~~Le~~cture 13 - The Determinants:

To know how determinants work, you need to recall the idea of linear transformation. What generally linear transformation does is → sometimes it stretches things out, sometimes squishes things in.

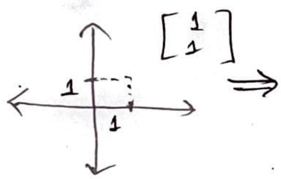
Determinant is a tool to measure exactly how much are things being stretched.

For example, think about this matrix → $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

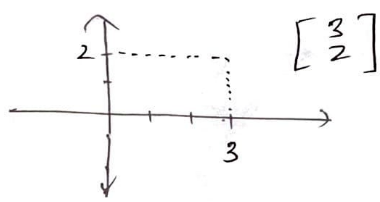
$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ → It scales $\hat{i}$ by a factor of 3.

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ → It scales $\hat{j}$ by a factor of 2.

Now if you multiply a vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with this matrix →



$\Rightarrow$



Area = 1×1 = 1

Area = 3×2 = 6

You can say that this matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ has scaled the area by a factor of 6.

This very special factor, the factor by which linear transformation changes any area, is called determinants.

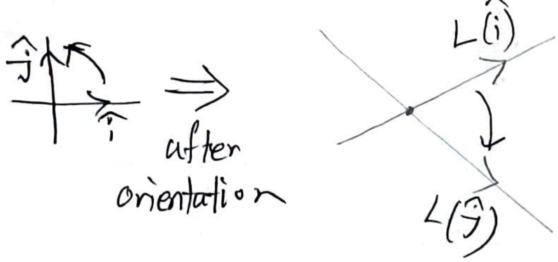
$$\det\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right) = 6, \quad \det\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\right) = 2, \quad \det\left(\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\right) = 1/2$$

$\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = 0$, why? Because it squishes everything on a line.

One might ask, then why the determinants are negative sometimes)

This has to do with the idea of orientation.

For example →



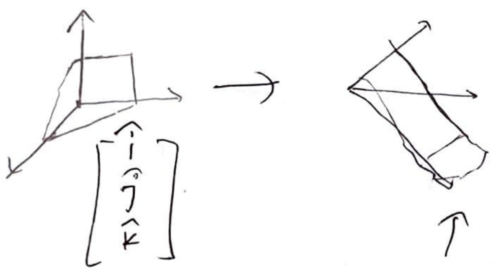
The orientation of space has inverted.

So whenever the orientation of space is inverted, the determinant will be negative.

$$\det\left(\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}\right) = -5$$

Now what about 3-dimensions? In 2D, determinants work with area, in 3D it works with volume.

So, determinants in 3D tells you how much volume is scaled.



Parallelepiped

Negative determinants or zero determinants works same as the 2D convention.

Okay, now some properties of determinants – (major 3 prop.)

① $\det(I) = 1$

② Exchanging the rows of the matrix reverse the sign of the determinant. So, as $\det(I) = 0$, $\det(P) =$ either 1 or -1

③ → is all about Linear Combination

III.a) Multiplying any row by a scalar factor has the below effect –

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

III.b) Adding with a row has below effect → $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \square + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

here's one thing clarify, $\det(A+B) \neq \det(A) + \det(B)$
this linearity is applicable only for rows.

Let's learn some more properties (all other properties come from the the first three):

(IV) If there are 2 equal rows in a matrix, the determinant is zero.

[Use property (II) for reasoning. how? the sign did change, but the matrix didn't. so the only way possible is the det = 0]

(V) While elimination, if $l \times$ row i is subtracted from row k, the determinant doesn't change.

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{\to 0}$$

(VI) A complete row of zeros leads to $\det(A) = 0$

we saw, $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ if $t=0$, then det = 0

(VII) For any lower or upper triangular matrix, the determinant is the product ~~multiplication~~ of the diagonal elements.

$$\det(U) = \begin{vmatrix} d_1 & \times & \times & \times \\ 0 & d_2 & & \times \\ 0 & 0 & \ddots & \times \\ 0 & 0 & 0 & d_n \end{vmatrix} = (d_1)(d_2) \cdots (d_n) \text{ [product of the pivots]}$$

Reasoning! For U, we can get $\begin{vmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_n \end{vmatrix} \Rightarrow d_1 \cdot d_2 \cdots d_n \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & & \\ 0 & & \ddots & \\ 0 & 0 & & 1 \end{vmatrix} = d_1 \cdot d_2 \cdots d_n$

(VIII)

from (VI) and (VII) we can say that $\boxed{\det(A) = 0 \text{ when } A \text{ is singular.}}$

so, $\boxed{\det A \neq 0 \text{ when } A \text{ is invertible.}}$

Now, Let's see how we can find out determinants.

Let's say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

By elemination $\rightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix} \Rightarrow$ so from property (VII) $\det = a \times (d - \frac{c}{a}b)$

$= ad - bc$

(IX) $\det(AB) = (\det A)(\det B)$

but $\det(A+B) \neq \det(A) + \det(B)$

From this can we find $\det A^{-1} = ?$

We know, ~~det~~ $AA^{-1} = I$

$\Rightarrow \det(AA^{-1}) = \det(I) = 1 = \det(A)\det(A^{-1})$

$\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

A question, Can we tell me what is $\underset{\downarrow}{\det A^2}$ and $\underset{\downarrow}{\det 2A}$?

Homework

$(\det A)^2$ $\qquad$ $2^n \det A$

(X) $\det(A^T) = \det(A)$

Can be factored into

LU

$\det(LU) = \underbrace{\det(L)}_{\downarrow} \det(U)$ $\rightarrow$ Because the diagonal is all 1s.

$= 1 \det(U)$

$= \det(U)$

$\det(U^T L^T) = \det(U^T)\det(L^T)$

$= \det(U^T) \cdot 1$

$= \det(U)$