2.1 Understand

1) $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 2T(\lfloor \frac{n}{4} \rfloor) + 16 & \text{if } n > 1 \end{cases}$

$a = 2$ $\quad \log_b a = \log_4 2 = \frac{1}{2}$

$b = 4$ $\quad d < 1/2$, so $T(n) \in \Theta(n^{1/2})$

$d = 0$

2) $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 4T(\lfloor \frac{n}{4} \rfloor) + 16n & \text{if } n > 1 \end{cases}$

$a = 4$ $\quad \log_b a = \log_4 4 = 1$

$b = 4$ $\quad d = 1$, so $T(n) \in \Theta(n \log n)$

$d = 1$

3) $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 8T(\lfloor \frac{n}{4} \rfloor) + 16n & \text{if } n > 1 \end{cases}$

$a = 8$ $\quad \log_b a = \log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$

$b = 4$

$d = 1$ $\quad d < \frac{3}{2}$, so $T(n) \in \Theta(n^{3/2})$

4) $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 2T(\lfloor \frac{n}{3} \rfloor) + 16n & \text{if } n > 1 \end{cases}$

$a = 2$ $\quad \log_b a = \log_3 2$

$b = 3$ $\quad d > \log_3 2$, so $T(n) \in \Theta(n)$

$d = 1$

5) $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 25T(\lfloor \frac{n}{3} \rfloor) + 16n^3 & \text{if } n > 1 \end{cases}$

$a = 25$ $\quad \log_b a = \log_3 25$

$b = 3$

$d = 3$ $\quad d > \log_3 25$, so $T(n) \in \Theta(n^3)$

2.2 Explore

1) <u>Input:</u> Let a and n be positive integers.
<u>Output:</u> a^n is the product such that a is multiplied by itself by n times.

2) Exponential (a, n) —
Let K = 1
For i = 1 to n
K = K · a
End For
return K

3) $a^n = a^{\frac{n}{2}} \cdot a^{\frac{n}{2}} = \left(a^{\frac{n}{2}}\right)^2$

4) $a^n = a^{\frac{n-1}{2}} \cdot a^{\frac{n-1}{2}} \cdot a = \left(a^{\frac{n-1}{2}}\right)^2 \cdot a$

5)

$$E(a,n) = \begin{cases} a & \text{if } n = 1 \\ E(a,\frac{n}{2}) \cdot E(a,\frac{n}{2}) & \text{if } n > 1 \text{ and } n \text{ is even} \\ a \cdot E(a,\frac{n-1}{2}) \cdot E(a,\frac{n-1}{2}) & \text{if } n > 1 \text{ and } n \text{ is odd} \end{cases}$$

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6) Exponential(a, n)

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Let k = a
If n is 1
  return k
If n > 1 and even
  k = E(a, n/2)
  return k • k
If n > 1 and odd
  k = E(a, (n-1)/2)
  return a • k • k
End
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7) $T(n) = T(\frac{n}{2}) + d$

$a = 1$ $\quad \log_b a = \log_2 1 = 0$

$b = 2$ $\quad d = 0$, so $T(n) \in \Theta(\log n)$

$d = 0$

Proof by <u>Masters Theorem</u>

2.3 Expand

1) Input: Two sorted lists A[1...n] and B[1...n] with n number of positive integers.

Output: Let m represent the index of the median of both A[1...n] and B[1...n] and M(A, B) represent the max(A[m], B[m]) when n = 1.

2) Median (A[1...n], B[1...n])

    Let C be an empty 1-d array
    Let i = 1, j = 1, and k = 1
    If A[i] < B[j], C[k++] = A[i]
        increment i
    else if A[i] > B[j], C[k++] = B[j]
        increment j
    else
        C[k++] = A[i]
        C[k++] = B[j]
        increment i and j
    Return $C[\lceil \frac{k+1}{2} \rceil]$

3)

$$M(A,B) = \begin{cases} \max(A[n], B[n]) & \text{if } n = 1 \\ M(A[m-1], B[m]) & \text{if } n = 2 \text{ \& } A[m] > B[m] \\ M(A[m], B[m-1]) & \text{if } n=2 \text{ \& } A[m] < B[m] \\ M(A[m \dots n], B[1 \dots m]) & \text{if } n > 2 \text{ \& } A[m] < B[m] \\ M(A[1 \dots m], B[m \dots n]) & \text{if } n > 2 \text{ \& } A[m] > B[m] \\ A[m] & \text{if } n > 1 \text{ \& } A[m] = B[m] \end{cases}$$

Let $m = \left\lceil \frac{n+1}{2} \right\rceil$

4) Median(A[1...n], B[1...n])

if $n = 1$

Return max(A[n], B[n])

if $n > 1$ and $A[m] = B[m]$

Return A[m]

if $n = 2$ and $A[m] > B[m]$

Return Median(A[m-1], B[m])

if $n = 2$ and $A[m] < B[m]$

Return Median(A[m], B[m-1])

if $n > 2$ and $A[m] < B[m]$

Return Median(A[m...n], B[1...m])

if $n > 2$ and $A[m] > B[m]$

Return Median(A[1...m], B[m...n])

5) $T(n) = T(\lceil \frac{n}{2} \rceil) + d$

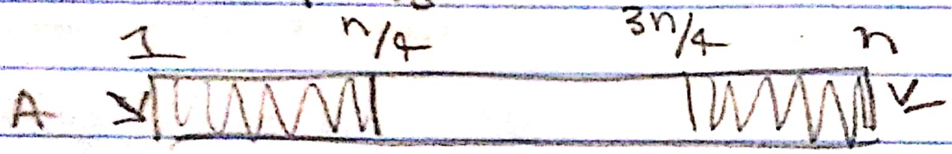
$a = 1$ $\qquad \log_b a = \log_2 1 = 0$

$b = 2$

$d = 0$ $\qquad d = 0$, so $T(n) \in \Theta(\log n)$ based on <u>Master Theorem</u>.

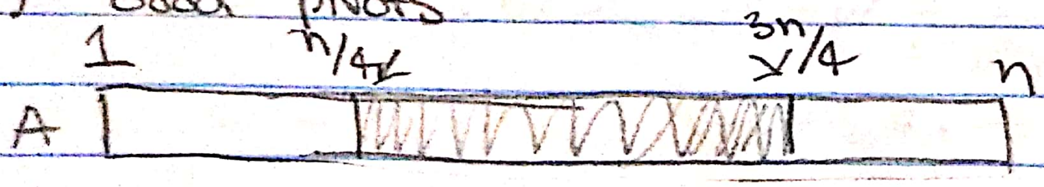
2.4 Challenge

1) Bad pivots



$$T(n) = \begin{cases} c & \text{if } n = 1 \\ T(n-1) + dn & \text{if } n > 1 \end{cases}$$

2) Good pivots



$$T(n) = \begin{cases} c & \text{if } n = 1 \\ T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{4} \right\rfloor\right) + dn & \text{if } n \geq 1 \end{cases}$$

3) $T(n) = \sum_{i=1}^{n} \frac{1}{n} (T(i-1) + T(n-i)) + (n-1)$

$T(n) = \frac{2}{n} \sum_{i=1}^{n} T(i-1) + n-1$

$nT(n) = 2 \sum_{i=1}^{n} T(i-1) + n(n-1)$

$(n-1)T(n-1) = 2 \sum_{i=1}^{n-1} T(i-1) + (n-1)(n-2)$

$nT(n) - (n-1)T(n-1) = 2T(n-1) + 2(n-1)$

$=$

$\frac{T(n)}{\cancel{n+1}} = \frac{T(n-1)}{n}(n+1) + \frac{2(n-1)}{n\cancel{(n+1)}}\cancel{(n+1)}$

$T(n) = \frac{T(n-1)(n+1)}{n} + \frac{2(n-1)}{n}$

4)

$$T(n) = \begin{cases} c & \text{if } n=1 \\ \frac{T(n-1)(n+1)}{n} + \frac{2(n-1)}{n} & \text{if } n>1 \end{cases}$$

Lemma: $T(n) \leq bn\log n$ for all $n > n_0$

Base case: $(n=2)$

$$T(2) = \frac{3T(1)}{2} + 1 \leq 2b = \frac{3c+2}{2} \leq 4b$$

$$\boxed{b \geq \frac{3c+2}{4}}$$

Inductive Hypothesis: $(n>2)$

Assume $T(k) \leq bk\log k$ for $k<n$

Induction Step:

$$T(n) = \frac{T(n-1)(n+1)}{n} + \frac{2(n-1)}{n}$$

$$\leq \frac{T(k)(n+1)}{n} + \frac{2k}{n} \quad \text{for } k=n-1<n$$

$$\leq b(n-1)\log(n-1) + \frac{2(n-1)}{n} \leq b(n-1)\log n + 2$$

$$\leq bn\log n - b\log n + 2 \leq bn\log n$$

$$b\log n \geq 2 \qquad b \geq \max\left(\frac{3c+2}{4}, \frac{2}{\log n}\right)$$

By induction, $T(n) \leq bn\log n$ $\boxed{b \geq \frac{2}{\log n}}$

when $n_0 = 2$