Solution Set #2

Discussions are allowed and encouraged, but please write your own answers.

1. (5pts each) Two random processes are given by

$$X(t) = A\cos(2\pi f_0 t + \theta)$$
$$Y(t) = A\sin(2\pi f_0 t + \theta),$$

where A, f_0 are given constants and θ is a uniform random variable over $[-\pi, \pi)$.

(a) Find the autocorrelation functions of $X(t)$ and $Y(t)$.

(b) Are $X(t)$ and $Y(t)$ wide-sense stationary? Justify your answer.

(c) Find the power spectral densities of $X(t)$ and $Y(t)$. Are they identical? Why?

Answer

(a) By definition,

$$\begin{aligned}
R_X(\tau) &= \mathbb{E}[X(t)X(t+\tau)] = \mathbb{E}\left[A\cos(2\pi f_0 t + \theta) \cdot A\cos(2\pi f_0(t+\tau) + \theta)\right] \\
&= \mathbb{E}\left[\frac{A^2}{2}\left(\cos(2\pi f_0(2t+\tau) + 2\theta) + \cos(2\pi f_0 \tau)\right)\right] \quad \text{by the trigonometric identity} \\
&= \frac{A^2}{2}\cos(2\pi f_0 \tau).
\end{aligned}$$

We can obtain $R_Y(\tau)$ similarly.

$$\begin{aligned}
R_Y(\tau) &= \mathbb{E}[Y(t)Y(t+\tau)] = \mathbb{E}\left[A\sin(2\pi f_0 t + \theta) \cdot A\sin(2\pi f_0(t+\tau) + \theta)\right] \\
&= \mathbb{E}\left[-\frac{A^2}{2}\left(\cos(2\pi f_0(2t+\tau) + 2\theta) - \cos(2\pi f_0 \tau)\right)\right] \quad \text{by the trigonometric identity} \\
&= \frac{A^2}{2}\cos(2\pi f_0 \tau).
\end{aligned}$$

(b) Yes, they WSS. For instance of $X(t)$,

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[A\cos(2\pi f_0 t + \theta)] = 0 \\
\mathrm{Var}(X(t)) &= \mathbb{E}[X^2(t)] = \mathbb{E}[A^2\cos^2(2\pi f_0 t + \theta)] = \mathbb{E}\left[\frac{A^2(1+\cos(4\pi f_0 t + 2\theta))}{2}\right] = \frac{A^2}{2},
\end{aligned}$$

and its autocorrelation is only dependent on τ.

(c) Since $R_X(\tau) = R_Y(\tau)$,

$$S_X(f) = \mathfrak{F}[R_X(\tau)] = \frac{A^2}{4}\left(\delta(f - f_0) + \delta(f + f_0)\right) = S_Y(f).$$

Intuitively, sinusoidal waves of a given frequency are different only in their phase; it means they are identical under a certain time shift. As power is the average over a period, the time shift does not change the power of sinusoidal waves.

2. (5 pts each) A band-limited analog signal is sampled at its Nyquist rate $f_s = 1/T_s$ and then quantized into L distinct levels. The L levels are binarized in the end. The obtained binary stream will be sent in real-time using a line coding scheme that we have learned in class.

(a) Show that one bit symbol duration can be at most $T \le \frac{T_s}{\log_2 L}$.

(b) Specify when the above equality holds, i.e., $T = \frac{T_s}{\log_2 L}$.

Answer

(a) Note that n bits can represent 2^n types of levels. Since each sampled point has L levels, we need $\lceil \log_2 L \rceil$ bits to represent each sampled point. As the sampling period is T_s, one bit can take at most $T \le T_s / \lceil \log_2 L \rceil$ time. Therefore,

$$T = \frac{T_s}{\lceil \log_2 L \rceil} \le \frac{T_s}{\log_2 L}.$$

(b) When $L = 2^i$ for some integer i, $\lceil \log_2 L \rceil = \log_2 L = i$. It means $T = \frac{T_s}{\log_2 L}$.

3. (5 pts each) A compact disc (CD) stores audio signals digitally. Assume that the audio signal bandwidth equals $15k$Hz.

(a) If the Nyquist samples are uniformly quantized into $L = 65536$ levels and then binarized, determine the number of binary digits required to encode a sample.

(b) Determine the number of binary digits per second(bit/s) required to encode the audio signal.

(c) For practical reasons, signals are sampled at a rate well above the Nyquist rate. Practical CDs use $44.1k$ samples per second. If $L = 65536$, determine the number of bits per second required to encode the signal.

Answer

(a) Since $65536 = 2^{16}$, it needs 16 bits to encode each sample.

(b) The Nyquist rate is $30k$Hz. Therefore, we need $30k \times 16 = 480k$ bits per second to encode an audio signal.

(c) Sampling rate is now $44.1k$Hz. Therefore, we need $44.1k \times 16 = 705.6k$ bits per second to encode an audio signal.

4. (5 pts each) Consider the unipolar RZ coding. See Lecture note 5, pp. 11–16. Suppose the symbol period is T.

(a) Determine values of a_k, their probability, and $p(t)$ for unipolar RZ.

(b) Compute correlation coefficients R_m.

(c) Compute the power spectrum of unipolar RZ coding. Why does it have discrete components?

Answer

See Example 5.3 in the textbook for full details.

(a) $a_k = 0$ with probability 0.5 and $a_k = A$ (or some positive constant) with probability 0.5. The shape of the component pulse is $p(t) = \prod(\frac{2t}{T})$.

(b) They can be obtained as follows.

$$R_m = \begin{cases} \frac{1}{2} \cdot A^2 + \frac{1}{2} \cdot 0 = \frac{A^2}{2} & \text{when } m = 0 \\ \frac{1}{4} \cdot A^2 + \frac{1}{4} \cdot A \cdot 0 + \frac{1}{4} \cdot 0 \cdot A + \frac{1}{4} \cdot 0 \cdot 0 = \frac{A^2}{4} & \text{when } m \neq 0 \end{cases}$$

(c) Since $P(f) = \frac{T}{2} sinc\left(\frac{T}{2}f\right)$,

$$\begin{aligned}
S_{URZ}(f) &= \frac{T}{4} sinc^2\left(\frac{T}{2}f\right)\left(\sum_{m=-\infty}^{\infty} R_m e^{-j2\pi mTf}\right) \\
&= \frac{T}{4} sinc^2\left(\frac{T}{2}f\right)\left(\frac{A^2}{2} + \frac{A^2}{4}\sum_{m=-\infty, m\neq 0}^{\infty} e^{-j2\pi mTf}\right) \\
&= \frac{T}{4} sinc^2\left(\frac{T}{2}f\right)\left(\frac{A^2}{4} + \frac{A^2}{4}\sum_{m=-\infty}^{\infty} e^{-j2\pi mTf}\right) \\
&\overset{(a)}{=} \frac{T}{4} sinc^2\left(\frac{T}{2}f\right)\left(\frac{A^2}{4} + \frac{A^2}{4}\sum_{m=-\infty}^{\infty} \frac{1}{T}\delta(f - m/T)\right) \\
&= \underbrace{\frac{A^2 T}{16} sinc^2\left(\frac{T}{2}f\right)}_{\text{continuous component}} + \underbrace{\frac{A^2}{16} sinc^2\left(\frac{T}{2}f\right)\sum_{m=-\infty}^{\infty} \delta(f - m/T)}_{\text{discrete component}},
\end{aligned}$$

where (a) follows since sum of the complex exponentials is in fact impulse train. The impulse train results in the discrete components.

5. (10 pts) We want to design a transmission system that sends data at $10k$bps over a channel of bandwidth $8k$Hz usign raised-cosine pulses. What is the maximum value of the roll-off factor β we can use?

Answer Note that the bandwidth of the raised-cosine is given as $\frac{1+\beta}{2T}$, where T is the sampling rate. Hence,

$$\frac{1+\beta}{2T} = \frac{1+\beta}{2 \times (10k)^{-1}} = 8k.$$

Solving the equation, we have $\beta = \frac{3}{5}$.

6. (5 pts each) Assume the following channel pulse response samples.

$p_c(-3T) = 0.001 \quad p_c(-2T) = -0.01 \quad p_c(-T) = 0.1 \quad p_c(0) = 1.0 \quad p_c(T) = 0.2 \quad p_c(2T) = -0.02 \quad p_c(3T) = 0.005.$

(a) Find the tap coefficients for a three-tap zero forcing equalizer.

(b) Find the output samples for $mT = -2T, -T, 0, T, 2T$.

Answer

(a) The channel response matrix is

$$[P_c] = \begin{bmatrix} 1.0 & 0.1 & -0.01 \\ 0.2 & 1.0 & 0.1 \\ -0.02 & 0.2 & 1.0 \end{bmatrix}$$

and its inverse matrix is

$$[P_c]^{-1} = \begin{bmatrix} 1.0217 & -0.1063 & 0.0209 \\ -0.2106 & 1.0423 & -0.1063 \\ 0.0626 & -0.2016 & 1.0217 \end{bmatrix}$$

The coefficient vector is the middle column, i.e., $[a_{-1}, a_0, a_1] = [-0.1063, \quad 1.0423, \quad -0.2106]$.

(b) The output samples are $p_{eq}(mT) = \sum_{n=-1}^{1} a_n p_c[(m-n)T]$. Therefore,

$$[p_{eq}] = [-0.021, \quad 0.0, \quad 1.0, \quad 0.0, \quad -0.0635].$$