Digital Signal Processing (Lecture Note 10)

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Structure for Discrete-Time Systems (Chap. 6)

Introduction

- For implementation, the difference equation or the system function representation must be converted to an algorithm or structure that can be realized in the desired technology.

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a z^{-1}}, \qquad |z| > |a|$$

$$h[n] = b_0 a^n u[n] + b_1 a^{n-1} u[n-1]$$

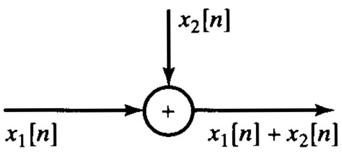
$$y[n] - a y[n-1] = b_0 x[n] + b_1 x[n-1]$$

  - If $x[n] = 0$ and $y[n] = 0$ for n<0, this system will be linear and time invariant and we can use the recurrence formula for computing the output and implementation.

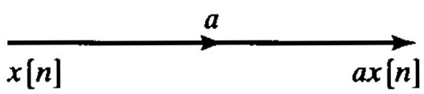
Block Diagram Representation

Basic Operations of linear constant-coefficient difference equation

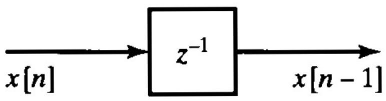
(1) Addition operator



(2) Multiplication operator

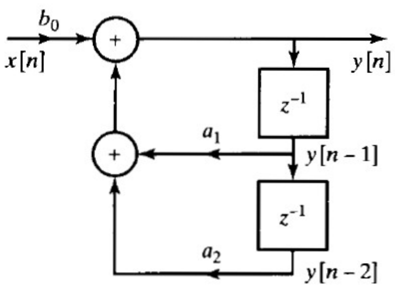


(3) Unit delay operator

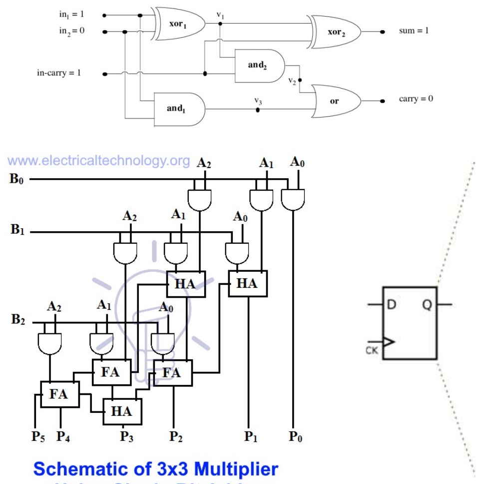


- Example: Block diagram representation of a Linear constant-coefficient difference equation

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n].$$



Corresponding DSP HW Components



Schematic of 3x3 Multiplier Using Single-Bit Adder

- Example: Represent the linear constant-coefficient difference equation by using block diagrams

y[n]=3y[n-3]+4y[n-2]-y[n-1]+15x[n]

Direct Form I

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}.$$

$$H(z) = H_2(z)H_1(z) = \left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

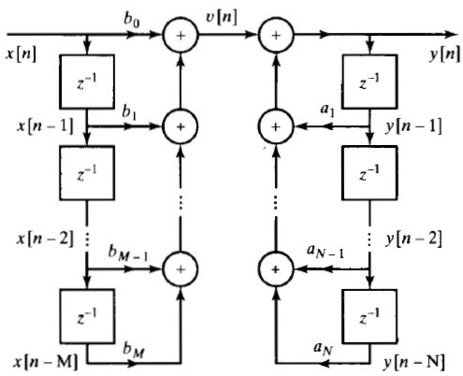
$$V(z) = H_1(z)X(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)X(z)$$

$$Y(z) = H_2(z)V(z) = \left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)V(z)$$

$$v[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n]$$

Block diagram representation for a general Nth-order difference equation



- Alternative Representation

Replace order of cascade LTI systems

$$H(z) = H_1(z)H_2(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)$$

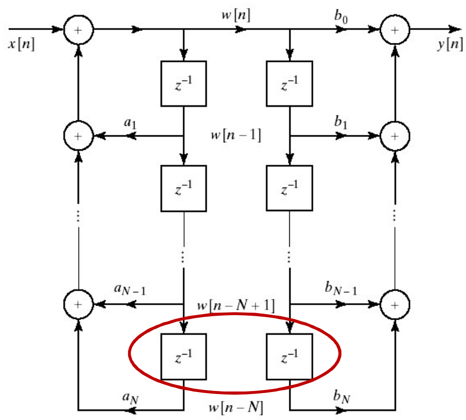
$$W(z) = H_2(z)X(z) = \left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)X(z)$$

$$Y(z) = H_1(z)W(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)W(z)$$

$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

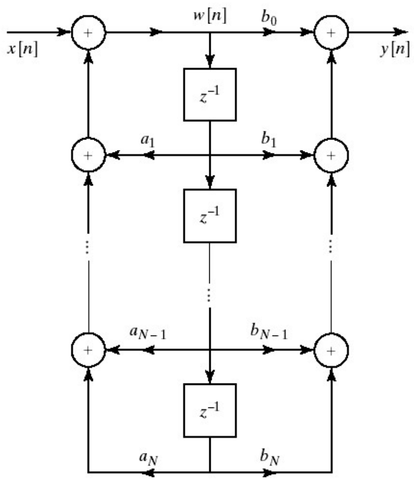
Alternative Block Diagram



w[n] is stored in the two chains of delay elements.

Direct Form II

- No need to store the same data twice in previous system
- So we can collapse the delay elements into one chain
- This is called Direct Form II or the **Canonical Form**
- Theoretically no difference between Direct Form I and II
- Implementation
  - Less memory in Direct II
  - Difference when using finite-precision arithmetic

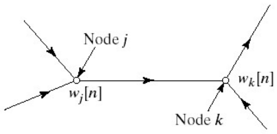


- Example: Block diagram representation of a Linear constant-coefficient difference equation with the following system Response

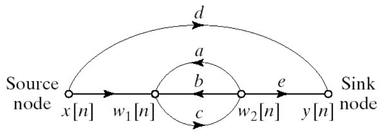
$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}.$$

Signal Flow Graph Representation

- Similar to block diagram representation
  - Notational differences
- A network of directed branches connected at nodes

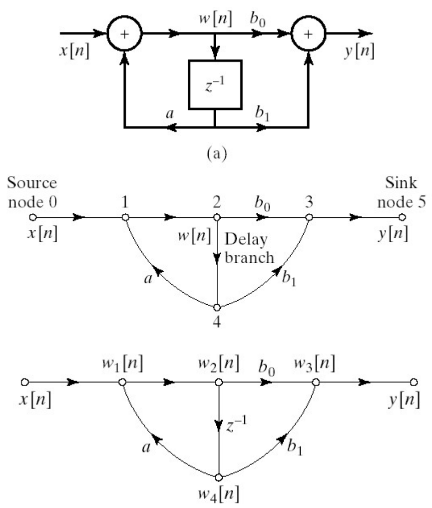


- Example representation of a difference equation



$$
\begin{aligned}
w_1[n] &= x[n] + aw_2[n] + bw_2[n] \\
w_2[n] &= cw_1[n] \\
y[n] &= dx[n] + ew_2[n]
\end{aligned}
$$

- Representation of Direct Form II with Signal Flow Graphs



$w_1[n] = aw_4[n] + x[n]$

$w_2[n] = w_1[n]$ ⟵ Renaming

$w_3[n] = b_0w_2[n] + b_1w_4[n]$

$w_4[n] = w_2[n-1]$ ⟵ Memory update

$y[n] = w_3[n]$ ⟵ Renaming

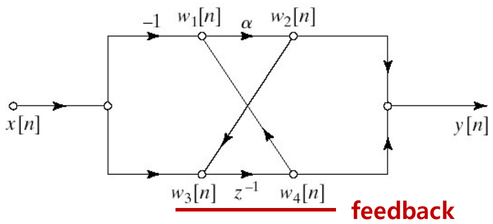
$w_1[n] = aw_1[n-1] + x[n]$

$y[n] = b_0w_1[n] + b_1w_1[n-1]$

Initial-rest conditions would be imposed in this case by defining $w_2[-1] = 0$ or $w_4[0] = 0$.

In many cases, this manipulation is difficult ➔ You can use the z-transform representation.

- Determination of the system function from a flow graph



feedback

This is not in direct form. Therefore, the set of difference equations represented by the graph can be written down by inspection of the graph.

$$w_1[n] = w_4[n] - x[n],$$
$$w_2[n] = \alpha w_1[n],$$
$$w_3[n] = w_2[n] + x[n],$$
$$w_4[n] = w_3[n-1],$$
$$y[n] = w_2[n] + w_4[n].$$

Z-transform

$$w_1(z) = w_4(z) - X(z)$$
$$w_2(z) = \alpha w_1(z)$$
$$w_3(z) = w_2(z) + X(z)$$
$$w_4(z) = w_3(z)z^{-1}$$
$$Y(z) = w_2(z) + w_4(z)$$

13

– We can eliminate $W_1(z)$ and $W_3(z)$,

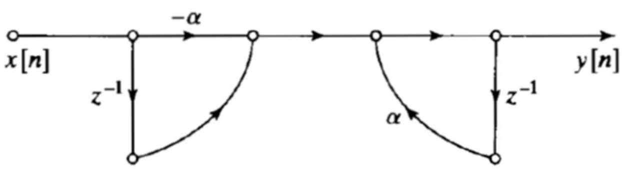
$$W_2(z) = \alpha(W_4(z) - X(z)),$$
$$W_4(z) = z^{-1}(W_2(z) + X(z)),$$
$$Y(z) = W_2(z) + W_4(z).$$

$$W_2(z) = \frac{\alpha X(z)(z^{-1} - 1)}{1 - \alpha z^{-1}}$$
$$W_4(z) = \frac{X(z)z^{-1}(1 - \alpha)}{1 - \alpha z^{-1}}$$
$$Y(z) = W_2(z) + W_4(z)$$

$$Y(z) = \left(\frac{\alpha(z^{-1} - 1) + z^{-1}(1 - \alpha)}{1 - \alpha z^{-1}}\right) X(z) = \left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}\right) X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

$$h[n] = \alpha^{n-1}u[n-1] - \alpha^{n+1}u[n]$$



Direct form I equivalent

One more multiplication and delay components are required.

14

Difference Between IIR and FIR Systems

- Infinite impulse response(IIR) system
  - at lease one nonzero pole of H(z) is not canceled by a zero
  - h[n] will not be of finite length.(not to be zero outside a finite interval)

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \qquad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}.$$

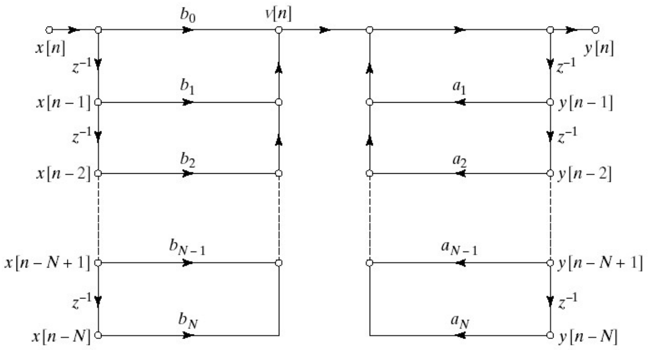
- Finite impulse response(FIR) system
  - h[n] will be of finite length (zero outside a finite interval)
  - there is no poles except for at zero

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

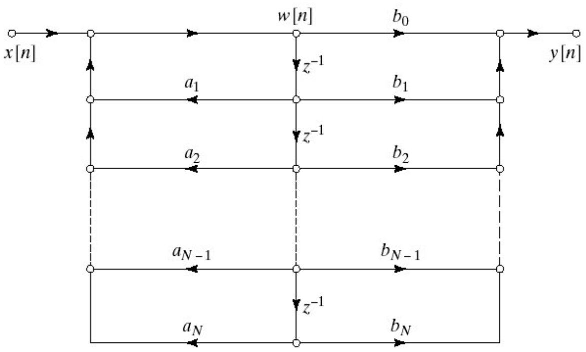
Basic Structures for IIR Systems: Direct

- Direct Form I and II

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}.$$



Direct form I

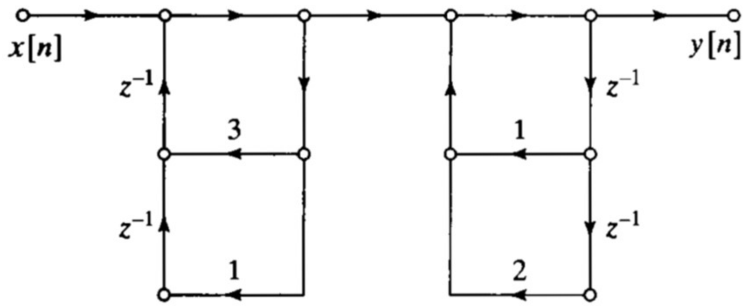


Direct form II

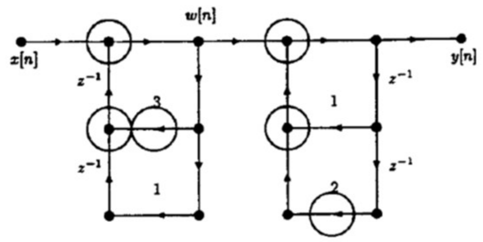
- Example 1) Illustrate Direct Form I and Direct Form II Structures

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}.$$

- Example 2) A linear time-invariant system is realized by the flow graph below.



  - Write the difference equation relating x[n] and y[n] for this flow graph.
  - What is the system function of the system?



(a)

$$w[n] = x[n] + 3w[n-1] + w[n-2]$$
$$y[n] = w[n] + y[n-1] + 2y[n-2]$$

(b)

$$W(z) = X(z) + 3z^{-1}W(z) + z^{-2}W(z)$$
$$Y(z) = W(z) + z^{-1}Y(z) + 2z^{-2}Y(z)$$

So

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{(1 - z^{-1} - 2z^{-2})(1 - 3z^{-1} - z^{-2})} = \frac{1}{1 - 4z^{-1} + 7z^{-3} + 2z^{-4}}.$$

Basic Structures for IIR Systems: Cascade

Most popular method:

- Factoring the numerator and denominator polynomials

General form for cascade implementation

$$H(z) = A\frac{\prod_{k=1}^{M_1}\left(1 - f_k z^{-1}\right)\prod_{k=1}^{M_2}\left(1 - g_k z^{-1}\right)\left(1 - g_k^* z^{-1}\right)}{\prod_{k=1}^{N_1}\left(1 - c_k z^{-1}\right)\prod_{k=1}^{N_2}\left(1 - d_k z^{-1}\right)\left(1 - d_k^* z^{-1}\right)}$$

First-order factors — Real

Second-order factors — Complex Conjugate

$$M = M_1 + 2M_2$$
$$N = N_1 + 2N_2$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}.$$

More practical form in 2nd order systems

$$H(z) = \prod_{k=1}^{N_S} \frac{b_{0k} + b_{1k}z^{-1} - b_{2k}z^{-2}}{1 - a_{1k}z^{-1} - a_{2k}z^{-2}}$$

By combining pairs of real factors and complex conjugate pairs into second-order factors

Assume $M \le N$

N_S: the largest integer contained in (N+1)/2

- If there are an odd number of **real zeros**, one of the coefficients b_{2k} will be zero.
- If there are an odd number of **real poles**, one of the coefficients a_{2k} will be zero.

– Individual second-order sections can be implemented using either of the direct from structures.

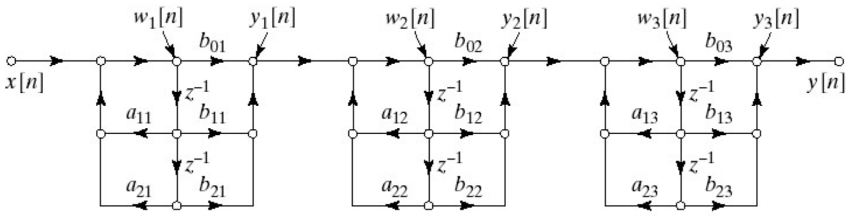
– The difference equations represented by a general cascade of direct from II second-order sections are of the form

$$y_0[n] = x[n],$$

$$w_k[n] = a_{1k}w_k[n-1] + a_{2k}w_k[n-2] + y_{k-1}[n], \quad k = 1, 2, \ldots, N_s,$$

$$y_k[n] = b_{0k}w_k[n] + b_{1k}w_k[n-1] + b_{2k}w_k[n-2], \quad k = 1, 2, \ldots, N_s,$$

$$y[n] = y_{N_s}[n].$$

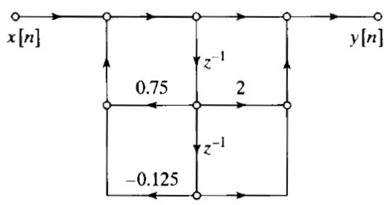


Cascade structure for a sixth-order system with a direct form II realization of each second-order subsystem

• Example 1)

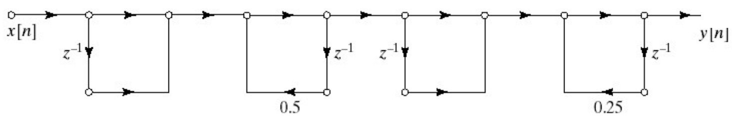
$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$

$$= \frac{(1 + z^{-1})}{(1 - 0.5z^{-1})} \frac{(1 + z^{-1})}{(1 - 0.25z^{-1})}$$

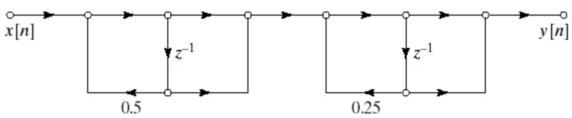


A second-order system with a direct form II realization

A cascade structure with first-order sections



Cascade of Direct Form I subsections

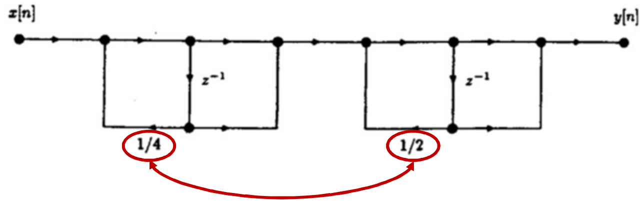


Cascade of Direct Form II subsections

- Example 2) For the system function

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

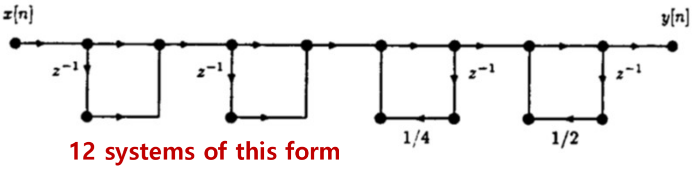
  – Draw the flow graphs of all possible realizations for this system as cascades of first-order systems.



$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}.$$

$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right)$$

$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right)$$



12 systems of this form

- Example 3)

A linear time-invariant system with system function

$$H(z) = \frac{0.2(1+z^{-1})^6}{\left(1-2z^{-1}+\frac{7}{8}z^{-2}\right)\left(1+z^{-1}+\frac{1}{2}z^{-2}\right)\left(1-\frac{1}{2}z^{-1}+z^{-2}\right)}$$

is to be implemented using a flow graph of the form shown in Figure P6.26-1.

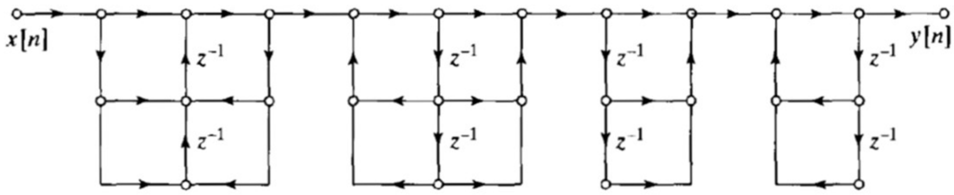


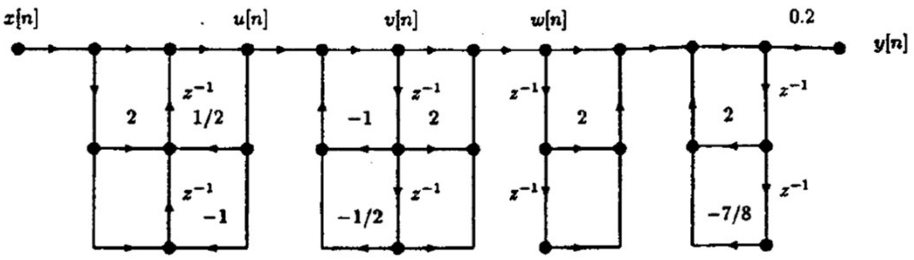
Figure P6.26-1

(a) Fill in all the coefficients in the diagram of Figure P6.26-1. Is your solution unique?
(b) Define appropriate node variables in Figure P6.26-1, and write the set of difference equations that is represented by the flow graph.

$$H(z) = \frac{0.2(1+z^{-1})^6}{\left(1-2z^{-1}+\frac{7}{8}z^{-2}\right)\left(1+z^{-1}+\frac{1}{2}z^{-2}\right)\left(1-\frac{1}{2}z^{-1}+z^{-2}\right)}$$

We can rearrange $H(z)$ this way:

$$H(z) = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}+z^{-2}} \cdot \frac{(1+z^{-1})^2}{1+z^{-1}+\frac{1}{2}z^{-2}} \cdot (1+z^{-1})^2 \cdot \frac{1}{1-2z^{-1}+\frac{7}{8}z^{-2}} \cdot 0.2$$



The solution is not unique; the order of the denominator 2nd-order sections may be rearranged.

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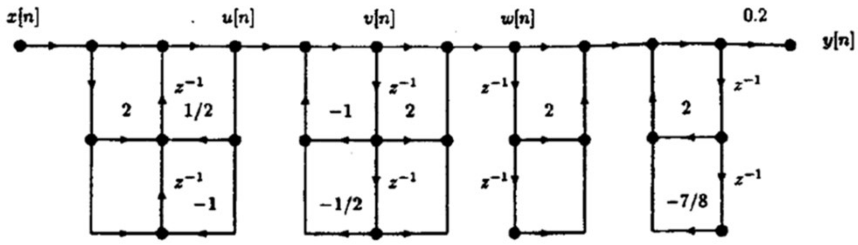
Define appropriate node variables and write the set of difference equations that is represented by the flow graph.



$$\begin{aligned}
u[n] &= x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}u[n-1] - u[n-2] \\
v[n] &= u[n] - v[n-1] - \frac{1}{2}v[n-2] \\
w[n] &= v[n] + 2v[n-1] + v[n-2] \\
y[n] &= w[n] + 2w[n-1] + w[n-2] + 2y[n-1] - \frac{7}{8}y[n-2].
\end{aligned}$$

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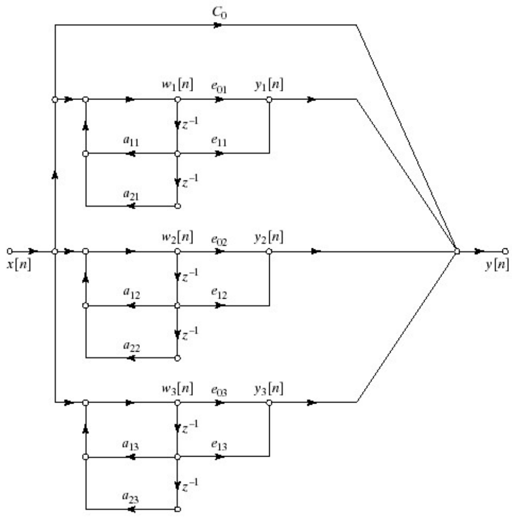
Basic Structures for IIR Systems: Parallel

Represent system function using partial fraction expansion

$$H(z)=\sum_{k=0}^{N_P} C_k z^{-k}+\sum_{k=1}^{N_P}\frac{A_k}{1-c_k z^{-1}}+\sum_{k=1}^{N_P}\frac{B_k\left(1-e_k z^{-1}\right)}{\left(1-d_k z^{-1}\right)\left(1-d_k^* z^{-1}\right)}$$

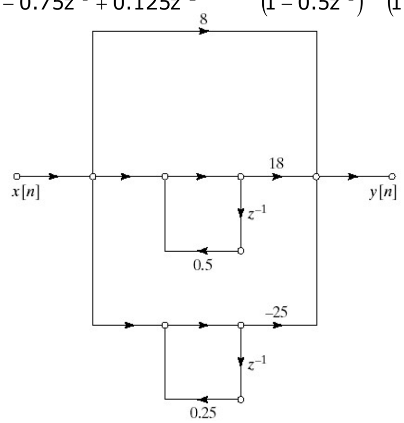
$$H(z)=\sum_{k=0}^{N_P} C_k z^{-k}+\sum_{k=1}^{N_S}\frac{e_{0k}+e_{1k}z^{-1}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}$$



- Example

Partial Fraction Expansion

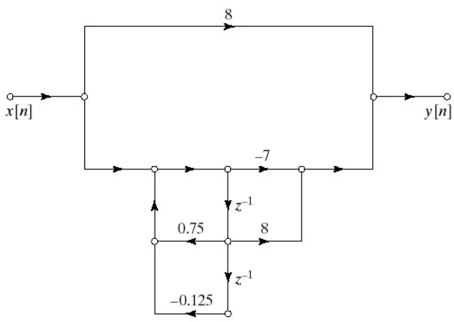
$$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}=8+\frac{18}{\left(1-0.5z^{-1}\right)}-\frac{25}{\left(1-0.25z^{-1}\right)}$$



Parallel-from structure using first-order systems

Combine poles to get

$$H(z)=8+\frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$$

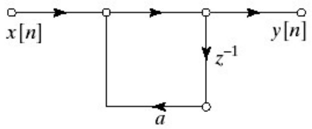


Parallel-from structure using a second-order system

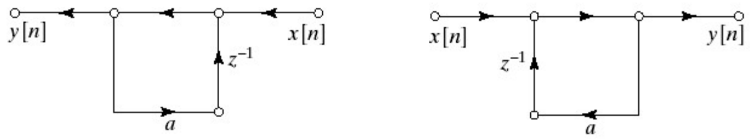
Transposed Forms

- Linear signal flow graph property:
  - Transposing doesn't change the input-output relation
- Transposing:
  - Reverse directions of all branches
  - Interchange input and output nodes

$$H(z) = \frac{1}{1 - az^{-1}}$$



Reverse directions of branches and interchange input and output



Examples

Causal linear time-invariant system

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{\left(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}\right)\left(1 + \frac{1}{4}z^{-1}\right)}$$

Draw the signal flow graphs for implementations of the system in each of the following forms:

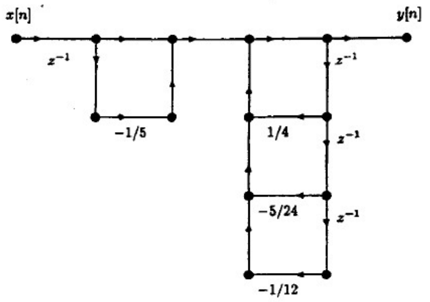
1) Direct form I
2) Direct form II
3) Cascade form using first and second-order direct form II sections
4) Parallel form using first and second-order direct form II sections
5) Transposed direct form II

- Direct form I and II

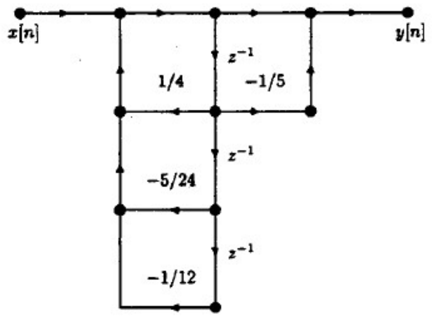
$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z(-3)}$$

$$b_0 = 1\ ,\ b_1 = -\frac{1}{5} \text{ and } a_1 = \frac{1}{4}\ ,\ a_2 = -\frac{5}{24}\ ,\ a_3 = -\frac{1}{12}.$$

Direct I



Direct II

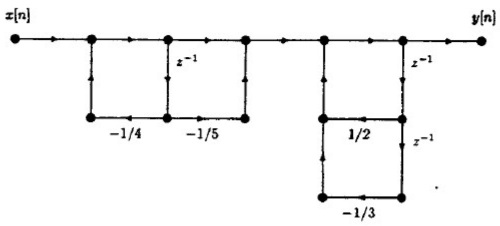


- Cascade and Parallel form

Cascade form

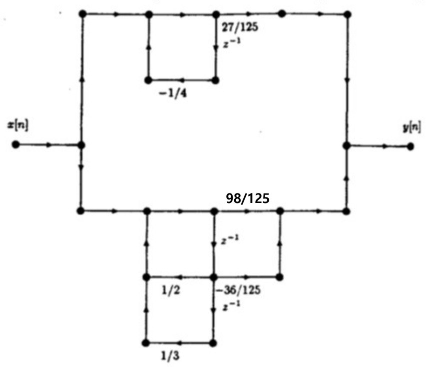
$$H(z) = \left(\frac{1-\frac{1}{5}z^{-1}}{1+\frac{1}{4}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}\right).$$

$$b_{01} = 1\ ,\ b_{11} = -\tfrac{1}{5}\ ,\ b_{21} = 0\ ,$$
$$b_{02} = 1\ ,\ b_{12} = 0\ ,\ b_{22} = 0 \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$

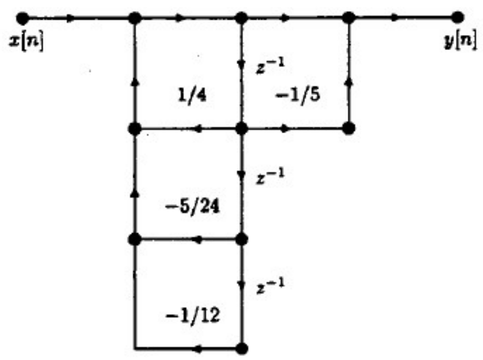


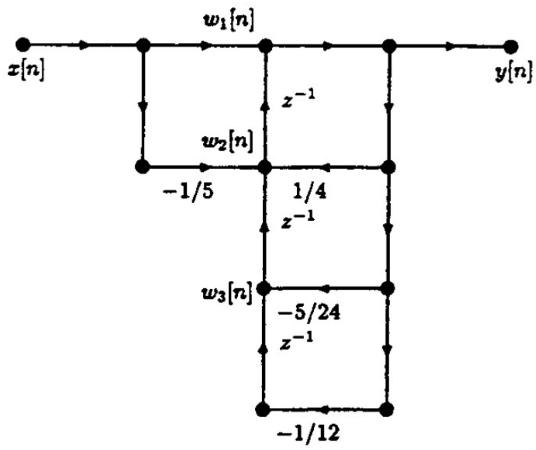
$$H(z) = \frac{\frac{27}{125}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{98}{125}-\frac{36}{125}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{3}z^{-2}}.$$

$$e_{01} = \tfrac{27}{125}\ ,\ e_{11} = 0\ ,$$
$$e_{02} = \tfrac{98}{125}\ ,\ e_{12} = -\tfrac{36}{125}\ , \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$



• Transposed direct form II





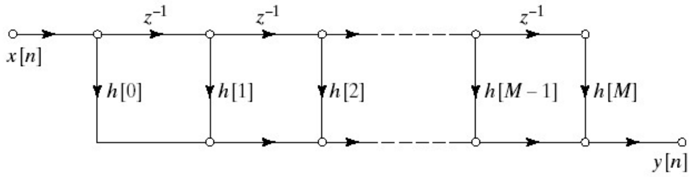
FIR systems

• Finite impulse response(FIR) system
 - h[n] will be of finite length (zero outside a finite interval)
 - there is no poles except for at zero

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

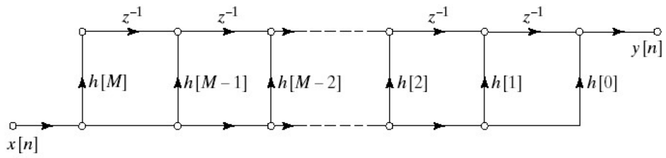
Basic Structures for FIR Systems: Direct

Special cases of IIR direct form structures



$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$

Transpose of direct form I gives direct form II



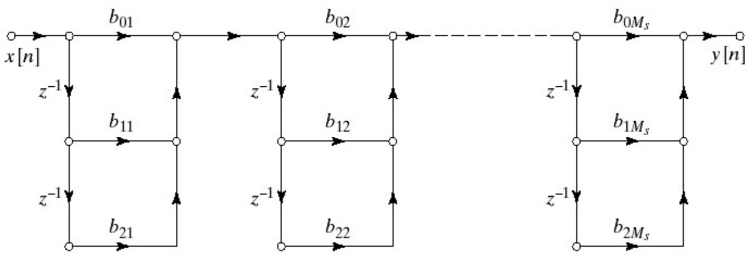
Both forms are equal for FIR systems

Basic Structures for FIR Systems: Cascade

- **Obtained by factoring the polynomial system function**

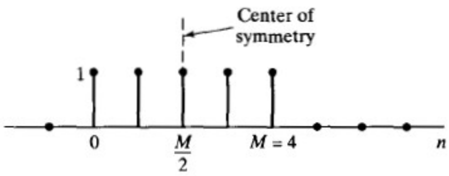
$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}=\prod_{k=1}^{M_S}\left(b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}\right)$$

M_S: the largest integer contained in (M+1)/2

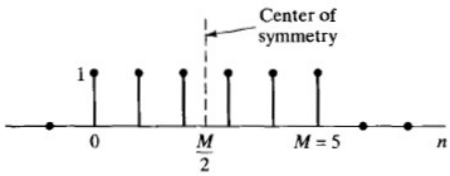


If M is odd, one of the coefficients b_{2k} will be zero.

Linear-Phase FIR Systems



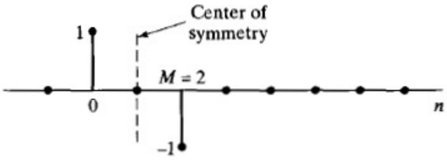
$h[n] = h[M-n], \qquad 0 \le n \le M,$



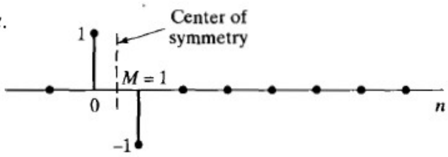
(b)

$$H_{\mathrm{id}}(e^{j\omega}) = e^{-j\omega\alpha}, \qquad |\omega| < \pi,$$

$$|H_{\mathrm{id}}(e^{j\omega})| = 1,$$
$$\sphericalangle H_{\mathrm{id}}(e^{j\omega}) = -\omega\alpha,$$
$$\mathrm{grd}[H_{\mathrm{id}}(e^{j\omega})] = \alpha.$$



$h[n] = -h[M-n], \qquad 0 \le n \le M,$

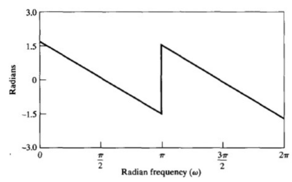


(d)

Structures for Linear-Phase FIR Systems

Causal FIR system with generalized linear phase are symmetric:

$$h[M-n] = h[n] \qquad n = 0,1,\ldots,M$$
$$h[M-n] = -h[n] \qquad n = 0,1,\ldots,M$$



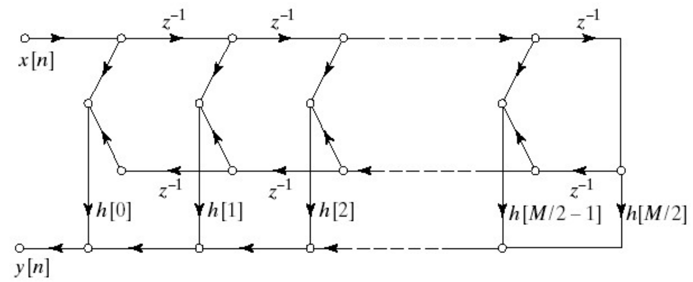
Symmetry means we can half the number of multiplications

Example: For even M and type I or type III systems:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=M/2+1}^{M} h[k]x[n-k]$$
$$= \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=0}^{M/2-1} h[M-k]x[n-M+k]$$
$$= \sum_{k=0}^{M/2-1} h[k](x[n-k] + x[n-M+k]) + h[M/2]x[n-M/2]$$

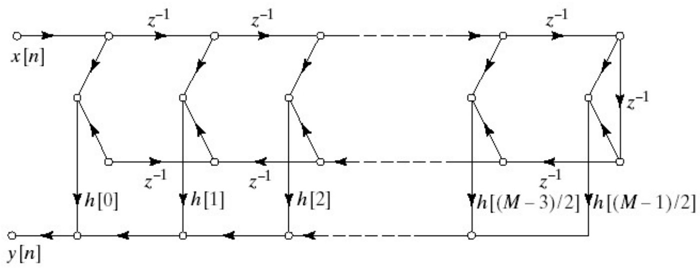
Structure for even M

$$\sum_{k=0}^{M/2-1} h[k](x[n-k]+x[n-M+k]) + h[M/2]x[n-M/2]$$



Structure for odd M

$$y[n] = \sum_{k=0}^{(M-1)/2} h[k](x[n-k]+x[n-M+k])$$



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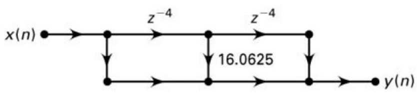
Example

$$H(z) = 1 + \left(16 + \frac{1}{16}\right) z^{-4} + z^{-8}$$

- Direct form

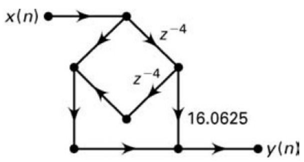
  y(n) = x(n) + 16.0625x(n − 4) + x(n − 8)



(a) Direct Form

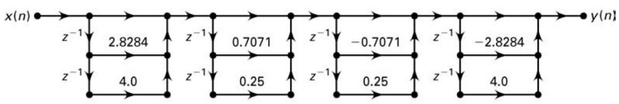
- Linear-phase form

  y(n) = [x(n) + x(n − 8)] + 16.0625x(n − 4)



- Cascade form

$H(z) = (1 + 2.8284z^{-1} + 4z^{-2})\,(1 - 2.8284z^{-1} + 4z^{-2})$
$(1 - 0.7071z^{-1} + 0.25z^{-2})\,(1 + 0.7071z^{-1} + 0.25z^{-2})$



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