Lecture 8

Complete solution of $Ax=b$

Let's take our previous matrix -

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

Our Augmented matrix would be -

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

One particular solution is $\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$. why?

$$\xrightarrow[R_3 - R_1\times 3]{R_2 - R_1\times 2} \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

$\to b_3 - b_2 - b_1 = 0$

$\to \boxed{b_3 = b_2 + b_1}$ This is exactly why

Solvability condition on the right hand side b.

$Ax=b$ is solvable when b is in $C(A)$.

* If a combination of rows of A gives zero row, then same combination of the entries of b must gives 0.

To find the complete solution to $Ax=b \to$

Step 1: Find $X_{particular}$. Set all free variables to 0 and solve $Ax=b$ for pivots

$$x_1 + 2x_3 = 1$$
$$2x_3 = 3$$

So the solution would be $\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$. $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

Step 2: $X_{complete} = X_{particular} +$ Any vector from the Nullspace

$\Rightarrow \quad X = X_p + X_n$

Prove: $AX_p = b$

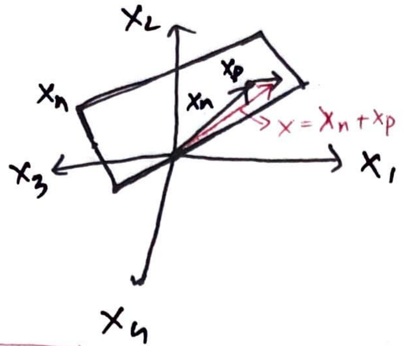
$\oplus \quad AX_n = 0$

$A(X_p+X_n) = (b+0) = b$

* Important $\to$ Difference between X_p and $X_{special}$? For X_p, all free variables are set to 0, But for $X_{special}$, one of them is set to 1.

$$\therefore \boxed{X_{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \left\{ c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Plot all solutions x in R^4!



$\boxed{\text{No Need to show}}$

We'll see the bigger picture now -

Think about a $m \times n$ matrix 'A' of rank r.
What's the relation between r & m and r & n.
$(r \le m)$ $(r \le n)$

Case 1: Full column rank, $\boxed{r=n}$ → n pivots.
No free variables.

$N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

solution of $Ax=b$ → ~~x is on~~ x is only x_p. Unique solution if it exists. One or zero solution. And the nullspace matrix is empty.

Example -
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$
, Can we have solution for all b?

~~Case 2: Full row rank,~~ ~~$\boxed{r=m}$~~

How many pivots? → m * No rows with zeros.
I can solve ~~$A(x)$~~ $Ax=b$ for

So, Full column matrix would have below properties -

1. All columns of A are pivot columns
2. There are no free variables or special solutions
3. The Nullspace contains only the zero vector, $x=0$
4. If $Ax=b$ has a solution (it might not) then it has only one solution

Case 2: Full row rank, $r=m$ and $m \leq n$

Every row has a pivot. So, the rows are linearly independent

Example:

$x+y+z=3$

$x+2y-z=4$

$$\hookrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \rightarrow \rightarrow \rightarrow \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{array}{cc} I & F \\ \hline \end{array}$$

So, $x_p = (2,1,0)$ and $x_n = c(-3,2,1)$

Every matrix A with full row rank $(r=m)$ has following properties

1. All rows have pivots, R has no zero rows.
2. $Ax=b$ has a solution for all right hand side b.
3. The column space is the whole space R^m.
4. There are $n-r=n-m$ special solutions in the $N(A)$.

Case 3: Full row and column rank, $r=m=n$

A is an invertible square matrix. The nullspace has dimension zero and $Ax=b$ has a unique solution for every b in R^m.

Summary

	$r=m=n$	$r=n<m$	$r=m<n$	$r<m, r<n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# of solutions for $Ax=b$	Exactly 1	$\boxed{0 \text{ or } 1}$	Infinitely Many	$\boxed{0 \text{ or infinitely many}}$

* Find all solutions depending on b_1, b_2, b_3.

$$\begin{aligned} x-2y-2z &= b_1 \\ 2x-5y-4z &= b_2 \\ 4x-9y-8z &= b_3 \end{aligned} \rightarrow \left[\begin{array}{ccc:c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -2 & 5b_1-2b_2 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots free

One step before this: $\begin{bmatrix} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1+b_2 \\ 0 & 0 & 0 & -2b_1-b_2+b_3 \end{bmatrix}$

If $-2b_1-b_2+b_3 \neq 0$, No solutions!

* Particular solution -

$Ax=b$

$$X_p = \begin{bmatrix} 5b_1-2b_2 \\ 2b_1-b_2 \\ 0 \end{bmatrix}$$

* Special solution:

$Ax=0$ [Free column = 1]

$$X_s = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

∴ All solutions, $X_{complete} = X_p + c \cdot X_s = \begin{bmatrix} 5b_1-2b_2 \\ 2b_1-b_2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Independence:

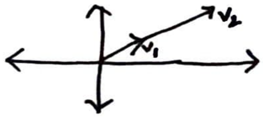
Suppose A is m by n with $m<n$. Then ~~there're~~ there are non-zero solutions to $Ax=0$ [Because more unknown than equations]

Reason: There will be free variables! Because some columns would be a combination of other columns.

Formal def^n → Vectors $x_1, x_2, \ldots, x_n$ are independent if no combination gives zero vector except the zero combination.

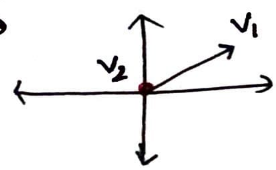
> If some other combination other than zero gives zero, then the vectors are dependent.

For example -



$2v_1 - v_2 = 0$

Interesting →



$0v_1 + 6v_2 = 0$

so, summarizing →

When $v_1, v_2, \ldots, v_n$ are columns of A :-

rank = n = all columns

* They are independent if nullspace of A is only the {zero vector}
* They " dependent if $Ac = 0$ for some non-zero c.

rank < n