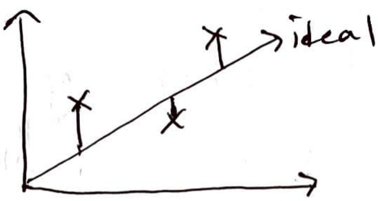
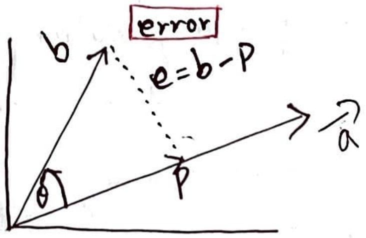
why projection? Because $Ax = b$ may have no sol^n so, we find $A\hat{x} = p$, the closest sol^n

Lecture 12 → Projection

Think about a housing problem, where you have to find the best fit line! After that you have to project lines on the ideal line.



Suppose, a line goes through the origin in the direction of $\hat{a}$,



The line from b to p is perpendicular to the vector a.

$$\boxed{p = \hat{x} a}$$

We're 3 goals with this equation, Find $\hat{x}$, find p and find P.

$e = b - \hat{x} a$

So, $a \cdot e = 0$

$\Rightarrow a \cdot (b - \hat{x} a) = 0$

$a \cdot b - \hat{x} a \cdot a = 0$

$$\Rightarrow \boxed{\hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{a^T b}{a^T a}}$$

2 special cases.
What if $\boxed{b = a}$ and $\boxed{b \perp a}$?

(*) Project $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ to find $p = \hat{x} a$. Also, find e and check if e is perpendicular to a.

Projection matrix P → what is the columnspace of this matrix? & rank?

$$\boxed{p = a\hat{x} = a \frac{a^T b}{a^T a} = Pb, \quad \text{so} \quad P = \frac{a a^T}{a^T a}}$$

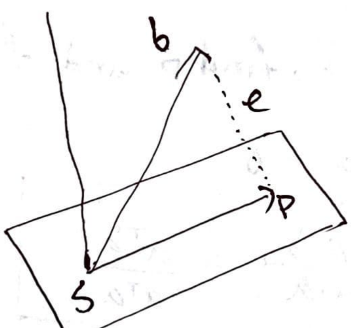
* Find the projection matrix $P = \frac{aa^T}{a^Ta}$ onto the line through $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

* If the vector a is doubled, matrix P stays the same.

* If the matrix is squared to P^2, projecting second time doesn't change anything. So, $P^2 = P$. Also $P^T = P$

Think about $I-P$. It should be a projection too. It produces the other side of the triangle, the perpendicular part of b. Also, $(I-P)b = b-p$ which is e in the left nullspace. When P projects onto one subspace, $I-P$ projects onto the perpendicular subspace.

Projection onto a subspace:



The dotted line goes from b to the nearest point $A\hat{x}$ in the subspace. This error vector $e = b-p = b - A\hat{x}$ is perpendicular to the subspace.

So, this error vector $b - A\hat{x}$ in fact makes a right angle with all the vectors $a_1, \dots, a_n$ of the rowspace.

$$a_1^T(b - A\hat{x}) = 0$$
$$a_2^T(b - A\hat{x}) = 0$$
$$\vdots$$
$$a_n^T(b - A\hat{x}) = 0$$

or $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b - A\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

collectively it can be written as -

$$A^T(b - A\hat{x}) = 0$$

Do you know in which space this is in? → e ⊥ C(A) → e in $N(A^T)$

This has a very famous form → $A^TA\hat{x} = A^Tb$

This symmetric matrix A^TA is n by n. It is invertible if the a's are independent.

Solution,

$$\hat{x} = \cancel{(A^TA} (A^TA)^{-1} A^Tb$$

Projection, $p = A\hat{x} = A(A^TA)^{-1} A^Tb$

Projection matrix, $P = \cancel{A\ pb}\ \frac{p}{b} = A(A^TA)^{-1}A^T$

~~Exam~~ The vector b is being split into the projection p ~~at~~ and the error $e = b - p$.

Example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$, find $\hat{x}$ and p and P.

1. compute $A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$ and $A^Tb \rightarrow \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

2. ~~Cor~~ Solve the equation $A^TA\hat{x} = A^Tb$ to find $\hat{x}$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad \text{gives} \quad \hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

3. Find $p = A\hat{x}$

4. To find $p = Pb$ for every b, Compute $P = A(A^TA)^{-1}A^T$

$$(A^TA)^{-1} = \frac{1}{\text{det.}} \begin{bmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

Multiply A times $(A^TA)^{-1}$ times A^T.

Home work: A^TA has certain properties. Read them, with proofs.

Least Squares Approximations:

It often happens that $Ax=b$ has no solution. It usually happens when there are more equations thatn unknowns.

We can not always get the error $e=b-Ax$ down to zero.

When e is zero, x is an exact solution to $Ax=b$.

When the length of e is as small as possible, $\hat{x}$ is a least squares solution.

We already learnt p (the projection). In fact, p and $\hat{x}$ are connected.

$p = A\hat{x}$.

The short ~~of~~ unofficial way to reach the solution is –

When $Ax=b$ has no solution, multiply by A^T and solve $A^TA\hat{x} = A^Tb$

Let's start with an example!