

## 2.1 Understand

1) Prove the following theorem:

Theorem 1:  $3n^2 - 2n - 4 \in \Theta(n^2)$

Direct proof: Suppose we have constants  $a, b$ , and  $n_0$  such that  $a, b, n_0 > 0$  and  $n \geq n_0$ . To prove this theorem,  $an^2 \leq 3n^2 - 2n - 4 \leq bn^2$  must be true.

Lemma:  $3n^2 - 2n - 4 \leq bn^2$  for all  $n \geq n_0$ .

1)  $3n^2 - 2n - 4 \leq 3n^2$ ,  $b=3$  when  $n \geq 0$

Because  $n \geq 0$ ,  $-2n - 4$  makes the function smaller, so  $b=3$  when  $n_0 = 1$  is a valid conclusion.

Lemma:  $3n^2 - 2n - 4 \geq an^2$  for all  $n > n_0$ .

1)  $3n^2 - 2n - 4 \geq 3n^2 - 2n^2$  ✓ because  $3n^2$  is a larger number than  $-2n$ .

2)  $3n^2 - 2n - 4 \geq 3n^2 - 2n^2$  ||  $-2n^2 \geq -2n - 4$

3)  $3n^2 - 2n - 4 \geq n^2$

4)  $a = 1$  when  $n_0 = 1$ .

$$\begin{cases} -2n^2 \geq -2n - 4 \\ n^2 \geq n + 2 \\ n \geq 2 \end{cases}$$

Conclusion: This theorem is valid.

Theorem 2:  $\log(n^2+1) \in \Theta(\log(n))$

Direct proof: Suppose we have constant  $a, b$ , and  $n_0$  such that  $a, b, n_0 > 0$  and  $n \geq n_0$ . To prove this theorem,  $a \log(n) \leq \log(n^2+1) \leq b \log(n)$  must be true.

Lemma:  $\log(n^2+1) \leq b \log(n)$  for all  $n > n_0$ .

1)  $\log(n^2+1) \leq \log(n^2+n^2)$

2)  $\log(n^2+1) \leq \log(2n^2)$

3)  $\log(n^2+1) \leq \log 2 + \log n^2$

4)  $\log(n^2+1) \leq 2 \log n + 1$

$$2\log n \geq 1 \\ \text{when } n \geq 2$$

5)  $\log(n^2+1) \leq 2\log n + 1 \leq 2\log n + 2\log n$

6)  $\log(n^2+1) \leq 4\log n$  as a result

$a = 4$  when  $n_0 = 1$  is valid.

Lemma:  $\log(n^2+1) \geq a\log(n)$  for all  $n > n_0$

1)  $\log(n^2+1) \geq \log(n^2)$

2)  $\log(n^2) = 2\log n$

3)  $\log(n^2+1) \geq \log(n^2) = 2\log n$

$a = 2$  when  $n_0 = 0$ .

Conclusion: This theorem is true.

Theorem 3:  $2^{n+2} + 2 \in 2^n$

Proof by limit test:

If  $\lim_{n \rightarrow \infty} \frac{2^{n+2} + 2}{2^n} = C$ , where  $C$  is an arbitrary nonzero constant

then Theorem 3 holds.

Proof:

1)  $\lim_{n \rightarrow \infty} \frac{2^{n+2} + 2}{2^n} = \lim_{n \rightarrow \infty} \frac{2^{n+2}}{2^n} + \frac{2}{2^n}$

2)  $\lim_{n \rightarrow \infty} \frac{2^{n+2}}{2^n} + \frac{2}{2^n} = \lim_{n \rightarrow \infty} \frac{4 \cdot 2^n}{2^n} + 0$

3)  $\lim_{n \rightarrow \infty} \frac{2^{n+2}}{2^n} + 2 = 4$

Conclusion: Theorem 3 holds.

2a)

$$\sum_{i=1}^n \left[ c + \sum_{j=1}^i i \right] = \sum_{i=1}^n c + \sum_{i=1}^n ni$$

$$cn + n \sum_{i=1}^n i = cn + n \left( \frac{n(n+1)}{2} \right)$$

$$cn + \frac{n^2(n+1)}{2} = cn + \frac{n^3 + n^2}{2} = \frac{n^3 + n^2 + 2cn}{2}$$

$$2b) \sum_{i=1}^n \left[ c + \sum_{j=1}^i j \right] = \sum_{i=1}^n c + \sum_{i=1}^n \frac{n(n+1)}{2}$$

$$cn + \sum_{i=1}^n \frac{n^2 + n}{2} = cn + \frac{n(n^2 + n)}{2} = cn + \frac{n^3 + n^2}{2}$$

$$= \frac{n^3 + n^2 + 2cn}{2}$$

$$2c) \sum_{i=1}^n \frac{2^i}{2^n} = \frac{1}{2^n} \sum_{i=1}^n 2^i = \frac{2^{n+1} - 2}{2^n} = \frac{2^{n+1}}{2^n} - \frac{2}{2^n} \\ = 2 - \frac{2}{2^n} = 2 - 2^{1-n}$$

remove is  $\Theta(1)$  worst-case to find minimum element.

$$3a) \sum_{i=1}^n c \cdot \Theta(k) \quad \text{where } k = \text{size of A} \\ \text{size of A} = n-i+1$$

$$\left[ \sum_{i=1}^n c \cdot (n-i+1) \right] = c \cdot \sum_{i=1}^n n-i+1 \\ n+n-1+n-2+\dots+1$$

$$= c \sum_{i=1}^n i = \frac{cn(n+1)}{2}$$

The worst-case runtime is  $\Theta(n^2)$

3b) remove is  $\Theta(1)$

$$\sum_{i=1}^n C \cdot \Theta(\log(k)) \text{ where } k \text{ is the size of your tree}$$

$$k = n-i+1$$

$$\left[ \sum_{i=1}^n C \cdot \log(n-i+1) \right] = C \sum_{i=1}^n \log(n-i+1) \stackrel{\log(n)+\log(n-1)+\dots+\log(1)}{=} 0$$

$$\left( \sum_{i=1}^{\log(n)} i \right) = C (\log(n) - \log(n) + 1)$$

worst-case runtime is  $\Theta(\log n)$ .

4a) RecursiveSearch( $A[a \dots b]$ )

If  $a = b$  return  $A[a]$

Let  $c = A[a]$

Let  $d = \text{RecursiveSearch}(A[a+1 \dots b])$

If  $a < b$  return  $\max(c, d)$

level	size	# of calls	cost	cost of comparing the value of two elements
0	$n$	1	$d$	
1	$n-1$	:	:	
2	$n-2$	:	:	
3	$n-3$	:	:	
:	:	:	:	
$i$	$n-i$			
:	:	:	:	
$K$	$n-k+1$			
		$k=n-1$		
		$K-1$		

$$\text{Summation: } C + d \sum_{i=0}^{K-1} 1 = C + dk = C + d(n-1)$$

$$= C + dn - d$$

$$T(n) \in \Theta(n)$$

## 2.2 Explore

1) Logarithmic  $\rightarrow$  Polynomial  $\rightarrow$  Exponential  $\rightarrow$  Beyond

$$f_1(n) = 2^{2^n} = 4^n > f_2(n) = 3^n$$

$$3^{\log n} < 3^n < 4^n \quad // n > \log n$$

$$10^9 + 25^2 < 3^{\log n} < 3^n < 4^n$$

$$10^9 + 25^2 < 3^{\log n} < 2^{\sqrt{n}} < 3^n < 4^n \quad \lim_{n \rightarrow \infty} \frac{\log n}{\sqrt{n}} = \frac{1}{2}$$

$$f_3(n) = \log(n \cdot n^n) = \log(n^{n+1})$$

$$= (n+1)\log n = n\log n + \log n$$

$$\frac{1}{n} = \frac{1}{n} \cdot \frac{2\sqrt{n}}{2\sqrt{n}} = \frac{1}{2\sqrt{n}}$$

$$\log n + 3 < \log(n \cdot n^n) \quad // n > \log n$$

$$\log(\log n + 3) < \log n + 3 < \log(n \cdot n^n)$$

$$f_4(n) = \left(\frac{n}{\log n}\right)^2 = \frac{n^2}{\log^2 n}$$

$$10^9 + 25 < \log(\log n + 3) < \log n + 3 < \log(n \cdot n^n) < 3^{\log n}$$

$$< \left(\frac{n}{\log n}\right)^2 < 6n^2 + 12n - 4 < 2^{\sqrt{n}} < 3^n < 2^{2^n}$$

2) Consider recurrence  $T(n)$ :

$$T(n) = \begin{cases} c & \text{if } n=0 \\ 2T(\lfloor \frac{n}{4} \rfloor) + 2\sqrt{n} & \text{if } n>0 \end{cases}$$

level	size	# nodes	cost
0	$n$	1	$2\sqrt{n}$
1	$\lfloor \frac{n}{4} \rfloor$	2	$2\sqrt{\lfloor \frac{n}{4} \rfloor}$
2	$\lfloor \frac{n}{16} \rfloor$	4	$2\sqrt{\lfloor \frac{n}{16} \rfloor}$
:	:	:	:
i	$\lfloor \frac{n}{4^i} \rfloor$	$2^i$	$2\sqrt{\lfloor \frac{n}{4^i} \rfloor}$
:	:	:	:
k	$\lfloor \frac{n}{4^k} \rfloor = 0$	$2^k$	$2\sqrt{\lfloor \frac{n}{4^k} \rfloor} = c$

$$\begin{aligned} c \cdot 2^k + \sum_{i=0}^{k-1} 2^i \cdot 2\sqrt{\lfloor \frac{n}{4^i} \rfloor} &= c \cdot 2^k + 2 \sum_{i=0}^{k-1} 2^i \cdot \sqrt{\lfloor \frac{n}{4^i} \rfloor} \\ &= c \cdot 2^k + 2\sqrt{n} \sum_{i=0}^{k-1} 2^i \cdot \frac{1}{2^i} = c \cdot 2^k + 2\sqrt{n} \sum_{i=0}^{k-1} 1 \\ &= c \cdot 2^k + 2k\sqrt{n} \end{aligned}$$

Because  $\lfloor \frac{n}{4^k} \rfloor = 0$ ,  $n < 4^k$  must be true.

$$\begin{aligned} \text{Suppose } n &= 4^{k-1}, 4^{k-1} = \frac{4^k}{4} \\ 4n &= 4^k \end{aligned}$$

$$k = \log_4(4n) = \frac{1}{2} \log_2(4n) = \frac{1}{2} \log_2 n + \frac{1}{2} \log_2 4$$

$$k = \frac{1}{2} \log_2 n + 1$$

$$2c \cdot 2^{\frac{1}{2} \log_2 n} + 2(\frac{1}{2} \log_2 n + 1)\sqrt{n}$$

$$= 2c \cdot \sqrt{n} + \sqrt{n} \log_2 n + 2\sqrt{n}$$

$$= \sqrt{n} \log_2 n + (2c+2)\sqrt{n}$$

$$T(n) \in \Theta(g(n))$$

where  $g(n) = \sqrt{n} \log_2 n$

$$3) T(n) = \begin{cases} C & \text{if } n=0 \\ 4T\left(\frac{n}{4}\right) + 12n & \text{if } n>0 \end{cases}$$

a) Lemma:  $T(n) \leq bn \log_2 n$  for all  $n > n_0$

Proof: (by induction on  $n$ )

Base case: ( $n=0$ )  $\rightarrow$  base case does not work.

$$T(0) = C \leq 0 \quad \times$$

Base case:  $n \leq 4$  ( $n \neq 1, n=2, n=3$ )

$$\text{When } n \leq 4, T\left(\frac{n}{4}\right) = T(0)$$

$$n \neq 1: 4C + 12 \leq 0, \boxed{C \leq -3} \quad (\text{$C \leq 0$ is invalid.})$$

$$n=2: 2b \geq T(2), \boxed{b \geq \frac{1}{2} T(2)}$$

$$n=3: 3b \log 3 \geq T(3), \boxed{b \geq \frac{T(3)}{3 \log 3}}$$

$$T(2) = 4C + 24 \quad b \geq \frac{4C + 24}{2} \geq \frac{4C + 36}{3 \log 3}$$

$$T(3) = 4C + 36 \quad \boxed{b \geq \frac{1}{2} T(2)}$$

Inductive Hypothesis:

Assume  $T(k) \leq bk \log k$  for  $k < n$

Inductive Step: ( $n > 3$ )

$$T(n) \leq 4T\left(\frac{n}{4}\right) + 12n, \text{ suppose } k = \lfloor \frac{n}{4} \rfloor < n \quad \checkmark$$

$$T(n) \leq 4T(k) + 12n = 4bk \log k + 12n$$

$$T(n) \leq 4b\lfloor \frac{n}{4} \rfloor \log_2\left(\lfloor \frac{n}{4} \rfloor\right) + 12n \leq 4b\left(\frac{n}{4}\right) \log_2\left(\frac{n}{4}\right)$$

$$\lfloor \frac{n}{4} \rfloor \leq \frac{n}{4}$$

$$T(n) \leq bn \log_2 n - bn \log_2 4 = bn \log_2 n - 2bn$$

$$T(n) \leq bn \log_2 n - 2bn \leq bn \log_2 n$$

$$-2bn \geq 0, b \geq 0, \text{ so } b \geq \frac{1}{2} T(2)$$

By induction,  $T(n) \leq bn \log_2 n$  for  $n_0 = 3$ .

b) Lemma:  $T(n) \geq \alpha n \log_2 n$  for all  $n > n_0$

Proof: (by induction)

Base case:  $n < 4$  ( $n \neq 1, n=2, n=3$ )

When  $n < 4$ ,  $T(\lfloor \frac{n}{4} \rfloor) = C$

$$n=1: T(1) = 4C + 12 \geq 0 \quad |C \geq -3 \times C \text{ can't be negative.}$$

$$n=2: T(2) \geq 2\alpha \quad |\alpha \leq \frac{1}{2} T(2)$$

$$n=3: T(3) \geq 3 \log_2 3 \alpha \quad |\alpha \leq T(2)$$

$$\alpha \leq \frac{T(3)}{3 \log_2 3} \leq \frac{1}{2} T(2)$$

$$|\alpha \leq \frac{T(3)}{3 \log_2 3}$$

Inductive hypothesis:

Assume  $T(k) \geq \alpha k \log_2 k$  for  $k \leq n$

Inductive step: ( $n > 3$ )

$$T(n) = 4T(\lfloor \frac{n}{4} \rfloor) + 12n \quad \text{suppose } k = \lfloor \frac{n}{4} \rfloor < n$$

$$T(n) = 4T(k) + 12n$$

$$T(n) \geq 4\alpha(\lfloor \frac{n}{4} \rfloor \log_2 \lfloor \frac{n}{4} \rfloor) + 12n \quad \lfloor \frac{n}{4} \rfloor \geq \frac{n}{4} - 1$$

$$T(n) \geq 4\alpha(\lfloor \frac{n}{4} \rfloor \log_2 (\lfloor \frac{n}{4} \rfloor)) + 12n \geq 4(\alpha(\frac{n}{4} - 1) \log_2 (\frac{n}{4} - 1)) + 12n$$

$$T(n) \geq \frac{4\alpha n}{4} \log_2 (\frac{n}{4} - 1) - 4 \log_2 (\frac{n}{4} - 1), \quad \log_2 (\frac{n}{4} - 1) \geq \log_2 (\frac{n}{4}) - 1$$

$$T(n) \geq \alpha n \log_2 (\frac{n}{4}) - 4 \log_2 (\frac{n}{4})$$

$$T(n) \geq \alpha n \log_2 n - 2\alpha - 4 \log_2 n + 8 \geq \alpha n \log_2 n$$

$$T(n) \geq \alpha n \log_2 n - 2\alpha - 4 \log_2 n + 8 \geq \alpha n \log_2 n$$

$$8 \geq 2\alpha - 4 \log_2, 4 \log_2 n + 8 \geq 2\alpha$$

$$\alpha \leq 2 \log_2 n + 4$$

$$\alpha \leq \min(\frac{1}{2} T(2), 2 \log_2 n + 4)$$

By induction,  $T(n) \geq \alpha n \log_2 n$  when  $n_0 = 3$ .

## 2.3 Exponent

1)

Theorem 4:  $\sum_{i=1}^n i^d \in \Theta(n^{d+1})$

Lemma:  $\sum_{i=1}^n i^d \leq b n^{d+1}$  for all  $n \geq n_0$

$$\sum_{i=1}^n i^d \leq \sum_{i=1}^n n^d = \frac{n^{d+1}}{n-1} - 1 \leq n^{d+1} - 1 \leq n^{d+1}$$

$n$  is a positive value, so we can conclude  
 $\frac{n^{d+1}-1}{n-1} \leq n^{d+1}-1$ . In this case,  $\sum_{i=1}^n i^d \in O(n^{d+1})$

when  $b=1$ . when  $n_0=0$ .

Lemma:  $\sum_{i=1}^n i^d \geq a n^{d+1}$  for all  $n \geq n_0$

$$\sum_{i=1}^n i^d \geq \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^d + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n i^d$$

$$\sum_{i=1}^n i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n (\lfloor \frac{n}{2} \rfloor + 1)^d = (\lfloor \frac{n}{2} \rfloor + 1)^d \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n 1$$

$$\sum_{i=1}^n i^d \geq (\lfloor \frac{n}{2} \rfloor + 1)^d \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n 1 = (\lfloor \frac{n}{2} \rfloor + 1)^d (n - (\lfloor \frac{n}{2} \rfloor + 1))$$

$$\geq (\frac{n}{2})^d (\lfloor \frac{n}{2} \rfloor + 1) \geq (\frac{n}{2})^d \lfloor \frac{n}{2} \rfloor + 1 \geq \frac{n}{2}$$

$$\geq (\frac{n}{2})^d (n - \frac{n}{2}) = (\frac{n}{2})^d (\frac{n}{2})$$

$$= \frac{n^{d+1}}{2^{d+1}} \quad \text{where } a = \frac{1}{2^{d+1}} \quad \text{when } n_0=0.$$

2)

$$\text{Theorem 5: } \sum_{i=1}^n (\log_2 i)^c \in \Theta(n(\log_2 n)^c)$$

Lemma:  $\sum_{i=1}^n (\log_2 i)^c \leq b n (\log_2 n)^c$  for all  $n > n_0$

$$\sum_{i=1}^n (\log_2 i)^c \leq \sum_{i=1}^n (\log_2 n)^c = n (\log_2 n)^c$$

where  $b = 1$  and  $n_0 = 0$ .

Lemma:

$$\sum_{i=1}^n (\log_2 i)^c \geq a n (\log_2 n)^c \text{ for all } n > n_0.$$

$$\sum_{i=1}^n (\log_2 i)^c \geq \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\log_2 i)^c + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n (\log_2 i)^c \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n (\log_2 i)^c$$

$$\sum_{i=1}^n (\log_2 i)^c \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n (\log_2 i)^c \geq \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n (\log_2 (\lfloor \frac{n}{2} \rfloor + 1))^c$$

$$\sum_{i=1}^n (\log_2 i)^c \geq (\log_2 (\lfloor \frac{n}{2} \rfloor + 1))^c \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^n 1 =$$

$$\geq (\log_2 (\lfloor \frac{n}{2} \rfloor + 1))^c (n - (\lfloor \frac{n}{2} \rfloor + 1))$$

$$\lfloor \frac{n}{2} \rfloor + 1 \geq \frac{n}{2}$$

$$= (\log_2 (\frac{n}{2}))^c (\frac{n}{2})$$

$$= (\log_2 n - 1)^c (\frac{n}{2}) \geq (\log_2 n)^c (\frac{n}{2})$$

$$\sum_{i=1}^n (\log_2 i)^c \geq \frac{1}{2} n (\log_2 n)^c$$

where  $a = \frac{1}{2}c$  and  $n_0 = 0$ .

3)

$$T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 2T\left(\lfloor \frac{n}{4} \rfloor\right) + 16 & \text{if } n > 1 \end{cases}$$

a.)

level	size	#	cost
0	n	1	16
1	$\lfloor \frac{n}{4} \rfloor$	2	:
2	$\lfloor \frac{n}{16} \rfloor$	4	:
:	:	:	:
:	$\lfloor \frac{n}{4^k} \rfloor$	$2^k$	:
:	:	:	:
K	$\lfloor \frac{n}{4^K} \rfloor \leq 1$	$2^K$	c

$$2^K \cdot c + \sum_{i=0}^{K-1} 2^i \cdot 16 = 2^K \cdot c + 16 \sum_{i=0}^{K-1} 2^i$$

$$= 2^K \cdot c + 16(2^K - 1)$$

Because  $\lfloor \frac{n}{4^K} \rfloor \leq 1$ , suppose  $n = 4^K$

$$k = \log_4 n = \frac{1}{2} \log_2 n$$

$$\frac{1}{2} \log_2 n \cdot c + 16(2^{\frac{1}{2} \log_2 n} - 1) = 2^{\log_2 n^{\frac{1}{2}}} \cdot c + 16(2^{\log_2 n^{\frac{1}{2}}} - 1)$$

$$= c\sqrt{n} + 16(\sqrt{n} - 1) = c\sqrt{n} + 16\sqrt{n} - 16$$

$T(n) \in \Theta(f(n))$  where  $f(n) = \sqrt{n}$

b) Lemma:  $T(n) \leq b\sqrt{n} - a$  for all  $n \geq n_0$

Proof (by induction):

Base case: ( $n=1$ )

$$T(1) = c \leq b-a$$

$$[b-a \geq c]$$

$$-a \geq c-b$$

$$a \leq b-c$$

Inductive Hypothesis:

Assume  $T(k) \leq b\sqrt{k} - a$  for  $k \leq n$

Inductive Step: ( $n \geq 1$ )

$$T(n) = 2T(\lfloor \frac{n}{4} \rfloor) + 16, \text{ suppose } k = \lfloor \frac{n}{4} \rfloor \leq n$$

$$T(n) \leq 2T(k) + 16 = 2(b\sqrt{\lfloor \frac{n}{4} \rfloor} - a) + 16$$

$$\leq 2(b\sqrt{\lfloor \frac{n}{4} \rfloor} - a) + 16 \leq 2(b\sqrt{\frac{n}{4}} - a) + 16, \lfloor \frac{n}{4} \rfloor < \frac{n}{4} \leq n$$

$$\leq b\sqrt{n} - 2a + 16 \leq b\sqrt{n} - a$$

$$16 \leq a \quad -2a + 16 \leq -a$$

$$16 \leq -a, a \geq 16$$

By induction,  $T(n) \leq b\sqrt{n} - a$  for  $n_0 = 1$

Lemma:  $T(n) \geq a\sqrt{n} - d$  for all  $n \geq n_0$

Proof (by induction):

Base case: ( $n=1$ )

$$T(1) = c \geq a-d$$

$$[a-d \leq c]$$

Inductive Hypothesis:

Assume  $T(k) \geq a\sqrt{k} - d$  for  $k \leq n$

Inductive Step: ( $n \geq 1$ )

$$T(n) = 2T(\lfloor \frac{n}{4} \rfloor) + 16, \text{ suppose } k = \lfloor \frac{n}{4} \rfloor \leq n$$

$$T(n) \geq 2T(k) + 16 = 2(a\sqrt{\lfloor \frac{n}{4} \rfloor} - d) + 16$$

$$\geq 2(a\sqrt{\lfloor \frac{n}{4} \rfloor} - d) + 16 \geq 2(a\sqrt{\frac{n}{4}} - d) + 16, \lfloor \frac{n}{4} \rfloor > \frac{n}{4}$$

$$\geq 2(a\sqrt{\frac{n}{4}} - d) + 16 = \frac{1}{2}a\sqrt{n} - 2d + 16 \geq a\sqrt{n} - d$$

$$16 \geq d$$

$$-2d + 16 \geq -d$$

$$-d \geq -16$$

By induction,  $T(n) \geq a\sqrt{n} - d$  for  $n_0 = 1$ .

## 2.4 Challenge

$$T(n) = \begin{cases} c & \text{if } n \leq 2 \\ T(\lfloor \sqrt{n} \rfloor) + d & \text{if } n > 2 \end{cases}$$

a)

level	size	H	cost
0	n	1	d
1	$\lfloor n^{1/2} \rfloor$	1	:
2	$\lfloor n^{1/4} \rfloor$	2	:
3	$\lfloor n^{1/8} \rfloor$	3	:
4	$\lfloor n^{1/16} \rfloor$	4	:
5	$\lfloor n^{1/32} \rfloor$	5	:
6	$\lfloor n^{1/64} \rfloor$	6	:
7	$\lfloor n^{1/128} \rfloor$	7	:
8	$\lfloor n^{1/256} \rfloor$	8	:
9	$\lfloor n^{1/512} \rfloor$	9	:
10	$\lfloor n^{1/1024} \rfloor$	10	:
11	$\lfloor n^{1/2048} \rfloor$	11	:
12	$\lfloor n^{1/4096} \rfloor$	12	:
13	$\lfloor n^{1/8192} \rfloor$	13	:
14	$\lfloor n^{1/16384} \rfloor$	14	:
15	$\lfloor n^{1/32768} \rfloor$	15	:
16	$\lfloor n^{1/65536} \rfloor$	16	:
17	$\lfloor n^{1/131072} \rfloor$	17	:
18	$\lfloor n^{1/262144} \rfloor$	18	:
19	$\lfloor n^{1/524288} \rfloor$	19	:
20	$\lfloor n^{1/1048576} \rfloor$	20	:
21	$\lfloor n^{1/2097152} \rfloor$	21	:
22	$\lfloor n^{1/4194304} \rfloor$	22	:
23	$\lfloor n^{1/8388608} \rfloor$	23	:
24	$\lfloor n^{1/16777216} \rfloor$	24	:
25	$\lfloor n^{1/33554432} \rfloor$	25	:
26	$\lfloor n^{1/67108864} \rfloor$	26	:
27	$\lfloor n^{1/134217728} \rfloor$	27	:
28	$\lfloor n^{1/268435456} \rfloor$	28	:
29	$\lfloor n^{1/536870912} \rfloor$	29	:
30	$\lfloor n^{1/1073741824} \rfloor$	30	:
31	$\lfloor n^{1/2147483648} \rfloor$	31	:
32	$\lfloor n^{1/4294967296} \rfloor$	32	:
33	$\lfloor n^{1/8589934592} \rfloor$	33	:
34	$\lfloor n^{1/17179869184} \rfloor$	34	:
35	$\lfloor n^{1/34359738368} \rfloor$	35	:
36	$\lfloor n^{1/68719476736} \rfloor$	36	:
37	$\lfloor n^{1/137438953472} \rfloor$	37	:
38	$\lfloor n^{1/274877906944} \rfloor$	38	:
39	$\lfloor n^{1/549755813888} \rfloor$	39	:
40	$\lfloor n^{1/1099511627776} \rfloor$	40	:
41	$\lfloor n^{1/2199023255552} \rfloor$	41	:
42	$\lfloor n^{1/439804651108} \rfloor$	42	:
43	$\lfloor n^{1/879609302216} \rfloor$	43	:
44	$\lfloor n^{1/1759218604432} \rfloor$	44	:
45	$\lfloor n^{1/3518437208864} \rfloor$	45	:
46	$\lfloor n^{1/7036874417728} \rfloor$	46	:
47	$\lfloor n^{1/14073748835456} \rfloor$	47	:
48	$\lfloor n^{1/28147497670912} \rfloor$	48	:
49	$\lfloor n^{1/56294995341824} \rfloor$	49	:
50	$\lfloor n^{1/112589990683648} \rfloor$	50	:
51	$\lfloor n^{1/225179981367296} \rfloor$	51	:
52	$\lfloor n^{1/450359962734592} \rfloor$	52	:
53	$\lfloor n^{1/900719925469184} \rfloor$	53	:
54	$\lfloor n^{1/1801439850938368} \rfloor$	54	:
55	$\lfloor n^{1/3602879701876736} \rfloor$	55	:
56	$\lfloor n^{1/7205759403753472} \rfloor$	56	:
57	$\lfloor n^{1/14411518807506944} \rfloor$	57	:
58	$\lfloor n^{1/28823037615013888} \rfloor$	58	:
59	$\lfloor n^{1/57646075230027776} \rfloor$	59	:
60	$\lfloor n^{1/11529215046005552} \rfloor$	60	:
61	$\lfloor n^{1/23058430092011104} \rfloor$	61	:
62	$\lfloor n^{1/46116860184022208} \rfloor$	62	:
63	$\lfloor n^{1/92233720368044416} \rfloor$	63	:
64	$\lfloor n^{1/184467440736088832} \rfloor$	64	:
65	$\lfloor n^{1/368934881472177664} \rfloor$	65	:
66	$\lfloor n^{1/737869762944355328} \rfloor$	66	:
67	$\lfloor n^{1/147573952588871064} \rfloor$	67	:
68	$\lfloor n^{1/295147905177742128} \rfloor$	68	:
69	$\lfloor n^{1/590295810355484256} \rfloor$	69	:
70	$\lfloor n^{1/1180591620710968512} \rfloor$	70	:
71	$\lfloor n^{1/2361183241421937024} \rfloor$	71	:
72	$\lfloor n^{1/4722366482843874048} \rfloor$	72	:
73	$\lfloor n^{1/9444732965687748096} \rfloor$	73	:
74	$\lfloor n^{1/18889465931375496192} \rfloor$	74	:
75	$\lfloor n^{1/37778931862750992384} \rfloor$	75	:
76	$\lfloor n^{1/75557863725501984768} \rfloor$	76	:
77	$\lfloor n^{1/151115727450003969344} \rfloor$	77	:
78	$\lfloor n^{1/302231454900007938688} \rfloor$	78	:
79	$\lfloor n^{1/604462909800001877376} \rfloor$	79	:
80	$\lfloor n^{1/1208925819600003754752} \rfloor$	80	:
81	$\lfloor n^{1/2417851639200007509504} \rfloor$	81	:
82	$\lfloor n^{1/4835703278400015019008} \rfloor$	82	:
83	$\lfloor n^{1/9671406556800030038016} \rfloor$	83	:
84	$\lfloor n^{1/19342813113600060076032} \rfloor$	84	:
85	$\lfloor n^{1/38685626227200120152064} \rfloor$	85	:
86	$\lfloor n^{1/77371252454400240304128} \rfloor$	86	:
87	$\lfloor n^{1/154742504908800480608256} \rfloor$	87	:
88	$\lfloor n^{1/309485009817600961216512} \rfloor$	88	:
89	$\lfloor n^{1/618970019635201922432024} \rfloor$	89	:
90	$\lfloor n^{1/1237940039270403844864048} \rfloor$	90	:
91	$\lfloor n^{1/2475880078540807689728096} \rfloor$	91	:
92	$\lfloor n^{1/4951760157081615379456192} \rfloor$	92	:
93	$\lfloor n^{1/9903520314163230758912384} \rfloor$	93	:
94	$\lfloor n^{1/19807040628326461517824768} \rfloor$	94	:
95	$\lfloor n^{1/39614081256652923035649536} \rfloor$	95	:
96	$\lfloor n^{1/79228162513305846071299072} \rfloor$	96	:
97	$\lfloor n^{1/158456325226611692142598144} \rfloor$	97	:
98	$\lfloor n^{1/316912650453223384285196288} \rfloor$	98	:
99	$\lfloor n^{1/633825300906446768570392576} \rfloor$	99	:
100	$\lfloor n^{1/126765060181289553714078552} \rfloor$	100	:

$$C + \sum_{i=0}^{k-1} d = C + d \sum_{i=0}^{k-1} 1 = C + dk$$

Because  $\lfloor n^{1/2^k} \rfloor \leq 2$ , suppose  $n^{1/2^k} - 1 \leq 2$   
 $n^{1/2^k} \leq 3$ ,  $\log_3 n = 1/2^k$ ,  $k = 2 \log_3 n$

$$k = \frac{2}{\log_3 2} \log_2 n$$

$$C + d \left( \frac{2}{\log_3 2} \log_2 n \right) \quad T(n) \in \Theta(f(n))$$

where  $f(n) = \log_2 n$

b) Lemma:  $T(n) \leq b \log n$  for all  $n > n_0$ .

Proof (by induction):

Base case: ( $n=2$ )

$$T(2) = c \leq b \quad [b \geq c]$$

Inductive hypothesis:

Assume  $T(k) \leq b \log k$

Inductive step: ( $n > 2$ )

$$T(n) = T(\lfloor \sqrt{n} \rfloor) + d$$

$$T(n) \leq T(k) + d$$

$$T(n) \leq b \log(\lfloor \sqrt{n} \rfloor) + d \leq b \log\left(\frac{n}{2}\right) + d$$

$$\leq b \log\left(\frac{n}{2}\right) + d = b \log n - b + d \leq b \log n$$

$$[b \geq d]$$

$$k = \lfloor \sqrt{n} \rfloor < n$$

$$\lfloor \sqrt{n} \rfloor < \frac{n}{2} < n$$

$$b \geq \max(c, d)$$

By induction,  $T(n) \leq b \log n$  when  $n > 2$ .

Lemma:  $T(n) \geq a \log n$  for all  $n > n_0$ .

Proof (by induction):

Base case: ( $n=2$ )

$$T(2) = c \geq a \quad [a \leq c]$$

Inductive hypothesis:

Assume  $T(k) \geq a \log k$  for  $k < n$

Inductive step: ( $n > 2$ )

$$n \geq \lfloor \sqrt{n} \rfloor \geq \sqrt{\frac{n}{4}}$$

$$T(n) = T(\lfloor \sqrt{n} \rfloor) + d$$

$$T(n) \geq T(k) + d = a \log(\lfloor \sqrt{n} \rfloor) + d \geq a \log\left(\sqrt{\frac{n}{4}}\right) + d$$

$$\geq a \log\left(\left(\frac{n}{4}\right)^{1/2}\right) + d = \frac{1}{2}a \log\left(\frac{n}{4}\right) + d$$

$$\geq \frac{1}{2}a \log_2 n - a + d \geq a \log_2 n$$

$$a \leq d$$

$$a \leq \min(c, d)$$

By induction,  $T(n) \geq a \log n$  when  $n > 2$ .