Digital Signal Processing (Lecture Note 7)

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Z Transform (Chap. 3)

Objectives of This Lecture

- Relationship between z-transform and Fourier transform
- Region of Convergence (ROC) for z-transform
- Characteristics of poles and zeros
- Stability and Causality in z-transform of the impulse response of a LTI system

Why Z-Transform?

- Fourier transform dose not exist for all sequences.
- The Fourier transform of the unit-sample response should converge if LSI systems are stable; absolutely summable or finite energy
- Z-transform is a generalization of the DTFT, which may exist for many signals that are not available in the DTFT.
  - DTFT cannot describe **transient responses of infinite length**, such as step functions, **systems' stability**, or **systems with nonzero initial conditions**.
  - DTFT is a special case of the Z-transform.
- While the DTFT is used to figure out which frequency components constitute an input or an output signal, Z-transform is used to describe functionality and responses to an input signal of a digital LTI system.
  - Laplace transform is an analog version of Z-transform.
  - **Digital filters are designed, expressed, applied and represented in terms of the Z-transform**.

Convergence of Fourier Transform

- Fourier Transform
  - $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

    $|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x(n)||e^{-j\omega n}|$

    $X(e^{j\omega})$ converges if $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

    Stable system -> $H(e^{j\omega})$ converges

Examples for Convergence of FT

- Does the Fourier transform of the following sequences converge?

(1) $x(n) = (½)^n u(n)$

$\sum_{n=-\infty}^{\infty} |x(n)| = 2$

(2) $x(n) = (2)^n u(n)$

$\sum_{n=-\infty}^{\infty} |x(n)| = \infty$

Z-Transform

- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$\rightarrow X(e^{j\omega}) \cdot r^{-n} = \sum_{n=-\infty}^{\infty} [x[n] \cdot r^{-n}] e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} [x[n]] \underbrace{(re^{j\omega})}_{z}{}^{-n}$$

The z-transform

$z = re^{j\omega}$

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ r=1

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \big|_{z=e^{j\omega}}$

Converges if $\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$

- Definition

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad X(z) = Z\{x(n)\}$$
$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \qquad x(n) \leftrightarrow X(z)$$

– Z-transform is an infinite power-series that exists for values of z in a certain region, called Region of Convergence (ROC).

- DTFT vs. Z-transform

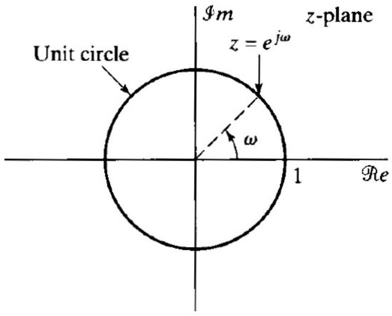
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \xrightarrow{z = re^{j\omega}} X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

- Existence (Convergence) of Z-transform

$$|X(z)| \le \sum_{n=-\infty}^{\infty} |x(n) r^{-n} e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

Unit Circle in the Complex Z-plane

- z is a complex variable.
- The contour corresponding to |z| = 1 referred to as the unit circle.
- The z-transform on the unit circle corresponds to the Fourier transform.
- z=$e^{j\omega}$, $r = 1$



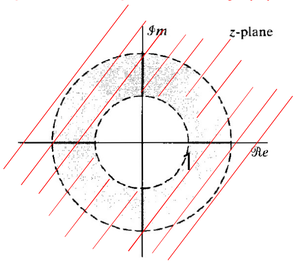
Complex z-plane

Region of Convergence (ROC)

- Region of Convergence of the z-transform:
  - **For a given sequence**, the set R of values of z for which the z-transform power series converges is called the ROC.

$$\sum_{n=-\infty}^{\infty} |x[n]|\,|z|^{-n} < \infty,$$

Convergence of the power series for a given sequence depends only |z|



ROC ?

Example: x[n]=u[n] is not absolutely summable.
r^{-n} u[n], absolutely summable if r>1
➔ the z-transform for the unit step exists with an ROC r = |z| >1

Right-sided Exponential Sequence

- Determining the Z-transform and the corresponding ROC of the causal sequence $x(n) = a^n u(n)$.

z-transform of x[n] : $X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$

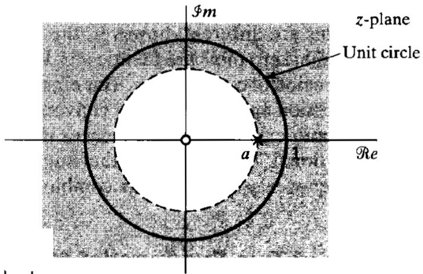
For convergence of X(z): $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$

For $|az^{-1}| < 1$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|.$$

Causal System

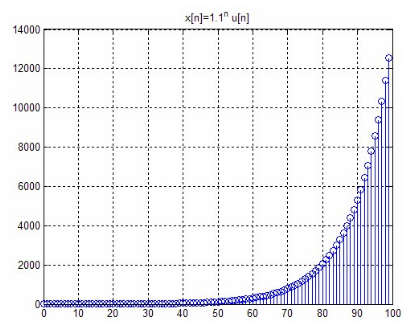
ROC

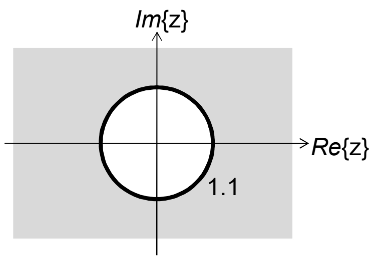


- For $x[n] = (1.1)^n \cdot u[n]$

$$X(z) = \frac{1}{1 - 1.1 \cdot z^{-1}}, \quad \text{for } \infty > |z| > |1.1|$$

  - This sequence does not have a DTFT if |z|≠1
  - However, it does have a Z-transform.
  - This sequence has an ROC that is outside of a circular area.





Left-Sided Exponential Sequence

- Consider the Anti-causal Sequence: $y[n] = -a^n \cdot u[-n-1]$

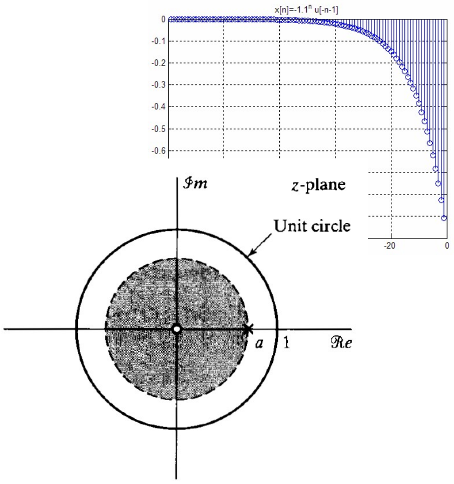
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n.$$

$$|a^{-1} z| < 1$$

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \boxed{\frac{1}{1 - a z^{-1}} = \frac{z}{z - a},} \qquad |z| < |a|.$$

cf: $x(n) = \alpha^n u(n) \longleftrightarrow X(z) = \frac{1}{1 - \alpha z^{-1}} \quad ROC: |z| > |\alpha|$

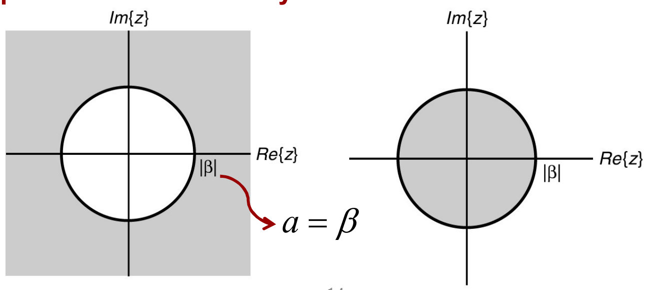


- The Z-transforms of the two sequences

$$x[n] = a^n \cdot u[n] \qquad y[n] = -a^n \cdot u[-n-1]$$

  are identical even though the two parent sequences are different.
- Only way a unique sequence can be associated with a Z-transform is by specifying its ROC.

DTFT of a sequence exist if and only if the ROC includes the unit circle due to $Z = e^{j\omega}$



Two-Sided Exponential Sequence

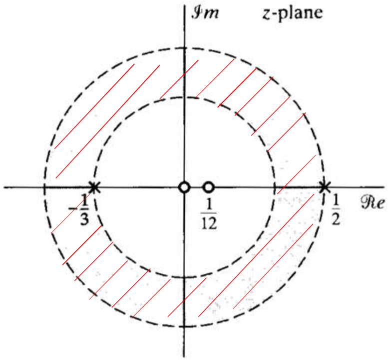
$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1].$$

$$\left(-\tfrac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3},$$

$$-\left(\tfrac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \tfrac{1}{2}.$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \tfrac{1}{3} < |z|, \quad |z| < \tfrac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$



Finite-Length Exponential Sequence

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

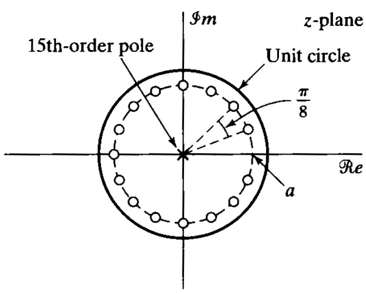
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a},$$

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty.$$ **If a is finite, |z|>0**

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1.$$

N=16



The zero at k=0 cancels the poles at z=a.
There are no poles other than at the origin.

Digital Transfer Function

- The Z-transform H(z) of the impulse response, if exist, is called the transfer function of the system
  - Not every LTI system possesses a transfer function, since not every sequence has a Z-transform.
  - Of particular interest are LTI systems that are stable in the sense of having the bounded-input, bounded-output (BIBO) property.

$$\sum_{n=-\infty}^{\infty} \left| h[n] \right| < \infty$$

  - **LTI system is <u>stable</u> if and only if the region of convergence of its transfer function <u>includes the unit circle</u>.**
  - **An LTI system is <u>causal</u> if and only if its impulse response h[n] is a <u>causal sequence</u>.**

  - All the singularities of **the transfer function** of a stable and causal LTI system **must be inside the unit circle**.
- Transfer function

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j], \quad a_0 = 1$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Z   Z

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$= z^{(N-M)} \frac{b_0 \prod_{l=1}^{M} (z - \alpha_l)}{a_0 \prod_{l=1}^{N} (z - \beta_l)}$$

Zeros

Poles

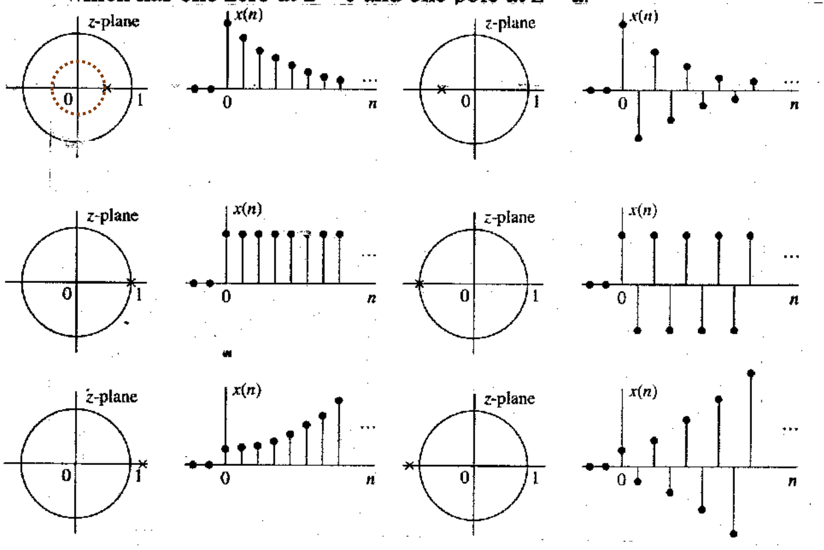
- If N>M, there are additional N-M zeros at z=0 (the origin in the Z-plane)
- If N<M, there are additional M-N poles at z=0
- Including trivial poles and zeros at z=0 or infinite, the number of poles and zeros are equal.
- Recall that for a system to be causal, its impulse response must satisfy h[n]=0, n<0, that is for a causal system, the impulse response is right sided in time. Based on this,
  - The ROC of a causal system extends outside of the outermost pole circle
  - The ROC of an anti-causal system whose h[n] is purely left-sided lies inside of the innermost pole circle
  - The ROC of a noncausal system whose h[n] is two-sided is bounded by two different pole circles

- For a system to be stable, its h[n] must be absolutely summable.
  - ✓ An LTI system is stable if and only if the ROC of its transfer function H(z) includes the unit circle.
- A causal system's ROC lies outside of a pole circle. If that system is also stable, its ROC must include unit circle.
  - ✓ **A causal system is stable, if and only if all poles are inside the unit circle**.
- **Similarly, an anti-causal system is stable, if and only if its poles lie outside the unit circle**.

- Filter design is simply the determination of appropriate filter coefficients, a(n) and b(n), that provide the desired spectral shaping.
- A digital filter is designed by placing appropriate number of
  - **zeros at the frequencies (z-values) to be suppressed**
  - **Poles at the frequencies to be amplified**

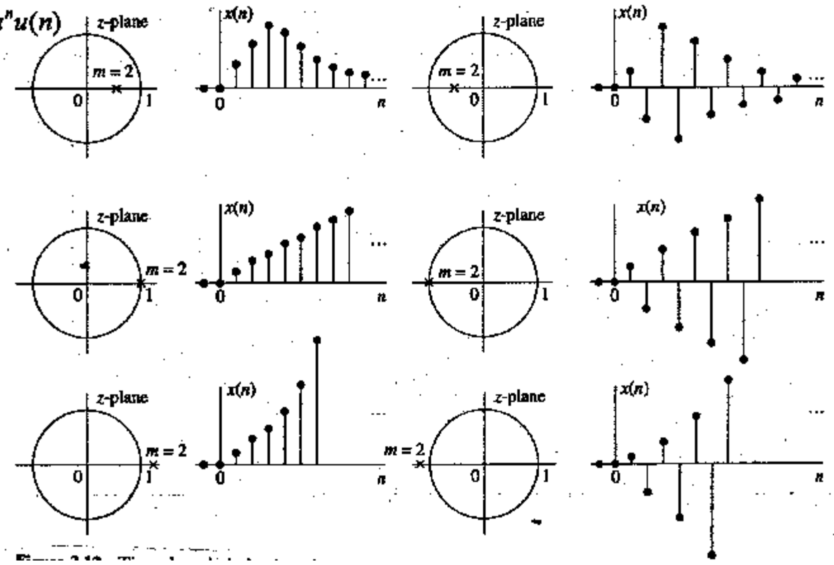
$$x(n) = a^n u(n) \xleftrightarrow{z} X(z) = 1/(1 - az^{-1}) \quad \text{ROC: } |z| > |a|$$

which has one zero at z = 0 and one pole at z = a.

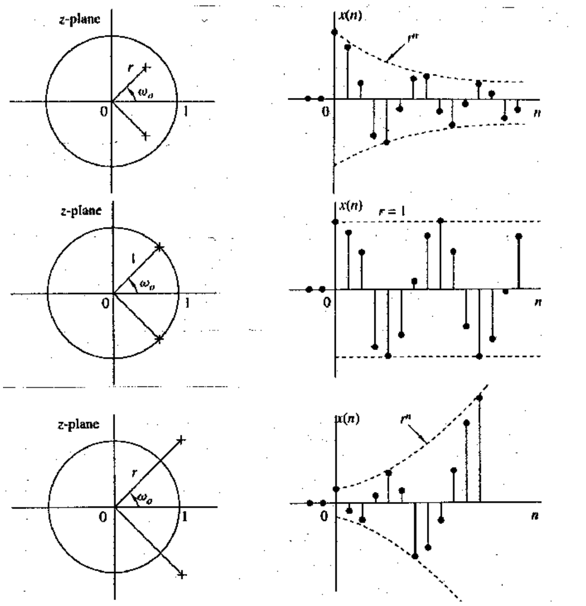


A real causal signal with a double pole has the form

$x(n) = na^n u(n)$



- A real causal signal with a pair of complex-conjugate poles:

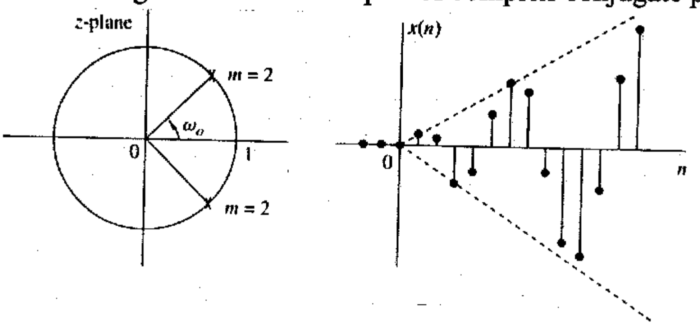


$x(n) = a^n \sin\omega_0 n \cdot u(n)$ or

$x(n) = a^n \cos\omega_0 n \cdot u(n)$

Note) Location of zeros only affects the signal phase in this case.

A real causal signal with a double pair of complex-conjugate poles:

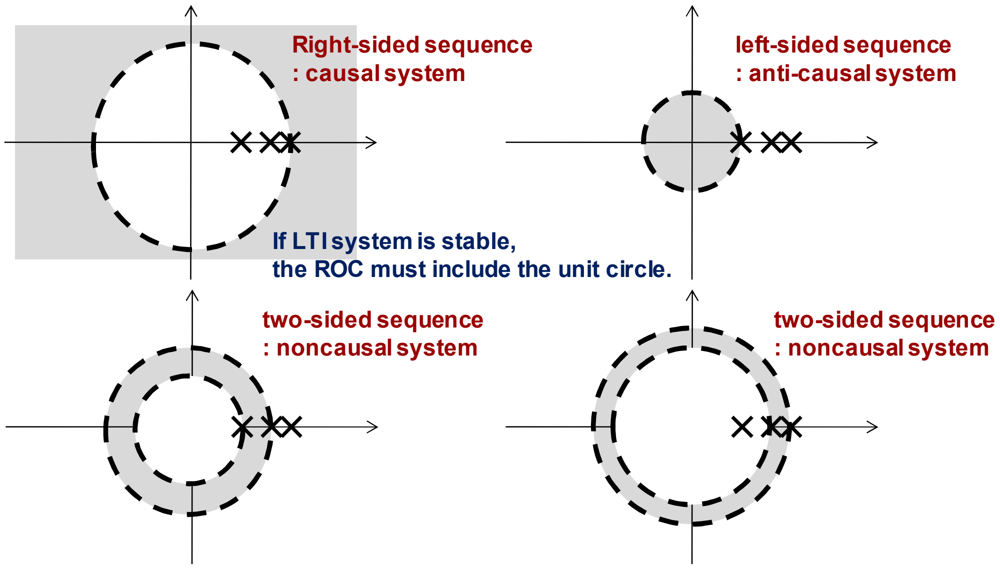


- System characteristics from pole locations:
  - The signal characteristics we observed for the different pole locations applies as well to causal LTI systems, since their impulse response is a causal signal.
  - ➔ If <u>**a pole of a system is outside the unit circle**</u>, the impulse response of the system becomes **unbounded and, consequently, the system is unstable**.

Properties of the ROC

- Property 1: the ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$
- Property 2: The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
- Property 3: the ROC cannot contain any poles.
- Property 4: If x[n] is a finite-duration sequence, i.e., a sequence that is zero except in a finite interval $-\infty \le N_1 < n < N_2 \le \infty$, then the ROC is the entire z-plane, except possibly z=0 or z=∞.
- Property 5: If x[n] is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the outermost (i.e., the largest magnitude) finite pole in X(z) to (and possibly including) z=∞.

- Property 6: If x[n] is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the innermost (i.e., the smallest magnitude) nonzero pole in X(z) to (and possibly including) z=0.
- Property 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.
- Property 8: The ROC must be a connected region.



Example 1

Sketch the pole-zero pattern of their z-transforms. Include an indication of the region of convergence.

$$\delta(n) + (1/2)^n u(n)$$

Example 2

Sketch the pole-zero pattern of their z-transforms. Include an indication of the region of convergence.

$$\left(\frac{1}{2}\right)^{|n|}$$

Example 3

Perform z-transform of these sequences and then determine the stability and causality of these system

h(n)=$(1/2)^n u(n)$

h(n)=$-(1/2)^n u(-n-1)$

Some Common Z-transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Inverse Z-Transform

- Z-domain representations ➔ time-domain representations
  - Inspection Method
  - Partial Fraction Expansion
  - Power series expansion

Inspection Method

- Consists simply of becoming familiar with, or recognizing "by inspection:"
  - Refer to common z-transform pairs

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}} \right), \qquad |z| > \frac{1}{2},$$

Partial Fraction Expansion: N>M

- X(z) may not be given explicitly in an available table

➔ Represent X(z) as a sum of simpler terms (use a partial fraction expansion)

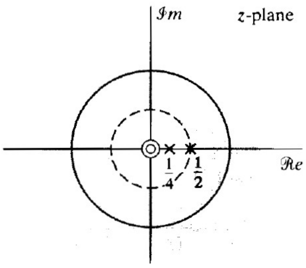
$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \quad \xrightarrow{\text{If } N>M} \quad X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}$$

Example

Inverse z-transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$



Partial Fraction Expansion: M≥N

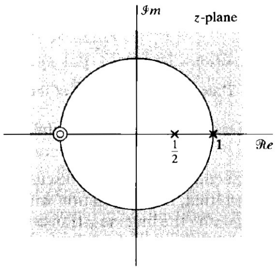
If M≥N

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \quad \Rightarrow \quad X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

Example 1

Inverse z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1.$$



$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)(1 - z^{-1})\right]_{z=1} = 8.$$

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ \hline & \quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}.$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

$$2 \overset{\mathcal{Z}}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1 - \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n],$$

$$\frac{1}{1 - z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} u[n].$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

Example 2

$$H(z) = \frac{\left(1 - \frac{1}{2}z^{-2}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$

- Determine the impulse response of the system?

- Determine the inverse z-transform of this sequence using partial fraction expansion method.

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}, \qquad x[n] \text{ stable}$$

Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots,$$

If the z-transform is represented as a power series in the above form, we can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}

- Finite-length sequences where X(z) may have no simpler form than a polynomial in z^{-1}

Example: Finite-Length Sequence

- Finite-length sequence

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}).$$

Inverse z-transform?

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}.$$

By inspection

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

Example: Power Series Expansion

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Taylor series expansion for log(1+x) with |x|<1

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$

Taylor Series Expansion

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots=\sum_{k=0}^{\infty}x^k \quad (|x|<1)$$

$$\frac{1}{1-x}=1-x+x^2-\ldots+(-x)^n+\ldots=\sum_{k=0}^{\infty}(-1)^k x^k \quad (|x|<1)$$

$$e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots=\sum_{k=0}^{\infty}\frac{x^k}{k!} \quad (|x|<\infty)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \quad (|x|<\infty)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \quad (|x|<\infty)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} \quad (-1<x\le 1)$$

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^n\frac{x^{2n+1}}{2n+1}+\ldots=\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1} \quad (|x|\le 1)$$

Power Series Expansion by Long Division

$$X(z)=\frac{1}{1-az^{-1}}, \qquad |z|>|a|.$$

X(z) is the ratio of polynomials

-> a power series by long division of the polynomial

$$\begin{array}{r|l} & 1+az^{-1}+a^2z^{-2}+\cdots \\ \hline 1-az^{-1} & 1 \\ & \underline{1-az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1}-a^2z^{-2}} \\ & \qquad\quad a^2z^{-2} \quad \cdots \end{array}$$

$$\frac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+\cdots.$$

$$x[n]=a^n u[n].$$

Example: Power Series Expansion

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

Z-transform Properties

- Linearity
- Time Shifting
- Multiplication by an exponential sequence
- Differentiation of X(z)
- Conjugation of a Complex Sequence
- Time Reversal
- Convolution of Sequences

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \mathrm{ROC} = R_x.$$

ROC is denoted by R_x

Linearity

$$ax_1[n] + bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2},$$

1. ROC will be the common region between ROCs for individual z-transforms if the poles of $aX_1(z)+bX_2(z)$ consist of all the poles of $X_1(z)$ and $X_2(z)$

2. ROC may be larger if the linear combination causes that some zeroes cancels some poles

Example

$$x[n] = a^n u[n] - a^n u[n-N].$$

ROC: |z| >|a| ROC: |z| >|a|

Time Shifting

$$x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}X(z),$$

ROC $= R_x$ (except for the possible addition or deletion of $z=0$ or $z=\infty$).

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}.$$

With the substitution of variables $m = n - n_0$,

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m},$$

$$Y(z) = z^{-n_0}X(z).$$

z^{-n_0} can alter the number of poles at z=0 or z =∞

Example

- Determine the corresponding sequence of the Z-transform

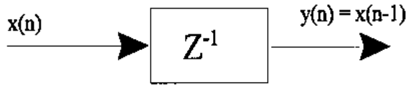
$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \tfrac{1}{4}.$$

Detail of Time Shift Property

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- When identified with a data sequence, such as x[n], z^{-n} represents an interval shift of n samples, or an associated time shift of nT_s seconds.
  - The equation indicates that every data sample in the sequence x[n] is associated with a unique power of *z*, and this power of *z* defines a sample's position in the sequence.
- This time shifting property of z^{-n} can be formally stated as:

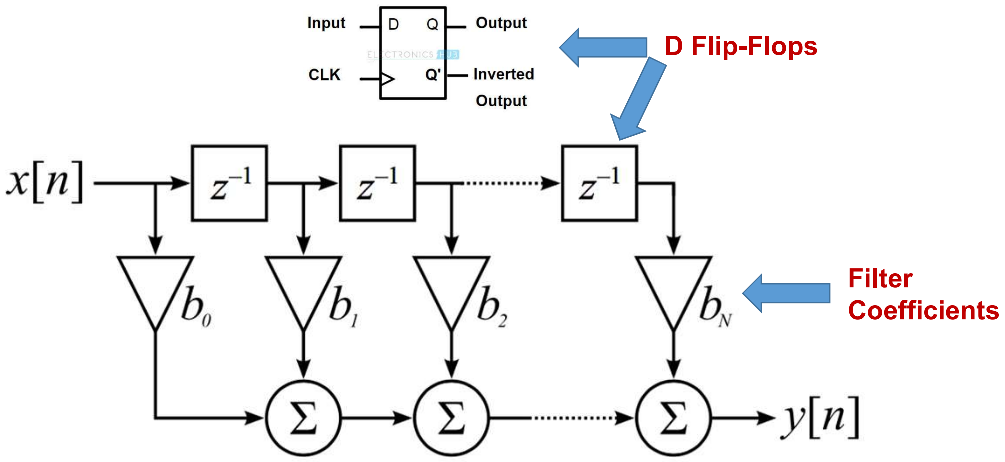
$$x(n-k) \overset{z}{\Longleftrightarrow} z^{-k} X(z)$$



  - It shows a unit delay process shifts the input by one data sample. Other powers of *z* could be used to provide larger shifts.

Example: FIR Filter Implementation

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = h[0] + h[1] z^{-1} + h[2] z^{-2} + \cdots + h[M] z^{-M}$$



Multiplication by an Exponential Sequence

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \mathrm{ROC} = |z_0| R_x.$$

$|z_0|R_x$ denotes that the ROC is R_x scaled by $|z_0|$

$$r_R < |z| < r_L \longrightarrow |z_0| r_R < |z| < |z_0| r_L$$

all the poles and zeroes are scaled by z_0

$$z = z_1 \longrightarrow z = z_0 z_1$$

Example

$$x[n] = r^n \cos(\omega_0 n) u[n].$$

Differentiation of X (z)

$$nx[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

Example: Second-order Pole

$$x[n] = na^n u[n] = n(a^n u[n]).$$

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right), \qquad |z| > |a|$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|.$$

$$na^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|.$$

Conjugation of a Complex Sequence

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

$$\mathcal{Z}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* = X^*(z^*)$$

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Time Reversal

$$x^*[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}.$$

- x[n] is real or not conjugated

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}.$$

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{l=-\infty}^{\infty} x[l]z^{l} = \sum_{l=-\infty}^{\infty} x[l](z^{-1})^{-l}$$

$$= X(z^{-1})$$

$$ROC: \quad r_1 < \left|z^{-1}\right| < r_2 \Rightarrow \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

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Example

$$x[n] = a^{-n}u[-n],$$

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1}z^{-1}}{1 - a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

Convolution of Sequences

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}.$$

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k],$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}.$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}.$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k}.$$

$$Y(z) = X_1(z)X_2(z),$$

Evaluating a Convolution Using Z-transform

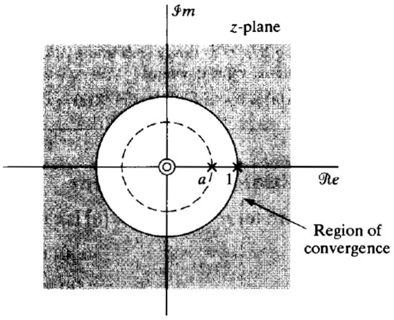
Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$.

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|,$$

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

$|a| < 1.$

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1.$$



$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right), \qquad |z| > 1.$$

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$