Chapter 7 - Discrete Probability

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Sample Spaces, events, probability distributions
Independence, conditional probability
7.3 Bayes theorem & applications
7.4 Random variables & expectation, linearity of expectations
7.4 Markov & Chebyshev's inequalities

Ask the question: *"What are the possible outcomes of this experiment?"*

Sample Space Ω the possible outcomes, set $\Omega = \{H, T\}^7$ (for a coin flipped 7 times)

Event

The probability of an event, E is the number of outcomes in E divided by the total number of outcomes;

$$P(E) = \frac{|E|}{|S|}$$

assuming all outcomes equally likely

Probability Distributions

$P : \Omega \longrightarrow [0,1]$ such that $\sum_{s \in \Omega} P(s) = 1$

- For event $E \subseteq \Omega$, the **complement event**, $\bar{E} = \Omega - E$

1 Suppose $E_0, E_1, E_2 \dots$ are a finite/countable sequence of pairwise disjoint events from the sample space Ω. In other words, $E_i \subseteq \Omega$, and $E_i \cap E_j = \emptyset$ for all $i, j \in \mathbb{N}$ then;

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

- Additionally, for each event $E \subseteq \Omega$, $P(\bar{E}) = 1 - P(E)$

Conditional Probability of E given F, denoted $P(E|F)$ is;

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Independent Probability

Independent if knowing whether 1 occured does not alter the probability of the others

$$P(A \cap B) = P(A)P(B)$$

Pairwise Independent if every pair of events are independent

Mutually Independent if for every subset $J \subseteq \{1, \dots, n\}$, $P\left(\bigcap_{j \in J} E_j\right) = \prod_{j \in J} P(E_j)$

- ★ mutual independence implies pairwise independence but not vice versa
- typically when we refer to two events as independent we mean mutually

Biased Coins & Bernoulli trials

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- Assumption that trials are mutually independent

1 The probability of exactly k successes in n (mutually independent) Bernoulli trials, with probability p of success and $q = (1-p)$ of failure in each trial is;

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}$$

THE BINOMIAL DISTRIBUTION

Proof. We can associate each B. trial with outcomes $\Omega = \{H, T\}^n$.

- Each sequence $s = (s_1, \dots s_n)$ with exactly k heads and $n-k$ tails occurs with probability $p^k q^{n-k}$. There are $\binom{n}{k}$ such sequences with exactly k heads

Random Variables

neither a variable nor random

Random Variable is a function $X: \Omega \to \mathbb{R}$ that assigns a real value to each outcome

- For random variable $X: \Omega \to \mathbb{R}$ we write $P(X = r)$ as short hand for the probability $P(\{s \in \Omega \mid X(s) = r\})$. The distribution of a random variable X is given by the set of pairs $\{r, P(X = r) \mid r$ is in the range of $X\}$

Biased Coins & Geometric Distribution

A random variable $X: \Omega \to \mathbb{N}$, is said to have a geometric distribution with parameter p, $0 \le p \le 1$, if for all positive integers $k \ge 1$, $P(X = k) = (1-p)^{k-1} p$

7.4.2 Monte Carlo Algorithms

Probabilistic Algorithms make random choices.

- **Monte Carlo Algorithms** Research!

3 If the probability that an element chosen at random from S does not have a particular property is less than 1, there exists an element in S with this property

THE PROBABILISTIC METHOD

7.3 BAYES THEOREM

$$p(F \mid E) = \frac{p(E \mid F)\,p(F)}{p(E \mid F)\,p(F) + p(E \mid \bar{F})\,p(\bar{F})}$$

Generalised; Suppose $E, F_1 \dots F_n$ events in a sample space with $\bigcup_i^n F_i = \Omega$ and $F_i \cap F_j = \emptyset$. Then:

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$$P(F_j|E) = \frac{P(E|F)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

7.4 EXPECTED VALUE & VARIENCE

Expected Value of a random variable is the sum over all elements in a sample space of the product of the probability of the element and the value of the random variable at this element

↳ The weighted average of the values of a random variable

expectation/mean

$$\boxed{E(X) = \sum_{s \in S} p(s)X(s)}$$ — box

1 If X is a random variable and $p(X=r)$ is the probability that $X=r$, so that $p(X=r) = \sum_{s \in S, X(s)=r} p(s)$, then; $E(X) = \sum_{r \in X(S)} p(X=r)r$

2 The expected number of successes when n mutually independent Bernoulli trials are done, where p is the propability of success on each trial is np.

3 If X_i, $i = 1, 2, 3 \dots n$ with n a postitive integer are random variables on S, and if a & b are real numbers then;

i) $E(X_1 + X_2 + \cdots X_n) = E(X_1) + E(X_2) + \cdots E(X_n)$

ii) $E(aX+b) = aE(X) + b$

LINEARITY OF EXPECTATION

7.4.5 The Geometric Distro

(Often) in the case of a random variable with infinitly many outcomes

Geometric Distribution A random variable has geometric distribution with parameter p if $p(X=k) = (1-p)^{k-1}$ for $k = 1, 2 \dots$, where $0 \leq p \leq 1$

7.4.6 Independence of Random Variables

The random variables X and Y on a sample space S are **independent** iff

$$p(X=r_1 \wedge Y=r_2) = p(X=r_1) \cdot p(Y=r_2)$$

↳ If X and Y are independent random variables on a sample space Ω Then; $E(XY) = E(X)E(Y)$

7.4.7 Varience

Varience Let X be a random variable on a sample space S. The

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Varience Let X be a random variable on a sample space S. The varience of X, denoted by $V(X)$ is:, $V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$

- standard deviation $\sigma(X) = \sqrt{V(X)}$

$$V(X) = E(X^2) - E(X)^2$$ — useful identity

Markovs Inequality

"What is the probability that the value of the r.v, X, is far from its expectation?"

For a non-negative random variable, $X: \Omega \to R$, where $X(s) \geq 0$ for all $s \in \Omega$, for any positive real number $a > 0$;

$$P(X \geq a) \leq \frac{E(X)}{a}$$

MARKOVS INEQ.

Let $X: \Omega \to R$ be any random variable, and let $r > 0$ be any positive real number, Then;

$$P(|X - E(X)| \geq r) \leq \frac{V(X)}{r^2}$$

CHEBYSHEVS INEQ.

Birthday Paradox

Suppose that each of $m \geq 1$ pigeons independently and uniformly at random enter 1 of $n \geq 1$ pigeon holes. If;

$$m \geq (1.1175 \cdot \sqrt{n}) + 1$$

then the probability that 2 pigeons go in the same pigeon hole is greater than ½.

HALFWAY POINT OF B.D PARADOX

Ramsey Numbers

Suppose that in a group of 6 people, every pair are either friends or enemies
→ Then, there are either 3 mutual friends or 3 mutual enemies.

For any positive integer, k, there is a positive integer n such that in any undirected graph with n or more vertices:

- either there are k vertices that are mutually adjacent, meaning they graph with n or more vertices:

- either there are k vertices that are mutually adjacent, meaning they form a k-clique
- or there are k vertices that are mutually non adjacent, forming a k-independent set

RAMSEY'S THEOREM

- For each integer $k \geq 1$, let $R(k)$ be the smallest integer $n \geq 1$ such that every undirected graph with n or more vertices has either a k-clique or a k independent set as an induced subgraph.
  - The number $R(k)$ are called diagonal Ramsey numbers.
  - $R(k) \leq 2^{2k-1}$