Target Localization Using Quantized Observations in Delay-Sensitive Networks

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I. Appendix

Here, we derive the squared position error bound (SPEB) of $\mathbf{p}^+$ in Proposition 1. The unknown parameters are collected as $\boldsymbol{\theta} = [(\mathbf{p}^+)^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$ and the log-likelihood function is

$$\ln f(\boldsymbol{z}, \boldsymbol{\theta}) = \ln f(\boldsymbol{z}|\boldsymbol{p}) + \ln f(\boldsymbol{p}^+|\boldsymbol{p}) + \ln f(\boldsymbol{p}), \quad (1)$$

where $f(\boldsymbol{x})$ denotes the probability density function (PDF) $f_{\mathbf{x}}(\boldsymbol{x})$ of the random vector $\mathbf{x}$ unless specified otherwise. Then, the Fisher information matrix (FIM) of $\boldsymbol{\theta}$ can be decomposed into three parts, as follows [1]:

$$\boldsymbol{J}_{\boldsymbol{\theta}}(\boldsymbol{b}) = \boldsymbol{J}_{\mathrm{ob}}(\boldsymbol{b}) + \boldsymbol{J}_{\mathrm{M}}(\boldsymbol{b}) + \boldsymbol{J}_{\mathrm{P}} \quad (2)$$

where $\boldsymbol{J}_{\mathrm{ob}}(\boldsymbol{b})$, $\boldsymbol{J}_{\mathrm{M}}(\boldsymbol{b})$, and $\boldsymbol{J}_{\mathrm{P}}$ represent FIMs from quantized observations, target motions, and a prior knowledge of the target, respectively. The expression of $\boldsymbol{J}_{\mathrm{ob}}(\boldsymbol{b})$ is

$$\boldsymbol{J}_{\mathrm{ob}}(\boldsymbol{b}) = -\mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \ln f(\boldsymbol{z}|\boldsymbol{p})\right\} \quad (3)$$

where $\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \ln f(\boldsymbol{z}|\boldsymbol{p})$ denotes $\partial^2 \ln f(\boldsymbol{z}|\boldsymbol{p})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\mathrm{T}}$ and (3) is rewritten since the observation is not associated with $\mathbf{p}^+$ as

$$\boldsymbol{J}_{\mathrm{ob}}(\boldsymbol{b}) = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \mathbb{E}_{\mathbf{p}}\left\{\boldsymbol{J}_{\mathrm{Q}}(\boldsymbol{b})\right\} \end{bmatrix} \quad (4)$$

where $\boldsymbol{O}$ denotes the 2×2 zero matrix and

$$\boldsymbol{J}_{\mathrm{Q}}(\boldsymbol{b}) = -\mathbb{E}_{\mathbf{z}|\mathbf{p}}\left\{\Delta_{\boldsymbol{p}}^{\boldsymbol{p}} \ln f(\boldsymbol{z}|\boldsymbol{p})\right\}. \quad (5)$$

The expression for $\boldsymbol{J}_{\mathrm{Q}}(\boldsymbol{b})$ is computed similarly to previous studies analyzing performance limits for unknown parameters from quantized observations [2]. It is provided as follows:

$$\boldsymbol{J}_{\mathrm{Q}}(\boldsymbol{b}) = \sum_{n=1}^{N_{\mathrm{a}}} \sum_{k=1}^{K} \gamma_n \rho_{n,k}(b_n) \frac{\partial s_k(\tau_n)}{\partial \boldsymbol{p}} \frac{\partial s_k(\tau_n)}{\partial \boldsymbol{p}^{\mathrm{T}}}, \quad (6)$$

where $\gamma_n = |\beta_n|^2/\sigma_n^2$ represents the signal-to-noise ratio (SNR) at n-th agent, $s_k(\tau_n)$ refers to the k-th element of the signal vector $\boldsymbol{s}(f_s;\tau_n)$, and $\rho_{n,k}(b_n)$ represents the quantized coefficient at k-th quantizer of n-th agent, calculated by

$$\rho_{n,k}(b_n) = \frac{1}{2\pi} \sum_{l=1}^{L_n} \frac{\left(e^{-r_{l+1}^2/2} - e^{-r_l^2/2}\right)^2}{\Phi(r_{l+1}) - \Phi(r_l)}, \quad (7)$$

$$r_l = \left[u_l - \beta_n s_k(\tau_n)\right]/\sigma_n, \quad l = 1, 2, \cdots, L_n - 1 \quad (8)$$

where $\Phi(x)$ denotes the tail distribution function of the standard norm distribution with $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$ [3]. Subsequently, the matrix $\boldsymbol{J}_{\mathrm{M}}(\boldsymbol{b})$ can be computed as follows:

$$\boldsymbol{J}_{\mathrm{M}}(\boldsymbol{b}) = -\mathbb{E}_{\mathbf{p},\mathbf{p}^+}\left\{\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \ln f(\boldsymbol{p}^+|\boldsymbol{p})\right\}, \quad (9)$$

which can be calculated as

$$\boldsymbol{J}_{\mathrm{M}}(\boldsymbol{b}) = \frac{1}{\alpha \Delta t(\boldsymbol{b})} \begin{bmatrix} \boldsymbol{I}_2 & -\boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & \boldsymbol{I}_2 \end{bmatrix}. \quad (10)$$

Furthermore, the FIM from the a priori knowledge of the target is independent of the bit-depth, which is given by

$$\boldsymbol{J}_{\mathrm{P}} = -\mathbb{E}_{\mathbf{p}}\left\{\Delta_{\boldsymbol{p}}^{\boldsymbol{p}} \ln f(\boldsymbol{p})\right\}. \quad (11)$$

Due to the a priori knowledge only relies on $\mathbf{p}$, we can obtain

$$\boldsymbol{J}_{\mathrm{P}} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \sigma_0^{-2}\boldsymbol{I}_2 \end{bmatrix} \quad (12)$$

Substituting (4), (10), and (12) into (2), we can obtain the partition of the FIM $\boldsymbol{J}_{\boldsymbol{\theta}}(\boldsymbol{b})$ is given by

$$\boldsymbol{J}_{\boldsymbol{\theta}}(\boldsymbol{b}) = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{A} \\ -\boldsymbol{A}^{\mathrm{T}} & \boldsymbol{A} + \boldsymbol{W} \end{bmatrix} \quad (13)$$

where $\boldsymbol{A} = 1/\left[\alpha \Delta t(\boldsymbol{b})\right] \boldsymbol{I}_2$ and $\boldsymbol{W} = \mathbb{E}\left\{\boldsymbol{J}_{\mathrm{Q}}(\boldsymbol{b})\right\} + \sigma_0^{-2}\boldsymbol{I}_2$. Then, the equivalent Fisher information matrix (EFIM) for estimating $\mathbf{p}^+$ is given by [4]

$$\boldsymbol{J}_{+,+}^{\mathrm{e}}(\boldsymbol{b}) = \boldsymbol{A} - \boldsymbol{A}(\boldsymbol{A} + \boldsymbol{W})^{-1}\boldsymbol{A}^{\mathrm{T}}. \quad (14)$$

According to the matrix inverse lemma, we can obtain the closed-from EFIM for estimating $\mathbf{p}^+$, which is given by

$$\boldsymbol{J}_{+,+}^{\mathrm{e}}(\boldsymbol{b}) = \left[\boldsymbol{W}^{-1} + \alpha \Delta t(\boldsymbol{b}) \boldsymbol{I}_2\right]^{-1} \quad (15)$$

Finally, the closed-form SPEB for estimating $\mathbf{p}^+$ is given as

$$\mathcal{P}(\mathbf{p}^+; \boldsymbol{b}) = \mathrm{tr}\left\{\left(\mathbb{E}_{\mathbf{p}}\{\boldsymbol{J}_{\mathrm{Q}}(\boldsymbol{b})\} + \sigma_0^{-2}\boldsymbol{I}_2\right)^{-1}\right\} + 2\alpha \Delta t(\boldsymbol{b}), \quad (16)$$

which completes the proof of Proposition 1.

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