EE403: Digital Communications

Lecture 10: Advanced Data Communications

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M-ary Communications

- The main focus of this chapter is M**-ary modulation schemes**, i.e., each symbol could take M different signals, and their **signal space (=signal constellation)** representations

- Notation

  – T_s: symbol time or duration

  – If binary modulation ($M = 2$), T_s = bit duration

- Recall that by (matched) filtering we can computes any operation such that

$$\int_T s(t)\phi(t)dt = \langle s(t), \phi(t) \rangle$$

Orthonormal Sets

- If a set of signals $\{\phi_i(t)\}_{i=1}^{K}$ satisfies
  - ϕ_i has unit energy over T_s, i.e.,

$$\|\phi_i\|^2 = \langle \phi_i, \phi_i \rangle = \int_{T_s} \phi_i^2(t)dt = 1$$

  - ϕ_i, ϕ_j are orthogonal each other if $i \neq j$, i.e.,

$$\langle \phi_i, \phi_j \rangle = \int_{T_s} \phi_i(t)\phi_j(t)dt = 0$$

  then it is said to be **orthonormal**

- It means, if $x(t) = \sum_{i=1}^{K} a_i\phi_i(t)$ was sent, a receiver can always detect $\{a_i\}_{i=1}^{K}$

Quadrature Multiplexing

- Consider two signals, $\phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$ and $\phi_2(t) = \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t)$
- ϕ_1, ϕ_2 have unit energy over T_s, i.e.,

$$\|\phi_1\|^2 = \langle \phi_1, \phi_1 \rangle = \frac{2}{T_s} \int_{T_s} \cos^2(2\pi f_c t) dt = \frac{2}{T_s} \int_{T_s} \frac{1 + \cos(4\pi f_c t)}{2} dt = 1$$

- In addition, they are (almost) orthogonal each other, i.e.,

$$\langle \phi_1, \phi_2 \rangle = \frac{2}{T_s} \int_{T_s} \cos(2\pi f_c t) \sin(2\pi f_c t) dt = \frac{2}{T_s} \frac{1}{2} \int_{T_s} \sin(4\pi f_c t) - \sin(0) dt \approx 0$$

(If T_s is an integer multiple of the period of $\sin(4\pi f_c t)$, it is exactly zero)

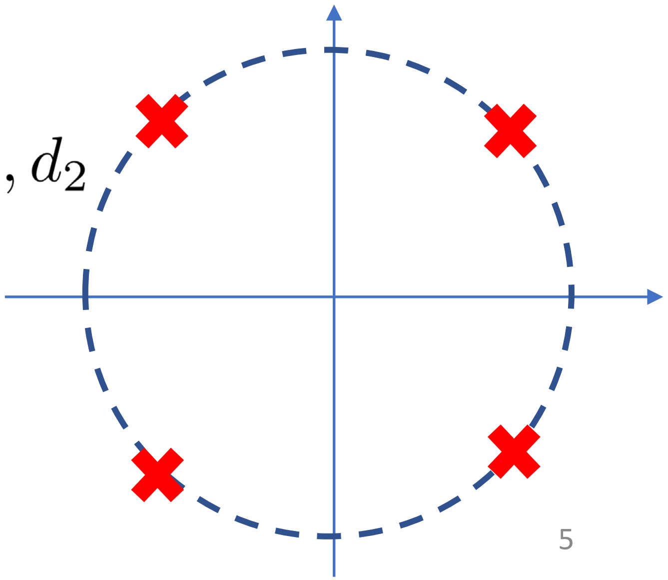
Quadrature Multiplexing

- Quadrature multiplexing:

$$x_c(t) = A[d_1(t)\cos(2\pi f_c t) + d_2(t)\sin(2\pi f_c t)] = R(t)\cos(2\pi f_c t + \theta_i(t))$$

where $R(t) = \sqrt{d_1^2(t) + d_2^2(t)}$ and $\theta_i = \tan^{-1}(d_2(t)/d_1(t))$

- Let $d_1, d_2 \in \{-1, +1\}$ (called **QPSK**, quadriphase-shift keying)
- If QPSK, θ_i takes ± 45 degrees and ± 135 degrees
- For any M-PSK, if $x_c(t)$ was sent, we can exactly find d_1, d_2



Quadrature Multiplexing

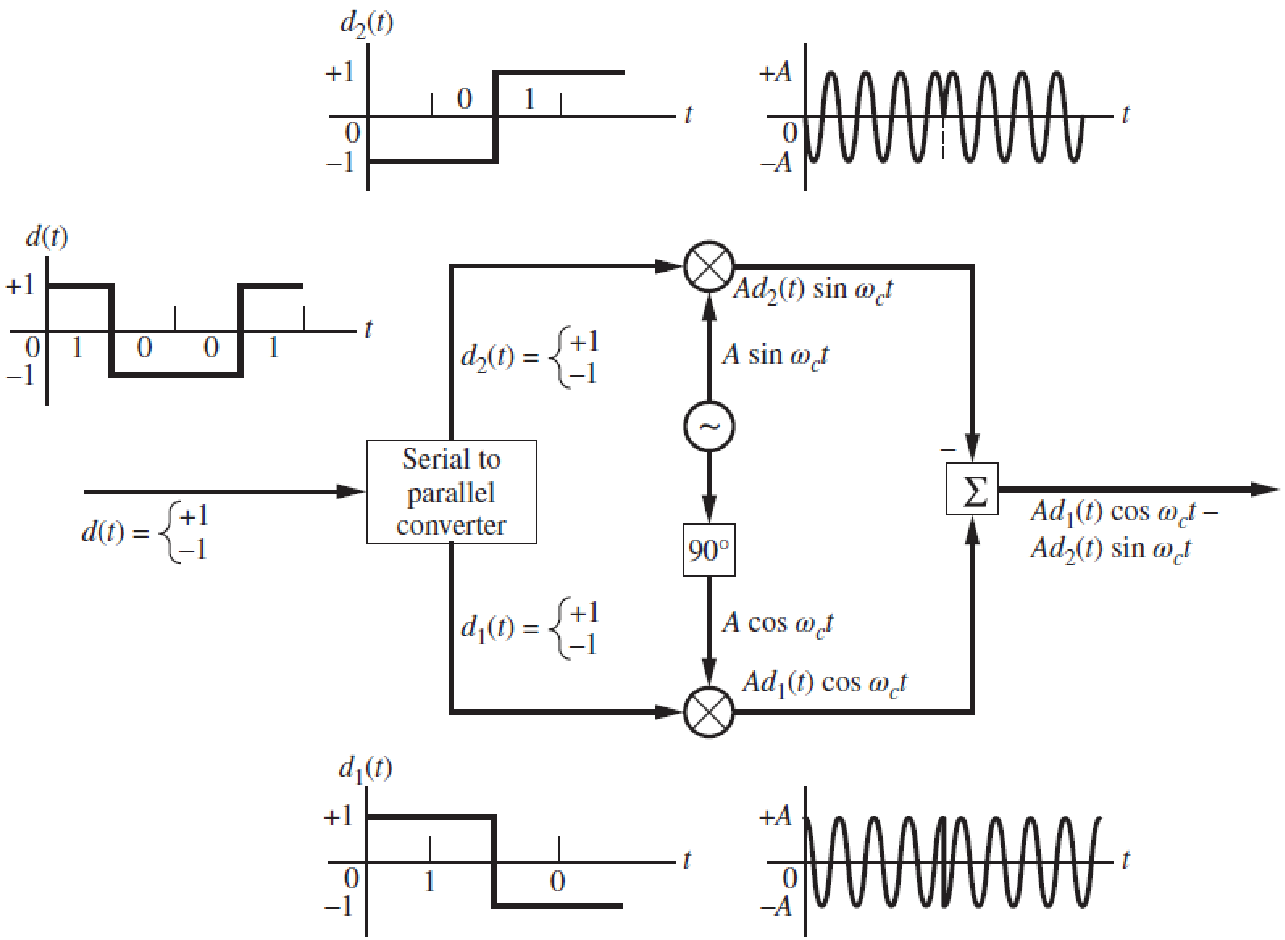


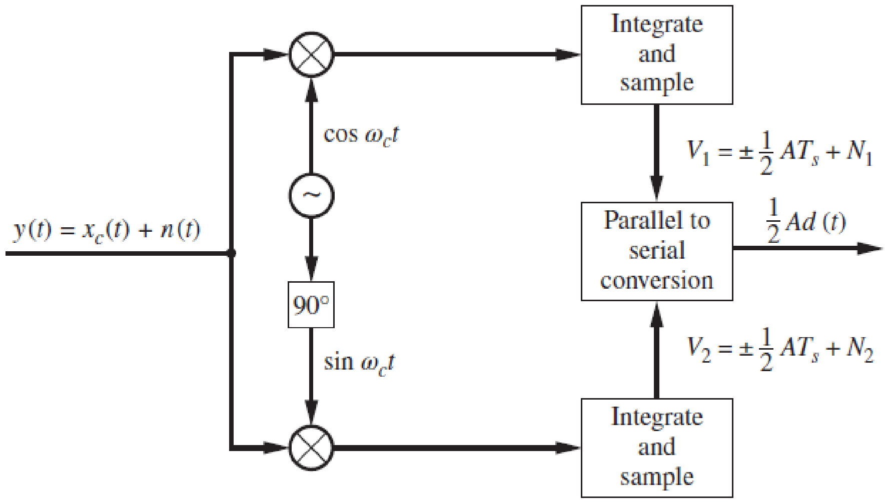
Figure 10.1
Modulator and typical waveforms for QPSK.

Quadrature Multiplexing

- Error analysis of QPSK
- A symbol is correctly detected only if d_1, d_2 are both correct, i.e.,

$$\Pr[\text{a symbol is correct}] = P_c = (1 - P_{E_1})(1 - P_{E_2})$$

- Suppose $y(t) = Ad_1(t)\cos(2\pi f_c t) - Ad_2(t)\sin(2\pi f_c t) + n(t)$
- Repeat what we did in Chap. 9 and take **inner product** directly (via filtering)



Quadrature Multiplexing

- The upper part output

$$V_1 = \int_{T_s} y(t)\cos(2\pi f_c t)dt = \int_{T_s} Ad_1(t)\cos^2(2\pi f_c t)dt + \int_{T_s} n(t)\cos(2\pi f_c t)dt$$
$$= \pm\frac{AT_s}{2} + N_1$$

- The bottom part output

$$V_2 = \int_{T_s} y(t)\sin(2\pi f_c t)dt = \pm\frac{AT_s}{2} + N_2$$

Quadrature Multiplexing

- N_1, N_2 are Gaussian with mean zero

$$
\begin{aligned}
\mathrm{Var}(N_1) = \mathrm{Var}(N_2) = \mathbb{E}[N_1^2] = \mathbb{E}\left[\int\int n(t)n(\alpha)\cos(2\pi f_c t)\cos(2\pi f_c \tau)dtd\tau\right] \\
&= \int\int \mathbb{E}[n(t)n(\alpha)]\cos(2\pi f_c t)\cos(2\pi f_c \tau)dtd\tau \\
&= \int\int \frac{N_0}{2}\delta(t-\tau)\cos(2\pi f_c t)\cos(2\pi f_c \tau)dtd\tau \\
&= \frac{N_0}{2}\int \cos^2(2\pi f_c t)dt = \frac{N_0 T_s}{4}
\end{aligned}
$$

Quadrature Multiplexing

- P_{E_i} is the error probability of $\mathcal{N}(-AT_s/2, N_0T_s/4)$ vs $\mathcal{N}(AT_s/2, N_0T_s/4)$: decision threshold is 0

Energy that each symbol has

$$P_{E_i} = Q\left(\sqrt{\frac{\boxed{A^2T_s}}{N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

- Symbol error probability

$$P_E = 1 - P_c = 1 - (1 - P_{E_1})^2 = 2P_{E_1} - P_{E_1}^2$$

$$\approx 2P_{E_1} = Q\left(\sqrt{\frac{A^2T_s}{N_0}}\right) = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Signal Space

- Let a set of signals $\{\phi_i(t)\}_{i=1}^{K}$ be orthonormal, i.e,

$$\langle \phi_i, \phi_j \rangle = \int_{T_s} \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

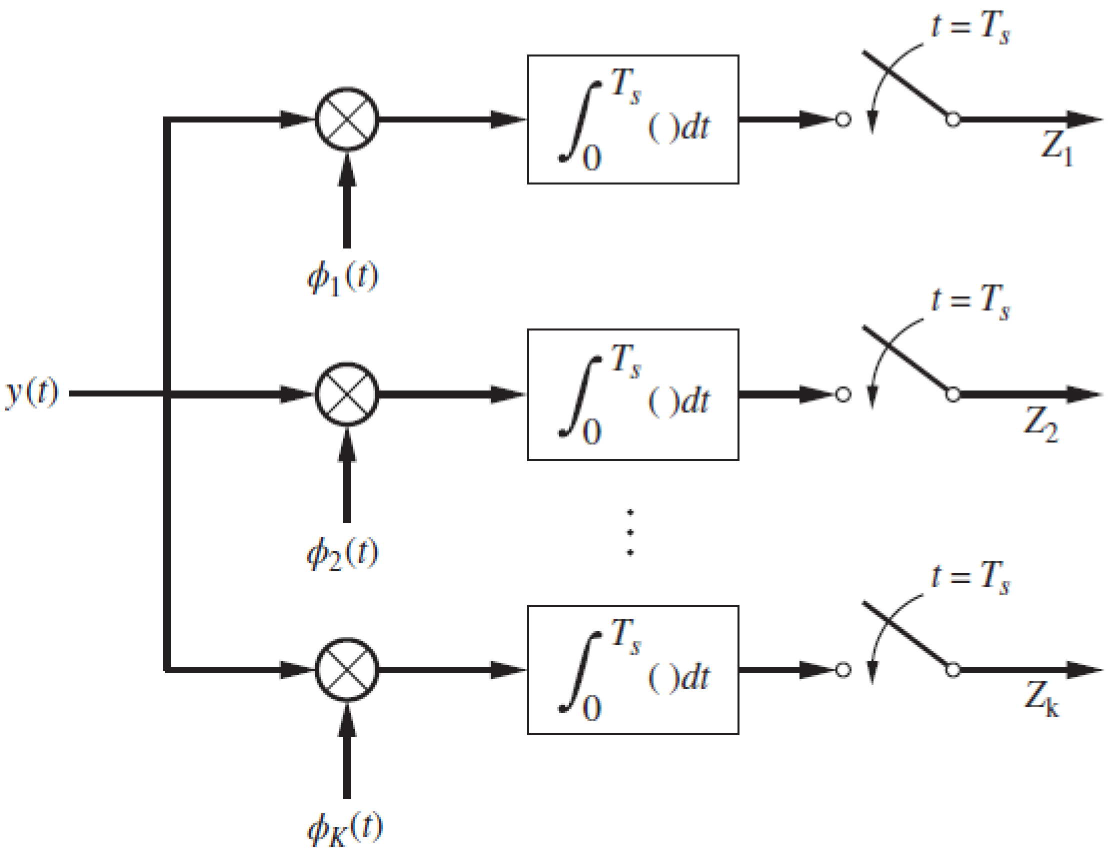
- We form M-ary signals $s_1(t), s_2(t), \ldots, s_M(t)$

$$s_i(t) = \sum_{j=1}^{K} a_{ij}\phi_j(t)$$

- For instance, QPSK: $M = 4, K = 2$
- Noise corrupted signal is received, $y(t) = s_i(t) + n(t)$

Signal Space

- The receiver consists of a bank of K correlator (=inner productor)



Note: $y(t) = s_i(t) + n(t)$ where $n(t)$ is white Gaussian noise.

Signal Space

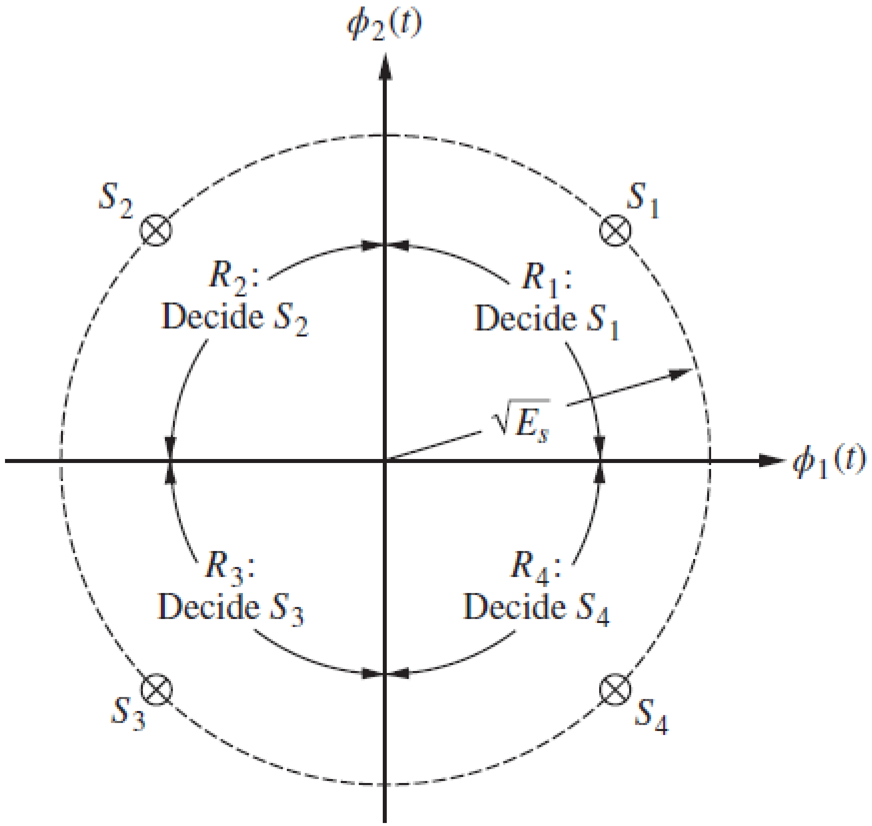
- As ϕ_j are orthonormal each other,

$$Z_j = a_{ij} + N_j$$

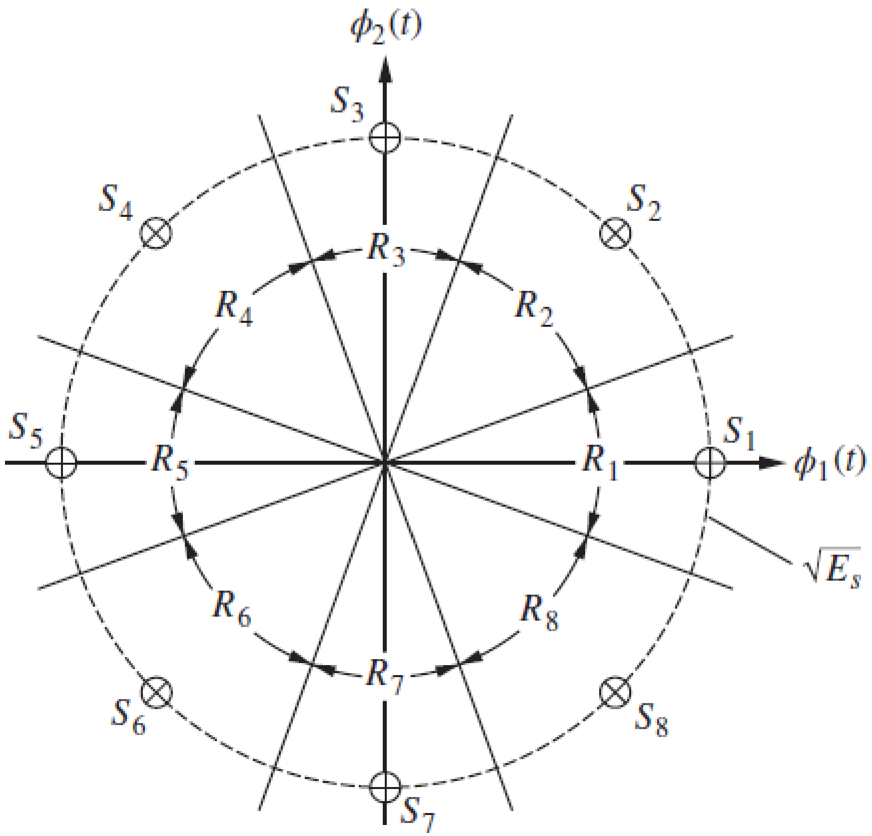
  where N_j is a Gaussian RV with mean zero and variance $N_0/2$

- In addition, $N_i, N_j, j \neq i$ are uncorrelated, i.e., $\mathbb{E}[N_i N_j] = 0$
- As they are Gaussian, uncorrelated $\Leftrightarrow$ independence
- **The signal space representation preserves all the information required to detect symbols**

Signal Space

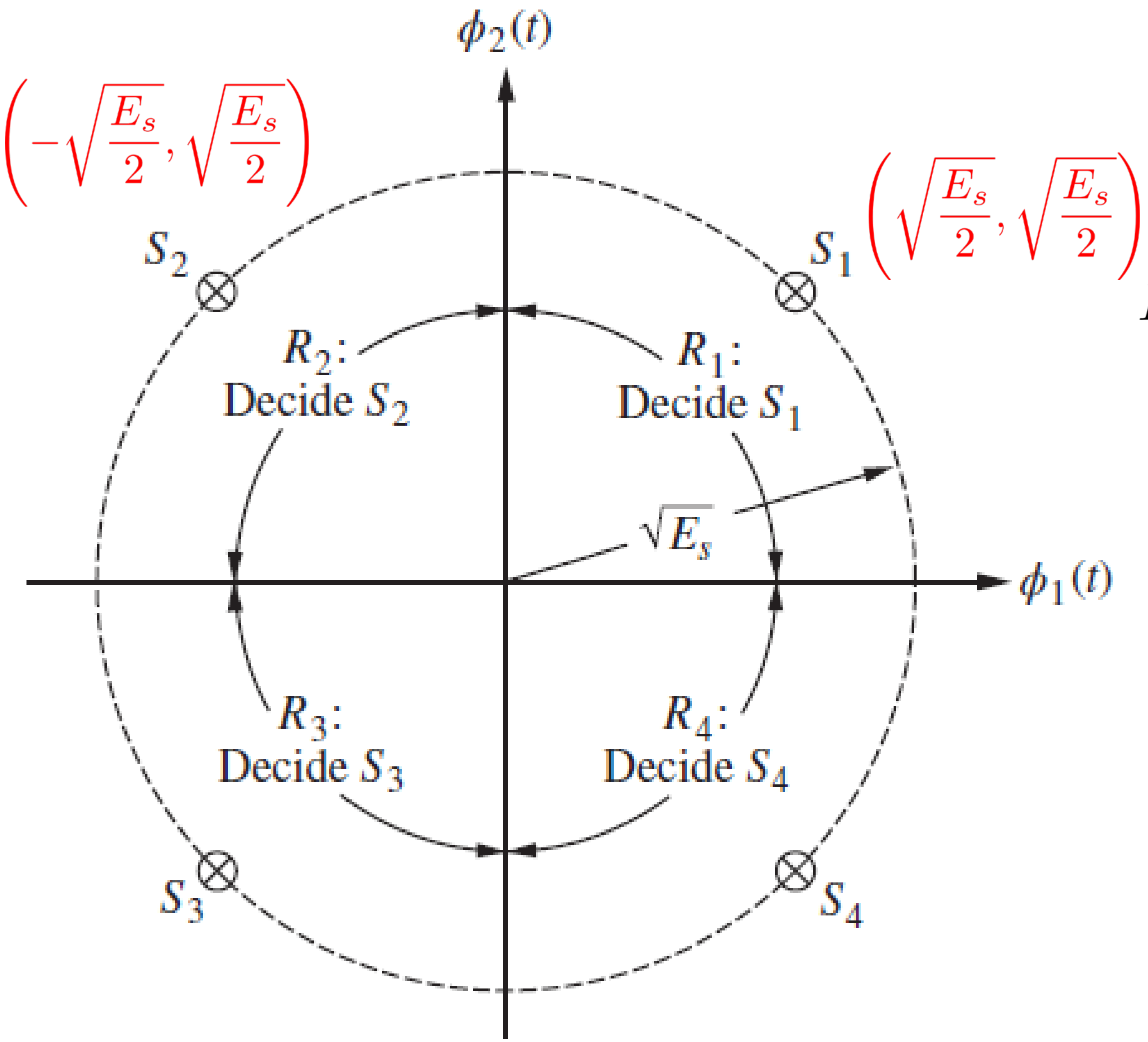


QPSK



8-PSK

Signal Space (QPSK)



Noise variance in each dimension $= N_0/2$

$$
\begin{aligned}
P_E &= \Pr[Z \in R_2 \cup R_3 \cup R_4 | S_1 \text{ sent}] \\
&\leq \Pr[Z \in R_2 \cup R_3 | S_1 \text{ sent}] + \Pr[Z \in R_3 \cup R_4 | S_1 \text{ sent}] \\
&= P_{E_1} + P_{E_2}
\end{aligned}
$$

$$
P_{E_1} = Q\left(\frac{\sqrt{E_s/2}}{\sqrt{N_0/2}}\right) = Q(\sqrt{E_s/N_0})
$$

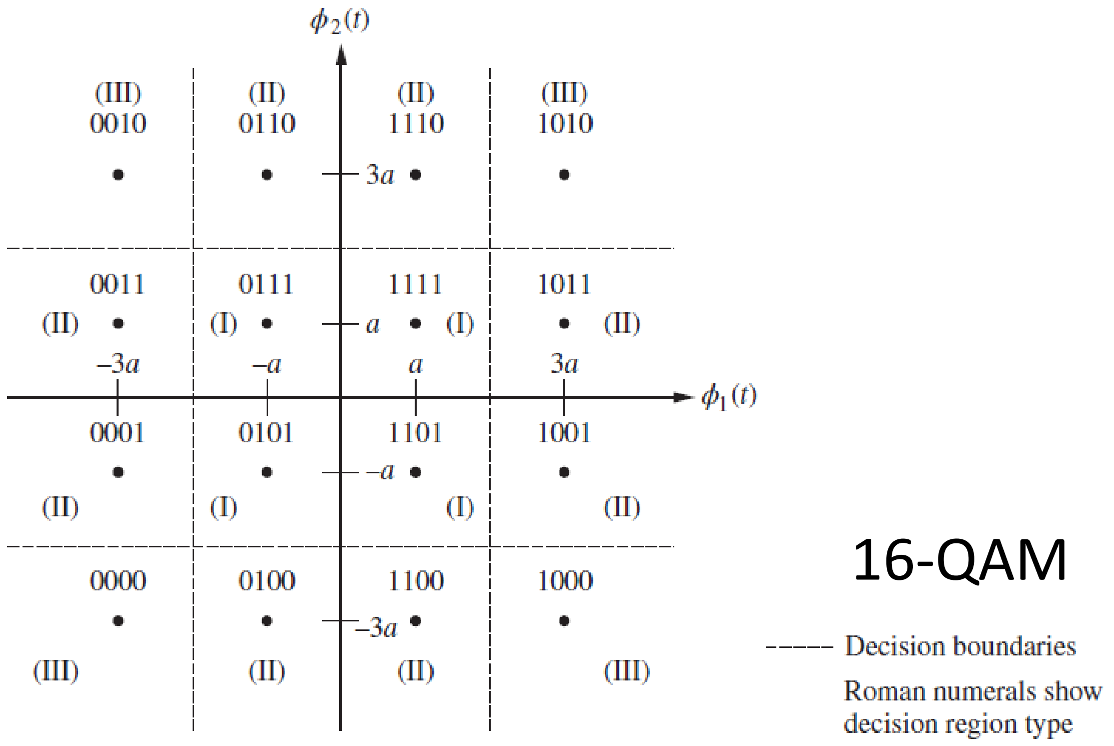
$$
\Rightarrow P_E = 2Q(\sqrt{E_s/N_0})
$$

Same as what we have previously!

Signal Space (QAM)

- Quadrature amplitude modulation (QAM): convey bits on the **amplitude** of carriers

$$s_i(t) = \sqrt{\frac{2}{T_s}}\left(A_i \cos(2\pi f_c t) + B_i \sin(2\pi f_c t)\right)$$



16-QAM

M-PSK vs M-QAM

- QPSK is indeed the same as 4-QAM
- As M-PSK only changes phase of signals, signal space gets denser and denser quickly with M $\Rightarrow$ High-order M-PSK is not widely used
- However, M-QAM changes phase and amplitude both, signal space gets denser less quickly with M $\Rightarrow$ High-order M-QAM is often used
- 4-QAM: 2 bits
- 16-QAM: 4 bits
- 64-QAM: 6 bits

Gray Code

- As tail of Gaussian pdf exponentially decreases, the most probable symbol error is mistaking an adjacent signal point
- Example 1: Two symbols $s_1 = (0000)$ and $s_2 = (1111)$ are adjacent, mistaking an adjacent symbol means 4 bit error
- Example 2: Two symbols $s_1 = (0000)$ and $s_2 = (0100)$ are adjacent, mistaking an adjacent symbol means 1 bit error only
- We wish to minimize the "bit difference" of adjacent symbols so that mistaking an adjacent symbol only gives one bit error
- Frank Gray finds how to encode bits in this way $\Rightarrow$ **Gray code**

Gray Code

Table 10.2 Gray Code for $M = 8$

Digit	Binary code	Gray code
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

Note: The encoding algorithm is given in Problem 9.32.