통신 2-1. Fourier Series & Transform

관련 과목	
관련 시험/과제	통신_HW1
강의 일자	
상태	정리중
강의자료	lecture2(1)_FS_FT_rev.pdf

Why We Learn FT?

- Why Fourier Transform matters in digital communication?

1) Signals should be modulated (=up-converted)

- BW 잘게 쪼개서 할당함
- 이유 1 : To utilize own nature of frequency bands. (reflection, diffraction, scattering, penetration, …) high/low frequency의 특성이 각각 있음. high는 빠르고 low는 회절이 잘 됨.
- 이유 2 : 여러 user들이 서로 다른 신호를 동시에 송수신 하기 위해 (e.g., freq-division multiplexing)

2) Digital information (=bit stream) is time-slotted, but what we can send is always continuous

- Sharp transition in continuous signal (ex. 0101...처럼 discontinous한 신호)은 BW가 무한대. 그대로 보내면 다른 유저들이 BW 사용을 못함.
  → Some smooth signal must be send (Chap. 5)

Dual Time-frequency Viewpoint

- 대부분의 무선 통신 시스템은 frequency modulation을 함.
  - Frequency up-conversion / down-conversion (FCC regulation)
- 보통 신호들이 frequency domain일 때 처리하기 쉬운 경우가 많음.
- 따라서 frequency domain에 대한 높은 이해도가 필요함.

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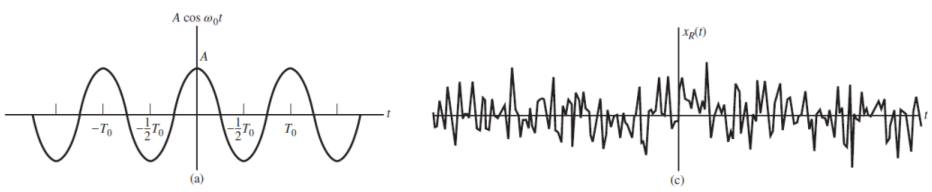
1. Functions

(1) Deterministic vs Random signals

- Deterministic signals (ex. Figure 2.1(a))
  - At any time t, the value of $x(t)$ is given without uncertainty
  - Model for known signals
  - 이 챕터에서 다루는 신호
- Random signals (ex. Figure 2.1(c))
  - At time t, the value of $x(t)$ is a random variable
  - Probabilistic model
  - Model for unknown signals (e.g., noise, coin-toss)
  - Chapter 6, 7에서 다룰 예정



(a) Deterministic (sinusoidal) signal. (c) Random signal.

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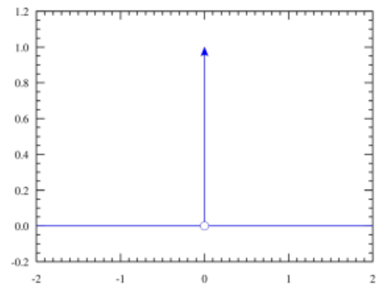
(2) Singular Functions

- *def)* A function is called singular if it has a finite number of discontinuities
  함수값이 discontinous한(끊겨있는) 구간의 개수가 유한한 함수

1) Delta Function, *δ*(*t*)

- = Unit impluse function, Dirac delta function
- *def)* a function that satisfies the following property for any test function *x*(*t*)



$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

- $\delta(t)$는 위와 같은 “property”에 의해서 정해지는 함수이며, input-output closed-form 특성이 없음.
  → 따라서 ordinary function이 아님. (*ordinary function의 예 : f(x) = 3x+5)
- 간단히 말하자면, $\delta(0) = \infty$, $\delta(t) = 0$ for all $t \neq 0$ 이다.
  - Infinitely high but infinitesimal peak at $t = 0$
  - 실제로 존재하지 않는 함수

— 함수 유도 예시 —

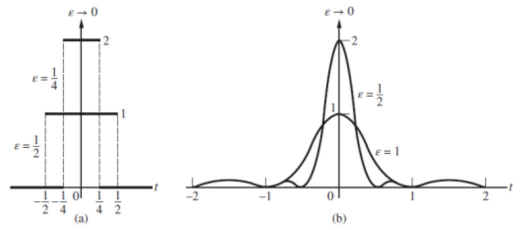
아래와 같은 함수에서 $\lim_{\epsilon\to\infty}$ 하는 것으로 $\delta(t)$를 정의할 수 있음.

- Example 1 of $\delta(t)$

$$\delta_\epsilon(t) = \frac{1}{2\epsilon}\Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & \text{otherwise} \end{cases}$$

- Example 2 of $\delta(t)$

$$\delta_{1\epsilon}(t) = \epsilon\left(\frac{1}{\pi t}\sin\frac{\pi t}{\epsilon}\right)^2$$



+) Gaussian 분포 with $\mu = 0, \sigma^2$ 으로 유도 가능 : $\delta(t) = \lim_{\sigma^2\to\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$

— Properties —

- Sifting : $x(t_0) = \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt$
  Pf.) Let $x(t_0) = c$, $x(t_0) = \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} c\,\delta(t-t_0)dt = c\int_{-\infty}^{\infty}\delta(t-t_0)dt = c \times 1 = c = x(t_0)$
  이후 properties 증명 - 과제일 수도 있고 아닐 수도 있음
- Time-scaling : $\delta(at) = \frac{1}{|a|}\delta(t)$
- Symmetry : $\delta(t) = \delta(t-t_0)$
- $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
- Differentiation : $\int_{-\infty}^{\infty} x(t)\delta^{(n)}(t-t_0)dt = (-1)^n x^{(n)}(t_0)$,
  where the superscript (n) denotes the n-th derivative
- $a_0\delta(t) + a_1\delta^{(1)}(t) + ... + a_n\delta^{(n)}(t) = b_0\delta(t) + b_1\delta^{(1)}(t) + \cdots + b_n\delta^{(n)}(t)$
  implies that $a_i = b_i$

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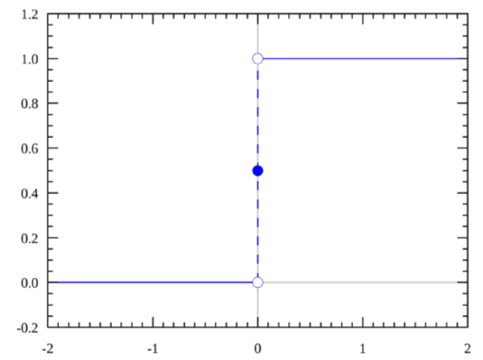
2) Step Function

- delta function 적분한 함수
- *def)* unit step function :

$$u(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda$$

- (Almost) equivalently, $u(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0 \end{cases}$

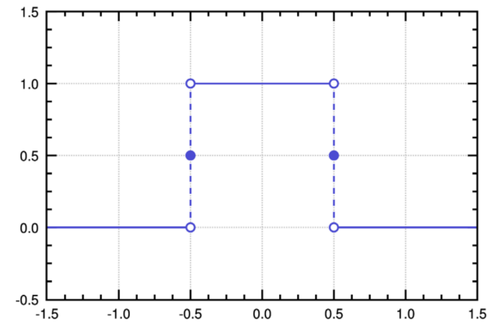


- The value at $t = 0$ can be arbitrary (in engineering) as it is a (Lebesgue) measure zero set
  t=0 일 때의 값은 공학적으로 하나도 중요하지 않음. 사람마다 1/2로 정의하기도 함.

3) Rectangular Function

- *def)* Unit rectangular function :

$$rect(t) = \prod(t) = \begin{cases} 1 & |t| \leq 1/2, \\ 0 & else \end{cases}$$



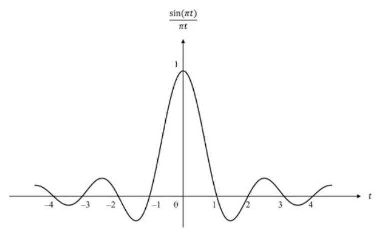
- The value at $t = \pm 1/2$ can be arbitrary (in engineering) as it is a (Lebesgue) measure zero set
  마찬가지로 공학적으로 전혀 안 중요한 값임.

(3) Sinc Function

- singular function 아님 (discontinuity가 없음. continous함)
- *def)* unit sinc function :

$$sinc(t) \triangleq \frac{sin(\pi t)}{\pi t}$$

  sin 함수와 1/t 함수의 곱이라고 생각하면 됨



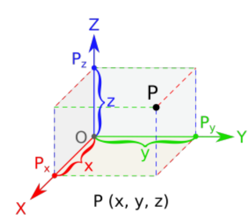
- rectangular function의 frequency counterpart 함수로써 자주 사용함.

2. Fourier Series

- 시간이나 공간에 대한 신호를 주파수 성분으로 분해하는 변환

(1) Signal Spaces and Basis Functions

- For instance of 3-dimensional Euclidean space,
  $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a canonical basis of $\mathbb{R}^3$
  → That is, **any** vector in $\mathbb{R}^3$ can be represented by a linear sum of basis vectors



- 이 개념을 general function spaces로 확장할 수 있음 → “inner product spaces”
  우리가 다룰 것은 아니니 이런 게 있다 정도로만 알고 넘어가면 됨 - Linear vector space, Inner product, Banach space, Hilbert space, … (in functional analysis), Measurable functions, Lesbesgue measure, …

- $L_p([a,b])$ functions : $L_p([a,b]),\ p \geq 1$ is the set of functions such that
  $\left(\int_a^b |f(t)|^p dt\right)^{1/p} < \infty$ (finite)

$L_1([a,b])$ is the set of functions satisfying $\int_a^b |f(t)|dt < \infty$
$L_2([a,b])$ is the set of functions satisfying $\left(\int_a^b |f(t)|^2 dt\right)^{1/2} < \infty$

- engineering에서 다루는 거의 모든 함수는 L_1 혹은 L_2 function이라고 보면 됨.

- A function $f : [a,b] \to \mathbb{C}$ is square-integrable [a,b] (i.e, $L_2([a,b])$) if $\int_a^b |f(t)|^2 dt < \infty$
→ Then, $\{e^{jnw_0t}\}_{n\in\mathbb{Z}}$ with $w_0 = \frac{2\pi}{b-a}$ is a **orthonormal basis** for $L_2([a,b])$
⇒ Fourier Series 관련됨

- A function $f : \mathbb{R} \to \mathbb{C}$ is square-integrable [a,b] (i.e, $L_2(\mathbb{R})$) if $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$
→ Then, $\{e^{j2\pi ft}\}_{f\in\mathbb{R}}$ is a **orthonormal basis** for $L_2(\mathbb{R})$
⇒ Fourier Transform 관련됨

- 즉, engineering에서 다루는 거의 모든 함수는 복소함수 e^{jnw_0t}의 linear combination으로 표현될 수 있음.
Why? → 오일러 공식(Euler's formula)에 의거

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(2) FS: Definition

- (Analysis) For x(t) defined on $[t_0, t_0 + T_0]$, the **Fourier coefficients** are :

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jnw_0t}dt \qquad (\text{where } w_0 = 2\pi f_0 = \frac{2\pi}{T_0})$$

- (Synthesis) $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{-jnw_0t}$
벡터 공간에서 어떤 벡터 x를 basis 벡터들의 linear sum으로 나타낼 때, 각 basis 벡터 앞에 coefficient가 붙음. 각 coefficient는 그 vector와 각 basis vector를 inner product 하여 구함.
→ 같은 원리라고 보면 됨.

- Fourier coefficient를 frequency component라고 부름.

`3/2 -> 3/7`

(3) FS: Examples

Table 2.1 Fourier Series for Several Periodic Signals

Signal (one period)	Coefficients for exponential Fourier series
1. Asymmetrical pulse train; period = T_0: $x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right), \tau < T_0$ $x(t) = x(t+T_0)$, all t	$X_n = \frac{A\tau}{T_0}\text{sinc}(nf_0\tau)e^{-j2\pi nf_0t_0}$ $n = 0, \pm1, \pm2, \ldots$
2. Half-rectified sinewave; period = $T_0 = 2\pi/\omega_0$: $x(t) = \begin{cases} A\sin(\omega_0 t), & 0 \le t \le T_0/2 \\ 0, & -T_0/2 \le t \le 0 \end{cases}$ $x(t) = x(t+T_0)$, all t	$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm2, \pm4, \cdots \\ 0, & n = \pm3, \pm5, \cdots \\ -\frac{1}{4}jnA, & n = \pm1 \end{cases}$
3. Full-rectified sinewave; period = $T_0' = \pi/\omega_0$: $x(t) = A\lvert\sin(\omega_0 t)\rvert$	$X_n = \frac{2A}{\pi(1-4n^2)}, n = 0, \pm1, \pm2, \ldots$
4. Triangular wave: $x(t) = \begin{cases} -\frac{4A}{T_0}t + A, & 0 \le t \le T_0/2 \\ \frac{4A}{T_0}t + A, & -T_0/2 \le t \le 0 \end{cases}$ $x(t) = x(t+T_0)$, all t	$X_n = \begin{cases} \frac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

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(4) FS: Convergence

- Suppose someone correctly computed $\{X_n\}_n$ for some x(t)

- Can we say $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0t}$?
$x(t)$ —FS→ $\{X_n\}$ —FS^{-1}→ $x(t)$? 이게 가능한 함수인 지 어떻게 확인할까?

- Naive approach : 직접 계산해보기. Calculate the inverse of the series and compare to *x*(*t*) → 복잡함.

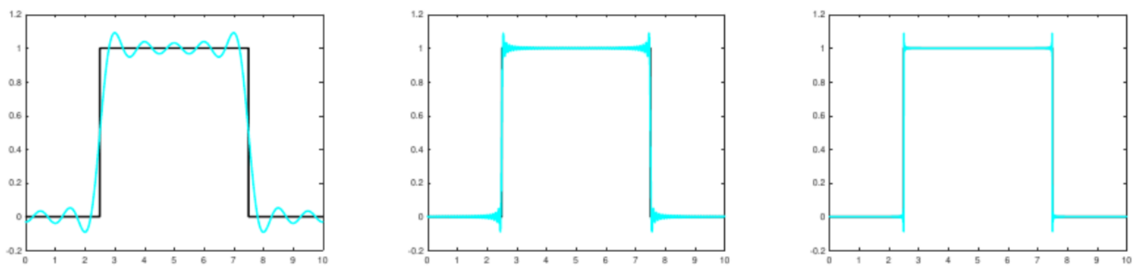
— Sufficient condition for convergence —

- condition 1, 2 둘 중 하나만 만족시키면 됨.
- 1) If $x(t) \in L_2([t_0, t_0+T_0])$ (finite energy),
  $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t}$ at any **continuous point *t* in** $[t_0, t_0+T_0]$
  discontinous한 point에서는 convergence를 얘기할 수 없음.
- 2) If $x(t)$ satisfies the Dirichlet condition, $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t}$ at any continuous point *t*
- The Dirichlet condition
  - *x*(*t*) is absolutely integrable over the interval,
  i.e., $\int_{t_0}^{t_0+T_0} |x(t)|dt < \infty$, or equivalently $L_1([t_0, t_0+T_0])$
  - The number of maxima and minima of *x*(*t*) is finite
  - The number of discontinuities is finite

- In engineering, (almost) every function is smooth and has finite energy
  - (almost) every function is in $L_2([a,b])$
  - or satisfies the Dirichlet condition
- **[결론]** We can recover the original *x*(*t*) from the Fourier coefficients with no doubt!

(5) Gibbs Phenomenon

- Unavoidable overshoot/undershoot at discontinuity
  - Even when the number of components(k) grows without bound
- x(t) 복원할 때 k를 어떻게 설정하든 discontinous한 point에서는 어쩔 수 없이 발생함.
  그래서 convergence를 보장할 수 없다는 것.



(a) 10 components (b) 100 components (c) 1000 components

(6) FS: Properties

- definition 이용하면 모두 쉽게 증명됨. 외우는 게 중요한 건 아니지만 유도할 수 있어야 함. 시험에서 table 제공.

- If $x(t)$ is real, $X_n^* = X_{-n}$
- DC component:

$$X_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j0w_0 t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)dt = \text{average of } x(t)$$

- AC component:

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)[\cos(nw_0 t) - j\sin(nw_0 t)]dt$$

- If $x(t)$ is real and even, that is $x(t) = x(-t)$, X_n is purely real and even
- If $x(t)$ is real and odd, that is $x(t) = -x(-t)$, X_n is purely imaginary and odd

modulation
중요한 개념!!

time scailing : fundamental frequency를 꼭 aw_0로 바꿔줘야 함.

- Let X_n be $x(t)$'s Fourier coefficients, Y_n be $y(t)$'s coefficients
- Time-translation: $x(t-t_0) \longleftrightarrow e^{-jnw_0t_0}X_n$
- Modulation: $x(t)e^{jkw_0t} \longleftrightarrow X_{n-k}$
- Time reversal: $x(-t) \longleftrightarrow X_{-n}$
- Time scailing: $x(at) \longleftrightarrow X_n$ with $w_0 \to aw_0$
- Differentiation: $x'(t) \longleftrightarrow jnw_0X_n$
- Integration: $\int_{-\infty}^{t} x(t)dt \longleftrightarrow \frac{X_n}{jnw_0}$ (assuming $X_0 = 0$)
- Periodic convolution: $x(t) * y(t) \longleftrightarrow TX_nY_n$

P : power

time domain에서도 구할 수 있고 freq domain에서도 구할 수 있음

- Parseval's theorem

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |X_n|^2$$

- In particular, $|X_n|^2$ is called **energy (spectral) density**

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(7) Fourier Series for ∞-period Signals

- Consider an **aperiodic(비주기적)** signal *x*(*t*) on the intire real line (-∞, ∞)
- It is indeed a special case of periodic signals with **infinite period**

$$\begin{aligned} x(t) &= \lim_{T_0\to\infty} \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \\ &= \lim_{T_0\to\infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(\lambda)e^{-j2\pi n f_0 \lambda} d\lambda\right] e^{j2\pi n f_0 t} \\ &\text{Then, } \frac{1}{T_0} \to df, nf_0 \to f \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\lambda)e^{-j2\pi f\lambda} d\lambda\right] e^{j2\pi f t} df \end{aligned}$$

- → Fourier transform 유도 가능

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3. Fourier Transform

(1) FT: Definition

- (Analysis) For *x*(*t*), the Fourier transform is

$$X(f) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t} dt$$

- 부호가 헷갈린다면 **복소함수의 inner product**임을 기억할 것 : $< f,\ g > = \int_a^b f(t)\overline{g(t)}dt$
- (Synthesis) For *x*(*t*), the inverse Fourier transform is

$$x(t) = \mathfrak{F}^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t} df$$

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(2) FT: Symmetry

- For real periodic *x*(*t*), $X_{-n} = X_n^*$ (FS)
- For real aperiodic *x*(*t*), $X(-f) = X^*(f)$ (FT)
  f → -f 결과와 X^* 결과 비교해보면 같음을 쉽게 증명 가능

- If x(*t*) is even, that is, $x(t) = x(-t)$, then *X*(*f*) is purely real and even
- If x(*t*) is odd, that is, $x(t) = -x(-t)$, then *X*(*f*) is purely imaginary

(3) Energy Spectral Density

- For periodic signal, $|X_n|^2$ is called power (spectral) density
- Similarly for aperiodic signal, we have energy (spectral) density $|X(f)|^2$
  $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$ (by Parseval's theorem)

(4) Signal Bandwidth

- The bandwidth of a signal represents the range of frequencies in the signal
- bandwidth가 높을수록 신호의 최소 주파수와 최대 주파수의 차이가 큰 것 (the variations in the frequencies 큼)
- 보통, the bandwidth of a real signal *x*(*t*)는 신호의 positive한 frequency 범위에서 정의함
- The bandwith is : BW = $f_{max} - f_{min}$
  - f_{min} : the highest positive frequency in *X*(*f*)
  - f_{max} : the lowest positive frequency in *X*(*f*)

(5) Fourier Series vs Transform

	Fourier Series	**Fourier Transform**
x(*t*)	**periodic** with period $T_0 = 1/f_0$	**aperiodic**
Analysis	$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0 t} dt$	$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$
Synthesis	$X_n = \sum_{n=-\infty}^{\infty} X_n e^{-jnw_0 t}$	$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$
Frequency	Has a fundamental frequency and harmonics (nf_0) n=0 : DC(constant), n=1 : fundamental n=2, 3 : second, third harmonic	No fundamental frequency Could contain all possible frequency
Energy density	Energy spectral density $\|X_n\|^2$ By Parseval's theorem $P = \frac{1}{T_0} \int_{T_0} \|x(t)\|^2 dt = \sum_{n=-\infty}^{\infty} \|X_n\|^2$	Energy spectral density $\|X(f)\|^2$ By Parseval's theorem $P = \int_{-\infty}^{\infty} \|x(t)\|^2 dt = \int_{-\infty}^{\infty} \|X(f)\|^2 df$

(6) FT: Properties

1) Linearity : $x(t) \leftrightarrow X(f)$ and $y(t) \leftrightarrow Y(f)$, then $ax(t) + by(t) \leftrightarrow aX(f) + bY(f)$

2) Duality : $X(t) \leftrightarrow x(-f)$ and $X(-t) \leftrightarrow x(f)$
Since the FT and inverse FT differ only by negative sign in the exponent

3) Translation in time-domain : $x(t - t_0) \leftrightarrow e^{-j2\pi f t_0} X(f)$

4) Scaling : $x(at) \leftrightarrow \frac{1}{|a|} X(\frac{f}{a})$
- If *a* > 1, then x(*at*) is a contracted form of x(*t*) (t에서 늘어남, f에서 압축)
- If *a* < 1, x(*at*) is an expanded version of x(*t*) (t에서 압축, f에서 늘어남)
- If we expand a signal in the time-domain, its FT representation contracts, and vice versa(반대도 마찬가지)
- This is exactly what we expect since contraction in time domain makes the changes more quickly, thus increasing its frequency

`3/7 → 3/14`

5) **Convolution** : $x(t) * y(t) \leftrightarrow X(f)Y(f)$ (밑에 자세히)

6) Multiplication : $x(t)y(t) \leftrightarrow X(f) * Y(f)$

7) Translation in freq-domain : $x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$
- Dual of the shift in time-domain
- In communication systems, this operation is called **modulation**, very important! (밑에 자세히)

8) Differentiation : Assume x(*t*) is differentiable, then : $x^{(n)}(t) \leftrightarrow (j2\pi f)^n X(f)$

9) Integration : Assume has no DC component, that is, $x(0) = 0$, then : $\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{X(f)}{j2\pi f}$

Name	Time-domain operation (signals assumed real)	Frequency-domain operation
Superposition	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(f) + a_2 X_2(f)$
Time delay	$x(t - t_0)$	$X(f)\exp(-j2\pi t_0 f)$
Scale change	$x(at)$	$\lvert a\rvert^{-1} X(f/a)$
Time reversal	$x(-t)$	$X(-f) = X^*(f)$
Duality	$X(t)$	$x(-f)$
Frequency translation	$x(t)\exp(j2\pi f_0 t)$	$X(f - f_0)$
Modulation	$x(t)\cos(2\pi f_0 t)$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
Convolution[4]	$x_1(t) * x_2(t)$	$X_1(f)X_2(f)$
Multiplication	$x_1(t)x_2(t)$	$X_1(f) * X_2(f)$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$X(f)/(j2\pi f) + \frac{1}{2}X(0)\delta(f)$

1) Convolution

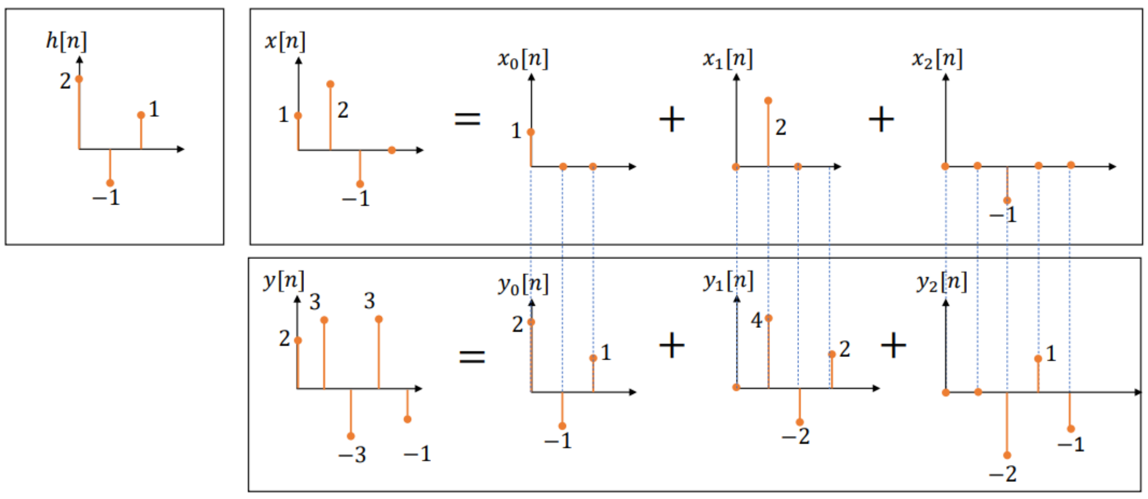
- *def)* (continuous-time)
  Convolution is used to obtain the output of an LTI system for a given input

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$$

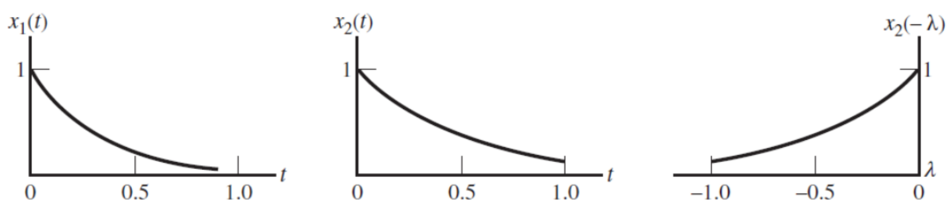
- *def)* (discrete-time)

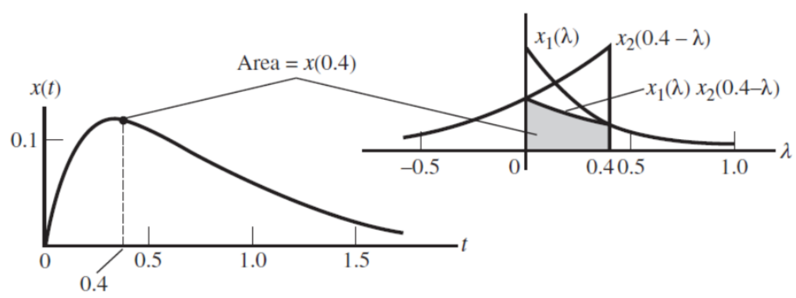
$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

$$y[n] = x_0 h[n-0] + x_1 h[n-1] + x_2 h[n-2] = x[0]h[n-0] + x[1]h[n-1] + x[2]h[n-2]$$
$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \implies x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$



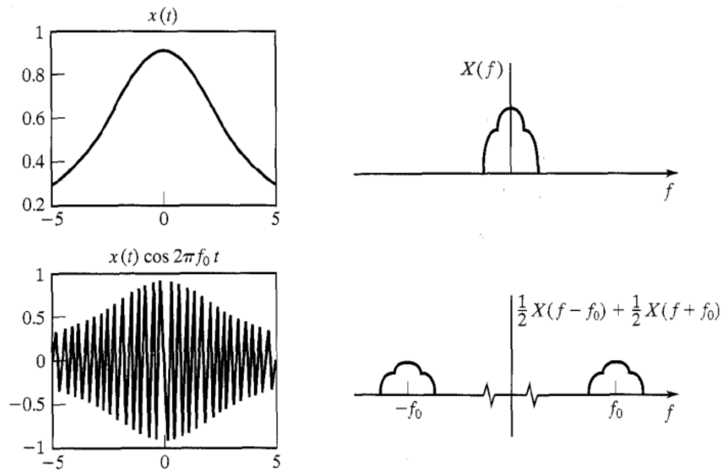
- [Example] $x_1(t) = e^{-at}u(t),\ x_2(t) = e^{-\beta t}u(t)$





2) Modulation

- For instance, $x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}X(f-f_0)+\frac{1}{2}X(f+f_0)$



- This is what we will learn in Chapter 3, amplitude modulation (AM).

(7) Generalization

- The Fourier transform of $\delta(t)$ is undefined as it is a “bad” function
- But we can formally calculate it via “generalization”

$$\mathfrak{F}[\delta(t)] = \mathfrak{F}\left[\lim_{\tau\to 0}\frac{1}{\tau}\prod\left(\frac{t}{\tau}\right)\right] = \lim_{\tau\to 0}\operatorname{sinc}(f\tau) = 1$$

(8) FT of Impulse Train

- Impulse train = sum of harmonics

$$y_s(t) = \sum_{m=-\infty}^{\infty}\delta(t-mT_s) = f_s\sum_{n=-\infty}^{\infty}e^{jn2\pi f_s t}$$
$$\longleftrightarrow Y_s(f) = \sum_{m=-\infty}^{\infty}e^{j2\pi mT_s f} = f_s\sum_{n=-\infty}^{\infty}\delta(f-nf_s)$$

- Proof : 과제 (📄 통신_HW1)

(9) FT of Periodic Signals

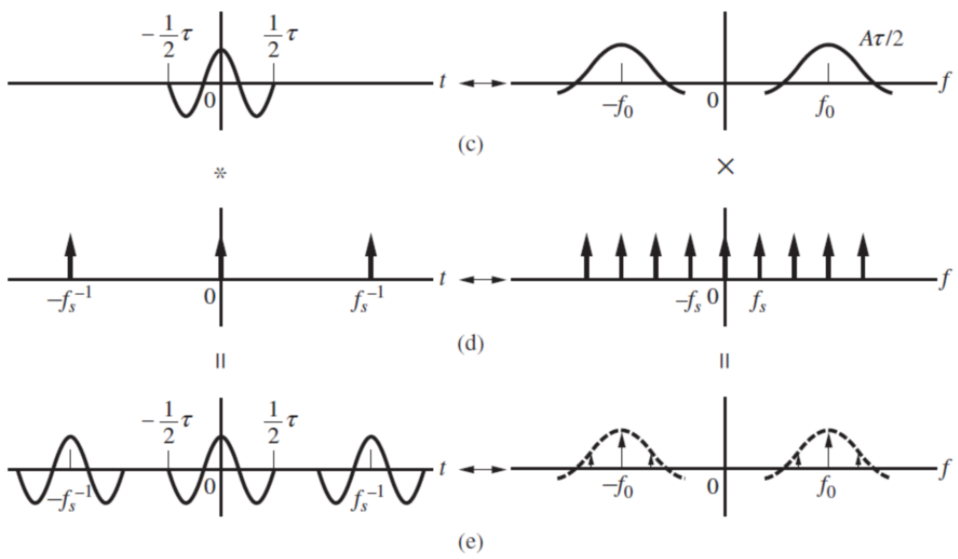
- How can we obtain the Fourier transform of a periodic signal?

  1) Decompose it into an infinite sum of finite-energy signals
     - Convolution with impulse train recovers the original signal

  2) For the unit component, apply FT

  3) The FT of impulse train is still impulse train. As the FT of convolution is multiplication, we can get the FT of the entire function

$$x(t) = \sum_{m=-\infty}^{\infty}p(t-mT_s) = \left[\sum_{m=-\infty}^{\infty}\delta(t-mT_s)\right]*p(t) = \sum_{m=-\infty}^{\infty}\delta(t-mT_s)*p(t)$$
$$\longleftrightarrow \quad \left[f_s\sum_{n=-\infty}^{\infty}\delta(f-nf_s)\right]P(f) = \sum_{n=-\infty}^{\infty}f_sP(nf_s)\delta(f-nf_s) = X(f)$$



(10) Poisson Sum Formula

$$x(t) = \sum_{m=-\infty}^{\infty} p(t - mT_s) \xrightarrow{\mathfrak{F}} \sum_{n=-\infty}^{\infty} f_s P(nf_s)\delta(f - nf_s) = X(f)$$

$$\xrightarrow{\mathfrak{F}^{-1}} \sum_{n=-\infty}^{\infty} f_s P(nf_s) e^{j2\pi n f_s t} = x(t)$$

- This means, $P(nf_s)$ are the Fourier series coefficients of $T_s \sum_{m=-\infty}^{\infty} p(t - mT_s)$

(11) FT Pairs

Signal	Fourier transform
$\Pi(t/\tau) = \begin{cases} 1, & \lvert t\rvert \le \tau/2 \\ 0, & \text{otherwise} \end{cases}$	$\tau\,\mathrm{sinc}(f\tau) = \tau \dfrac{\sin(\pi f \tau)}{\pi f \tau}$
$2W\,\mathrm{sinc}(2Wt)$	$\Pi(f/2W)$
$\Lambda(t/\tau) = \begin{cases} 1 - \lvert t\rvert/\tau, & \lvert t\rvert \le \tau \\ 0, & \text{otherwise} \end{cases}$	$\tau\,\mathrm{sinc}^2(f\tau)$
$W\,\mathrm{sinc}^2(Wt)$	$\Lambda(f/W)$
$\exp(-\alpha t)u(t),\ \alpha > 0$	$1/(\alpha + j2\pi f)$
$t\exp(-\alpha t)u(t),\ \alpha > 0$	$1/(\alpha + j2\pi f)^2$
$\exp(-\alpha\lvert t\rvert),\ \alpha > 0$	$2\alpha/\left[\alpha^2 + (2\pi f)^2\right]$
$\exp\left[-\pi(t/\tau)^2\right]$	$\tau\exp\left[-\pi(\tau f)^2\right]$
$\delta(t)$	1
1	$\delta(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
$u(t)$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
$1/(\pi t)$	$-j\,\mathrm{sgn}(f);\ \mathrm{sgn}(f) = \begin{cases} 1, & f > 0 \\ -1, & f < 0 \end{cases}$
$\sum_{m=-\infty}^{\infty}\delta(t - mT_s)$	$f_s\sum_{n=-\infty}^{\infty}\delta(f - nf_s);\ f_s = 1/T_s$

(12) Auto-correlation Function

— ϕ(τ) —

- Auto-correlation function for energy signals : $\phi(\tau) = \int_{-\infty}^{\infty} x(\lambda)x(\lambda+\tau)d\lambda = x(\tau) * x(-\tau)$
- or equivalently, $\phi(\tau) = \mathfrak{F}^{-1}[X(f)X^*(f)] = \mathfrak{F}^{-1}[X(f)] * \mathfrak{F}^{-1}[X^*(f)]$
- $\phi(\tau)$ measures the similarity between $x(t)$ and $x(t+\tau)$
- $\phi(0) = \int \lvert x(t)\rvert^2 dt = \int \lvert X(f)\rvert^2 df$ by Parseval's theorem
- Note that $\phi(\tau) = \int \lvert X(f)\rvert^2 e^{j2\pi f\tau} df$ 에서 $\phi(\tau)$ and $\lvert X(f)\rvert^2$ are FT pairs
  More generally, see gen'ed(generalized) Parseval's theorem.

— Generalized Parseval's Theorem —

- (Generalized) Parseval's theorem : $\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$
- If $f(t) = g(t)$: $\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(f)|^2 df$

- Recall the FT of the delta function : $\delta(t) = \int e^{j2\pi f t} df$
- Proof :

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(t)g^*(t)dt &= \int_t \left(\int_{f_1} F(f_1)e^{j2\pi f_1 t} df_1\right) \left(\int_{f_2} G(f_2)e^{j2\pi f_2 t} df_2\right)^* dt \\
&= \int_{f_1}\int_{f_2}\int_t F(f_1)G^*(f_2)e^{j2\pi(f_1-f_2)t} dt df_2 df_1 \\
&= \int_{f_1}\int_{f_2} F(f_1)G^*(f_2)\left(\int_t e^{j2\pi(f_1-f_2)t} dt\right) df_2 df_1 \\
&= \int_{f_1} F(f_1)\left(\int_{f_2} G^*(f_2)\delta(f_1-f_2) df_2\right) df_1 \\
&= \int_{f_1} F(f_1)G^*(f_1) df_1
\end{aligned}
$$

— R(τ) —

- **Auto-correlation function** for **power** signals :

$$R(\tau) =< x(t), x(t+\tau) > \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(\lambda)x(\lambda+\tau)d\lambda$$

- If periodic, $R(\tau) = \frac{1}{T}\int_T x(\lambda)x(\lambda+\tau)d\lambda$
- $R(\tau)$ measures the similarity between $x(t)$ and $x(t+\tau)$, too.
- Power spectral density (PSD) : $S(f) \triangleq \mathfrak{F}[R(\tau)]$
- If $x(t)$ is periodic, $S(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0)$
- $S(f)$ represents the signal power at frequency f

— Properties of $R(\tau)$,$\phi(\tau)$ —

- $R(0) \geq |R(\tau)|$ for all τ
  - Why? Use $(x(\lambda) - x(\lambda-\tau))^2 \geq 0$ and the definition of $R(\tau)$
- $R(\tau) = R(-\tau)$, that is, $R(\tau)$ is even.
- $S(f)$ is non-negative

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