

# EE403: Digital Communications

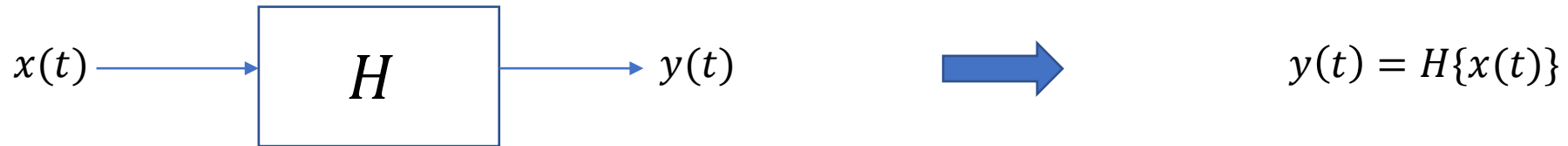
## Lecture 2: Signal and Systems

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March 16, 2022

# Signal and Linear Systems

- In system modeling, actual components such as resistors, capacitors, inductors, ... are not of our concern
- We view a system in terms of input-output operation



# Signal and Linear Systems

- **Linear** systems: Superposition principle is met

$$y(t) = H\{ax_1(t) + bx_2(t)\} = H\{ax_1(t)\} + H\{bx_2(t)\} = ay_1(t) + by_2(t)$$

- **Time-invariant**: Delayed input produces an output with the same delay

$$H\{x(t)\} = y(t) \quad \Rightarrow \quad H\{x(t - t_0)\} = y(t - t_0)$$

- Our focus is only on **linear time-invariance (LTI)** systems
  - Otherwise, analysis is challenging

# Convolution

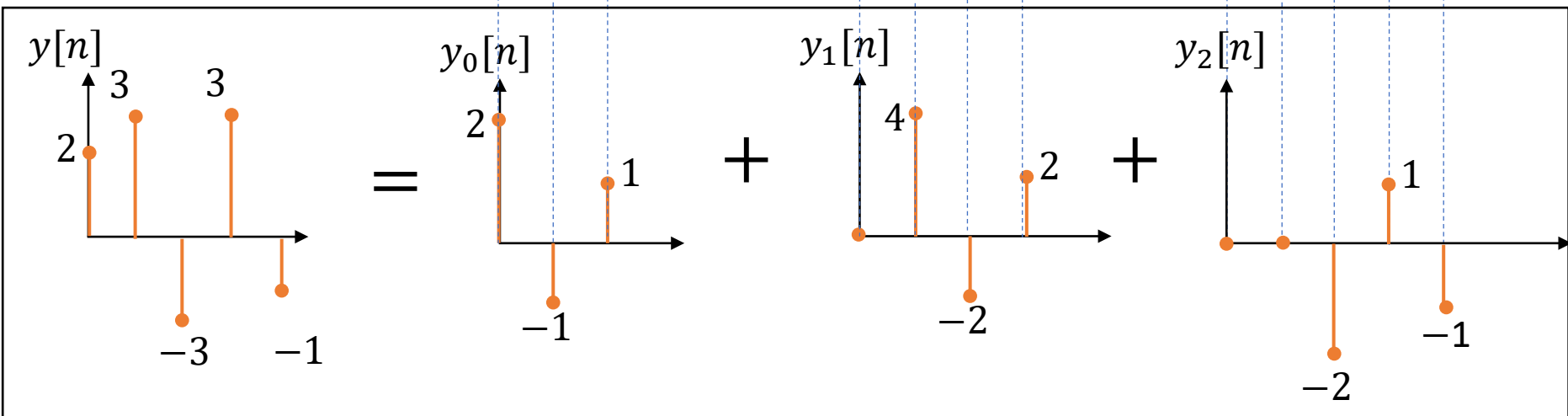
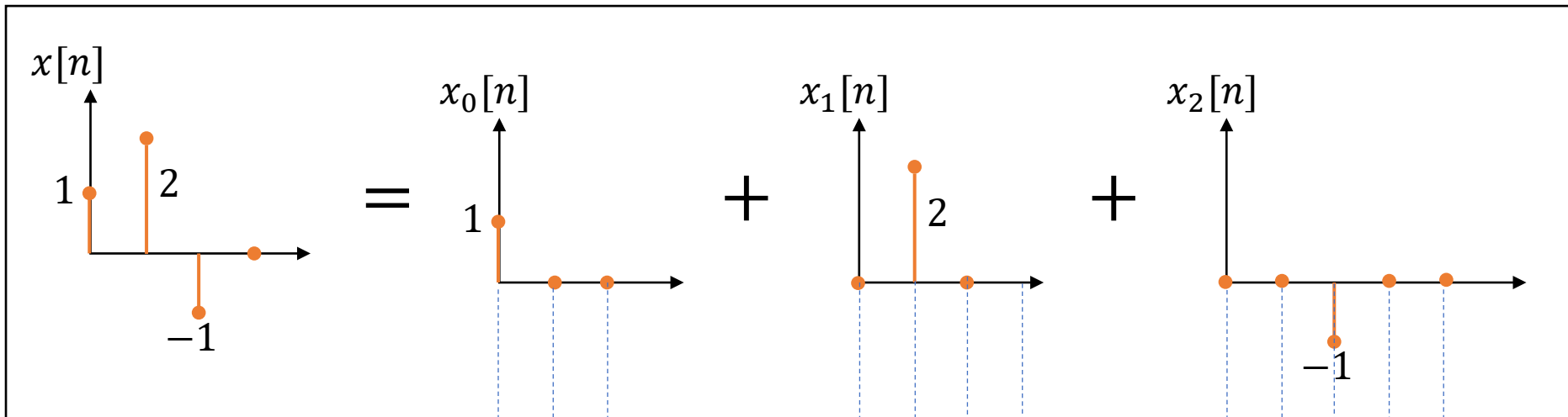
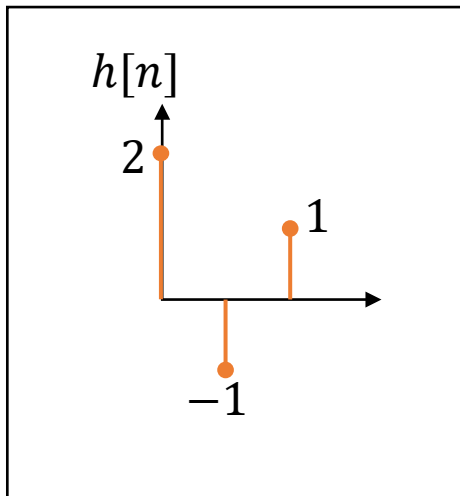
- Definition (continuous-time)

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t - \lambda) d\lambda$$

- Convolution is used to obtain the output of an LTI system for a given input

- Definition (discrete-time)

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n - k]$$



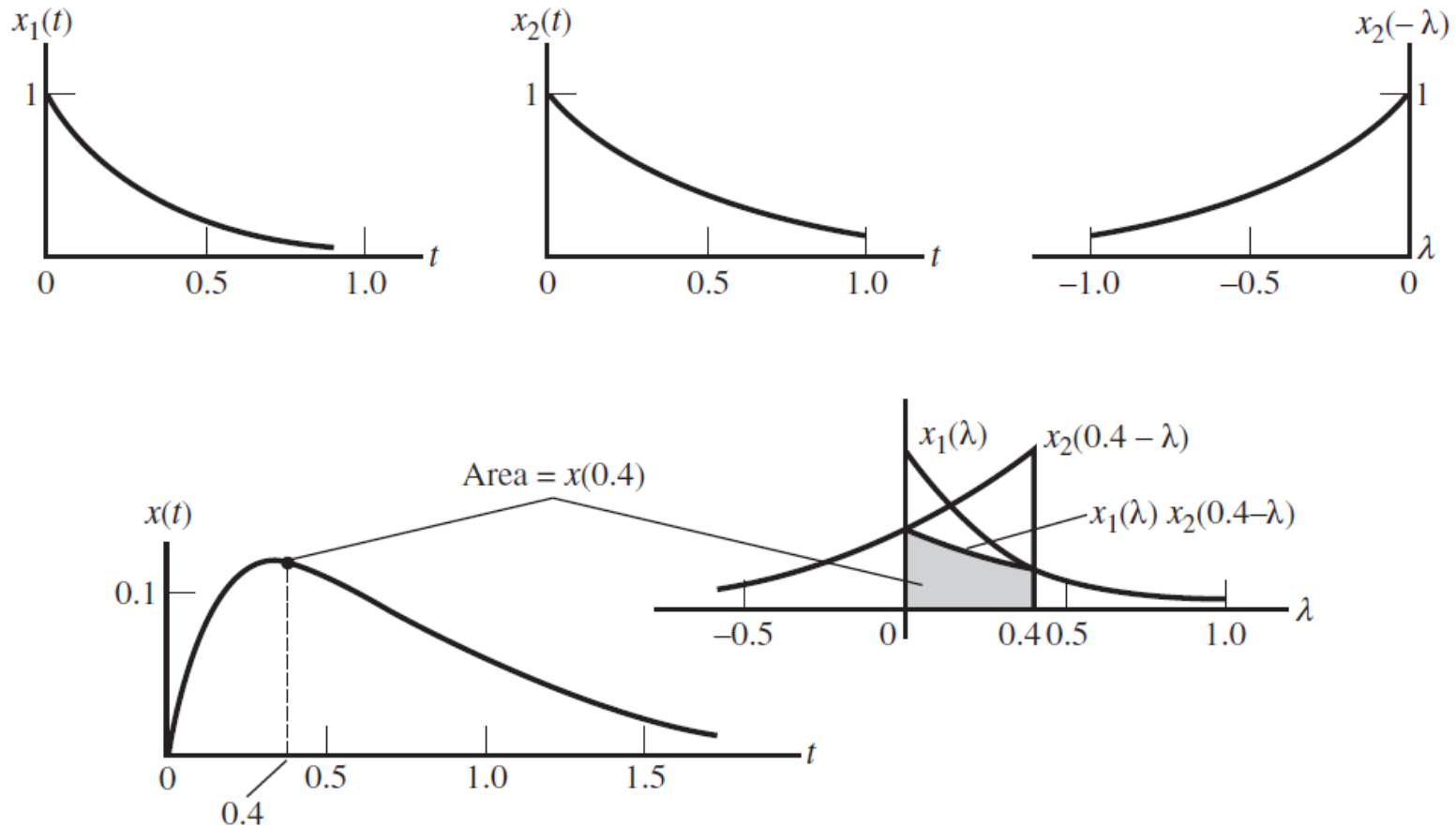
$$y[n] = x_0 h[n - 0] + x_1 h[n - 1] + x_2 h[n - 2] = x[0] h[n - 0] + x[1] h[n - 1] + x[2] h[n - 2]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n - k] = x[n] * h[n] \implies x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) h(t - \lambda) d\lambda$$

# Convolution

- Example

$$x_1(t) = e^{-\alpha t}u(t), x_2(t) = e^{-\beta t}u(t)$$



# Impulse Response

- Q: How to completely describe an LTI system? (=predictable)
- A: by the impulse response,  $h(t) \triangleq H\{\delta(t)\}$ 
  - Response due to an impulse input applied at  $t = 0$
- Then, for any arbitrary input  $x(t)$ ,

$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty} x(\lambda)\delta(t - \lambda)d\lambda \\ \implies y(t) &= H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\lambda)\delta(t - \lambda)d\lambda\right\} \\ &= \int_{-\infty}^{\infty} x(\lambda)H\{\delta(t - \lambda)\}d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda = x(t) * h(t)\end{aligned}$$

# Transfer Function

- For linear systems, output is the **convolution** of input and system response
- Transfer function (in frequency domain) or frequency response:  $H(f)$

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(f) = X(f)H(f)$$

- In general, it is easier to compute  $X(f)H(f)$  than  $x(t) * h(t)$



# Eigenfunction and Eigenvalue

- Recall the eigenvalue and eigenvector in matrix algebra
- Similarly, if a waveform does not change its shape after passing through an LTI system, it is called eigenfunction
- $H\{g(t)\} = cg(t)$ 
  - $c$ : eigenvalue
  - $g(t)$ : eigenfunction
- Recall that any periodic input can be represented by a sum of complex exponentials (Fourier series)

# Eigenfunction and Eigenvalue

- For an arbitrary LTI system,  $g(t) = Ae^{st}$  is an eigenfunction, where  $s$  is any complex number

- Special case:  $s = j2\pi f$

- Proof:

$$\begin{aligned} y(t) &= h(t) * g(t) = \int h(\lambda) Ae^{s(t-\lambda)} d\lambda = \left[ \int h(\lambda) e^{-s\lambda} d\lambda \right] Ae^{st} = \\ &= H(s) Ae^{st} = cg(t) \end{aligned}$$

# Responses to Periodic Inputs

- Since any periodic input can be represented by a sum of complex exponentials, its output will be the sum of complex exponentials, too

- For a unit component  $x(t) = Ae^{j2\pi f_0 t}$ ,

$$y(t) = \int h(\lambda) Ae^{j2\pi f_0(t-\lambda)} d\lambda = \int h(\lambda) e^{-j2\pi f_0 \lambda} d\lambda \cdot Ae^{j2\pi f_0 t} = H(f_0) Ae^{j2\pi f_0 t}$$

- That is, for general periodic  $x(t) = \sum X_n e^{j2\pi n f_0 t}$ ,

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(n f_0) e^{j2\pi n f_0 t}$$

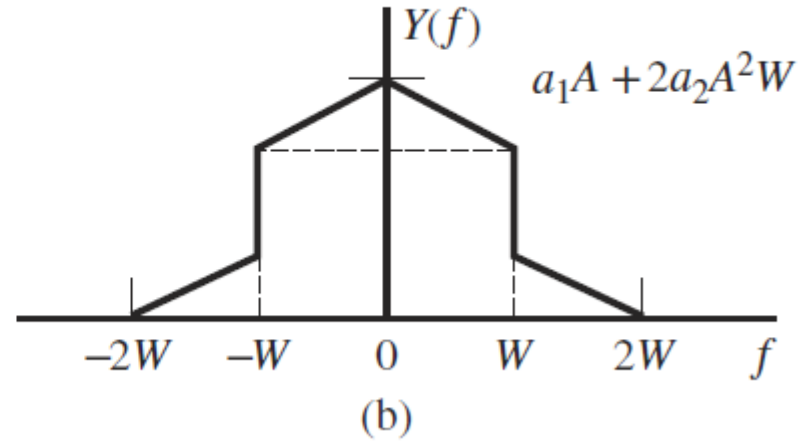
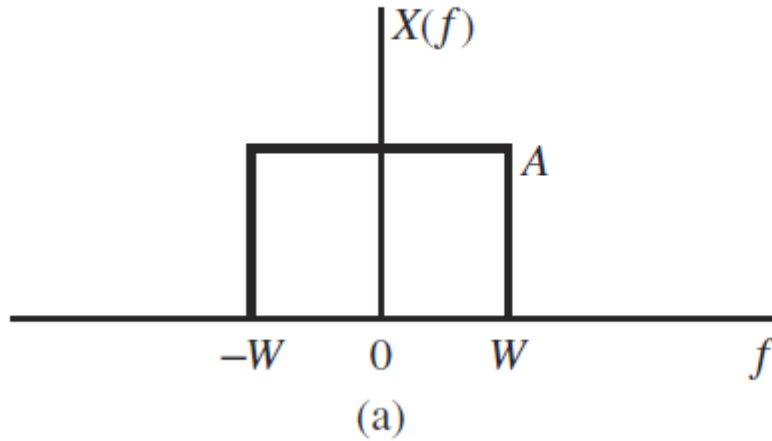
# Nonlinear Distortion

- What if a system is not LTI?
  - The response has different frequency components from the input
- Example:  $y(t) = a_1 x(t) + a_2 x^2(t)$  and  $x(t) = A_1 \cos(w_1 t) + A_2 \cos(w_2 t)$

$$\begin{aligned} y(t) = & \underbrace{a_1 [A_1 \cos(w_1 t) + A_2 \cos(w_2 t)]}_{\text{desired}} \\ & + \underbrace{\frac{a_2}{2} (A_1^2 + A_2^2) + \frac{a_2}{2} [A_1^2 \cos(2w_1 t) + A_2^2 \cos(2w_2 t)]}_{\text{harmonic}} \\ & + \underbrace{a_2 A_1 A_2 [\cos((w_1 + w_2)t) + \cos((w_1 - w_2)t)]}_{\text{intermodulation}} \end{aligned}$$

# Nonlinear Distortion

- As the system is second-order, the max frequency is doubled
- This is a general result for arbitrary input  $x(t)$ 
  - Example: Consider  $X(f) = \Pi(f/2W)$



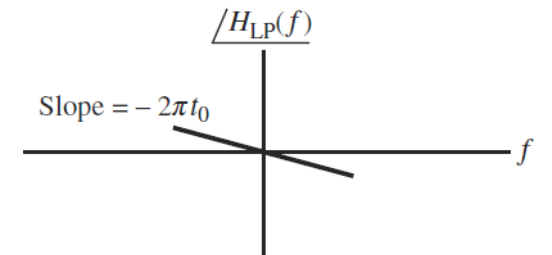
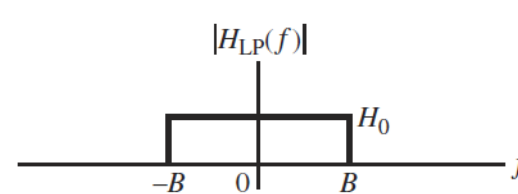
- If third-order, then tripled, and so forth...

# Ideal Filters

- Idealized filters must have **constant** amplitude and **linear** phase response

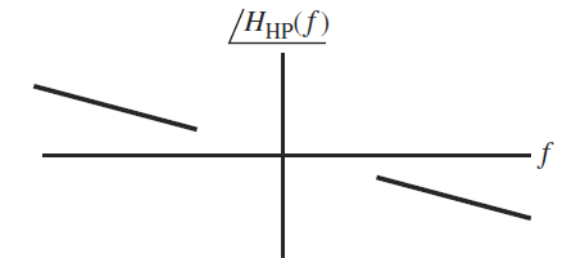
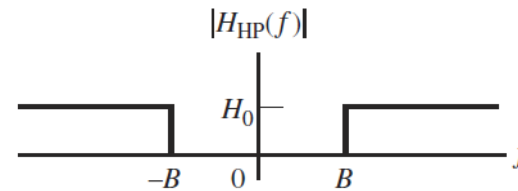
- Low-pass

- $H_{LP}(f) = H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$



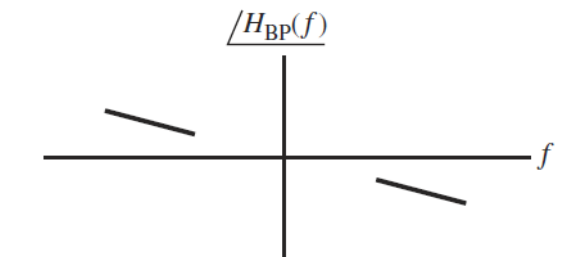
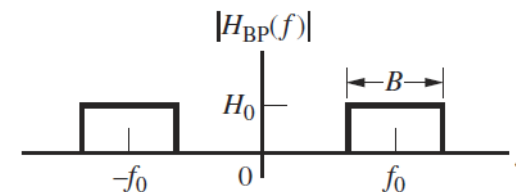
- High-pass

- $H_{HP}(f) = H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$

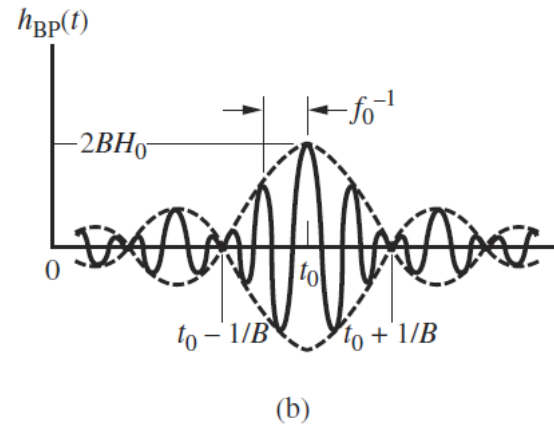
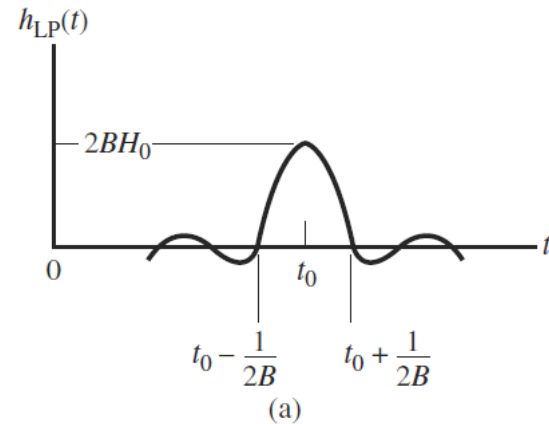


- Band-pass

- $H_{BP}(f) = H_0 \left( \Pi\left(\frac{f-f_0}{B}\right) + \Pi\left(\frac{f+f_0}{B}\right) \right) e^{-j2\pi f t_0}$



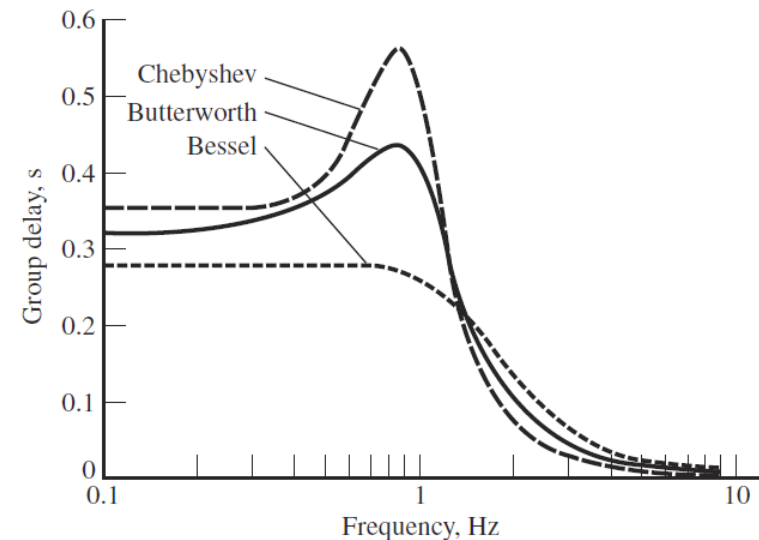
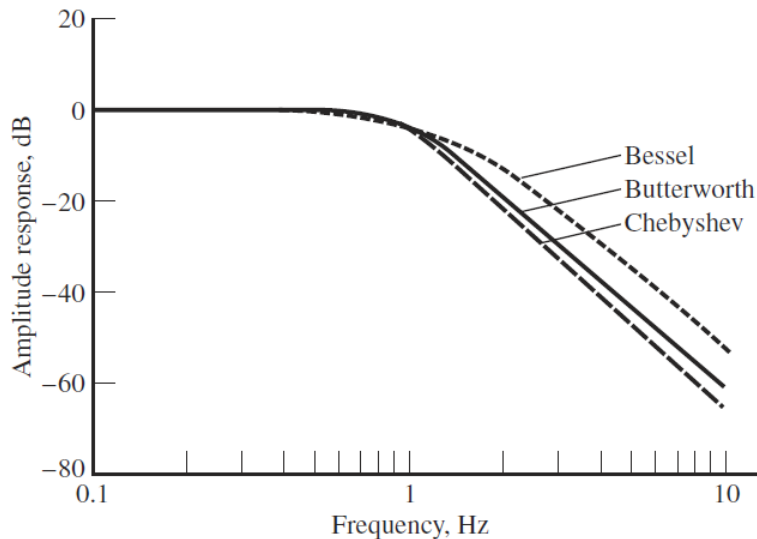
# Ideal Filters



- The ideal low-pass filter is
$$h_{LP}(t) = 2BH_0 \text{sinc}(2B(t - t_0))$$
- The ideal band-pass filter is
$$h_{BP}(t) = 2BH_0 \text{sinc}(B(t - t_0)) \cos(2\pi f_0(t - t_0))$$
- ➔ infinite-length responses, so impossible to implement in practice
  - Called an infinite impulse response (**IIR**) filter
  - We need a finite impulse response (**FIR**) filter

# Realizable Filters

- Several practical filters to approximate them
- Low-pass filters: Butterworth, Chebyshev, Bessel, ...
- Band-pass filters: From LP filters, transform the passband



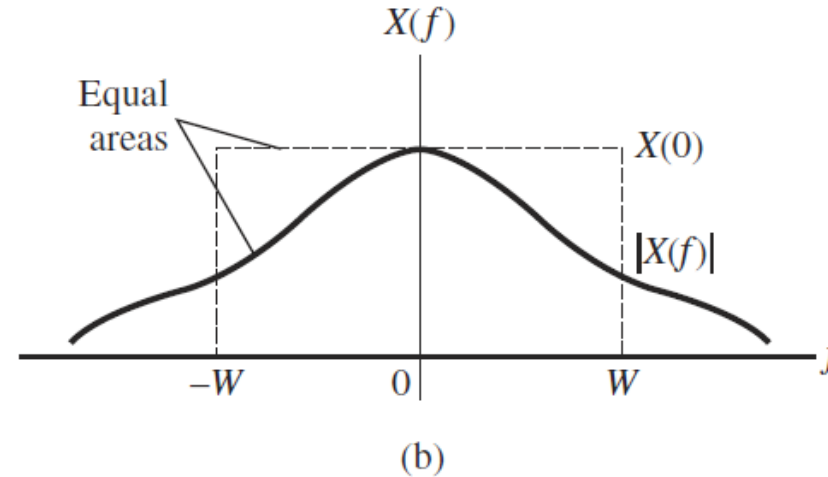
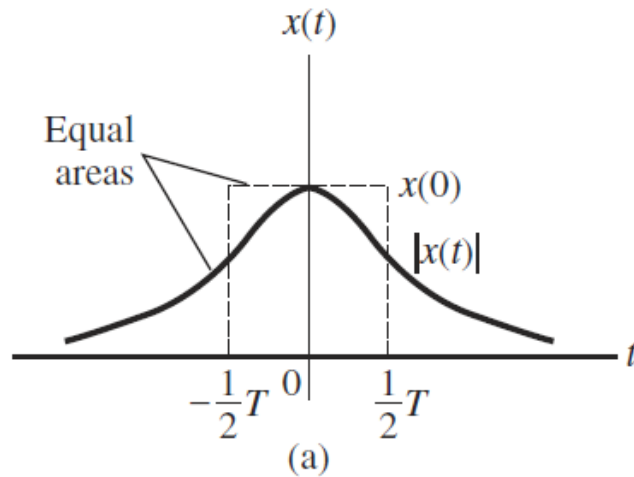


# Time-bandwidth Product

- A **narrow** duration ( $T$ ) signal in time must have **wider** bandwidth ( $W$ ) in frequency, and vice versa

$$WT \geq \text{constant}$$

- Like the uncertainty principle in physics



# Time-bandwidth Product

- Let us model  $x(t)$  by a rectangle pulse with equal area

$$Tx(0) = \int_{-\infty}^{\infty} |x(t)|dt \geq \int_{-\infty}^{\infty} x(t)dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi 0t}dt = X(0)$$

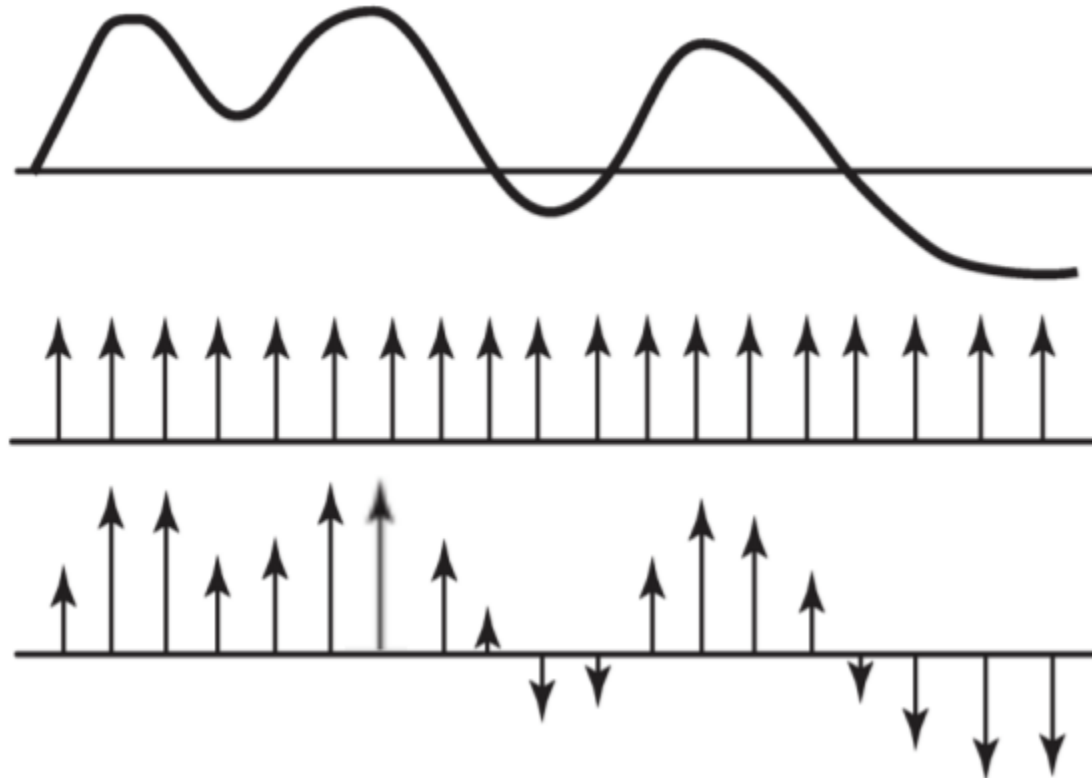
- Let us model  $X(f)$  by a rectangle response with equal area

$$2WX(0) = \int_{-\infty}^{\infty} |X(f)|df \geq \int_{-\infty}^{\infty} X(f)df = \int_{-\infty}^{\infty} X(f)e^{-j2\pi f0}df = x(0)$$

- Collecting the pair of inequalities,  $2W \geq \frac{x(0)}{X(0)} \geq \frac{1}{T} \Rightarrow WT \geq \frac{1}{2}$
- A **narrow** pulse width in time must have **wider** frequency spectrum

# Sampling Theory

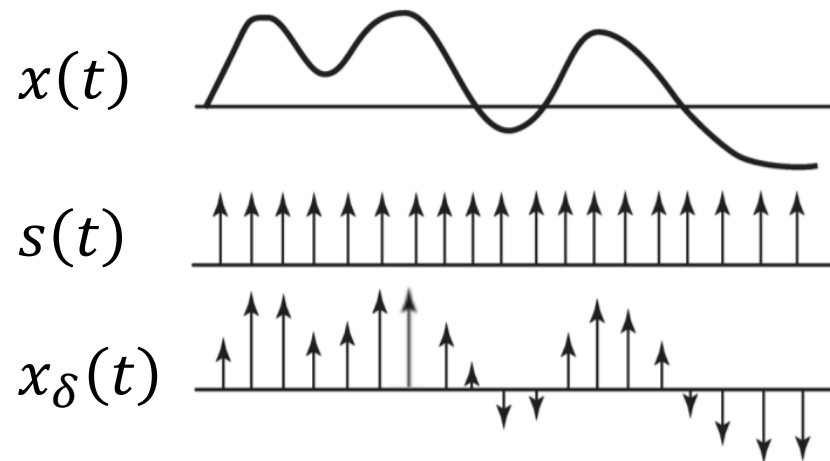
- Sampling is used in numerous applications, e.g., digital signal processing (DSP), computer simulation, ...



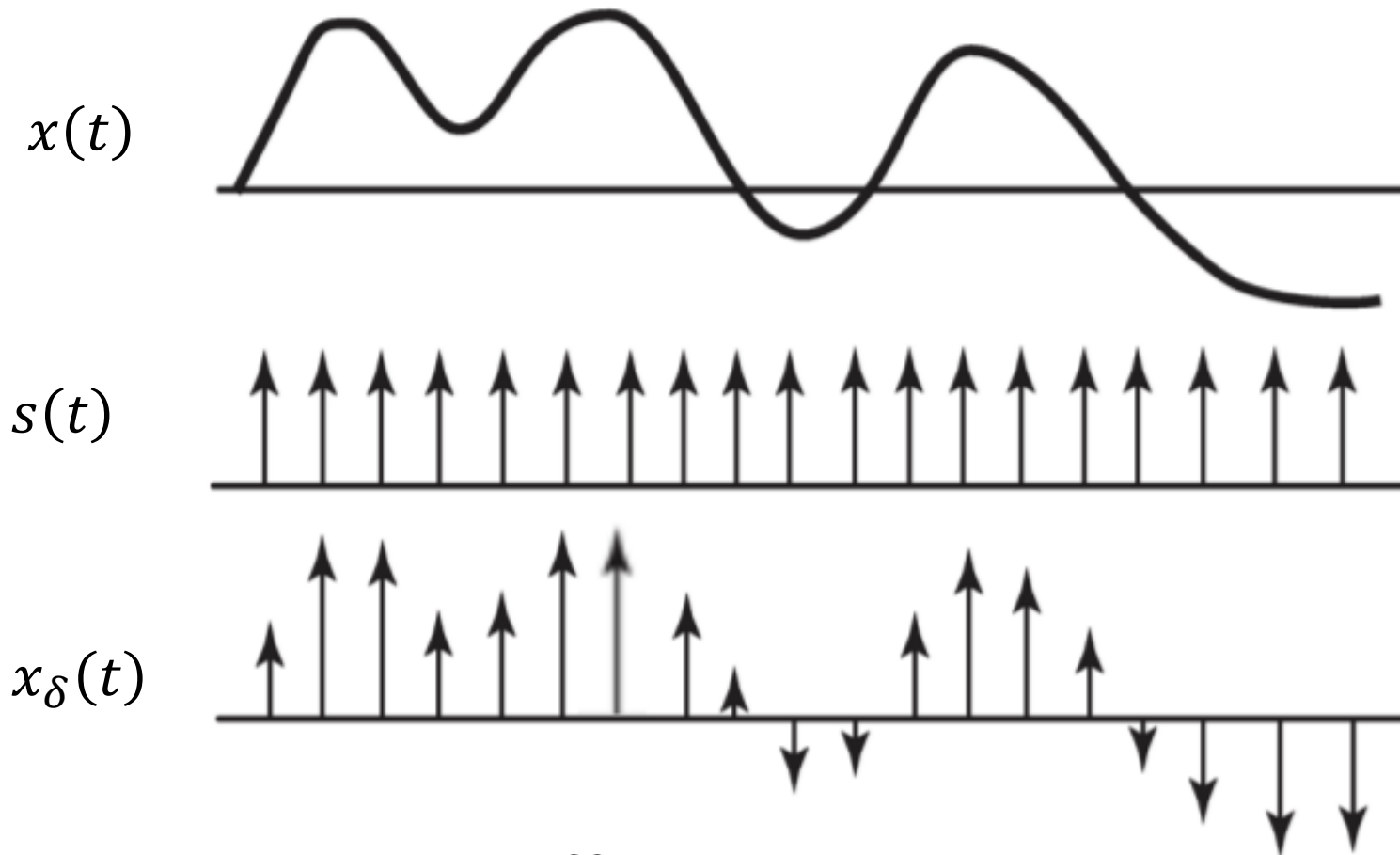
# Sampling Theory

- Let the sampling period be  $T_s$
- Ideal sampling uses impulse train  $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$
- The sampled signal (in continuous-time) is

$$x_\delta(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$



# Sampling Theory



$$x_\delta(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

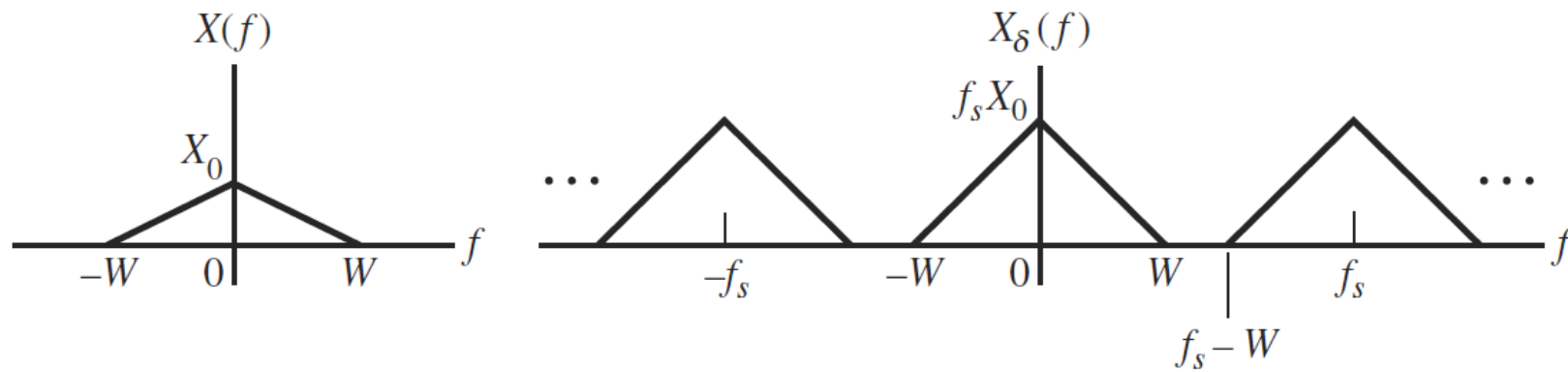
# Sampling Theory

- In frequency domain,

$$\begin{aligned} X_\delta(f) &= X(f) * S(f) = X(f) * \left[ f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \\ &= f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \end{aligned}$$

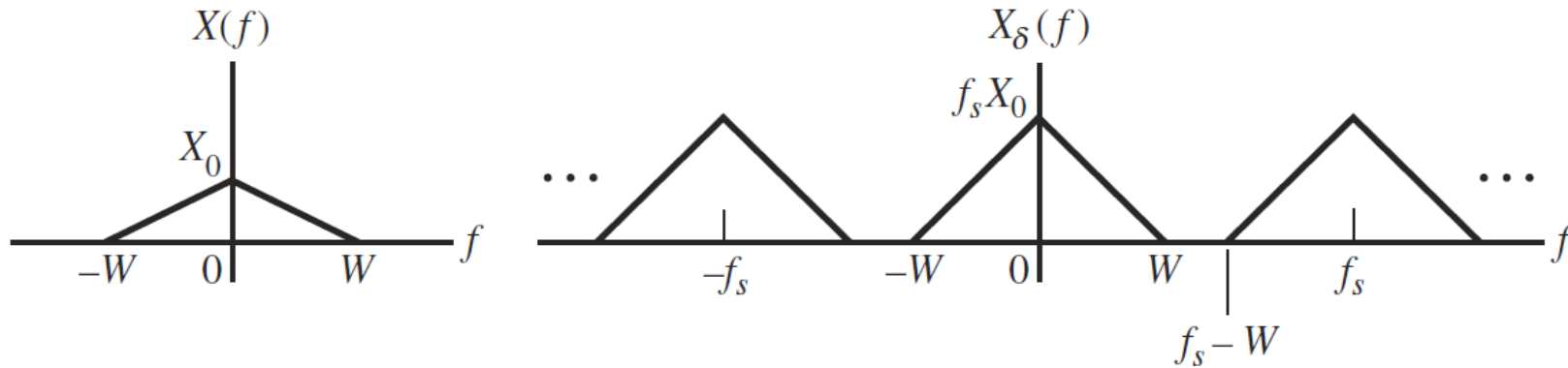
- Sampling  $\iff$  duplication of shifted spectrum at every  $nf_s$
- Perfect recovery  $\iff$  only one copy of spectra after proper processing

# Sampling Theory



# Aliasing

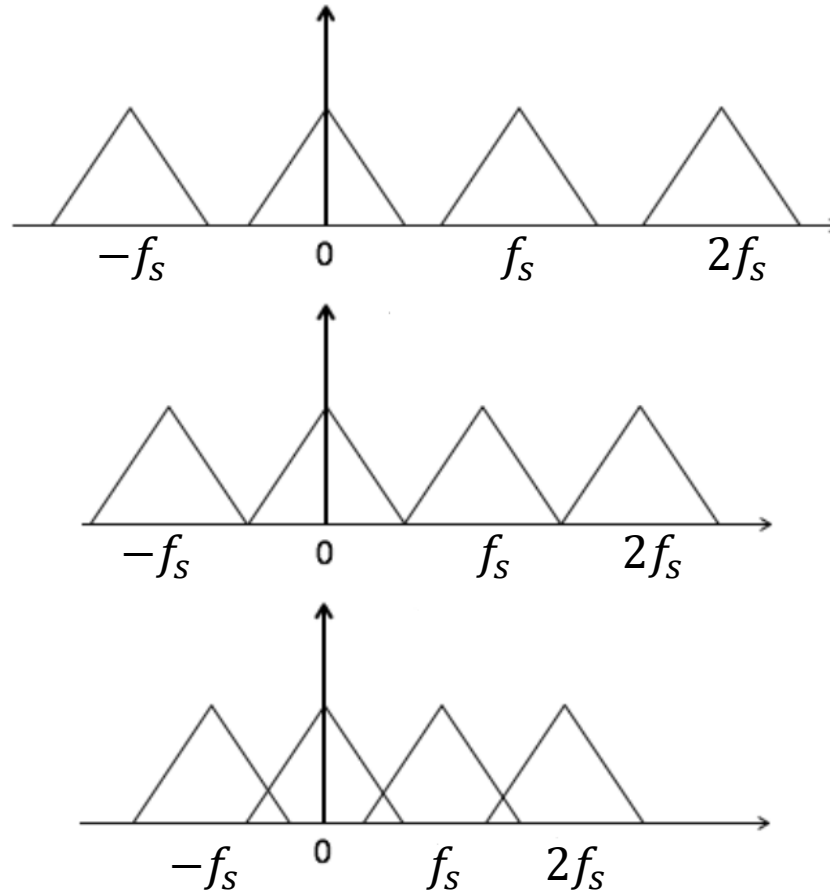
- Let the signal be bandwidth-limited with bandwidth  $W$
- Aliasing: If  $f_s < 2W$ , then the replicas of  $X(f)$  overlap in frequency domain



- (No aliasing in the above as  $f_s > 2W$ )



# Aliasing



Oversampling:  $f_s > 2W$

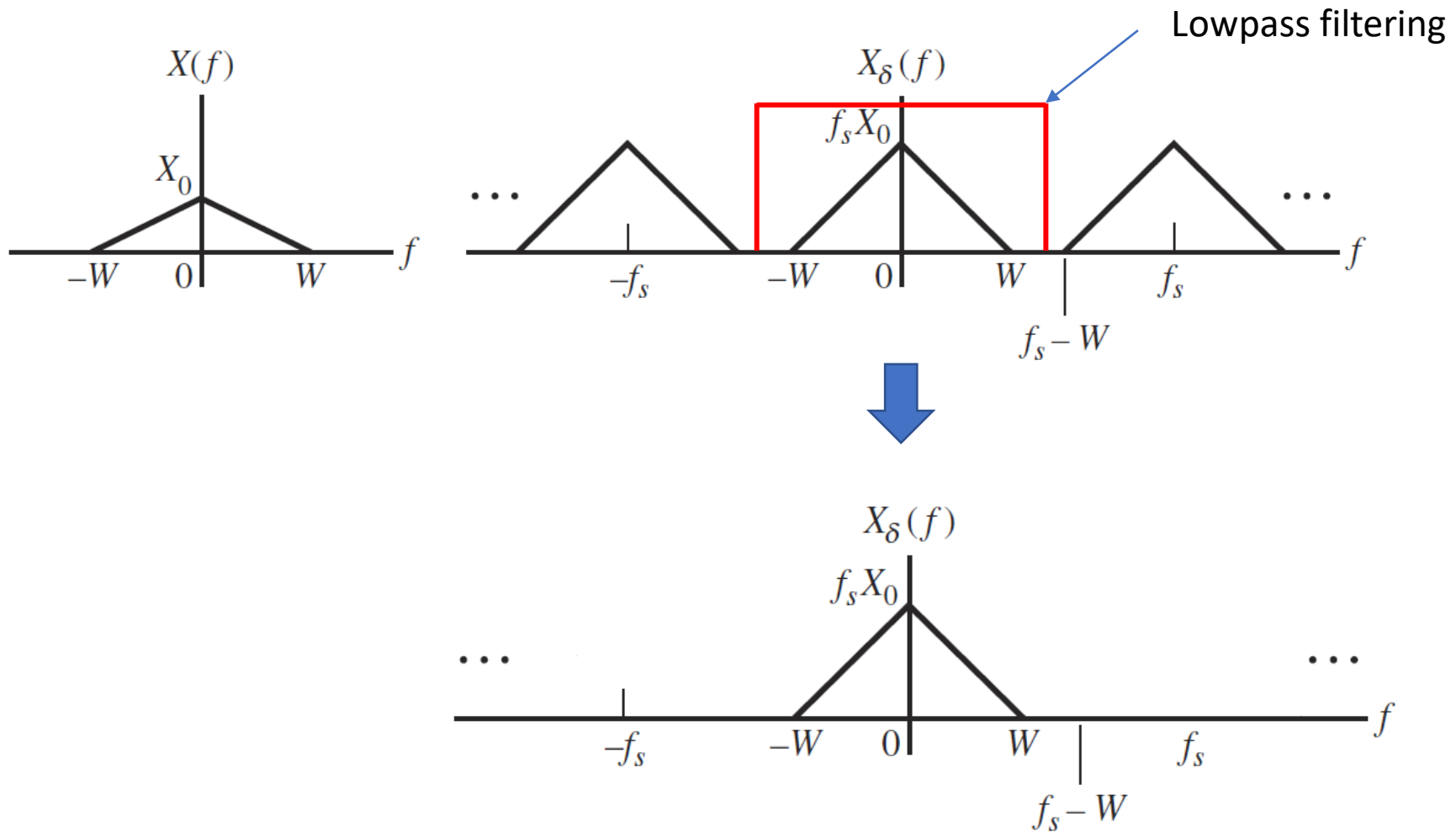
Perfect sampling:  $f_s = 2W$

Undersampling:  $f_s < 2W$

# Nyquist Sampling Theorem

- Thm: Let  $x(t)$  be a band-limited signal with bandwidth  $W$ . Then,  $x(t)$  can be perfectly recovered from its samples  $\{x(nT_s)\}_n$  if sampling rate  $f_s = \frac{1}{T_s} \geq 2W$
- $2W$ : Nyquist rate
- Undersampling if  $f_s < 2W$
- Oversampling if  $f_s > 2W$
- How to recover?
  - Pass  $x_\delta(t)$  through an ideal lowpass filter with bandwidth  $B$ , where  $W < B < f_s - W$

# Nyquist Sampling Theorem



# Nyquist Sampling Theorem

- Ideal reconstruction filter (lowpass filter)

$$H(f) = \Pi\left(\frac{f}{2B}\right), \quad W \leq B \leq f_s - W$$

- Less ideal reconstruction filter (lowpass filter)

$$H(f) = H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}, \quad W \leq B \leq f_s - W$$

- Passing  $x_\delta(t)$  through this filter,  $Y(f) = f_s H_0 X(f) e^{-j2\pi f t_0}$
- That is, in time-domain,  $y(t) = f_s H_0 x(t - t_0)$

# Nyquist Sampling Theorem

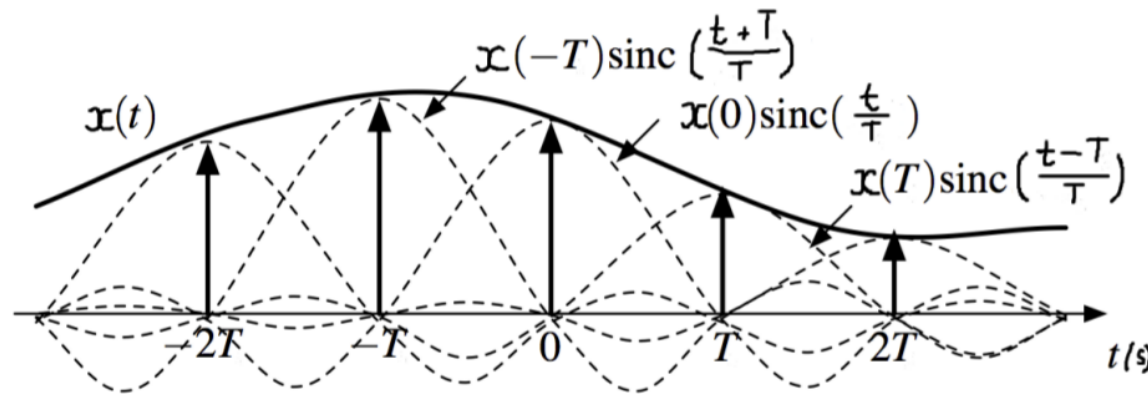
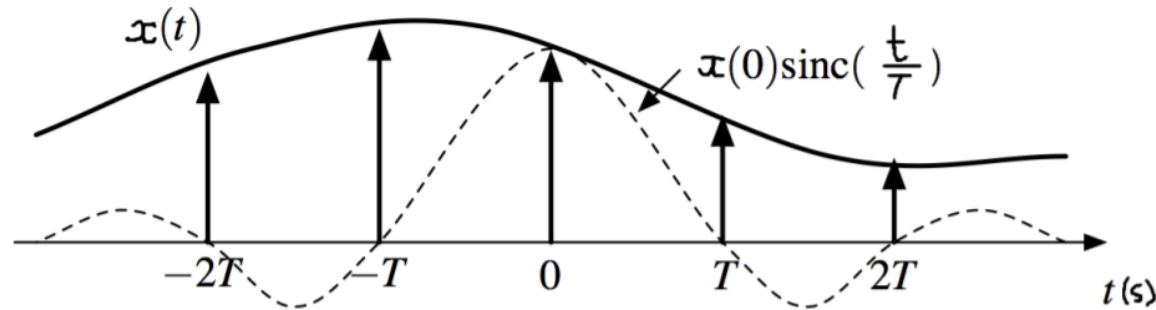
- Alternative expression: Let  $h(t)$  be the impulse response of the ideal low-pass filter

$$\begin{aligned} y(t) &= \underbrace{\left[ \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) \right]}_{\text{sampled input}} * \underbrace{h(t)}_{\text{impulse response}} = \sum_{n=-\infty}^{\infty} x(nT_s) h(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} x(nT_s) [2BH_0 \text{sinc}(2B(t - t_0 - nT_s))] \\ &= 2BH_0 \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}(2B(t - t_0 - nT_s)) \end{aligned}$$

- A weighted sum of shifted sinc functions with weights  $x(nT_s)$ !

# Nyquist Sampling Theorem

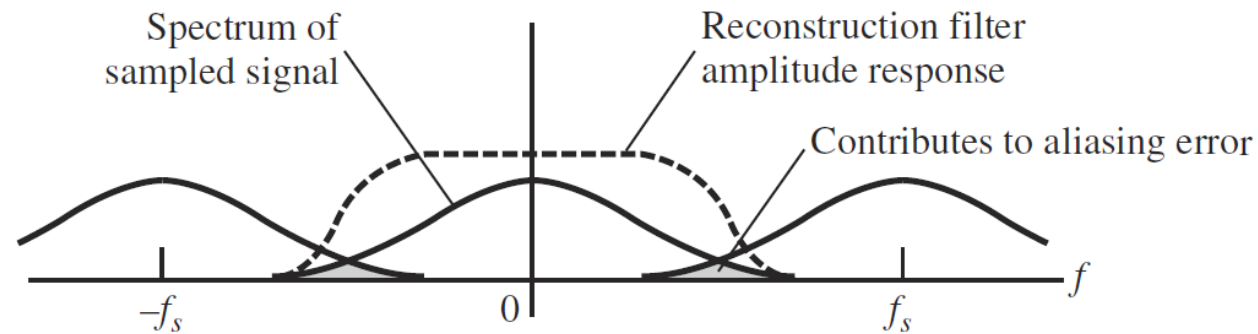
- Let  $B = 0.5f_s$ ,  $H_0 = T_s$ ,  $t_0 = 0$ . Then,  $y(t) = \sum_{-\infty}^{\infty} x(nT_s)\text{sinc}(f_s t - n)$



Orthogonal basis!

# Reconstruction Errors

- Spectrum aliasing either because  $x(t)$  is not bandlimited or  $f_s < 2W$



- Nonideal lowpass filter

