Lecture 7: The Nullspace of A: Solving $Ax=0$

Recap: vector space Requirements –

1. $v+w$ and cv are in the space,

all combination of $cv+dw$ are in the space, for v and w any vectors in the space

And a subspace was a space inside a vectorspace.

what are all the subspaces of R^3?

suppose you've 2 subspaces – P and L.

P: a plane in R^3 through origin

L: a line in R^3 through the origin

Is $\boxed{P \cup L}$ a subspace?

Is $\boxed{P \cap L}$ a subspace? → Intersection is in fact $\begin{bmatrix}0\\0\\0\end{bmatrix}$

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column space of A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Column space of A is a subspace of R^4.
↳ all linear combinations of the columns.

For this A, Does $Ax=b$ have a solution for every b?

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

for which b's I can solve x?

~~give~~ ** show examples like $b = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\5\\7\\9\end{bmatrix}$ etc and reach the conclusion. **

I can solve $Ax=b$ exactly when my b is in the columnspace of A, $C(A)$.

And, that's why we're interested this much in columnspace.

Another question, Does all the columns contribute something new to $C(A)$? can we throw ~~away~~ away column 3?

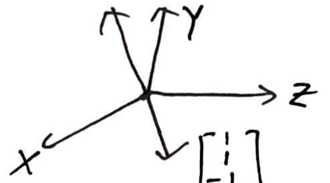
The Nullspace of A:

The nullspace of a matrix A is all solutions X for which $Ax = 0$

For previous example, $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In this case, the nullspace is a subspace of R^3.

Let's solve this guy. Tell me one solution without looking at the equation. It would be $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
Another solution can be $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Another one? $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$?
In fact the solution is all the vectors of form $c \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$
Do I have a subspace out of this?



YES! A line in R^3!

Now why we are calling it a space? Check if the solutions to $Ax = 0$ always gives a subspace.
If we say it in reversed manner, if $Av = 0$ and $Aw = 0$ then obviously $A(v+w) = 0$. Also $A(cv) = 0$ or $A(cw) = 0$

Okay, for $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ in the previous example, does it create a subspace? In fact in this case one of the solutions is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Another solution is $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Their addition and multiplication goes out of the claimed vectorspace.
Still not convinced? Is zero vector included in this vector space? No. So it's not a vector space.

Now, How do we find out the nullspace?

Let's take a matrix for this purpose.

$$A = \begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix} \xrightarrow[R_3=R_3-R_1\times 3]{R_2=R_2-R_1\times 2} \begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix} \xrightarrow{R_3=R_3-R_2} \begin{bmatrix}1&2&2&4\\0&0&2&4\\0&0&0&0\end{bmatrix}$$

Echelon Form (Staircase)

Rank = # of pivots = 2. Now for this matrix, how many pivot columns or free columns are there?

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}$$

pivot columns

free columns

Why do we call it free columns? We can assign any value to those unknowns freely. For example:

$$X = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$$

$$x_1+2x_2+2x_3+2x_4=0$$
$$2x_3+4x_4=0$$

-2 $\quad 1 \quad 0 \quad 0$

So, X could be $X = c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$. But is it the only solution?

No! We could use $x_2=0$ and $x_4=1$ and $X = d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$

So, the nullspace would be

$$c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

Two special solutions

So, the nullspace contains exactly the linear combination of all the special solutions. So, How many special solutions can be there? There ~~are~~ is one for every free variable!

So, how many free variables? free variables $= n - r =$ column $-$ rank
$= 4 - 2 = 2$

I'd like to take one more step. We got our echelon form. I want to clean up that ~~x~~ matrix even more.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

I want to get R = Reduced row echelon form

I've one question. Why there're ~~no pivots~~ zeros on ~~column 2 and 4~~ row 3 ?
Because ~~Those are dependent~~ row 3 is a combination of row 1 and 2, and elimination discovered that fact.

Now, To get row echelon form → get zeros above and below the pivots

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Get one more step → make the pivots 1.

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

We can immediately get R in matlab by this command rref(A)

So, $Rx = 0$

If you look closely to R -

$$\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 2 & -2 \\ 0 & 2 \\ 0 & 0 \end{matrix}$$

Pivot columns   free columns

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$