Solution Set #1

Discussions are allowed and encouraged, but please write your own answers.

1. (5 pts each) Using the property of the unit impulse, calculate the following.

   (a) $\int_{-\infty}^{\infty} \cos(6t)\delta(t-3)dt$

   (b) $\int_{-\infty}^{\infty} \frac{10\delta(t-3)}{1+t}dt$

Answer

(a) By the sifting property, it is $\cos(6t)$ at $t = 3$, i.e., $\cos(18)$.

(b) Similarly, $\frac{5}{2}$.

2. (5 pts each) Prove the following three properties of the Fourier series.

   (a) If $x(t)$ is real, $X_n^* = X_{-n}$

   (b) If $x(t)$ is real and even, that is $x(t) = x(-t)$, X_n is purely real and even

   (c) If $x(t)$ is real and odd, that is $x(t) = -x(-t)$, X_n is purely imaginary and odd

Answer

(a)

$$X_n^* = \left(\frac{1}{T_0}\int_{T_0} x(t)e^{-jnw_0t}dt\right)^* = \frac{1}{T_0}\int_{T_0} x(t)e^{jnw_0t}dt = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(-n)w_0t}dt = X_{-n}.$$

(b)

$$\begin{aligned}
X_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jnw_0t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(nw_0t)dt - \underbrace{\frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(nw_0t)dt}_{=0 \text{ as } x(t) \text{ is even}} \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(nw_0t)dt \qquad \text{real as it is an integration of a real function} \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(-nw_0t)dt = X_{-n} \qquad \text{since } \cos(\cdot) \text{ is even}
\end{aligned}$$

(c)

$$
\begin{aligned}
X_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jnw_0 t}dt = \underbrace{\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(nw_0 t)dt}_{=0 \text{ as } x(t) \text{ is odd}} - \frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(nw_0 t)dt \\
&= -\frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(nw_0 t)dt \qquad \text{imaginary as the integration is real} \\
&= \frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(-nw_0 t)dt. \\
X_{-n} &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{jnw_0 t}dt = -\frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(-nw_0 t)dt \\
&= \frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(nw_0 t)dt = -X_n \qquad \text{from the last equation of } X_n
\end{aligned}
$$

3. (5 pts each) Let $y_s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s)$.
   (a) Prove that $y_s(t) = f_s \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}$.
   (b) Prove that $y_s(t) \longleftrightarrow Y_s(f) = \sum_{m=-\infty}^{\infty} e^{j2\pi m T_s f} = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$

Answer See Example 2.14 of the textbook.

4. (5 pts each) Find the Fourier transforms of the signals below. Assume $A, \tau > 0$.
   (a) $x_1(t) = A\exp(-t/\tau)u(t)$
   (b) $x_2(t) = A\exp(t/\tau)u(-t)$
   (c) $x_3(t) = x_1(t) - x_2(t)$
   (d) $x_4(t) = x_1(t) + x_2(t)$

Answer

(a)

$$
\begin{aligned}
X_1(f) &= \int_0^{\infty} A\exp(-t/\tau)e^{-j2\pi f t}dt = A\int_0^{\infty} e^{-(1/\tau + j2\pi f)t}dt \\
&= \left.\frac{Ae^{-(1/\tau+j2\pi f)t}}{-(1/\tau + j2\pi f)}\right|_0^{\infty} = \frac{A}{1/\tau + j2\pi f} = \frac{A\tau}{1 + j2\pi f\tau}
\end{aligned}
$$

(b) Since $x_2(t) = x_1(-t)$, using the time reversal property, $X_2(f) = X_1^*(f) = \frac{A\tau}{1-j2\pi f\tau}$

(c) Since $x_3(t) = x_1(t) - x_2(t)$, we can easily obtain the following by linearity. Detail calculations are omitted.

$$X_3(f) = \frac{-j4A\pi f\tau^2}{1 + (2\pi f\tau)^2}$$

(d) Since $x_4(t) = x_1(t) + x_2(t)$, we can easily obtain the following by linearity. Detail calculations are omitted.

$$X_4(f) = \frac{2A\tau}{1 + (2\pi f\tau)^2}$$

5. (5 pts each) Find the inverse Fourier transforms of the spectra below. For (a), do not use the Fourier transform table. You may use the table for (b) and (c).

(a) $X_1(f) = \prod(f/2B)$

(b) $X_2(f) = 2\cos(2\pi f)\prod(f)\exp(-j4\pi f)$

(c) $X_3(f) = \left[\prod\left(\frac{f+4}{2}\right) + \prod\left(\frac{f-4}{2}\right)\right]\exp(-j8\pi f)$

Answer

(a)

$$\begin{aligned}
x_1(t) &= \int_{-\infty}^{\infty} \prod\left(\frac{f}{2B}\right) e^{j2\pi ft} df = \int_{-B}^{B} e^{j2\pi ft} df \\
&= \int_{-B}^{B} \cos(2\pi ft) + j\sin(2\pi ft) df = \int_{-B}^{B} \cos(2\pi ft) df \\
&= \frac{1}{2\pi t}\sin(2\pi ft)\bigg|_{-B}^{B} = \frac{1}{\pi t}\sin(2\pi Bt) \\
&= \frac{2B}{\pi(2Bt)}\sin(\pi(2Bt)) = 2B\mathrm{sinc}(2Bt)
\end{aligned}$$

(b) The inverse of the unit rectangular function is $\mathrm{sinc}(t)$. Therefore, by the time translation property and convolution-multiplication duality, we have

$$\begin{aligned}
x_2(t) &= \mathrm{sinc}(t-2) * [\delta(t-1) + \delta(t+1)] \\
&= \mathrm{sinc}(t-3) + \mathrm{sinc}(t-1)
\end{aligned}$$

(c) The inverse of $\prod(f/2)$ is $2\mathrm{sinc}(2t)$. By the modulation and time translation properties,

$$\begin{aligned}
\prod(f/2) &\longleftrightarrow 2\mathrm{sinc}(2t) \\
\prod\left(\frac{f+4}{2}\right) + \prod\left(\frac{f-4}{2}\right) &\longleftrightarrow 4\mathrm{sinc}(2t)\cos(2\pi 4t) \\
\left[\prod\left(\frac{f+4}{2}\right) + \prod\left(\frac{f-4}{2}\right)\right]\exp(-j8\pi f) &\longleftrightarrow 4\mathrm{sinc}(2(t-4))\cos(2\pi 4(t-4))
\end{aligned}$$

6. (5 pts each)

(a) Show that the Fourier transform of $x(t) * y(t) * z(t)$ is $X(f)Y(f)Z(f)$, where $X(f), Y(f), Z(f)$ are Fourier transforms of $x(t), y(t), z(t)$, respectively. Do not use Fourier transform tables.

(b) What is the Fourier transform of $x(at+b), a \neq 0$? Represent your answer in terms of $X(f)$. Do not use Fourier transform tables.

Answer

(a)

$$\begin{aligned}\mathfrak{F}[x(t) * y(t)] &= \int_{-\infty}^{\infty} x(t) * y(t) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} y(t-\tau) e^{-j2\pi f (t-\tau)} dt}_{\text{letting } t' := t-\tau, \text{ it is } Y(f)} e^{-j2\pi f \tau} d\tau \\ &= Y(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau = Y(f) X(f).\end{aligned}$$

Since $x(t) * y(t) * z(t) = [x(t) * y(t)] * z(t)$, repeating the argument again, we can prove the statement.

(b)

$$\begin{aligned}\int_{-\infty}^{\infty} x(at+b) e^{-j2\pi f t} dt &= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f (\tau - b)/a} \frac{1}{|a|} d\tau \\ &= e^{j2\pi f b/a} \frac{1}{|a|} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau / a} d\tau \\ &= e^{j2\pi f b/a} \frac{1}{|a|} X(f/a).\end{aligned}$$