2.1 Understand

1) Prove the following theorem:

Theorem 1: $3n^2 - 2n - 4 \in \Theta(n^2)$

Direct proof: Suppose we have constants $a, b,$ and n_ϵ such that $a, b, n_\epsilon > 0$ and $n \geq n_\epsilon$. To prove this theorem, $an^2 \leq 3n^2 - 2n - 4 \leq bn^2$ must be true.

Lemma: $3n^2 - 2n - 4 \leq bn^2$ for all $n > n_\epsilon$

1) $3n^2 - 2n - 4 \leq 3n^2$, $b = 3$ when $n > 0$

Because $n > 0$, $-2n - 4$ makes the function smaller, so $b = 3$ when $n_\epsilon = 1$ is a valid conclusion.

Lemma: $3n^2 - 2n - 4 \geq an^2$ for all $n > n_\epsilon$.

1) $3n^2 - 2n - 4 \geq 3n^2 - 2n^2 - 4$ ← $2n^2$ is a larger negative number than $-2n$.
2) $3n^2 - 2n - 4 \geq 3n^2 - 2n^2$ || $-2n^2 \leq -2n - 4$
3) $3n^2 - 2n - 4 \geq n^2$ $n^2 \geq n + 2$
4) $a = 1$ when $n_\epsilon = 1$. $n \geq 2$

Conclusion: This theorem is valid.

Theorem 2: $\log(n^2+1) \in \Theta(\log(n))$

Direct proof: Suppose we have constant $a, b,$ and n_ϵ such that $a, b, n_\epsilon > 0$ and $n \geq n_\epsilon$. To prove this theorem, $a\log(n) \leq \log(n^2+1) \leq b\log(n)$ must be true.

Lemma: $\log(n^2+1) \leq b\log(n)$ for all $n > n_\epsilon$

1) $\log(n^2+1) \leq \log(n^2+n^2)$
2) $\log(n^2+1) \leq \log(2n^2)$
3) $\log(n^2+1) \leq \log 2 + \log n^2$
4) $\log(n^2+1) < 2\log n + 1$

$2\log n \geq 1$
when $n \geq 2$

5) $\log(n^2+1) \leq 2\log n + 1 \leq 2\log n + 2\log n$

6) $\log(n^2+1) \leq 4\log n$ as a result

$b=4$ when $n_\epsilon = 1$ is valid.

Lemma: $\log(n^2+1) \geq a\log(n)$ for all $n > n_\epsilon$

1) $\log(n^2+1) \geq \log(n^2)$

2) $\log(n^2) = 2\log n$

3) $\log(n^2+1) \geq \log(n^2) = 2\log n$

$a=2$ when $n_\epsilon = 0$.

Conclusion: This theorem is true.

Theorem 3: $2^{n+2}+2 \in 2^n$

Proof by Limit test:

If $\lim_{n\to\infty} \frac{2^{n+2}+2}{2^n} = c$, where c is an arbitrary nonzero constant

then theorem 3 holds.

Proof:

1) $\lim_{n\to\infty} \frac{2^{n+2}+2}{2^n} = \lim_{n\to\infty} \frac{2^{n+2}}{2^n} + \frac{2}{2^n}$

2) $\lim_{n\to\infty} \frac{2^{n+2}}{2^n} + \frac{2}{2^n} = \lim_{n\to\infty} \frac{4\cdot\cancel{2^n}}{\cancel{2^n}} + 0$

3) $\lim_{n\to\infty} \frac{2^{n+2}+2}{2^n} = 4$

Conclusion: theorem 3 holds.

2a) $\sum_{i=1}^{n}\left[c+\sum_{j=1}^{n} i\right] = \sum_{i=1}^{n} c + \sum_{i=1}^{n} ni$

$cn + n\sum_{i=1}^{n} i = cn + n\left(\frac{n(n+1)}{2}\right)$

$cn + \frac{n^2(n+1)}{2} = cn + \frac{n^3+n^2}{2} \boxed{= \frac{n^3+n^2+2cn}{2}}$

2b) $\sum_{i=1}^{n}\left[c+\sum_{j=1}^{n} j\right] = \sum_{i=1}^{n} c + \sum_{i=1}^{n} \frac{n(n+1)}{2}$

$cn + \sum_{i=1}^{n} \frac{n^2+n}{2} = cn + \frac{n(n^2+n)}{2} = cn + \frac{n^3+n^2}{2}$

$\boxed{= \frac{n^3+n^2+2cn}{2}}$

2c) $\sum_{i=1}^{n} \frac{2^i}{2^n} = \frac{1}{2^n}\sum_{i=1}^{n} 2^i = \frac{2^{n+1}-2}{2^n} = \frac{2^{n+1}}{2^n} - \frac{2}{2^n}$

$= 2 - \frac{2}{2^n} = 2 - 2^{1-n}$

$\boxed{= 2 - 2^{1-n}}$

remove is $\Theta(1)$, worst-case to find minimum element.

3a) $\sum_{i=1}^{n} c \cdot \Theta(k)$ where k = size of A

size of $A = n-i+1$

$\boxed{\sum_{i=1}^{n} c\cdot(n-i+1)} = c\sum_{i=1}^{n} n-i+1$

$n+n-1+n-2+\ldots+1$

$= c\sum_{i=1}^{n} i = \frac{cn(n+1)}{2}$

The worst-case runtime is $\Theta(n^2)$

3b) remove is $\Theta(1)$

$$\sum_{i=1}^{n} c \cdot \Theta(\log(k))$$ where k is the size of your tree.

$k = n - i + 1$

$$\boxed{\sum_{i=1}^{n} c \cdot \log(n-i+1)} = c \cdot \sum_{i=1}^{n} \log(n-i+1)$$

$\log(n) + \log(n-1) + \ldots + \underbrace{\log(1)}_{=0}$

$$c \sum_{i=1}^{\log(n))} i = c(\log(n) - \log(n) + 1)$$

<u>worst-case runtime is $\Theta(\log n)$.</u>

4a) RecursiveSearch(A[a...b])

If a = b return A[a]

Let c =' A[a].

Let d = Recursive Search (A [a+1...b]) $\leftarrow n$

If a < b return max(c, d)

4b)

level	size	# of calls	cost
0	n	1	d
1	n-1	.	.
2	n-2	.	.
3	n-3	.	.
⋮	⋮		
i	n-i		
⋮	⋮		
K	n-k=1		c
	k=n-1		

cost ← cost of comparing the value of two elements

Summation: $c + d \sum_{i=0}^{k-1} 1 = c + dk = c + d(n-1)$

$= c + dn - d$

$$\boxed{T(n) \in \Theta(n)}$$

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2.2 Explore

1) Logarithmic → Polynomial → Exponential → Beyond

$f_1(n) = 2^{2n} = 4^n > f_4(n) = 3^n$

$3^{\log n} < 3^n < 4^n$ // $n > \log n$

$10^9 + 25^2 < 3^{\log n} < 3^n < 4^n$

$10^9 + 25^2 < 3^{\log n} < 2^{\sqrt{n}} < 3^n < 4^n$

$\lim_{n \to \infty} \frac{\log n}{\sqrt{n}} \to \frac{\infty}{\infty}$

$\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \frac{1}{n} \cdot \frac{2\sqrt{n}}{1} = ($

$f_5(n) = \log(n \cdot n^n) = \log(n^{n+1})$

$= (n+1)\log n = n\log n + \log n$

$\log n + 3 < \log(n \cdot n^n)$ // $n > \log n$.

$\log(\log n + 3) < \log n + 3 < \log(n \cdot n^n)$

$f_2(n) = \left(\frac{n}{\log n}\right)^2 = \frac{n^2}{\log^2 n}$

$10^9 + 25 < \log(\log n + 3) < \log n + 3 < \log(n \cdot n^n) < 3^{\log n}$
$< \left(\frac{n}{\log n}\right)^2 < 6n^2 + 12n - 4 < 2^{\sqrt{n}} < 3^n < 2^{2n}$

2) Consider recurrence $T(n)$:

$$T(n) = \begin{cases} c & \text{if } n = 0 \\ 2T(\lfloor \frac{n}{4} \rfloor) + 2\sqrt{n} & \text{if } n > 0 \end{cases}$$

level	size	# of nodes	cost
0	n	1	$2\sqrt{n}$
1	$\lfloor \frac{n}{4} \rfloor$	2	$2\sqrt{\lfloor \frac{n}{4} \rfloor}$
2	$\lfloor \frac{n}{16} \rfloor$	4	$2\sqrt{\lfloor \frac{n}{16} \rfloor}$
⋮	⋮	⋮	⋮
i	$\lfloor \frac{n}{4^i} \rfloor$	2^i	$2\sqrt{\lfloor \frac{n}{4^i} \rfloor}$
⋮	⋮	⋮	⋮
k	$\lfloor \frac{n}{4^k} \rfloor = 0$	2^k	$2\sqrt{\lfloor \frac{n}{4^k} \rfloor} = c$

$$c \cdot 2^k + \sum_{i=0}^{k-1} 2^i \cdot 2\sqrt{\lfloor \frac{n}{4^i} \rfloor} = c \cdot 2^k + 2\sum_{i=0}^{k-1} 2^i \cdot \sqrt{\lfloor \frac{n}{4^i} \rfloor}$$

$$= c \cdot 2^k + 2\sqrt{n} \sum_{i=0}^{k-1} 2^i \cdot \frac{1}{2^i} = c \cdot 2^k + 2\sqrt{n} \sum_{i=0}^{k-1} 1$$

$$= c \cdot 2^k + 2k\sqrt{n}$$

Because $\lfloor \frac{n}{4^k} \rfloor = 0$, $n < 4^k$ must be true.

Suppose $n = 4^{k-1}$, $4^{k-1} = \frac{4^k}{4}$

$4n = 4^k$

$k = \log_4(4n) = \frac{1}{2}\log_2(4n) = \frac{1}{2}\log_2 n + \frac{1}{2}\log_2 4$

$k = \frac{1}{2}\log_2 n + 1$

$$2c \cdot 2^{(\frac{1}{2})\log_2 n} + 2(\tfrac{1}{2}\log_2 n + 1)\sqrt{n}$$

$$= 2c \cdot \sqrt{n} + \sqrt{n}\log_2 n + 2\sqrt{n}$$

$$= \sqrt{n}\log_2 n + (2c+2)\sqrt{n}$$

$\boxed{T(n) \in \Theta(g(n)) \text{ where } g(n) = \sqrt{n}\log_2 n}$

3) $T(n) = \begin{cases} c & \text{if } n = 0 \\ 4T(\lfloor \frac{n}{4} \rfloor) + 12n & \text{if } n > 0 \end{cases}$

a) Lemma: $T(n) \leq bn\log_2 n$ for all $n > n_0$

Proof: (by induction on n)

Base case: $(n = 0)$ ← Base case does not work.

$T(0) = c \leq 0$ X

Base case: $n < 4$ ($\cancel{n=1}$, $n=2$, $n=3$)

When $n < 4$, $T(\lfloor \frac{n}{4} \rfloor) = T(0)$,

~~n=1~~: $4c + 12 \leq 0$, $\boxed{c \leq -3}$ X ($c \leq 0$ is invalid)

n=2: $2b \geq T(2)$, $\boxed{b \geq \frac{1}{2}T(2)}$

n=3: $3b\log 3 \geq T(3)$, $\boxed{b \geq \frac{T(3)}{3\log 3}}$

$T(2) = 4c + 24$ $\qquad b \geq \frac{4c+24}{2} \geq \frac{4c+36}{3\log 3}$

$T(3) = 4c + 36$ $\qquad \boxed{b \geq \frac{1}{2}T(2)}$

Inductive Hypothesis:

Assume $T(k) \leq bk\log k$ for $k < n$

Inductive Step: $(n \geq 3)$

$T(n) \leq 4T(\lfloor \frac{n}{4} \rfloor) + 12n$, suppose $k = \lfloor \frac{n}{4} \rfloor < n$ ✓

$T(n) \leq 4T(k) + 12n = 4bk\log k + 12$

$T(n) \leq 4b\lfloor \frac{n}{4} \rfloor \log_2(\lfloor \frac{n}{4} \rfloor) + 12n \leq 4b(\frac{n}{4})\log_2(\frac{n}{4})$

$\lfloor \frac{n}{4} \rfloor \leq \frac{n}{4}$

$T(n) \leq bn\log_2 n - bn\log_2 4 = bn\log_2 n - 2bn$

$T(n) \leq bn\log_2 n - 2bn \leq bn\log_2 n$

$2bn \geq 0$, $b \geq 0$, so $b \geq \frac{1}{2}T(2)$

By induction, $T(n) \leq bn\log_2 n$ for $n_0 = 3$.

b) Lemma: $T(n) \geq an\log_2 n$ for all $n > n_0$

Proof: (by induction)

Base case: $n < 4$ ($\cancel{n=1}$, $n=2$, $n=3$)

When $n<4$, $T(\lfloor \frac{n}{4} \rfloor) = c$

$\cancel{n=1}$: $T(1) = 4c+12 \geq 0$ $\boxed{c \geq -3}$ X c can't be negative

$n=2$: $T(2) \geq 2a$ $\boxed{a \leq \frac{1}{2}T(2)}$

$n=3$: $T(3) \geq 3\log_2 3\ a$ $\boxed{a \leq \frac{T(3)}{3\log_2 3}}$

$$a \leq \frac{T(3)}{3\log_2 3} \leq \frac{1}{2}T(2)$$

$$\boxed{a \leq \frac{T(3)}{3\log_2 3}}$$

Inductive Hypothesis:

Assume $T(k) \geq ak\log_2 k$ for $k<n$

Inductive Step: $(n > 3)$

$T(n) = 4T(\lfloor \frac{n}{4} \rfloor) + 12n$ suppose $k = \lfloor \frac{n}{4} \rfloor < n$

$T(n) = 4T(k) + 12n$

$T(n) \geq 4a(\lfloor \frac{n}{4} \rfloor \log_2 \lfloor \frac{n}{4} \rfloor) + 12n$ $\lfloor \frac{n}{4} \rfloor \geq \frac{n}{4} - 1$

$T(n) \geq 4a(\lfloor \frac{n}{4} \rfloor \log_2 (\lfloor \frac{n}{4} \rfloor)) + 12n \geq 4(a(\frac{n}{4}-1)\log_2(\frac{n}{4}-1)) + 12n$

$T(n) \geq \frac{4an}{4}\log_2(\frac{n}{4}-1) - 4\log_2(\frac{n}{4}-1)$, $\log_2(\frac{n}{4}-1) \geq \log_2(\frac{n}{4}) - 1$

$T(n) \geq an\log_2(\frac{n}{4}) - 4\log_2(\frac{n}{4})$

$T(n) \geq an\log_2 n - 2a - 4\log_2 n + 8 \geq an\log_2 n$

$T(n) \geq an\log_2 n - 2a - 4\log_2 n + 8 \geq an\log_2 n$

$8 \geq 2a - 4\log_2$, $4\log_2 n + 8 \geq 2a$

$a \leq 2\log_2 n + 4$

$a \leq \min(\frac{1}{2}T(2), 2\log_2 n + 4)$

By induction, $T(n) \geq an\log_2 n$ when $n_0 = 3$.

2.3 Expand

1)

Theorem 4: $\sum_{i=1}^{n} i^d \in \Theta(n^{d+1})$

Lemma: $\sum_{i=1}^{n} i^d \leq b n^{d+1}$ for all $n > n_0$

$$\sum_{i=1}^{n} i^d \leq \sum_{i=1}^{n} n^d = \frac{n^{d+1}-1}{n-1} \leq n^{d+1}-1 \leq n^{d+1}$$

n is a positive value, so we can conclude $\frac{n^{d+1}-1}{n-1} \leq n^{d+1}-1$. In this case, $\sum_{i=1}^{n} i^d \in O(n^{d+1})$

when $b=1$. when $n_0=0$.

Lemma: $\sum_{i=1}^{n} i^d \geq a n^{d+1}$ for all $n > n_0$

$$\sum_{i=1}^{n} i^d \geq \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^d + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} i^d$$

$$\sum_{i=1}^{n} i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} i^d \geq \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} \left(\lfloor \tfrac{n}{2} \rfloor+1\right)^d = \left(\lfloor \tfrac{n}{2} \rfloor+1\right)^d \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} 1$$

$$\sum_{i=1}^{n} i^d \geq \left(\lfloor \tfrac{n}{2} \rfloor+1\right)^d \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} 1 = \left(\lfloor \tfrac{n}{2} \rfloor+1\right)^d \left(n-\left(\lfloor \tfrac{n}{2} \rfloor+1\right)\right)$$

$$\lfloor \tfrac{n}{2} \rfloor+1 \geq \tfrac{n}{2}$$

$$\geq \left(\tfrac{n}{2}\right)^d \left(n-\tfrac{n}{2}\right) = \left(\tfrac{n}{2}\right)^d \left(\tfrac{n}{2}\right)$$

$$= \frac{n^{d+1}}{2^{d+1}}$$ where $a = \frac{1}{2^{d+1}}$ when $n_0=0$.

2)

Theorem 5: $\sum_{i=1}^{n} (\log_2 i)^c \in \Theta(n(\log_2 n)^c)$

Lemma: $\sum_{i=1}^{n} (\log_2 i)^c \leq bn(\log_2 n)^c$ for all $n \geq n_0$

$$\sum_{i=1}^{n} (\log_2 i)^c \leq \sum_{i=1}^{n} (\log_2 n)^c = n(\log_2 n)^c$$

where $b = 1$ and $n_0 = 0$.

Lemma:

$\sum_{i=1}^{n} (\log_2 i)^c \geq an(\log_2 n)^c$ for all $n \geq n_0$.

$$\sum_{i=1}^{n} (\log_2 i)^c \geq \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\log_2 i)^c + \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (\log_2 i)^c \geq \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (\log_2 i)^c$$

$$\sum_{i=1}^{n} (\log_2 i)^c \geq \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (\log_2 i)^c \geq \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (\log_2(\lfloor \tfrac{n}{2} \rfloor+1))^c$$

$$\sum_{i=1}^{n} (\log_2 i)^c \geq (\log_2(\lfloor \tfrac{n}{2} \rfloor+1))^c \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} 1 =$$

$$\geq (\log_2(\lfloor \tfrac{n}{2} \rfloor+1))^c (n-(\lfloor \tfrac{n}{2} \rfloor+1))$$

$$\lfloor \tfrac{n}{2} \rfloor + 1 \geq \frac{n}{2}$$

$$= (\log_2(\tfrac{n}{2}))^c (\tfrac{n}{2})$$

$$= (\log_2 n - 1)^c (\tfrac{n}{2}) \geq (\log_2 n)^c (\tfrac{n}{2^c})$$

$$\sum_{i=1}^{n} (\log_2 i)^c \geq \frac{1}{2^c} n(\log_2 n)^c$$

where $a = \frac{1}{2^c}$ and $n_0 = 0$.

3)

$$T(n) = \begin{cases} c & \text{if } n \le 1 \\ 2T(\lfloor \frac{n}{4} \rfloor) + 16 & \text{if } n > 1 \end{cases}$$

a)

level	size	#	cost
0	n	1	16
1	$\lfloor \frac{n}{4} \rfloor$	2	⋮
2	$\lfloor \frac{n}{16} \rfloor$	4	⋮
⋮	⋮	⋮	⋮
⋮	$\lfloor \frac{n}{4^i} \rfloor$	2^i	⋮
⋮	⋮	⋮	⋮
k	$\lfloor \frac{n}{4^k} \rfloor \le 1$	2^k	c

$$2^k \cdot c + \sum_{i=0}^{k-1} 2^i \cdot 16 = 2^k \cdot c + 16 \sum_{i=0}^{k-1} 2^i$$

$$= 2^k \cdot c + 16(2^k - 1)$$

Because $\lfloor \frac{n}{4^k} \rfloor \le 1$, suppose $n = 4^k$

$k = \log_4 n = \frac{1}{2}\log_2 n$

$$2^{\frac{1}{2}\log_2 n} \cdot c + 16(2^{\frac{1}{2}\log_2 n} - 1) = 2^{\log_2 n^{1/2}} \cdot c + 16(2^{\log_2 n^{1/2}} - 1)$$

$$= c\sqrt{n} + 16(\sqrt{n} - 1) = c\sqrt{n} + 16\sqrt{n} - 16$$

$$\boxed{T(n) \in \Theta(f(n)) \text{ where } f(n) = \sqrt{n}}$$

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b) Lemma: $T(n) \le b\sqrt{n} - a$ for all $n \ge n_0$

Proof (by induction):

Base case: $(n=1)$ $\qquad -a \ge c-b$

$T(1) = c \le b-a$ $\qquad \boxed{b-a \ge c}$ $\qquad a \le b-c$

Inductive Hypothesis:

Assume $T(k) \le b\sqrt{k} - a$ for $k < n$

Inductive Step: $(n > 1)$

$T(n) = 2T(\lfloor \frac{n}{4} \rfloor) + 16$, suppose $k = \lfloor \frac{n}{4} \rfloor < n$

$T(n) \le 2T(k) + 16 = 2(b\sqrt{\lfloor \frac{n}{4} \rfloor} - a) + 16$

$\le 2(b\sqrt{\lfloor \frac{n}{4} \rfloor} - a) + 16 \le 2(b\sqrt{\frac{n}{4}} - a) + 16$, $\lfloor \frac{n}{4} \rfloor < \frac{n}{4} < n$

$\le b\sqrt{n} - 2a + 16 \le b\sqrt{n} - a$

$16 \le a$ $\qquad -2a + 16 \le -a$

$16 \le -a$, $a \ge 16$

By induction, $T(n) \le b\sqrt{n} - a$ for $n_0 = 1$

Lemma: $T(n) \ge a\sqrt{n} - d$ for all $n \ge n_0$

Proof (by induction):

Base case: $(n=1)$

$T(1) = c \ge a-d$ $\qquad \boxed{a-d \le c}$

Inductive Hypothesis:

Assume $T(k) \ge a\sqrt{k} - d$ for $k < n$

Inductive Step: $(n > 1)$

$T(n) = 2T(\lfloor \frac{n}{4} \rfloor) + 16$, suppose $k = \lfloor \frac{n}{4} \rfloor < n$

$T(n) \ge 2T(k) + 16 = 2(a\sqrt{\lfloor \frac{n}{4} \rfloor} - d) + 16$

$\ge 2(a\sqrt{\lfloor \frac{n}{4} \rfloor} - d) + 16 \ge 2(a\sqrt{\frac{n}{16}} - d) + 16$, $\lfloor \frac{n}{4} \rfloor \ge \frac{n}{16}$

$\ge 2(\frac{a\sqrt{n}}{4} - d) + 16 = \frac{1}{2}a\sqrt{n} - 2d + 16 \ge a\sqrt{n} - d$

$16 \ge d$ $\qquad -2d + 16 \ge -d$

$-d \ge -16$

By induction, $T(n) \ge a\sqrt{n} - d$ for $n_0 = 1$.

2.4 Challenge

$T(n) = \begin{cases} c & \text{if } n \le 2 \\ T(\lfloor \sqrt{n} \rfloor) + d & \text{if } n > 2 \end{cases}$

a)

level	size	#	cost
0	n	1	d
1	$\lfloor n^{1/2} \rfloor$	:	:
2	$\lfloor n^{1/4} \rfloor$	:	:
:	:	$\downarrow$	$\downarrow$
i	$\lfloor n^{1/2^i} \rfloor$		
:	:		
k	$\lfloor n^{1/2^k} \rfloor \le 2$		, c

$$c + \sum_{i=0}^{k-1} d = c + d \sum_{i=0}^{k-1} 1 = c + dk$$

Because $\lfloor n^{1/2^k} \rfloor \le 2$, suppose $n^{1/2^k} - 1 \le 2$

$n^{1/2^k} \le 3$, $\log_3 n = 1/2 k$, $k = 2\log_3 n$

$$k = \frac{2}{\log_3 2} \log_2 n$$

$c + d\left(\frac{2}{\log_3 2} \log_2 n\right)$ $T(n) \in \Theta(f(n))$

where $f(n) = \log_2 n$

b) Lemma: $T(n) \le b\log n$ for all $n > n_0$

Proof (by induction):

Base case: $(n=2)$

$T(2) = c \le b$ $\boxed{b \ge c}$

Inductive Hypothesis:

Assume $T(k) \le b\log k$

Inductive Step: $(n > 2)$

$T(n) = T(\lfloor\sqrt{n}\rfloor) + d$ $\qquad k = \lfloor\sqrt{n}\rfloor < n$

$T(n) \le T(k) + d$ $\qquad \lfloor\sqrt{n}\rfloor < \frac{n}{2} < n$

$T(n) \le b\log(\lfloor\sqrt{n}\rfloor) + d \le b\log(\frac{n}{2}) + d$

$\le b\log(\frac{n}{2}) + d = b\log n - b + d \le b\log n$

$\boxed{b \ge d}$

$b \ge \max(c, d)$

By induction, $T(n) \le b\log n$ when $n > 2$.

Lemma: $T(n) \ge a\log n$ for all $n > n_0$

Proof (by induction):

Base case: $(n=2)$

$T(2) = c \ge a$ $\boxed{a \le c}$

Inductive Hypothesis:

Assume $T(k) \ge a\log k$ for $k < n$

Inductive Step: $(n > 2)$ $\qquad n \ge \lfloor\sqrt{n}\rfloor \ge \sqrt{\frac{n}{4}}$

$T(n) = T(\lfloor\sqrt{n}\rfloor) + d$

$T(n) \ge T(k) + d = a\log(\lfloor\sqrt{n}\rfloor) + d \ge a\log(\sqrt{\frac{n}{4}}) + d$

$\ge a\log((\frac{n}{4})^{1/2}) + d = \frac{1}{2}a\log(\frac{n}{4}) + d$

$\ge \frac{1}{2}a\log_2 n - a + d \ge a\log_2 n$

$a \le d$

$a \le \min(c, d)$

By induction, $T(n) \ge a\log n$ when $n > 2$.

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