Digital Signal Processing (Lecture Note 4)

Jin Ho Chang

Professor

Dept. of Electrical Engineering and Computer Science

Discrete Time Fourier Transform (DTFT)

Discrete Time Fourier Transform

- Similar to continuous time signals, discrete time sequences can also be periodic or non-periodic, resulting in discrete-time Fourier series or discrete – time Fourier transform, respectively.
  - Most signals in engineering applications are non-periodic ➔ focusing on discrete-time Fourier transform (DTFT).
- Facts
  - The sum of x[n], weighted with continuous exponentials, is continuous ➔ the DTFT X(ω) is continuous (non-discrete)
  - Since X(ω) is continuous, x[n] is obtained as a continuous integral of X(ω).
  - X[n] is obtained as an integral of X(ω), where the integral is over an interval of 2π. ➔ This is the first clue that DTFT is periodic with 2π in frequency domain.

- Let x[n] be a discrete-time signal whose values can be real or complex.
- Definition

$$X(\Theta) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Theta n}$$ (θ: Digital angular frequency)

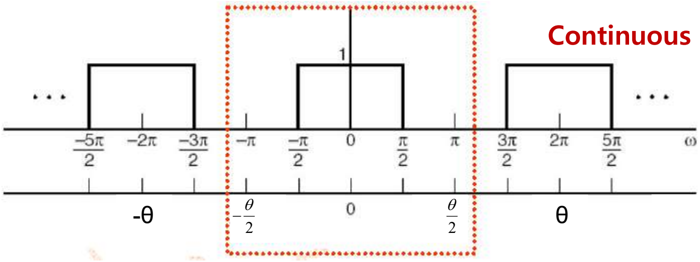
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\text{if } t \rightarrow nT \text{ and } \theta = \omega T = \frac{\omega}{f_s},\ X(\theta) = \sum_{n=-\infty}^{\infty} x(nT) e^{-jn\omega T} = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta}$$

  - θ is unitless
  - $X(\Theta) = X(\Theta + 2\pi)$: Periodic function with a period of 2π

- Implications of the periodicity property

$$X(\Theta) = X(\Theta + 2\pi)$$



Continuous Spectrum

What if the spectrum is also in discrete form?

- Inverse DT (Discrete-Time) Fourier transform

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Theta)e^{j\Theta n}d\Theta$$

$$\int_{\Theta_0}^{\Theta_0+2\pi} X(\Theta)e^{j\Theta m}d\Theta = \int_{\Theta_0}^{\Theta_0+2\pi}\sum_{n=-\infty}^{\infty} x(n)e^{-j\Theta(n-m)}d\Theta$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} x(n)\int_{\Theta_0}^{\Theta_0+2\pi} e^{-j\Theta(n-m)}d\Theta = \int_{\Theta_0}^{\Theta_0+2\pi} X(\Theta)e^{j\Theta m}d\Theta$$

$$\boxed{\int_{\Theta_0}^{\Theta_0+2\pi} e^{-j\Theta(n-m)}d\Theta = \begin{cases} 2\pi, & n = m \\ 0, & n \neq m \end{cases}}$$

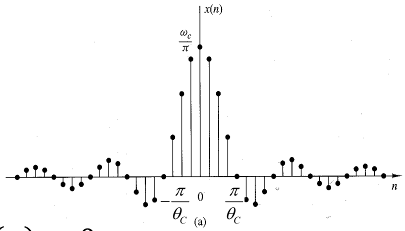
$$x(m) = \frac{1}{2\pi}\int_{2\pi} X(\Theta)e^{j\Theta m}d\Theta$$

Note:

$$\delta(n) \leftrightarrow 1 \quad \Leftrightarrow \quad x(n) = \frac{1}{2\pi}\int_{2\pi} e^{j\Theta n}d\Theta = \delta(n)$$

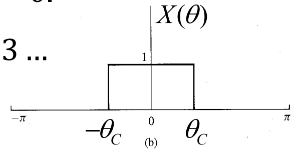
- DTFT Example 1: Sinc function

$$X(\Theta)=\begin{cases}1, & |\Theta| \le \Theta_c \\ 0, & \Theta_c < |\Theta| \le \pi\end{cases}$$



$if\ \theta_C n = k\pi, x(n) = 0.$

$\therefore n = \dfrac{k\pi}{\theta_C}, k = 1,2,3 \ldots$



$$\boxed{\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{-\Theta_c}^{\Theta_c} e^{j\Theta n} d\Theta \\ &= \frac{(e^{j2\Theta_c n} - e^{-j2\Theta_c n})}{2\pi j n} \\ &= \frac{\sin\Theta_c n}{n\pi} \end{aligned}}$$

: not absolute summable

$if\ x(n) = 0,$

$$\frac{\frac{d}{dn}(sin(\theta_C n))}{\frac{d}{dn}(n\pi)} = \frac{\theta_C cos(\theta_C n)}{\pi}\bigg|_{n=0}$$

$$= \frac{\theta_C}{\pi}$$

- DTFT Example 2: Truncation
  - Gibbs phenomenon due to truncation.

$$X_N(\Theta) = \sum_{n=-N}^{N} \frac{\sin\Theta_c n}{\pi n} e^{-j\Theta n}$$

  - FT of a sinc function exists, but the infinite series does not converge uniformly for all θ

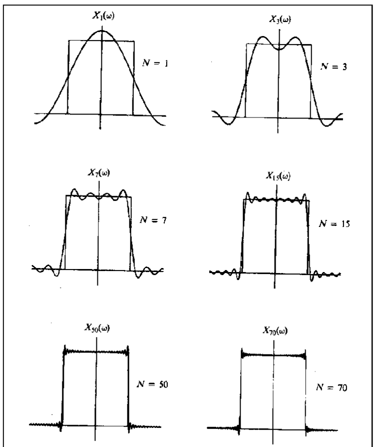


Figure 4.14 Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity.

Properties of DTFT

- Bandwidth of DT signals
  - A DT signal is said to be band-limited if
    $X(\omega)=0 \ \ for \ \ all \ |\omega| \geq \omega_c < \pi$
  - Periodic, repeated spectrum
  - Low-Pass signal.
    - ✓Symmetric signal -> Real signal
  - Multiplication by a complex exponential
    - ✓Shifted spectrum
      Unsymmetric spectrum
    - ✓Complex signal
  - Modulation
    $x(n)\cos\omega_0 n \overset{F}{\Longleftrightarrow} \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$

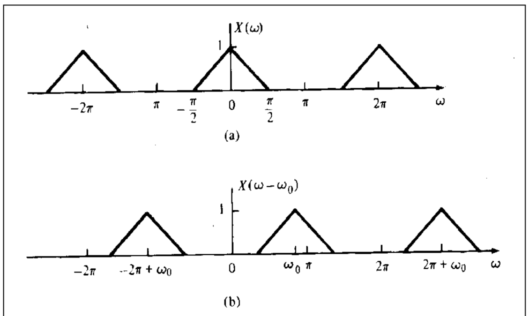


Figure 4.35 Illustration of the frequency-shifting property of the Fourier transform.

- Frequency Response of DT systems
  - LTI system and convolution sum

$$L[e^{j\Theta n}]=H(\Theta)e^{j\Theta n}$$

$$L[\delta(n)] = L\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\Theta n}d\Theta\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} L[e^{j\Theta n}]d\Theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\Theta)e^{j\Theta n}d\Theta = h(n)$$

$$L[x(n)] = L\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)L[\delta(n-k)]$$

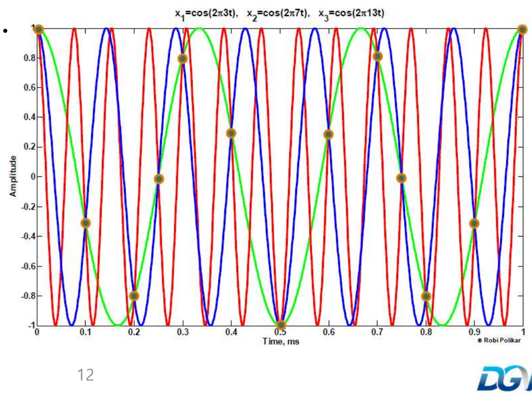
$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad : convolution \ sum$$

$$\boxed{Y(\Theta)=H(\Theta)X(\Theta) \quad S_{yy}(\Theta)=|Y(\Theta)|^2=|H(\Theta)|^2 S_{xx}(\Theta)=|H(\Theta)|^2|X(\Theta)|^2}$$

Sampling Theory (Chap. 4)

Sampling of Continuous Signals

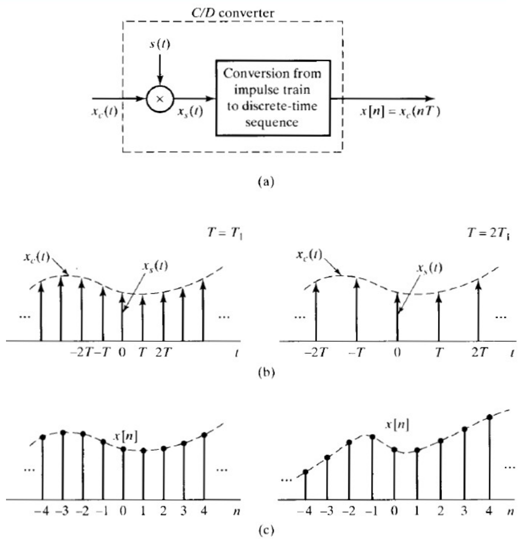
- In sampling, identical discrete-time signals may result from the sampling of more than one distinct continuous-time function. In fact, there exists an infinite number of continuous-time signals, which when sampled lead to the same discrete-time signal.



Shannon's Sampling Theorem

- A continuous time signal x(t) with frequencies no higher than $\omega_{max}=2\pi f_{max}$ can be reconstructed exactly, precisely and uniquely from its samples $x[n]=x(nT_s)$, if the samples are taken at a sampling rate (frequency) of $fs=1/Ts$ or ($\omega_{max}=2\pi/Ts$) that is greater than $2f_{max}$.
- The frequency $\omega_s/2$ (or f_s/x or f_{max}) is called the Nyquist frequency (or folding frequency), as it determines the minimum sampling frequency required.
- The minimum required sampling frequency is called the Nyquist rate.
- In other words, if a continuous time signal is sampled at a rate that is at least twice as high (or higher) as the highest frequency in the signal, then it can be uniquely reconstructed from its samples
- Aliasing can be avoided if a signal is sampled at or above the Nyquist rate.

Ideal Sampling



Ideal sampling signal: impulse train (an analog signal)

$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT),\quad T\text{: sampling period}$$

Analog (continuous-time) signal: $x_c(t)$

Sampled (continuous-time) signal: $x_s(t)$

$$x_s(t)=x_c(t)s(t)=x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$=\sum_{n=-\infty}^{\infty}x_c(t)\delta(t-nT)=\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$$

- $x_s(t)$ is, formally, a continuous-time signal, since it is defined for all t.
  - It is clear that the information it conveys about x(t) is limited to the values x(nT), since $x_s(t)$ is identically zero at all other points.
- Two different ways to look at the sampled signal:
  - To consider it as a sequence of numbers x[n]=x(nT), or as a discrete-time signal.
    - ✓x[n] is referred to as **point sampling** of x(t).
  - To consider it as a continuous-time signal $x_s(t)$.
    - ✓$x_s(t)$ is referred to **as impulse sampling** of x(t).

Point sampling of x(t)

Impulse sampling of x(t)

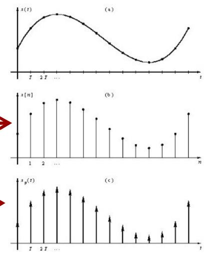


Figure 3.1 The sampling operation: (a) the continuous-time signal $x(t)$; (b) the point-sampled sequence $x[n]$; (c) the impulse-sampled signal $x_p(t)$.

Impulse sampling is convenient for mathematical derivations, since known results from continuous-time signal analysis can be used.

- Fourier Transform of Impulse Sampled Signal

In the text book, X(jΩ)≡X(ω)

$$X_p(\omega)=\int_{-\infty}^{\infty}x_p(t)e^{-j\omega t}dt$$

  - 1st Relationship

$$X_p(\omega)=\int_{-\infty}^{\infty}x_p(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)\right]e^{-j\omega t}dt$$

$$=\sum_{n=-\infty}^{\infty}x(nT)\int_{-\infty}^{\infty}\delta(t-nT)e^{-j\omega t}dt=\sum_{n=-\infty}^{\infty}x(nT)e^{-jn\omega T}$$

  - 2nd Relationship

$$X_p(\omega)=\frac{1}{2\pi}P_T(\omega)*X^F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^F(\lambda)\left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\lambda-\frac{2\pi k}{T}\right)\right]d\lambda$$

$$=\frac{1}{T}\sum_{k=-\infty}^{\infty}X^F\left(\omega-\frac{2\pi k}{T}\right)$$

  - From 1st and 2nd Relationships

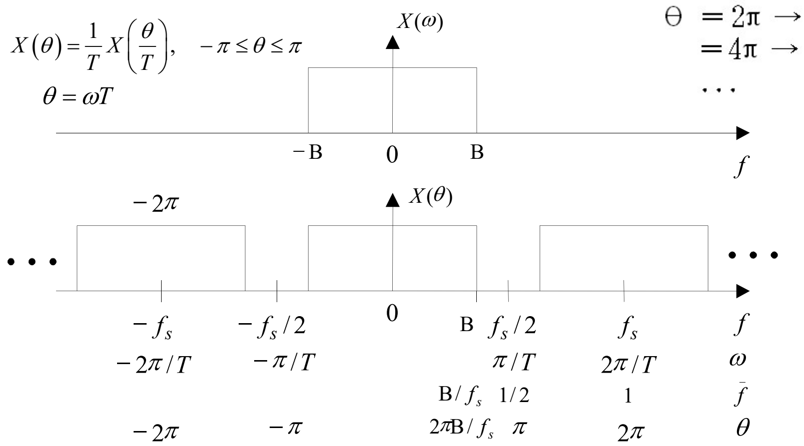
$$X_p(\omega)=\int_{-\infty}^{\infty}x_p(t)e^{-j\omega t}dt=\underline{\sum_{n=-\infty}^{\infty}x(nT)e^{-jn\omega T}=\frac{1}{T}\sum_{n=-\infty}^{\infty}X\left(\omega-\frac{2\pi n}{T}\right)}$$

- Analog Frequency vs. Digital Frequency
  - $\Theta = 2\pi \bar{f}$ where $\bar{f} = f/f_s$ **: normalized frequency**
  - $\Theta = n \cdot 2\pi$ means $2\pi f/f_s = n \cdot 2\pi \Rightarrow \dfrac{f}{f_s} = n$

$$X(\theta) = \frac{1}{T} X\left(\frac{\theta}{T}\right), \quad -\pi \le \theta \le \pi$$

$$\theta = \omega T$$

$\Theta = 2\pi \rightarrow f = f_s$
$= 4\pi \rightarrow f = 2f_s$
$\cdots$



DGIST 대구경북과학기술원 Daegu Gyeongbuk Institute of Science & Technology

- Condition under which the replicas (repeated spectra) do not overlap
- x(t) must be band limited, that is,

$$X(\omega) = 0 \text{ for } |\omega| \ge \omega_m$$

$$f_s \ge 2 \cdot f_m \quad or \quad \omega_m/\pi$$

  - Bandwidth of x(t) : ω_m or f_m
  - Nyquist rate (= twice the highest frequency) : $2f_m$
  - Aliasing: The phenomenon that happens when

$$X(\Theta) = X(\omega T) = \frac{1}{T} X(\omega) \text{ not hold.} \qquad \frac{1}{T}X(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{T}\right)\Bigg|_{n=0}$$

The replicas overlap and $f_s \ge 2 \cdot f_m$ not hold either.

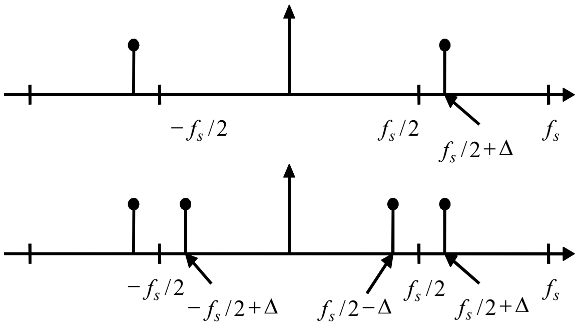
We can never reconstruct $X(\omega)$ or $x(t)$ from $X(\Theta)$ or $x(nT)$

DGIST 대구경북과학기술원 Daegu Gyeongbuk Institute of Science & Technology

- Sampling Theorem
  - A **band limited** CT signal, with highest frequency B Hz, can be uniquely recovered from its samples if the sampling rate satisfies

$$f_s \geq f_{Nyquist} = 2B$$

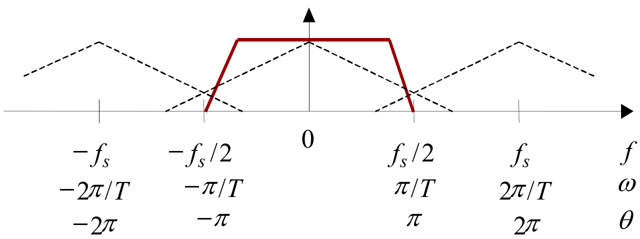
  - Frequency Aliasing



- Common sampling rule
  - Practical sampling rate to achieve a necessary bandgap.

$$f_s \geq 2 \cdot f_m + \delta \quad where\ \delta \geq 0$$
$$\delta \geq 0.1 \cdot (2 \cdot f_m)$$

  - **Anti-aliasing filter is commonly used in front of ADC.**



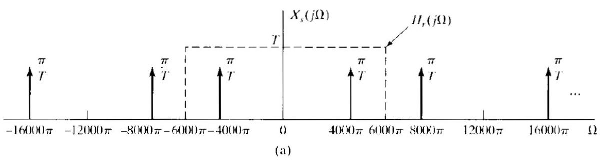
Example

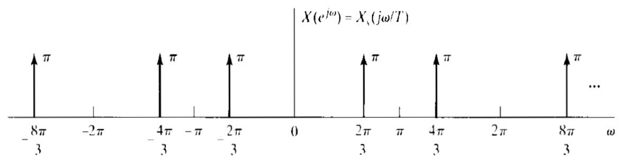
$x_c(t) = \cos(4000\pi t)$ **Sampling period:** $T = 1/6000$

We obtain $x[n] = x_c(nT) = \cos(4000\pi Tn) = \cos(\omega_0 n)$. $\omega_0 = 4000\pi T = 2\pi/3$.

$\Omega_s = 2\pi/T = 12000\pi$ **The highest frequency of the signal:** $\Omega_0 = 4000\pi$

The Fourier transform of $x_c(t)$ ⇒ $X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$.





$$X_s(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Omega_s = 12{,}000\pi$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \quad \omega = \Omega T$$

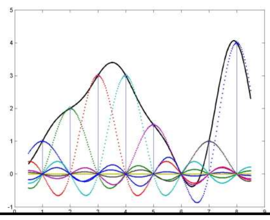
$\Omega_0 = 4000\pi$ ⇒ $\omega_0 = 4000\pi T = 2\pi/3$.

Shannon's Reconstruction Theorem

- A band-limited signal x(t) whose bandwidth is smaller than $\pm\pi/T$ can be exactly reconstructed from its samples $\{x(nT), -\infty < n < \infty\}$, using the formula

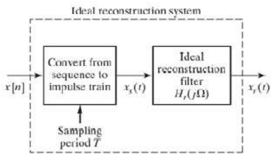
$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

  - This equation has a great theoretical importance, but a limited practical utility, because it requires to interpolate an infinite number of samples, which means that the observation interval must be of infinite length.

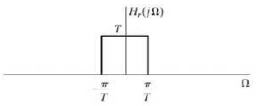


Ideal Reconstruction

Ideal lowpass (bandpass) filter



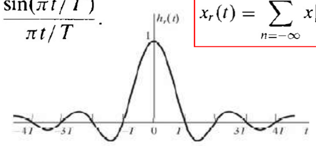
(a)



(b)

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.}$$



(c)

Ideal low-pass reconstruction filter:

$$H_r(j\Omega) = \begin{cases} T & -\pi/T < \Omega \le \pi/T \\ 0 & otherwise \end{cases} \qquad h_r(t) = \frac{\sin(\pi t/T)}{(\pi t/T)}$$

$$x_c(t) \rightarrow sampling \rightarrow x_s(t) = \sum x(nT)\delta(t-nT) \rightarrow seq.-convr. \rightarrow x[n]$$

$$x[n] \rightarrow impulse-convr. \rightarrow x_s(t) = \sum x[n]\delta(t-nT) \rightarrow recon. \rightarrow x_r(t)$$

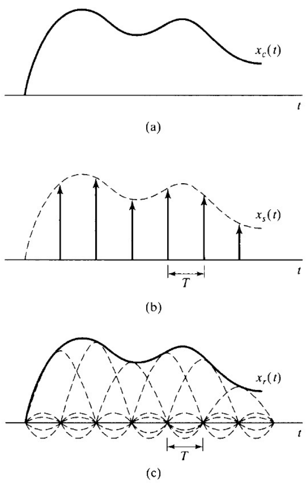
$$x_r(t) = x_s(t) * h_r(t) = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} x[n]\delta(\lambda - nT)h_r(t-\lambda) \right\} d\lambda$$

$$= \sum_{n=-\infty}^{\infty} \left\{ x[n]\int_{-\infty}^{\infty} \delta(\lambda - nT)h_r(t-\lambda)d\lambda \right\} = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

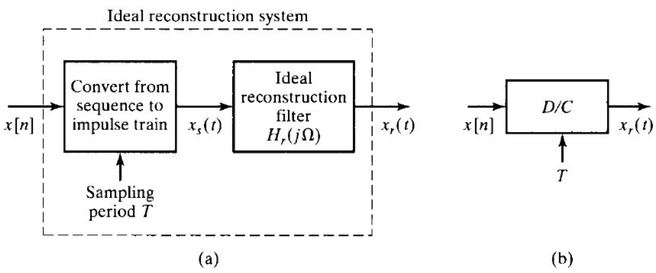
$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]H_r(j\Omega)e^{-j\Omega Tn} = H_r(j\Omega)\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}\}$$

$$= H_r(j\Omega)X(e^{jw})\Big|_{w=\Omega T} = H_r(j\Omega)X(e^{j\Omega T}) = H_r(j\Omega)X(j\Omega)$$

Note that x_s (t) is an analog signal (impulses).



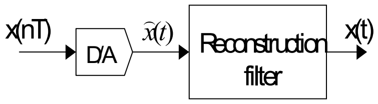
DAC converter



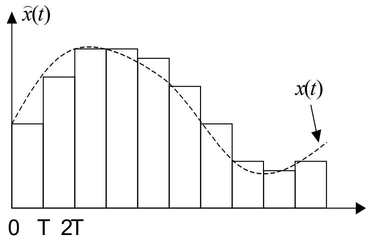
$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.}$$

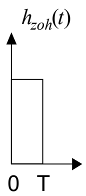
Noncausal (ideal)

- **Practical approach**: With a real DAC
  - Note that $x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$ can't exist in reality.
  - Zero-order Hold (ZOH) using DAC (Digital-to-Analog Converter)



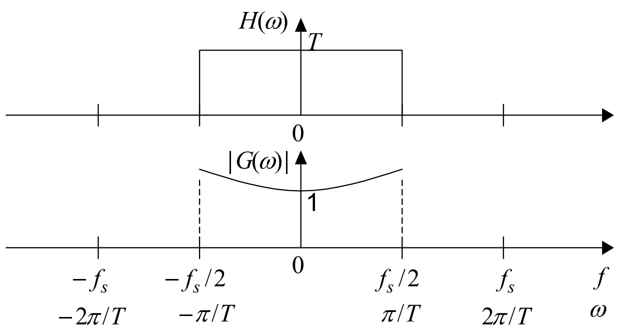
$$h_{opt}(t) = rect\left(\frac{t-T/2}{T}\right) \longleftrightarrow H_{zoh}(\omega) = T \cdot \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j0.5\omega T}$$





Note: Can you exactly describe the frequency spectrum of the DAC output?

- Practical approach: With a real DAC
  - Amplitude equalizer



Changing Sampling Rate

- Discrete-time signal $x[n]$ is obtained by extracting the information from a continuous-time signal $x_c(t)$ every T seconds:

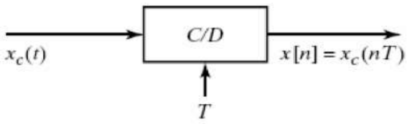


Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.

- Often necessary to change the sampling rate of a discrete-time signal, such that

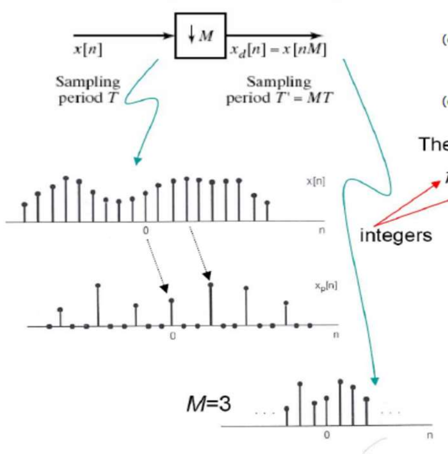
$$x'[n] = x_c(nT') \qquad \text{where} \quad T \neq T'$$

- Approach 1:
  - Reconstruct $x_c(t)$ from $x[n]$, then resample $x_c(t)$ with period T' to obtain $x'[n]$
    - Problems in practice due to non-ideal analog reconstruction filters, ADC and DAC

- Approach 2:
  - Changing the sampling rate using only discrete-time operations

Sampling Rate Reduction (Decimation)

Figure 4.20 Representation of a compressor or discrete-time sampler.



Time − domain	DTFT
$x[n]$ (original sequence)	$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$
$x[nM]$ (down-sampled sequence)	$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right)$

The summation index can be expressed as

$$r = i + kM, \qquad where \quad i = 0,1,2,\ldots,M-1$$

integers

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{k=-\infty}^{\infty}\sum_{i=0}^{M-1} X_c\left(j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right)$$

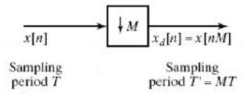
$$= \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right)\right]$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega - 2\pi i)/M}\right) \qquad \text{Scaling and shifting}$$

M copies of the periodic DTFT $X(e^{j\omega})$, frequency-scaled by M and shifted by integer multiples of 2π.

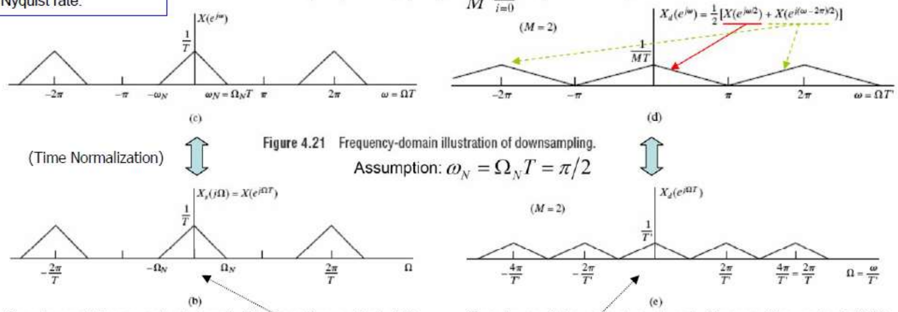
Graphical Interpretation

- Graphical interpretation: with $M = 2$



Note: Sampling here is 2x the Nyquist rate.

$$X(e^{j\omega}) \rightarrow X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega - 2\pi i)/M}\right)$$



(Time Normalization)

Figure 4.21 Frequency-domain illustration of downsampling.

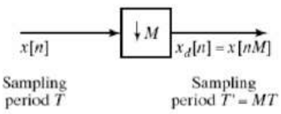
Assumption: $\omega_N = \Omega_N T = \pi/2$

Spectrum of the sampled signal with sampling period of T

Spectrum of the sampled signal with sampling period of $2T$

Exact reconstruction possible in both cases!

- Graphical interpretation: with $M = 3$



$$X(e^{j\omega}) \rightarrow X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega - 2\pi i)/M}\right)$$

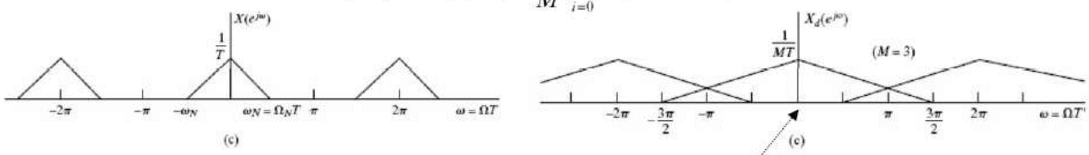


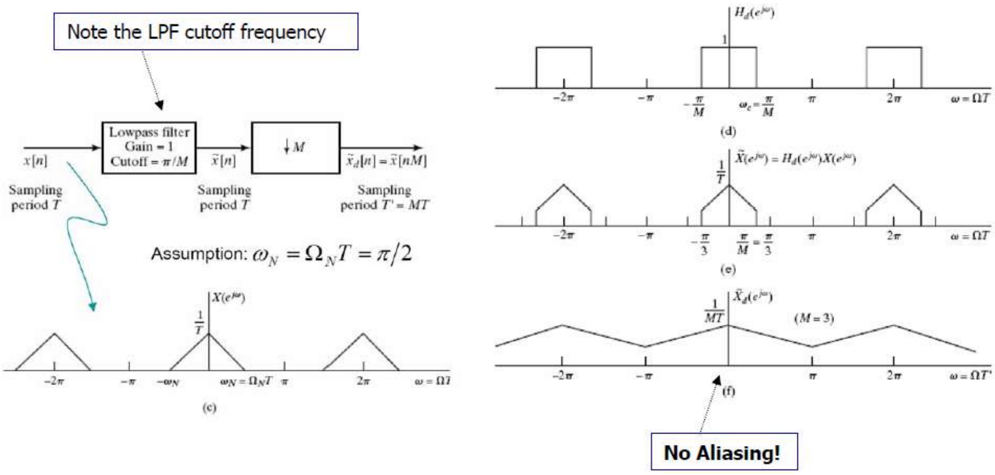
Figure 4.22 (a)–(c) Downsampling with aliasing.

Assumption: $\omega_N = \Omega_N T = \pi/2$

Exact reconstruction not possible due to aliasing!

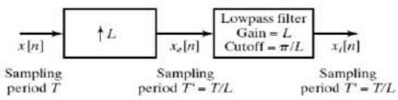
How to avoid the aliasing?

- Aliasing due to downsampling can be avoided with prefiltering (ie. reduce the BW before downsampling)



Increasing Sampling Rate (Interpolation)

Figure 4.24 General system for sampling rate increase by L.



- The expression of expander $x_e[n]$:

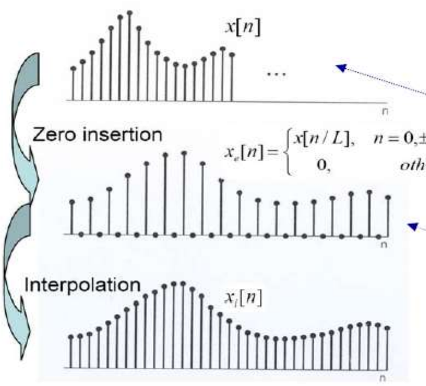
$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & otherwise \end{cases} = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

- This gives

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X\left(e^{j\omega L}\right)$$ Frequency Scaling

L=2



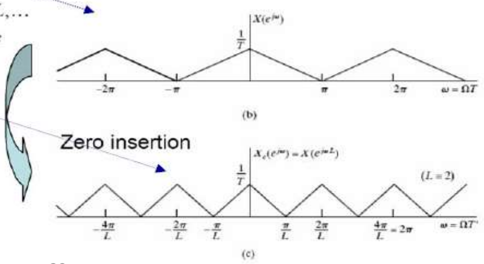
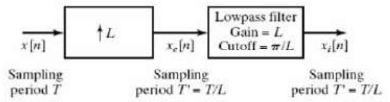
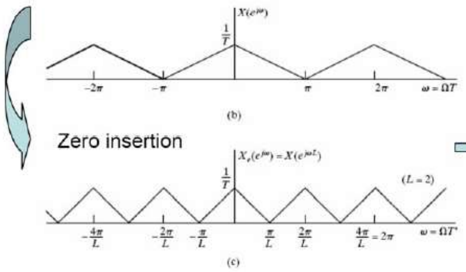


Figure 4.24 General system for sampling rate increase by L.



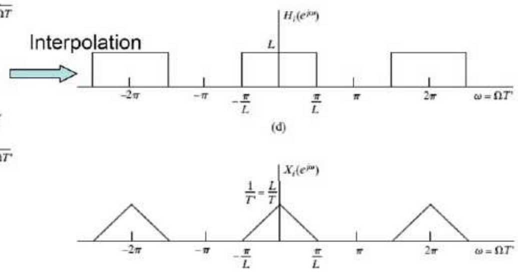


•The output signal $x_i[n]$ can be obtained by:

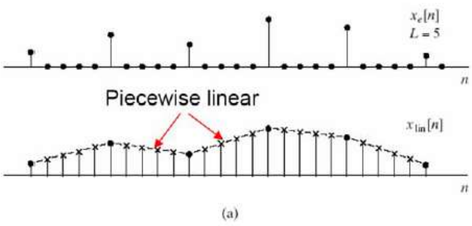
$$X_i\left(e^{j\omega}\right)=H_i\left(e^{j\omega}\right)X\left(e^{j\omega L}\right)$$

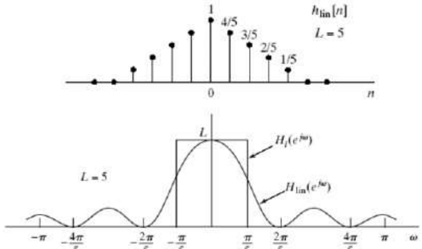
$$x_i[n]=x_e[n]*h_i[n]=\sum_{k=-\infty}^{\infty}x[k]h_i[n-kL]$$

$$=\sum_{k=-\infty}^{\infty}x[k]\frac{\sin[\pi(t-kL)/L]}{[\pi(t-kL)/L]}$$



- But, in practice, the ideal LPF cannot be implemented.
- Linear interpolation is a simple and often used approximate technique (although it is generally not very accurate).



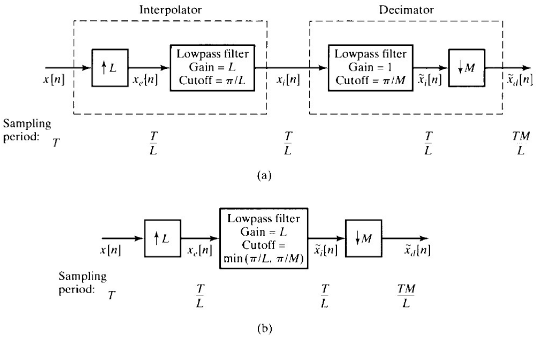


$$h_{lin}[n]=\begin{cases}1-|n|/L, & |n|<L\\ 0 & otherwise\end{cases}$$

$$x_{lin}[n]=x_e[n]*h_{lin}[n]=\sum_{k=-\infty}^{\infty}x[k]h_{lin}[n-kL]$$

Examples

Each of the following parts lists an input signal $x[n]$ and the upsampling and downsampling rates L and M for the system in Figure 4.28. Determine the corresponding output $\tilde{x}_d[n]$.



a) $x[n] = \sin(2\pi n/3)/\pi n, \quad L = 4, \quad M = 3$

In the frequency domain,

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < 2\pi/3 \\ 0, & 2\pi/3 < |\omega| < \pi \end{cases}$$

After the sampling rate change,

$$\tilde{X}_d(e^{j\omega}) = \begin{cases} 4/3, & |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| < \pi \end{cases}$$

which leads to

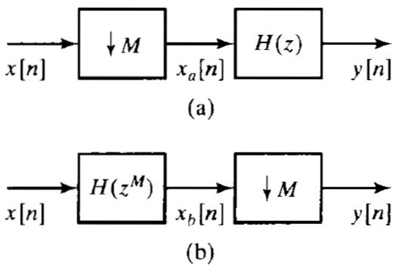
$$x[n] = \frac{4}{3}\frac{\sin(\pi n/2)}{\pi n}$$

b) $x[n] = \sin(3\pi n/4), \quad L = 3, \quad M = 5$

Ans) Upsampling by 3 and low-pass filtering x[n]=sin(3πn/4) results in sin(πn/4). Downsampling by 5 gives us $\tilde{x}_d[n]$=sin(5πn/4)

Multirate Signal Processing

- **Interchange of Filtering and Downsampling**



$$X_b(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega}),$$

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X_b(e^{j(\omega/M - 2\pi i/M)}).$$

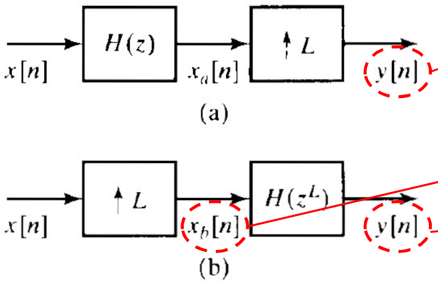
$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)})H(e^{j(\omega - 2\pi i)}).$$

$$H(e^{j(\omega-2\pi i)}) = H(e^{j\omega}).$$

$$Y(e^{j\omega}) = H(e^{j\omega})\frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)})$$
$$= H(e^{j\omega})X_a(e^{j\omega}).$$

Multirate Signal Processing

- **Interchange of Filtering and Upsampling**



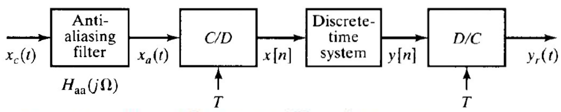
$$Y(e^{j\omega}) = X_a(e^{j\omega L}) = X(e^{j\omega L})H(e^{j\omega L}).$$

$$X_b(e^{j\omega}) = X(e^{j\omega L}),$$

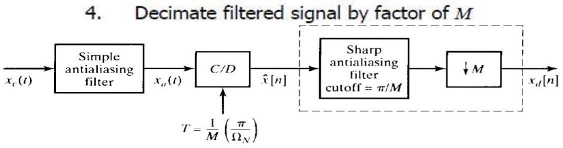
$$Y(e^{j\omega}) = H(e^{j\omega L})X_b(e^{j\omega}),$$

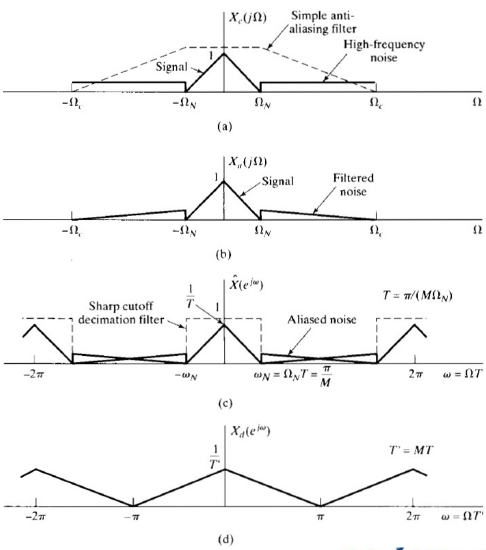
ADC – Anti-Aliasing Filter Implementation

- Pre-filtering required before ADC to avoid aliasing



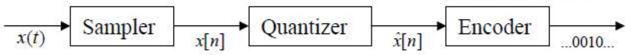
- Requires sharp-cutoff analog anti-aliasing filters. Such filters:
  - Are difficult to implement
  - Are expensive
  - Typically have non-linear phase response
- Alternative approach:
  1. Use simple analog anti-aliasing filter
  2. Sample at higher rate than $2\Omega_N$, eg at $2M\Omega_N$
  3. Implement sharp cutoff DT filter
  4. Decimate filtered signal by factor of M

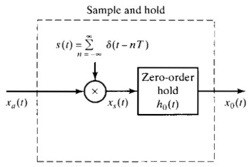




Quantization

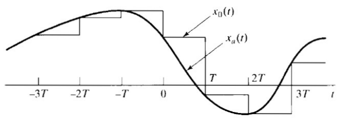
- So far, we have mainly talked about discrete-time systems
  - the sampled signals are represented by real numbers
- In practice, we represent these samples as digital numbers
  - sequences of 0's and 1's
- Discrete-time to digital mapping performed via quantization and encoding





(a)

- Quantizer is a non-linear system whose purpose is to transform the input sample $x[n]$ into one of a finite set of prescribed values
- Many-to-one mapping
- Linear quantization – Fixed step size Δ
- Non-linear quantization – Variable step size
- Encoder converts discrete voltages at quantizer output into binary representations



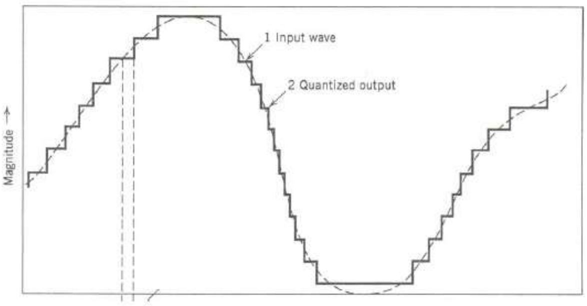
(b)

$$x_0(t) = h_0(t) * \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT),$$

$$h_0(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise}. \end{cases}$$

Uniform Quantization

- Uniform Quantization
  - Quantization levels are equally spaced.
  - Called uniform quantization or PCM (pulse code modulation)



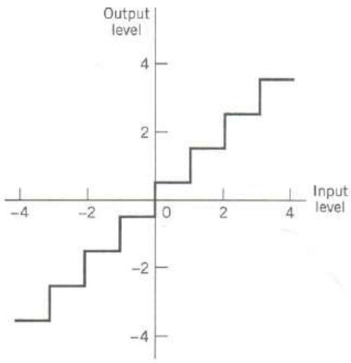


Illustration of linear quantization

Non-uniform Quantization

- Non-uniform Quantization
  - Companding: closer quantization levels for smaller signal value

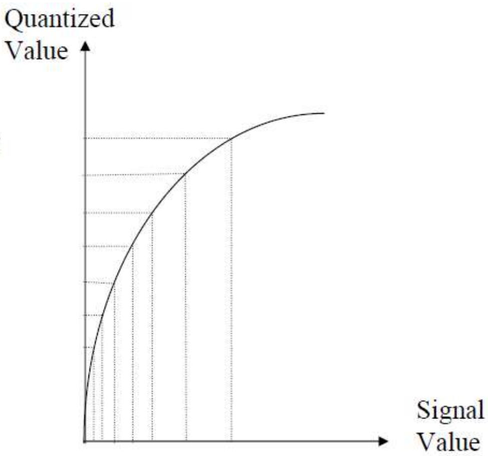
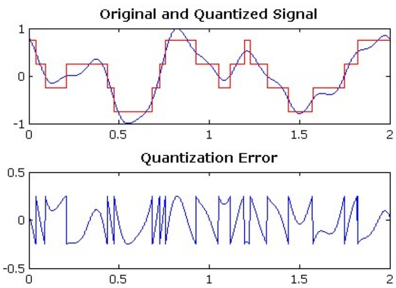
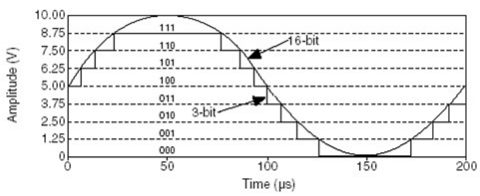


Illustration of non-linear quantization

Quantization Error (revisitation)

- The number of bits used for conversion sets a lower limit on the resolution, and also determines the quantization error that can be thought of as a noise process added to the signal.
- 8-Bit vs. 12-Bit ADC
  - 12-Bit ADC has less quantization error than 8-Bit ADC





Quantization Considerations

- Sample size is the number of bits per sample. Selection of sample size determines:
  - Number of quantization levels
    - For sample size B, there are $L=2^B$ quantization levels
  - Signal-to-quantization-noise-ratio (SQNR)
- The more bits:
  - the more accurate the sample
  - the higher the SQNR
  - the more storage space required
- Typically examples
  - 8 bits for speech, image and video, 16 bits for audio signals
- Let $x_{\max}$ and $-x_{\max}$ denote the max and min quantization levels respectively. The quantization step size is then

$$\Delta = \frac{2x_{\max}}{L} = \frac{x_{\max}}{2^{B-1}}$$

Signal-to-Quantization-Noise Ratio (SQNR)

- Quantization error is the difference between the actual analog value at a sample time and the selected quantization level
- Limited to the range between+ Δ /2 and - Δ /2
- The SQNR is an important measure of the distortion induced by the quantization process
- When the input signal is sine wave, it is assumed that the quantization error uniformly distributed when Δ is small, the SQNR is approximated as

$$\boxed{SQNR \approx 6.02\,B + 1.76\ (dB)}$$

B: sample size , typically 8 or 16 bits

- Larger SQNR, better quality
- 16-bit quantization : SQNR = 98.1 dB
- 15-bit quantization: SQNR= 92.1 dB.
- Each additional bit reduces the quantization error by about 6 dB