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Digital Signal Processing (Lecture Note 8)

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EE401

Transform Analysis of LTI systems (Chap. 5)

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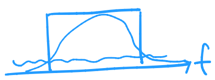
Ideal Digital Filters

- Filters are closely related to spectral analysis since the <u>goal of filtering is to reshape the spectrum</u> to one's advantage.
- Most noise is broadband (for example, white noise) and most signals are narrowband
  - Filters, that appropriately reshape a waveform's spectrum, will almost always provide some improvement in SNR. filtering 하면 noise의 power ↓. → SNR이 좋아진다. (증가한다)
  - As a general concept, a basic filter can be viewed as a <u>linear process</u> in which the input signal's spectrum is reshaped in some well-defined manner.
- An ideal filter is of rectangular function shape, which cannot be realized. ➔ Why?



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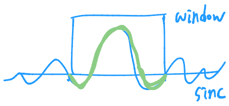
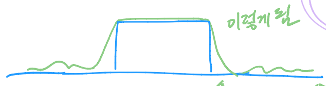
→ t domain에서 sinc function을 구현하려면 -∞부터 ∞까지 신호가 있어야(?) 하는데 그게 불가.

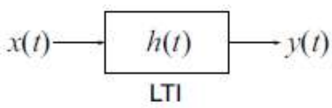


w[n]·h[n] 해서 사용. → H(ω) * W(ω).



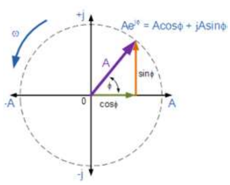
왜가 변하는지 다시 듣기

LTI Systems with Sinusoidal Inputs



$$x(t) = A\cos(\omega_0 t + \theta_0)$$

Amplitude Frequency Phase



$$\begin{aligned} x(t) &= A\cos(\omega_0 t + \theta_0) \\ X(\omega) &= \pi A\left[e^{-j\theta_0}\delta(\omega+\omega_0) + e^{j\theta_0}\delta(\omega-\omega_0)\right] \end{aligned}$$

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \pi A\left[e^{-j\theta_0}H(\omega)\delta(\omega+\omega_0) + e^{j\theta_0}H(\omega)\delta(\omega-\omega_0)\right] \\ &= \pi A\left[e^{-j\theta_0}H(-\omega_0)\delta(\omega+\omega_0) + e^{j\theta_0}H(\omega_0)\delta(\omega-\omega_0)\right] \\ &= \pi A\left[e^{-j(\theta_0+\angle H(\omega_0))}|H(\omega_0)|\delta(\omega+\omega_0)\right. \\ &\qquad \left. + e^{j(\theta_0+\angle H(\omega_0))}|H(\omega_0)|\delta(\omega-\omega_0)\right] \end{aligned}$$

$$H(\omega) = |H(\omega)| \cdot e^{j\angle(H(\omega))}$$

From Dr. Maxim Raginsky's Lecture Note , Duke University

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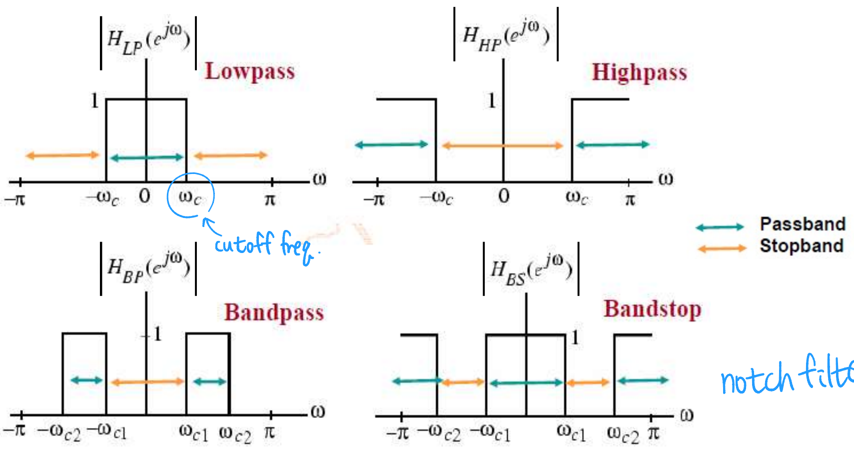
그냥 다시듣자...

$$y(t) = FT^{-1}\left[Y(\omega)\right]$$
$$= A\left|H(\omega_0)\right|\cos\left(\omega_0 t + \theta_0 + \angle H(\omega_0)\right)$$

$\boxed{A \rightarrow A\left|H(\omega_0)\right|}$ $\boxed{\theta_0 \rightarrow \theta_0 + \angle H(\omega_0)}$

- Ideal Filter
  - Designed to pass signal components of certain frequencies **without distortion**, which has a frequency response equal to 1 at these frequencies, and has a frequency response equal to 0 at all other frequencies.
  - **Passband ➔** the range of frequencies where the frequency response takes the value of one.
  - **Stopband ➔** the range of frequencies where the frequency response takes the value of zero.
  - **Cutoff Frequency ➔** the transition frequency from a passband to stopband region.

- It is convenient to look at filters in the frequency domain in order to specify:
  - Filters' amplitude responses: $|H(\omega_0)|$
  - Filters' phase responses: $\angle H(\omega_0)$



notch filter

Next, we need to specify the phase response $\angle H(\omega)$ of the filter. We will call a filter $H(\omega)$ **ideal** if

$$|H(\omega)| = \begin{cases} 1, & \text{if } \omega \text{ is in the passband} \\ 0, & \text{if } \omega \text{ is in the stopband} \end{cases}$$

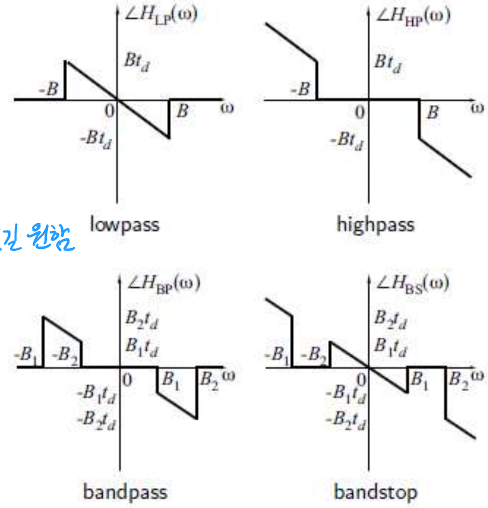
and

$$\angle H(\omega) = \begin{cases} -\omega t_d, & \text{if } \omega \text{ is in the passband} \\ 0, & \text{if } \omega \text{ is in the stopband} \end{cases}$$

→ ideal한 filter의 phase. 이렇게 되길 원함

where $t_d > 0$ is some constant.

The reason for calling such filters "ideal" will become clear shortly.



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- Let's see what happens when we feed a sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \theta_0)$$

into an ideal filter H(ω).

$$y(t) = A|H(\omega_0)|\cos(\omega_0 t + \theta_0 + \underbrace{\angle H(\omega_0)}_{-\omega_0 t_d})$$

– Since $|H(\omega)| = 1$ when ω is in the passband and 0 when ω is in the stopband, while $\angle H(\omega_0) = -\omega t_a$ when ω is in the passband and 0 otherwise.

$$y(t) = \begin{cases} A\cos(\omega_0(t - t_d) + \theta_0) & \text{, if } \omega_0 \text{ is in the passband} \\ 0 & \text{, if } \omega_0 \text{ is in the stopband} \end{cases}$$

– If the frequency of the sinusoid is in the passband of the filter, the output y(t) of the filter is a **time-delayed** version of the input:

$$y(t) = x(t - t_d)$$

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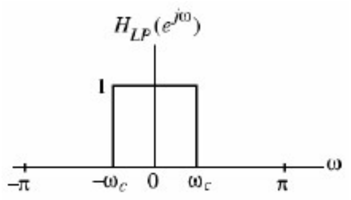
- An ideal filter does not distort the input signal, only delay it provided the input frequency is in the passband.
- Let us consider in detail the ideal lowpass filter:

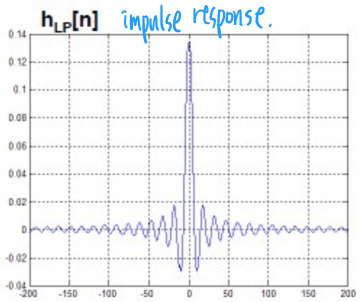
$$|H_{LP}(\omega)| = A \cdot rect(\omega) \quad \text{and} \quad \angle H_{LP}(\omega) = -\omega t_d$$

$$H_{LP}(\omega) = A \cdot rect(\omega) \cdot e^{-j\omega t_d}$$

$$h_{LP}(t) = FT^{-1}\left[A \cdot rect(\omega) \cdot e^{-j\omega t_d}\right] = \frac{A}{2\pi} \sin c\left[\frac{A}{2\pi}(t - t_d)\right]$$

  – The frequency response $H_{LP}(\omega)$ is band-limited, therefore the impulse response $h_{LP}(t)$ **cannot be time-limited**.

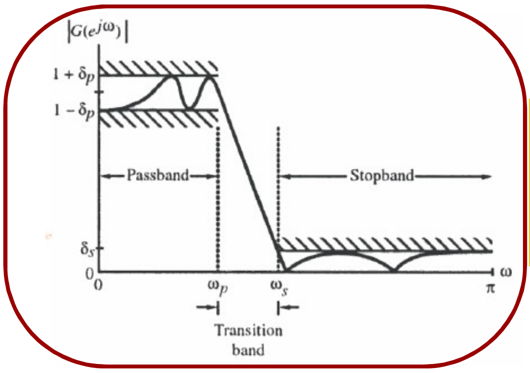




- The impulse response of an ideal filter:
  – It is not summable because of its infinite length of response
  – The corresponding transfer function is not bounded-input bounded-output stable.
  – It is not causal, i.e., doubly infinite length.

  ➔ **The ideal filters cannot be realized with finite dimensional LTI filter.** 구현하기가 불가능함

- In order to develop a stable and realizable filter transfer function
  - The ideal frequency response specifications are relaxed by including **a transition band** between the passband and the stopband
  - This permits the magnitude response **to decay slowly** from its maximum value in the passband to the zero value in the stopband
  - The magnitude response is allowed **to vary by a small amount** both in the passband and the stopband



How could we design this stable and realizable filter transfer function ?

7:51 PM

Z-transform of a LTI system

- LTI system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$
$$Y(z) = X(z)H(z)$$

- System function: z-transform of an output sequence over z-transform of the input sequence

$$H(z) = \frac{Y(z)}{X(z)}$$

$\rightarrow h[n]$ *z-transform*

System function $H(z)$: z-transform of the impulse response, $h[n]$

Revisiting Transfer Function in Z-Plane

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}$$

$$=z^{(N-M)}\frac{b_0\prod_{l=1}^{M}(z-\alpha_l)}{a_0\prod_{l=1}^{N}(z-\beta_l)}$$

Zeros

Poles

- Filter design is simply the determination of appropriate filter coefficients, a(n) and b(n), that provide the desired spectral shaping.
- A digital filter is designed by placing appropriate number of
  - **zeros at the frequencies (z-values) to be suppressed**
  - **Poles at the frequencies to be amplified**

TF로 뭘 알수 있는가

↓

Magnitude, phase.

Magnitude and Phase Response

$$\left|H\left(e^{j\omega}\right)\right|=\left|\frac{b_0}{a_0}\right|\frac{\prod_{l=1}^{M}\left|\left(e^{j\omega}-\alpha_l\right)\right|}{\prod_{l=1}^{N}\left|\left(e^{j\omega}-\beta_l\right)\right|}$$

주파수 특성을 보려면 z-transform 해야 함

$$\arg H\left(e^{j\omega}\right)=\arg\left(\frac{b_0}{a_0}\right)+\omega(N-M)+\sum_{l=1}^{M}\arg\left(e^{j\omega}-\alpha_l\right)-\sum_{l=1}^{N}\arg\left(e^{j\omega}-\beta_l\right)$$

- The magnitude response at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors.
- The phase response at a specific value of ω is obtained by adding the phase of the term b_0/a_0, and the linear-phase term ω(N-M) to the sum of the angles of the zero vectors minus the angles of the pole vectors.

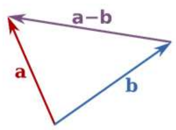
- From the magnitude and phase responses of the transfer function of an LTI digital filter,
  - It is learned that the examination of pole and zero locations leads to developing an approximate plot of frequency response of the LTI digital filter.
- Frequency response has
  - the smallest magnitude at zero locations
  - the largest magnitude at pole locations
- As a result,
  - To highly attenuate signal components in a specified frequency range, zeros should be placed very close to or on the unit circle in this range.
  - To highly emphasize signal components in a specified frequency range, poles should be placed very close to or on the unit circle in this range.

Graphical Interpretation

- In $H(z)=\frac{b_0}{a_0}z^{(N-M)}\frac{\prod_{l=1}^{M}(z-\alpha_l)}{\prod_{l=1}^{N}(z-\beta_l)}$, α_l,β_l are generally a complex number and thus those can be rewritten by $\alpha_l = p_l e^{j\phi_l}, \beta_l = q_l e^{j\phi_l}$

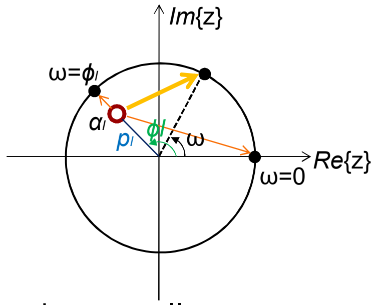
on the unit circle

- The term $\left(e^{j\omega}-p_l e^{j\phi_l}\right)$ represents a vector in the z-plane
  - Starting at the point $z = p_l e^{j\phi_l}$
  - Ending at the point $z = e^{j\omega}$



- As ω varies from 0 to 2π, the tip of the vector moves counterclockwise tracing the unit circle.

- The magnitude vector corresponding to the factor α_l will be the smallest at $\omega = \phi_l$. If α_l is on the unit circle, then this magnitude will be zero.
- Therefore, if α_l is a zero, i.e., a numerator factor, the overall magnitude vector of H(ω) will be small at frequencies around ϕ_l, and will be exactly zero if α_l is on the unit circle, causing H(ϕ_l)=0.



- Conversely, if α_l is a pole, i.e., a denominator factor, the overall magnitude vector of H(ω) will be large at frequencies around ϕ_l, and will go to infinity if α_l is on the unit circle.
- This is why the zeros and the poles that are at or close to the unit circle have a larger impact on the overall frequency response than are further away from the unit circle.
- The phase of H(ω) can be computed in a similar way, except , we now add each phase component corresponding to zeros, and subtract those corresponding to the poles.

요 페이지 설명 교재 해설은 시험문제 내었다

Example

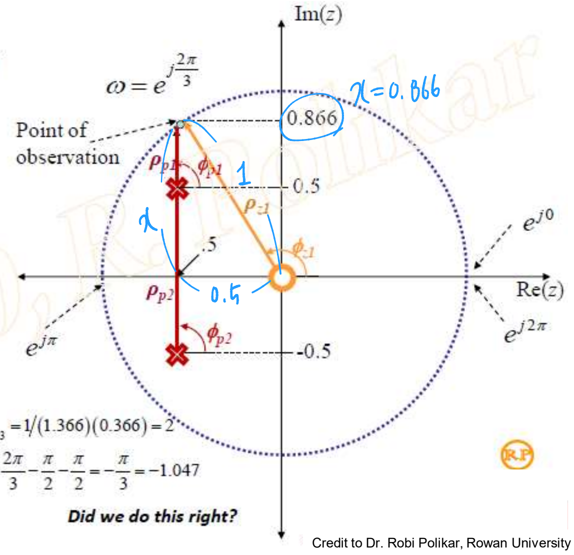
$$H(z) = \frac{z^{-1}}{1+z^{-1}+0.5z^{2}} = \frac{z}{z^2+z+0.5}$$

$$= \frac{z}{(z+0.5+j0.5)(z+0.5-j0.5)}$$

One zero at z = 0

One pair of conjugate poles at z = -0.5000 ± j0.5000

Then,



$$\left.\begin{aligned} \rho_{z_1} &= \sqrt{(0.5)^2+(0.866)^2} = 1 \\ \rho_{p_1} &= 0.5+0.866 = 1.366 \\ \rho_{p_2} &= 0.866-0.5 = 0.366 \\ \phi_{z_1} &= 2\pi/3, \phi_{p_1} = \phi_{p_2} = \pi/2 \end{aligned}\right\}$$

$$|H(\omega)|_{\omega=2\pi/3} = 1/(1.366)(0.366) = 2$$

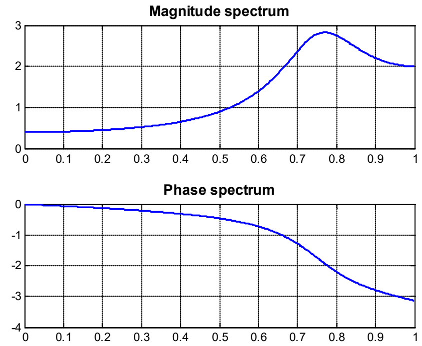
$$\measuredangle H(\omega)|_{\omega=2\pi/3} = \frac{2\pi}{3} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{3} = -1.047$$

Did we do this right?

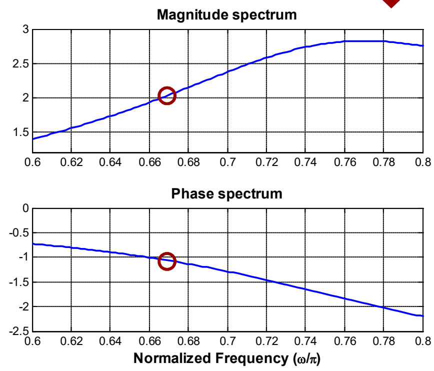
Credit to Dr. Robi Polikar, Rowan University

손으로 계산은 X, 어떻게 구하는지 알아보았다

$$H(z)=\frac{z^{-1}}{1+z^{-1}+0.5z^{-2}}=\frac{z}{z^2+z+0.5}$$



observation frequency: 2/3~0.667



$$\left|H\left(\frac{2\pi}{3}\right)\right|=2$$

$$\angle H\left(\frac{2\pi}{3}\right)=-\frac{\pi}{3}=-1.047$$

사용할 수 있는 matlab 함수들

- **[h,w] = freqz(b,a,N,'whole')**
  - Calculates the frequency response of a filter whose constant coefficient linear difference equation coefficients are given as b and a, using N number of points around the unit circle. If 'whole' is included, it returns a frequency base of w from 0 to 2π, otherwise, from 0 to π.
- **[hz, hp, hi]=zplane(zeros, poles)**
  - Plots the zeros specified in column vector z and the poles specified in column vector p in the current figure window.
  - The symbol 'o' represent a zero and the symbol 'x' represents a pole. The plot includes the unit circle for reference.
  - Hz is a vector of handles to the zeros lines, hp is a vector of handles to the poles lines, and hi is a vector of handles to the axes/unit circle line.

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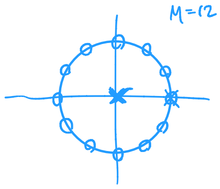
M-point Moving-Average FIR Filter

$$h[n] = \begin{cases} \frac{1}{M}, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M\left(1-z^{-1}\right)} = \frac{z^{M}-1}{M\left[z^{M-1}(z-1)\right]}$$

– The transfer function has M zeros on the unit circle at

$$z = e^{j\frac{2\pi k}{M}}, \quad 0 \le k \le M-1$$



– There are M-1 poles at z=0 and an single pole at z=1

– The pole at z=1 exactly cancels the zero at z=1.

– The ROC is the entire z-plane except z=0

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– Moving Average Filter

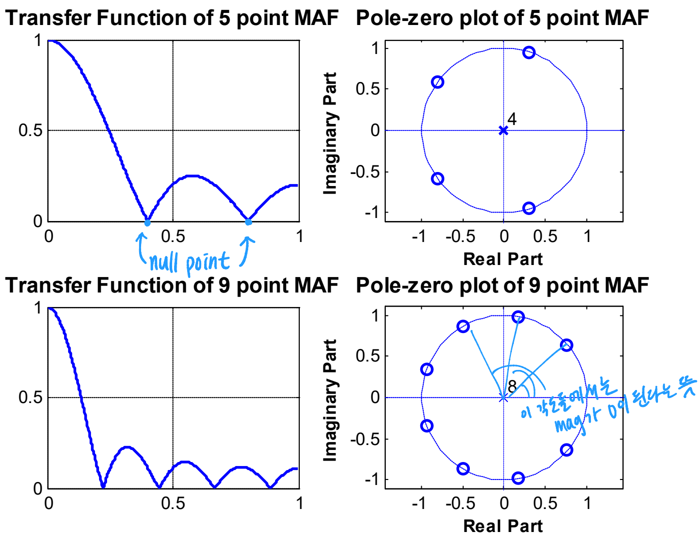
```
b1=1/5*[1 1 1 1 1]; a1=1;
b2=1/9*[1 1 1 1 1 1 1 1 1]; a2=1;
[H1 w]=freqz(b1, 1, 512);
[H2 w]=freqz(b2, 1, 512);

subplot(221)
plot(w/pi, abs(H1)); grid
title('Transfer Function of 5 point MAF')

subplot(222)
zplane(b1,a1);
title('Pole-zero plot of 5 point MAF')

subplot(223)
plot(w/pi, abs(H2)); grid
title('Transfer Function of 9 point MAF')

subplot(224)
zplane(b2,a2);
title('Pole-zero plot of 9 point MAF')
```



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Phase of Filters

- A frequency selective system (filter) with frequency response

$$H(\omega) = |H(\omega)| \angle H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

changes the amplitude of all frequencies in the signal by a factor of |H(ω)|, and adds a phase of θ(ω) to all frequencies.

– Both the amplitude change and the phase delay are functions of ω.

– The phase θ(ω) can be expressed in terms of time, which is called the **phase delay**. The phase delay at a particular frequency ω_0 is given as

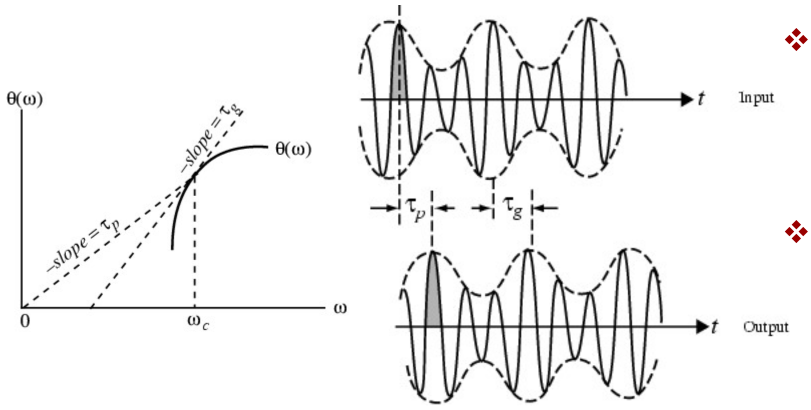
Phase delay in cont. time

$$\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0} \qquad \left(\text{c.f.} \Rightarrow t_\theta = \frac{\theta}{2\pi f} = \frac{\theta}{\omega} \right)$$

- If an input system consists of many frequency components (which most practical signals do), we can also define **group delay**, the phase shift by which the envelope of the signal shifts.
- This can also be considered as the average phase delay, in seconds (or in samples), of the filter as a function of frequency, given by

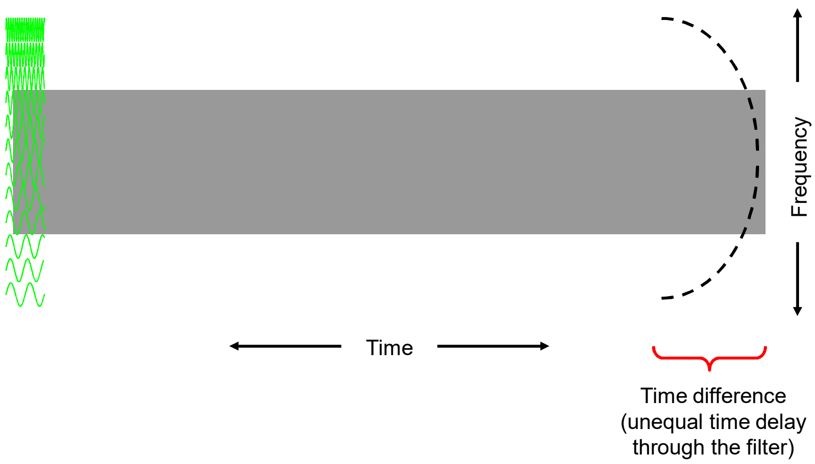
$$\tau_g(\omega_c) = \left. \frac{d\theta(\omega)}{d\omega} \right|_{\omega=\omega_c}$$

- Both are slopes of the phase function, just defined slightly differently



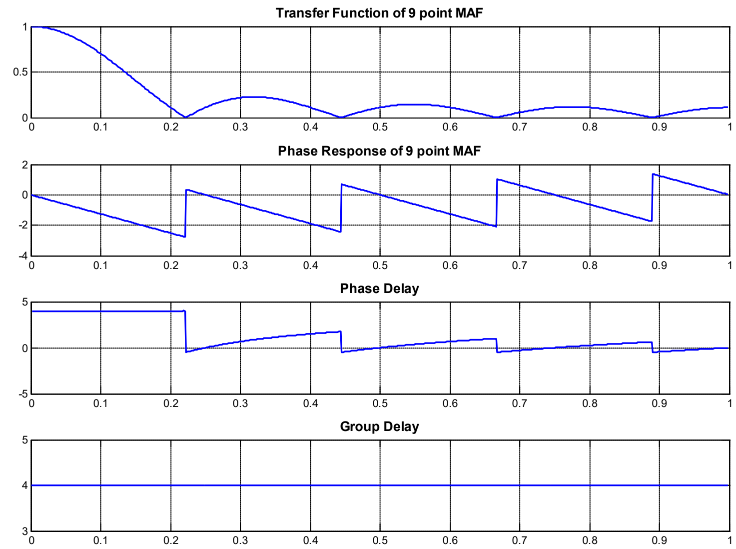
- ❖ **Phase distortion is measured using a parameter called envelope delay distortion, or group delay distortion.**
- ❖ **If group delay exists, signals at some frequencies travel faster than signals at other frequencies.**

- Group Delay: An Analogy
  - The "track" is a filter's passband, the "ditches" are the filter's band edges.



Credit to Dr. Ron Hranac, Cisco Systems

- [gd, w] = grpdelay(b, a, N, fs)
  - Returns the N-point group delay of the digital filter H(z) given its numerator and denominator coefficients in vectors "b" and "a".
  - grpdelay returns both gd, the group delay, which has units of samples, and w, a vector containing the N frequency points in radians.
  - grpdelay evaluates the group delay at N points equally spaced around the upper half of the unit circle, so w contains N points between 0 and π.
  - Specifies a positive sampling frequency fs in hertz. It returns a length N vector containing the actual frequency points at which the group delay is calculated, also in hertz.
    - ✓ f contains N points between 0 and fs/2.
  - [gd, w]=grpdelay(b,a,N,'whole') use N points around the whole unit circle (from 0 to 2π, or from 0 to fs).



Zero-Phase Filter

- One way to avoid any phase distortion is to make sure the frequency response of the filter does not delay any of the spectral components.
  - Such a transfer function is said to have **a zero-phase characteristic**.
- A zero-phase transfer function has no phase component, that is, the spectrum is purely real (no imaginary component) and non-negative
- However, it is NOT possible to design a causal digital filter with a zero phase. Why?
- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be implemented by relaxing the causality requirement.

- A zero-phase filtering scheme can be obtained by the following procedure:
  - Process the input data (finite length) with a causal real-coefficient filter H(z).
  - Time reverse the output of this filter and process by the same filter
  - Time reverse once again the output of the second filter
  - The resulting filter will have an effective order that is twice that of H(z).

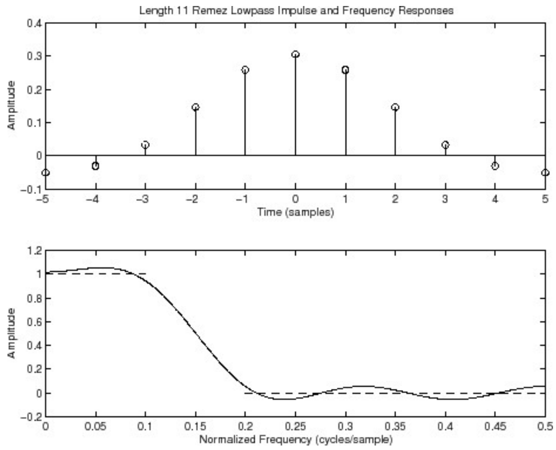
$$V(\omega)=H(\omega)X(\omega) \qquad W(\omega)=H(\omega)U(\omega)$$

$$U(\omega)=V^*(\omega) \qquad Y(\omega)=W^*(\omega)=H^*(\omega)U^*(\omega)$$

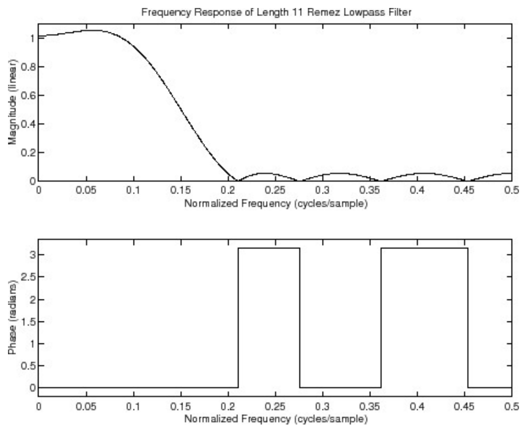
$$\left(\text{if } u[n]=v[-n]\right) \qquad \left(\text{if } y[n]=w[-n]\right)$$

$$Y(\omega)=H^*(\omega)V(\omega)=H^*(\omega)H(\omega)X(\omega)=|H(\omega)|^2 X(\omega)$$

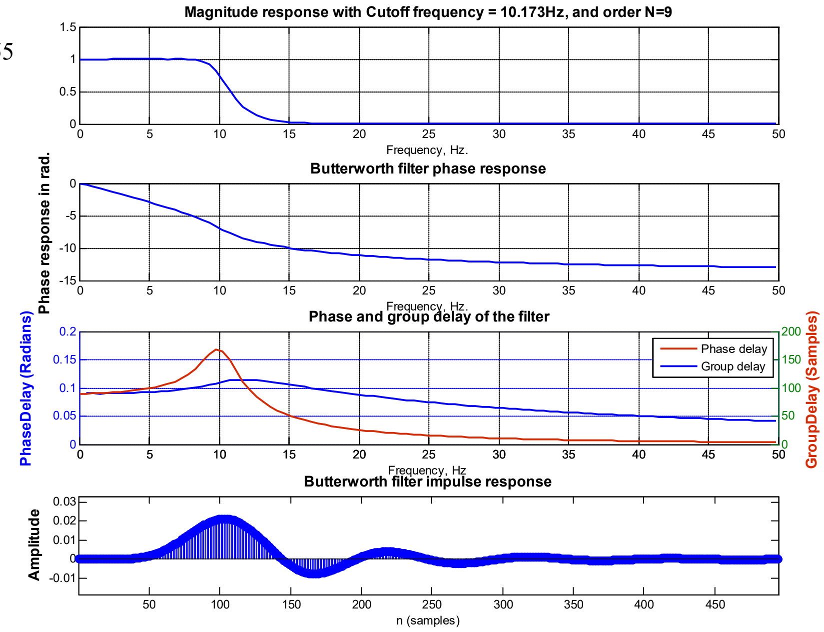
- Y = filtfilt(b, a, x) performs zero-phase digital filtering by processing the input data in both the forward and reverse directions.
  - After filtering in the forward direction, it reverses the filtered sequence and runs it back through the filter.
  - The result has precisely zero-phase distortion, the magnitude is the square of the filter's magnitude response, and the filter order is double the order of the filter **specified by b and a**.



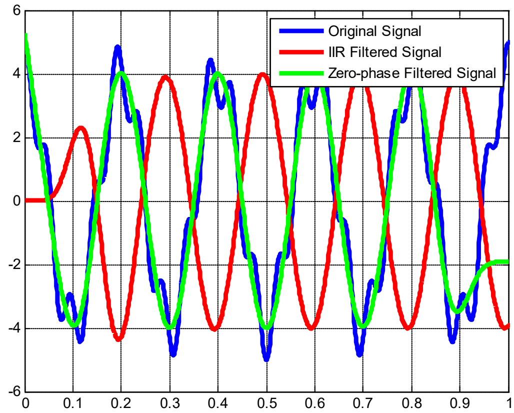
- This figure shows the magnitude and phase responses of the zero-phase filter.
  - The phase response is zero throughout the passband and transition band.
  - Each zero-crossing in the stopband results in a phase jump of π radians, so that the phase alternates between zero and π in the stopband.



$$\tau_p = \frac{3}{2\pi \cdot 5} = 0.0955$$




```
t = 0:0.001:1;
x=4*cos(2*pi*5*t)+cos(2*pi*21*t);
x1 = filter(b, a, x); %Filter the signal with the nonlinear phase Butterworth filter
x2 = filtfilt(b,a,x); %Noncausalzero phase filtering
plot(t, x, 'b', t, x1, 'r', t, x2, 'g','LineWidth', 3); grid
legend('Original Signal', 'IIR Filtered Signal', 'Zero-phase Filtered Signal')
```



Linear Phase

- A zero-phase filter cannot be implemented for real-time application.
- For a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has (preferably) a unity magnitude and a linear-phase characteristic in the frequency band of interest.

$$H(\omega) = e^{-j\alpha\omega} \Rightarrow \left|H(\omega)\right| = 1 \quad \angle H(\omega) = \theta(\omega) = -\alpha\omega$$

  - This phase characteristic is linear for all ω in [0 2π].
  - The phase delay at any given frequency ω_0 is
  - If we have linear phase, that is, θ(ω)=-αω, then the total delay at any frequency ω_0 is

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$$\boxed{\tau_0 = -\frac{\theta(\omega_0)}{\omega_0} = -\frac{\alpha\omega_0}{\omega_0} = \alpha}$$

  - Note that this is identical to the group delay evaluated at ω_0

$$\tau_g(\omega_0) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_0}$$

  - If the phase spectrum is linear, the phase delay is independent of the frequency, and it is the same constant α for all frequencies.
    - ✓ All frequencies are delayed by α seconds, or equivalently, the entire signal is delayed by α seconds.
    - ✓ Since the entire signal is delayed by a constant amount, there is no distortion.
  - If the filter does not have linear phase, different frequency components are delayed by different amounts, causing significant distortion.

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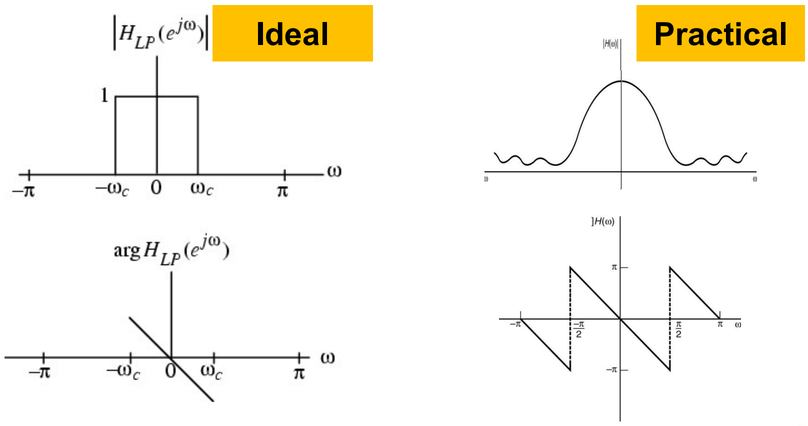
- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest.



- Generalized Linear Phase
  - Consider $$H(\omega)=e^{-j\alpha\omega}G(\omega)$$
    , where G(ω) is real, i.e., no phase
  - The exponential term introduces a phase delay, that is, normally independent of frequency.
    - ✓ If G(ω) is positive, the phase term is θ(ω)=-αω, hence the system has linear phase.
    - ✓ If G(ω)<0, then a 180˚ (π rad) phase term is added to the phase spectrum. Therefore, the phase response is θ(ω)=-αω+ π, the phase delay is no longer independent of frequency.
    - ✓ From this, H'(ω)=-H(ω), which is still linear phase ➔ No distortion. The negative signs simply flips the signal.
    - ✓ As long as the system response does not change signs for different values of ω, it can be written as a linear phase transfer function. ➔ **Generalized Linear Phase (GLP)**.
  - In general, a system whose group delay is constant α, for all passband frequencies is considered as a GLP filter.