# Lab7 Report

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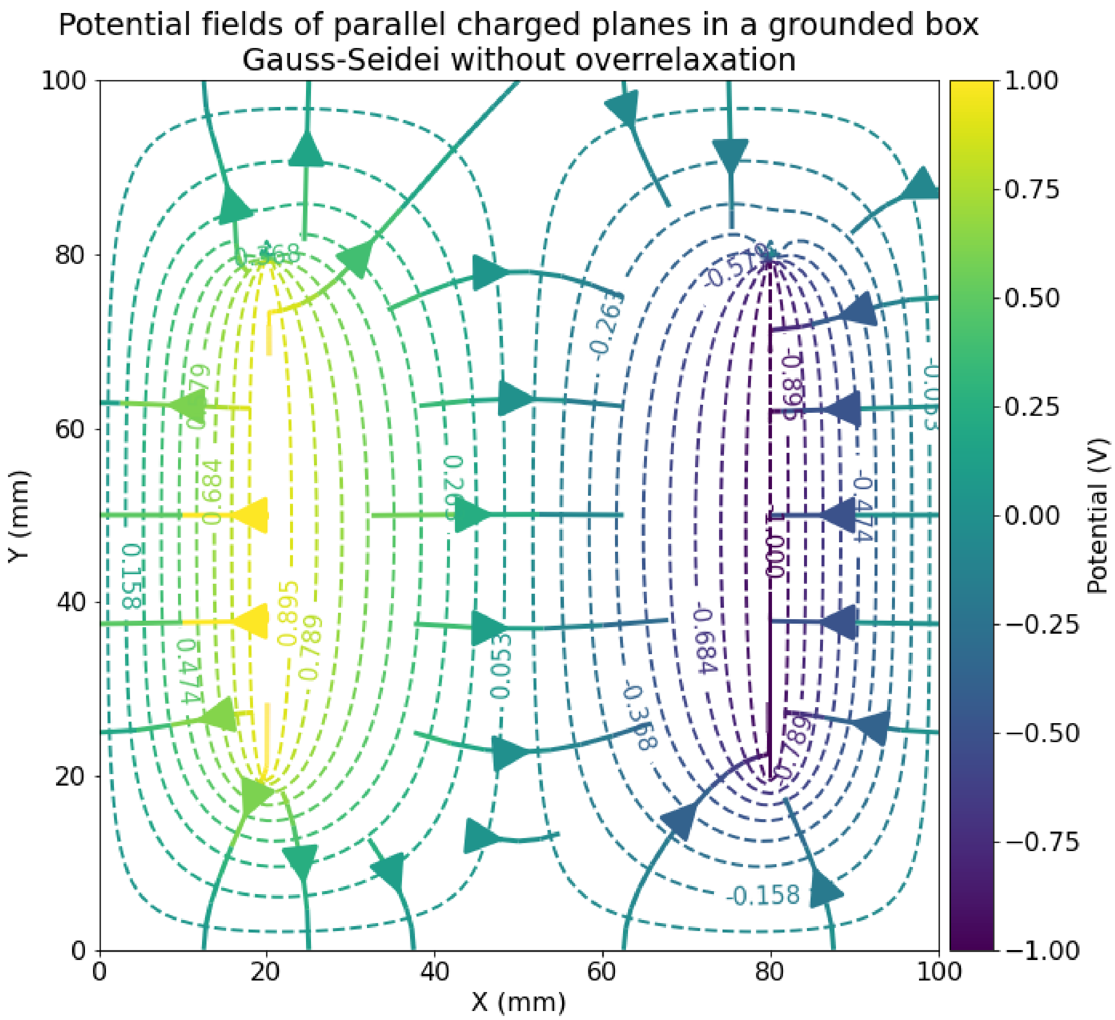
# 1 Question 1

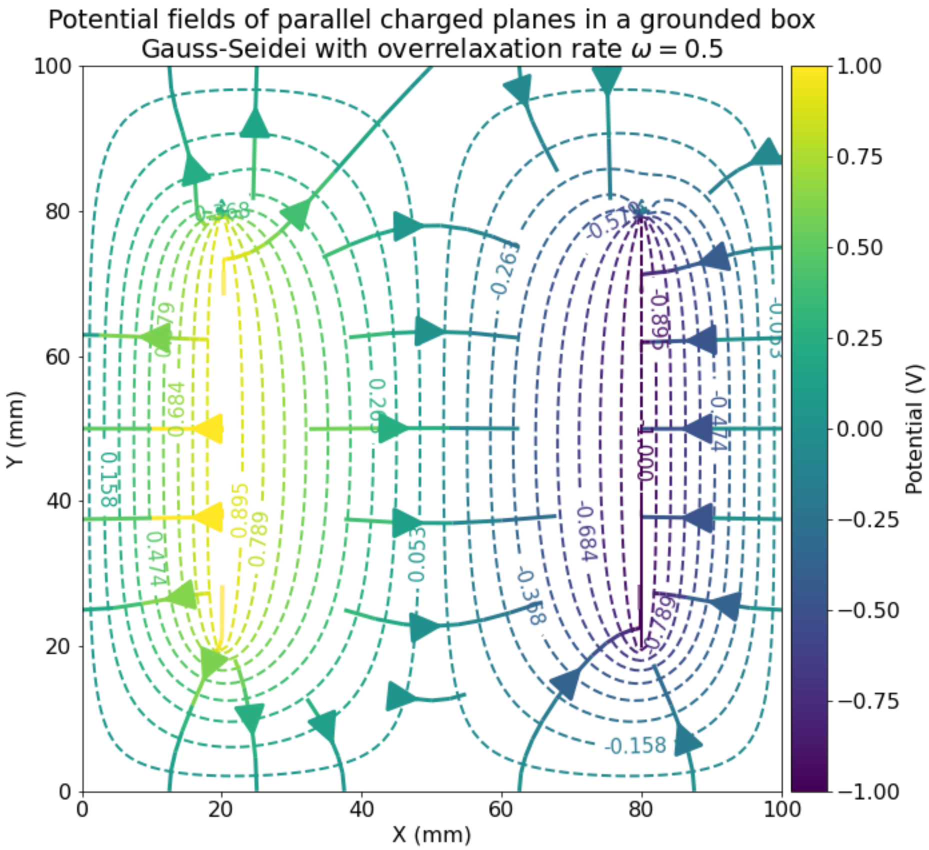
## 1.1 Q1 (a): Gauss-Seidel method without overrelaxation

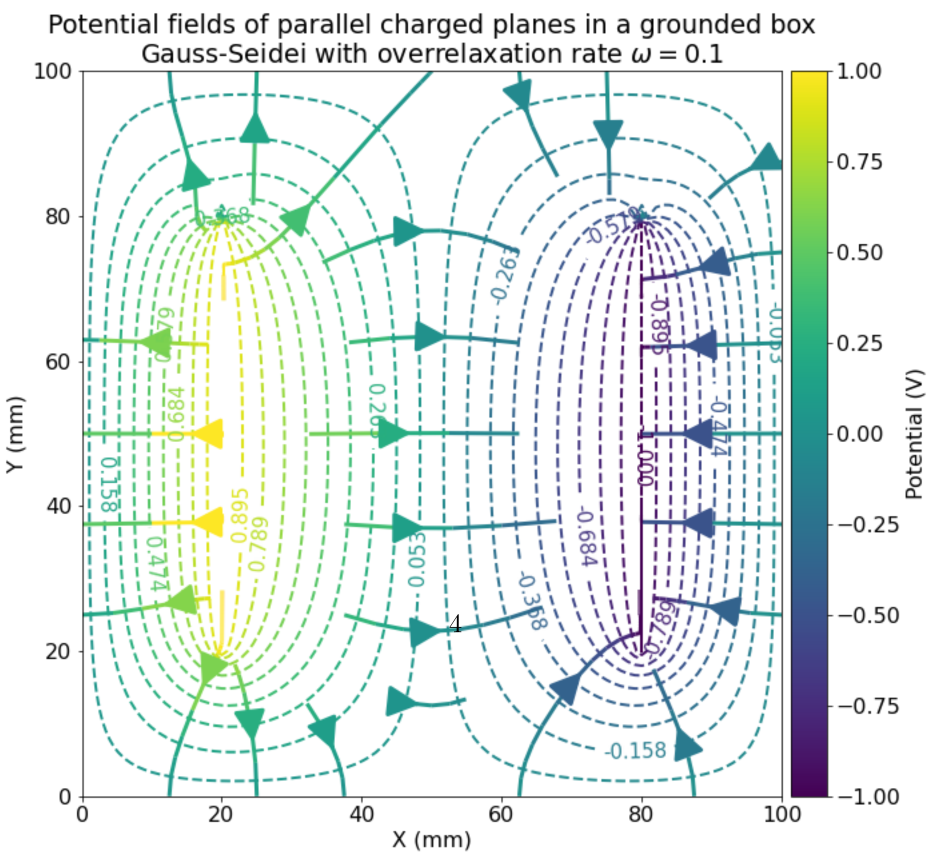



**Restatement of the question:** Two charged panels are placed in a grounded box. The panel on left is charged to have $+1V$ potential, and the one on right is charged to have $-1V$. The grounded box is assumed to be $100mm \times 100mm$ large. And the two panels are placed vertically, on $20mm$ and $80mm$ from the left wall of the ground box.

**Comments on the figure:** The contour plot of the potential fields and the streamplot of the electric field are plotted above. You can see that the potential fields have max and min values of $\pm 1V$ around the two charged panels, and have value of zero near the grounded box. The electric field lines are perpendicular to the potential fields, which makes physical sense since $\vec{E} = -\nabla V$.

## 1.2 Q1 (b): Gauss-Seidel method with overrelaxation



**Restatement of the question:** The code for Gauss-Seidel method is asked to be modified to include overrelaxation. Describe what you notice.

**Comments on the result:** The code for the Gauss-Seidel scheme method was modified to include overrelaxation. And the outputs are attached above, corresponding to the cases with overrelaxation rate $\omega = 0.5$ and $\omega = 0.1$, respectively. You can see that the results are almost identical to the one without overrelaxation. However, when running the script, the speed of convergence is obviously different now. In order to investigate this, all the 3 scripts were timed and the raw outputs are attached below.

```
In [1]: runfile('C:/Users/ellen/Desktop/PHY Couses/PHY407/Lab 8/Lab08_Q1a.py',
Users/ellen/Desktop/PHY Couses/PHY407/Lab 8')
Gauss-Seidel without overrelaxation took 41.023873 seconds

Figures now render in the Plots pane by default. To make them also appear inli
Console, uncheck "Mute Inline Plotting" under the Plots pane options menu.

In [2]: runfile('C:/Users/ellen/Desktop/PHY Couses/PHY407/Lab 8/Lab08_Q1b.py',
Users/ellen/Desktop/PHY Couses/PHY407/Lab 8')
Gauss-Seidel with overrelaxation (omega=0.5) took 28.057570 seconds

In [3]: runfile('C:/Users/ellen/Desktop/PHY Couses/PHY407/Lab 8/Lab08_Q1b.py',
Users/ellen/Desktop/PHY Couses/PHY407/Lab 8')
Gauss-Seidel with overrelaxation (omega=0.1) took 49.737011 seconds
```

| Overrelaxation rate $\omega$ | Time (second) |
|---|---|
| 0 (without overrelaxation) | 41.02 |
| 0.1 | 49.74 |
| 0.5 | 28.06 |
| 0.8 | 13.35 |
| 0.9 | 6.53 |
| 0.99 | 35.70 |

Table 1: Timing comparison between Gauss-Seidel schemes with and without overrelaxation

Other values were also experimented, and the results are summarized in Table 1. You can see that, the relationship between the speedup of the code and the $\omega$ is very non-linear. Although a general rule is that larger values of $\omega$ could lead to faster convergence speed, a too large value could indeed cause a slowdown of the code. Thus, the appropriate value of $\omega$ really relies on sensitivity experiments on it.

## 2 Question 2

### 2.1 Q2a, Derivation of the 1D shallow water equation into the flux-conservative form

From Equation 6 of the lab handout, for $u$, we have:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\left(u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x}\right)\\ &= -\left(\frac{1}{2}\frac{\partial u^2}{\partial x} + g\frac{\partial \eta}{\partial x}\right)\end{aligned}$$

So

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + g\eta\right) \tag{1}$$

For $\eta$, we have:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\big[u(\eta - \eta_b)\big] \quad \text{where } h = \eta - \eta_b \tag{2}$$

From Equation (1) and (2) we have:

$$\frac{\partial}{\partial t}\vec{u} = \frac{\partial}{\partial t}\begin{bmatrix} u \\ \eta \end{bmatrix} = -\begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + g\eta\right) \\ \frac{\partial}{\partial x}\big[u(\eta - \eta_b)\big] \end{bmatrix} = -\frac{\partial \vec{F}(u,\eta)}{\partial x} \tag{3}$$

where,

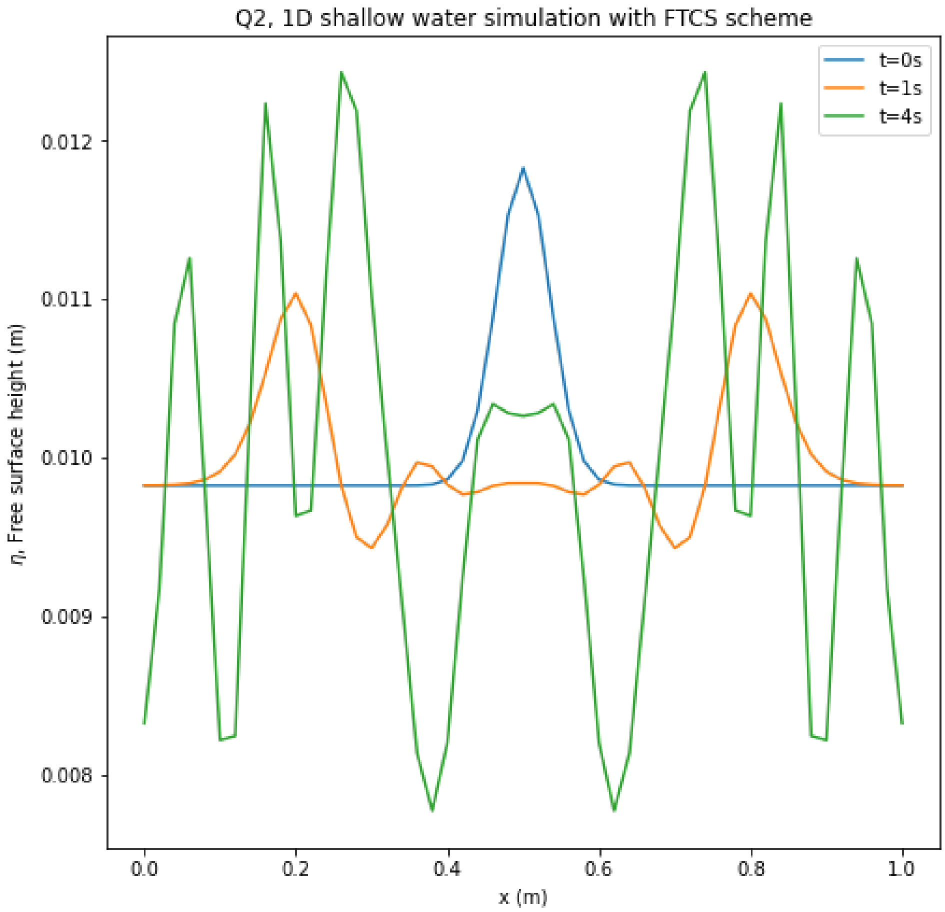
$$\vec{F}(u,\eta) = \begin{bmatrix} \frac{1}{2}u^2 + g\eta \\ (\eta - \eta_b)u \end{bmatrix}$$

Then, we apply the FTCS scheme:

$$\begin{aligned}u_j^{n+1} &= u_j^n - \frac{\Delta t}{2\Delta x}\left(\frac{1}{2}(u_{j+1}^n)^2 - \frac{1}{2}(u_{j-1}^n)^2 + g\eta_{j+1}^n - g\eta_{j-1}^n\right)\\ \eta_j^{n+1} &= \eta_j^n - \frac{\Delta t}{2\Delta x}\Big(u_{j+1}^n(\eta_{j+1}^n - \eta_{b,j+1}) - u_{j-1}^n(\eta_{j-1}^n - \eta_{b,j-1})\Big)\end{aligned}$$

### 2.2 Q2b, FTCS implementation and free surface height plot

So here, I apply the FTCS scheme to the set up from the lab handout. I plot the free surface height $\eta(x,t)$ at $t = 0$s, $t = 1$s, and $t = 4$s.




As you can see from the plot, at $t = 4$s, the wave form clearly diverges and becomes chaotic.
I also made an animation named 'Q2.mp4', where you can see how the simulation diverges straightforwardly.

### 2.3 Q3c, von Neumann stability analysis of the FTCS for the 1D shallow water equations

Similarly, from Equation 6 of the lab handout:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - g\frac{\partial \eta}{\partial x}$$

We plug in the condition $u = 0$:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

And

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial x}\big[u(\eta - \eta_b)\big] \\
&= -\frac{\partial}{\partial x}(u\eta) \quad (\text{since } \eta_b = 0) \\
&= -\eta\frac{\partial u}{\partial x} - u\frac{\partial \eta}{\partial x} \quad (\text{chain rule}) \\
\frac{\partial \eta}{\partial t} &= -H\frac{\partial u}{\partial x} \quad (\text{plug in } (u,\eta) = (0,H))
\end{aligned}$$

So we reached Equation 10 in the lab handout:

$$\begin{cases}
\dfrac{\partial u}{\partial t} = -g\dfrac{\partial \eta}{\partial x} \\
\dfrac{\partial \eta}{\partial t} = -H\dfrac{\partial u}{\partial x}
\end{cases}$$

We apply the FTSC scheme to it:

$$\begin{cases}
u(t+\Delta t) = u(t) - g\dfrac{\Delta t}{2\Delta x}\big[\eta(x+\Delta x) - \eta(x-\Delta x)\big] \\
\eta(t+\Delta t) = \eta(t) - H\dfrac{\Delta t}{2\Delta x}\big[u(x+\Delta x) - u(x-\Delta x)\big]
\end{cases}$$

Apply Fourier transform:

$$\begin{cases}
c_u(t+\Delta t)e^{ikx} = c_u(t)e^{ikx} - g\dfrac{\Delta t}{2\Delta x}\big(e^{ikx+ik\Delta x} - e^{ikx-ik\Delta x}\big)c_\eta(t) \\
c_\eta(t+\Delta t)e^{ikx} = c_\eta(t)e^{ikx} - H\dfrac{\Delta t}{2\Delta x}\big(e^{ikx+ik\Delta x} - e^{ikx-ik\Delta x}\big)c_u(t)
\end{cases}$$

We can cancel out $e^{ikx}$ terms and combine $e^{ik\Delta x} - e^{-ik\Delta x} = 2\sin k\Delta x$ , and get:

$$\begin{cases}
c_u(t+\Delta t) = c_u(t) - g\dfrac{\Delta t}{\Delta x}\big(\sin k\Delta x\big)c_\eta(t) \\
c_\eta(t+\Delta t) = c_\eta(t) - H\dfrac{\Delta t}{\Delta x}\big(\sin k\Delta x\big)c_u(t)
\end{cases}$$

Written in matrix form:

$$\begin{pmatrix} c_u(t+\Delta t) \\ c_\eta(t+\Delta t) \end{pmatrix} = \begin{pmatrix} 1 & -\,gh \\ -Hh & 1 \end{pmatrix}\begin{pmatrix} c_u(t) \\ c_\eta(t) \end{pmatrix}$$

where,

$$h = \frac{\Delta t}{\Delta x}\sin k\Delta x$$

So we let

$$\mathbf{A} = \begin{pmatrix} 1 & -\,gh \\ -Hh & 1 \end{pmatrix}$$

and calculate the eigen values of the matrix by:

$$\det\big(\mathbf{A} - \lambda\mathbf{I}\big) = 0$$

We get

$$\begin{aligned}(1-\lambda)^2 + gHh^2 &= 0\\ 1-\lambda &= \pm ih\sqrt{gH}\\ \lambda &= 1 \pm ih\sqrt{gH}\\ &= 1 \pm i\frac{\Delta t}{\Delta x}\sqrt{gH}\sin k\Delta x\end{aligned}$$

So for both solutions, we have

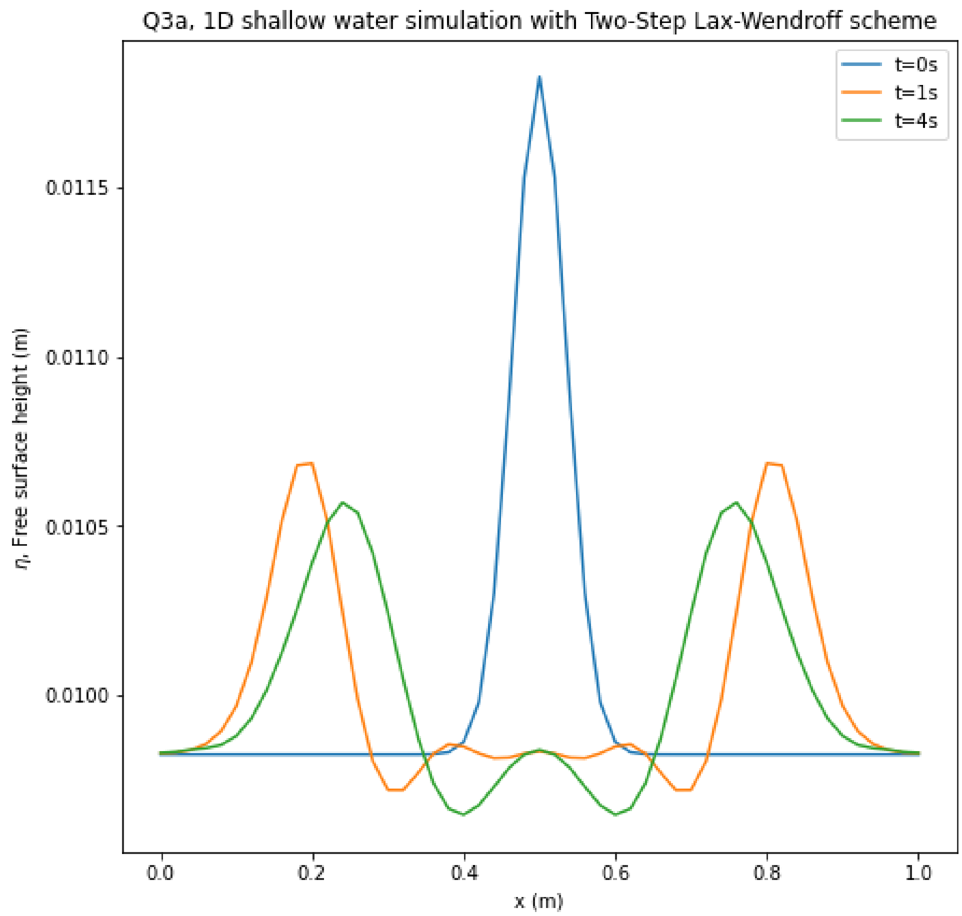
$$|\lambda| = \sqrt{1 + \left(\frac{\Delta t}{\Delta x}\right)^2 gH\sin^2(k\Delta x)}$$

We can conclude that FTCS scheme is not stable, since $g$ and $H$ are both positive constants. $|\lambda|$ will always be greater than 1. The Fourier component of order $k$ will grow exponentially.
This derivation result can be confirmed from my simulation. As the time evolving, the wave form tends to becomes extremely wavy. This phenomenon corresponds to the exponentially growing $k$ components.

# 3 Question 3

## 3.1 Q3a, Implementation of the Two-Step Lax-Wendroff scheme and free surface height plot

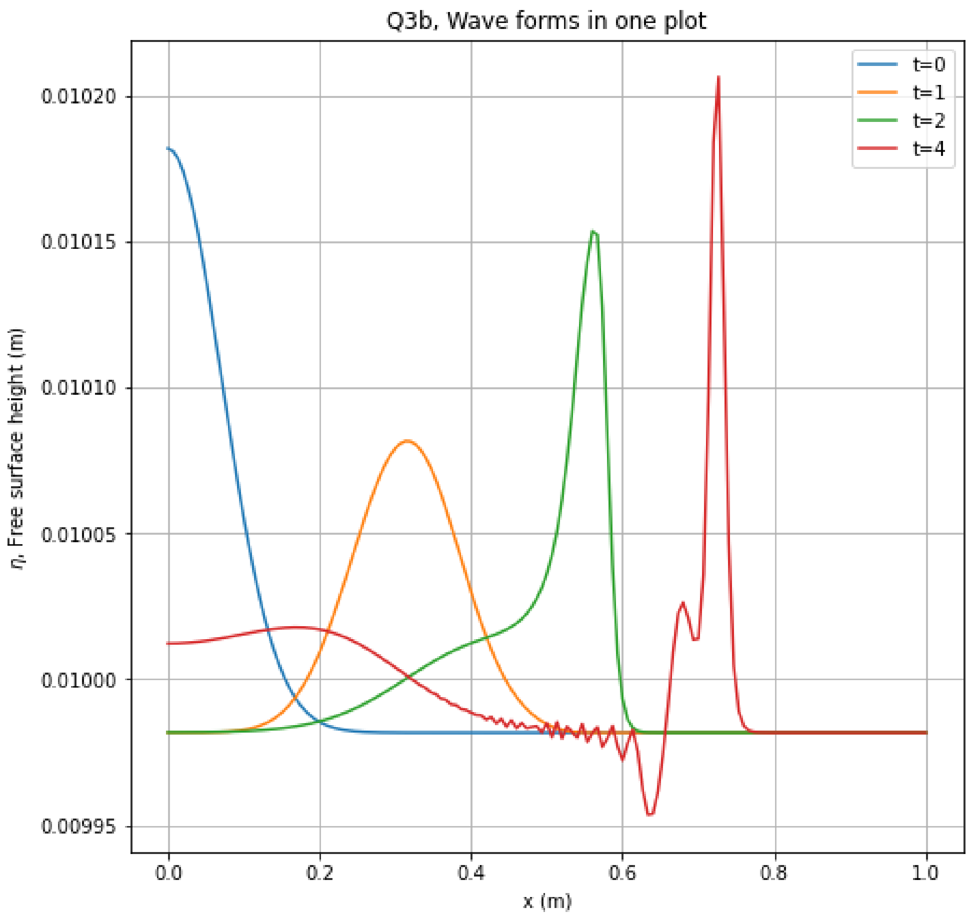
So I apply the Two-Step Lax-Wendroff scheme to the same setting as Question 2 and plot the free surface height $\eta(x, t)$ at $t = 0$s, $t = 1$s, and $t = 4$s.



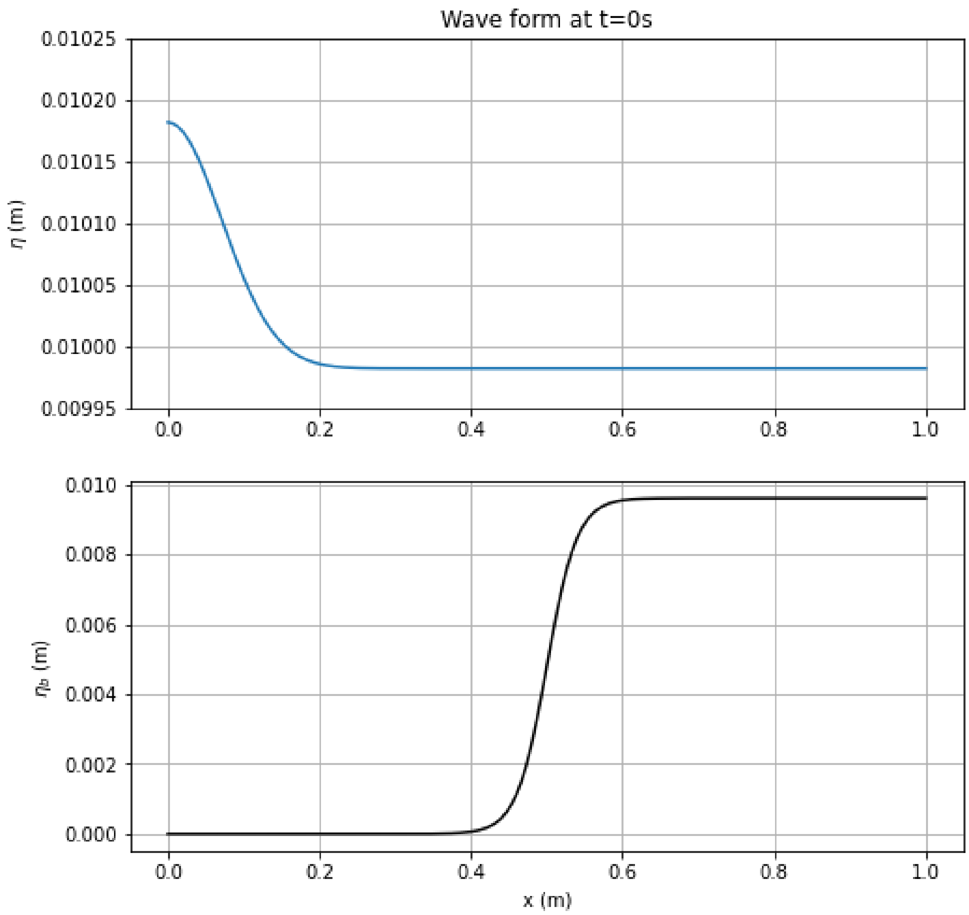

As you can see from the graph. The simulation result at $t = 1$s is pretty similar to the FTCS scheme result. However, at $t = 4$s, the Two-Step Lax-Wendroff scheme does not diverge. It provides a reasonable wave form.
I made an animation named 'Q3a.mp4', where you can see the wave form evolves peacefully with the Two-Step Lax-Wendroff scheme.

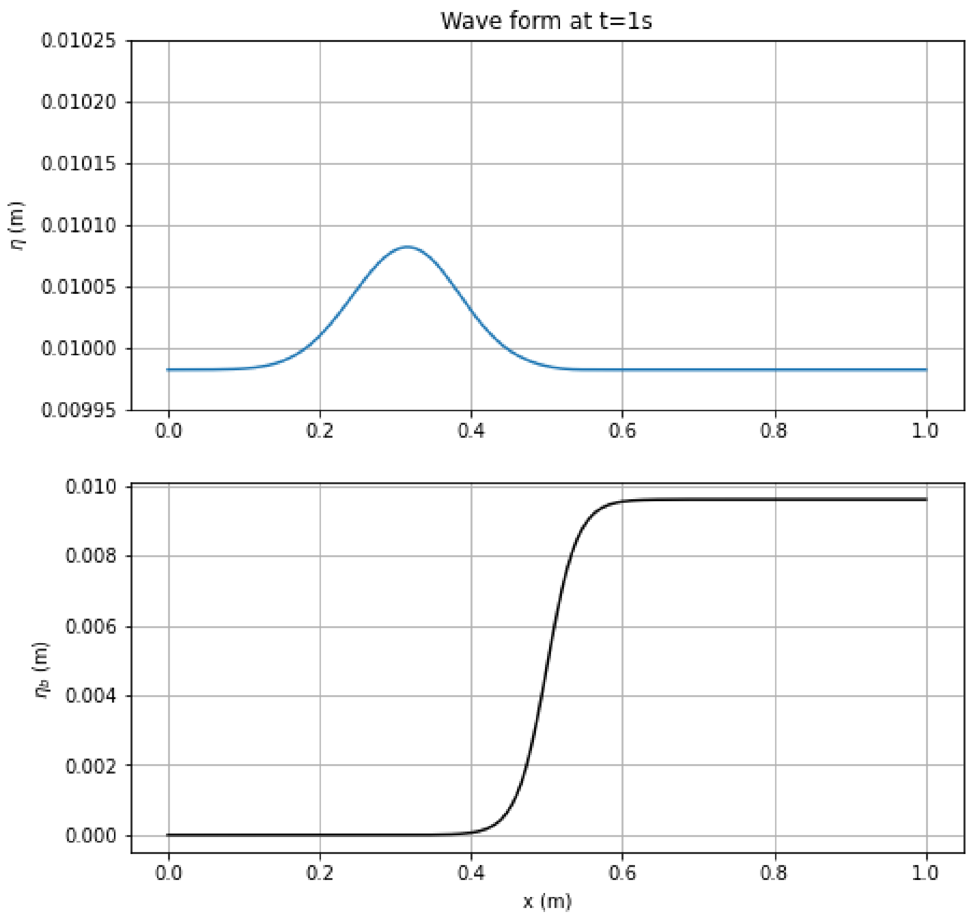
## 3.2 Q3b, Apply the variable bottom topography, 1D tsunami simulation and free surface height plots

With the setting from the lab handout, I first plot the free surface height $\eta(x,t)$ at $t = 0$s, $t = 1$s, $t = 2$s, and $t = 4$s in the plot without the bottom topography.

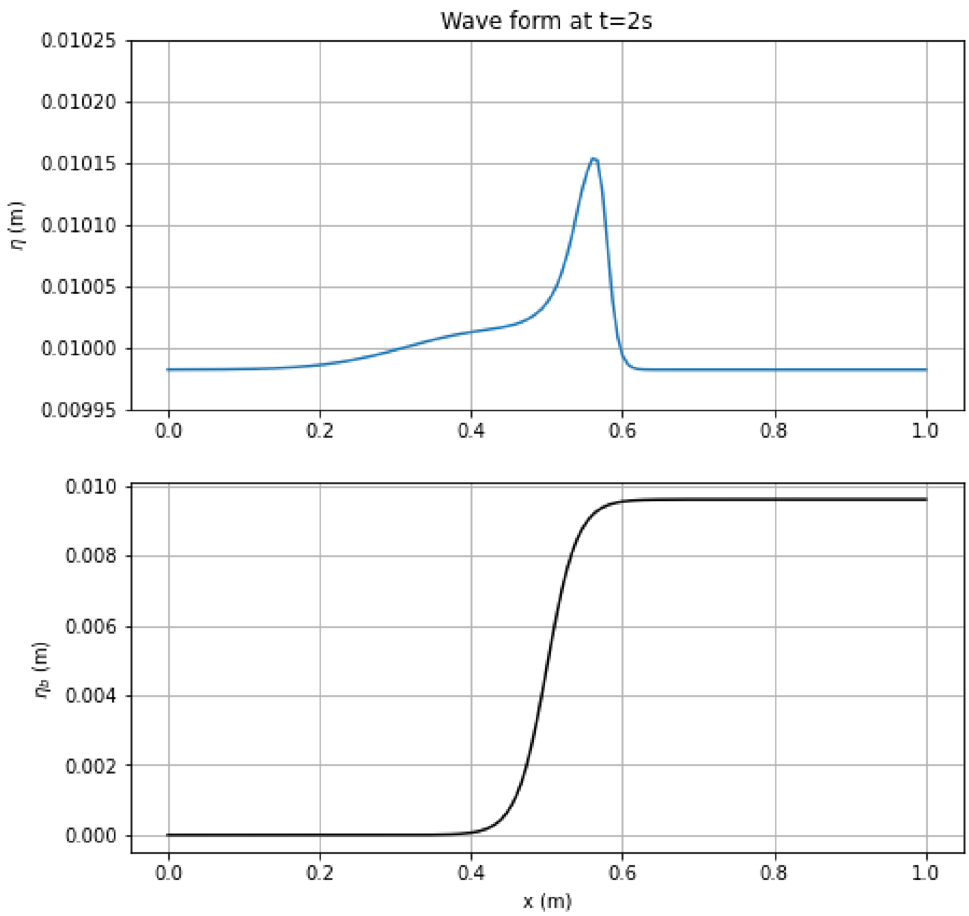

Q3b, Wave forms in one plot
t=0
t=1
t=2
t=4
0.01020
0.01015
0.01010
0.01005
0.01000
0.00995
η, Free surface height (m)
0.0
0.2
0.4
0.6
0.8
1.0
x (m)

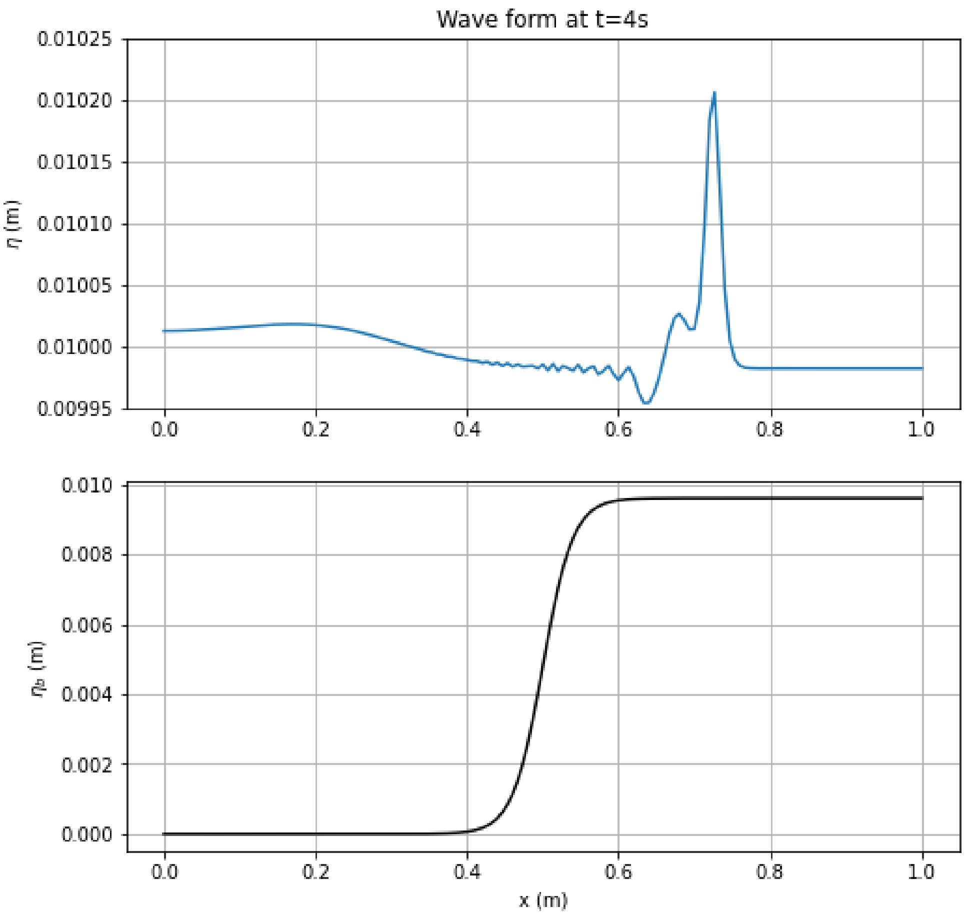
The following are separate wave form plots with bottom topography:





Wave form at t=1s
0.01025
0.01020
0.01015
0.01010
0.01005
0.01000
0.00995
$\eta$ (m)
0.0
0.2
0.4
0.6
0.8
1.0
0.010
0.008
0.006
0.004
0.002
0.000
$\eta_b$ (m)
x (m)


Wave form at t=2s
0.01025
0.01020
0.01015
0.01010
0.01005
0.01000
0.00995
η (m)
0.0
0.2
0.4
0.6
0.8
1.0
0.010
0.008
0.006
0.004
0.002
0.000
ηb (m)
x (m)




From the plots, we can see that the wave form starts as a shallow, wide bump in the deep water area moving right towards the shallower area, aka the continental shelf.

When the wave hits the shelf, where the bottom topography changes rapidly, its height increases greatly. The wave also becomes narrower, no longer a shallow, wide bump; but a thin, tall wall, which a real tsunami looks like.

From the graph at $t = 4$s, we can see that when the wave completely enters the shallow area, the free surface height plot becomes wavy. It is no longer as smooth as previous plots. However, I don't know if this is due to simulation error or there is some real physics behind it.

I made an animation named 'Q3b.mp4'. From the animation, I can tell the the speed of the wave deceases greatly when it enters the shallow area.