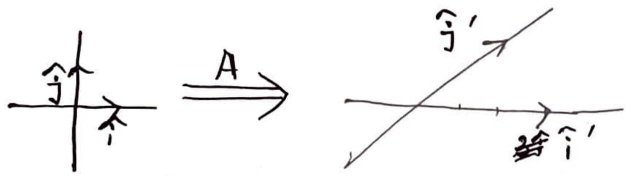
Lecture 17 → Eigenvalues and Eigenvectors -

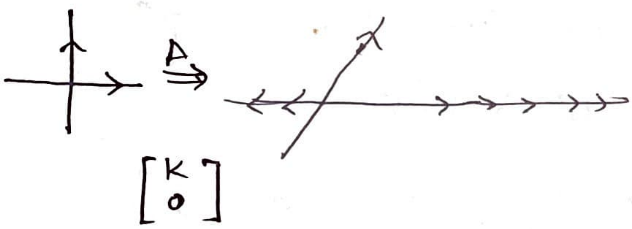
To visualize eigenvalues and eigenvectors, you need to have a solid visual understanding about →

(I) Linear Transformation done by matrices
(II) Determinants
(III) change of Basis

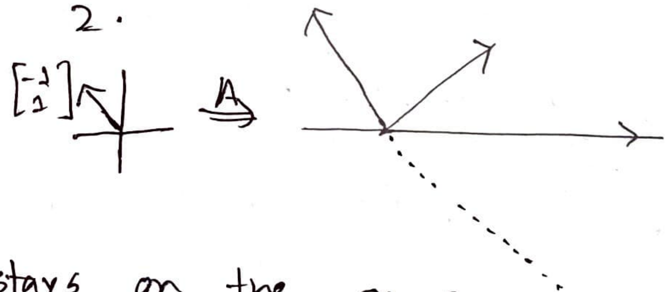
Consider a linear transformation done by this matrix → $\overset{\nearrow A}{\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}}$
If you multiply any vector with this matrix, that ~~matrix~~ vector would be knocked off of its span →



But some vectors do remain on its ~~s~~ span. For example → Any vector on the x axis. It only gets stretched on the x ~~a~~ axis.



Another example can $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. By multiplying with matrix, it only gets stretched by a factor of 2.



These special vectors, which stays on the same span after the transformation are called eigenvectors. By the factor the eigenvectors get ~~sq~~ squished or stretched, is called eigenvalue.

can eigenvalues be negative?

<u>Ans</u>: Yes! If some eigenvectors ~~have~~ has eigenvalues $-1/2$, that means the orientation of the basis gets flipped and the vector gets squished by $1/2$.

Let's figure out the conceptual ideas of computation →

As the eigenvectors do not change span, so,

$$A\vec{v} = \lambda\vec{v}$$

Eigenvectors

Eigenvalue.

But, the left hand side is matrix x vector, and the right hand side is scalar x vector. Let's simplify it a bit further.

$$A\vec{v} = (\lambda I)\vec{v}$$

$$\Rightarrow \underbrace{(A-\lambda I)}_{\text{This is a matrix.}}\vec{v} = 0$$

Now, $\vec{v}$ is a non-zero vector. so, to be $(A-\lambda I)\vec{v}=0$, $(A-\lambda I)$ has to be a matrix that squishes space into a lower dimension, such as into a straight Line or a point.

That squishification corresponds to, $\det(A-\lambda I)=0$

Now, Let's find out the method of calculation to find λ, x.

How to solve $Ax=\lambda x$?

Rewriting the equation, $(A-\lambda I)x=0$

$$\det(A-\lambda I)=0$$

(i) Find λ first.

(ii) Then find the nullspace, which will be x.

Let's take this example –

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(i) This matrix is symmetric.

(ii) constant along the diagonal.

I took this special matrix to show certain properties of eigenvalues.

we know, $\det(A-\lambda I)=0$

$$\det\left(\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=0$$

$$\Rightarrow \begin{vmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix}=0 \Rightarrow (3-\lambda)^2-1=0 \Rightarrow \lambda^2-6\lambda+8=0$$

$$\downarrow$$

$$\lambda=4,\ \lambda=2.$$

For the eigen vector $\rightarrow$ $(A-\lambda I)\vec{x}=0$

$$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}[x]=0 \qquad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}[x]=0$$

$\lambda=4$ $\rightarrow$ $x=\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda=2$ $\rightarrow$ $x=\begin{bmatrix}-1\\1\end{bmatrix}$

Let's take another example - (similar kind of matrix)

$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\cancel{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$ $\rightarrow$ for the λ, $\cancel{\det \begin{vmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix}=0 \Rightarrow}$

$$\downarrow$$

$$\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-1=0 \Rightarrow \lambda=+1 \text{ \& } -1$$

For the eigenvector $\rightarrow$

$$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}[x]=0 \qquad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}[x]=0$$

$\lambda=1$ $\rightarrow$ $x=\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda=-1$

$x=\begin{bmatrix}-1\\1\end{bmatrix}$

Summary of the whole solution process:

1. compute the determinant of $A-\lambda I$.
2. Find the roots of this polynomial, by solving $\det(A-\lambda I)=0$
3. For each eigenvalue λ, solve $(A-\lambda I)x=0$, to find an eigenvector x.

Now, I want to point out some facts →

① For $n \times n$ matrix, the product of the 'n' eigenvalues equals the determinant.

(ii) The sum of 'n' eigenvalues equals the sum of n diagonal entries of the matrix, this ~~is~~ is called 'Trace'.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}$$

Now, Notice carefully the last two examples. I added $3I$ to the second matrix and got the first one? Now, what changes happen to the eigenvalues & eigenvectors if I add any number of 'I' to any matrix?

Let's see,

$$Ax = \lambda x$$

$$\Rightarrow (A + 3I)x = Ax + 3Ix$$

$$= \lambda x + 3Ix$$

$$= (\lambda + 3)x$$

So, if 'KI' is added to any matrix, the eigenvalue is increased by K and the eigenvector remains the same.

A common misconception,

If

$$Ax = \lambda x \qquad \text{and} \quad B \text{ has eigenvalues } \alpha$$

$$Bx = \alpha x$$

$\Rightarrow (A+B)x = (\lambda + \alpha)x$, but its wrong. Why? Because A and B might not have same eigenvector for x.

Let's take a ~~r~~ rotation matrix.

90° rotation, $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace $= 0 = \lambda_1 + \lambda_2$

det $= 1 = \lambda_1 \lambda_2$

the equation would be, $\lambda^2 + 1 = 0$

$\therefore$ ~~$\lambda = \pm\sqrt{\ }$~~ $\lambda = \pm\sqrt{-1} = +i$ and $-i$

It's very intuitive. After ~~a~~ you rotate a vector (with rotation matrix), that vector can't be in its own span anymore unless you imagine it, that's where those imaginary numbers come.

* So the ~~s~~ more symmetric the matrix be, the better your eigenvalues would be.

Let's take a triangular matrix $\rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\lambda_1 + \lambda_2 = 6$

$\lambda_1 \lambda_2 = \cancel{8}\,9$

$\therefore \lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow \lambda_1 = 3, \lambda_2 = 3$

The eigenvectors would be, $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ⊗

$\downarrow$

Only one ~~F~~ independent eigenvector.

So, For triangular matrices, you'd ~~only have~~ not get 'n' eigenvalues and eigenvectors.

$\boxed{\text{Question}}$ Tell me the eigenvalues of (i) projection matrix

(ii) Reflection "

(iii) Rotation "