Lecture - 1 → The ~~geometry~~ essence of Linear Algebra

* Why linear algebra?

→ Linear algebra is simple - solving linear systems.

* What is a vector?

→ The rudiment of Linear Algebra.

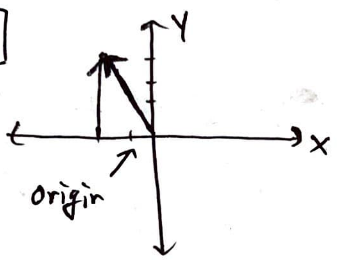
When the word 'vector' is said, I want you to think about an arrow in a co-ordinate system that starts from the origin. For 2D,

$$\begin{cases} 2x + 3y = 5 \\ 3x - 5y = 7 \end{cases}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

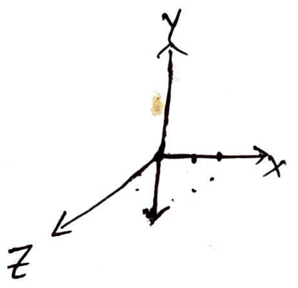
$\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Every pair of numbers give you only one vector.



The co-ordinates of a vector basically tells you how to get to the tip of the vector from its tail at the origin.

For 3D, you add another axis called Z - axis.

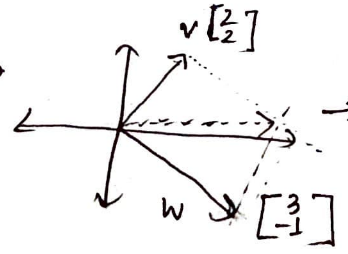


$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

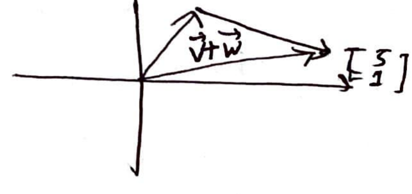
Every topic in linear algebra is gonna centered around two operation.

1. Vector Addition
2. " Multiplication by numbers

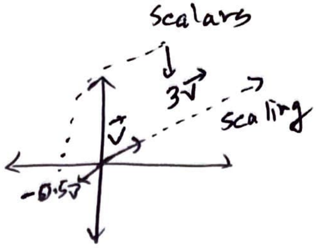
$\vec{v} + \vec{w}$ →



→



Multiplication →



$\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$2\vec{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Now think about

$3\hat{i} + (-2)\hat{j}$

addition

Scalars

$\hat{i}$ and $\hat{j}$ are basis vectors

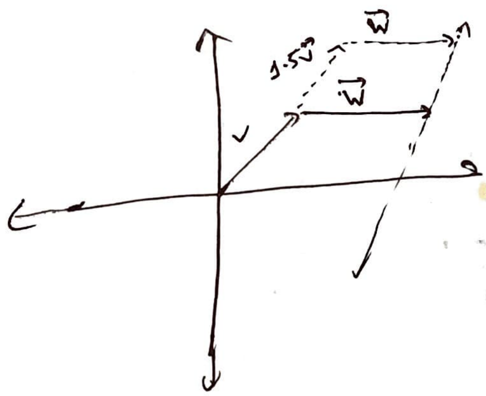
Why thinking geometrically? It gives a clear understanding of what operations we are doing.

→ data to information!

Going back to the basis vectors. Anytime that ~~we~~ you're scaling two vectors and adding them, its called the linear combination of vectors.

$a\vec{v} + b\vec{w}$ → $\cancel{[2x + 3y = 5]}$

Why does it called linear? If you fix one of the vectors and let the other vector move freely, the tip of the resulting vector draws a straight line.



Every possible point on the plane!

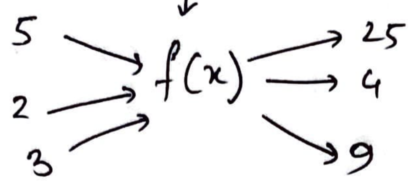
What if you let both the vectors move freely? You get a plane! Another possibility is there? The origin!

The set of all the possible vectors that you can reach with all the linear combination of vectors is called the span of those vectors.

What happens in 3D space? First think with 2 vectors.

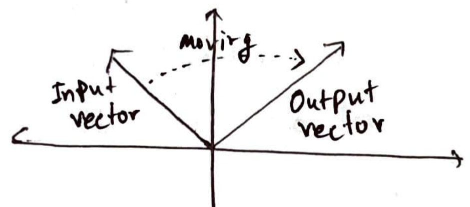
Homework → First 2 videos of 3blue1brown.

The Idea of linear $\underbrace{\text{transformation}}_{\text{a fancy word for function}}$ & matrices!



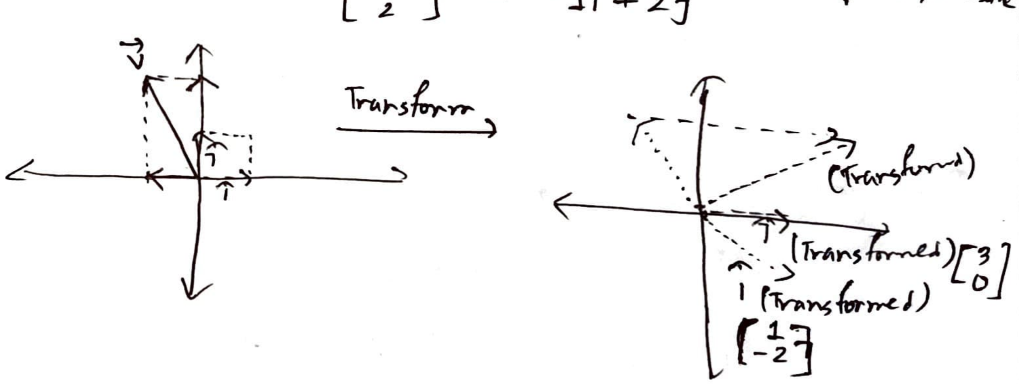
In case of Linear Algebra, $\begin{bmatrix}5\\7\end{bmatrix} \rightarrow \overset{\text{Transformation}}{L(\vec{v})} \rightarrow \begin{bmatrix}2\\-3\end{bmatrix}$

So, why using transformation instead of function? to give you an abstract idea of 'movement';



2 main conditions of Lin. Transformation → ① All lines must remain lines ② The origin stays the same

For example, Consider the vector $\vec{v} = \begin{bmatrix}-1\\2\end{bmatrix} \rightarrow -1\hat{i} + 2\hat{j}$



$$\text{Transformed } \vec{v} = -1(\text{Transformed } \hat{i}) + 2(\text{Transformed } \hat{j})$$
$$= -1\begin{bmatrix}1\\-2\end{bmatrix} + 2\begin{bmatrix}3\\0\end{bmatrix}$$
$$= \begin{bmatrix}5\\2\end{bmatrix}$$

For more general formula, $\begin{bmatrix}x\\y\end{bmatrix} \rightarrow x\begin{bmatrix}1\\-2\end{bmatrix} + y\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}1x+3y\\-2x+0y\end{bmatrix}$

In this way, I give you any vector and you can tell where the transformed vector land.

What I'm trying to say is, in 2D linear transformation, the whole transformation can be described by just 4 numbers –

The 2 co-ordinates where $\hat{i}$ lands

The 2 co-ordinates where $\hat{j}$ lands

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \quad \text{"2×2 Matrix"}$$

where $\hat{i}$ lands

where $\hat{j}$ lands

If you're given a 2x2 matrix describing a linear transformation and a specific vector and you wanna know where that linear transformation takes that vector, take the co-ordinates of the vector, multiply them with the columns of the matrix, then add together

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ -3 \end{bmatrix}$$

Transformation

vector $\vec{v}$

Transformed $\vec{v}$

For general case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} \rightarrow \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

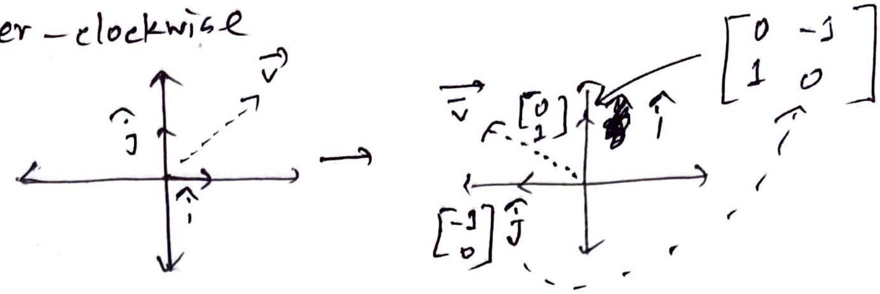
Where first basis vector lands?

Any vector

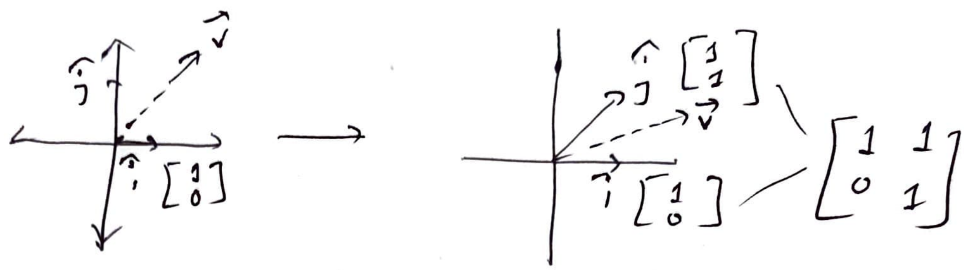
Matrix - vector multiplication !!

Let's practice some fun linear transformation!

1. 90° rotation counter-clockwise



2. Shear



So, Matrices are language to describe transformations! Everytime you see matrix, you can interpret it as certain ~~information~~ ~~in~~ transformation of space!

Now, some questions -

(i) What is the picture of all combinations $c\vec{u}$? → a line through (0,0,0)

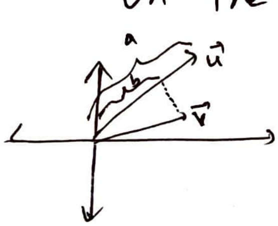
(ii) What is " " " " " $c\vec{u} + d\vec{v}$? → a plane " "

(iii) What " " " " " " $c\vec{u} + d\vec{v} + e\vec{w}$? [c, d, e are scalar]
↳ a three D space thru origin

Dot products & Duality:

What information do we get from dot products?

Geometrically, dot product of two vectors are basically →
length of one vector * length of the projection of the second vector on the first vector



$\vec{u} \cdot \vec{v} = a * b$

So, (1) $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. $v \cdot w = 1 \cdot 4 + 2 \cdot 5 = 14$, order makes no difference.

(2) If they are perpendicular, the dot product is zero. why?

(3) The length of $v = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ is $\|v\| = \sqrt{1^2+2^2+5^2}$

(4) The angle θ between v and w is $\cos\theta = \dfrac{v \cdot w}{\|v\| \, \|w\|}$

(5)

we get

$\|v\| \cdot \|w\| \cos\theta = \vec{v} \cdot \vec{w}$

$\Rightarrow \|v\| \cdot \|w\| \geq |\vec{v} \cdot \vec{w}|$, As $\cos\theta$ is never greater than 1.

↓

Schwarz Inequality

and Triange InEquality $\|v+w\| \leq \|v\| + \|w\|$

Question: Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and check both inequalities.

(*) Math exam question in china →

If a ship has 26 sheep and 10 goats onboard, how old is the ship's captain?