EE403: Digital Communications

Lecture 9: Digital Communications with Noise

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April 6, 2022

Announcement

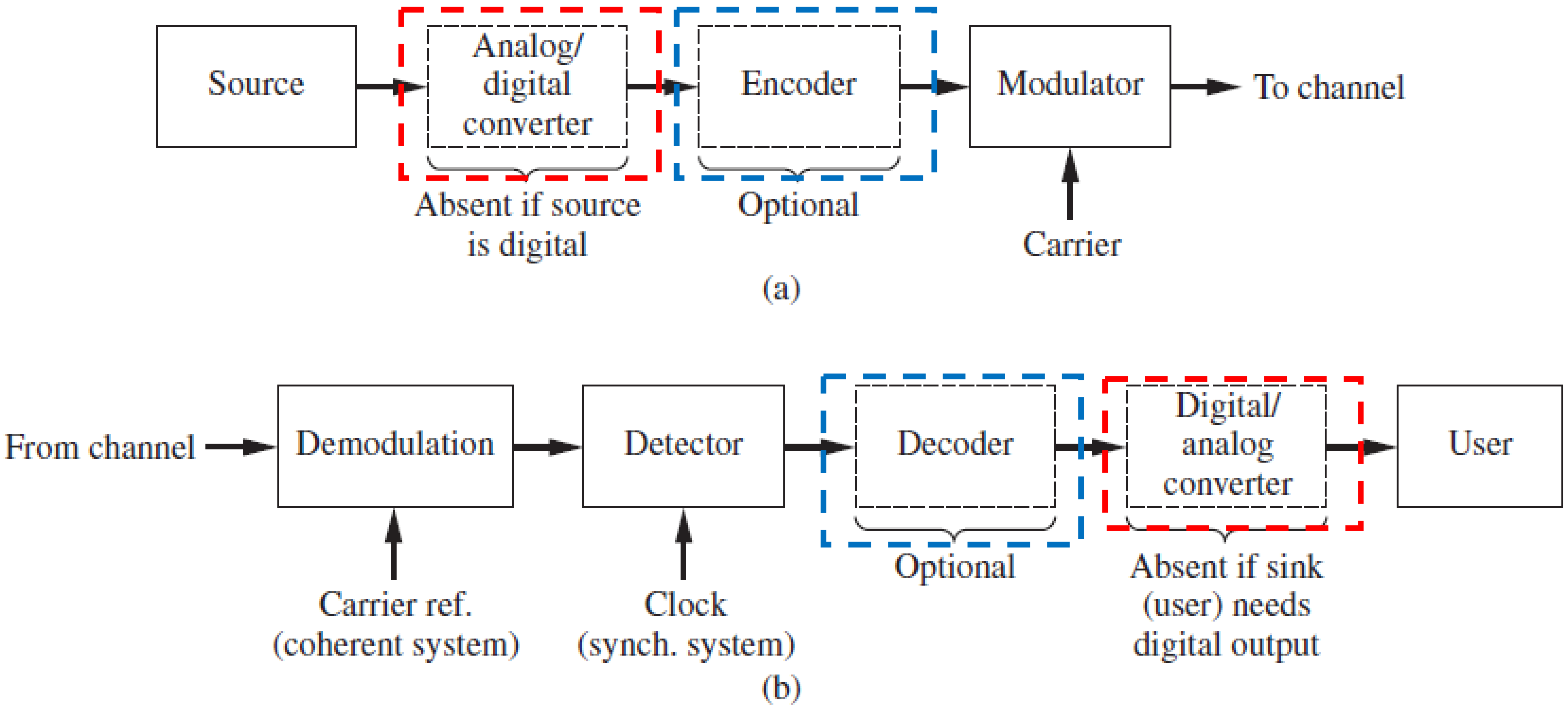
- Midterm
  - 4/18 Mon, 10:30-11:59, E3-112
  - Closed book

- In person class next week
  - The univ recommends to offer in person lectures for small-size classes
  - Mon, Wed, 10:30-11:59, E3-112

- Office hour will be also in person by default
  - Every Monday, 10:00-10:30, E3-311 or walk-ins/emails are welcome
  - Email me if you want to discuss online

- Homework 2 will be posted soon
  - No late submissions are allowed
  - They will be graded for educational purpose, but not counted in the final grade

Digital Communications Systems



Absent in modern comm systems

Essential in modern comm systems (not covered in this Chap)

Figure 9.1
Block diagram of a digital data transmission system. (a) Transmitter. (b) Receiver.

Digital Communications Systems

- (assumption) bits are equally likely
  - If not, bits can be mapped to equally likely bits, which is called source coding, e.g., Huffman code
  - Encoder also maps to another bit streams to be robust against noise, called channel coding, e.g., Turbo code, LDPC code

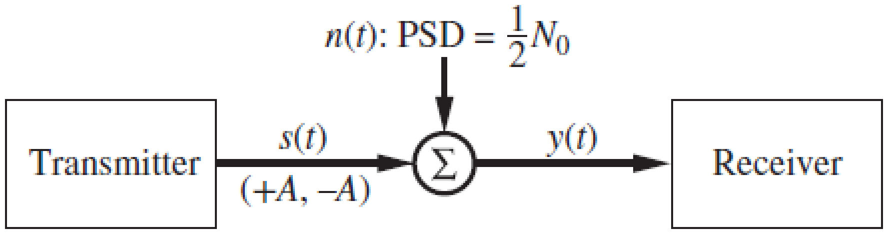
- Coherent systems: Rx knows Tx frequency and phase exactly
  - Otherwise, called noncoherent

- Synchronous: Rx knows start/end time of each symbol exactly
  - Otherwise, called asynchronous

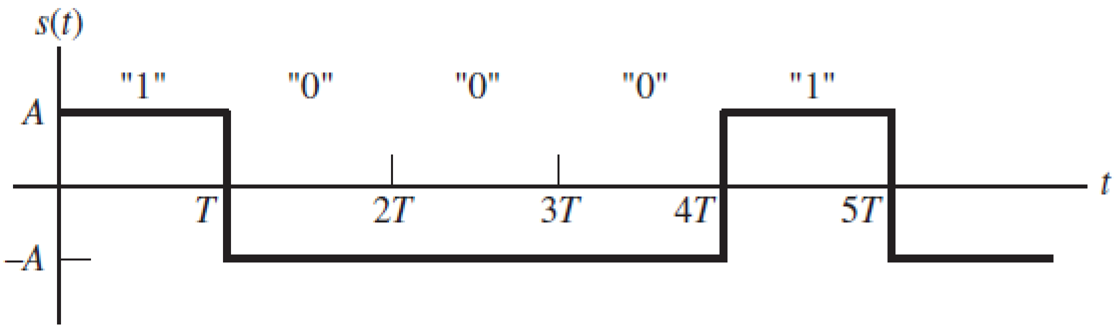
- Unlike Chap. 5, our focus is a system in the presence of **noise**
  - Wireless communication, the signal power is often close to the noise power
  - How to minimize error probability (P_E)?

Baseband Transmission with Noise

- Given continuous signal with (white Gaussian) noise, how to distinguish "0" and "1"?
- What we learned in Chap. 5 is: 1) sample at mT and 2) say "0" if $y(mT) < 0$, say "1" otherwise $\Rightarrow$ it does not work well with noisy channel



(a)



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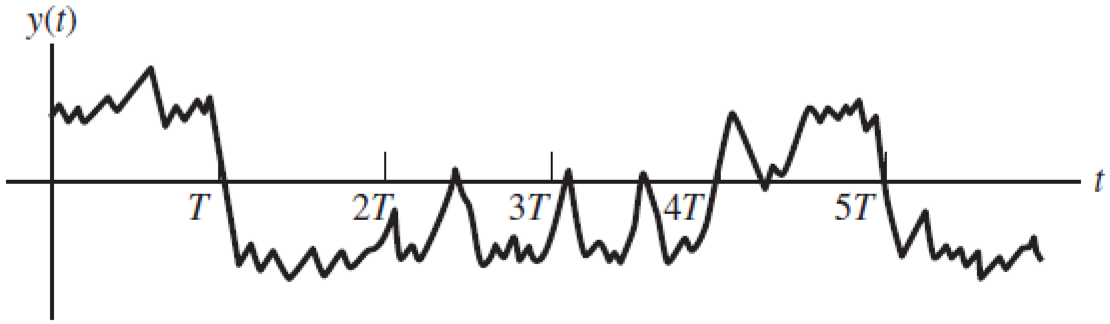
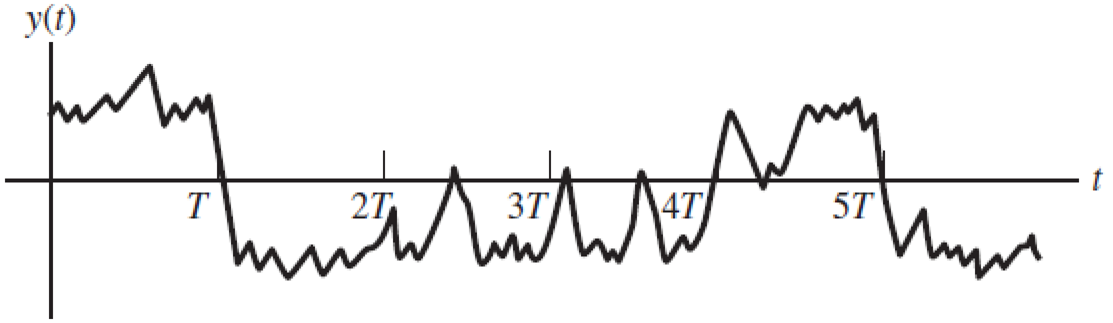


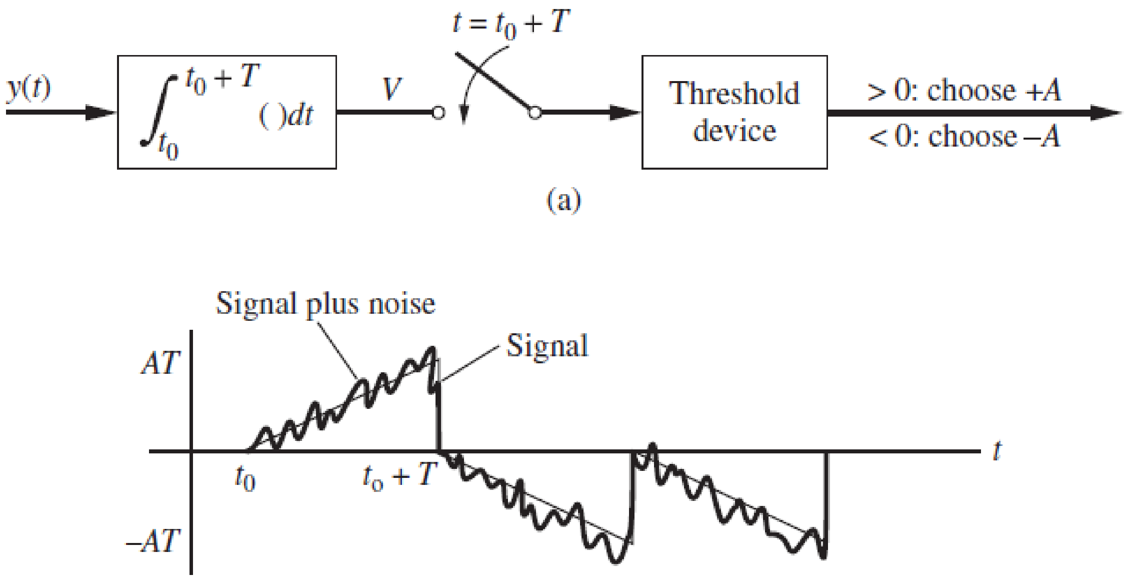
(c)

Baseband Transmission with Noise

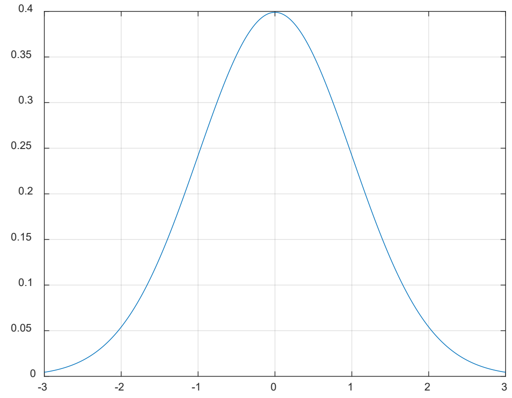
- A better way is to use "integrate-and-dump" detector
- Calculate the area under each symbol and say "0" if the area > 0. Say "1" otherwise
- After each symbol, reset (=dump) the calculation



(c)



Q-function



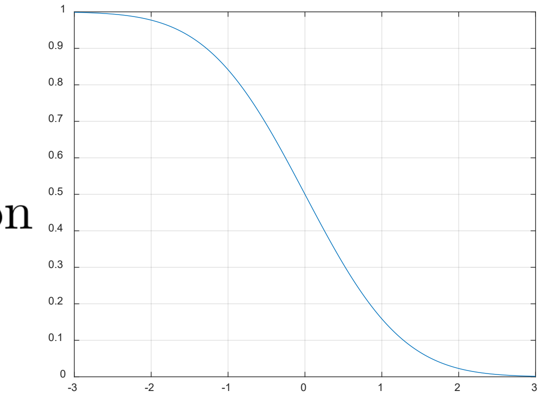
- A Gaussian RV has a pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dt$$

  - Often denoted by $\mathcal{N}(\mu, \sigma^2)$
  - When $\mu = 0$, $\sigma^2 = 1$, it is called **standard** Gaussian or standard normal.

- The Q-function is the tail distribution of the standard Gaussian distribution. Formally,

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) dt$$



- The Q-function is non-analytic, i.e., does not have a closed-form expression
- A good approximation for large x is $Q(x) \approx \frac{e^{-x^2/2}}{x\sqrt{2\pi}}$

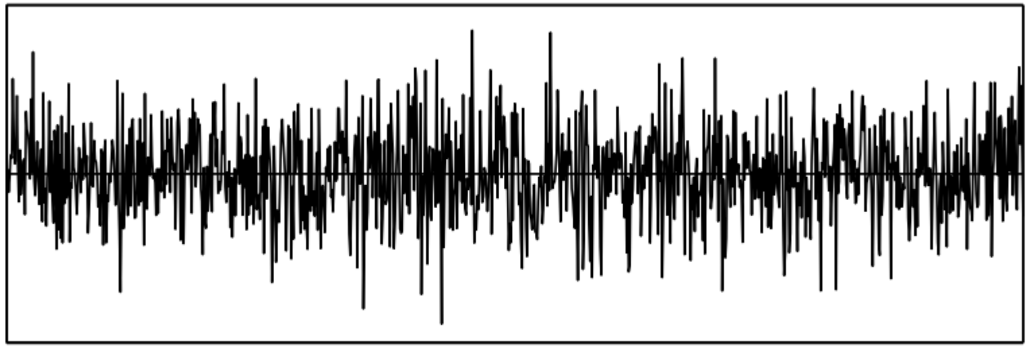
Baseband Transmission with Noise

- Performance analysis: The output at the end of each symbol is

$$V = \int_{t_0}^{t_0+T} s(t) + n(t)dt = \begin{cases} +AT + N & \text{if } +A \text{ is sent} \\ -AT + N & \text{if } -A \text{ is sent,} \end{cases}$$

  where $N = \int_{t_0}^{t_0+T} n(t)dt$ is a random variable

- White Gaussian noise: the most basic/essential noise in communication
  - Stationary and ergodic
  - $n(t)$ is Gaussian
  - $\mathbb{E}[n(t)] = 0$ for all t
  - $\mathbb{E}[n(t)n(\tau)] = \frac{N_0}{2}\delta(t-\tau)$



Baseband Transmission with Noise

- As $n(t)$ is a Gaussian process, its time average is zero $\Rightarrow \mathbb{E}[N] = 0$
- Variance

$$Var(N) = \mathbb{E}[N^2] = \mathbb{E}\left[\left(\int_{t_0}^{t_0+T} n(t)dt\right)^2\right] = \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} \mathbb{E}[n(t)n(\tau)]dtd\tau$$

$$= \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} \frac{N_0}{2}\delta(t-\tau)dtd\tau = \int_{t_0}^{t_0+T} \frac{N_0}{2}d\tau = \frac{N_0T}{2}$$

- $\int n(t)dt$ is a sum of (infinitesimal) independent Gaussians $\Rightarrow N$ is Gaussian too

$$f_N(z) = \frac{1}{\sqrt{\pi N_0 T}} e^{-\frac{z^2}{N_0 T}}$$

Baseband Transmission with Noise

- If A was sent, $AT + N$ is Gaussian with mean AT and variance $\frac{N_0 T}{2}$
- Decision threshold is 0
- Error probability

$$P(\text{say "-A"} \mid \text{"A" sent}) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0 T}} e^{-\frac{(z-AT)^2}{N_0 T}} dz$$

$$= \int_{\sqrt{2A^2T/N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \triangleq Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

- $Q(x) \triangleq \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ is the Q-function: It is the tail CDF of the standard Gaussian RV

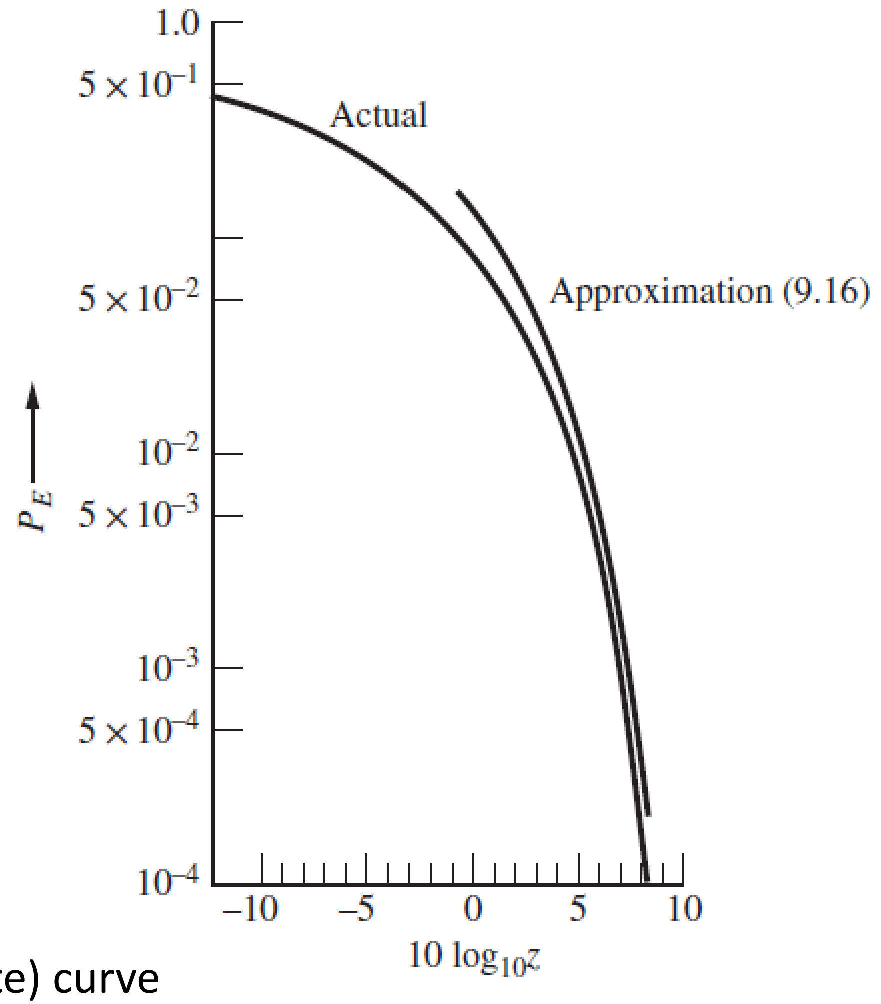
Baseband Transmission with Noise

- P(say "-A" | "A" sent) is the same by symmetry
- $P_E = P(+A)P(-A|+A) + P(-A)P(+A|-A) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$
- Energy in each pulse (=each bit) is $A^2T \triangleq E_b$
- Noise PSD $= N_0$
  - Noise PSD is N_0 or $N_0/2$?
  - N_0 for single sideband
  - $N_0/2$ for double sideband $\Rightarrow$ PSD$=N_0$ considering positive/negative band both
- Signal-to-noise-ratio (SNR) $= \frac{E_b}{N_0} \Rightarrow P_E = Q(\sqrt{2SNR}) = Q(\sqrt{2E_b/N_0})$

Baseband Transmission with Noise

- As $Q(x) \approx \frac{e^{-x^2/2}}{x\sqrt{2\pi}}$ when x is large

$$P_E \approx \frac{e^{-E_b/N_0}}{2\sqrt{\pi E_b/N_0}}$$



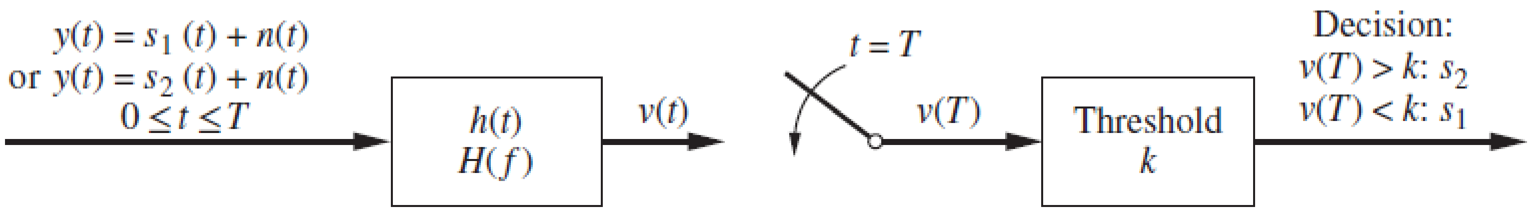
It is called **BER** (bit error rate) curve

Transmission with Arbitrary Shapes

- Instead of square pulse, let us consider arbitrary pulse shapes
- Let $s_1(t)$ be the shape of bit "1"
- Let $s_2(t)$ be the shape of bit "0"
- Energies $E_1 = \int_0^T s_1^2(t)dt$ and $E_2 = \int_0^T s_2^2(t)dt$
- Consider a filter-based detector with impulse response $h(t), H(f)$

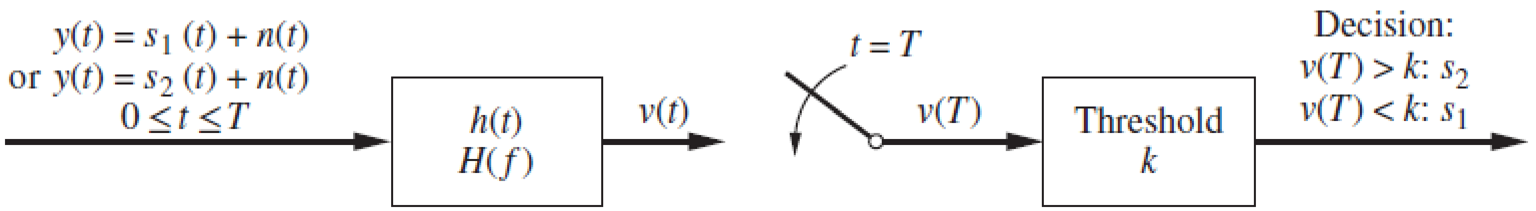
$$y(t) = s_1(t) + n(t) \quad \text{or} \quad y(t) = s_2(t) + n(t)$$

- After the filter, it makes a decision based on the threshold



Transmission with Arbitrary Shapes

- Let $s_{01}(T)$ be the filter output when the input is clean $s_1(t)$
- Let $s_{02}(T)$ is for the clean $s_2(t)$. Assume $s_{01}(T) < s_{02}(T)$
- Let $n_0(t)$ be the noise at output
- Noise PSD at output $= S_{n_0}(f) = \frac{1}{2}N_0|H(f)|^2$
- $\Rightarrow$ noise power $= \sigma_0^2 = \int_{-\infty}^{\infty} \frac{1}{2}N_0|H(f)|^2 df$
- $V = s_{01}(T) + N \quad$ or $\quad s_{02}(T) + N$
- It means, V is mean $s_{01}(T)$ or $s_{02}(T)$ Gaussian with variance σ_0^2



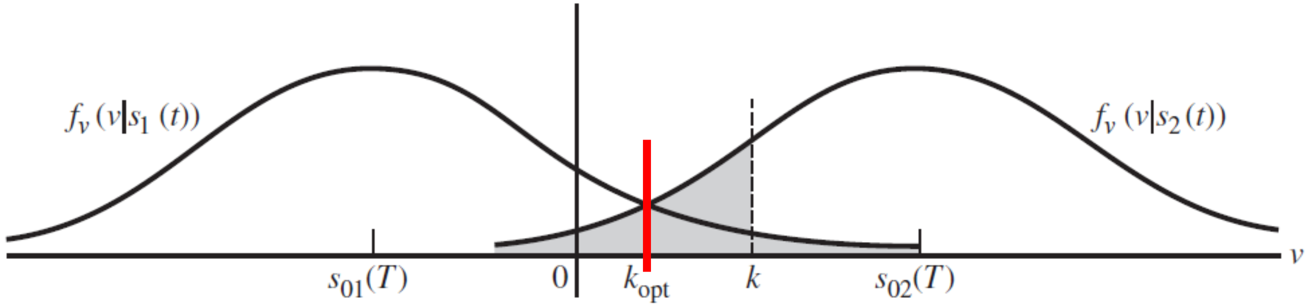
Transmission with Arbitrary Shapes

- Error analysis

$$P(s_2|s_1) = \int_k^{\infty} f_V(v|s_1(t))dt = \int_k^{\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(v-s_{01}(T))^2}{2\sigma_0^2}}$$

$$P(s_1|s_2) = \int_{-\infty}^{k} f_V(v|s_2(t))dt = \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(v-s_{02}(T))^2}{2\sigma_0^2}}$$

- Optimal threshold is $k_{opt} = \frac{1}{2}[s_{01}(T) + s_{02}(T)]$
- $\Rightarrow P(s_2|s_1) = P(s_1|s_2) = Q(\frac{s_{02}(T) - s_{01}(T)}{2\sigma_0})$



Matched Filter

- Let $\zeta = \frac{s_{02}(T) - s_{01}(T)}{\sigma_0}$, which we have to maximize to minimize P_e

$$\zeta^2 = \frac{(s_{02}(T) - s_{01}(T))^2}{\sigma_0^2} = \left. \frac{(s_{02}(t) - s_{01}(t))^2}{\sigma_0^2} \right|_{t=T}$$

- Note that

$$\begin{aligned} s_{02}(t) - s_{01}(t) &= s_2(t) * h(t) - s_1(t) * h(t) \\ &= (s_2(t) - s_1(t)) * h(t) = \int_{-\infty}^{\infty} (S_2(f) - S_1(f)) H(f) e^{j2\pi f t} df \end{aligned}$$

- Using $\sigma_0^2 = \frac{N_0}{2} \int |H(f)|^2 df$,

$$\zeta^2 = \frac{\left| \int_{-\infty}^{\infty} (S_2(f) - S_1(f)) H(f) e^{j2\pi f T} df \right|^2}{\frac{N_0}{2} \int |H(f)|^2 df}$$

Matched Filter

- Cauchy-Schwartz inequality for general vector spaces

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

  - The equality holds when $x = cy$ for some scalar c
  - The Euclidean space is an instance of vector spaces
  - For functions in L_2, it is $\left|\int x(t)y^*(t)dt\right| \leq \left(\int |x(t)|^2 dt\right)^{1/2} \left(\int |y(t)|^2 dt\right)^{1/2}$

- Then, $x \leftarrow H(f), y^* \leftarrow red$

$$\zeta^2 = \frac{\left|\int_{-\infty}^{\infty} (S_2(f) - S_1(f))H(f)e^{j2\pi fT} df\right|^2}{\frac{N_0}{2}\int |H(f)|^2 df}$$

$$\leq \frac{\int |S_2(f) - S_1(f)|^2 df \int |H(f)|^2 df}{\frac{N_0}{2}\int |H(f)|^2 df} = \frac{2\int |S_2(f) - S_1(f)|^2 df}{N_0}$$

- The equality holds when $H(f) = c(S_2^*(f) - S_1^*(f))e^{-j2\pi fT}$ for some (complex) constant c

Matched Filter

- $H(f) = c(S_2^*(f) - S_1^*(f))e^{-j2\pi fT}$ for some (complex) constant c
- Its time response is (letting $c = 1$)

$$h(t) = \int (S_2^*(f) - S_1^*(f))e^{-j2\pi fT}e^{j2\pi ft}df$$

$$= \int \underbrace{(S_2(-f) - S_1(-f))}_{\text{Time reversal}}\underbrace{e^{-j2\pi fT}e^{j2\pi ft}}_{\text{Time shift}}df$$

- Therefore, $h(t) = s_2(T-t) - s_1(T-t)$
- That is, the optimal receiver must have two parallel filters whose impulses responses are time reversal of pulse shapes (**"matched"**)

Matched Filter

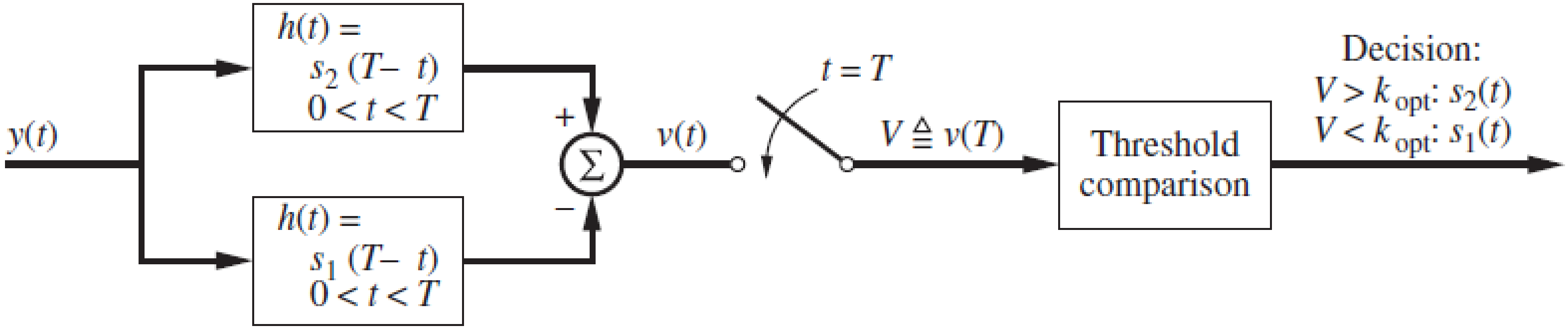


Figure 9.9
Matched-filter receiver for binary signaling in white Gaussian noise.

Matched Filter

- Error analysis: We know that $P_E = Q(\zeta/2)$, where

$$\zeta_{max} = \sqrt{\frac{2\int |S_2(f) - S_1(f)|^2 df}{N_0}}$$

$$= \sqrt{\frac{2}{N_0}}\sqrt{\int |s_2(t) - s_1(t)|^2 dt} = \sqrt{\frac{2}{N_0}}\sqrt{\int s_2^2(t) - 2s_2(t)s_1(t) + s_1^2(t)dt}$$

$$\triangleq \sqrt{\frac{2}{N_0}}\sqrt{E_1 + E_2 - 2\sqrt{E_1E_2}\rho_{12}} \quad \text{by defining } \rho_{12} = \frac{1}{\sqrt{E_1E_2}}\int s_1(t)s_2(t)dt$$

$$\frac{\zeta_{max}}{2} = \sqrt{\frac{1}{N_0}}\sqrt{\frac{E_1 + E_2 - 2\sqrt{E_1E_2}\rho_{12}}{2}} = \sqrt{\frac{1}{N_0}}\sqrt{E_b - \sqrt{E_1E_2}\rho_{12}}$$

$$= \sqrt{\frac{E_b}{N_0}}\sqrt{1 - \frac{\sqrt{E_1E_2}\rho_{12}}{E_b}}$$

1. When $\rho_{12} = -1$ (when s_1, s_2 are the most dissimilar, called antipodal), P_E is minimized
2. When $\rho_{12} = -1$, $\frac{\zeta_{max}}{2} = \sqrt{\frac{2E_b}{N_0}}$

Matched Filter

- The smallest error probability is at $\zeta_{\max}$ and the error is

$$Q(\zeta_{max}/2) = Q\left(\sqrt{\frac{E_b}{N_0}}\sqrt{1 - \frac{\sqrt{E_1 E_2}\rho_{12}}{E_b}}\right)$$

where

- $E_b = \frac{E_1+E_2}{2}$
- $\rho_{12} = \frac{1}{\sqrt{E_1 E_2}} \int_{-\infty}^{\infty} s_1(t) s_2(t) dt$

Matched Filter

- If the matched filter is used,

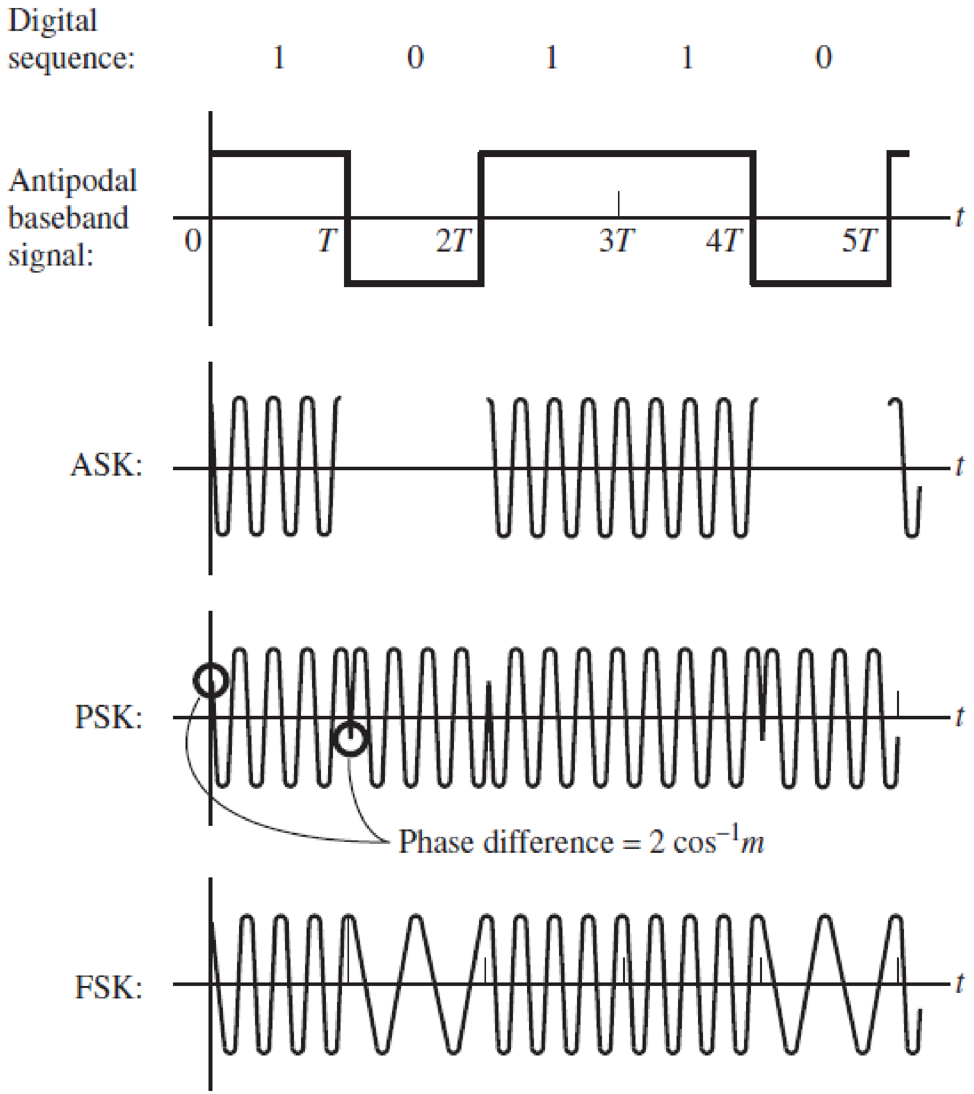
$$
\begin{aligned}
s_{01}(T) &= \int_{-\infty}^{\infty} h(\lambda)s_1(T-\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} [s_2(T-\lambda) - s_1(T-\lambda)]s_1(T-\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} s_2(u)s_1(u)du - \int_{-\infty}^{\infty} s_1^2(u)du \\
&= \sqrt{E_1E_2}\rho_{12} - E_1, \\
s_{02}(T) &= E_2 - \sqrt{E_1E_2}\rho_{12}
\end{aligned}
$$

- Since the threshold is $k_{opt} = \frac{1}{2}[s_{01}(T) + s_{02}(T)] = \frac{1}{2}(E_2 - E_1)$
- Signal shape does not matter, only signal energy determines the threshold

X-shift keying

Table 9.1 Possible Signal Choices for Binary Digital Signaling

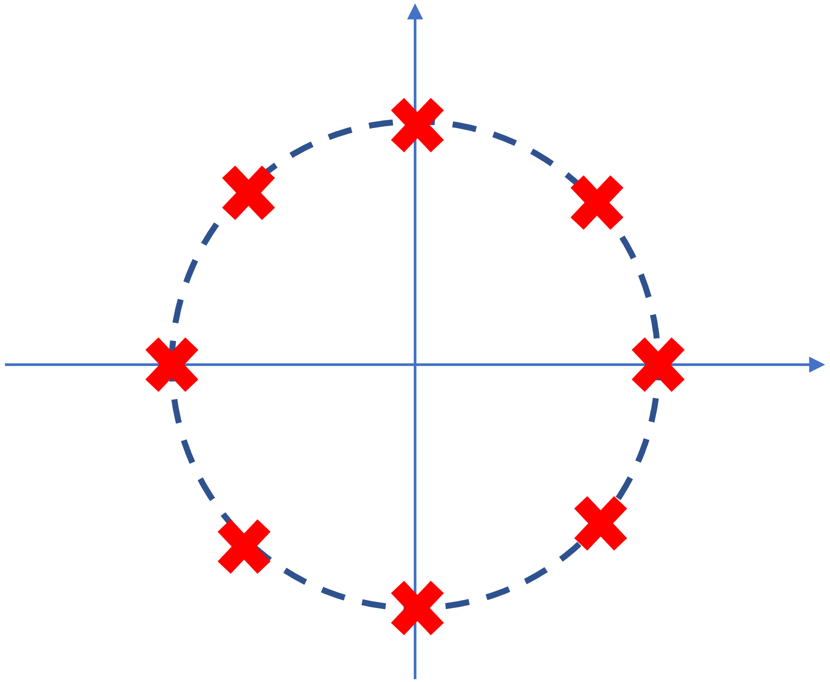
Case	$s_1(t)$	$s_2(t)$	Type of signaling
1	0	$A\cos(\omega_c t)\,\Pi\left(\frac{t-T/2}{T}\right)$	Amplitude-shift keying
2	$A\sin(\omega_c t + \cos^{-1} m)\Pi\left(\frac{t-T/2}{T}\right)$	$A\sin(\omega_c t - \cos^{-1} m)\Pi\left(\frac{t-T/2}{T}\right)$	Phase-shift keying with carrier ($\cos^{-1} m \triangleq$ modulation index)
3	$A\cos(\omega_c t)\,\Pi\left(\frac{t-T/2}{T}\right)$	$A\cos\left[(\omega_c + \Delta\omega)\,t\right]\Pi\left(\frac{t-T/2}{T}\right)$	Frequency-shift keying



M-ary PSK

- Transmitting pulses with $M = 2^i$ different carrier phases

$$s_i(t) = A\cos(w_c t + i\frac{2\pi}{M})$$



Error Analysis: ASK

- ASK: $s_1(t) = 0$, $s_2(t) = A\cos(w_c t)$

$$Q(\zeta_{max}/2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

  where $E_b = \frac{E_1+E_2}{2} = \frac{E_2}{2}$

- s_1, s_2 are not opposite signals; not antipodal. It leads to the absence of $\sqrt{2}$ factor

- $k_{opt} = \frac{E_1+E_2}{2} = \frac{0+\frac{A^2T}{2}}{2} = \frac{A^2T}{4}$

Error Analysis: FSK

- PSK: $s_1(t) = A\cos(w_c t)$, $s_2(t) = A\cos((w_c + \Delta w)t)$
- For simplicity, let $w_c = \frac{2\pi n}{T}$ and $\Delta w = \frac{2\pi m}{T}$ with $m \neq n \Rightarrow$ Then both will go through an integer number of cycles during T

$$\begin{aligned}\sqrt{E_1 E_2}\rho_{12} &= \int s_1(t) s_2(t) dt = \int_0^T A^2 \cos(w_c t)\cos((w_c + \Delta w)t) dt \\ &= \frac{A^2}{2}\int_0^T \cos(\Delta w t) + \cos((2w + \Delta w)t) dt \\ &= 0\end{aligned}$$

- It leads to

$$Q(\zeta_{max}/2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The error prob is the same as ASK, but ASK needs higher peak power

- $k_{opt} = 0$

M-ary PAM

If $p(t)$ is sinusoidal, it is indeed ASK

- Baseband M-ary PAM (Pulse-Amplitude Modulation): A signal set is

$$s_i(t) = A_i p(t), i = 1, 2, \ldots, M$$

- Receiver of interest: given $y(t) = s_i(t) + n(t)$,

$$Y = \int_0^T (s_i(t) + n(t))p(t)dt = A_i + N$$

where $N = \int_0^T n(t)p(t)dt$ is a Gaussian RV with mean 0 and variance $N_0/2$

M-ary PAM

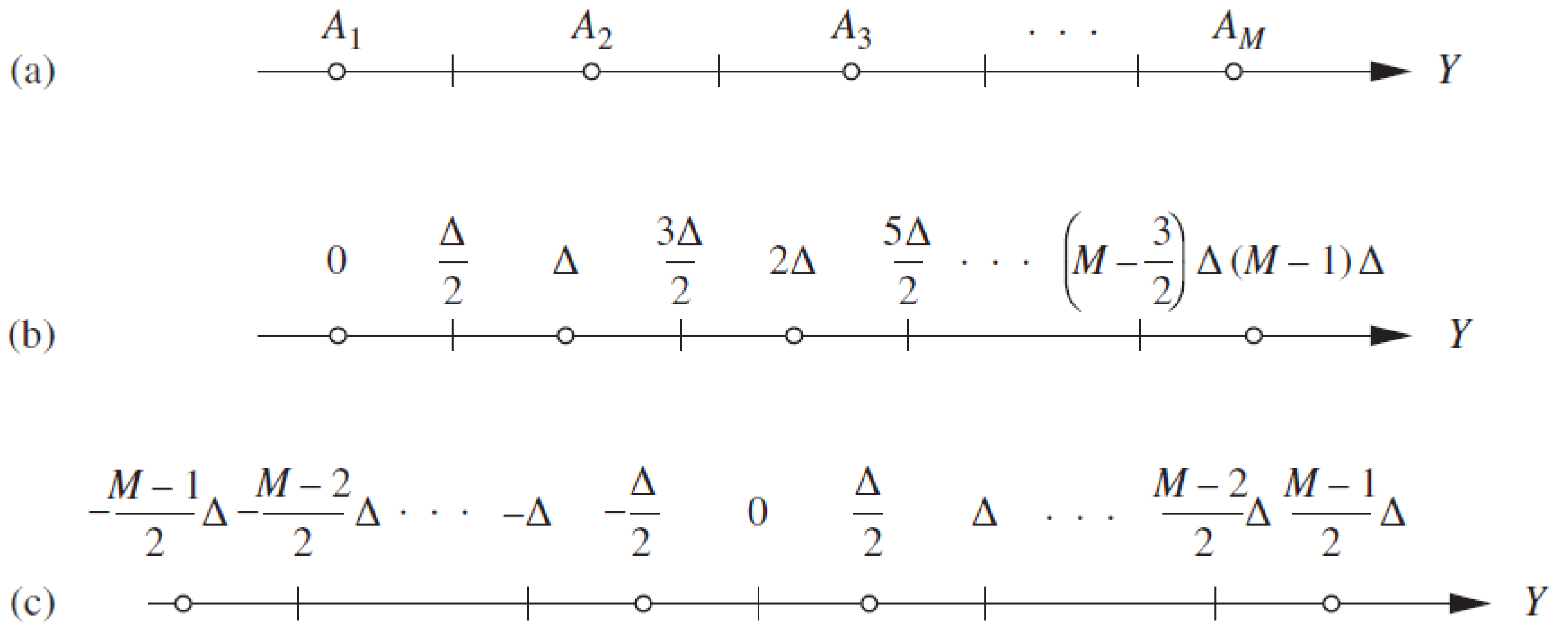


Figure 9.21

(a) Amplitudes and thresholds for PAM. (b) Nonnegative-amplitude equally spaced case. (c) Antipodal equally spaced case.

M-ary PAM

If $p(t)$ is sinusoidal, it is indeed ASK

- For beginning/end symbols, $j = 1, M$, the probability of error is

$$P(\text{error}|A_j \text{ sent}) = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

- For middle symbols, $2 \leq j \leq M - 1$ the probability of error is

$$\begin{aligned} P(\text{error}|A_j \text{ sent}) &= 1 - \int_{\Delta/2}^{\Delta/2} \frac{1}{\sqrt{\pi N_0}} e^{-t^2/N_0} dt \\ &= 2\int_{\Delta/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-t^2/N_0} dt = 2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) \end{aligned}$$

- As symbols are equally likely, total error is

$$P_E = \frac{2M-2}{M} Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

M-ary PAM

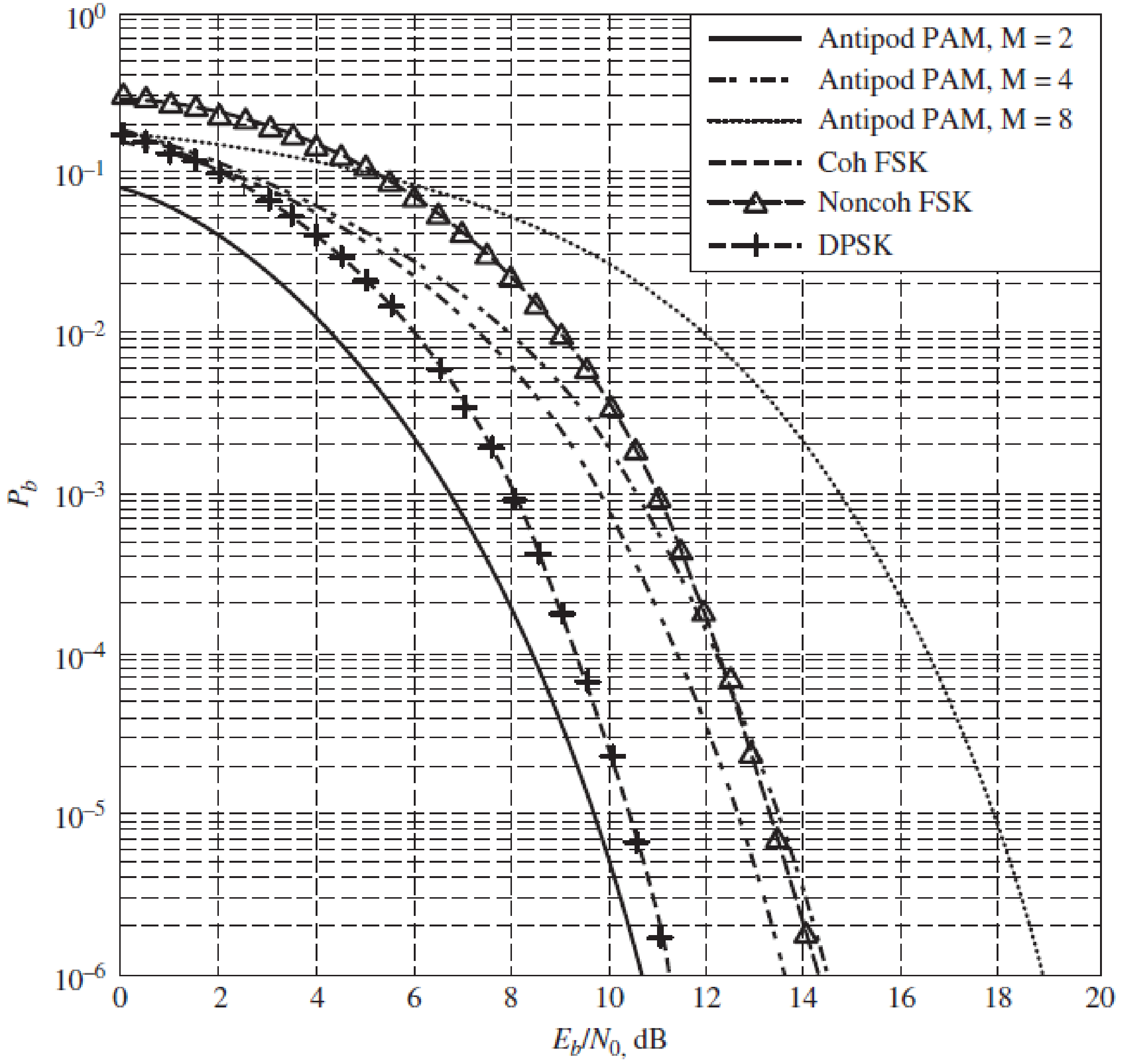
- Energy calculation: Non-negative equally spaced case

$$E_{ave} = \frac{1}{M}\sum_{j=1}^{M} E_j = \frac{1}{M}\sum_{j=1}^{M}((j-1)\Delta)^2 = \frac{\Delta^2}{M}\sum_{j=1}^{M}(j-1)^2$$
$$= \frac{\Delta^2}{M}\frac{(M-1)M(2M-1)}{6} = \frac{(M-1)(2M-1)\Delta^2}{6}$$

- Combining with the previous,

$$P_E = \frac{2M-2}{M} Q\left(\sqrt{\frac{3E_{ave}}{(M-1)(2M-1)N_0}}\right)$$

- Also note that $E_b = \frac{E_{ave}}{\log_2 M}$, assuming $M = 2^i$ for some integer



Higher order PAM shows worse BER performance; but each symbol contains more bits

➔ Good for BW limited channel and high Tx power regime