Lecture 2 & 3 → The geometry of Linear Algebra

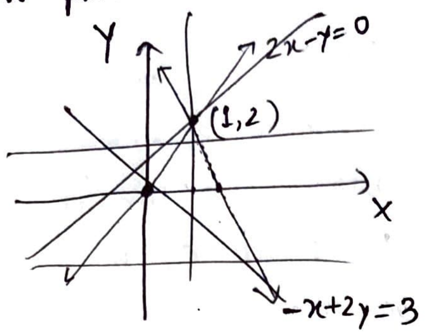
[2+3]

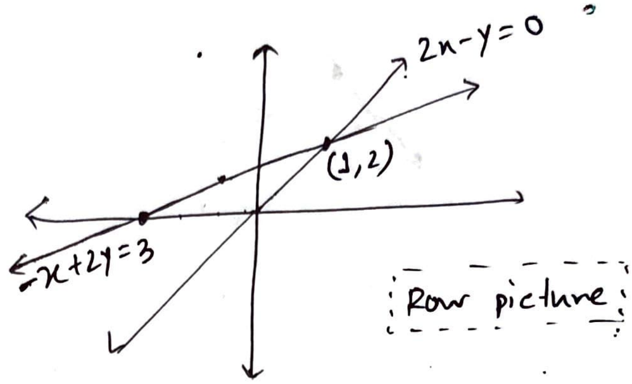
Take a linear system of 2 equations -

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \quad x = b$$

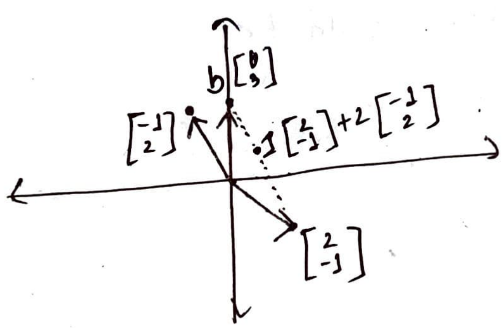
Row picture →





Column picture →

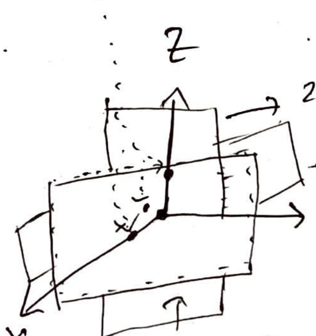
$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



Let's switch to 3 equations and 3 unknowns -

$$\begin{aligned} 2x - y \quad &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned} \Rightarrow A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row picture →



→ $2x - y = 0$

← $-3y+4z=4$

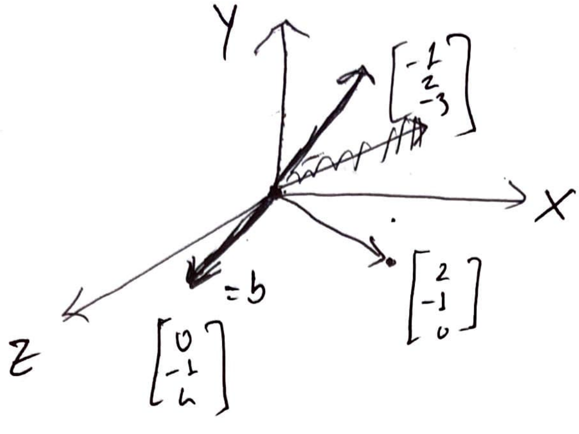
→ Two planes meet on a straight line

→ Three planes meet on a point.

The solution of $-x+2y-z=-1$ fills a plane.

Column picture -

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$



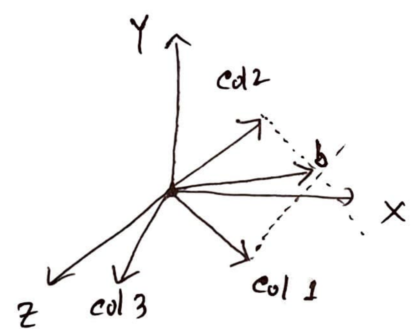
what this equation is asking us to do is to combine these 3 vectors, with a right combination (x, y, z) to produce 'b'.

$x = 0,\ y = 0,\ z = 1$

~~The~~ Imagine we have a different right hand side.

$b = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, For this, the solution would be, $x = 1, y = 1, z = 0$

so the column picture in this case would be -



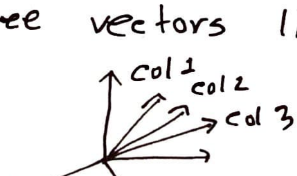
Now, the big question, can I solve $Ax = b$ for every b?

In linear algebra, the question should be - do the linear combi. of the columns fill the 3-D space?

For this matrix A, the answer is YES!!

For certain matrices, the answer could be NO! when it could go wrong → means when I can not generate a random b?

If all the three vectors lie on the same plane.



can we get this b?

~~3~~ similarly, if column 3 is a combination of column 1 and column 2, we don't get anything new from their combination. so, they can't generate any random vector b. so, that would be a singular case, the matrix would be invertible.

Think about 9 dimensions. Imagine that we have a vector of 9 components, 9 equations and 9 unknowns. If we have a vector with 9 columns, can we hit all the vectors b? some times yes, some times no. If the 8th column is same as the 9th, means dependent, then we'll get a ~~8-1~~ 8-D plane on a 9 Dimensional plane.

So, our familiar equation, $Ax = b$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ = How do you do this matrix-vector multiplication?

1. Normal Row-operation

$2 \cdot 1 + 5 \cdot 2$

$1 \cdot 1 + 3 \cdot 2$

2. Column operation → Linear combination of the columns.

$$1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

so, Ax is a combination of columns of A.

<u>Question:</u> Solve $\begin{cases} 2x+y=3 \\ x-2y=-1 \end{cases}$ and find out its "row picture" and "column picture".

<u>Extra:</u> the matrix form would be, $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$ax = b$, $x = \frac{b}{a}$, $x = a^{-1} b$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Elimination with matrices: * Also known as Gaussian Elimination

→ The method of solution would be elimination, the way every software package solves linear systems.

→ For good matrices, elimination works well. But it fails sometimes, when?

The idea of elimination can be summarized by 4 points –

(i) Elimination goes from A to a triangular U by a sequence of elimination steps that generates a matrix E.

(ii) The triangular system is solved by back substitution working bottom to top.

(iii) In matrix language A is factored into LU = (lower triangular) (upper triangular)

(iv) Elimination succeds if A is invertible. (But it may need row exchange).

Lets take an example –

$$2x+4y-2z = 2$$
$$4x+9y-3z = 8$$
$$-2x-3y+7z = 10$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

The goal of elimination is to produce an upper triangular system. How can we do it?

$$\begin{aligned} aw + bx + cy + dz &= k_1 \\ b'x + c'y + d'z &= k_2 \\ c'y + dz &= k_3 \\ d'z &= k_4 \end{aligned}$$

Pivot = First non-zero in the row that does the elimination

Pivots can't be zero. [Explain pivots later]

$$\begin{matrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & +7 \end{matrix} \rightarrow \begin{matrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{matrix}$$

→ step 1 [$R_2 - R_1 \times 2$]

→ step 2 [$R_2 - R_1 \times (-1)$]

$\downarrow R_2 - R_1 \times 2$

$$\begin{matrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{matrix} \xrightarrow{R_3 - R_1 \times (-1)} \begin{matrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{matrix}$$

$\downarrow R_3 - R_2$

$$\begin{matrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{matrix}$$

$\boxed{U}$

→ Now what does failure mean? → Getting pivots with zero.

If the first pivot was zero, did we have to give up?

No! Just exahange the rows.

If we had a 3 at (3,3) position, what would happen?
The elimination process would fail.

→ Backward substitution

Bring the right hand side.

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}}_{\text{Augmented matrix}} \xrightarrow{R_2 - R_1 \times 2} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$\Big\downarrow \; R_3 - R_1 \times (-1)$$

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

$$\Big\downarrow \; R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & \cancel{11}\,8 \end{array}\right]$$

$$\underbrace{}_{U} \qquad \underbrace{}_{c}$$

So, $Ax=b$ becomes $Ux=c$

Final equation –

$$2x + 4y - 2z = 2 \qquad \rightarrow x = -1$$
$$y + z = 4 \qquad \rightarrow y = 2$$
$$4z = \cancel{11}\,8 \qquad \rightarrow z = 2$$

Backward substitution is basically simple steps solving the equations in reverse order.

* Eliminating with matrices

Let's take a tour back to our previous class.
What was the output of a matrix-vector multiplication?

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\times col\,1 + 4\times col\,2 + 5\times col\,3$$ = combination of the columns of the matrix

Output → matrix × column = column

What if we multiply a matrix with a row?

$$\underset{1\times 3}{\begin{bmatrix}1 & 2 & 5\end{bmatrix}}\ \underset{3\times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} = \begin{array}{l} 1\times row\,1 \\ +\,2\times row\,2 \\ +\,5\times row\,3 \end{array}$$

Output → row × matrix = row

So, what's the matrix that does the first step $(R_2 - R_1\times 2)$ in the previous example?

$$\underset{E_{21}}{\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}\begin{bmatrix}2 & 4 & -2\\ 4 & 9 & -3\\ -2 & -3 & 7\end{bmatrix} = \begin{bmatrix}2 & 4 & -2\\ 0 & 1 & 1\\ -2 & -3 & 7\end{bmatrix}$$ Step 1

$$\underset{E_{31}}{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}}\begin{bmatrix}2 & 4 & -2\\ 0 & 1 & 1\\ -2 & -3 & 7\end{bmatrix} = \begin{bmatrix}2 & 4 & -2\\ 0 & 1 & 1\\ 0 & 1 & 5\end{bmatrix}$$ Step 2

$$\underset{E_{32}}{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}}\begin{bmatrix}2 & 4 & -2\\ 0 & 1 & 1\\ 0 & 1 & 5\end{bmatrix} = \begin{bmatrix}2 & 4 & -2\\ 0 & 1 & 1\\ 0 & 0 & 4\end{bmatrix}$$

NOW, put all the matrices together to get one matrix that does all the work!

So, ~~Step 1~~ →

$$\left(E_{32}\left(E_{31}\left(E_{21}A\right)\right)\right) = U \Rightarrow \left(E_{32}E_{31}E_{21}\right)A = U$$

step 3 step 2 step 1

Elementary/Elimination matrix

Associativity Law (But we can't change the matrix order)

Permutation and Inverse → Next class In sha ALLAH.