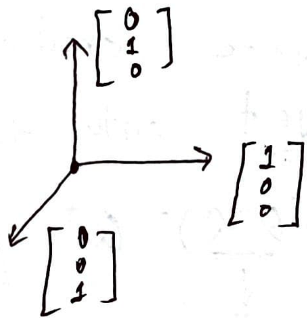
Lecture 14: Orthogonal matrices and gram-schimdt

Columns are definitely independent if they are perpendicular unit vector.



These perpendicular unit vectors are called ortho-normal vectors.

Now, Our today's topic is orthogonal vectors. Let's note orthogonal ~~vectors~~ or basis as $q_1, q_2, \dots q_n$. Orthogonal ~~vectors~~ baisise are also called Orthonormal vectors. For Orthonormal vectors -

$$q_i^T q_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Let's put these matrices into a matrix,

$$Q = \begin{bmatrix} \uparrow & & \uparrow \\ q_1 & \cdots & q_n \\ \downarrow & & \downarrow \end{bmatrix} \qquad Q^T Q = \begin{bmatrix} \leftarrow q_1^T \rightarrow \\ \vdots \\ \leftarrow q_n^T \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow \\ q_1 & \cdots & q_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Q can be called as an orthonormal matrix, isn't it?

But the convention is, we only use 'Orthogonal matrix' when the Orthonormal matrix is square.

If Q is square then, $Q^T Q = I$, that tells us $Q^T = Q^{-1}$

Example: Perm, $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow Q^T Q = I$

Some Examples of Q -

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ can also be rectangular, $Q = 1/3 \begin{bmatrix} 1 & \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$

Suppose, you have a columnspace created by a Q.
Now, you want to project onto this columnspace.

Projection matrix, $P = Q\underbrace{(Q^TQ)}_{I}^{-1}Q^T$ [Remember A?]

$= QQ^T$

$= I$ (if Q is a square matrix)

So, our previous equation was, $A^TA\hat{x} = A^Tb$

Now, $Q^TQ\hat{x} = Q^Tb$

$\hookrightarrow \hat{x} = Q^Tb$

So, $\boxed{\hat{x}_i = q_i^Tb}$

And the projection, $p = Q\hat{x} = QQ^Tb$

So, $p = \begin{bmatrix} q_1 & \cdots & q_n \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} q_1^Tb \\ \vdots \\ q_n^Tb \end{bmatrix} = q_1(q_1^Tb) + \cdots + q_n(q_n^Tb)$

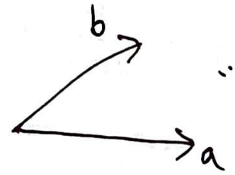
Gram Schmidt Process is a way to create orthonormal vectors.

You've 3 ~~orthonormal~~ independent vectors $\rightarrow a, b, c$.

1. We intend to construct 3 orthogonal vectors A, B, C.
2. Then by dividing A, B, C by their lengths, we get the orthonormal vec

~~Gram-Schmidt: Begin by choosing $A = a$.~~

about 2 independent vectors – a, b

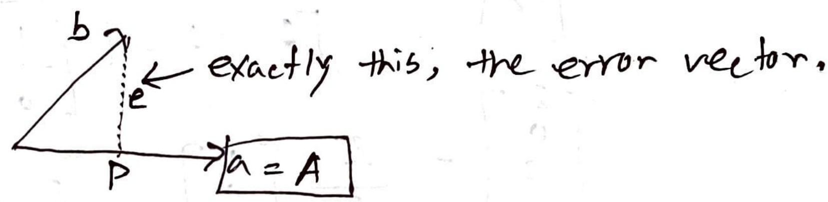


$\rightarrow$ we need A, B $\rightarrow$ we need $\frac{A}{\|A\|}, \frac{B}{\|B\|}$

Let, ~~a=A~~ $a = A$.

The problem is with b. Why? Because we need orthonormal vectors and a is not orthogonal to b.

So, we need the ~~x~~ orthogonal portion of the vector b on. And what is that?



e is definitely a part of b, cause, $b = p + e$.

So, we suppose $e = B$.

Now, $e = b - A\hat{x} = b - a\hat{x} = b - \frac{a^T b}{a^T a} a$ ~~$= b - A^T B$~~

$$\boxed{B = b - \frac{A^T b}{A^T A} A}$$

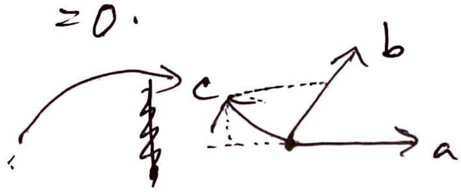
Now, they must be orthogonal, so $A^T B = 0$

$$\hookrightarrow A^T B = A^T b - A^T \frac{A^T b}{A^T A} A$$

$$= \frac{(A^T A)(A^T b) - \cancel{A(A^T A)}(A^T A)(A^T b)}{A^T A}$$

$$= 0.$$

If you've a third vector, c.



very easy!

$$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$$

Last step,

$$q_1 = \frac{A}{\|A\|}, \quad q_2 = \frac{B}{\|B\|}, \quad q_3 = \frac{C}{\|C\|}$$

<u>Example:</u> $a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$

$a = A.$

$$B = b - \frac{A^Tb}{A^TA}A = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \frac{\cancel{[2 \ 0 \ -2]}\,[1 \ -1 \ 0]\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}}{[1 \ -1 \ 0]\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

<u>Check:</u> $A^TB = 0$ as required.

Next step, $C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B = c - \frac{6}{2}A + \frac{6}{6}B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

So, $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ $\quad q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

So, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -1/\sqrt{2} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -2/\sqrt{6} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Factorization of $A = QR$, HW