}((j-1)\Delta)^2 = \frac{\Delta^2}{M}\sum_{j=1}^{M}(j-1)^2$$

$$= \frac{\Delta^2}{M}\frac{(M-1)M(2M-1)}{6} = \frac{(M-1)(2M-1)\Delta^2}{6}$$

- Combining with the previous,

$$P_E = \frac{2M-2}{M}Q\left(\sqrt{\frac{3E_{ave}}{(M-1)(2M-1)N_0}}\right)$$

- Also note that $E_b = \frac{E_{ave}}{\log_2 M}$, assuming $M = 2^i$ for some integer

Higher order PAM shows worse BER performance; but each symbol contains more bits
➔ Good for BW limited channel and high Tx power regime