3) $T(n) = \begin{cases} c & \text{if } n = 0 \\ 4T(\lfloor \frac{n}{4} \rfloor) + 12n & \text{if } n > 0 \end{cases}$

a) Lemma: $T(n) \le bn\log_2 n$ for all $n > n_0$

Proof: (by induction on n)

Base case: $(n = 0)$ ← Base case does not work.

$T(0) = c \le 0$ X

Base case: $n < 4$ ($n \ne 1, n = 2, n = 3$)

When $n < 4$, $T(\lfloor \frac{n}{4} \rfloor) = T(0)$,

$\cancel{n=1}$: $4c + 12 \le 0$, $\boxed{c < -3}$ X ($c \le 0$ is invalid)

$n = 2$: $2b \ge T(2)$, $\boxed{b \ge \frac{1}{2} T(2)}$

$n = 3$: $3b\log 3 \ge T(3)$, $\boxed{b \ge \frac{T(3)}{3\log 3}}$

$T(2) = 4c + 24$ $\qquad b \ge \frac{4c+24}{2} > \frac{4c+36}{3\log 3}$

$T(3) = 4c + 36$ $\qquad \boxed{b \ge \frac{1}{2} T(2)}$

Inductive Hypothesis:

Assume $T(k) \le bk\log k$ for $k < n$

Inductive Step: $(n > 3)$

$T(n) \le 4T(\lfloor \frac{n}{4} \rfloor) + 12n$, suppose $k = \lfloor \frac{n}{4} \rfloor < n$ ✓

$T(n) \le 4T(k) + 12n = 4bk\log_2 k + 12$

$T(n) \le 4b\lfloor \frac{n}{4} \rfloor \log_2(\lfloor \frac{n}{4} \rfloor) + 12n \le 4b(\frac{n}{4})\log_2(\frac{n}{4}) + 12n$

$\lfloor \frac{n}{4} \rfloor \le \frac{n}{4}$

$T(n) \le bn\log_2 n - bn\log_2 4 = bn\log_2 n - 2bn + 12n$

$T(n) \le bn\log_2 n - 2bn + 12n \le bn\log_2 n$

$= bn\log_2 n - 2n(b - 6) \le bn\log_2 n$

$b \ge 6$, $\boxed{b = \max(6, \frac{1}{2}T(2))}$

By induction, $T(n) \le bn\log_2 n$ for $n_0 = 3$.

b) Lemma: $T(n) \ge a n \log_2 n$ for all $n > n_0$

Proof: (by induction)

Base case: $n < 4$ ($n \neq 1$, $n=2$, $n=3$)

When $n < 4$, $T(\lfloor \frac{n}{4} \rfloor) = c$ $\boxed{c \ge -3}$ X c can't be negative.

$n \neq 1$: $T(1) = 4c + 12 \ge 0$

$n=2$: $T(2) \ge 2a$ $\boxed{a \le \frac{1}{2} T(2)}$

$n=3$: $T(3) \ge 3 \log_2 3 \, a$ $\boxed{a \le \frac{T(3)}{3\log_2 3}}$

$$a \le \frac{T(3)}{3\log_2 3} \le \frac{1}{2} T(2)$$

$$\boxed{a \le \frac{T(3)}{3\log_2 3}}$$

Inductive Hypothesis:

Assume $T(k) \ge a k \log_2 k$ for $k < n$

Inductive Step: $(n > 3)$

$T(n) = 4T(\lfloor \frac{n}{4} \rfloor) + 12n$ suppose $k = \lfloor \frac{n}{4} \rfloor < n$

$T(n) \stackrel{?}{=} 4T(k) + 12n$

$T(n) \ge 4a(\lfloor \frac{n}{4} \rfloor \log_2 \lfloor \frac{n}{4} \rfloor) + 12n$ $\lfloor \frac{n}{4} \rfloor \ge \frac{n}{4} - 1$

$T(n) \ge 4a(\lfloor \frac{n}{4} \rfloor \log_2 (\lfloor \frac{n}{4} \rfloor)) + 12n \ge 4(a(\frac{n}{4} - 1)\log_2(\frac{n}{4} - 1)) + 12n$

$T(n) \ge \frac{4an}{4} \log_2(\frac{n}{4} - 1) - 4a\log_2(\frac{n}{4} - 1) + 12n$

$\log_2(n/4 - 1) \ge \log_2(n/4) - 1$

$T(n) \ge an\log_2(\frac{n}{4}) - 4a\log_2(\frac{n}{4}) + 12n$,

$T(n) \ge an\log_2 n - 2a - 4a\log_2 n + 8a + 12n \ge an\log_2 n$

$T(n) \ge an\log_2 n + 6a + 12n - 4a\log_2 n \ge an\log_2 n$

$6a + 12n - 4a\log_2 n \ge 0$

$6a - 4a\log_2 n \ge -12n$, $a(6 - 4\log_2 n) \ge -12n$

$\boxed{a = \min(\frac{T(3)}{3\log_2 3}, \frac{-12n}{6 - 4\log_2 n})}$ $a \ge \frac{-12n}{6 - 4\log_2 n}$

By induction, $T(n) \ge ak\log_2 k$ for $n_0 = 3$.