Problem Set: Calculus of Variations (Euler–Lagrange)

Instructions. Show all steps clearly. Unless stated otherwise, variations vanish at fixed endpoints.

Problem 1

Find the function $y(x)$ on the interval $[0, 1]$ with fixed endpoints $y(0) = 0$ and $y(1) = 1$ that minimises the functional

$$J[y] = \int_0^1 \big(y'(x)\big)^2 \, dx.$$

Problem 2

Minimise the functional

$$J[y] = \int_0^1 \big((y'(x))^2 + 4\,y(x)^2\big) \, dx$$

subject to the boundary conditions $y(0) = 0$, $y(1) = 1$. Solve exactly (symbolic steps) and evaluate the integration constants numerically where appropriate.

Problem 3

Consider the arc-length functional

$$J[y] = \int_0^1 \sqrt{1 + \big(y'(x)\big)^2} \, dx,$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. Using the calculus of variations, show that the extremal is a straight line and compute it explicitly.

Problem 4

Let

$$J[y] = \int_0^{\pi/2} \big((y'(x))^2 - y(x)^2\big) \, dx,$$

with boundary conditions $y(0) = 0$ and $y(\frac{\pi}{2}) = 1$. Note that the Lagrangian is independent of x; use the Beltrami (first-integral) identity to obtain a first-order equation, solve it, and determine the solution satisfying the boundary conditions.

Problem 5

Minimise

$$J[y] = \int_0^1 \big(y'(x)\big)^2 \, dx,$$

subject to the boundary condition $y(0) = 0$ while the endpoint at $x = 1$ is *free* (i.e. $y(1)$ is not prescribed). Find the minimiser and state the natural boundary condition that determines it.

Problem 6

(Constrained variational problem — isoperimetric type.) Minimise

$$J[y] \;=\; \int_0^1 \big(y'(x)\big)^2\,dx$$

subject to fixed endpoints $y(0) = 0$, $y(1) = 0$ and the integral constraint

$$\int_0^1 y(x)\,dx \;=\; m,$$

where m is a given constant (assume the constraint is feasible). Use a Lagrange multiplier to find the extremal $y(x)$ in terms of m. Then give the explicit result for the numerical value $m = \frac{1}{6}$.