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#Lecture 9 →

☐ spanning a space

we already know it right? vectors $v_1, v_2 \ldots\ldots v_d$ span a space means the space consists of all combinations of those vectors.
For example, the columns of a matrix span the column space.

☐ Basis for a space is a sequence of vectors $v_1, v_2, v_3 \ldots\ldots v_d$ with 2 properties –

1. They are independent
2. They span the space.

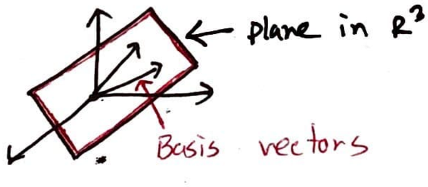
Examples

For R^3, one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$. why?

But this isn't the only basis? Another basis can be $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$.
Now if I took only two out of: these ~~th~~ three, would that be a basis?

So, In R^n, n vectors give basis if the $n \times n$ matrix with those columns is invertible.

Now, If I took $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\2\\5\end{bmatrix}$, would they be a basis? Yes!
They would be a basis for a subspace created by the plane (all linear combinations) in R^3.



So, Basis is not unique. There can be a million possibilities of basis.
But one thing common for all the basis are –

Every basis for the space has the same number of vectors.

So, In R^6, if we got 7 vectors, that is too many for the basis, ~~we~~ if we got 5 vectors, that would ~~f~~ not be enough. So, 6 is just the perfect number.
And, this perfect number is the dimension of the space.

Example:

Any matrix, $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

(i) what is the span of these column vectors? → $C(A)$

(ii) Are the columns independent? → No. Because $N(A)$ has a non-zero solution.

(iii) what ~~are~~ is the basis of the columnspace? → The first two columns / column 1&3

(iv) What is the dimension of the columnspace? → rank(A) = 2 = # of pivots = dimensions

$$\boxed{\dim C(A) = \text{rank}}$$

This isn't the dimension of the matrix, but only the columnspace.

(v) What is the dimension of the nullspace?

→ Give me two special solutions → $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ → And These two are the basis of the nullspace.

And, $\boxed{\dim N(A) = \#\text{ of free columns} = n - r = 2}$

The four subspaces of R.

Row Reduced Form

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we are already familiar with 2 subspaces. Columnspace and Nullspace - these two ~~spaces~~ subspaces come from A. The other two subspaces, namely rowspace and left nullspace comes from A^T.

Let's know about all the subspaces through a specific example - Let's say we have a 3x5 matrix A and its reduced to its row-echelon form R.

$$R = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccl} \textcircled{1} & 3 & 5 & 0 & 7 & \leftarrow \text{Pivot row} \\ 0 & 0 & 0 & \textcircled{1} & 2 & \leftarrow \text{Pivot row} \\ 0 & 0 & 0 & 0 & 0 & \\ \uparrow & & & \uparrow & & \\ \text{Pivot Column} & & & \text{Pivot Column} & & \end{array}$$

$m = 3$
$n = 5$
$r = 2$

1. **The column space $C(A)$**

As the rank = 2, dimension is also 2.

The pivot columns 1 & 4 form a basis for $C(R)$.

So, The dimension of the column space is rank r, & the pivot columns form a basis.

2. The Null Space $N(A)$

First, think about the special solutions of A. that solves $Ax=0$. There are 3 free columns. So, setting the free columns to $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ gives the solution

$$x_{special} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So, the dimension is $n-r = 5-2 = 3$, and these 3 $x_{special}$ are a basis for the nullspace.

3. The row space $C(A^T)$

Think about R^T. What is the rank? 2. And this is the dimension. And the ~~for~~ first and the second row form a basis. Because the third row doesn't add anything with the rowspace.

So,

The dimension of the row-space is $r=2$, The non-zero rows form a basis.

4. The Left-Nullspace $N(A^T)$

Let's say,

$$\underset{\uparrow\ \text{special solution}}{Ay} = 0$$

Now, if you transpose the whole ~~thei~~ thing -

$$(Ay)^T = 0^T$$

$$\Rightarrow \underset{\uparrow\ \text{Null space comes at left.}}{y^T} \underset{\uparrow\ \text{Rowspace}}{A^T} = 0$$

So, Left Nullspace is basically the nullspace of the rowspace

So, $R^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 0 \end{bmatrix}$. So, How many special solution? Only 1.

$\qquad\qquad\qquad\qquad\uparrow$
Free column.

So, the basis would be only one vector. Which is? $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Dimesion = 1.

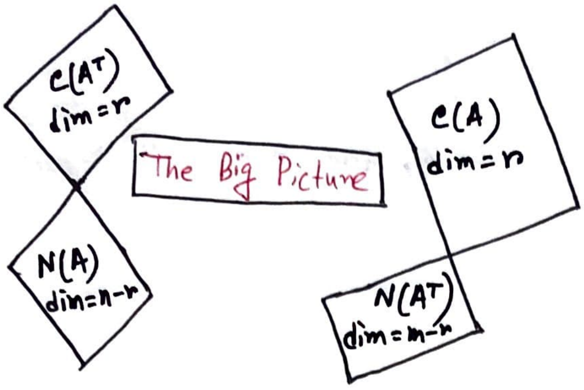
So,

If A is $m \times n$ matrix of rank r, its left nullspace has dimesion $m-r$

summarizing - of a matrix m×n,

In R^n, the row space and nullspace have dimensions r and n-r.
In R^m, the column space and left-nullspace have dimensions r and m-r.

so, the big picture would be -



The Four Subspaces of A:

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}} \Rightarrow \underset{R}{\begin{bmatrix} \textcircled{1} & 0 & 1 & 1 \\ 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

But their dimension is same.

see, the columnspace has changed! $C(A) \neq C(R)$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is gone and you can't get it back by reversing the operations. But, their rowspace is same! Because, you only changed the rows by some row-operations!

$$EA = R \Rightarrow A = E^{-1}R$$

① A has the same dimension as R and same basis. In fact, A has the same row-space as R.

② The column space of A has dimension r → rank. For Every matrix, the following is true -

The number of independent columns equals the number of independent rows.

Reason: The same combinations of the columns are zero for both A & R. $Ax=0$ exactly when $Rx=0$.

③ A has the ~~same~~ nullspace as R. Same dimension $n-r$ and ~~same~~ basis.

Reason!

③ A has the same nullspace as R.

$$\left[\begin{array}{ccc|c} - & - & - & b_1 \\ - & - & - & b_1-2b_2 \\ - & - & - & b_3-2b_1-b_2 \end{array}\right]$$

Because nullspace is related to rowspace. Does elimination change the rowspace? If you think carefully, you can understand elimination steps actu~~t~~ally ~~rea~~ reveal which rows are dependent. So, as it doesn't change the rowspace, it wouldn't change the nullspace as well.

So, the dimension would be $n-r$ and the basis would be same.

④ The left nullspace is related to columnspace. So, it wouldn't be the same as well. The dimension would be $m-r$.

Example 1: $A = [1 \ 2 \ 3]$

$m = 1$, $n = 3$, rank $= 1$

→ Row space is a line in R^3.

⊛ ~~Geo~~ start with this matrix → $A = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix}$

Column space:

$\text{Span} = \left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$, Basis $= \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, Dimension $= 1$

Nullspace:

$R = \left[\begin{array}{ccc:c} 1 & -1/2 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $\text{Span} = \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$, Basis, Dimension $= 2$.

$C(A^T)$:

$A^T = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ -3 & 6 \end{bmatrix}$, $\text{span} = \left\{ \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \right\}$, Basis, Dimension $= 1$.

↑

But this is the row of our original matrix. so, we can also say that this is the span of the row vector of A, or the rowspace of A^T! so, $C(A^T)$ is called the rowspace of A!

$N(A^T)$:

$\text{rref}(A^T) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\text{span} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, Basis, Dimension $= 1$.

Now, If the nullspace of A^T is y,

$$A^T y = 0$$

$$\Rightarrow (A^T y)^T = 0^T$$

$$\Rightarrow y^T A = 0^T$$

so, this is also called the left nullspace of A!

$\text{Rank}(A) = \text{Rank}(A^T)$ → will always be the case.