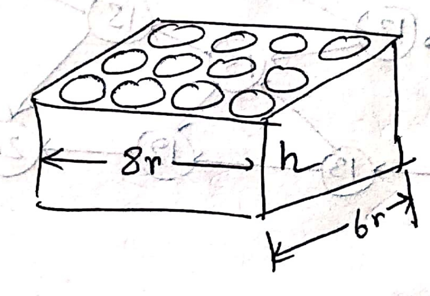
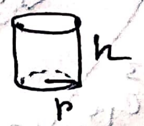
$\boxed{4}$ Optimization problem involves

- Problem data (parameters)
- Decision variable
- objective function
- Constraints

$\boxed{5}$





$S_1 = (2\pi r^2 + 2\pi r h) \times 12$
$= (2\pi r(r+h)) \times 12$

Surface Area of 12 cans

$S_2 = 2 \times 8r.h$
$+ 2 \times 6r.h$
$+ 2 \times 8r \times 6r$

$$\min \quad S_1 c_1 + S_2 c_2$$

Non-linear optimization problem

s. t:

$\pi r^2 h \geq V_0$

$h \leq D_0$

$\cancel{6r \leq D_0} \longrightarrow$ Redundant $\because 8r \leq D_0$

$8r \leq D_0$

$h, r \geq 0$

Reformulation: $\boxed{10}$

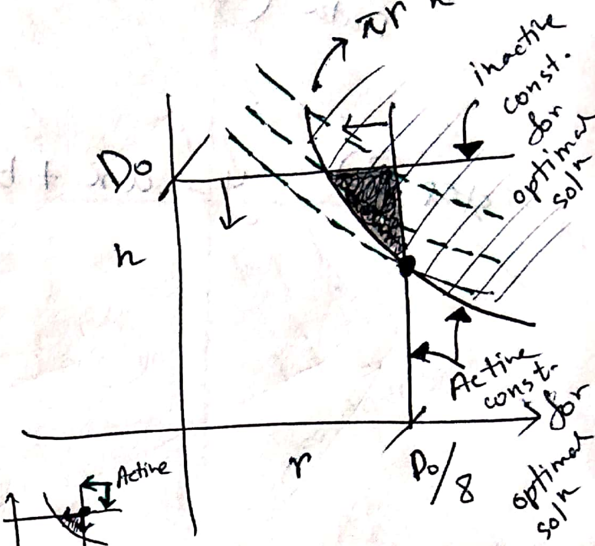
$$\min_{r,h} \quad (24\pi c_1 + 96 c_2) r^2 + (24\pi c_1 + 28 c_2) rh$$

$$\text{s.t.} \quad \pi r^2 h \geq V_0$$

$$r \leq \frac{D_0}{8}$$

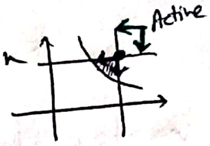
$$h \leq D_0$$

$$r, h \geq 0$$



(*) Active constraints are necessary for developing algorithms

For example: which direction should be chosen to find a better sol^n



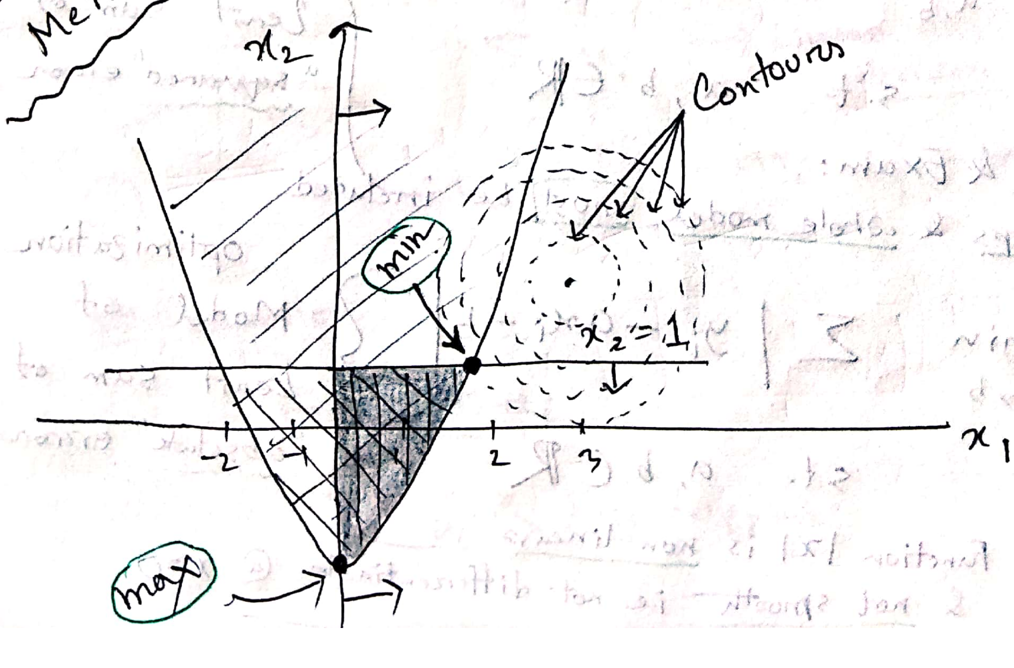
$\boxed{38}$ $\min \quad f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 2)^2$

$$g_1(x_1, x_2) = x_1^2 - x_2 - 3 \leq 0$$

$$g_2(x_1, x_2) = x_2 - 1 \leq 0$$

$$g_3(x_1, x_2) = -x_1 \leq 0$$

** Geometric Method



Parameter Estimation

step ① $(x_1, y_1) \ldots\ldots \ldots (x_n, y_n)$

Problem Data (collected from exp.; These will act like parameters)

step (2) $y = ax + b$; Decision variables a, b [Though seems like Parameter]

$$\hat{y}_i = a x_i + b$$

↳ estimator of y_i

$$\varepsilon_i = y_i - \hat{y}_i \quad [\text{Error}]$$

step (3) $\min \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - a x_i - b)^2$

step (4) constraints : N/A (Unconstrained)

$a, b \in \mathbb{R}$

Whole Model

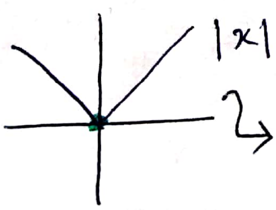
$$\therefore \min_{a,b} \sum (y_i - a x_i - b)^2 \quad \text{s.t. } a, b \in \mathbb{R}$$

Optimization Model of Least sum of "squared" error

** For HW & Exam: all 4 steps & whole model **must** be included

$$\min_{a,b} \sum |y_i - a x_i - b| \quad \text{s.t. } a, b \in \mathbb{R}$$

Optimization Model of Least sum of "absolute" error



↳ Function $|x|$ is non-linear & not smooth i.e. not differentiable @ $x=0$

$\boxed{\text{SIE 545}}$

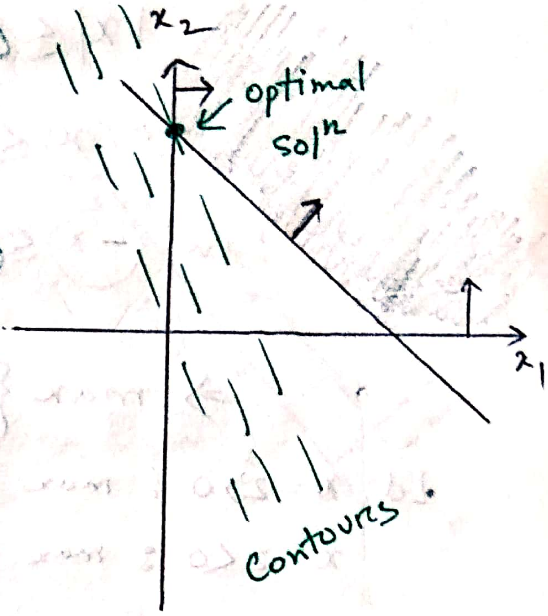
Lecture 3

Review of last class:

$$\min \quad 3x_1 + 2x_2$$

$$x_1 + x_2 \geq 1$$

$$x_1 \geq 0 \; ; \; x_2 \geq 0$$



Formulation of a non-Linear Problem into a Linear prob

Optimization model of Least sum of absolute errors $\rightarrow$

$$\min_{a,b} \sum |y_i - ax_i - b| \Rightarrow \text{non-Linear obj. func.}$$

$$\text{s.t. } a, b \in \mathbb{R}$$

Using Re-formulation —

Lets, convert it into a Linear problem

Approach ①

$$\min_{u_i, a, b} \sum u_i \longrightarrow \text{Linear obj. func. with new decision variables } (u_i)$$

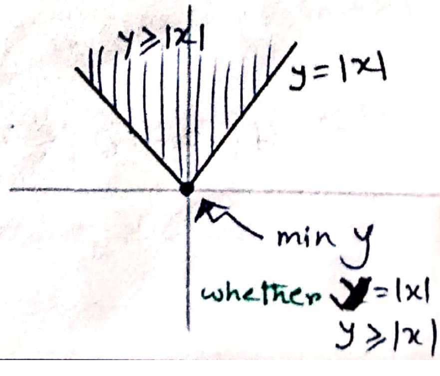
$$u_i^* = |y_i - ax_i - b| \longrightarrow \text{non-linear (Additional contraint)}$$

$\equiv$ (equivalent)

Approach ②

$$\min \sum u_i$$

$$u_i^* \geq |y_i - ax_i - b|$$



** Prove that they give the same opt. result

Optimization Problem Formulation Steps:

1. Problem data i.e., parameters
2. Decision variables
3. Objective function
4. Constraints
5. Optimization model

Optimization Problem

$$\underset{x}{\text{minimize}} \quad f(x) \longrightarrow \text{objective function}$$

$$\text{subject to} \quad g(x) \leq 0 \longrightarrow \text{Inequality constraint}$$

$$h(x) = 0 \longrightarrow \text{Equality constraint}$$

Can be Linear/Non-linear

Reformulation Techiques for 'Non-linear and Non-differentiable' functions:

Case 1: $a \geq |x|$; $a \in \mathbb{R}^+$ e.g., $a = 3$

Case 2: $\min\limits_{x} |x|$

Case 3: $a \leq \min\{x_1, x_2, \dots, x_n\}$

Case 4: $a \geq \max\{x_1, x_2, \dots, x_n\}$

Easy to Handle

Case 5: $a \leq |x|$

Case 6: $\max\limits_{x} |x|$

Case 7: $a \geq \min\{x_1, x_2, \dots, x_n\}$

Case 8: $a \leq \max\{x_1, x_2, \dots, x_n\}$

Difficult to Handle

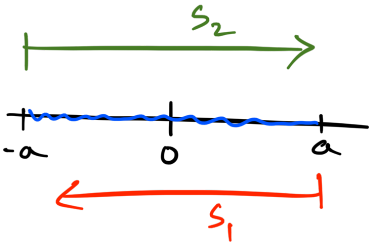
Easy to Handle cases:

Case 1: (constraint) $a \geq |x| \Rightarrow$

$a \geq x^{*}$; $x \geq 0$

$a \geq -x$; $x \leq 0$

$\hookrightarrow -a \leq x^{*}$



$x \in S_1 \cap S_2$

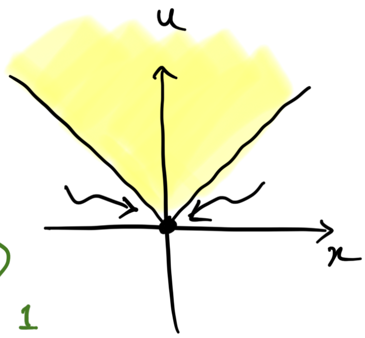
Example:

$$\min_{x} \; 2x+3 \quad \text{s.t.} \quad 3 \geq |x| \quad \Leftrightarrow \quad \min_{x} \; 2x+3 \quad \text{s.t.} \quad 3 \geq x, \; 3 \geq -x$$

Case 2: (obj. function) $\min |x| \Leftrightarrow \min_{x,u} u$ s.t. $u = |x|$ $\Leftrightarrow \min_{x,u} u$ s.t. $u \geq |x|$ (Case 1)

$\Updownarrow$

$$\min_{x,u} u \quad \text{s.t.} \quad u \geq x, \; u \geq -x$$



Case 3: (const.) $u \leq \min\{x_1, x_2, \dots, x_n\} \rightsquigarrow 1$ const. $\Rightarrow u$ is smaller than all the x_is.

$\Leftrightarrow u \leq x_i$; $i = 1, 2, \dots, n \rightsquigarrow n$ const.

Case 4: $u \geq \max\{x_1, x_2, \dots, x_n\} \rightsquigarrow 1$ const. $\Rightarrow u$ is greater than all the x_is.

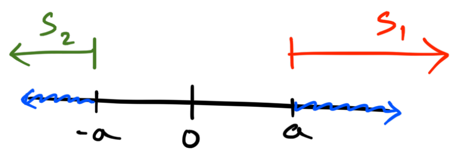
$\Leftrightarrow u \geq x_i$; $i = 1, 2, \dots, n \rightsquigarrow n$ const.

Case 5: $a \le |x| \Rightarrow$

$a \le x^{*}$; $x \ge 0$

$a \le -x$; $x \le 0$

$\hookrightarrow -a \ge x^{*}$



$\Rightarrow x \in S_1 \cup S_2$

OR

we need a binary variable $d \in \{0,1\}$ to deal with above 'OR' situation

Let, $d = \begin{cases} 1 & ; \ x \ge 0 \\ 0 & ; \ o/w \end{cases}$

$\therefore \quad a - M(1-d) \le x$

$\qquad a - Md \le -x$

Case 6: $\max\limits_{x} |x| \iff \max\limits_{x,u} u \iff \max\limits_{x,u} u$

s.t. $u = |x|$ $\qquad$ s.t. $u \le |x|$

Case 5

$\Updownarrow$

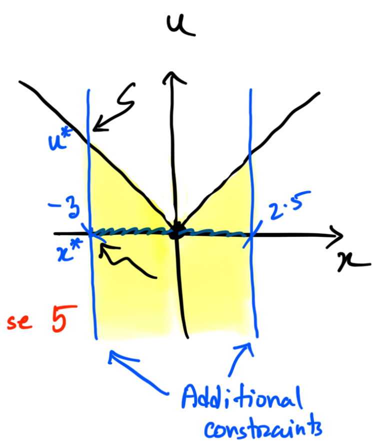
$$\max_{x,u} \quad u$$

s.t. $u - M(1-d) \le x$

$u - Md \le -x$

$d \in \{0,1\}$

$d = \begin{cases} 1 & ; \ x \ge 0 \\ 0 & ; \ o/w \end{cases}$



$\max\limits_{x} |x|$

s.t. $x \le 2.5$

$x \ge -3$

$\iff$

$$\max_{x,u} \quad u$$

s.t. $u - M(1-d) \le x$

$u - Md \le -x$

$d \in \{0,1\}$

$d = \begin{cases} 1 & ; \ x \ge 0 \\ 0 & ; \ o/w \end{cases}$

$x \le 2.5$

$x \ge -3$

Case 7: $a \geq \min\{x_1, x_2, \dots, x_n\}$ $\Rightarrow$ u is greater than some of the x_is.
↳ we need binary variable $z_i \in \{0, 1\}$ for each of x_i

$$\Rightarrow z_i = \begin{cases} 1 & ; \; x_i = \min\{x_1, x_2, \dots, x_n\} \\ 0 & ; \; o/w \end{cases}$$

$$\Rightarrow a + M(1 - z_i) \geq x_i \quad ; \quad \forall i = 1, 2, \dots, n$$

$$\sum_{i=1}^{n} z_i = 1$$

Final Formulation: $a \geq \min\{x_1, x_2, \dots, x_n\}$

$$\Updownarrow$$

$$a + M(1 - z_i) \geq x_i \quad ; \quad \forall i = 1, 2, \dots, n$$

$$\sum_{i=1}^{n} z_i = 1$$

$$z_i = \begin{cases} 1 & ; \; x_i = \min\{x_1, x_2, \dots, x_n\} \\ 0 & ; \; o/w \end{cases}$$

$$z_i \in \{0, 1\}$$

Case 8: $a \leq \max\{x_1, x_2, \dots, x_n\}$ $\Rightarrow$ u is smaller than some of the x_is.
↳ we need binary variable $z_i \in \{0, 1\}$ for each of x_i

$$\Rightarrow z_i = \begin{cases} 1 & ; \; x_i = \max\{x_1, x_2, \dots, x_n\} \\ 0 & ; \; o/w \end{cases}$$

$$\Rightarrow a - M(1 - z_i) \leq x_i \quad ; \quad \forall i = 1, 2, \dots, n$$

$$\sum_{i=1}^{n} z_i = 1$$

Final Formulation: $a \leq \max\{x_1, x_2, \dots, x_n\}$

$$\Updownarrow$$

$$a - M(1 - z_i) \leq x_i \quad ; \quad \forall i = 1, 2, \dots, n$$

$$z_i \in \{0, 1\} \; ; \; \sum_{i=1}^{n} z_i = 1 \; ; \; z_i = \begin{cases} 1 & ; \; x_i = \max\{x_1, x_2, \dots, x_n\} \\ 0 & ; \; o/w \end{cases}$$

Composite Cases:

- $u = |x| \iff \begin{cases} u \geq |x| & \text{[Case 1]} \\ u \leq |x| & \text{[Case 5]} \end{cases} \iff \begin{cases} u - M(1-d) \leq x \leq u \\ u - Md \leq -x \leq u \\ d \in \{0,1\} \\ d = \begin{cases} 1 ; & x \geq 0 \\ 0 ; & o/w \end{cases} \end{cases}$

- $u = \min\{x_1, x_2, \dots, x_n\} \iff \begin{cases} u \leq \min\{x_1, x_2, \dots, x_n\} & \text{[case 3]} \\ u \geq \min\{x_1, x_2, \dots, x_n\} & \text{[case 7]} \end{cases}$

$$\Updownarrow$$

$$\begin{cases} u \leq x_i & ; \forall i = 1, 2, \dots, n \\ u + M(1-z_i) \geq x_i & ; \forall i = 1, 2, \dots, n \\ \sum_{i=1}^{n} z_i = 1 \\ z_i = \begin{cases} 1 ; & x_i = \min\{x_1, x_2, \dots, x_n\} \\ 0 ; & o/w \end{cases} \\ z_i \in \{0, 1\} \end{cases}$$

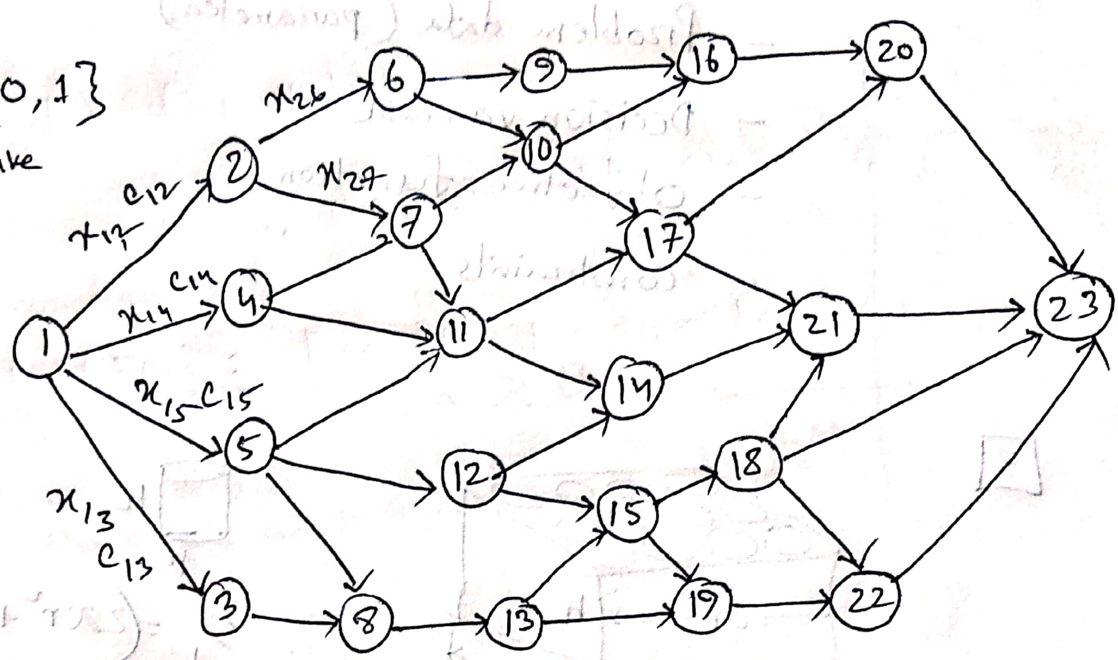
- $u = \max\{x_1, x_2, \dots, x_n\} \iff \begin{cases} u \geq \max\{x_1, x_2, \dots, x_n\} & \text{[case 4]} \\ u \leq \max\{x_1, x_2, \dots, x_n\} & \text{[case 8]} \end{cases}$

$$\Updownarrow$$

$$\begin{cases} u \geq x_i & ; \forall i = 1, 2, \dots, n \\ u - M(1-z_i) \leq x_i & ; \forall i = 1, 2, \dots, n \\ \sum_{i=1}^{n} z_i = 1 \\ z_i = \begin{cases} 1 ; & x_i = \max\{x_1, x_2, \dots, x_n\} \\ 0 ; & o/w \end{cases} \\ z_i \in \{0, 1\} \end{cases}$$

$\boxed{70}$ Problem data → Transportation Network.

$x_{ij} \in \{0,1\}$ whether take the path or not



c_{ij} → Distance values from i to j

$$\min \sum_{(i,j) \in E \text{ (Edges)}} x_{ij} c_{ij}$$

s.t. $x_{12} + x_{13} + x_{14} + x_{15} = 1$

$\cancel{x_{12} (x_{26} + x_{27}) = 1}$

For 2 $\underbrace{\boxed{x_{12}}}_{\text{in}} = \underbrace{\boxed{x_{26} + x_{27}}}_{\text{out}}$ [Balance the flow]

⋮

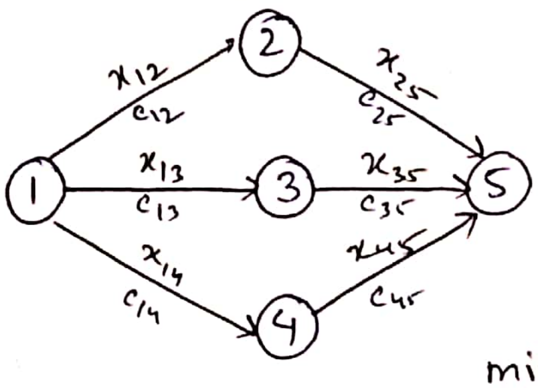
For 11 $x_{4,11} + x_{5,11} + x_{7,11} = x_{11,14} + x_{11,17}$

⋮

For 23 $x_{20,23} + x_{21,23} + x_{18,23} + x_{22,23} = 1$

Shortest Path Problem:

$$x_{ij} \in \{0, 1\}; \quad i,j \in E$$



$$x_{ij} = \begin{cases} 1 ; & (i,j) \in P \\ 0 ; & o/w \end{cases}$$

$$\min \sum_{i,j \in E} x_{ij} c_{ij} \quad \text{i.e} \Rightarrow x_{12}c_{12} + x_{13}c_{13} + x_{14}c_{14} + x_{25}c_{25} + x_{35}c_{35} + x_{45}c_{45}$$

s.t.

$$x_{12} + x_{13} + x_{14} = 1$$

$$x_{25} + x_{35} + x_{45} = 1$$

$$x_{12} - x_{25} = 0$$

$$x_{13} - x_{35} = 0$$

$$x_{14} - x_{45} = 0$$

$$x(a) = \{0, 1\}, \quad a \in A$$

Compact form:

$$\min_x \; c^T x$$

$$Mx = b$$

$$x \in \{0,1\}^{|A|}$$

Here,

$$c = \begin{bmatrix} c_{12} \\ c_{13} \\ c_{14} \\ c_{25} \\ c_{35} \\ c_{45} \end{bmatrix}; \quad x = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{25} \\ x_{35} \\ x_{45} \end{bmatrix}; \quad M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\# dv = \# \text{Edges}$$

Least median absolute error

(y_i, x_i) ; $i = 1, \dots, n$

$\varepsilon_i = |y_i - (ax_i + b)|$; $i = 1, \dots, n$

$n \Rightarrow$ odd ; median index , $p = \dfrac{n+1}{2}$

Sorted : $\varepsilon_{i_1} \leq \varepsilon_{i_2} \leq \dots \leq \varepsilon_{i_p} \leq \varepsilon_{i_{p+1}} \dots \leq \varepsilon_{i_n}$

Elaborate Discussion of some of the Previous Topics

consider distance of x from origin less than a

$|x| \leq a$

+ve $x \leq a$ if $x > 0$ — ①

+ve $-x \leq a$ if $x < 0$ — ②

$\overset{a}{\longleftrightarrow}$ $x(-)$ 0 $x(+)$

$\hookrightarrow -a \leq x \leq a$ — ①

$-x \leq a$ — ②

$\Rightarrow \max\{x, -x\} = |x| \leq a$

Let, $x = 2 > 0$: $\max\{2, -2\} = 2 = |2| \leq a$ [①]

$x = -2 < 0$: $\max\{-2, -(-2)\} = 2 = |-2| \leq a$ [②]

Same idea

Applying this idea in App #2 : $[u_i \equiv a ;\ |y_i - ax_i - b| \equiv x]$

Linear $\left\{ \min\limits_{\substack{a, b, u_i \\ (i=1 \ldots n)}} \sum u_i \ ; \ i = 1, \ldots\ldots, n \right.$

non-Linear $\left\{ \ u_i \geq |y_i - ax_i - b| \right.$

$\Updownarrow$

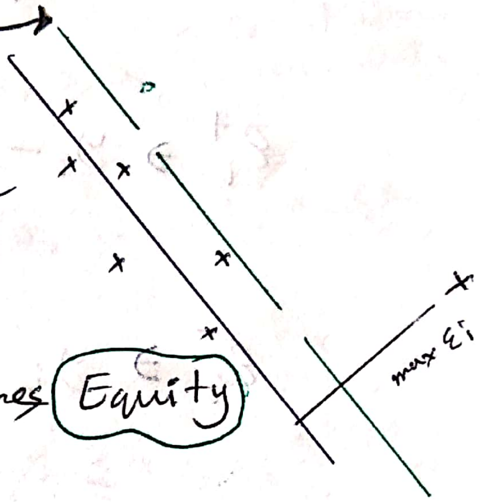
Linear $\left\{ \begin{array}{ll} u_i \geq y_i - ax_i - b ; & i = 1 \ldots n \\ u_i \geq -(y_i - ax_i - b) & i = 1 \ldots \end{array} \right.$

Least max absolute error

$$|\varepsilon_1| \; \dots \; |\varepsilon_n|$$

$$\min \max_{i=1}^{n} |\varepsilon_i| \quad \text{s.t. } a, b \in \mathbb{R}$$

Results to f2. line instead of Blue line



i.e. it ensures Equity

$\downarrow$ Linearization

$$\min_{a,b \in \mathbb{R}} u \Rightarrow [\text{not } u_i \,;\, i = 1, \dots, n]$$

$$\text{s.t. } u = \max_{i=1\dots n} |\varepsilon_i|$$

$$\Updownarrow$$

$$u \geq \max_{i=1\dots n} |\varepsilon_i| \rightarrow \max\{|\varepsilon_i|, \dots |\varepsilon_n|\}$$

$$\Rightarrow u \geq |\varepsilon_i| \,;\, i = 1, \dots, n$$

$$\Rightarrow u \geq |\varepsilon_1| \,;\, u \geq |\varepsilon_2| \,;\, \dots\dots \, u \geq |\varepsilon_n|$$

$$\Rightarrow u \geq \varepsilon_1 \,;\, u \geq -\varepsilon_1 \, \dots\dots \, u \geq \varepsilon_n \,;\, u \geq -\varepsilon_n$$

$$\Updownarrow$$

$$u \geq y_i - (ax_i + b)$$

$$u \geq -y_i + ax_i + b$$

Final Formulation

$$\min u$$

$$\text{s.t. } u \geq y_i - (ax_i + b)$$

$$u \geq -y_i + (ax_i + b)$$

$$a, b \in \mathbb{R}$$

$(x_i, y_i) \quad i = 1, \dots, n$

$y_i \rightarrow$ demand

$\varepsilon_i^+ \ni \underbrace{y_i - (ax_i + b)}_{\text{estimate}} > 0 \Rightarrow$ Underestimate [Can cause black out for electricity] $\ast\ast$ More severe

$\varepsilon_i^- \ni y_i - (ax_i + b) < 0 \Rightarrow$ Over estimate [waste of money] $\ast$

$$\min_{a,b \in \mathbb{R}} \sum_{i=1}^{n} \omega_1 \varepsilon_i^+ + \omega_2 \varepsilon_i^- \qquad \omega_1, \omega_2 \geq 0$$

$\varepsilon_i^+ = \max\{y_i - (ax_i + b), 0\}$

$\varepsilon_i^- = \max\{-y_i + ax_i + b, 0\}$

non-Linear ; e.g. $\max\{x, 0\}$

The formulation on R.H.S. ensures that ε_i (either $\varepsilon^+/\varepsilon^-$) is not repeated by including 0 in the set. Example:

Let, $y_i - (ax_i + b) = -2$

$\therefore \varepsilon_i^+ = \max\{-2, 0\} = 0$ [makes $\omega_1 \varepsilon_i^+ = 0$]

$\varepsilon_i^- = \max\{-(-2), 0\} = 2$ [$\omega_2 \varepsilon_i^-$ exists only]

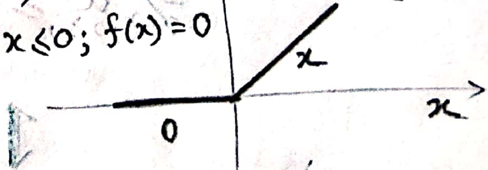
Note: Any Step-function is non-Linear

Linearization:-

$x > 0;\ f(x) = x$

$x \leq 0;\ f(x) = 0$

$\max\{x, 0\}$



$$\Leftrightarrow \min_{a,b \in \mathbb{R}} \sum_{i=1}^{n} \omega_1 u_i^+ + \omega_2 u_i^-$$

$u_i^+ = \max\{y_i - (ax_i + b), 0\}$

$u_i^- = \max\{-y_i + (ax_i + b), 0\}$

$\Updownarrow$ Relaxation

$u_i^+ \geq y_i - (ax_i + b)\ ;\ u_i^+ \geq 0$

$u_i^+ \geq \max\{y_i - (ax_i + b), 0\}$

$u_i^- \geq \max\{-y_i + (ax_i + b), 0\} \Leftrightarrow u_i^- \geq -y_i + (ax_i + b)\ ;\ u_i^- \geq 0$

$\boxed{31}$

$\min\{x_1, \ldots x_n\} \geq a$

$\Rightarrow$ each of x_i is greater than a

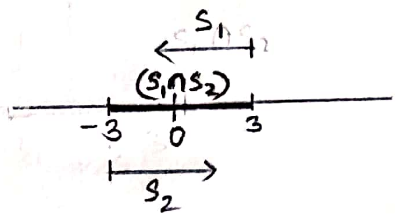
$x_i \geq a, \ldots\ldots, x_n \geq a$ } Set of Linear constraints

$\max\{x_1, \ldots\ldots, x_n\} \leq a$ i.e. $x_i \leq a, \ldots\ldots, x_n \leq a$

$\Rightarrow$ each of x_i is less than a

□ $|x| \leq \cancel{3}^{a} \Rightarrow \underbrace{x \leq 3}_{S_1} \Rightarrow -\cancel{3}^{-a} \leq x \leq \cancel{3}^{a} \; (S_1 \cap S_2)$

$\underbrace{-x \leq 3}_{S_2}$



Diff. cases

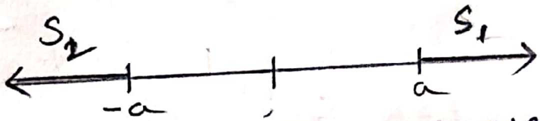
$x \in S_1, S_2$ simultaneously

$|x| \geq \cancel{3}^{a}$ $\overset{\text{non-negative}}{a \geq 0}$ [Hard to deal!]

if $x \geq 0$; $x \geq \cancel{3}^{a}$ S_1

o/w (if $x \leq 0$) ; $-x \geq \cancel{3}^{a}$ S_2

or, $x \leq -\cancel{3}^{-a}$



$x \in S_1$ OR S_2 i.e. $S_1 \cup S_2$

Disjoint sets

Since, $x \in S_1$ OR $x \in S_2$; we need a binary variable "d" $\in \{0,1\}$

$d = \begin{cases} 1; & \text{if } x \geq 0 \Rightarrow x \geq a \\ 0; & \text{o/w} \Rightarrow -x \geq a \end{cases}$

$x \geq a \xrightarrow[\text{relation}]{\text{to hold this}}$ if $d=1$, $x \geq a + \cancel{g(d=1)}^{0}$

$d = 0$, $\underbrace{x \geq a + g(d=0)}$

this should be redundant

i.e. $g(0) = -M$

$\therefore g(d) = -M(1-d)$

$-x \geq a$ to hold this relation $\rightarrow$

If

$d = 0$; $-x \geq a + \cancel{\bar{g}(d=0)}^{0}$

$d = 1$; $\underbrace{-x \geq a + \bar{g}(d=1)}$,

this should be redundant

i.e. $\bar{g}(1) = -M$ $\therefore \bar{g}(d) = -Md$

$\therefore |x| \geq a \Leftrightarrow \begin{cases} x \geq a - M(1-d) \\ -x \geq a - Md \end{cases}$

where

MILP problem

$d \in \{0,1\}$ & $d = \begin{cases} 1; & x \geq 0 \\ 0; & \text{o/w} \end{cases}$

Easy to Deal with

1. $\min |x| \Leftrightarrow \min u \;;\; u \geq |x|$
(obj. func.)
$\hookrightarrow u \geq x$
$u \geq -x$

2. $|x| \leq a$ (Const.) $\Leftrightarrow$ $x \leq a$, $-x \leq a$

3. $\min \{x_1, \ldots x_n\} \geq a \Leftrightarrow x_i \geq a \;;\; i = 1 \ldots n$

4. $\max \{x_1, \ldots x_n\} \leq a \Leftrightarrow x_i \leq a \;;\; i = 1 \ldots n$

NOT Easy to Deal with

5. $\max |x|$

6. $|x| \geq a \rightarrow$ $x \geq a \;;\; \text{if } x \geq 0$ OR $-x \geq a \;;\; o/w$ $\Leftrightarrow$ $\boxed{\begin{array}{l} x \geq a - M(1-d) \\ -x \geq a - Md \\ d \in \{0,1\} \\ d = \begin{cases} 1; & x \geq 0 \\ 0; & o/w \end{cases} \end{array}}$

from Last class

7. $\min \{x_1, \ldots x_n\} \leq a$

8. $\max \{x_1, \ldots 0\} \geq a$

Case 5: $\max |x| \Leftrightarrow \max u$ s.t. $u \leq |x|$ [$u \geq |x|$ is not suitable since $(\max u) \rightarrow \infty$]

$\Updownarrow$

Case 6: $|x| \geq a$ [solve/reformulate accordingly]

Case 7: $\min\{x_1, x_2, x_3, \ldots, x_n\} \leq a \Leftrightarrow \exists k \in \{1,2,\ldots\ldots n\}$

$\Leftrightarrow \quad | \; x_k \leq a$

i.e. at least one element x_k is $\leq a$

Now, for each x_i we need a binary variable z_i

if $x_i = \min\{x_1 \ldots x_n\}$: $z_i = 1 \Rightarrow x_i \leq a + \cancel{g(z_i=1)}^{\,0}$

o/w $\quad : z_i = 0 \Rightarrow \underbrace{x_i \leq a + \cancel{g(z_i=0)}^{\,M}}_{\text{This should be redundant}}$ i.e $g(0) = M$

Therefore,

$$x_i \leq a + (1 - z_i) M$$

* $\sum z_i = 1$ [∵ At least one x_i should be $\leq a$]

where, $z_i \in \{0,1\}$

$$z_i = \begin{cases} 1, & \text{if } x_i = \min\{x_1, \ldots\ldots x_n\} \\ 0, & \text{o/w} \end{cases}$$

*Note, that, there can be cases where $\sum z_i \geq 1$ is true i.e. more than one x_i's are $\leq a$! But $\sum z_i = 1$ will give the same optimal solⁿ where $\sum z_i \geq 1$ can create additional/unnecessary restrictions.

Case 8: $\max\{x_1, x_2, \ldots\, x_n\} \geq 0$ [Try]

(case 2)

$a \geq |x| \iff a \geq x \ldots$ ①
$a \geq -x \ldots$ ②

$a = |x|$

$a \leq |x| \iff a - M(1-d) \leq x \ldots$ ③
$a - Md \leq -x \ldots$ ④

(case 6)

$\Updownarrow$

where, $d \in \{0, 1\}$

$d = \begin{cases} 1; & \text{if } x \geq 0 \\ 0; & \text{o/w} \end{cases}$

4 constraints

① ~ ④

① & ③ can be combined as:

② & ④

$x \leq a \leq x + M(1-d)$ [From ① & ③]

$-x \leq a \leq -x + Md$ [From ② & ④]