Digital Signal Processing (Lecture Note 3)

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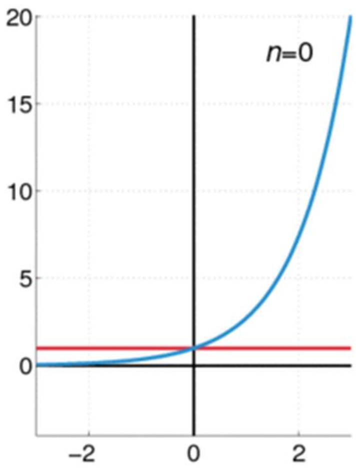
Fourier Series

Taylor Series

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots.$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

The exponential function e^x (in blue), and the sum of the first n+1 terms of its Taylor series at 0 (in red)



Fourier Series

- From Fourier series analysis, we know that **any periodic waveform** can be **represented by a series of sinusoids** that are at the same frequency as, or multiples of, the waveform frequency.
- One of advantages is that we can represent both continuous and **discontinuous** functions such as the square wave or saw-tooth wave forms.
  - Such functions are **not expandable** in Taylor series because these functions have a number of discontinuities and thus their derivatives are not defined.
- Fourier series is a **bridge** through which we can walk to Fourier transformation.

Definition of Fourier Series

- Sometimes we may need to express a function f(x) as an infinite series of sine and cosine function;

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

  - The series in the equation is called a trigonometric series or Fourier series.
  - It turns out that expressing a function as a Fourier series is sometimes more advantageous than expanding it as a power series.
    - ✓ Expressing heartbeat is an example.

Coefficients a_n and b_n

- Assuming that the trigonometric series converge and has a continuous function f(x) as its sum on [-π, π], that is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad -\pi \leqslant x \leqslant \pi$$

  - For Finding a_0,

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,dx$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\,dx$$

But

$$\int_{-\pi}^{\pi} \cos nx\,dx = \frac{1}{n} \sin nx \Big]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0$$

because n is an integer. Similarly, $\int_{-\pi}^{\pi} \sin nx\,dx = 0$. So

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$ Signal Average

– To determine a_n for n≥1, multiply both sides of the Fourier series equation by cos(mx), where m is an integer and m≥1 and integrate term-by-term from –π to π:

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]\cos mx\,dx$$

$$= a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + \sum_{n=1}^{\infty}b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = a_m\pi$$

$$\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = 0 \qquad \text{for all } n \text{ and } m$$

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m \end{cases}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2, 3, \ldots}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, 3, \ldots}$$

General Form of Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

where, $\omega_0 = \frac{2\pi}{T}$ and T is the period of f(t)

Symmetry Properties

Things to watch out for when computing the Fourier coefficients:

- if $x(t)$ is an even function, i.e., $x(t) = x(-t)$ for all t, then all its sine Fourier coefficients are zero:

$$b_k = \frac{2}{T}\int_{-2/T}^{T/2} x(t)\sin(k\omega_0 t)dt = 0$$

- if $x(t)$ is an odd function, i.e., $x(t) = -x(-t)$, then all its cosine Fourier coefficients are zero:

$$a_k = \frac{2}{T}\int_{-2/T}^{T/2} x(t)\cos(k\omega_0 t)dt = 0,$$

and

$$a_0 = \frac{1}{T}\int_{-2/T}^{T/2} x(t)dt = 0$$

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Energy Spectrum

$$P(t) = a_0 + a_1\cos\left(\frac{\pi t}{L}\right) + b_1\sin\left(\frac{\pi t}{L}\right) + a_2\cos\left(\frac{2\pi t}{L}\right) + b_2\sin\left(\frac{2\pi t}{L}\right) + \cdots$$

- The nth term of the Fourier series, that is,

$$\boldsymbol{a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)}$$

  - It is called the nth harmonic of P(t).
- The amplitude of the nth harmonic is

$$\mathrm{A_n} = \sqrt{\mathrm{a_n^2 + b_n^2}}$$

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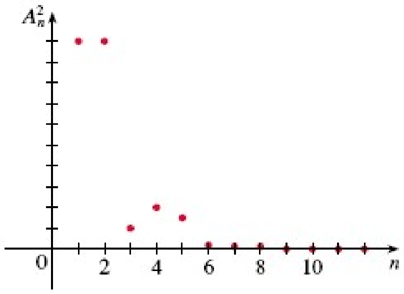
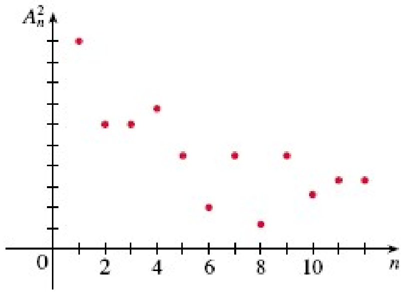
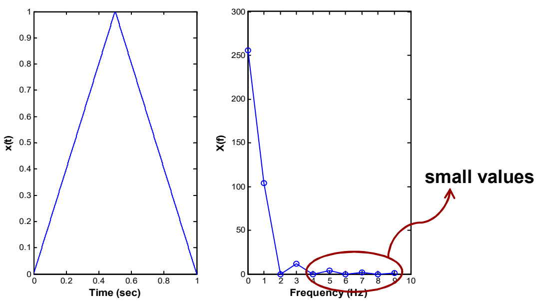
- $A_n^2 = a_n^2 + b_n^2$ is sometimes called energy of the *n*th harmonic.
- The graph of the sequence of $\{A_n^2\}$ is called the energy spectrum of the P(t) and shows at a glance the relative sizes of the harmonics.





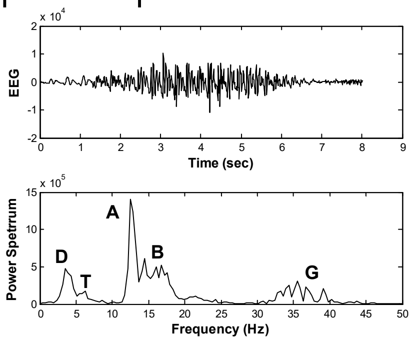
Spectrum Information

- Spectrum information is usually presented as a frequency plot
  - A plot of sine and cosine amplitude vs. component number (or the equivalent frequency).
  - To convert from component number to frequency, $f = \frac{m}{T}$ where *T* is a period of the fundamental and *m* is the component number.



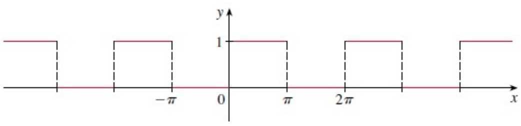
Why bother with spectrum information?

- New information can be obtained
- Example: Segment of an EEG signal and the resultant power spectrum.



Type	Frequency	Location	Normally	Pathologically
Delta	up to 4 Hz	frontally in adults, posteriorly in children (high amplitude waves)	• adults slow wave sleep • in babies	• subcortical lesions • diffuse lesions • metabolic encephalopathy hydrocephalus • deep midline lesions
Theta	4 – 7 Hz		• young children • drowsiness or arousal in older children and adults • idling	• focal subcortical lesions • Metabolic encephalopathy • deep midline disorders • some instances of hydrocephalus
Alpha	8 – 12 Hz	posterior regions of head, both sides, higher in amplitude on dominant side.	• relaxed/reflecting • closing the eyes	• coma
Beta	12 – 30 Hz	both sides, symmetrical distribution, most evident frontally; low amplitude waves	• alert/working • active, busy or anxious thinking, active concentration	• benzodiazepines
Gamma	30 – 100 Hz +		• certain cognitive or motor functions	

Example 1



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

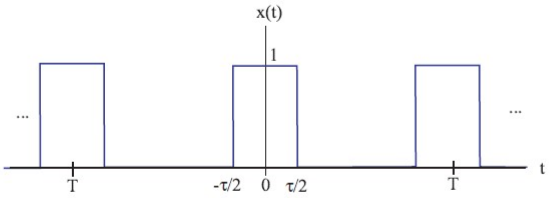
$$a_n = 0$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\boxed{S_n(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin(2k-1)x}$$

$$S_n(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \cdots + \frac{2}{n\pi}\sin nx$$

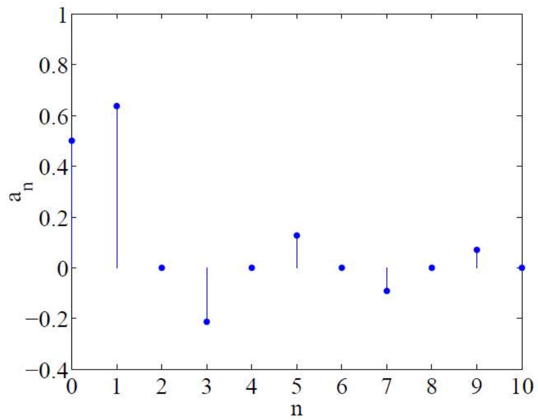
Example 2



$$\boxed{\hat{x}(t) = a_0 + \sum_{n=0}^{N} a_n cos\left(n\Omega_0 t\right)}$$

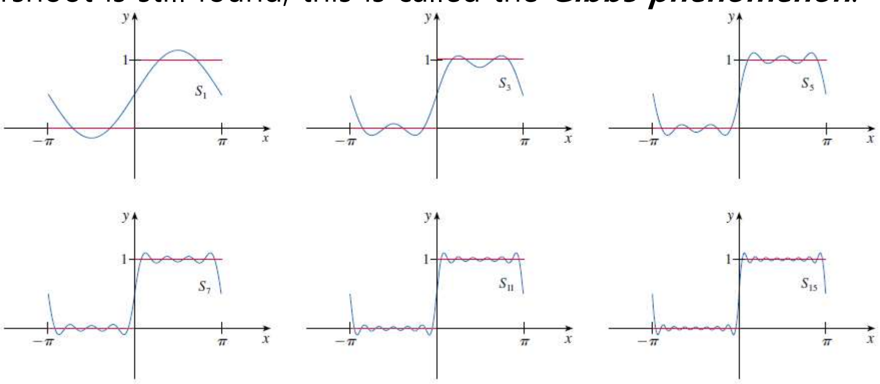
$$a_n = \frac{2\tau}{T}\frac{sin\left(n\Omega_0\tau/2\right)}{n\Omega_0\tau/2}, \quad n = 1, 2, \ldots$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \tau/T$$



Gibbs Phenomenon

– When a sudden change of amplitude occurs in a signal and the attempt is made to represent it by a **finite** number of terms in a Fourier series, the overshoot at the corners (at the points of abrupt change) is always found. As the number of terms is increased, the overshoot is still found; this is called the ***Gibbs phenomenon***.



Complex Form of Fourier Series

- It allows the magnitude and phase of each frequency component to be easily calculated.
- From Euler's formula $\cos\theta = \frac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right)$ $\sin\theta = \frac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)$

$$\cos(n\omega_0 t) = \frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2} \qquad \sin(n\omega_0 t) = \frac{e^{jn\omega_0 t}-e^{-jn\omega_0 t}}{j2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\omega_0 t}-e^{-jn\omega_0 t}}{j2}\right]$$

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$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - jb_n}{2}\right] e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \left[\frac{a_n + jb_n}{2}\right] e^{-jn\omega_0 t}$$

- a_n and b_n are only defined for positive values of n
- Let's sum over negative integers in the second summation:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - jb_n}{2}\right] e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} \left[\frac{a_{-n} + jb_{-n}}{2}\right] e^{jn\omega_0 t}$$

- Let's assume that a_n and b_n are defined for both positive and negative n. In this case, we find that

$$a_n = a_{-n} \qquad\qquad b_n = -b_{-n}$$

$$\begin{aligned} a_{-n} &= \tfrac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(-n\Omega_0 t)\,dt \\ &= \tfrac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\Omega_0 t)\,dt \\ &= a_n \end{aligned} \qquad \begin{aligned} b_{-n} &= \tfrac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(-n\Omega_0 t)\,dt \\ &= -\tfrac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\Omega_0 t)\,dt \\ &= -b_n \end{aligned}$$

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$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - jb_n}{2}\right]e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty}\left[\frac{a_n - jb_n}{2}\right]e^{jn\omega_0 t}$$

- Define

$$c_0 \equiv a_0 \qquad c_n \equiv \frac{a_n - jb_n}{2}$$

- And then we have a final result:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}$$

- This is called the **complex form of the Fourier Series**.
- Because $a_n = a_{-n}$ and $b_n = -b_{-n}$,

$$c_{-n} = c_n^* \longrightarrow \begin{array}{l} |c_{-n}| = |c_n| \\ \angle c_{-n} = -\angle c_n \end{array}$$

- We must find formulas for finding the c_n given x(t).

$$\int_0^T e^{jk\omega_0 t}\,dt = \begin{cases} T, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Consider

$$f(t)e^{-jk\omega_0 t} = \sum_{n=\infty}^{\infty} c_n e^{jn\omega_0 t} e^{-jk\omega_0 t}$$

$$\int_0^T f(t)e^{-jk\omega_0 t}\,dt = \int_0^T \sum_{n=\infty}^{\infty} c_n e^{jn\omega_0 t} e^{-jk\omega_0 t}\,dt$$

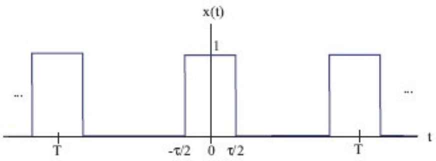
$$\int_0^T f(t)e^{-jk\omega_0 t}\,dt = \sum_{n=\infty}^{\infty} c_n \int_0^T e^{j(n-k)\omega_0 t}\,dt = Tc_n$$

$$\boxed{c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt}$$

$$a_n = c_n + c_{-n} \quad \text{for } n = 0, 1, 2\cdots$$

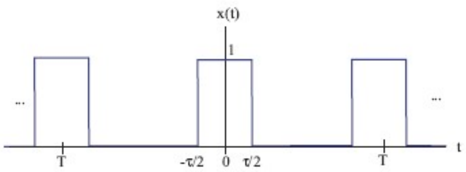
$$b_n = j(c_n - c_{-n}) \quad \text{for } n = 1, 2\cdots$$

Example

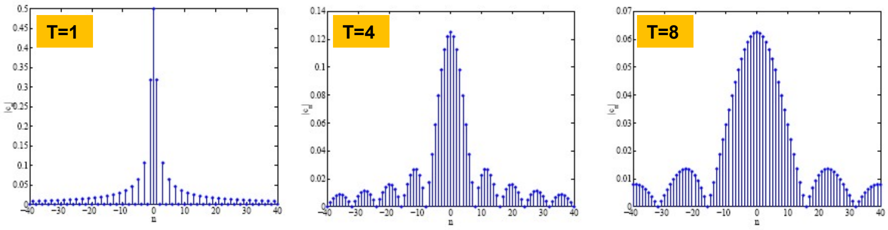


Fourier Transform

What happens as T increase?



$$c_n = \frac{\tau}{T}\frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\frac{n\omega_0\tau}{2}} = \frac{\tau}{T}\frac{\sin\left(\frac{\pi n\tau}{T}\right)}{\frac{\pi n\tau}{T}} = \tau\frac{\sin\left(\frac{\pi n\tau}{T}\right)}{\pi n\tau}$$



- What happens if T is getting very large?
  - As the period increases, successive Fourier Series coefficients represent more closely spaced frequencies because
  - When the period goes toward infinity, C_n becomes a continuous waveform;

$$\lim_{T\to\infty}\left[2\pi\left(\frac{n}{T}\right)\right] = 2\pi f$$

$$\lim_{T\to\infty}\left(\frac{1}{T}\right) = df = \frac{d\omega}{2\pi}$$

$$c_n\big|_{T\to\infty} = C(\omega)d\omega = \lim_{T\to\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt\right] = \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t) = \int_{-\infty}^{\infty} C(\omega)e^{j\omega t}dt$$

Aperiodic Functions

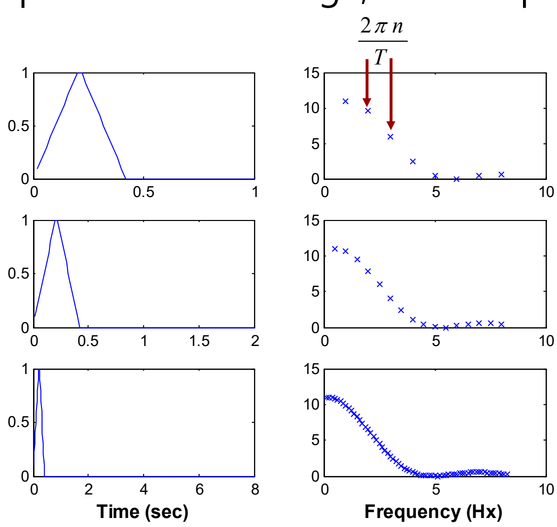
- If the function is not periodic, it can still be accurately decomposed into sinusoids if it is aperiodic
  - It exists only for a well-defined period of time.
  - The only difference is that, theoretically, the sinusoidal components can exist at all frequencies, not just multiple frequencies or harmonics.
- Recall

$$\lim_{T\to\infty}\left[2\pi\left(\frac{n}{T}\right)\right]=2\pi f$$

$$\lim_{T\to\infty}\left(\frac{1}{T}\right)=df=\frac{d\omega}{2\pi}$$

$$c_n\big|_{T\to\infty}=C(\omega)d\omega=\lim_{T\to\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-jn\omega_0 t}dt\right]=\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

- As the period gets longer, approaching an aperiodic function, the spectral shape does not change, but the points get closer together.



Definition of Fourier Transform

- Fourier Transform

$$FT\left[x(t)\right] \quad \boxed{X(j\omega) = \mathcal{F}(x(t)) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt}$$

- Inverse Fourier Transform

$$FT^{-1}\left[x(t)\right] \quad \boxed{x(t) = \mathcal{F}^{-1}(X(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\ \omega)\, e^{j\omega t}\, d\omega}$$

The variable ω is called a ***continuous frequency*** variable

- Sufficient Existence Condition

$$\lim_{T\to\infty}\int_{-T}^{T} |x(t)|\, dt < \infty$$

Crucial Facts

- The frequency domain representation provides how much which frequencies exist in the signal
- The FS is **discrete in frequency domain** because it is the discrete set of exponentials, i.e., integer multiples of ω_0, that make up the signal.
- The FT is **continuous in frequency domain** because exponentials with continuous frequencies are required to reconstruct a non-periodic signal.
- From the duality property of FT, the same rule is applied to time signals.
  - If $FT\left[f(t)\right]=F(\omega)$, then $FT\left[F(t)\right]=2\pi f(-\omega)$

Properties of Fourier Transform

- Linearity

Linearity Property: If $\mathcal{FT}[f(t)] = F(\omega)$ and $\mathcal{FT}[g(t)] = G(\omega)$, then

$$\mathcal{FT}[\alpha f(t) + \beta g(t)] = \alpha F(\omega) + \beta G(\omega)$$

Proof: Integration is a linear operation. That is
$\int_{-\infty}^{\infty}[\alpha f(t) + \beta g(t)]e^{-i\omega t}dt = \alpha\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt + \beta\int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt.$

- Duality

Dual Property If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[F(t)] = 2\pi f(-\omega)$$ *Yes, weird! Watch out for the $-\omega$*

Proof: We know that $f(t)$ is the $\mathcal{FT}^{-1}[F(\omega)]$ — that is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

But t and ω are just symbols, so replace t with $(-\omega)$ and ω with t.
NB! Replacing ω with t does NOT flip the limits of integration. So

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{it(-\omega)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt$$

$\Rightarrow 2\pi f(-\omega) = \mathcal{FT}[F(t)]$ *"comme il faut", as Mrs Fourier would say*

- Similarity Property

Parameter Scaling or Similarity Property: If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

Proof: The appearance of $|a|$ hints that this proof needs to consider the ranges $a > 0$ and $a < 0$ separately.
For $a > 0$:

$$\mathcal{FT}[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-i\omega t}dt\ .$$

Write $p = at$, and note that the signs on the limits do NOT change because a is positive. Then

$$\mathcal{FT}[f(at)] = \frac{1}{a}\int_{-\infty}^{\infty} f(p)e^{-i(\omega/a)p}dp = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

For $a < 0$:
Substitute $p = -|a|t$, and remember to change signs on the limits — when $t = \infty$, $p = -\infty$:

$$\mathcal{FT}[f(at)] = \frac{1}{-|a|}\int_{\infty}^{-\infty} f(p)e^{-i(\omega/a)p}dp = \frac{1}{|a|}\int_{-\infty}^{\infty} f(p)e^{-i(\omega/a)p}dp$$
$$= \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

So, for both cases, one can write

$$\mathcal{FT}[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

- Time Shifting

Parameter Shifting Property: If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[f(t-a)] = \exp(-i\omega a)F(\omega)$$

Proof:

$$\mathcal{FT}[f(t-a)] = \int_{-\infty}^{\infty} f(t-a)e^{-i\omega t}dt\ .$$

Substitute $p = t - a$. Then

$$\begin{aligned}\mathcal{FT}[f(t-a)] &= \int_{-\infty}^{\infty} f(p)e^{-i\omega(p+a)}dp = e^{-i\omega a}\int_{-\infty}^{\infty} f(p)e^{-i\omega p}dp \\ &= e^{-i\omega a}F(\omega)\end{aligned}$$

- Frequency Shifting

Frequency Shifting Property: If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[f(t)\exp(\pm i\omega_S t)] = F(\omega \mp \omega_S)$$

Proof:

$$\begin{aligned}\mathcal{FT}[f(t)\exp(\pm i\omega_S t)] &= \int_{-\infty}^{\infty} f(t)e^{\pm i\omega_S t}e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-i(\omega \mp \omega_S)t}dt \\ &= F(\omega \mp \omega_S)\end{aligned}$$

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- Amplitude Modulation by a Cosine

Amplitude modulation by a cosine If $\mathcal{FT}[f(t)] = F(\omega)$ then

$$\mathcal{FT}[f(t)\cos\omega_0 t] = \frac{1}{2}[F(\omega-\omega_0) + F(\omega+\omega_0)]$$

Proof: Write

$$\cos\omega_0 t = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

then use the Frequency shifting property.

- Amplitude Modulation by a Sine

$$FT\left[f(t)\sin(\omega_0 t)\right] = \frac{1}{j2}\left[F(\omega-\omega_0) - F(\omega+\omega_0)\right]$$

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- Differentiation wrt time

Differentiation Property in time: If $\mathcal{FT}\left[f(t)\right] = F(\omega)$, then

$$\mathcal{FT}\left[\frac{d^n}{dt^n}f(t)\right] = (i\omega)^n F(\omega)$$

Proof: It is so tempting to start by writing $\mathcal{FT}\left[\frac{d^n}{dt^n}f(t)\right] = \int_{-\infty}^{\infty}\left[\frac{d^n}{dt^n}f(t)\right]e^{-i\omega t}dt$. Resist. Instead, write

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

Now differentiate w.r.t. t. Because the integral is w.r.t. ω, the differentation can move through the integral sign:

$$\begin{aligned}\frac{d^n}{dt^n}f(t) &= \frac{1}{2\pi}\frac{d^n}{dt^n}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{d^n}{dt^n}e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)(i\omega)^n e^{i\omega t}d\omega\end{aligned}$$

This last expression says that

$$\frac{d^n}{dt^n}f(t) = \mathcal{FT}^{-1}\left[F(\omega)(i\omega)^n\right] \quad \Rightarrow \mathcal{FT}\left[\frac{d^n}{dt^n}f(t)\right] = F(\omega)(i\omega)^n .$$

- Differentiation wrt frequency

Differentiation Property in frequency: If $\mathcal{FT}\left[f(t)\right] = F(\omega)$, then

$$\mathcal{FT}\left[(-it)^n f(t)\right] = \frac{d^n}{d\omega^n}F(\omega)$$

- Time-domain Convolution

$$z(t) = x(t) * y(t) \Leftrightarrow Z(\omega) = X(\omega)Y(\omega)$$

- Time-domain Multiplication

$$z(t) = x(t)y(t) \Leftrightarrow Z(\omega) = \frac{1}{2\pi}X(\omega) * Y(\omega)$$

- Parseval's Theorem
  - Power computed in either domain equals the power in the other.

$$\int_{-\infty}^{\infty}\left|f(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\omega$$

- Symmetry
  - The FT of a real signal is Hermitian.

$$S(-\omega) = S^*(\omega)$$

$S^*(\omega)$ The complex conjugate of S and S(k) is **Hermitian Function**

  - **<u>The real part is even and the imaginary part is odd</u>**

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} s(t) e^{-2\pi i \omega t} dx \\ &= \int_{-\infty}^{\infty} \left[s_e(t) + s_o(t) \right] \cdot \left[\cos(2\pi\omega t) - i \sin(2\pi\omega t) \right] dt \\ &= \int_{-\infty}^{\infty} s_e(t) \cos(2\pi\omega t)\, dt - j \int_{-\infty}^{\infty} s_o(t) \sin(2\pi\omega t)\, dt \end{aligned}$$

  - In order to compute the FT of a real signal, it suffices to know one half-plane. The other half-plane can be computed using the property.
  - If a function is real and even, its FT is real and even, whereas if a function is real and odd, its FT is imaginary and odd.

- Symmetry (cont.)
  - If x(t) is real valued,

$$X(-\omega) = X^*(\omega) \quad \text{Hermitian Conjugate Symmetry}$$

$$\Re\{X(-\omega)\} = \Re\{X(\omega)\}$$

$$\Im\{X(-\omega)\} = -\Im\{X(\omega)\}$$

$$|X(-\omega)| = |X(\omega)|$$

$$\angle X(-\omega) = -\angle X(\omega)$$

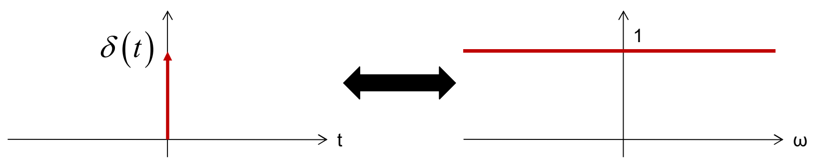
- Conjugation

$$y(t) = x^*(t) \Leftrightarrow Y(\omega) = X^*(-\omega)$$

$$\text{If } g(t) \Leftrightarrow G(\omega),\ \mathrm{g}^*(t) \Leftrightarrow G^*(-\omega)$$

Essential FT Pairs

- Dirac Delta and Sinusoid Functions



$$\delta(t-t_0)=0 \quad \text{for } t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\,dt = e^{-j\omega \cdot 0} = 1$$

$$FT\left[\cos(\omega_0 t)\right] = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)\cdot e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}+e^{-j(\omega+\omega_0)t}\,dt$$

$$= \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

Using Dual property of FT

$$FT\left[\sin(\omega_0 t)\right] = \frac{1}{j2}\int_{-\infty}^{\infty}\left(e^{j\omega_0 t}-e^{-j\omega_0 t}\right)\cdot e^{-j\omega t}\,dt$$

$$= \frac{1}{j2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}-e^{-j(\omega+\omega_0)t}\,dt$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$

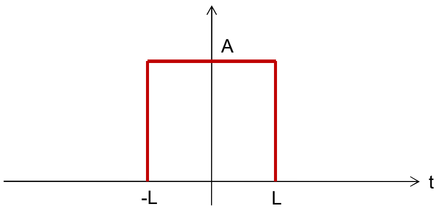
Using Dual property of FT

Dual Property If $\mathcal{FT}[f(t)] = F(\omega)$, then

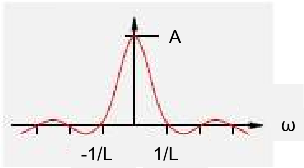
$\mathcal{FT}[F(t)] = 2\pi f(-\omega)$ *Yes, weird! Watch out for the* $-\omega$

- Rectangular and Sinc functions

$$f(t) = A \cdot rect\left(\frac{t}{2L}\right) = \begin{cases} A & for \quad |t| \le L \\ 0 & for \quad |t| > L \end{cases}$$



$$f(t) = A \sin c(Lt) = \begin{cases} A\dfrac{\sin(\pi L t)}{\pi L t} & for \quad t \ne 0 \\ A & for \quad t = 0 \end{cases}$$



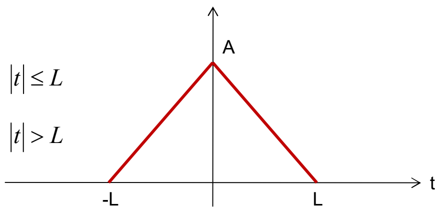
– Fourier Transform of the Functions

$$f(t) \Longleftrightarrow F(\omega)$$

$$A \cdot rect\left(\frac{t}{2L}\right) \qquad 2AL \cdot \sin c\left(\frac{L\omega}{\pi}\right)$$

- Triangular function

$$f(t) = A \cdot tri\left(\frac{t}{2L}\right) = \begin{cases} A\left(1 - \frac{|t|}{L}\right) & for \quad |t| \le L \\ 0 & for \quad |t| > L \end{cases}$$



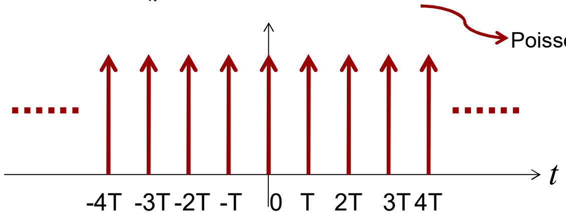
$$f(t) \Longleftrightarrow F(\omega)$$

$$A \cdot tri\left(\frac{t}{2L}\right) \qquad AL \cdot \sin c^2\left(\frac{L\omega}{2\pi}\right)$$

- Pulse Train

$$p_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{\frac{j2\pi nt}{T}}$$

Poisson Sum Formula



Proof : Fourier Series of $p_T(t)$

$$c_n = \frac{1}{T}\int_{-1/T}^{1/T} p_T(t) e^{-jn\omega_0 t} dt = \frac{1}{T}\int_{-1/T}^{1/T} \sum_{k=-\infty}^{\infty} \delta(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}$$

$$p_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Dual form of the Poisson formula is

$$\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right) = \frac{T}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\omega T} \qquad t \leftarrow \omega, \ \ T \leftarrow \frac{2\pi}{T}$$

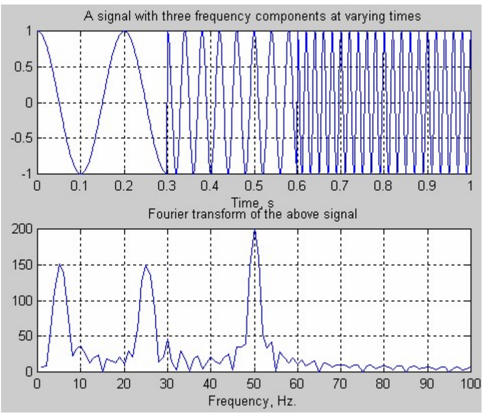
Fourier Transform of $p_T(t)$

$$P(\omega) = \int_{-\infty}^{\infty} p_T(t) e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt$$
$$= \sum_{n=-\infty}^{\infty} e^{jn\omega T} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

➔The FT of an impulse train in t is an impulse train in ω, up to scale factor.
➔ If the period of the given function is T, the period of the transform is 2π/T.

Limitation of FT

- Perfect knowledge of what frequencies exist, but no information about where these frequencies are located in time.



From lecture note of Dr. Robi Polikar, Rowan University, USA