Lecture 6 :

Permutations – Do row exchanges, identity matrices with reordered rows

What are all the 3x3 matrices that exchange rows?

(i) $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

↑ Does No row exchange; P_{12}; P_{23}

(iv) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

P_{13}; $P_{21}P_{32}$; $P_{32}P_{21}$

So, for a $n \times n$ matrix, there would be $n!$ permutations!

Now, question is, what is the inverse of P_{12}? $\Rightarrow P_{21}$. In fact the same thing!

So, the thing is $P^{-1} = P^T$

So, How many 4x4 P's? $\Rightarrow 24$.

The reality is, If we tell the ~~math~~ matlab to solve a linear system, it looks for very small pivots. If the pivots are really really small, that is numerically bad. so, matlab does some row exchanges for that. so $A = LU$ becomes $PA = LU$.

we can write, $P^TP = I$

Transposes:

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$

General formula for Transposes –

~~$(A^T)_J$~~ $(A^T)_{ij} = A_{ji}$

while working with Transposes, a lot of symmetric matrices show up. What are symmetric matrices?

~~$A = A^T$~~ $A = A^T$ Example?

$$\begin{bmatrix} 3 & 4 & 9 \\ 4 & 2 & 7 \\ 9 & 7 & 4 \end{bmatrix}$$

In our previous example of matrix A, How do we get a symmetric matrix? Let's say,

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} \quad R^T = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix},$$ ~~so R^TR is sys~~

R^TR is symmetric, always. Why? Because -

$$\cancel{R^T} \quad (R^TR)^T = R^T (R^T)^T = R^TR$$

That ~~cd~~ concludes our 2nd chapter.

3rd chapter - In this chapter we would see the highest level of Linear Algebra, instead of individual columns, we look at ~~'sap~~ 'spaces' of vectors.

$\boxed{\text{Vector spaces}}$

What does the word 'spaces' mean? It means a bunch of vectors or a space of vectors.

For example, R^2 is vector space.

R^2 = all 2 dim. real vectors. such as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}$

= the x-y plane

If I take one point, let's say $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ away, this wouldn't be a vector space anymore.

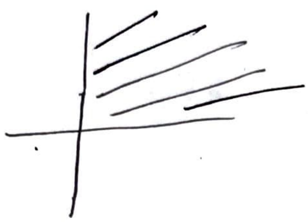
R^3 = all 3 dim. real vectors. such as $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

R^n = all column vectors with n components

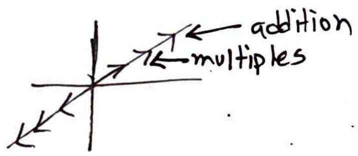
The characteristic of a vector space should be, if you multiply any vector with a scalar or if you add two vectors, you should be inside the vector space, means the result stays in the space.

To create a real vector-space, you should be able to produce vectors by addition and multiplication (which follows certain rules like commutative $v+w=w+v$, distributive $c(v+w)=cv+cw$) by real numbers.

not a vector space example?

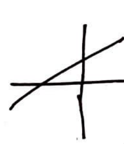


so tell me a vector space that is inside R^2 and closed by multiplication and addition? well, this particular vector space is called a sub-space.



$\boxed{\text{so a subspace could be a line through the Origin.}}$

Why not this line a vector-space?



** Because [0,0,0] has to be in the sub-vector space. **

so, what are all the possible subspaces of R^2?
(i) all of R^2 (ii) a line through the origin (iii) zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so, a subspace of a vector space is a set of vectors that satisfies two requirements: If v and w are vectors in the subspace and c is any scalar, then

(i) v+w is in the subspace
(ii) cv is in the subspace

All possible subspaces of R^2 →
1. all of R^2
2. any line through [
3. Zero vector

Now, what are all the possible subspaces of R^3?

What about R^3?

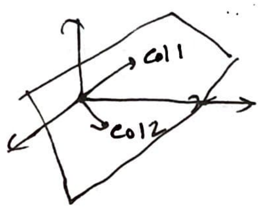
Let's create subspaces out of matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$ These are columns in R^3

Now I just can't put two column in a subspace and call it a subspace: what else do we have to have to call it a subspace?

I must be able to add those things. so, the sum of those columns. And, multiplications.

If we summarize, we've to be able to take all the linear combinations of these 2 columns. And these linear combinations form their subspace. And, this subspace is called, the columnspace of matrix A : C(A). The column-space of A is basically a plane, through the origin.



* Describe the column spaces of the following matrices

(i) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

↓

Every vector is a combination of the column of I

↓

R^2

(ii) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

↓

a line through the origin

(iii) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

↓

Every b is attainable

↓

R^2

Let's solve a problem-

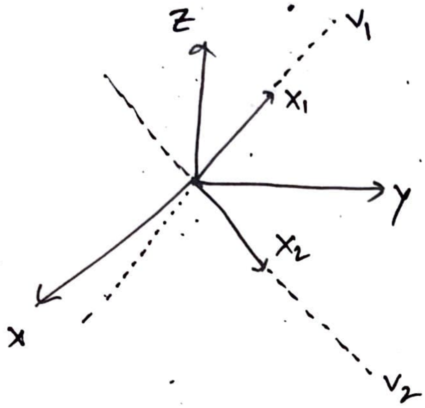
$x_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

1. Find V_1 = subspace generated by x_1, V_2 = subspace generated by X_2
   Describe $V_1 \cap V_2$

2. Find V_3 = Subspace generated by $\{x_1, x_2\}$. Is V_3 equal to $V_1 \cup V_2$?
   Find a subspace ~~≠~~ S of V_3 so that $x_1 \notin S$, $x_2 \notin S$

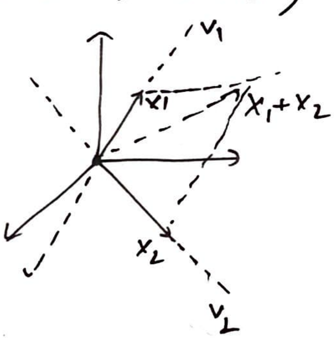
3. What is $V_3 \cap \{xy \text{ plane}\}$?

1.



$V_1 \cap V_2 = \{0\}$ ← a subspace

2. $V_1 \cup V_2$ is clearly the union of two lines. Is it a subspace?



$x_1 + x_2 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$, which is not in the $X_1 \cup X_2$ so its not a subspace.

So, V_3 must be a plane generated by all the linear combinations of x_1 & x_2.

S can be $x_1 + x_2$, not unique though.

3. A straight line whose z co-ordinate is 0.
   So, $V_3 \cap \{XY \text{ plane}\} =$ ~~The line~~ V_2