Lecture 16: Finding out the determinants:

Normally, computer finds the determinants from the pivots.
There are 3 methods in total → pivots, big formula, cofactors.

The pivot formula:

You perform elimination on the matrix. After the elimination

$$\det(A) = (\det L)(\det U) = (1)(d_1 d_2 d_3 \cdots d_n)$$

If any row-exchanges happened -

$$\det(A) = \pm (d_1 d_2 \cdots d_n)$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 6 & -4 & 4 \end{vmatrix} \Rightarrow - \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 6 & -4 & 4 \end{vmatrix} \Rightarrow - \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -6 & 4 \end{vmatrix}$$

$$\Rightarrow - \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 7 \end{vmatrix}$$

$$\det(A) = -(1)(3)(2)(7) = -42.$$

$$B = \begin{bmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2 + \cdots + R_5} \cancel{x+4y} \quad x+4y \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix}$$

$$\Rightarrow (x+4y) \begin{bmatrix} 1 & y & y & y & y \\ 1 & x & y & y & y \\ 1 & y & x & y & y \\ 1 & y & y & x & y \\ 1 & y & y & y & x \end{bmatrix} \Rightarrow (x+4y) \begin{bmatrix} 1 & y & y & y & y \\ 0 & x-y & 0 & 0 & 0 \\ 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & x-y \end{bmatrix} = (x+4y)(x-y)^4$$

The big Formula:

In computation with the pivots, we changed the entries of the matrix.

The big formula derives a single explicit formula directly from the entries a_{ij}. This formula has $n!$ terms.

Let's take a 2 by 2 example -

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \underset{\text{Applying Linearity}}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = ad - bc$$

Thus, 3×3 matrix has $3!$ terms, ... 11×11 has 40 million terms

Determinant by cofactors:

I could show it using the big formula. But I want to show this method separately. So you take every element from the first row, and you ignore the respective row and column. The lower rows are defined as cofactors.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{\begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix}} \\ a_{31} & \end{vmatrix}}_{C_{11}} + \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{C_{12}} + \underbrace{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{C_{13}}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots\cdots + a_{1n}C_{1n}$$

The sign of the co-factor is a major issue -

The co-factors along row 1 are, $C_{1j} = (-1)^{1+j} \det M_{1j}$

This is called minor of the element $1, j$

Example:

$$A = \begin{bmatrix} \cancel{x} & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & x \end{bmatrix} = \cancel{x \cdot x^4 + y \cdot x} \begin{vmatrix} 0 & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ y & 0 & 0 & x \end{vmatrix}$$

A^T

$$\begin{bmatrix} x & 0 & 0 & 0 & y \\ y & x & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 \\ 0 & 0 & y & x & 0 \\ 0 & 0 & 0 & y & x \end{bmatrix} = x \cdot x^4 + y \cdot y^4 = x^5 + y^5$$

Cramer's Rule, Inverse Matrix, Volume:

This lesson is the application of determinants. The previous lectures were for discussing the properties and formula of the determinant.

Now, I want to start this segment with a proof -

False Expansion Theorem -

If A is an $n \times n$ matrix and $i \neq k$ then,

$$a_{i1}C_{k1} + \dots + a_{in}C_{kn} = 0$$

Proof: From the previous part we know -

$$a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} = \det(A)$$

$$a_{21}C_{21} + \dots + a_{2n}C_{2n} = \det(A)$$

Generalising, $a_{i1}C_{k1} + \dots + a_{in}C_{kn} = \det(A)$ [when $i = k$]

Now, Let B be the matrix obtained from A by replacing the k-th row of A with the ~~it~~ i-th row of A.

So, B is identical to A except that the k-th row of B is same as the i-th row of A.

So, two rows of B are same. So, according to the properties of determinants, det B = 0.

Now, if we were to compute the det B by expanding along its k-th row

$$b_{k1}C^*_{k1} + \dots + b_{kn}C^*_{kn} = \det B = 0$$

C^*_{ij} is the co-factor of b_{ij} in B.

But k-th row of B = i-th row of A.

So,

$$a_{i1}C^*_{k1} + \dots + a_{in}C^*_{kn} = 0$$

The only thing that A and B ~~B~~ differ is in the k-th row. But that doesn't change the co-factors of C^*_{k1}, so we can replace –

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \dots + a_{in}C_{kn} = 0$$

[Proved]

Let's see a short example –

Suppose you have → $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ [2nd row of ~~B~~ = 1st row of B]

Using the 2nd row, $a \cdot \underset{C_{21}}{\underset{\uparrow}{(-b)}} + b\underset{C_{22}}{\underset{\uparrow}{(a)}} = 0$

Also, in A, using the 2nd row,

$c \cdot \underset{\substack{\text{using the} \\ \text{co-factor of} \\ C_{11}}}{\underset{\uparrow}{(-d)}} + d(c) = 0$

Now, finally we can say,

The ith row of A.

$$\det A = \vec{a}_i^{\,T} \cdot \vec{c}_i \quad [~~when~~$$

and, $0 = \vec{a}_i^{\,T} \cdot \vec{c}_k^{\,T} \quad [i \neq k]$

We can rewrite this -

$$\left(A\,(\text{cof } A)^T\right)_{ii} = a_i^T \cdot c_i^T$$

and $\left(A\,(\text{cof } A)^T\right)_{ik} = 0 \quad [i \neq k]$

I want to give anothe example here -

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{cof } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$A\,(\text{cof } A)^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

this means the matrix $A(\text{cof } A)^T$ has det A on its diagonal entries, and all other entries are 0. That is, it looks like the Identity matrix, except with det A instead of 1 on the diagonal.

So,

$$A(\text{cof } A)^T = (\det A)\, I$$

So, if A is invertible, Dividing both sides by det A -

$$\frac{A\,(\text{cof } A)^T}{(\det A)} = I = AA^{-1}$$

$$\therefore A^{-1} = \left[\frac{1}{(\det A)}\,(\text{cof } A)^T\right]$$

Let's see a quick example → $B = \begin{bmatrix} 7 & 1 & 2 \\ 4 & -2 & -5 \\ 9 & 8 & -3 \end{bmatrix}$

$\text{cof } B = \begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}$ $\det B = 439$

$$B^{-1} = \frac{1}{\det B} (\text{cof } B)^T = \frac{1}{439} \begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}^T = \frac{1}{439} \begin{bmatrix} 46 & 27 & 1 \\ -33 & -48 & 47 \\ 50 & -47 & -18 \end{bmatrix}$$

Our 2nd application is,

$Ax = b$

$\hookrightarrow x = A^{-1}b$

$\hookrightarrow$ we have a formula now, $A^{-1} = \dfrac{1}{\det A} (\text{cof } A)^T$

CRAMER'S RULE:

$x_1 = \dfrac{\det B_1}{\det A}$ → $\begin{bmatrix} b & \text{(n-1) other columns of } A \end{bmatrix}$

$x_2 = \dfrac{\det B_2}{\det A}$

$\vdots$

$x_n = \dfrac{\det B_n}{\det A}$

Key Idea:

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$\hookrightarrow$ Taking determinants

$(\det A)(\cancel{\det}\, x_1) = \det B_1$

Example: $A = \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $x = ?$

$|B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix}$ $|B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix}$ $|B_3| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix}$ $A = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix}$

$x_1 = \dfrac{|B_1|}{|A|}$ $x_2 = \dfrac{|B_2|}{|A|}$ $x_3 = \dfrac{|B_3|}{|A|}$