EE403: Digital Communications

Lecture 2: Signal and Systems

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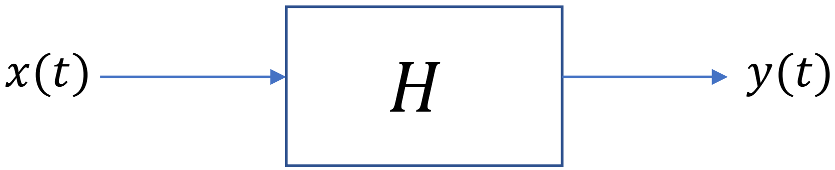
Signal and Linear Systems

- In system modeling, actual components such as resistors, capacitors, inductors, … are not of our concern
- We view a system in terms of input-output operation



$$y(t) = H\{x(t)\}$$

Signal and Linear Systems

- **Linear** systems: Superposition principle is met

$$y(t) = H\{ax_1(t) + bx_2(t)\} = H\{ax_1(t)\} + H\{bx_2(t)\} = ay_1(t) + by_2(t)$$

- **Time-invariant**: Delayed input produces an output with the same delay

$$H\{x(t)\} = y(t) \quad \Rightarrow \quad H\{x(t - t_0)\} = y(t - t_0)$$

- Our focus is only on **linear time-invariance (LTI)** systems
  - Otherwise, analysis is challenging

Convolution

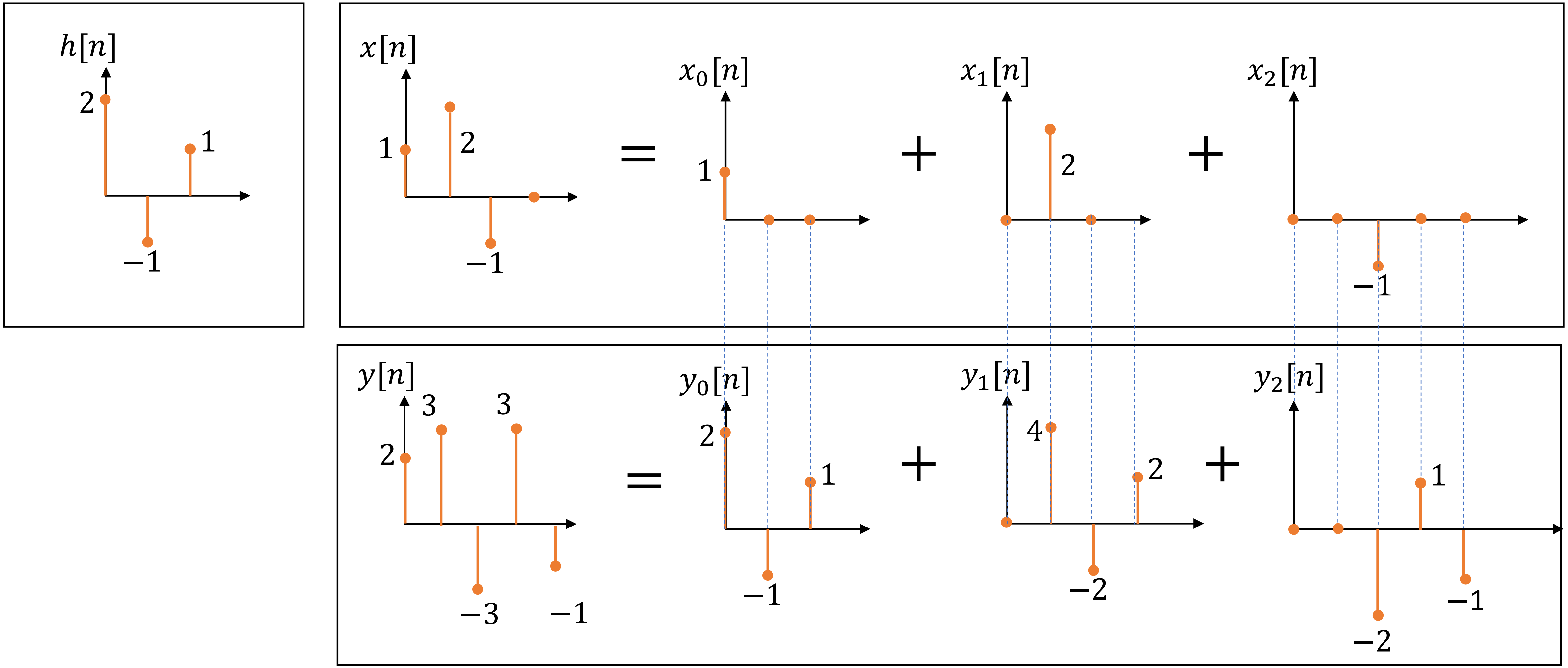
- Definition (continuous-time)

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$

  – Convolution is used to obtain the output of an LTI system for a given input

- Definition (discrete-time)

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]$$



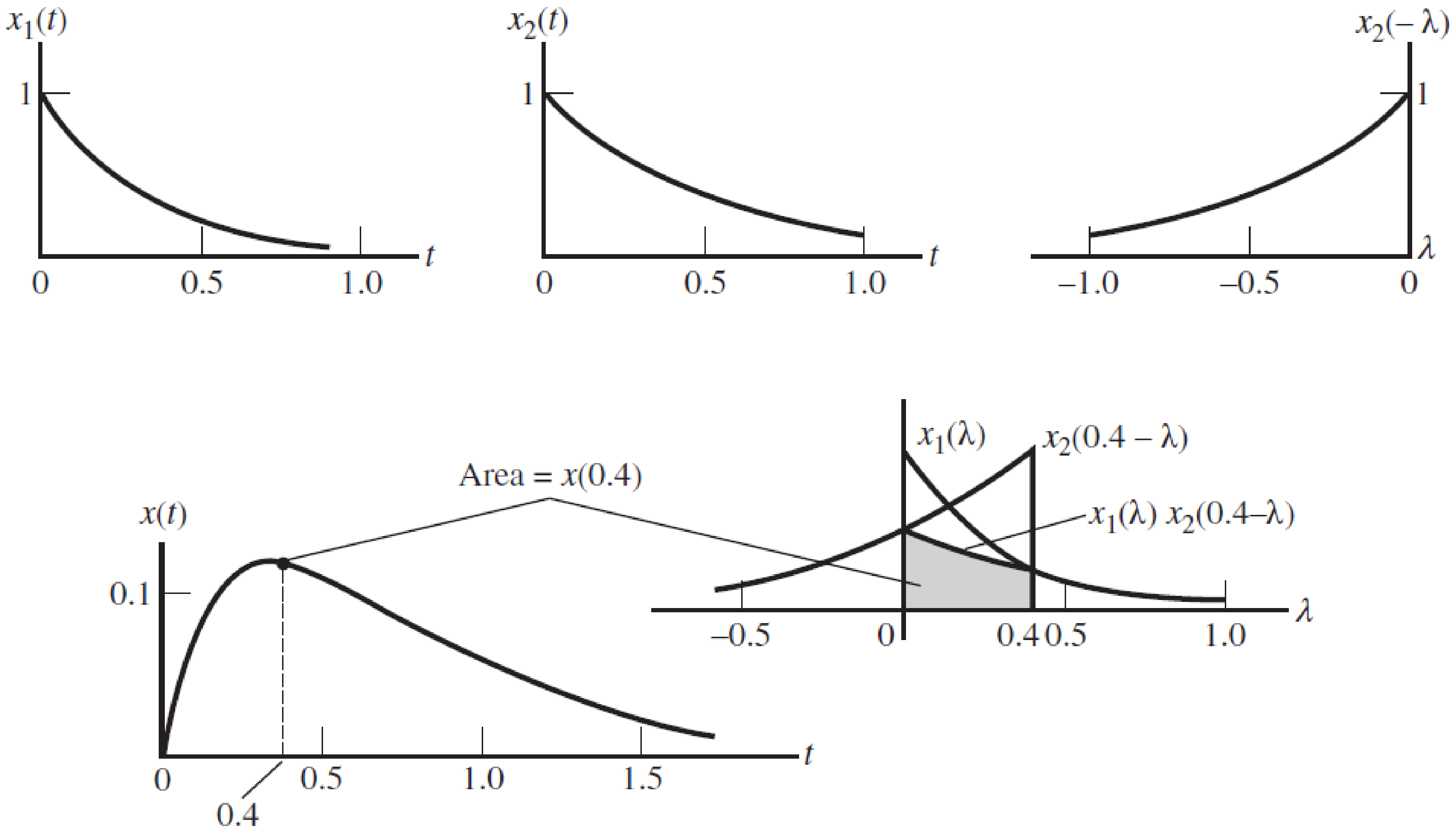
$$y[n] = x_0 h[n-0] + x_1 h[n-1] + x_2 h[n-2] = x[0]h[n-0] + x[1]h[n-1] + x[2]h[n-2]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \implies x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

Convolution

- Example

$$x_1(t) = e^{-\alpha t}u(t), x_2(t) = e^{-\beta t}u(t)$$



Impulse Response

- Q: How to completely describe an LTI system? (=predictable)
- A: by the impulse response, $h(t) \triangleq H\{\delta(t)\}$
  - Response due to an impulse input applied at $t = 0$
- Then, for any arbitrary input $x(t)$,

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda \\
\Longrightarrow \; y(t) = H\{x(t)\} &= H\left\{\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda\right\} \\
&= \int_{-\infty}^{\infty} x(\lambda)H\left\{\delta(t-\lambda)\right\}d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = x(t) * h(t)
\end{aligned}
$$

Transfer Function

- For linear systems, output is the **convolution** of input and system response

- Transfer function (in frequency domain) or frequency response: $H(f)$

$$y(t) = x(t) * h(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(f) = X(f)H(f)$$

  - In general, it is easier to compute $X(f)H(f)$ than $x(t) * h(t)$

Eigenfunction and Eigenvalue

- Recall the eigenvalue and eigenvector in matrix algebra

- Similarly, if a waveform does not change its shape after passing through an LTI system, it is called eigenfunction

- $H\{g(t)\} = cg(t)$
  - c: eigenvalue
  - $g(t)$: eigenfunction

- Recall that any periodic input can be represented by a sum of complex exponentials (Fourier series)

Eigenfunction and Eigenvalue

- For an arbitrary LTI system, $g(t) = Ae^{st}$ is an eigenfunction, where s is any complex number
  - Special case: $s = j2\pi f$
- Proof:

$$y(t) = h(t) * g(t) = \int h(\lambda) A e^{s(t-\lambda)} d\lambda = \left[\int h(\lambda) e^{-s\lambda} d\lambda \right] A e^{st} =$$

$$= H(s) A e^{st} = c g(t)$$

Responses to Periodic Inputs

- Since any periodic input can be represented by a sum of complex exponentials, its output will be the sum of complex exponentials, too

- For a unit component $x(t) = Ae^{j2\pi f_0 t}$,

$$y(t) = \int h(\lambda) A e^{j2\pi f_0 (t-\lambda)} d\lambda = \int h(\lambda) e^{-j2\pi f_0 \lambda} d\lambda \cdot \; A e^{j2\pi f_0 t} = H(f_0) A e^{j2\pi f_0 t}$$

- That is, for general periodic $x(t) = \sum X_n e^{j2\pi n f_0 t}$,

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(n f_0) e^{j2\pi n f_0 t}$$

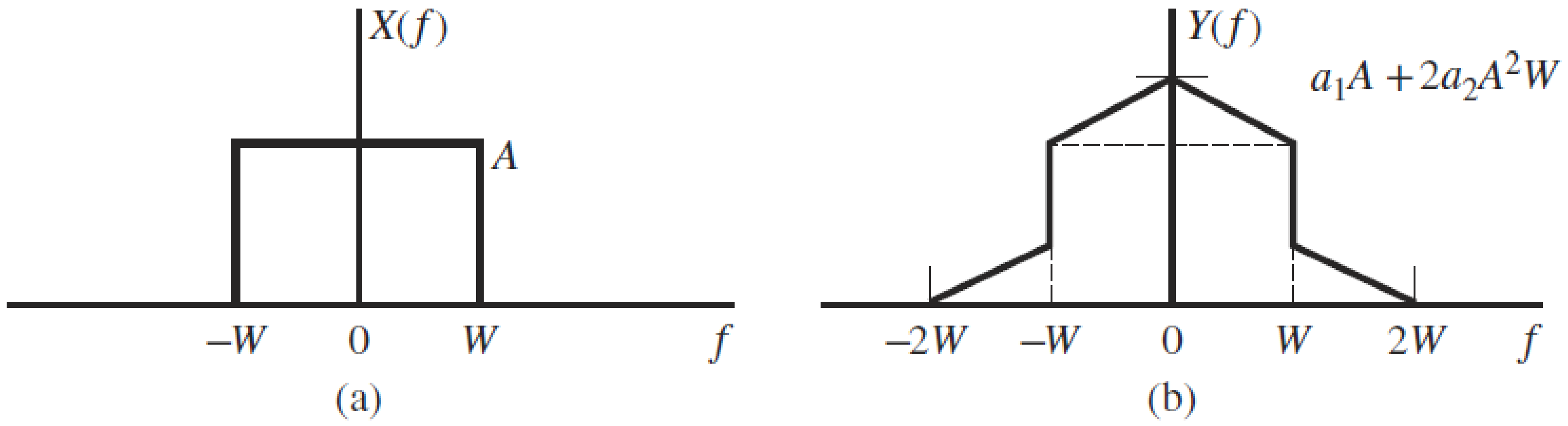
Nonlinear Distortion

- What if a system is not LTI?
  - The response has different frequency components from the input
- Example: $y(t) = a_1 x(t) + a_2 x^2(t)$ and $x(t) = A_1 \cos(w_1 t) + A_2 \cos(w_2 t)$

$$
\begin{aligned}
y(t) = & \underbrace{a_1 \left[A_1 \cos(w_1 t) + A_2 \cos(w_2 t)\right]}_{\text{desired}} \\
& + \underbrace{\frac{a_2}{2}(A_1^2 + A_2^2) + \frac{a_2}{2}\left[A_1^2 \cos(2w_1 t) + A_2^2 \cos(2w_2 t)\right]}_{\text{harmonic}} \\
& + \underbrace{a_2 A_1 A_2 \left[\cos((w_1 + w_2)t) + \cos((w_1 - w_2)t)\right]}_{\text{intermodulation}}
\end{aligned}
$$

Nonlinear Distortion

- As the system is second-order, the max frequency is doubled
- This is a general result for arbitrary input $x(t)$
  - Example: Consider $X(f) = \prod(f/2W)$

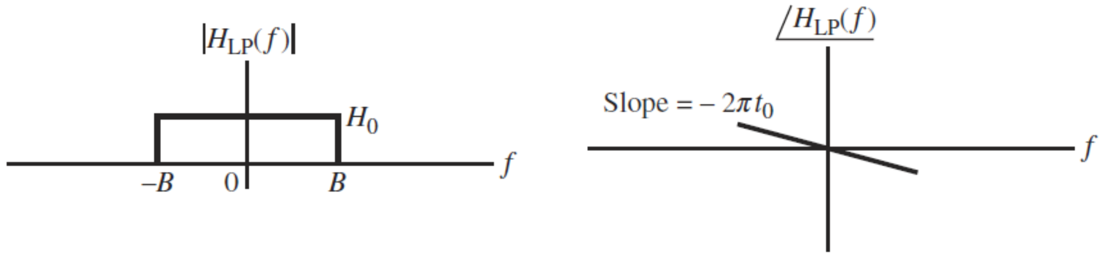


- If third-order, then tripled, and so forth...

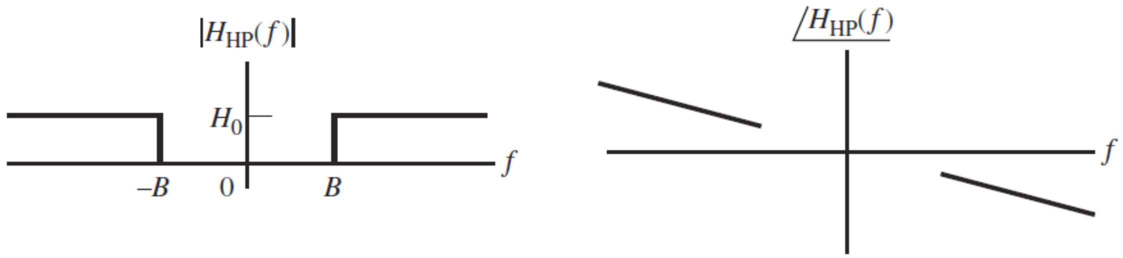
Ideal Filters

- Idealized filters must have **constant** amplitude and **linear** phase response

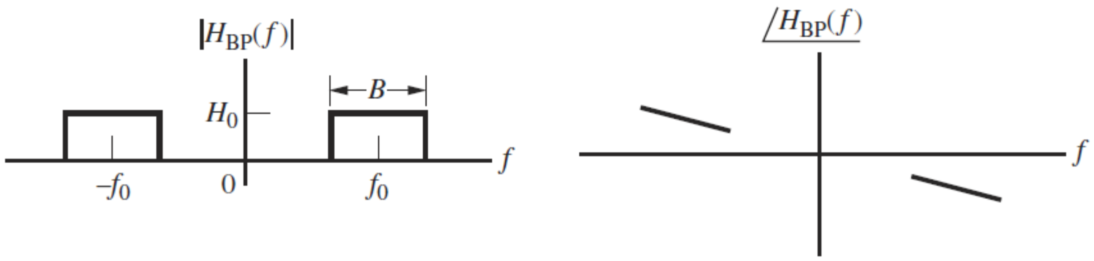
- Low-pass
  - $H_{LP}(f) = H_0 \prod\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$



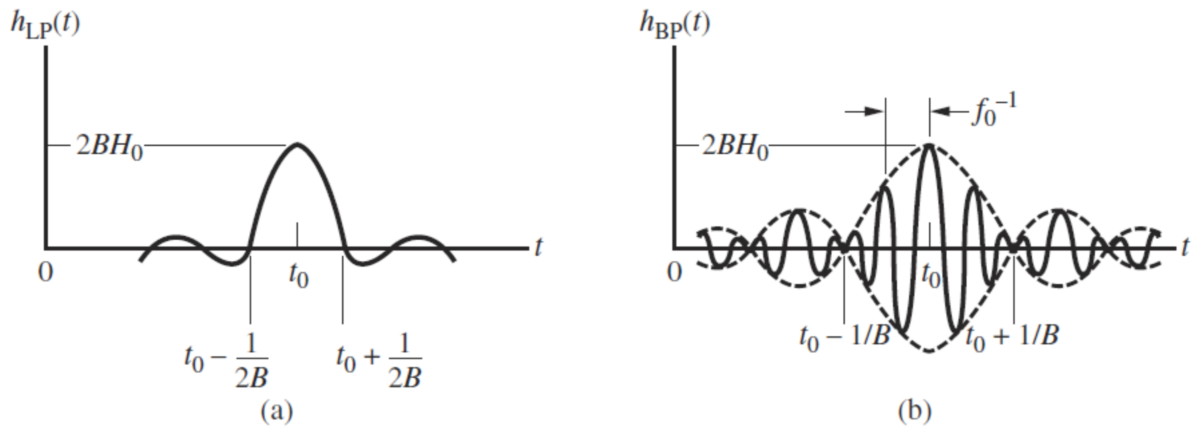
- High-pass
  - $H_{HP}(f) = H_0 \prod\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$



- Band-pass
  - $H_{BP}(f) = H_0 \left(\prod\left(\frac{f-f_0}{B}\right) + \prod\left(\frac{f+f_0}{B}\right)\right) e^{-j2\pi f t_0}$



Ideal Filters



- The ideal low-pass filter is

$$h_{LP}(t) = 2BH_0 \mathrm{sinc}\big(2B(t - t_0)\big)$$

- The ideal band-pass filter is

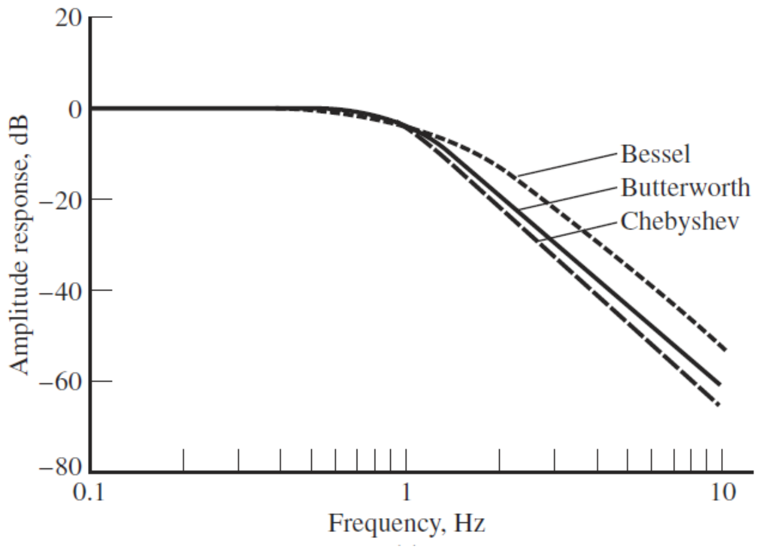
$$h_{BP}(t) = 2BH_0 \mathrm{sinc}\big(B(t - t_0)\big)\cos(2\pi f_0(t - t_0))$$

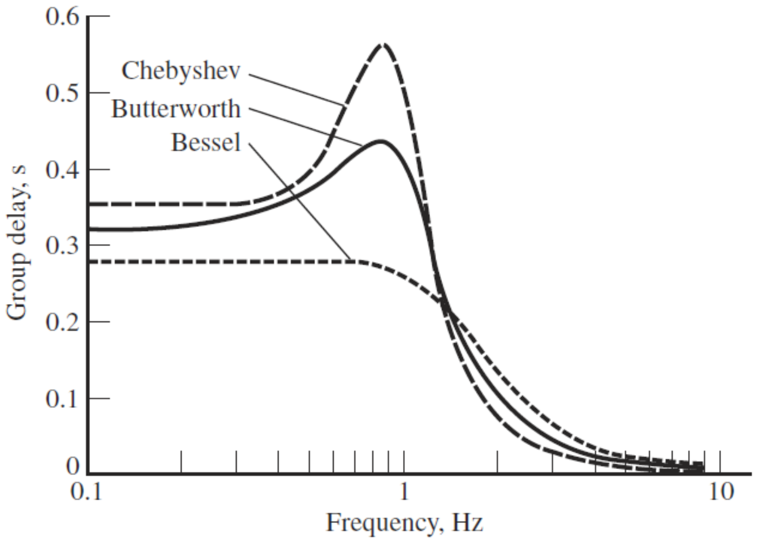
- ➔ infinite-length responses, so impossible to implement in practice
  - Called an infinite impulse response (**IIR**) filter
  - We need a finite impulse response (**FIR**) filter

Realizable Filters

- Several practical filters to approximate them

- Low-pass filters: Butterworth, Chebyshev, Bessel, …
- Band-pass filters: From LP filters, transform the passband



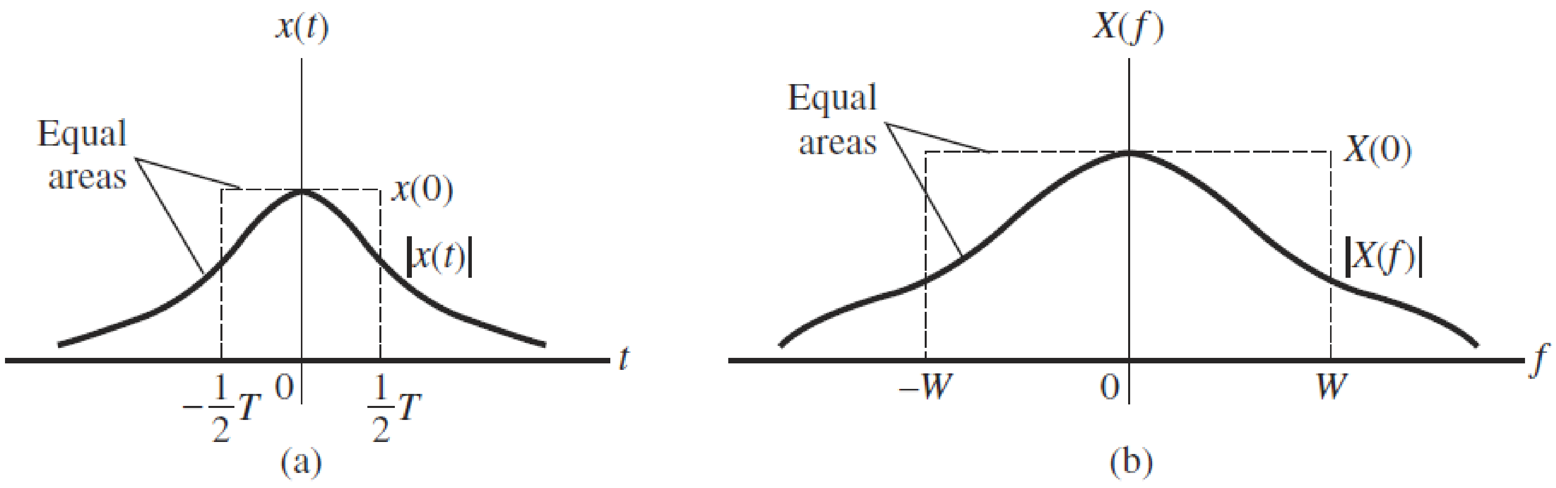


Time-bandwidth Product

- A **narrow** duration (T) signal in time must have **wider** bandwidth (W) in frequency, and vice versa

$$WT \geq \text{constant}$$

  - Like the uncertainty principle in physics



Time-bandwidth Product

- Let us model $x(t)$ by a rectangle pulse with equal area

$$Tx(0) = \int_{-\infty}^{\infty} |x(t)| dt \geq \int_{-\infty}^{\infty} x(t) dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi 0 t} dt = X(0)$$

- Let us model $X(f)$ by a rectangle response with equal area

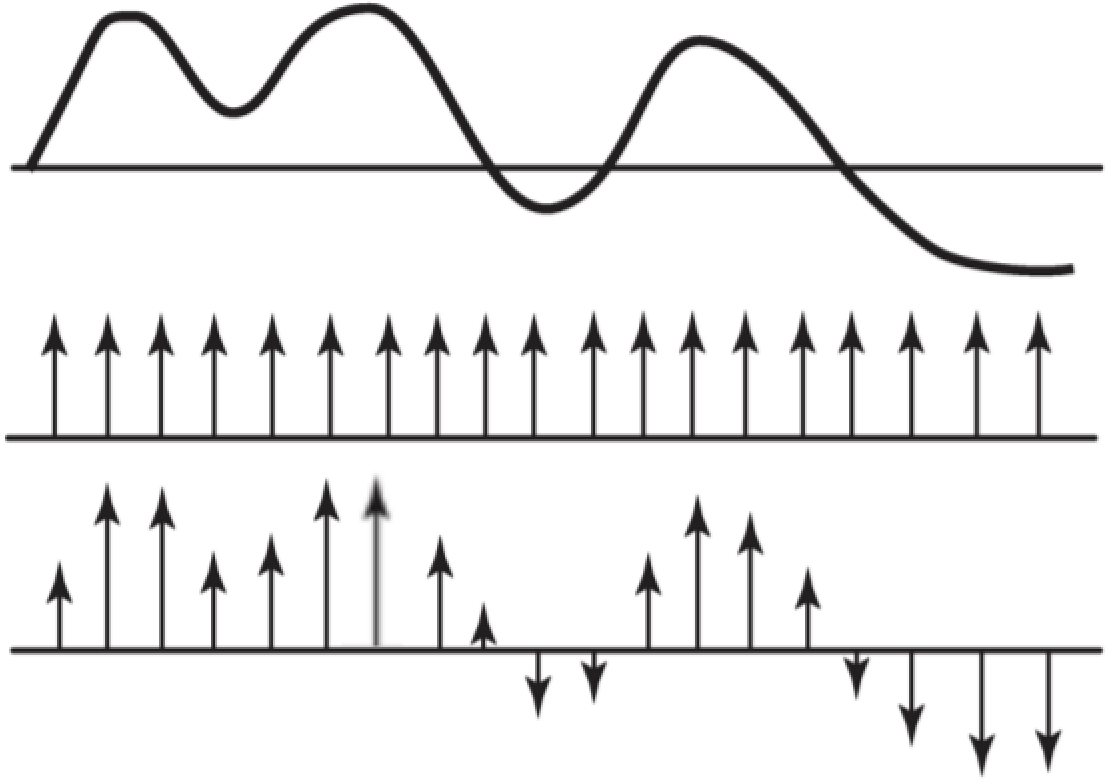
$$2WX(0) = \int_{-\infty}^{\infty} |X(f)| df \geq \int_{-\infty}^{\infty} X(f) df = \int_{-\infty}^{\infty} X(f) e^{-j2\pi f 0} dt = x(0)$$

- Collecting the pair of inequalities, $2W \geq \frac{x(0)}{X(0)} \geq \frac{1}{T}$ ➔ $WT \geq \frac{1}{2}$

- A **narrow** pulse width in time must have **wider** frequency spectrum

Sampling Theory

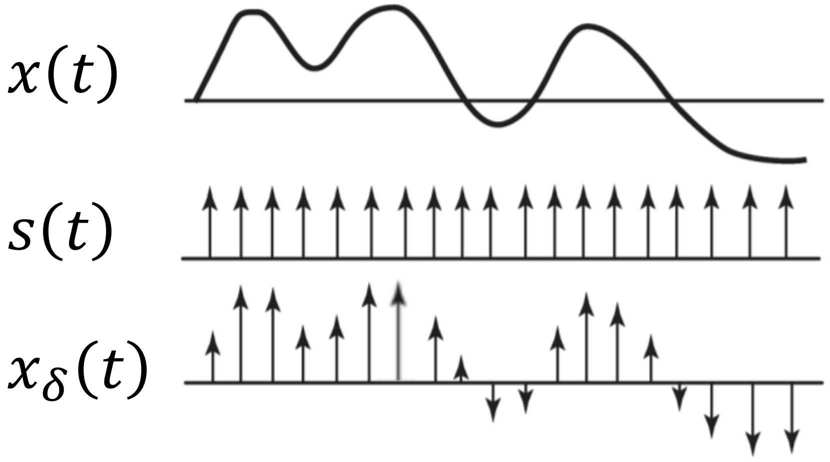
- Sampling is used in numerous applications, e.g., digital signal processing (DSP), computer simulation, . . .



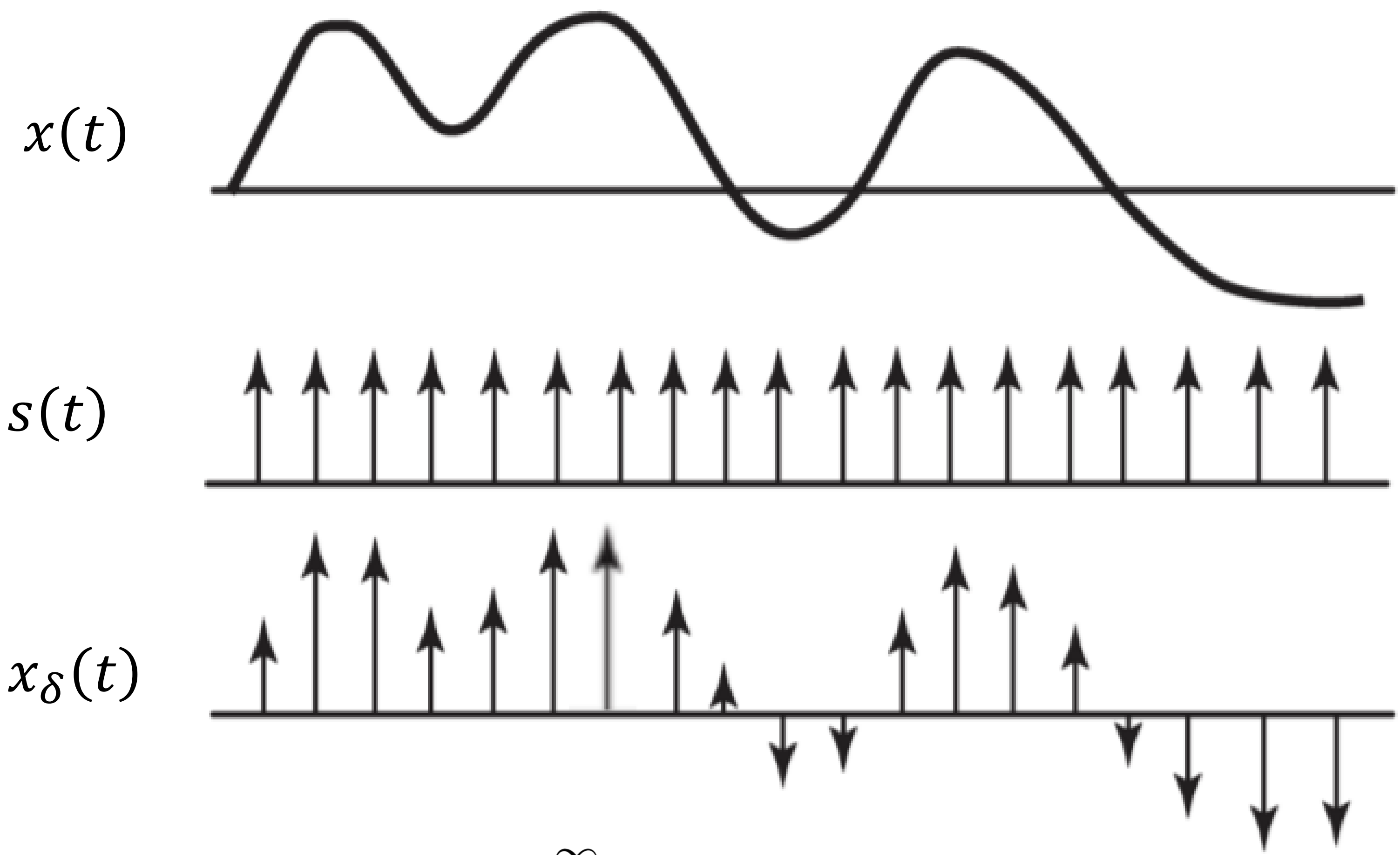
Sampling Theory

- Let the sampling period be T_s
- Ideal sampling uses impulse train $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$
- The sampled signal (in continuous-time) is

$$x_\delta(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$



Sampling Theory



$$x_\delta(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Sampling Theory

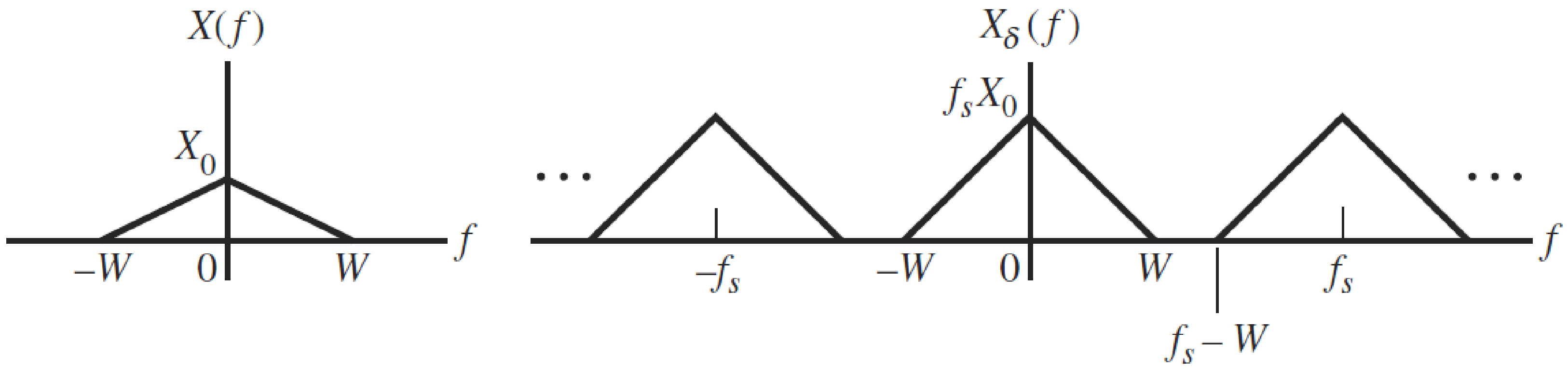
- In frequency domain,

$$X_\delta(f) = X(f) * S(f) = X(f) * \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right]$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

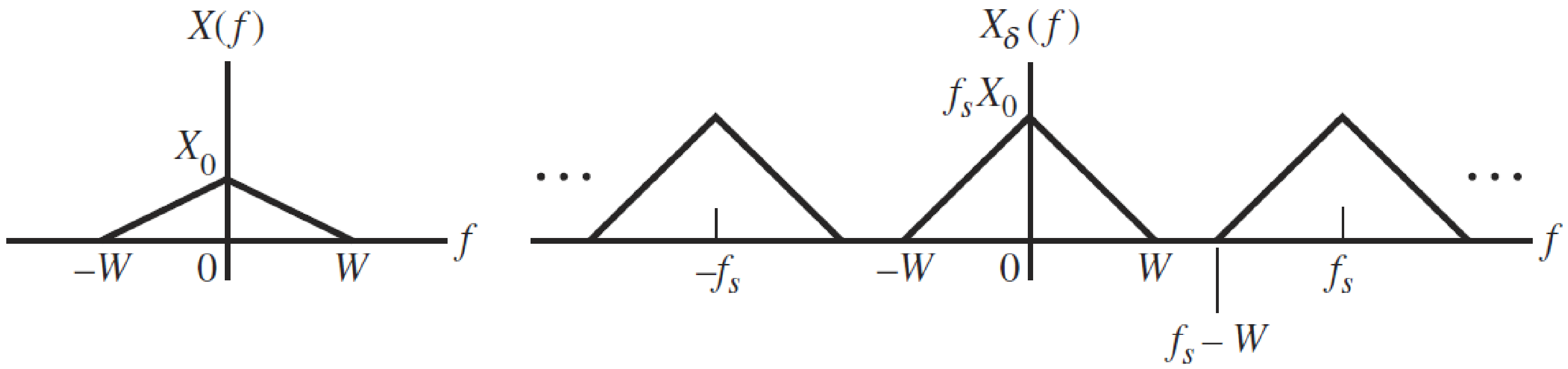
- Sampling $\iff$ duplication of shifted spectrum at every nf_s
- Perfect recovery $\iff$ only one copy of spectra after proper processing

Sampling Theory



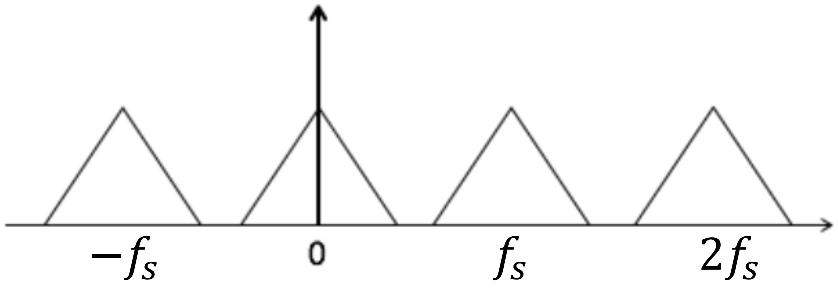
Aliasing

- Let the signal be bandwidth-limited with bandwidth W
- Aliasing: If $f_s < 2W$, then the replicas of $X(f)$ overlap in frequency domain

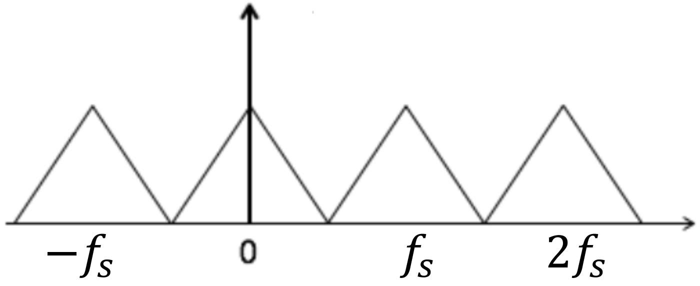


- (No aliasing in the above as $f_s > 2W$)

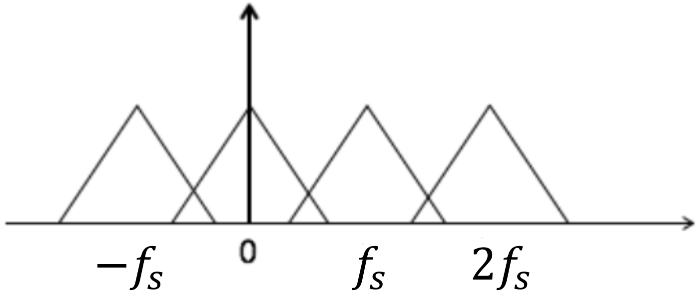
Aliasing



Oversampling: $f_s > 2W$



Perfect sampling: $f_s = 2W$

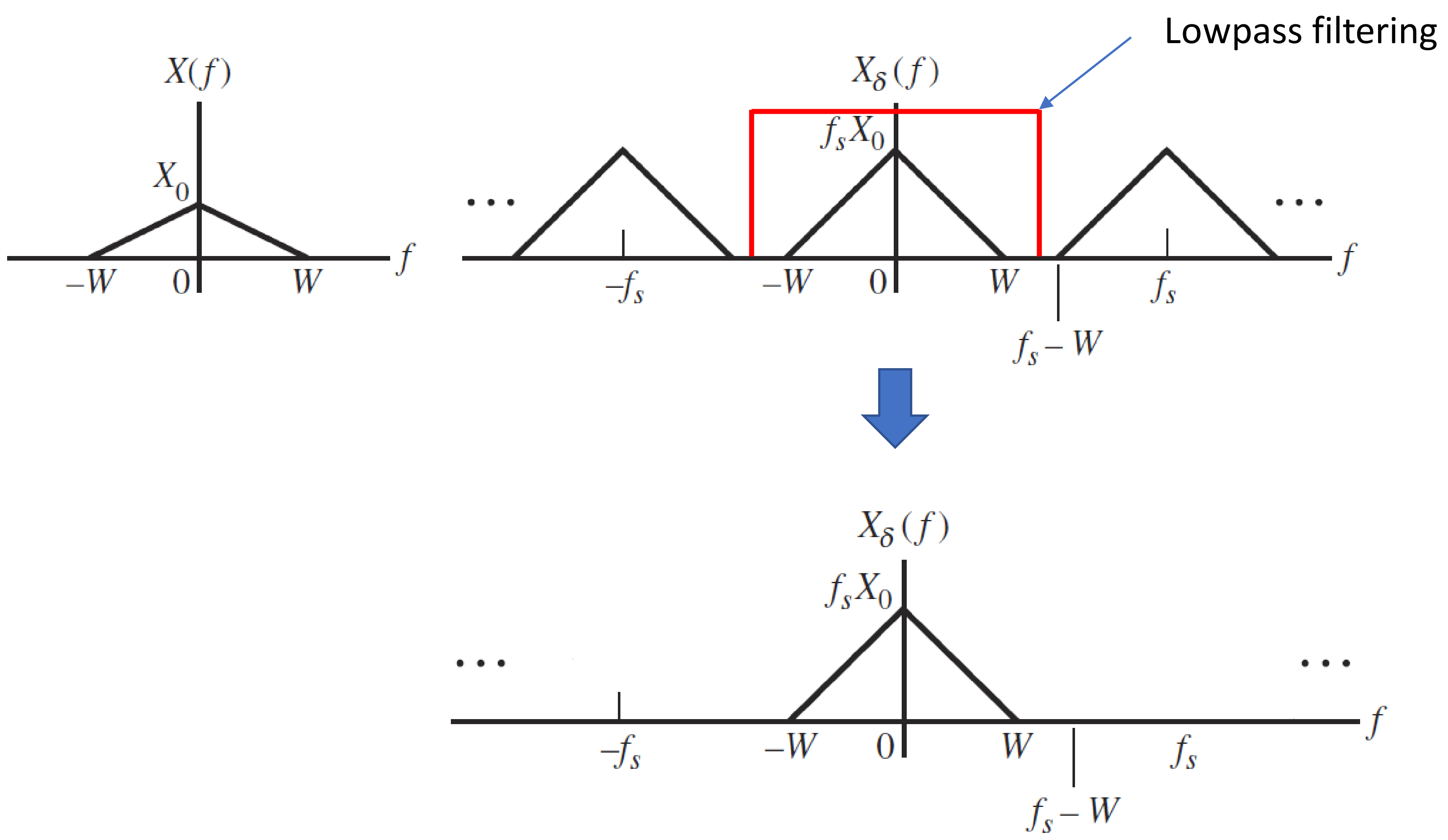


Undersampling: $f_s < 2W$

Nyquist Sampling Theorem

- Thm: Let $x(t)$ be a band-limited signal with bandwidth W. Then, $x(t)$ can be perfectly recovered from its samples $\{x(nT_s)\}_n$ if sampling rate $f_s = \frac{1}{T_s} \geq 2W$
- $2W$: Nyquist rate
- Undersampling if $f_s < 2W$
- Oversampling if $f_s > 2W$
- How to recover?
  - Pass $x_\delta(t)$ through an ideal lowpass filter with bandwidth B, where $W < B < f_s - W$

Nyquist Sampling Theorem



Nyquist Sampling Theorem

- Ideal reconstruction filter (lowpass filter)

$$H(f) = \prod\left(\frac{f}{2B}\right), \quad W \leq B \leq f_s - W$$

- Less ideal reconstruction filter (lowpass filter)

$$H(f) = H_0 \prod\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}, \quad W \leq B \leq f_s - W$$

- Passing $x_\delta(t)$ through this filter, $Y(f) = f_s H_0 X(f) e^{-j2\pi f t_0}$
- That is, in time-domain, $y(t) = f_s H_0 x(t - t_0)$

Nyquist Sampling Theorem

- Alternative expression: Let $h(t)$ be the impluse response of the ideal low-pass filter

$$y(t) = \underbrace{\left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right]}_{\text{sampled input}} * \underbrace{h(t)}_{\text{impulse response}} = \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s)$$

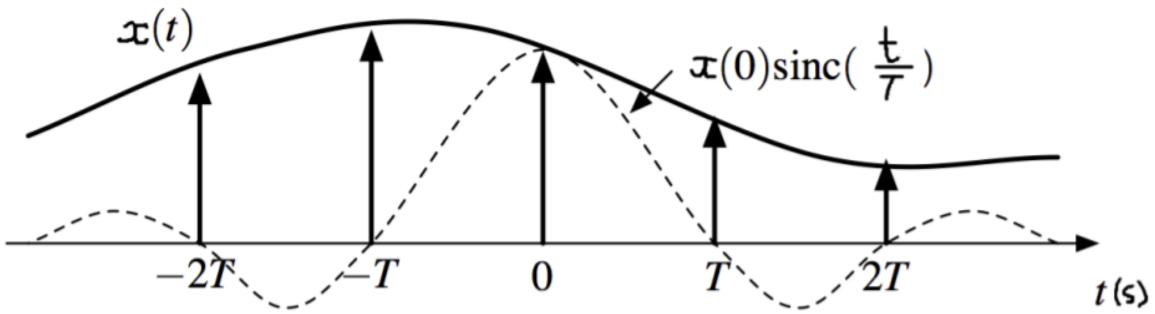
$$= \sum_{n=-\infty}^{\infty} x(nT_s)\left[2BH_0 \text{sinc}(2B(t - t_0 - nT_s))\right]$$

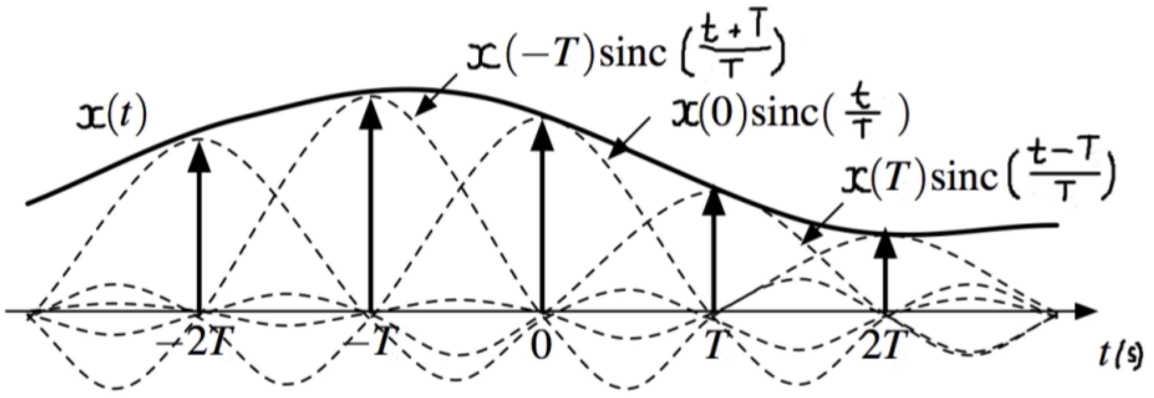
$$= 2BH_0 \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}(2B(t - t_0 - nT_s))$$

- A weighted sum of shifted sinc functions with weights $x(nT_s)$!

Nyquist Sampling Theorem

- Let $B = 0.5 f_s, H_0 = T_s, t_0 = 0$. Then, $y(t) = \sum_{-\infty}^{\infty} x(nT_s)\text{sinc}(f_s t - n)$

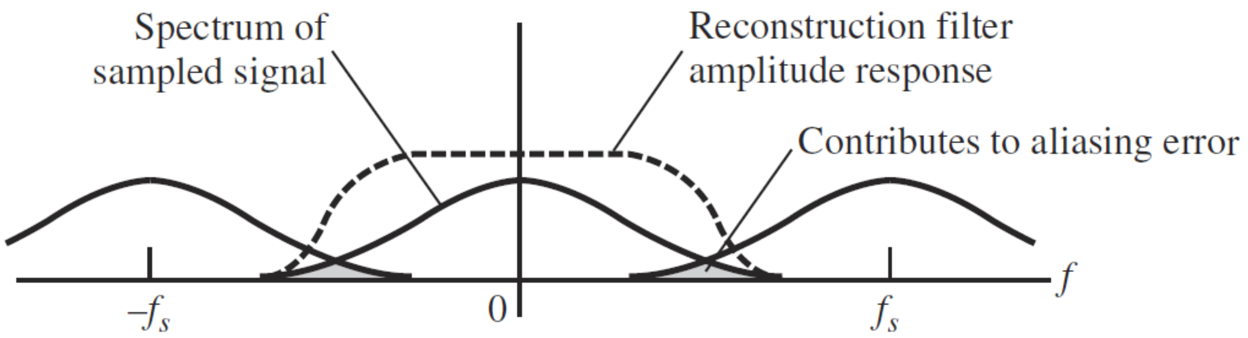




Orthogonal basis!

Reconstruction Errors

- Spectrum aliasing either because $x(t)$ is not bandlimited or $f_s < 2W$



- Nonideal lowpass filter