Lecture 11 → Orthogonality

Orthogonality is a synonymn for perpendicularity.
Two vectors are Orthogonal if their dot product is zero.
→ $v \cdot w = 0$ Or $v^T w = 0$

For a right triangle -

$$\|v\|^2 + \|w\|^2 = \|v+w\|^2 \leftarrow \text{Pythagorus}$$

$$\Rightarrow v^T v + w^T w = (v+w)^T (v+w) = v^T v + v^T w + w^T v + w^T w$$

For example → $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$\|x\|^2 = 14$ $\|y\|^2 = 5$ $\|x+y\|^2 = 19$

$v^T w + w^T v = 0$
$\Rightarrow 2v^T w = 0$
$\Rightarrow v^T w = 0$

So, the orthogonality of vectors is clear. What about the orthogonality of subspaces?

⇒ Two subspaces V and W are orthogonal if every vector v in V ~~and~~ is perpendicular to every vector w in W :-

$v^T w = 0$ for all v in V and all w in W.

Example → The floor of our room and the line where two walls meet.

Not an example → Two walls look perpendicular but they are not orthogonal.

⇒ Orthogonality is impossible when dim V + dim W > dim of whole space

An interesting fact about the four fundamental subspaces is that → zero is the only point where the nullspace meets the row space. More than that, the nullspace and rowspace of A meet at 90°. This key fact comes directly from → $Ax = 0$.

→ Every vector x in the nullspace is perpendicular to every row of A, as $Ax = 0$.

The nullspace $N(A)$ and the row space $C(A^T)$ are orthogoral subspace of R^n.

Why x is perpendicular to the rows? look at $Ax = 0$ →

$$Ax = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow (\text{row } 1) \cdot x \text{ is zero} \\ \\ \leftarrow (\text{row } m) \cdot x \text{ is zero} \end{array}$$

Every row has a zero dot product with x. Then x is also perpendicular to every combination of the rows. The whole rowspace $C(A^T)$ is orthogoral to $N(A)$ in R^n.

Similarly, The left nullspace $N(A^T)$ and the column space ~~CA~~ $C(A)$ are orthogoral in R^m.

Previously I said, if dim V + dim W > dim of whole space, they can't be orthogoral.

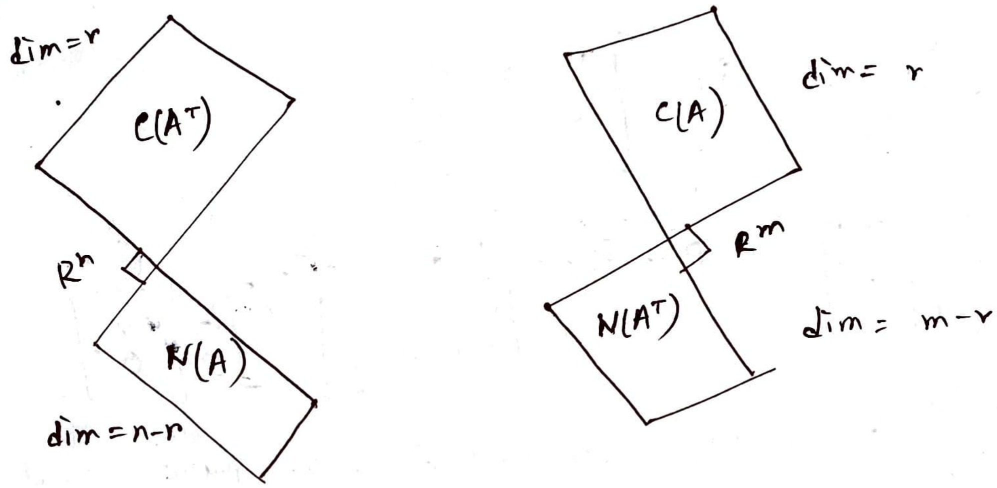
The fundamental subspaces make the dimensions right. Two orthogoral subspaces combinedly fill the whole space.

In fact, in R^3, two subspaces would be orthogoral if they have dimensions 2 and 1, or 3 and 0.

Two subspaces thus are not only orthogoral, they are orthogoral complements.

Definition: The orthogoral complement of a subspace V contains every vector that is perpendicular to V.

An important picture →

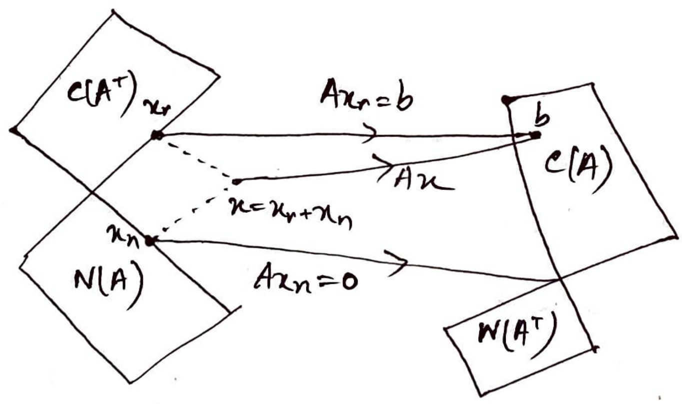


The point of "complements" is that every x can be split into a row-space component x_r and a null space component x_n. When A multiplies $x = x_r + x_n$, Figure ~~4.3~~ shows what happens-

the nullspace component goes to zero, $Ax_n = 0$

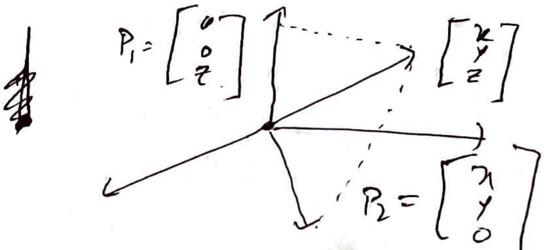
The rowspace component goes to the column space, $Ax_r = Ax$

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ split $x = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ into $x_r + x_n = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



Projection:

1. What are the projections of $b = (2, 3, 4)$ onto the z axis and the xy-plane?



2. What matrices produce those projections onto a line and a plane?

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p_1 = P_1 b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad p_2 = P_2 b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

In this case, P_1 & P_2 are perpendicular. xy-plane and z-axis are orthogonal subspaces.

Every vector b in the whole space is the sum of its parts in two subspaces.

So, $p_1 + p_2 = b$

and $P_1 + P_2 = I$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$