Lecture 4

Permutation matrix. → exchange rows.

Suppose you have a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. what matrix makes it $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$?

→ ~~Do the thing~~ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

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→ Easey way → Do the thing to the identity matrix.

~~The~~ Think in case of a 3x3 matrix.

what if I want to exchange columns of a matrix?

$\boxed{} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$? what matrix does this?

↑

Can I put something here? NO!

Matrices are multiplied at the left to do row operation

To do a column multiplication, we have to put ~~a~~ ~~to~~ a matrix on right.

So, ~~to~~ exchange columns, exchange columns of the identity matrix.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

Inverses:

What was our ~~first~~ E_{21}?

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now I wanna get a matrix that undoes this operation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Inverses (square Matrices):

Not All matrices are invertible. If it is invertible then there is some matrix A^{-1} that $A^{-1}A = I$
This is also true, $AA^{-1} = I$

If a matrix has an inverse, its called non-singular, invertible matrix.
Let's know about singular or non-invertible matrices.

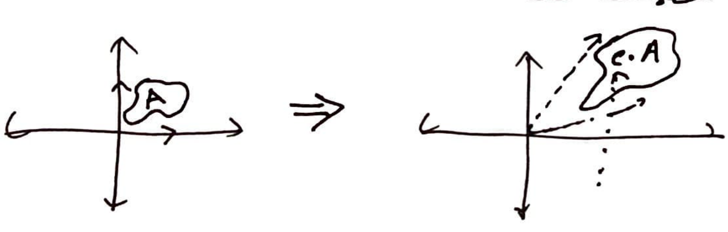
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

why isn't this matrix invertible?

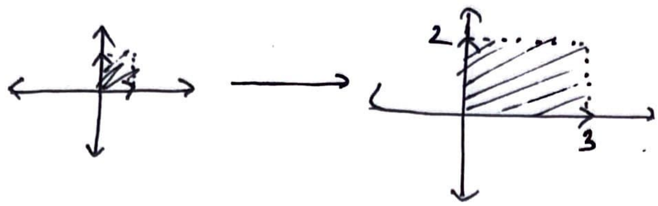
1. Determinant is 0.

Now we need to know about determinants!
Recall on the concept of Linear Transformation.
If you think about the idea of linear transformation, you already know by this point that, a matrix does a linear transformation, and it basically stretches and squizzes the space.

So, one thing thats pretty useful to understand this transformation is how much it stretches/squizzes the space
More specificly, Finding the factor by which the previous area is increased or decreased.

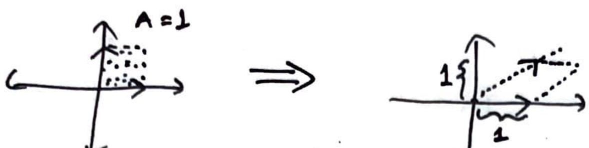


For example, think about this matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$



Scaled/Increased by a factor of 6.

Think about, shearing. A shear matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



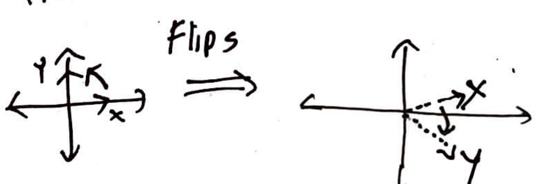
Area = 1

Now, this special scaling factor, the factor by which a linear transformation changes any area, is called ~~a~~ the determinant of that transformation.

Question: If the determinant of this transformation $\begin{bmatrix} 0 & 2 \\ -1.5 & 1 \end{bmatrix}$ is 3, what do you understand from it? what about 0.5?

If the determinant of any transformation $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is 0, it means it squizzes all of space on to a line.

Determinant also allows (-ve) values. This has to do with the idea of orientation.



$$\boxed{\det(M_1 M_2) = \det(M_1)\det(M_2)}$$

Why?

For example, Flipping a paper on the other side.

what about 3D Transformation?

→ In case of 3D, Determinant tells you how much volume gets scaled.

→ what about zero and (-ve) determinant in 3D space?

right hand rule

Now, let's go back to our previous matrix -

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

please interpret this.

~~so, I can't find~~

so, for inverses, whatever linear transformation a matrix A does, A^{-1} does its opposite and take the space to its previous form.

A matrix with ~~zero~~ 0 determinant squizzes the space into a line, so inverse can't take it back to its previous shape, because you can't make anything out of 0. That's why they can't have inverses.

Now take a different matrix, $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

it's inverse form should be, $AA^{-1} = I$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the concept of linear combination of matrix-vector multiplication,

$$A \times \text{column } j \text{ of } A^{-1} = \text{column } j \text{ of } I.$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution? Gauss-Jordan elimination → solve 2 eqns at once -

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \begin{matrix} \vdots \\ \vdots \end{matrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \xrightarrow{R_2 - R_1 \times 2} \begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix} \xrightarrow{\text{Upward elimination}} \begin{bmatrix} 1 & 0 & \vdots & 7 & -3 \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix}$$

$I \qquad A^{-1}$

Use the pivots to do upward elimination

check if A^{-1} is actual inverse

The essence here is,

$$E\,[A \quad I] = [I \quad ?] \Rightarrow [EA \quad EI] = [I \quad ?]$$

$$EA = I \qquad\qquad EI = A^{-1}I$$

$$\hookrightarrow E = A^{-1} \qquad\qquad = A^{-1}$$

$EA = I$ tells us $E = A^{-1}$

So, $\cancel{E^{-1}I =}$ $EI = A^{-1}$