Midterm Exam

Name:

Student ID:

Problem	Score
#1	/10
#2	/10
#3	/15
#4	/10
#5	/15
#6	/20
Total	/80

EE403 Digital Communications — Midterm
DGIST, 2022 Spring — April 20, 2022

1. (2 pts each) These are 'True/False' questions. Mark your answer. You do not need to justify your answer. You will get 2 points if your answer is correct, and 0 if the answer is wrong or you do not answer.

(a) **(True / False)** The sum of two Gaussian random variables is still Gaussian.

independent

(b) **(True / False)** Truncating a sinc function at left and right tails both, i.e., multiplying a sinc function by a rectangular function, we can obtain a finite-time and finite-bandwidth signal.

a signal cannot be bounded in time & freq both

(c) **(True / False)** For a bandlimited signal $x(t)$ of bandwidth B, we can always recover it from samples of sampling rate greater than $2B$.

(d) **(True / False)** Unlike the Q-function that has no closed-form expression, the first and second derivatives of it have closed-form expressions.

(e) **(True / False)** A zero-forcing equalizer of Alice's communication device and a zero-forcing equalizer of Bob's one could have different tap coefficients.

2. (5 pts each)

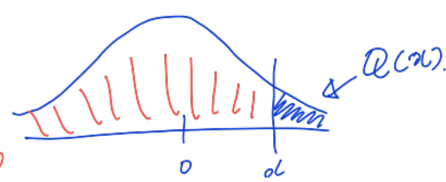
(a) Let X be a Gaussian random variable with mean μ and variance σ^2. Write $\Pr[X > z]$ in terms of the Q-function.

$$\Pr[X > z] = \int_z^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$\frac{x-\mu}{\sigma} = t \qquad \frac{1}{\sigma}dx = dt$$

$$= \int_{\frac{z-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = Q\left(\frac{z-\mu}{\sigma}\right)$$

(b) Prove $Q(x) = 1 - Q(-x)$.



$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= 1 - \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= 1 - \int_{-x}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \quad \text{by symmetry of } N(0,1)$$

$$= 1 - Q(-x)$$

3. (5, 10 pts)

(a) The Nyquist's pulse shaping criterion says that if a pulse shape $p(t)$ satisfies a certain condition, it achieves zero intersymbol interference (ISI) at sampling points mT. State clearly the Nyquist's pulse-shaping criterion.

If $p(t)$'s FT satisfies

$$\sum_{k=-\infty}^{\infty} P\left(f+\frac{k}{T}\right) = \text{const} \quad \text{for all } f$$

then

$$p(mT) = \begin{cases} 1 & \text{when } m=0 \\ 0 & m\neq 0 \end{cases}$$

(b) The frequency spectrum of the raised cosine is given as follows. Show that it has no ISI. It is okay to assume $T = 1$ for simplicity. [Caution: This problem might be time-consuming. If you cannot quickly come up with how to solve, skip it and come back later.]

$$P_{RC}(f) = \begin{cases} T & |f| \leq \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1+\cos\left[\frac{\pi T}{\beta}\left(|f|-\frac{1-\beta}{2T}\right)\right]\right\} & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases}$$

$$= \begin{cases} 1 & |f| \leq \frac{1-\beta}{2} \\ \frac{1}{2}\left\{1+\cos\left[\frac{\pi}{\beta}\left(|f|-\frac{1-\beta}{2}\right)\right]\right\} & \frac{1-\beta}{2} \leq |f| \leq \frac{1+\beta}{2} \\ 0 & |f| > \frac{1+\beta}{2} \end{cases} \qquad \text{if } T=1$$

Consider arbitrary $f_0 > 0$. WLOG, suppose that $k_0 < f_0 < k_0+1$ for some integer k_0.

at f_0, only two pulses are nonzero,

$P_{RC}(f_0 - k_0)$, $P_{RC}(f_0 - (k_0+1))$

$$\Rightarrow \quad P_{RC}(f_0-k_0) + P_{RC}(f_0-(k_0+1))$$

$$= \frac{1}{2}\left[1+\cos\left(\underbrace{\frac{\pi}{\beta}\left(f_0-k_0-\frac{1-\beta}{2}\right)}_{=A}\right)\right] + \frac{1}{2}\left[1+\cos\left(\underbrace{\frac{\pi}{\beta}\left(k_0+1-f_0-\frac{1-\beta}{2}\right)}_{=B}\right)\right]$$

$$= 1+\frac{1}{2}\left[\cos A + \cos B\right] = 1+\frac{1}{2}\cdot 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\frac{A+B}{2} = \frac{\pi}{2\beta}\left(\cancel{f_0} - \cancel{k_0} - \frac{1-\beta}{2} + \cancel{k_0}+1 - \cancel{f_0} - \frac{1-\beta}{2}\right) = \frac{\pi}{2\beta}\left(1-(1-\beta)\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad (\quad) = 1+\frac{1}{2}\cdot 2\cdot \overset{\to 0}{\cancel{\cos\left(\frac{\pi}{2}\right)}}\cos(\quad) = 1. \qquad \text{const!}$$

4. (5 pts each)

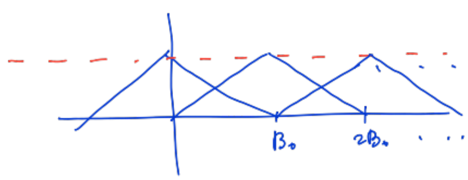
(a) The pulse-shaping spectrum that we wish to use is as follows. Assume that the period of our sampling device is T.

$$P(f) = \begin{cases} \frac{\sqrt{E}}{2B_0}\left(1 - \frac{|f|}{B_0}\right) & |f| < B_0 \\ 0 & \text{otherwise} \end{cases}$$

Is it possible to achieve zero intersymbol interference (ISI) using this pulse? If possible, specify the value of such B_0. If not, explain why.

As the sampling period is T, the amount of shift is $\frac{1}{T} \cdot k$.

$\Rightarrow$ for $\sum_{k=-\infty}^{\infty} P(f + \frac{k}{T}) = \text{const}$ to be true,

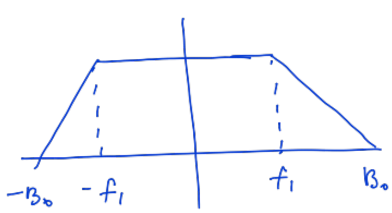


$\frac{1}{T} = B_0$.

(b) The pulse-shaping spectrum that we wish to use is as follows. Assume that the period of our sampling device is T.

$$P(f) = \begin{cases} \frac{\sqrt{E}}{2B_0} & |f| < f_1 \\ \frac{\sqrt{E}}{2B_0}\left(1 - \frac{|f| - f_1}{B_0 - f_1}\right) & f_1 \leq |f| < B_0 \\ 0 & \text{otherwise} \end{cases}$$

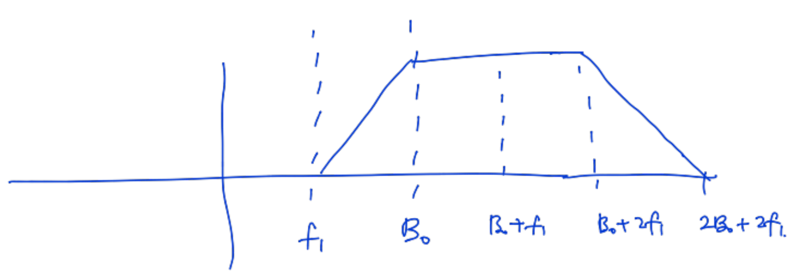
Is it possible to achieve zero intersymbol interference (ISI) using this pulse? If possible, specify the value of such B_0 in terms of $f_1 > 0$ and T. If not, explain why.



for $\sum_{k=-\infty}^{\infty} P(f + \frac{k}{T})$ to be flat,

$\frac{1}{T} = B_0 + f_1$

$\Rightarrow B_0 = \frac{1}{T} - f_1$.



5. (5 pts each) Fig. 1 is the pulse waveform of the non-return-to-zero (NRZ) line coding. Recall that the coded signal is

$$X(t) = \sum_{k=-\infty}^{\infty} a_k p(t-kT) = \left(\sum_{k=-\infty}^{\infty} a_k \delta(t-kT)\right) * p(t).$$

For simplicity, let the symbol duration (=sampling period) be 1, i.e., $T = 1$ and bits are equally likely.

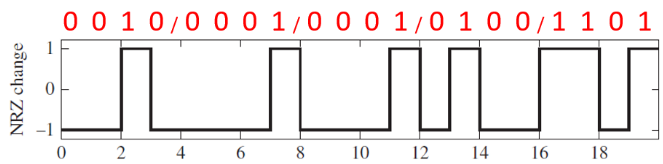


Figure 1: NRZ line coding

(a) Write down the expressions of $p(t)$ and its **power spectral density**. Assume the max magnitude of $p(t)$ is $A > 0$, i.e., $\max |p(t)| = A$.

$p(t) = A\,\Pi\left(\frac{t}{T}\right) = A\,\Pi(t) \xrightarrow{FT} P(f) = A\,\text{sinc}(f)$

$\Rightarrow \quad S_p(f) = A^2 \text{sinc}^2(f)$

(b) Compute the autocorrelation of a_k process, assuming each bit is independent and identically distributed.

$R_0 = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$

$R_m = \frac{1}{4}\,1\cdot 1 + \frac{1}{4}\,1\cdot(-1) + \frac{1}{4}(-1)\cdot 1 + \frac{1}{4}\cdot(-1)\cdot(-1)$

$= 0 \quad \forall m \neq 0.$

(c) Using (a) and (b), obtain the power spectrum density of the NRZ coding.

$S_{NRZ}(f) = S_p(f)\,R_0 = A^2 \text{sinc}^2(f)$

6. (5 pts each) Consider a binary communication system utilizing the two signal waveforms shown below. The noise in the channel $n(t)$ is zero-mean white Gaussian noise with power spectral density $\frac{N_0}{2}$.

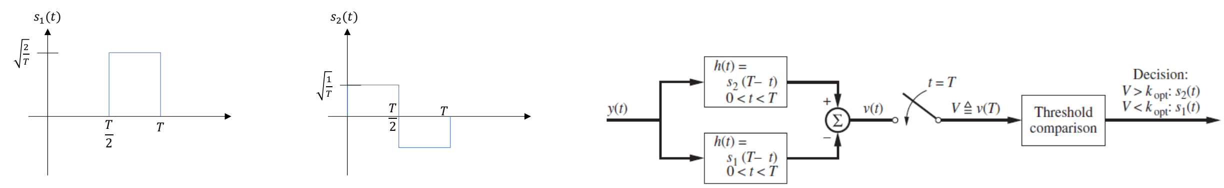
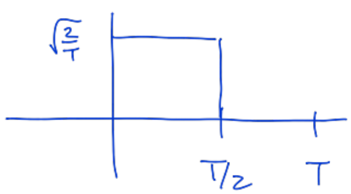
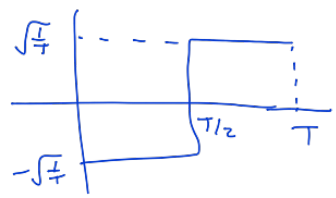


Figure 2: Left: Signal waveforms, right: filter-based receiver

(a) Sketch the matched filters corresponding to $s_1(t), s_2(t)$.

matched filter for $S_1(t)$ for $S_2(t)$





(b) Instead of the matched filters in (a), we use the following filters, $\phi_1(t), \phi_2(t)$, which are suboptimal. In other words, $h(t)$ of the top filter is $\phi_1(t)$ and $h(t)$ of the bottom filter is $\phi_2(t)$. Compute $s_{0i}(T) = (\phi_2(t) - \phi_1(t)) * s_i(t)\Big|_{t=T} = \int_{-\infty}^{\infty} (\phi_2(\lambda) - \phi_1(\lambda)) s_i(T-\lambda) d\lambda$ for $i = 1, 2$.

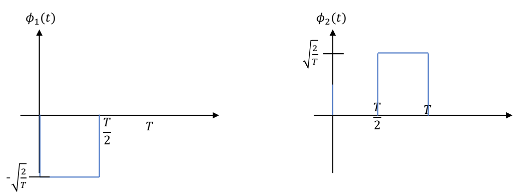
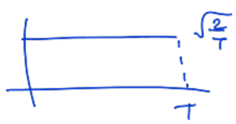


Figure 3: My suboptimal filters

$\phi_2(t) - \phi_1(t) =$



flat

$$\Rightarrow S_{o1}(T) = \sqrt{\frac{2}{T}} \int_0^T s_1(T-\lambda)\, d\lambda = \sqrt{\frac{2}{T}} \int_0^{T/2} \sqrt{\frac{2}{T}}\, d\lambda = \frac{2}{T} \cdot \frac{T}{2} = 1$$

$$S_{o2}(T) = \sqrt{\frac{2}{T}} \int_0^T s_2(T-\lambda)\, d\lambda = \sqrt{\frac{2}{T}} \int_0^{\frac{T}{2}} -\sqrt{\frac{1}{T}}\, d\lambda + \sqrt{\frac{2}{T}} \int_{\frac{T}{2}}^{T} \sqrt{\frac{1}{T}}\, d\lambda$$

$$= 0$$

(c) Specify the distributions of $N = (\phi_2(t) - \phi_1(t)) * n(t)\Big|_{t=T} = \int_{-\infty}^{\infty}(\phi_2(\lambda) - \phi_1(\lambda))n(T-\lambda)d\lambda$.

$N = \sqrt{\frac{2}{T}} \int_0^T n(t)\,dt$, $n(t)$ is a Gaussian process

$\Rightarrow$ N is Gaussian.

$\Rightarrow$ $E[N]=0$.

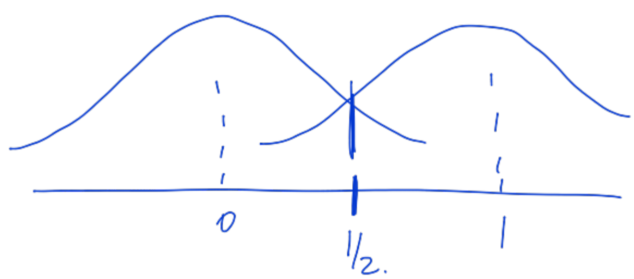
$Var[N] = \left(\sqrt{\frac{2}{T}}\right)^2 \cdot \frac{N_0}{2} T = N_0$.

$\Rightarrow$ N is Gaussian with mean 0, Var N_0.

(d) Compute the smallest error probability and state its decision rule when the receiver in Fig. 1 is used with $\phi_1(t), \phi_2(t)$. Assume bits are equally likely.

if $s_1(t)$ was sent, $V \sim N(1, N_0)$

s_2 " $V \sim N(0, N_0)$



the optimal threshold is $\frac{1}{2}$. $\Rightarrow$ $\begin{cases} \text{say } s_1 & \text{if } V > \frac{1}{2} \\ s_2 & \text{if } V < \frac{1}{2} \end{cases}$

$$P(\text{say } s_1 \mid s_2 \text{ sent}) = \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right)dx$$

$\frac{x}{\sqrt{N_0}} = t$.

$$= \int_{\frac{1}{2\sqrt{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)dt$$

$$= Q\left(\frac{1}{2\sqrt{N_0}}\right)$$

similarly, $P(\text{say } s_2 \mid s_1 \text{ sent}) = Q\left(\frac{1}{2\sqrt{N_0}}\right)$ $\Rightarrow$ $P_e = \frac{1}{2}(\quad) + \frac{1}{2}(\quad) = Q\left(\frac{1}{2\sqrt{N_0}}\right)$