

An Introduction to Small-world Networks: Exploring Theory & Real-world Applications

A brief introduction to the history, theory and mathematical underpinnings small-world networks with an in-depth review of three real-world applications of this theory. This paper discusses small-world topology in the brain and nervous system, then in public transportation networks and finally in artificial intelligence. It conclude discussing the extensive scale of the small-world phenomenon.

Key Words: small-world phenomenon, complex-system, efficiency, network-analysis.

Any group or system of interconnecting people or things is a network. Your brain is a network, roads are a network, the internet is a network. Small-world networks (SWN) display very interesting properties - they encompass 'cliques' of densely interconnected sub networks between nodes, while simultaneously allowing information to traverse the entire network efficiently. Small worldliness could be shown in many naturally occurring networks (Basett & Bullmore, 2017).

First investigated in sociology, Stanley Milgram sent 96 packages to random people on the opposite coast of America with instructions to get the package to a target friend his. (Milgram, 1967). The package contained only the targets name, occupation & address, with instructions to send the package to someone they personally knew who might be socially closer. Remarkably many of the packages reached their targets, and with a surprisingly short 'chain' length (5.9 on average). Most interesting was how the network could be navigated easily on the individual level using little, mostly local information (Travers & Milgram, 1969).

Thirty years later, Watts and Strogatz created a model to capture this phenomenon (Watts & Strogatz, 1998). Starting with a ring lattice of nodes locally connected, giving a high 'clustering-

coefficient' (cliquishness), ' C ', they randomly rewired nodes non-locally. They found they didn't need to rewire many links before the average shortest path length ' L ' dropped dramatically.

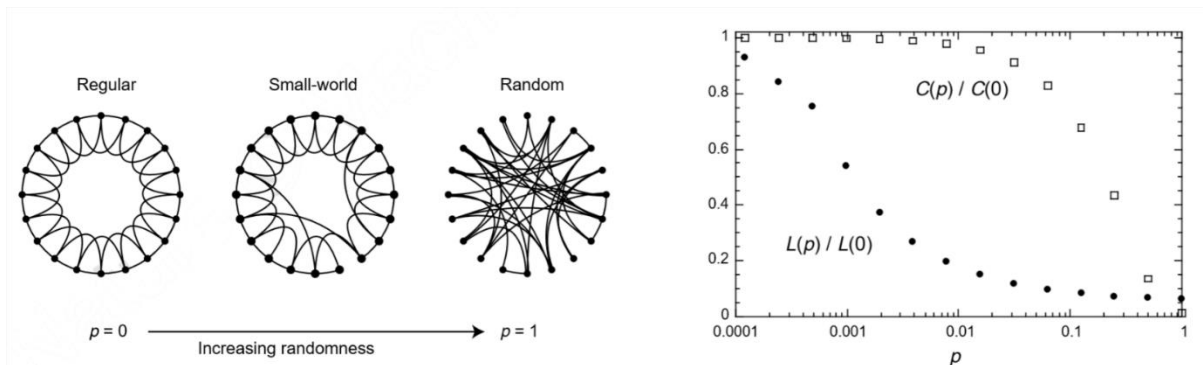


Figure 1: SWN are a mix between the extremes of regular and random networks defined by high clustering coefficient and short average path length. Note in the right-hand diagram a logarithmic horizontal scale has been used to resolve the rapid drop in $L(p)$, corresponding to the onset of the SWN phenomenon (Watts & Strogatz, 1998)

This is the small world phenomena. Watts and Strogatz speculated correctly that it would hold for many natural and technological networks. Because this architecture can be defined mathematically, SWN's have been fundamental in understanding many diverse applications (Bassett & Bullmore, 2017). This paper will look at three real world applications of this theory.

According to Bassett & Bullmore's 2006 paper 'Small-world brain networks', brain anatomical connectivity is sparse, locally clustered, and with a few long-range connections mediating short path lengths between any pair of regions – a small world network. (Bassett & Bullmore, 2006)

Furthermore, they showed this pattern of topology at micro and macro scale.

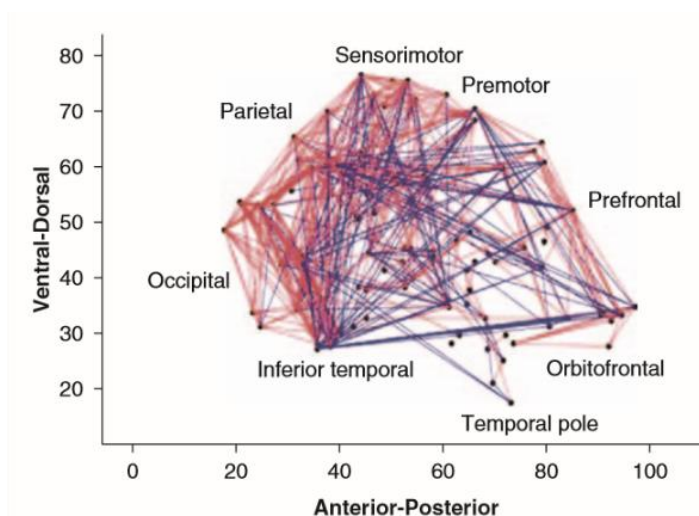


Figure 2: map showing functional connectivity using nonmetric functional scaling, a method that places node in close proximity if connection between them is strong, and far apart if weak.

They stated by analysing historical neuronographic data showing cat and macaque monkey cortical networks share small world characteristics (Hilgetag & Kaiser, 2004). They then looked at the first demonstration of small-world properties in human brain functional neural networks derived from fMRI data (Salvador, Suckling,

Coleman, & Pickard, 2005). Their analysis showed dense local connections between regions with clustering negatively correlated with distance from node. This intriguingly produced a new way to classify regions of the brains as either highly specialised (locally clustered) or more integrative.

Given this strong evidence for SWN they inquired as to why this architecture had evolved. They postulate that SWN's could be highly evolutionarily economical by minimising wiring costs while supporting high dynamic complexity. Artificial intelligence research supports this showing small world structure offers non trivial advantages in terms of both communication and computation in neural networks (Simard, Nadeau, & Kroger, 2005). In a review paper ten years later, the same authors expand on how demonstrations of small-world topology has been replicated across a wide range of neuroscience studies (Bassett & Bullmore, 2017).

Latora & Marchiori in their paper titled "Is the Boston subway a Small-world network?" investigated a pragmatic example of a complex system and tried to apply the concept of small-world networks (Latora & Marchiori, 2002). However, they found the Watts & Strogatz variables 'L' and 'C' were not well defined for their system. C is poorly defined because many nodes only have one neighbour so the clustering coefficient for a certain station could be 0/0. For similar reasons this would lead to $L = \infty$ in many cases for nodes not fully connected to the network. They solved this problem by proposing an alternative formalism. They introduced a new variable ' ϵ ' for efficiency and defined it as $\epsilon_{ij} = 1/d_{ij}$, where d is the path distance. This avoided the problem of path length being infinite as if $L = \infty \rightarrow \epsilon = 0$. They also took into account the capacity of the transport system to give edges a weighted value.

They were surprised to discover just how efficient the network was for long distance transportation they actual path was only 37% less efficient than the 'ideal' situation where there would be a direct tunnel between every station. However, the fault tolerance was very low – this means a temporary problem on a single node (station) would have dramatic effects on the wider network – particularly in the local area. In network terms this means the equivalent of the clustering coefficient being low

with poor connectivity to local areas, and resultantly failing to optimise to the benefits of small world topology. The researches questioned whether it would be better to have more tunnels (edges) in the network to improve this 'local efficiency'. However, they discover this would be incredibly cost inefficient. The current system was highly cost effective having 63% local efficiency of a fully tunnelled network for just 0.2% of the cost. They concluded that it would be a very poor investment to secure high fault tolerance in this way.

The solution came upon considering that the underground is not a closed system with other transportation link such as the over-ground bus network providing strong local connectivity and safeguard against node failure. They re-ran the mathematical analysis to discover that the combined public transportation network displayed impressive long distance and local efficiency – an exemplar small-world network. Overall, they concluded their introduction of an efficiency measure ' ϵ ' allowed a more generalisable definition of small world network to be applied.

Erkaymaz, Ozer and Matjaz studied the application of small-world networks in artificial intelligence in their paper "Impact of small-world network topology on the conventional artificial neural network for the diagnosis of diabetes" (Erkaymaz, Ozer, & Matjaz, 2017). They were investigating the performance of small-world feedforward neural networks in detecting diabetes, a computational diagnosis technique that is widely used in medical practice. In particular they were comparing the Watts-Strogatz model for creating SWN's to the Newman-Watts model. Usually SWN's are constructed on the Watts-Strogatz rewiring algorithm, this study also investigates the effects of the later developed Newman & Watts rewiring algorithm which differs only ever so slightly in the fact that the rewiring process does not replace any of the old edges, only add to them (Newman & Watts, 1999).

It had already been previously shown that small-world feedforward artificial neural networks (FFANN) have better learning performance than regular or random ones (Li, Xu, Zhang, & Wang, 2013). The researchers of this paper also did a recent study where they showed SW-FFANN to

outperform the conventional FFANN in diagnosis on diabetes (Erkaymaz & Ozer, 2016). For their new research they used a four layered FFANN involving 8 input, one output neuron and two hidden layers and compared a regular FFANN, a Watts-Strogatz SW-FFANN and a Newman-Watts SW-FFANN.

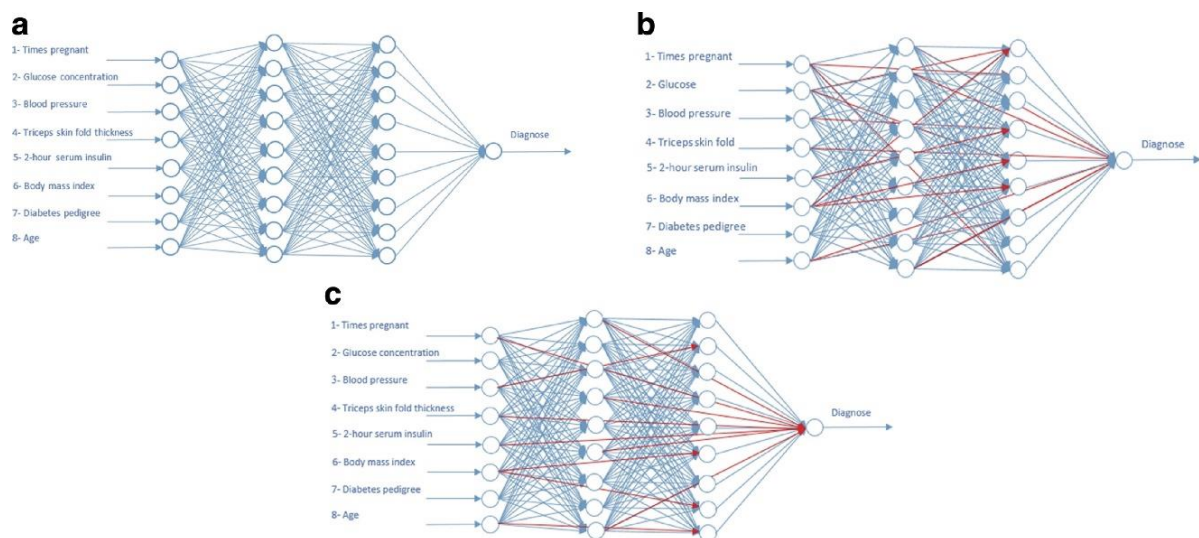


Figure 3: The tested networks; (a) the conventional FFANN, (b) Watts-Strogatz SW-FFANN, (c) Newman-Watts SW-FFANN

They tested the performance of the network with varying rewiring number (RN) for both algorithms. It was found that for both models' peak performance was around 64/65 RN. The conventional neural network diagnosed diabetes with an accuracy of 83.3%, the Watts-Strogatz model outperformed this with an accuracy of 91.6% and the Newman-Watts performed better still with an accuracy of 93.06% - proving to be the best model ever created for this dataset (Pima Indians Diabetic Dataset, 768 samples – normal: 500, diabetic: 268).

The authors hope this will encourage more work in intelligent algorithms for decision making.

Although tested on a very specific dataset the authors are confident their findings are generalisable to areas such as modelling memory in the brain, sports analytics etc. they also plan to test other rewiring algorithms in future.

In conclusion, since its conceptual discovery, small-worldliness has become ubiquitous characteristic of many complex systems (Basett & Bullmore, 2017). Understanding a networks architecture is

highly valuable as its structure shapes its function - defining how information flows through the network. The ability to mathematically model small-worldliness has allowed it to become highly pragmatic. The research space has exploded as SWN's have become fundamental in many diverse applications far beyond the limited scope touched in this essay; such as game theory (Li & Cao, 2009), epidemiology (Moore & Newman, 2000) and linguistics (Cancho & Sole, 2001). I think it's fair to say from wider reading that small-world network phenomenon has spearheaded a new wider 'science of networks' and looks to find its way into ever more domains of enquiry (Schnettler, 2009). I think it's likely that this process will bring more sophistication to the field as it matures. For example, weighted graphs are capable of retaining more biologically relevant information and so more appropriate to the increasingly complex data on brain connectivity. This new science of networks remains a fast-growing area of research and I predict more progress as more people apply small world models to their own fields.

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