~~Lecture~~ 18 → Diagonalizing a Matrix:

We saw in the previous lecture that, $Ax = \lambda x$. So, when x is an eigenvector, multiplication by A is just multiplication by a number λ. All the difficulties of matrices are swept away. So, if we want to multiply this vector x by A 100 times, we can actually follow the eigenvectors separately. It is like having a diagonal matrix with no off-diagonal interconnections. That's where the diagonalization comes.

Suppose, we have a n by n matrix A, where there are n linearly independent eigenvectors $x_1, x_2 \dots x_n$. Let's put them into the columns of an eigenvector matrix S.

$$\text{So, } S = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

What happens if you multiply with A?

$$AS = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & \\ & \vdots & & \lambda_n \end{bmatrix}$$

diagonal eigenvalue matrix (Λ)

$$= S\Lambda$$

$$\therefore AS = S\Lambda$$

$$\Rightarrow S^{-1}AS = \Lambda$$

$$\text{Or, } A = S\Lambda S^{-1}$$

[S has to be invertible, i.e. ~~the~~ all the eigenvectors must be independent]

So, what would be A^2?

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1}$$

$$= S\Lambda^2 S^{-1}$$

$$\text{So, } A^k = S\Lambda^k S^{-1}$$

So, For diagonalization, there's a condition.

"Eigenvector S has to be invertible."

When do we get the S invertible? When all the eigenvalues are different.

If any two of the eigenvalues are same, then there would be repetation in the eigenvector, and S wouldn't be invertible.

Example: Let's say, we have this matrix $A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$

As this is a triangular matrix, $\lambda_1 = 1$, $\lambda_2 = 6$

so, for $\lambda_1 = 1$, $A - \lambda I = \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and for $\lambda_2 = 6$, $A - \lambda I = \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ Powers of A,

$$\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}^k}_{\Lambda^k} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{S^{-1}} = \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}$$

Matrix Powers, A^k

You'll see a lot of use of difference equation, $u_{k+1} = Au_k$, where in each step it is multiplied by A. The final solution is, $u_k = A^k u_0$.

For example, let's say you have a vector $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. You need to multiply this vector 100 times by this matrix, $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

Remember, u^{100} and A^{100} are not the same.

So, you can do it in 3 easy steps-

① Write u_0 as a combination $c_1 x_1 + \dots + c_n x_n$ of the eigenvector

② Now, $A^{100} \cancel{u_0} = A^k u_0 = c_1 A^k x_1 + c_2 A^k x_2 + \dots + c_n A^k x_n$

$$= c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots + c_n \lambda_n^k x_n$$

$$= \Lambda^k S c \quad \left[\text{cause, } \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = Sc \right]$$

Combine all the eigenvectors $c_1(\lambda_1)^k x_1 + \cdots\cdots + c_n(\lambda_n)^k x_n$ into u_k.

~~Let's find~~ F_4.

We'll see 2 examples.

1. Fibonacci Numbers:

Let's say, we want to find the 100th fibonacci number, F_{100}.
For fibonacci number, we know, every new fibonacci ~~number~~ number is the sum of the previous two terms.

The sequence $0, 1, 1, 2, 3, 5, 8, 13 \ldots$ $\boxed{\text{Comes from}}$ $F_{k+2} = F_{k+1} + F_k$

[Fact → ~~Fibonacci~~ Plants and trees grow in a spiral pattern and all ~~b~~ follows the rule of Fibonacci. Please search youtube]

Now, let's solve this with eigenvalues and eigenvectors.
Let's make a vector with the terms of Fibonacci –

$$\cancel{u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}$$

If you have two term (x, y) ($x \to F_{k+1}$, $y \to F_k$) and a matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ →

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x \end{bmatrix} \quad \Bigg| \quad \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_k + F_{k-1} \\ F_k + 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = A\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

Similarly, If you have two term of Fibonacci series, namely F_k and F_{k-1} →

$$A\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} F_k + F_{k-1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = u_{k+1}$$

$$u_k \to \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = A\left(A\begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix}\right) = A^2\begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} = A^k\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = A^k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now, $u_{k+1} = A u_k$

Every step multiplies by $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. After 100 steps, we reach $u_{100} = A^{100} u_0$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ldots\ldots u_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

This problem is just right for eigenvalues! Subtract λ from the diagonal of A:

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$ leads to $\det(A - \lambda I) = \lambda^2 - \lambda - 1$

solving with the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

so, $\lambda_1 = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ $\qquad$ $\lambda_2 = \dfrac{1-\sqrt{5}}{2} \approx -0.618$

Let's follow the steps-

① Express u_0 as $c_1 x_1 + c_2 x_2$

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \approx \begin{bmatrix} -0.618 & 1 \\ 1 & -1.618 \end{bmatrix} \qquad \begin{bmatrix} 1.618 & 1 \\ 1 & 0.618 \end{bmatrix}$$

$\lambda_1 = 1.618$ $\qquad$ $\lambda_2 = -0.618$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $\qquad$ $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$$\cancel{\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1+\lambda_2}\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + \frac{1}{\lambda_1+\lambda_2}\begin{bmatrix} \lambda_2 \\ -1 \end{bmatrix}} \quad \frac{1}{\lambda_1+\lambda_2}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ -1 \end{bmatrix}\right)$$

$$u_0 = c_1 x_1 + c_2 x_2 = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} =$$

Let's find out the eigenvectors -

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \approx \begin{bmatrix} -0.618 & 1 \\ 1 & -1.618 \end{bmatrix} \qquad \begin{bmatrix} 1.618 & 1 \\ 1 & 0.618 \end{bmatrix}$$

$\lambda_1 = 1.618$ $\qquad$ $\lambda_2 = -0.618$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $\qquad$ $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} = \cancel{\frac{1}{\lambda_1 \lambda_2}}$$

$$\therefore \Lambda = S^{-1} A S = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$$

$$\Lambda = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} \lambda_1+1 & \lambda_2+1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$$

$$= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1+1-\lambda_2\lambda_1 & \lambda_2+1-\lambda_2^2 \\ \lambda_1^2-\lambda_1-1 & \lambda_1\lambda_2-\lambda_2-1 \end{bmatrix}$$

But, do you remember that $\lambda^2-\lambda-1=0$?

so, $\lambda_1^2-\lambda_1-1=0$

and $\lambda_2^2-\lambda_2-1=0$ or $\lambda_2+1-\lambda_2^2=0$

$$= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1+1-\lambda_1\lambda_2 & 0 \\ 0 & -(\lambda_2+1-\lambda_1\lambda_2) \end{bmatrix}$$

Now $\lambda_1\lambda_2 = \dfrac{(1+\sqrt{5})(1-\sqrt{5})}{2\cdot 2} = \dfrac{(1)^2-(\sqrt{5})^2}{4} = \dfrac{1-5}{4} = -1$

$$= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1+2 & 0 \\ 0 & -(\lambda_2+2) \end{bmatrix}$$

$\lambda_1+2 = \dfrac{1+\sqrt{5}}{2}+2 = \dfrac{5+\sqrt{5}}{2}$

$\lambda_2+2 = \dfrac{1-\sqrt{5}}{2}+2 = \dfrac{5-\sqrt{5}}{2}$

$\dfrac{1}{\lambda_1-\lambda_2} = \dfrac{1}{\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}} = \dfrac{2}{1+\sqrt{5}-1+\sqrt{5}}$

$= \dfrac{1}{\sqrt{5}}$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-5}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Now,

$$A^k = (S\Lambda S^{-1})^k = S\Lambda \cancel{S^{-1}} \cdot \cancel{S}\Lambda S^{-1} \cdots\cdots S\Lambda S^{-1} = S\Lambda^k S^{-1}$$

So,

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = A^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = S \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k \end{bmatrix}$$

$$\begin{pmatrix} \lambda_1^k & -\lambda_2 \lambda_1^k \\ -\lambda_2^k & \lambda_1 \lambda_2^k \end{pmatrix}$$

$$\therefore \quad F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$

$$\therefore F_{100} = \frac{(1+\sqrt{5})^{100} - (1-\sqrt{5})^{100}}{\cancel{2^{100}}\ 2^{100}(\sqrt{5})}$$

<u>Another Example:</u> Find F_k where $F_{k+2} = F_{k+1} + 3F_k$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

Compute ~~A~~ $A^{50} u_0$.

$\lambda_1 = 2$, $\quad$ $\lambda_2 = -1$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$S = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$S^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

$$\therefore \Lambda = \cancel{SA}\ S^{-1}AS = \tfrac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \tfrac{1}{3} \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore A^{50} = S\Lambda^{50} S^{-1} = \tfrac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^{50} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^{50} & 2^{50} \\ 1 & -2 \end{bmatrix}$$