Problem 1

$\mathcal{L}(y, y') = (y'(x))^2.$ $\quad \frac{\partial \mathcal{L}}{\partial y} = 0.$ $\quad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 2y''$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0 \Rightarrow 2y'' = 0$$

$$y' = C$$

$$y = Cx + D$$

$C(0) + D = 0$
$C + D = 1$ $\Rightarrow$ $C = 1$, $D = 0.$ $\quad y(x) = x$ $\quad$ (straight line in $[0,1]$)

Problem 2

$\mathcal{L}(y, y') = y'(x)^2 + 4y(x)^2.$ $\quad \frac{\partial \mathcal{L}}{\partial y} = 8y$ $\quad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y''} = 2y''$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0 \Rightarrow 8y - 2y'' = 0$$

$$4y - y'' = 0$$

characteristic equation

$4 - r^2 = 0 \Rightarrow r = \pm 2$

$y(x) = C_1 e^{2x} + C_2 e^{-2x}$

$y(0) = C_1 + C_2 = 0.$ $\quad C_1 = -C_2$

$y(1) = C_1 e^{2x} - C_1 e^{-2x} = 1$

$$C_1 = \frac{1}{e^{2x} - e^{-2x}}$$

$$y(x) = \frac{1}{e^{2x} - e^{-2x}} e^{2x} + \frac{1}{e^{-2x} - e^{2x}} e^{-2x}$$

Problem 3.

$\mathcal{L} = \sqrt{1+y'(x)^2}\,dx$. $\frac{\partial \mathcal{L}}{\partial y} = 0$

$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0 \Rightarrow \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial y'} = \text{constant}.$

let this constant be k. Thus $\frac{\partial \mathcal{L}}{\partial y'} = k$

$$\frac{\partial \mathcal{L}}{\partial y'} = y'(1+(y')^2)^{-1/2} = k$$

$$y' = k\sqrt{1+(y')^2}$$

$$(y')^2 = k^2 + k^2(y')^2$$

$$(y')^2(1-k^2) = k^2$$

$$(y')^2 = \frac{k^2}{(1-k^2)}$$

$$(y') = \frac{\pm k}{\sqrt{1-k^2}} = \text{constant. Let } \frac{\pm k}{\sqrt{1-k^2}} = A$$

$y = Ax + B$

$y(0) = B = 0 \qquad y(1) = A = 1. \qquad y(x) = x$

Problem 4.

Beltrami identity: $\mathcal{L} - y'(x)\frac{\partial \mathcal{L}}{\partial y'(x)} = C$

$(y')^2 - y^2 - y'(2y') = C$

$$(y')^2 + y^2 + C = 0$$

$$\frac{(y')^2}{-y^2-C} = 1$$

$$\frac{y'}{\sqrt{-y^2-C}} = \pm 1$$

$$\frac{y'}{\sqrt{-y^2-C}}\,dy = \pm dx$$

$\arcsin\left(\frac{y}{\sqrt{-C}}\right) = \pm x + D$

~~$y = \sqrt{-C}\sin x + D$~~

$y(x) = \sqrt{-C}\,\sin(x+D)$

Boundary conditions

$y(0) = \sqrt{-C}\,\sin D = 0$

$D = n\pi$

set $n=0$.

$y(\pi/2) = \sqrt{-C}\,\sin(\frac{\pi}{2}) = 1$

$\sqrt{-C} = 1$

$y(x) = \sin x$

Problem 5.

$\mathcal{L} = (y')^2$ gives $y'' = 0$.

$y' = A$, $y = Ax + B$.

$y(0) = B = 0$. If $x = 1$ is free, then $\frac{\partial \mathcal{L}}{\partial y'} \eta \big|_{x=1} 0$

$\frac{\partial \mathcal{L}}{\partial y'} \Big|_{x=1} = 0$

$y(x) = 0$.

$2y'(1) = 0$

$A = 0$

Interpretation: What curve starting from $y(0) = 0$ has the least amount of change? Answer: $y(x) = 0$.

Problem 6.

$J[y] = \int ((y')^2 + \lambda y)\, dx$. $\frac{\partial \mathcal{L}}{\partial y} = \lambda$, $\frac{\partial \mathcal{L}}{\partial y'} = 2y'$, $\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y} = 2y''$

$\lambda - 2y'' = 0$

$y'' = \frac{\lambda}{2}$, $y = \frac{\lambda}{4}x^2 + Ax + B$.

$y(0) = B = 0$ $y(1) = \frac{\lambda}{4} + A = 0$. $\frac{\lambda}{4} = -A$

$y(x) = \frac{\lambda}{4}x^2 - \frac{\lambda}{4}x$

$\int_0^1 y(x)\, dx = \left[\frac{\lambda}{12}x^3 - \frac{\lambda}{8}x^2\right]\Big|_0^1 = m$

$\frac{\lambda}{12} - \frac{\lambda}{8} = m = \frac{1}{6}$.

$\frac{\lambda}{12} - \frac{\lambda}{8} = \frac{2\lambda - 3\lambda}{24} = -\frac{\lambda}{24} = \frac{1}{6}$

$\lambda = -4$.

$y(x) = x - x^2$.

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