Digital Signal Processing (Lecture Note 2)

Jin Ho Chang

Professor

Dept. of Electrical Engineering and Computer Science

대구경북과학기술원
Daegu Gyeongbuk
Institute of Science & Technology

Electronic Noise

대구경북과학기술원
Daegu Gyeongbuk
Institute of Science & Technology

Electronic Noise

- It always presents in signals, which arises from the transducer and associated electronics and is intermixed with the signal being measured.
  - Johnson (or Thermal) Noise
  - Shot Noise
  - Flicker Noise (or 1/f Noise)
  - Burst Noise
- This noise should be removed and a medical instrumentation with a very low noise level is thought of as a state of the art device.

Johnson Noise (1)

- It is produced by resistance sources, and the amount of noise generated is related to the resistance and to the temperature:

$$V_J = \sqrt{4kTRB} \text{ volts}$$

  - R is the resistance in ohms, T is the temperature in degrees Kelvin, and k is Boltzman's constant (1.38×10^{-23} J/°K). B is the bandwidth, or range of frequencies, that is allowed to pass through the measurement system.
  - The system bandwidth is determined by the analog filter characteristics in the system.
  - In the case of Johnson noise current,

$$I_J = \sqrt{4kT\,B/R} \text{ amps}$$

Johnson Noise (2)

- Since Johnson noise is spread evenly over all frequencies, it is not possible to calculate a noise voltage or current without specifying bandwidth (B), the frequency range.
- Since the bandwidth is not always known in advance, it is common to describe a relative noise: the noise that would occur if the bandwidth were 1.0 Hz.
- Such relative noise specification can be identified by the unusual units required: $volts/\sqrt{Hz}$ or $amps/\sqrt{Hz}$.

• Other Useful Units

- dBm (or dBmW): the power ratio in decibels of the measured power referenced to one mW.
- N dBm = $10\cdot\log_{10}\left(V_{rms}^2/50\Omega\right)+30dB=20\cdot\log_{10}(V_{rms})+13dB$
- N dBm/Hz = $10\cdot\log_{10}\left(\frac{V_{rms}^2}{50\Omega\times\sqrt{Hz}}\right)=20\cdot\log_{10}\left(\frac{V_{rms}}{\sqrt{Hz}}\right)+13dB$

Shot Noise

• It is defined as a current noise and is caused by current fluctuations across a forward-biased in p-n junctions.

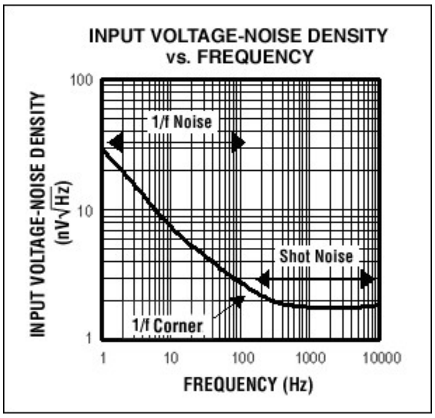
$$I_s=\sqrt{2qI_dB}\ \text{amps}$$

- q is the charge on an electron (1.662×10^{-19} coulomb), and I_d is the baseline semiconductor current.
- A relative noise can be specified in $amps/\sqrt{Hz}$.

• Thermal and Shot noises can be regarded as white noise that is a random signal with a flat power spectral density. In other words, the signal contains equal power within a fixed bandwidth at any center frequency.

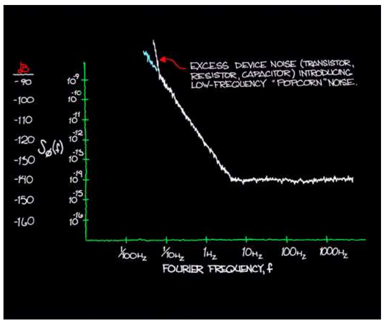
Flicker Noise (or 1/f Noise)

- This noise occurs in almost all electronic devices at low frequencies.
- Its amplitude is inversely proportional to frequency: dominant noise source at frequencies less than 200 Hz.

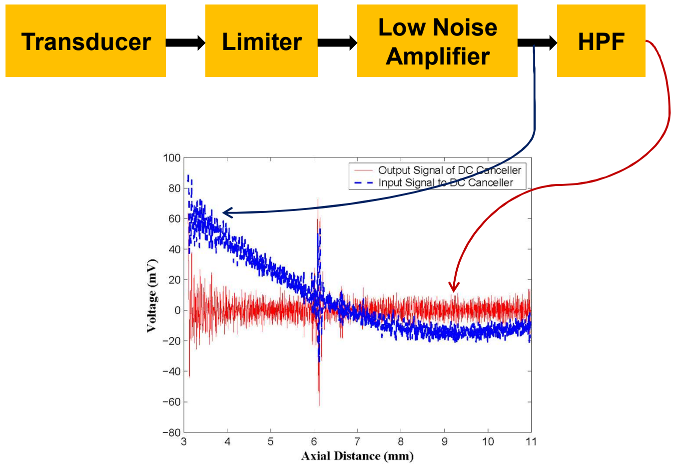


Burst Noise

- This noise is also called popcorn noise, impulse noise, or bi-stable noise.
- It occurs in semiconductors and is an abrupt step-like shift in offset voltage lasting for several milliseconds.
- Its amplitude ranges less than one microvolt to several hundred microvolts.
- An amplifier may exhibit several pops per second during one observation period and then remain popless for several minutes.



Effect of Flicker and Burst Noises



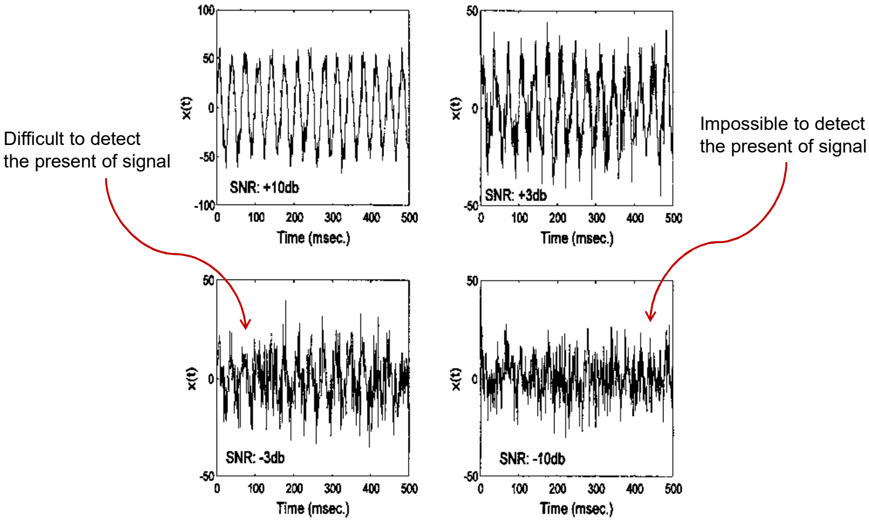
Signal-to-Noise Ratio (SNR)

- Most waveforms consist of signal plus noise mixed together.
  - The signal is that portion of the waveform of interest while the noise is everything else.
- SNR is a quantity indicating the relative amount of signal and noise appearing in a waveform.
  - It is simply the ratio of signal to noise, both measured in RMS (root-mean-squared) amplitude.
  - It is frequently expressed in dB

$$f_{rms} = \sqrt{\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |f(t)|^2 dt}$$

$$SNR = 20 \times \log_{10}\left(\frac{Signal_{RMS}}{Noise_{RMS}}\right)$$

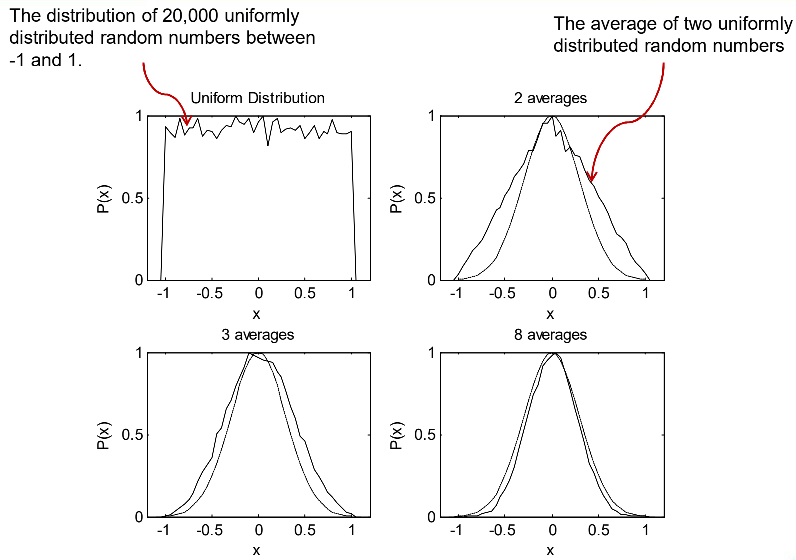
Effect of Different SNR Values



Characteristics of Noise

- Noise is usually represented as a random variable.
  - Since the variable is random, describing it as a function of time is not very useful.
  - It is more common to discuss other properties of noise such as its probability distribution, range of variability, or frequency characteristics.
- While noise can take on a variety of different probability distributions, the Central Limit Theorem implies that most noise will have a Gaussian or normal distribution.
  - The Central Limit Theorem says that when noise is generated by a large number of independent sources, it will have a Gaussian probability distribution regardless of the probability distribution characteristics of the individual sources.

Central Limit Theorem



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Ensemble Averaging

- The probability of a Gaussianly distributed variable x

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

- The mean value of a discrete array of N samples

$$\bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k$$

  – Frequently, the mean will be subtracted from the data sample to provide data with zero mean value.

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- The sample variance

$$\sigma^2 = \frac{1}{N-1}\sum_{k=1}^{N}(x_k - \bar{x})^2$$

  - σ is the standard deviation.
  - Normalizing the standard deviation or variance by 1/N-1 produces the best estimate of the variance (unbiased estimator), if x is a sample from a Gaussian distribution.
  - Alternatively, normalizing the variance by 1/N produces the second moment of the data around x, which is the equivalent of the RMS value of the data if the data have zero as the mean.
- When multiple measurements are made and if multiple random variables are added together,
  - The mean of sampling distribution of mean has same mean as original population.

$$\overline{x}_k = \overline{x}$$

- The variance of sampling distribution of mean is the population variance divided by sample size

$$\sigma_k^2 = \frac{\sigma^2}{N}$$

- The standard deviation of sampling distribution of mean is

$$\sigma_k = \frac{\sigma}{\sqrt{N}}$$

- Averaging noise from different sensors, or multiple observations from the same source, will reduce the standard deviation of the noise by the square root of the number of averages.
- The equation representing the standard deviation indicates that averaging of measured signals can be a simple, but powerful signal processing method for reducing noise when multiple observations of the signal are possible.

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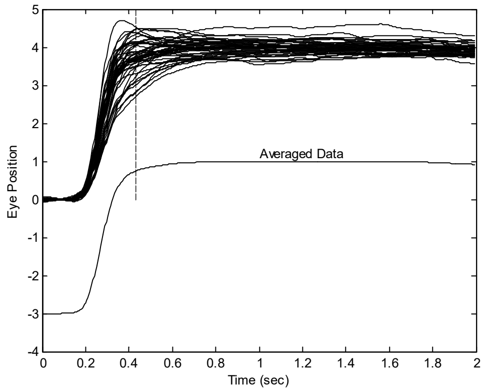
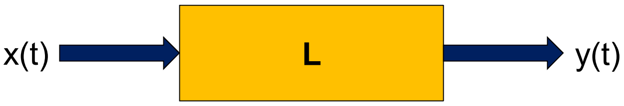
- In ensemble averaging, a group of time responses are averaged together on a point-by-point basis
  - An average signal is constructed by taking the average, for each point in time, over all signals in the ensemble.
  - Example: An ensemble of individual eye movement responses to a step change in stimulus



Linear Time Invariant (LTI) Systems

LTI Systems

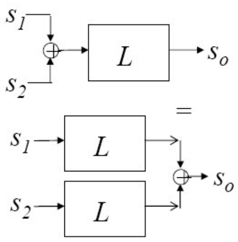
- Linearity : superposition (i.e., scaling & additive properties)



$y(t) = L\{x(t)\}$

  - Finding the mathematical relationship between in- and output is called ***modeling***.
  - Superposition

Linear systems fulfill *superposition principle*:



$$L\{c_1 s_1 + c_2 s_2\} = c_1 L\{s_1\} + c_2 L\{s_2\} \quad \forall c_1, c_2 \in \Re$$

where s_1, s_2 are arbitrary signals

- for example, consider an amplifier with gain A:

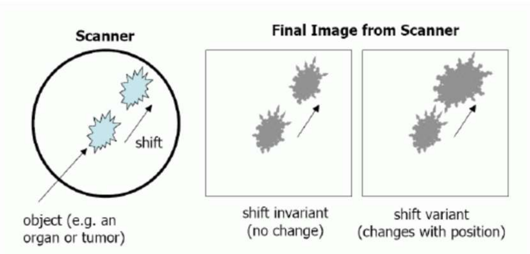
$$L\{c_1 s_1 + c_2 s_2\} = A(c_1 s_1 + c_2 s_2)$$
$$= c_1 A s_1 + c_2 A s_2 = c_1 L\{s_1\} + c_2 L\{s_2\}$$

- Shift(Time)-invariant system
  - The time shift of input sequence causes a corresponding shift in the output sequence

Time-invariance (shift-invariance = LSI):

- properties of L do not change over time (spatial position), that is:

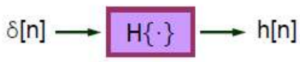
$$s_o(x) = L\{s_i(x)\} \text{ then } s_o(x - X) = L\{s_i(x - X)\}$$



- Linear shift(time)-invariant system:
  - It has both linear and time invariant property.

System function

h[n] = H{δ[n]}



- Linearity

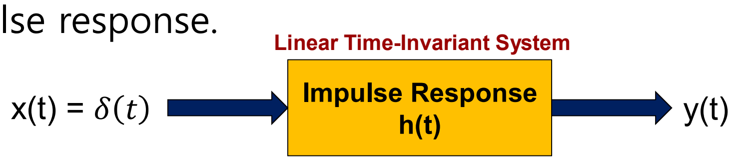
$$H\{a\cdot x_1[n]+b\cdot x_2[n]\}=a\cdot H\{x_1[n]\}+b\cdot H\{x_2[n]\}$$

- Time invarient

$$H\{x(n-N)\}=y(n-N)$$

Impulse Response

- The linear system can be completely characterized by its impulse response.



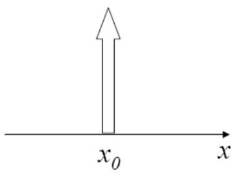
  - Impulse Response: A system's response to a Dirac impulse.

$$\delta(x-x_0)=0 \quad \text{for } x\neq x_0$$

$$\int_{-\infty}^{+\infty}\delta(x-x_0)dx=1$$

- an important property is its sifting property:

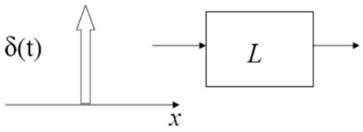
$$\int_{-\infty}^{+\infty}s(x)\delta(x-x_0)dx=s(x_0)$$

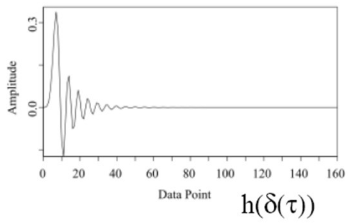


a "needle" spike of infinite height at $x=x_0$

Convolution

A system's response to a Dirac impulse is called *impulse response h*:





Start with:

$$s_i(x) = \int_{-\infty}^{+\infty} s_i(\xi)\delta(x-\xi)\,d\xi$$

Then write:

$$s_o(x) = L\{s_i\} = \int_{-\infty}^{+\infty} s_i(\xi) L\{\delta(x-\xi)\}\,d\xi = \int_{-\infty}^{+\infty} s_i(\xi) h(x-\xi)\,d\xi$$

The expression

$$s_o(x) = \int_{-\infty}^{+\infty} s_i(\xi) h(x-\xi)\,d\xi = s_i * h$$

is called *convolution*, defined as:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi) s_2(x-\xi)\,d\xi = \int_{-\infty}^{+\infty} s_1(x-\xi) s_2(\xi)\,d\xi$$

Procedure:

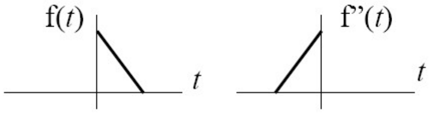
for each *x* do:

1: mirror s_2 about $\xi = 0$ (change ξ το $-\xi$)
2: translate mirrored s_2 by $\xi = x$
3: multiply s_1 and mirrored s_2
4: integrate the resulting signal

See next slides for an example and detailed explanation…

Mirroring:

- when you take a function f(t) and mirror it about the y-axis then you get a new function f"(t) = f($-t$)



For convolution:

- you have two functions: $f_1(t)$ and $f_2(t)$
- you would like to compute:

$$f(x)=\int_{-\infty}^{+\infty} f_1(t) f_2(x-t)\,dt$$

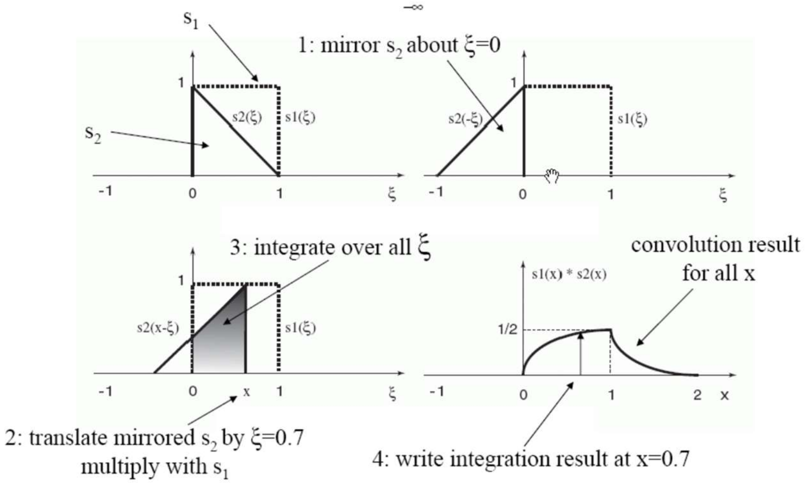
- but in this form: t increases in f_1 and decreases in f_2, which is not convenient
- to fix this, you mirror $f_2(x-t)$ into $f_2"(t-x) = f_2(-(x-t))$
- now the convolution writes:

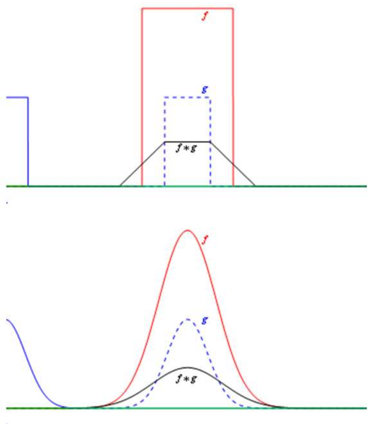
$$f(x)=\int_{-\infty}^{+\infty} f_1(t) f_2^{"}(-(x-t))\,dt=\int_{-\infty}^{+\infty} f_1(t) f_2^{"}(t-x)\,dt$$

- at this point you need $f_2"(t)$ which is obtained by mirroring $f_2(t)$: $f_2"(t) = f_2(-t)$
- now you can do the intuitive right-sliding of f_2" for growing x

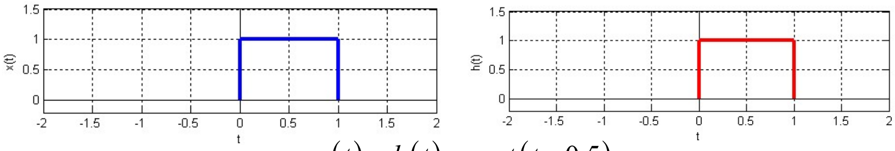
Example x=0.7:

$$s_1(x)*s_2(x)=\int_{-\infty}^{+\infty} s_1(\xi)s_2(x-\xi)\,d\xi$$



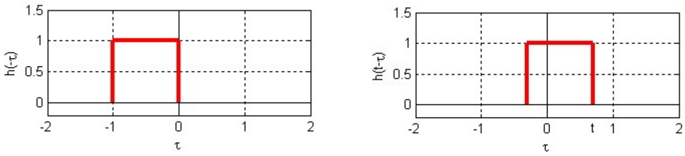


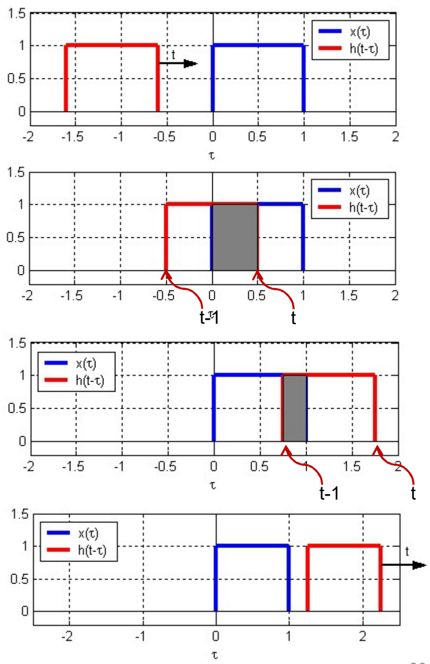
- Example



$$x(t) = h(t) = rect(t-0.5)$$

$$\text{,where } rect(t) = \begin{cases} 1, & |t| < 0.5 \\ 0.5, & |t| = 0.5 \\ 0 & |t| > 0.5 \end{cases}$$



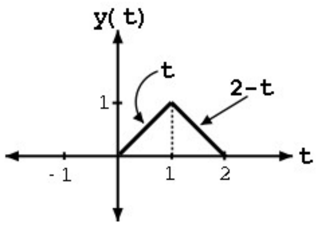


for $0 < t < 1$

$$x(t) * h(t) = \int_0^t 1 dt = t$$

for $1 < t < 2$

$$x(t) * h(t) = \int_{t-1}^1 1 dt = 2 - t$$



Convolution Properties

Also defined for multi-dimensional signals:

$$s_1(x,y) * s_2(x,y) = \int_{-\infty}^{+\infty} s_1(x-\xi, y-\zeta) s_2(\xi,\zeta) d\xi d\zeta$$

Some important properties:

- commutativity:

$$s_1 * s_2 = s_2 * s_1$$

- associativity:

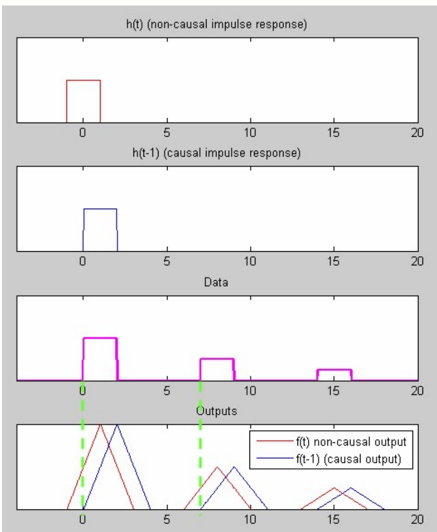
$$(s_1 * s_2) * s_2 = s_1 * (s_2 * s_3)$$

- distributivity:

$$s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3$$

Causal vs. Non-Causal Systems

- Causal
  - The present value of the output signal depends only on the present or the past values of the input signal.
  - The system cannot anticipate the input.
- Non-causal
  - The output signal of an non-causal system depends on one or more future values of the input signal.



Stability & Causality in LTI Systems

$x(n)$-> L[ ] -> y (n)

General:
y(n)=L[x(n)]

In a LSI system,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

Convolution Sum

Stability

General if x(n) bounded
ie. |x(n)| $< \infty \; all \; n$
Then y(n) bounded
ie. |y(n)| $< \infty \; all \; n$

In a LSI system,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

h(n)= 2^n u(n) - unstable
h(n)= $(½)^n$ u(n) - stable

Causality

y(n) for n=n_1 depends on x(n) only for n<=n_1

In a LSI system,

h(n)=0, n<0

h(n)=$(2)^n$ u(-n)

non-casual and stable

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