Predicate Logic

Dr Paolo Guagliardo

THE UNIVERSITY of EDINBURGH
informatics

Fall 2020 (v20.1.0)

Logic in general

Logics are formal languages for

- representing **what we know** about the world
- **reasoning** about this knowledge (draw conclusions from it)

Two components:

Syntax defines the sentences in the language

Semantics defines the **meaning** of the sentences

Used in many areas of Computer Science:

- Artificial Intelligence
- Semantic Web
- Software & Hardware verification
- **Databases**
- ... many many others

Motivation for Predicate Logic

Atomic formulas of propositional logic are **too atomic**

- statements that may be true or false
- but have no internal structure

First-order (or **predicate**) **logic** (FOL) overcomes this limitation

- atomic formulas are statements about relationships between objects

Predicates and constants

Consider the statements:

- *Mary is female*
- *John is male*
- *Mary and John are siblings*

In propositional logic these are just atomic propositions:

- `mary-is-female`
- `john-is-male`
- `mary-and-john-are-siblings`

In first-order logic atomic statements use **predicates**, with constants as arguments:

- Female(mary)
- Male(john)
- Sibling(mary, john)

Variables and quantifiers

Consider the statements:

- *Everybody is male or female*
- *A male is not a female*

In FOL predicates may have variables as arguments, whose value may be bound by **quantifiers**:

- $\forall x \; \big(\, \mathsf{Male}(x) \vee \mathsf{Female}(x) \,\big)$
- $\forall x \; \big(\, \mathsf{Male}(x) \rightarrow \neg \mathsf{Female}(x) \,\big)$

Syntax of FOL: terms

Countably infinite supply of

$$\begin{array}{rl} \text{variable symbols :} & x, y, z, \ldots \\ \text{constant symbols :} & a, b, c, \ldots \\ \text{predicate symbols :} & P, Q, R, \ldots \qquad \text{(with associated arities)} \end{array}$$

$$\begin{array}{lrl} \textbf{Term} & t := x & \text{variable} \\ & \mid a & \text{constant} \end{array}$$

Syntax of FOL: formulas

Formula	$\phi := P(t_1, \ldots, t_n)$	atomic formula
	$\mid \neg\phi$	negation
	$\mid \phi \wedge \phi$	conjunction
	$\mid \phi \vee \phi$	disjunction
	$\mid \phi \rightarrow \phi$	implication
	$\mid \forall x\ \phi$	universal quantification (if x occurs **free** in ϕ)
	$\mid \exists x\ \phi$	existential quantification (if x occurs **free** in ϕ)

Free variables

Variables that are not in the scope of any quantifier
A variable that is not free is **bound**

Example

$$\forall x\ \big(\, R(y, z) \wedge \exists y\ (\neg P(y, x) \vee R(y, z))\,\big)$$

Variables in blue are free, the others are bound

Semantics of FOL: Interpretations

A formula may be true (or false) w.r.t. a given **interpretation**

First-order structure $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$

- Δ non-empty domain of objects (universe)
- $\cdot^{\mathcal{I}}$ gives meaning to constant/predicate symbols

$$a^{\mathcal{I}} \in \Delta$$
$$R^{\mathcal{I}} \subseteq \Delta^n$$

Variable assignment ν

Maps each variable to an object in Δ

Notation: $\nu[x/d]$ is the same as ν except that $x \mapsto d$

Semantics of FOL: terms

Interpretation of terms under $(\mathcal{I}, \nu)$

$$\begin{aligned} x^{\mathcal{I},\nu} &= \nu(x) \\ a^{\mathcal{I},\nu} &= a^{\mathcal{I}} \end{aligned}$$

Semantics of FOL: formulas

$(\mathcal{I}, \nu) \models \phi$ means interpretation $(\mathcal{I}, \nu)$ satisfies formula ϕ

$$
\begin{array}{lcl}
\mathcal{I}, \nu \models P(t_1, \ldots, t_n) & \iff & (t_1^{\mathcal{I},\nu}, \ldots, t_n^{\mathcal{I},\nu}) \in P^{\mathcal{I}} \\
\mathcal{I}, \nu \models \neg\phi & \iff & \mathcal{I}, \nu \not\models \phi \\
\mathcal{I}, \nu \models \phi \wedge \psi & \iff & \mathcal{I}, \nu \models \phi \text{ and } \mathcal{I}, \nu \models \psi \\
\mathcal{I}, \nu \models \phi \vee \psi & \iff & \mathcal{I}, \nu \models \phi \text{ or } \mathcal{I}, \nu \models \psi \\
\mathcal{I}, \nu \models \phi \rightarrow \psi & \iff & \text{if } \mathcal{I}, \nu \models \phi \text{ then } \mathcal{I}, \nu \models \psi \\
\mathcal{I}, \nu \models \forall x\ \phi & \iff & \text{for every } d \in \Delta\colon \mathcal{I}, \nu[x/d] \models \phi \\
\mathcal{I}, \nu \models \exists x\ \phi & \iff & \text{there exists } d \in \Delta \text{ s.t. } \mathcal{I}, \nu[x/d] \models \phi
\end{array}
$$

Equality

Equality is a special predicate

$t_1 = t_2$ is true under a given interpretation

if and only if

t_1 and t_2 refer to the same object

That is,

$$\mathcal{I}, \nu \models t_1 = t_2 \quad \iff \quad t_1^{\mathcal{I},\nu} = t_2^{\mathcal{I},\nu}$$

Examples

Take any first order structure $\mathcal{I}$ such that

$$\begin{aligned}
\Delta &= \{d_1, \ldots, d_n\} \text{ for } n > 1 \\
a^{\mathcal{I}} &= d_1 \\
b^{\mathcal{I}} &= d_1 \\
\mathsf{Block}^{\mathcal{I}} &= \{d_1\} \\
\mathsf{Red}^{\mathcal{I}} &= \Delta
\end{aligned}$$

and any assignment ν such that $x \mapsto d_1$ and $y \mapsto d_2$

Does $(\mathcal{I}, \nu)$ satisfy

- $\mathsf{Block}(a) \land \mathsf{Block}(b) \land \neg(a = b)$?
- $\mathsf{Block}(x) \land \mathsf{Red}(y) \land (x = y)$?
- $\forall x \, \big(\mathsf{Block}(x) \to \mathsf{Red}(x)\big)$?
- $\mathsf{Block}(c) \lor \neg\mathsf{Block}(c)$?

Satisfiability and validity

An interpretation $(\mathcal{I}, \nu)$ is a **model** of ϕ if $(\mathcal{I}, \nu) \models \phi$

A formula is

satisfiable	if it has a model
unsatisfiable	if it has no models
falsifiable	if there is some interpretation that is not a model
valid	(i.e., a tautology) if every intepretation is a model

Equivalence

Two formulas are **logically equivalent** (written $\phi \equiv \psi$) if they have the same models

That is, for all interpretations $(\mathcal{I}, \nu)$

$$\mathcal{I}, \nu \models \phi \quad \Longleftrightarrow \quad \mathcal{I}, \nu \models \psi$$

Questions:

- Are $P(x)$ and $P(y)$ logically equivalent?
- What about $\forall x\ P(x)$ and $\forall y\ P(y)$?

Universal quantification

Everyone taking DBS is smart:

$$\forall x\ \big(\,\text{Takes}(x, \text{dbs}) \rightarrow \text{Smart}(x)\,\big)$$

Typically $\rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x\ \big(\,\text{Takes}(x, \text{dbs}) \wedge \text{Smart}(x)\,\big)$$

means "Everyone takes DBS and everyone is smart"

Existential quantification

Someone takes DBS and fails:

$$\exists x \ \big(\mathsf{Takes}(x, \mathsf{dbs}) \land \mathsf{Fails}(x, \mathsf{dbs})\big)$$

Typically $\land$ is the main connective with $\exists$

Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x \ \big(\mathsf{Takes}(x, \mathsf{dbs}) \rightarrow \mathsf{Fails}(x, \mathsf{dbs})\big)$$

is true if there is anyone who does not take DBS

Properties of quantifiers

- $\forall x \, \forall y \ \phi$ is the same as $\forall y \, \forall x \ \phi$
- $\exists x \, \exists y \ \phi$ is the same as $\exists y \, \exists x \ \phi$
- $\exists x \, \forall y \ \phi$ is **not the same** as $\forall x \, \exists y \ \phi$

Example

$$\exists x \, \forall y \ \mathsf{Loves}(x, y)$$

means "There is somebody who loves everyone in the world"

$$\forall y \, \exists x \ \mathsf{Loves}(x, y)$$

means "Everyone is loved by somebody (not necessarily the same)"

Quantifier duality

Each can be expressed using the other:

$$\forall x \text{ Likes}(x, \text{cake}) \equiv \neg\exists x \neg\text{Likes}(x, \text{cake})$$

Everybody likes cakes is the same as saying

There is nobody who does not like cake

$$\exists x \text{ Likes}(x, \text{broccoli}) \equiv \neg\forall x \neg\text{Likes}(x, \text{broccoli})$$

Somebody likes broccoli is the same as saying

Not everybody does not like broccoli

Equivalences (1)

Commutativity	$\phi \vee \psi \equiv \psi \vee \phi$
	$\phi \wedge \psi \equiv \psi \wedge \phi$
Associativity	$(\phi \vee \psi) \vee \chi \equiv \phi \vee (\psi \vee \chi)$
	$(\phi \wedge \psi) \wedge \chi \equiv \phi \wedge (\psi \wedge \chi)$
Distributivity	$\phi \wedge (\psi \vee \chi) \equiv (\phi \wedge \psi) \vee (\phi \wedge \chi)$
	$\phi \vee (\psi \wedge \chi) \equiv (\phi \vee \psi) \wedge (\phi \vee \chi)$
Idempotence	$\phi \vee \phi \equiv \phi$
	$\phi \wedge \phi \equiv \phi$
Absorption	$\phi \vee (\phi \wedge \psi) \equiv \phi$
	$\phi \wedge (\phi \vee \psi) \equiv \phi$

Equivalences (2)

Double Negation	$\neg\neg\phi \equiv \phi$
De Morgan	$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$
	$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$
Implication	$\phi \to \psi \equiv \neg\phi \vee \psi$

Equivalences (3)

$$
\begin{aligned}
(\forall x\ \phi) \wedge \psi &\equiv \forall x\ (\phi \wedge \psi) && \text{if } x \text{ is not free in } \psi \\
(\forall x\ \phi) \vee \psi &\equiv \forall x\ (\phi \vee \psi) && \text{if } x \text{ is not free in } \psi \\
(\exists x\ \phi) \wedge \psi &\equiv \exists x\ (\phi \wedge \psi) && \text{if } x \text{ is not free in } \psi \\
(\exists x\ \phi) \vee \psi &\equiv \exists x\ (\phi \vee \psi) && \text{if } x \text{ is not free in } \psi \\
(\forall x\ \phi) \wedge (\forall x\ \psi) &\equiv \forall x\ (\phi \wedge \psi) \\
(\exists x\ \phi) \vee (\exists x\ \psi) &\equiv \exists x\ (\phi \vee \psi)
\end{aligned}
$$

$$
\begin{aligned}
\neg\forall x\ \phi &\equiv \exists x\ \neg\phi \\
\neg\exists x\ \phi &\equiv \forall x\ \neg\phi
\end{aligned}
$$