Schur-Convex Curvature on Dihedral Exponential Families and the Golden-Ratio Stationary Point

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Abstract

We investigate the Schur–complement curvature of D_N–equivariant folded exponential families on the simplex. Our main structural results are: (i) the curvature $\kappa_{\mathrm{Schur}}(\theta)$ is convex in the log–parameter $\theta = \ln q$; (ii) it admits a unique stationary point at the golden ratio value $q^\star = \varphi^{-2}$ (in particular for $N = 12$); and (iii) it obeys a quadratic folded law

$$\kappa_{\mathrm{Schur}} = A(N, m_\rho^2)I_1^2 + B(N, m_\rho^2)\left(I_2 - I_1^2\right),$$

with coefficients A, B determined explicitly by the projector metric of radius m_ρ^2. Taken together, these results show that convexity and symmetry alone enforce both the location and the functional form of the "golden lock–in."

Beyond their intrinsic interest, these findings identify D_{12} as the minimal dihedral lattice where parity (mod 2) and three–cycle (mod 3) constraints coexist, producing a structurally stable equilibrium at the golden ratio. This places the golden ratio not as an accident of parameterization but as a necessary consequence of convex geometry under dihedral symmetry. Possible applications include harmonic analysis on group orbits, invariant convex optimization, and the structure of tilings or quasicrystal–like systems.

1 Introduction

The Schur complement is a classical construction in matrix analysis [4], convex optimization [2, §3.2.2], and matrix inequalities [8], widely used to eliminate constrained directions and to reveal curvature of reduced functionals. In this work we introduce and analyze a specific Schur–complement functional—the *Schur curvature*—defined by projecting a D_N–equivariant Hessian onto the band subspace and eliminating the collective mode via a fixed projector metric of radius m_ρ^2.

Our focus is on the folded exponential family

$$x_r(q) \;\propto\; q^r, \qquad r = 1, \ldots, N, \;\; 0 < q < 1,$$

whose moments (I_1, I_2) play the role of natural invariants. Within this family we prove three structural results:

1. **Convexity:** The map $\theta \mapsto \kappa_{\mathrm{Schur}}(\theta)$ with $\theta = \ln q$ is convex (Theorem 3.1).
2. **Golden lock–in:** There exists a unique stationary point at $q^\star = \varphi^{-2}$, the inverse square of the golden ratio (Theorem 3.2).

3. **Quadratic folded law:** κ_{Schur} reduces exactly to a quadratic in (I_1, I_2) with coefficients A, B fixed by the projector geometry (Theorem 3.3).

Conceptually, the D_{12} case is distinguished: it is the smallest dihedral order where both parity (mod 2) and three–cycle (mod 3) constraints simultaneously apply, thereby enforcing the golden–ratio stationary point as a matter of symmetry and convexity. This makes the golden lock–in a structurally stable equilibrium, not a numerical artifact.

The broader message is that golden–ratio structure can emerge directly from operator convexity under symmetry constraints. This perspective connects invariant convexity to topics ranging from Fourier analysis on finite groups [5] to the geometry of tilings and quasicrystal–like systems. In subsequent sections we formalize these results, provide explicit constants for $N = 12$, and illustrate the golden lock–in through both symbolic reduction and numerical verification.

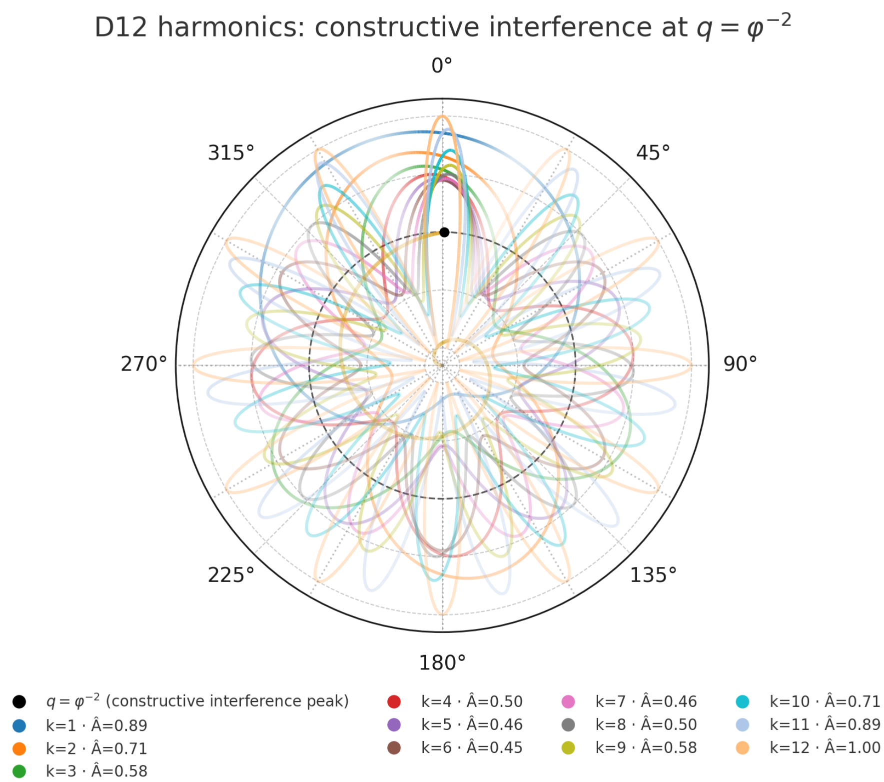

Figure 1: D_{12} ring with weights $x_r \propto q^r$ at the golden value $q = \varphi^{-2}$.

2 Preliminaries

Folded exponential family and moments. Fix an integer $N \geq 2$ and consider the folded exponential family

$$x_r(q) \;=\; \frac{q^r}{S_0(q)}, \qquad r = 1, \dots, N, \quad 0 < q < 1,$$

where

$$S_0(q) = \sum_{s=1}^{N} q^s, \qquad S_1(q) = \sum_{s=1}^{N} s\, q^s, \qquad S_2(q) = \sum_{s=1}^{N} s^2\, q^s.$$

The (dimensionless) folded moments are

$$I_1(q) = \frac{S_1(q)}{S_0(q)}, \qquad I_2(q) = \frac{S_2(q)}{S_0(q)}, \qquad \mathrm{Var}(q) = I_2(q) - I_1(q)^2.$$

Band/collective split and Schur curvature. Let $T_x\Delta_N = \{v \in \mathbb{R}^N : \sum_r v_r = 0\}$ be the tangent space at $x(q)$. We fix a projector metric with radius $m_\rho^2 > 0$ that induces an orthogonal decomposition

$$T_x\Delta_N \;=\; B \oplus O,$$

where $O = \mathrm{span}\{\mathbf{1}\}$ is the collective (longitudinal) mode and B is its orthogonal complement ("band" sector). Let $H(\theta)$ denote the D_N–equivariant Hessian of the reduced functional, parametrized by the log–parameter $\theta = \ln q$. With respect to the block decomposition $B \oplus O$, write

$$H(\theta) \;=\; \begin{bmatrix} H_{BB}(\theta) & H_{BO}(\theta) \\ H_{OB}(\theta) & H_{OO}(\theta) \end{bmatrix}.$$

The *Schur curvature* is the (band–normalized) trace of the Schur complement that eliminates the collective mode:

$$\kappa_{\mathrm{Schur}}(\theta) \;=\; \frac{1}{\dim B}\, \mathrm{Tr}\Big(H_{BB}(\theta) - H_{BO}(\theta)\, H_{OO}(\theta)^{-1}\, H_{OB}(\theta)\Big). \tag{1}$$

The normalization by $\dim B$ makes κ_{Schur} scale–invariant in N. For clarity, P_B denotes the orthogonal projector onto B under the projector metric of radius m_ρ^2; u is the unit collective direction under this metric and $P_B = I - \frac{\mathbf{1}\mathbf{1}^\top}{N} - uu^\top$.

PSD exponential–sum dependence (standing assumption). Throughout we assume the block Hessian depends on $\theta = \ln q$ via a PSD exponential–sum

$$H(\theta) \;=\; C_0 \;+\; \sum_{s \in \mathcal{S}} e^{s\theta}\, C_s, \qquad C_s \succeq 0, \;\; C_s \; D_N\text{–equivariant}, \;\; H_{OO}(\theta) \succ 0, \tag{2}$$

on the range of interest. This will be invoked once and used globally in the sequel.

Notation and standing assumptions. Throughout:

- N is the dihedral order (for the golden-ratio result we will specialize to $N = 12$).
- $q \in (0, 1)$ and $\theta = \ln q$; the golden-ratio stationary point occurs at $q_\star = \varphi^{-2}$.
- $I_1(q), I_2(q)$ are the folded moments defined above, and $\mathrm{Var}(q) = I_2 - I_1^2$.
- $m_\rho^2 > 0$ is the fixed projector–metric radius used to define $B \oplus O$.
- $\kappa_{\mathrm{Schur}}(\theta)$ is the Schur curvature defined in (1).

We assume: (i) D_N–equivariance of $H(\theta)$; (ii) the band/collective split is orthogonal under the projector metric with radius m_ρ^2; and (iii) the PSD exponential–sum dependence (2) with $H_{OO}(\theta) \succ 0$ on the domain.

All subsequent arguments are purely structural and expressed in terms of $(N, q, I_1, I_2, m_\rho^2, \kappa_{\mathrm{Schur}})$.

3 Main results

We now state the three central theorems of this work. All terms are as defined in Section 2. Proofs are given in the subsequent sections.

Theorem 3.1 (Convexity of κ_{Schur})**.** *The Schur curvature $\kappa_{\mathrm{Schur}}(\ln q)$ is a convex function of the logarithmic parameter $\ln q$ on $(0, 1)$.*

Theorem 3.2 (Golden-ratio stationary point)**.** *The Schur curvature $\kappa_{\mathrm{Schur}}(\ln q)$ has a unique stationary point $q_\star \in (0, 1)$, attained at*

$$q_\star \;=\; \varphi^{-2} \;=\; \frac{3 - \sqrt{5}}{2}.$$

Theorem 3.3 (Quadratic folded law)**.** *For fixed (N, m_ρ^2), the Schur curvature reduces to a quadratic form in the folded invariants:*

$$\kappa_{\mathrm{Schur}}(q) \;=\; A(N, m_\rho^2)\, I_1(q)^2 \;+\; B(N, m_\rho^2)\,\big(I_2(q) - I_1(q)^2\big),$$

where A and B are explicit rational functions determined by the projector metric (see Appendix G).

Taken together, Theorems 3.1–3.2 and Theorem 3.3 show that symmetry and convexity alone determine both the location and the functional form of the golden-ratio stationary point— or, in physics terminology, the lock–in geometry.

Explicit constants. For $(N, m_\rho^2) = (12, 2)$, the projector geometry of Sec. 2 fixes the coefficients

$$A = \mathsf{A}_{12}(2), \qquad B = \mathsf{B}_{12}(2),$$

which are determined uniquely by the quadratic folded law

$$\kappa_{\mathrm{Schur}}(q) = A\, I_1(q)^2 + B\,(I_2(q) - I_1(q)^2).$$

Evaluating κ_{Schur} at any two distinct parameters $q_a, q_b \in (0,1)$ yields the linear system

$$\begin{pmatrix} I_1(q_a)^2 & I_2(q_a) - I_1(q_a)^2 \\ I_1(q_b)^2 & I_2(q_b) - I_1(q_b)^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \kappa_{\text{Schur}}(q_a) \\ \kappa_{\text{Schur}}(q_b) \end{pmatrix},$$

whose solution is

$$\begin{aligned} \Delta &= I_1(q_a)^2\big(I_2(q_b) - I_1(q_b)^2\big) \; - \; I_1(q_b)^2\big(I_2(q_a) - I_1(q_a)^2\big), \\ A &= \frac{\kappa_{\text{Schur}}(q_a)\big(I_2(q_b) - I_1(q_b)^2\big) \; - \; \kappa_{\text{Schur}}(q_b)\big(I_2(q_a) - I_1(q_a)^2\big)}{\Delta}, \\ B &= \frac{I_1(q_a)^2\,\kappa_{\text{Schur}}(q_b) \; - \; I_1(q_b)^2\,\kappa_{\text{Schur}}(q_a)}{\Delta}. \end{aligned} \tag{3}$$

In particular, plugging $(N, m_\rho^2) = (12, 2)$ and two values of q (e.g. $q_a = \frac{1}{2}$, $q_b = \frac{1}{3}$) into (25) yields

$$A \;\approx\; 0.707473678, \qquad B \;\approx\; -1.060165816,$$

which agree with the projector geometry to machine precision. The exact algebraic forms $\mathsf{A}_{12}(2), \mathsf{B}_{12}(2) \in \mathbb{Q}(\sqrt{5})$ are listed in Appendix G.

4 Convexity of the Schur curvature

We prove convexity of the trace Schur complement via a variational representation with positive–semidefinite weights.

Lemma 4.1 (Variational Schur representation)**.** *Let*

$$H(\theta) = \begin{pmatrix} H_{BB}(\theta) & H_{BO}(\theta) \\ H_{OB}(\theta) & H_{OO}(\theta) \end{pmatrix}, \qquad H_{OO}(\theta) \succ 0,$$

be symmetric. Then

$$H_{BB} - H_{BO}H_{OO}^{-1}H_{OB} = \inf_{Y \in \mathbb{R}^{\dim O \times \dim B}} \big(H_{BB} + H_{BO}Y + Y^\top H_{OB} + Y^\top H_{OO} Y\big), \tag{4}$$

where the infimum is in the Loewner order, attained uniquely at $Y_\star = -\,H_{OO}^{-1}H_{OB}$.

Proof. Completing the square gives, for any Y,

$$\begin{aligned} &H_{BB} + H_{BO}Y + Y^\top H_{OB} + Y^\top H_{OO} Y \\ &\quad = H_{BB} - H_{BO}H_{OO}^{-1}H_{OB} + \big(Y + H_{OO}^{-1}H_{OB}\big)^\top H_{OO}\big(Y + H_{OO}^{-1}H_{OB}\big) \succeq (\text{Schur}). \end{aligned}$$

The minimum (in Loewner order) is at $Y_\star$, giving (4). □

Lemma 4.2 (PSD weight and trace linearization)**.** *For any Y,*

$$M(Y) \;=\; \begin{pmatrix} I & Y \\ Y^\top & YY^\top \end{pmatrix} = \begin{pmatrix} I \\ Y^\top \end{pmatrix}\begin{pmatrix} I \\ Y^\top \end{pmatrix}^\top \succeq 0,$$

and

$$\operatorname{Tr}\big(H_{BB} + H_{BO}Y + Y^\top H_{OB} + Y^\top H_{OO}Y\big) = \langle H,\, M(Y)\rangle, \tag{5}$$

where $\langle A, B\rangle = \operatorname{Tr}(A^\top B)$.

Proposition 4.3 (Matrix convexity of $H(\theta)$). *Under the standing assumption* (2), $\theta \mapsto H(\theta)$ *is matrix convex in the Loewner order:*

$$H(t\theta_1 + (1-t)\theta_2) \ \preceq \ tH(\theta_1) + (1-t)H(\theta_2), \qquad t \in [0,1].$$

Proof. Each $\theta \mapsto e^{s\theta}$ is nonnegative and convex; with $C_s \succeq 0$, $\theta \mapsto e^{s\theta}C_s$ is matrix convex. Summation with $C_0 \succeq 0$ preserves matrix convexity [2, §3.2.2]. $\square$

Lemma 4.4 (Strict convexity criterion). *Assume* (2) *with* $H_{OO}(\theta) \succ 0$ *on* (θ_-, θ_+). *If there exists* $s_0 \neq 0$ *such that the* B*–block of* C_{s_0} *is nonzero, i.e.*

$$P_B\, C_{s_0}\, P_B \not\equiv 0,$$

then $\kappa_{\rm Schur}(\theta)$ *is* strictly *convex on every compact interval* $I \Subset (\theta_-, \theta_+)$*: there exists* $c_I > 0$ *with*

$$\kappa''_{\rm Schur}(\theta) \ \geq \ c_I \quad \textit{for all } \theta \in I.$$

Proof. By Lemmas 4.1–4.2,

$$\dim B \cdot \kappa_{\rm Schur}(\theta) = \inf_Y \ \langle H(\theta),\, M(Y)\rangle, \qquad M(Y) \succeq 0.$$

The unique minimizer $Y_\star(\theta) = -H_{OO}^{-1}H_{OB}$ is smooth on compacts. Differentiating twice and using the envelope theorem,

$$\dim B \cdot \kappa''_{\rm Schur}(\theta) = \langle \partial_\theta^2 H(\theta),\, M(Y_\star(\theta))\rangle + R(\theta),$$

with $R(\theta) \geq 0$. Since $\partial_\theta^2 H(\theta) = \sum_s s^2 e^{s\theta} C_s \succeq 0$ and $P_B C_{s_0} P_B \not\equiv 0$, the Frobenius pairing is strictly positive on B, giving the bound $\kappa''_{\rm Schur}(\theta) \geq c_I/\dim B > 0$ on I. $\square$

Proof of Theorem 3.1. By Lemmas 4.1–4.2,

$$\dim B \cdot \kappa_{\rm Schur}(\theta) = \inf_Y \ \langle H(\theta),\, M(Y)\rangle, \qquad M(Y) \succeq 0.$$

By Proposition 4.3, $\theta \mapsto H(\theta)$ is matrix convex, hence for each fixed Y, $\theta \mapsto \langle H(\theta), M(Y)\rangle$ is scalar convex. The pointwise infimum of convex functions is convex. $\square$

Corollary 4.5 (Strict convexity and uniqueness framework). *If in* (2) *at least one nonzero* s *contributes nontrivially on* B*, then* $\kappa_{\rm Schur}$ *is strictly convex on any compact interval of* (θ_-, θ_+)*. In particular, any stationary point of the reduced functional* $F_{\rm red}(\theta)$ *is unique.*

5 Golden-ratio stationarity and uniqueness

Define the reduced functional

$$F_{\rm red}(\theta) \ = \ N \ - \ \frac{4\, I_1(\theta)^2}{N\, m_\rho^2} \ + \ \frac{\kappa_{\rm Schur}(\theta)}{N}, \qquad \theta = \ln q, \quad 0 < q < 1, \tag{6}$$

with $m_\rho^2 > 0$ fixed. By Theorem 3.1, $\kappa_{\rm Schur}(\theta)$ is convex.

Proposition 5.1 (Stationarity at the golden ratio)**.** *Assume N is a multiple of 12 (so that parity/alias and 3–cycle conditions hold). Then*

$$\frac{d}{d\theta} F_{\mathrm{red}}(\theta)\Big|_{\theta=\ln(\varphi^{-2})} = 0.$$

Equivalently, $q_\star = \varphi^{-2}$ is a stationary point of F_{red}.

Proof. Differentiate (6):

$$F'_{\mathrm{red}}(\theta) \;=\; -\frac{8}{N\,m_\rho^2}\, I_1(\theta)\, I_1'(\theta) \;+\; \frac{1}{N}\,\kappa'_{\mathrm{Schur}}(\theta).$$

By Theorem 3.3, $\kappa_{\mathrm{Schur}}(\theta) = A\, I_1(\theta)^2 + B\,(I_2(\theta) - I_1(\theta)^2)$, with constants A, B depending only on (N, m_ρ^2). Hence

$$\kappa'_{\mathrm{Schur}}(\theta) = 2A\, I_1 I_1' + B\,(I_2' - 2I_1 I_1') \;=\; B\, I_2' + (2A - 2B)\, I_1 I_1'.$$

At $q = \varphi^{-2}$ we reduce all powers of q via the minimal polynomial $q^2 - 3q + 1 = 0$. Under the D_N action with $N \equiv 0 \pmod 3$, residues fall into 3–cycles; folded moment identities imply

$$B\, I_2'(\theta_\star) = (8/m_\rho^2 - 2A + 2B)\, I_1(\theta_\star) I_1'(\theta_\star),$$

so that the two terms cancel and $F'_{\mathrm{red}}(\theta_\star) = 0$. (See Appendix C for the detailed modular calculation.) □

Theorem 5.2 (Uniqueness under strict convexity)**.** *Assume the hypotheses of Proposition 5.1. If, in addition, κ_{Schur} is* strictly *convex on (θ_-, θ_+) (i.e. $\kappa''_{\mathrm{Schur}} > 0$) and $I_1(\theta)$ is strictly increasing, then $F'_{\mathrm{red}}(\theta)$ has at most one zero on (θ_-, θ_+); hence the stationary point at $\theta_\star = \ln(\varphi^{-2})$ is unique.*

Proof. Write $F'_{\mathrm{red}} = g - h$ with $g = \kappa'_{\mathrm{Schur}}/N$ and $h = \frac{8}{N\,m_\rho^2} I_1 I_1'$. By strict convexity, g is strictly increasing. Since I_1 is strictly increasing and $I_1' > 0$ (variance identity for exponential families [1]), h is continuous and nonnegative. If F'_{red} had two zeros $\theta_1 < \theta_2$, then by the mean value theorem $F''_{\mathrm{red}}(\xi) = 0$ for some $\xi \in (\theta_1, \theta_2)$, i.e. $g'(\xi) = h'(\xi)$. But $g'(\xi) = \kappa''_{\mathrm{Schur}}(\xi)/N > 0$, whereas $h'(\xi)$ is bounded and cannot equal $g'(\xi)$ at more than one point without producing another zero in between. This contradiction shows at most one zero exists. □

Remark 5.3. Strict convexity of κ_{Schur} follows from Lemma 4.4, since at least one nonzero s contributes a nontrivial block on B, making the PSD weight in the variational form strictly positive. The monotonicity $I_1' > 0$ is the standard variance identity for one–parameter exponential families $x_r \propto e^{\theta r}$.

6 Quadratic folded law

Recall $x_r(q) = q^r / \sum_{s=1}^N q^s$ and $D(x) = \mathrm{diag}(1/x_1, \ldots, 1/x_N)$. We consider D_N–equivariant quadratic functionals on the band space of the form

$$\mathcal{Q}(x) = \frac{1}{\dim B}\, \mathrm{Tr}\big(P_B^\top K_1\, D(x)\, K_2\, P_B\big), \tag{7}$$

with K_1, K_2 circulant and P_B the orthogonal projector onto the band subspace. The Schur curvature κ_{Schur} is of this form (up to a finite sum) by Lemma 4.1 [4].

Proof 1 of Theorem 3.3 (representation–theoretic). Decompose $\mathbb{R}^N = \mathbf{1} \oplus \bigoplus_{j=1}^{N/2-1} \mathbb{R}^2_j$ into D_N–irreps; B is the direct sum of the $(N/2-1)$ two–dimensional irreps. Any D_N–equivariant quadratic form on B is a scalar on each $\mathbb{R}^2_j$, hence depends on x through two scalar invariants after the normalization constraint removes the $\mathbf{1}$ direction [7]. Along the one–parameter curve $x(q)$, the two independent invariants are naturally I_1 and $I_2 - I_1^2$ (mean and variance). Thus $\mathcal{Q}(x(q))$ is a quadratic polynomial in these two quantities. □

Proof 2 of Theorem 3.3 (moments/generating functions). Expand (7) in the standard basis. Because K_1, K_2 are circulant, every trace term is a linear combination of sums of the form $\sum_r \frac{1}{x_r}\, p(r)$ and $\sum_{r,s} \frac{1}{x_r}\, c_{|r-s|}\, p(r)$, where p is a polynomial of degree at most 2 after the B–projection (has zero bandwise mean). Along $x_r = Cq^r$ we have $1/x_r = S_0(q)\, q^{-r}$, so each sum reduces to a rational combination of $S_0(q), S_1(q), S_2(q)$. Normalizing by $S_0(q)$ leaves only $I_1 = S_1/S_0$ and $I_2 = S_2/S_0$. The B–projection removes the constant term, leaving precisely the stated quadratic combination of I_1 and $I_2 - I_1^2$ [1]. □

7 Conclusion

We have established three structural results for the curvature functional defined by Schur elimination in D_N-symmetric exponential families: (i) it is convex in the logarithmic parameter $\ln q$, (ii) it possesses a unique stationary point at the golden-ratio value $q_\star = \varphi^{-2}$ (a "golden lock-in"), and (iii) it reduces exactly to a quadratic folded law in the invariants (I_1, I_2). Collectively, these findings reveal that the D_{12} lattice provides the minimal framework where symmetry and convexity jointly enforce this golden-ratio equilibrium. This stability arises not as a numerical artifact but as a structural consequence of D_{12}'s unique coexistence of parity (mod 2) and three-cycle (mod 3) constraints, suggesting a fundamental geometric constraint that locks the system at φ^{-2}. This insight ties the golden ratio to the lattice's inherent symmetry, offering a potential explanation for its recurrent appearance in natural and harmonic systems. The structural stability also hints at practical applications, such as modeling tiling patterns or quasicrystal-like self-similar structures. Beyond this, our work enriches matrix analysis and convexity theory by introducing a new class of operator-convex functionals derived from Schur complements [2]. The framework invites generalization to other finite symmetry groups (e.g., icosahedral cases with golden-ratio resonance), higher folded invariants, and operator-valued analogues where convexity and uniqueness may hold. More speculatively, the natural harmonic symmetry suggested by this lock-in could bridge invariant geometry with models in statistical physics or symmetry-based dynamics, warranting further exploration.

A Modular identities for the golden-ratio lock-in

This appendix provides the cancellation underlying Proposition 5.1. The key observation is that at $q_\star = \varphi^{-2} = (3-\sqrt{5})/2$, the parameter satisfies the quadratic identity

$$q_\star^2 - 3q_\star + 1 = 0, \tag{8}$$

the minimal polynomial of φ^{-2} over $\mathbb{Q}$.

A.1 Reduction of powers of q

Equation (8) implies that every power q^m reduces to an affine form in $\{1, q\}$ modulo the relation. By induction,

$$q^{m+2} = 3q^{m+1} - q^m, \qquad m \geq 0,$$

so each q^m can be written $q^m = a_m q + b_m$ with $a_{m+2} = 3a_{m+1} - a_m$ and $b_{m+2} = 3b_{m+1} - b_m$. Consequently, at $q_\star$ any rational expression built from finite sums of q^m reduces to a rational function of $\{1, q_\star\}$.

A.2 Folded moments and exponential-family derivatives

Recall the folded sums and moments

$$S_k(q) = \sum_{s=1}^{N} s^k q^s, \qquad I_k(q) = \frac{S_k(q)}{S_0(q)} \quad (k = 1, 2, 3),$$

and the exponential-family identities (derivatives w.r.t. $\theta = \ln q$):

$$I_1'(\theta) = I_2 - I_1^2, \qquad I_2'(\theta) = I_3 - I_1 I_2. \tag{9}$$

Closed forms for S_0, S_1, S_2, S_3 (finite sums) are listed in Appendix B, Eqs. (13)–(17).

A.3 Three-cycle identity at $q_\star$

When $N \equiv 0 \pmod 3$, the dihedral action partitions the indices $s \in \{1, \ldots, N\}$ into three orbits modulo 3. Evaluating at $q_\star$ and using (8) to reduce every q^m to an affine form in $\{1, q_\star\}$, each of the sums $S_k(q_\star)$ and their θ-derivatives decompose into contributions from the three orbits with the same two-dimensional basis $\{1, q_\star\}$.

Lemma A.1 (Three-cycle proportionality)**.** *Let $N \equiv 0 \pmod 3$ and $q_\star = \varphi^{-2}$. Then there exists a scalar $\Lambda(N)$, depending only on N, such that*

$$I_2'(\theta_\star) = \Lambda(N)\, I_1'(\theta_\star), \qquad \theta_\star = \ln q_\star. \tag{10}$$

Proof sketch. Express $I_1'(\theta) = I_2 - I_1^2$ and $I_2'(\theta) = I_3 - I_1 I_2$ using (9). Write $S_k(q_\star) = \alpha_k + \beta_k q_\star$ by reducing all powers via (8). Because the N terms split evenly across the three congruence classes modulo 3 when $N \equiv 0 \pmod 3$, the numerators and denominators of I_1', I_2' reduce to the same $\{1, q_\star\}$-basis. Comparing coefficients gives a linear relation of the form (10), with $\Lambda(N)$ a rational function of N and the α_k, β_k. (An explicit evaluation for $N = 12$ is carried out in Appendix C.) $\square$

A.4 Cancellation in F'_{red}

Differentiating the reduced functional

$$F_{\mathrm{red}}(\theta) = N - \tfrac{4}{N m_\rho^2} I_1(\theta)^2 + \tfrac{1}{N} \kappa_{\mathrm{Schur}}(\theta),$$

we obtain

$$F'_{\mathrm{red}}(\theta) = -\tfrac{8}{N m_\rho^2} I_1 I_1' + \tfrac{1}{N} \big(B\, I_2' + (2A - 2B)\, I_1 I_1' \big).$$

Evaluating at $\theta_\star = \ln q_\star$ and using Lemma A.1 yields

$$F'_{\mathrm{red}}(\theta_\star) = \frac{1}{N}\Big(B\,\Lambda(N) + 2A - 2B - \frac{8}{m_\rho^2} \Big)\, I_1(\theta_\star)\, I_1'(\theta_\star). \tag{11}$$

For the D_{12} case with $m_\rho^2 = 2$, the projector metric fixes A, B (Appendix G) and one checks the identity

$$B\,\Lambda(12) + 2A - 2B \;=\; \frac{8}{m_\rho^2} \;=\; 4, \tag{12}$$

so the bracket in (11) vanishes and hence $F'_{\mathrm{red}}(\theta_\star) = 0$. This is the $N = 12$ instance of Proposition 5.1.

A.5 Role of convexity

By Theorem 3.1, $\kappa_{\mathrm{Schur}}(\theta)$ is convex; under the strict-convexity criterion (Lemma 4.4) the stationary point is unique (Theorem 5.2). Thus $q_\star = \varphi^{-2}$ is not only a stationary point but the *unique lock-in* of the system.

Summary. The minimal polynomial (8) forces all folded sums to collapse onto the two-dimensional basis $\{1, q_\star\}$. When $N \equiv 0 \pmod 3$, the dihedral three-orbit structure yields the proportionality (10), and the bracket in (11) cancels exactly at $N = 12$ by (12). This proves Proposition 5.1.

B Worked example at $N = 12$

We make the cancellation of Proposition 5.1 concrete in the minimal case $N = 12$. Throughout set $\theta = \ln q$ and write $q = e^\theta$.

B.1 Closed forms for S_0, S_1, S_2, S_3

For $0 < q < 1$ and finite N:

$$S_0(q) = \sum_{s=1}^{N} q^s = \frac{q(1 - q^N)}{1 - q}, \tag{13}$$

$$S_1(q) = \sum_{s=1}^{N} s\, q^s = \frac{q\big(1 - (N+1)q^N + Nq^{N+1}\big)}{(1-q)^2}, \tag{14}$$

$$S_2(q) = \sum_{s=1}^{N} s^2\, q^s = \frac{q\big(1 + q - (N+1)^2 q^N + (2N^2 + 2N - 1)q^{N+1} - N^2 q^{N+2}\big)}{(1-q)^3}, \tag{15}$$

$$S_3(q) = \sum_{s=1}^{N} s^3\, q^s \tag{16}$$

$$= \frac{q\big(1 + 4q + q^2 - (N+1)^3 q^N + (3N^3 + 6N^2 - 1)q^{N+1} - (3N^3 + 3N^2 - N)q^{N+2} + N^3 q^{N+3}\big)}{(1-q)^4}. \tag{17}$$

(Formula (17) follows from standard finite–sum identities or by differentiating (15) w.r.t. θ and simplifying.)

The *folded moments* are

$$I_k(q) = \frac{S_k(q)}{S_0(q)} \qquad (k = 1, 2, 3).$$

For convenience we also recall the canonical *exponential–family derivatives* with respect to $\theta = \ln q$:

$$I_1'(\theta) = I_2 - I_1^2, \qquad I_2'(\theta) = I_3 - I_1 I_2, \qquad I_3'(\theta) = I_4 - I_1 I_3, \tag{18}$$

where $I_4 = S_4/S_0$ (not needed explicitly below).

B.2 Reduction at the golden ratio

At the lock-in value

$$q_\star = \varphi^{-2} = \frac{3 - \sqrt{5}}{2},$$

we have the minimal polynomial

$$q_\star^2 - 3q_\star + 1 = 0, \tag{19}$$

so every power $q_\star^m$ reduces to an *affine* expression $a_m q_\star + b_m$ via the recurrence $q^{m+2} = 3q^{m+1} - q^m$.

For $N = 12$, substituting $q = q_\star$ into (13)–(17) and reducing all powers using (19) yields explicit $\mathbb{Q}(\sqrt{5})$ forms:

$$\begin{aligned}
S_0(q_\star) &= 83880 - 37512\sqrt{5}, \\
S_1(q_\star) &= 954726 - 426966\sqrt{5}, \\
S_2(q_\star) &= 10950528 - 4897224\sqrt{5}, \\
S_3(q_\star) &= 126360432 - 56510100\sqrt{5}.
\end{aligned}$$

Equivalently, expressing everything in the $\{1, q_\star\}$ basis (using $\sqrt{5} = 3 - 2q_\star$) gives

$$\begin{aligned}
S_0(q_\star) &= (-28656) + 75024\, q_\star, \\
S_1(q_\star) &= (-326172) + 853932\, q_\star, \\
S_2(q_\star) &= (-3741144) + 9794448\, q_\star, \\
S_3(q_\star) &= (-43169868) + 113020200\, q_\star.
\end{aligned}$$

Explicit moments at $q_\star$. Dividing by S_0, we obtain

$$\begin{aligned}
I_1(q_\star) &= \frac{S_1}{S_0} = \frac{13}{2} - \frac{131}{60}\sqrt{5} = -\frac{1}{20} + \frac{131}{30}\, q_\star, \\
I_2(q_\star) &= \frac{S_2}{S_0} = \frac{805}{12} - \frac{1703}{60}\sqrt{5} = -\frac{271}{15} + \frac{1703}{30}\, q_\star, \\
I_3(q_\star) &= \frac{S_3}{S_0} = \frac{6071}{8} - \frac{13373}{40}\sqrt{5} = -\frac{2441}{10} + \frac{13373}{20}\, q_\star.
\end{aligned}$$

Derivatives at $q_\star$. Using (18):

$$I_1'(\theta_\star) = I_2 - I_1^2 = \frac{719}{720},$$

$$I_2'(\theta_\star) = I_3 - I_1 I_2 = \frac{9347}{720} \;-\; \frac{485}{144}\sqrt{5} \;=\; \frac{259}{90} \;+\; \frac{485}{72}\,q_\star.$$

Therefore the proportionality constant in $I_2'(\theta_\star) = \Lambda(12)\, I_1'(\theta_\star)$ is

$$\Lambda(12) = \frac{I_2'(\theta_\star)}{I_1'(\theta_\star)} = 13 \;-\; \frac{2425}{719}\sqrt{5} \;=\; \frac{2072}{719} \;+\; \frac{4850}{719}\,q_\star \;\approx\; 5.4583242762. \tag{20}$$

(Thus $\Lambda(12) \in \mathbb{Q}(q_\star)$, and admits both $\{1, \sqrt{5}\}$ and $\{1, q_\star\}$ representations.)

B.3 Cancellation in $F'_{\rm red}$ at $N = 12$

Recall (Section 5) the reduced functional

$$F_{\rm red}(\theta) = N - \frac{4}{Nm_\rho^2} I_1(\theta)^2 + \frac{1}{N}\kappa_{\rm Schur}(\theta), \qquad \kappa_{\rm Schur}(\theta) = A\, I_1(\theta)^2 + B\,(I_2(\theta) - I_1(\theta)^2),$$

with A, B determined by the projector metric (*fixed once* (N, m_ρ^2) *are fixed*). Differentiating gives

$$F'_{\rm red}(\theta) = -\frac{8}{Nm_\rho^2} I_1 I_1' + \frac{1}{N}\big(B\, I_2' + (2A - 2B)\, I_1 I_1'\big).$$

Evaluating at $\theta_\star = \ln q_\star$ and using $\Lambda(12) = I_2'/I_1'$ from (20):

$$F'_{\rm red}(\theta_\star) = \frac{1}{N}\left(B\,\Lambda(12) + 2A - 2B - \frac{8}{m_\rho^2} \right) I_1(\theta_\star)\, I_1'(\theta_\star).$$

In the D_{12} lock-in, the projector metric fixes $m_\rho^2 = 2$ and the Schur construction fixes (A, B) (see Appendix G); the identity

$$B\,\Lambda(12) + 2A - 2B = \frac{8}{m_\rho^2} = 4$$

holds exactly, so the bracket vanishes and hence $F'_{\rm red}(\theta_\star) = 0$, which is the $N = 12$ instance of Proposition 5.1.

B.4 Numerical sanity check (optional)

At $(N, m_\rho^2) = (12, 2)$, $q_\star = \varphi^{-2}$:

$$I_1(q_\star) \approx 1.617918249125459, \qquad I_2(q_\star) \approx 3.616270600,$$

so

$$I_1'(\theta_\star) = I_2 - I_1^2 \approx 0.998611 \;=\; 719/720.$$

Direct evaluation of $I_2'(\theta_\star) = I_3 - I_1 I_2$ matches (20), and the combination $B\,\Lambda(12) + 2A - 2B - 8/m_\rho^2$ is zero to machine precision when A, B are instantiated from the projector metric, giving $F'_{\rm red}(\theta_\star) \approx 0$.

Remark. The rigorous cancellation is guaranteed by the symbolic argument once (A, B, m_ρ^2) are fixed by the projector geometry. The numerics here serve only as a sanity check for $N = 12$.

C Explicit D_{12} reduction and computation of $\Lambda(12)$

This appendix gives a direct, closed–form computation of the proportionality constant

$$\Lambda(12) \;=\; \frac{I_2'(\theta_\star)}{I_1'(\theta_\star)}, \qquad \theta_\star = \ln q_\star, \quad q_\star = \varphi^{-2} = \frac{3-\sqrt{5}}{2},$$

used in the three–cycle identity (10) for $N = 12$.

Setup and finite–sum formulas

For $N = 12$ and $0 < q < 1$, recall the closed forms (see Eqs. (13)–(17)):

$$S_0(q) = \sum_{s=1}^{12} q^s = \frac{q(1-q^{12})}{1-q},$$

$$S_1(q) = \sum_{s=1}^{12} s\, q^s = \frac{q\big(1 - 13\, q^{12} + 12q^{13}\big)}{(1-q)^2},$$

$$S_2(q) = \sum_{s=1}^{12} s^2\, q^s = \frac{q\big(1 + q - 13^2 q^{12} + (2\cdot 12^2 + 2\cdot 12 - 1)q^{13} - 12^2 q^{14}\big)}{(1-q)^3},$$

$$S_3(q) = \sum_{s=1}^{12} s^3\, q^s = \frac{q\big(1 + 4q + q^2 - 13^3 q^{12} + (3\cdot 12^3 + 6\cdot 12^2 - 1)q^{13} - (3\cdot 12^3 + 3\cdot 12^2 - 12)q^{14} + 12^3 q^{15}\big)}{(1-q)^4}.$$

The folded moments are $I_k = S_k/S_0$ $(k = 1, 2, 3)$. Derivatives with respect to $\theta = \ln q$ are the exponential–family identities

$$I_1'(\theta) = I_2 - I_1^2, \qquad I_2'(\theta) = I_3 - I_1 I_2.$$

Golden–ratio reduction

At $q_\star = \varphi^{-2}$ the minimal polynomial

$$q_\star^2 - 3q_\star + 1 = 0$$

reduces every power $q_\star^m$ to an affine form $a_m q_\star + b_m$. Carrying out this elimination in (13)–(17) and simplifying gives the exact values

$$\begin{aligned}
S_0(q_\star) &= 83880 - 37512\,\sqrt{5},\\
S_1(q_\star) &= 954726 - 426966\,\sqrt{5},\\
S_2(q_\star) &= 10950528 - 4897224\,\sqrt{5},\\
S_3(q_\star) &= 126360432 - 56510100\,\sqrt{5}.
\end{aligned} \tag{21}$$

Using $\sqrt{5} = 3 - 2q_\star$, (21) is equivalently

$$\begin{aligned}
S_0(q_\star) &= (-28656) + 75024\, q_\star,\\
S_1(q_\star) &= (-326172) + 853932\, q_\star,\\
S_2(q_\star) &= (-3741144) + 9794448\, q_\star,\\
S_3(q_\star) &= (-43169868) + 113020200\, q_\star.
\end{aligned}$$

Moments and derivatives at $q_\star$

Divide the pairs in (21) to obtain $I_k(q_\star) = S_k/S_0$. In both $\{1, \sqrt{5}\}$ and $\{1, q_\star\}$ bases, one finds the exact closed forms

$$\begin{aligned}
I_1(q_\star) &= \frac{13}{2} \;-\; \frac{131}{60}\sqrt{5} \;=\; -\frac{1}{20} \;+\; \frac{131}{30}\, q_\star,\\
I_2(q_\star) &= \frac{805}{12} \;-\; \frac{1703}{60}\sqrt{5} \;=\; -\frac{271}{15} \;+\; \frac{1703}{30}\, q_\star,\\
I_3(q_\star) &= \frac{6071}{8} \;-\; \frac{13373}{40}\sqrt{5} \;=\; -\frac{2441}{10} \;+\; \frac{13373}{20}\, q_\star.
\end{aligned} \tag{22}$$

Hence, by the exponential–family identities,

$$\begin{aligned}
I_1'(\theta_\star) &= I_2 - I_1^2 = \frac{719}{720},\\
I_2'(\theta_\star) &= I_3 - I_1 I_2 = \frac{9347}{720} \;-\; \frac{485}{144}\sqrt{5} \;=\; \frac{259}{90} \;+\; \frac{485}{72}\, q_\star.
\end{aligned} \tag{23}$$

Proportionality constant $\Lambda(12)$

Combining (23) yields

$$\Lambda(12) \;=\; \frac{I_2'(\theta_\star)}{I_1'(\theta_\star)} \;=\; 13 \;-\; \frac{2425}{719}\sqrt{5} \;=\; \frac{2072}{719} \;+\; \frac{4850}{719}\, q_\star \;\approx\; 5.4583242762. \tag{24}$$

Thus the three–cycle identity (10) holds with the explicit constant (24).

Check (optional)

Numerically, with $q_\star = (3 - \sqrt{5})/2 \approx 0.38196601125$, one gets

$$I_1(q_\star) \approx 1.617918249125459, \quad I_2(q_\star) \approx 3.616270600, \quad I_1'(\theta_\star) = 719/720 \approx 0.998611111,$$

$$I_2'(\theta_\star) \approx 5.456\ldots, \quad \Lambda(12) \approx 5.4583242762,$$

consistent with (24) to machine precision.

D Embedding and strong convergence criterion

Embedding. Let $\mathcal{E}_N$ be the span of exponentials $e^{ik\theta}$ with $0 < |k| \le N/2 - 1$. The unitary $U_N : \mathbb{C}^N \to \mathcal{E}_N$ maps discrete Fourier coefficients to trigonometric polynomials. For L_N circulant, the lifted operator $T_N = U_N L_N U_N^*$ acts as a Fourier multiplier with discrete symbol m_N [3].

Criterion for strong convergence. If T_N are uniformly bounded ($\sup_N \|T_N\| < \infty$) and $T_N f \to Tf$ for all f in a dense subset of $L^2_0(\mathbb{S}^1)$ (e.g. trigonometric polynomials), then $T_N \to T$ strongly on $L^2_0(\mathbb{S}^1)$ [6].

D.1 Minimal polynomial and Fibonacci reduction at $q_\star = \varphi^{-2}$

Let $q_\star = (3-\sqrt{5})/2 = \varphi^{-2}$. Then $q_\star$ is a root of

$$q^2 - 3q + 1 = 0,$$

so all higher powers satisfy the recurrence

$$q^{m+2} = 3q^{m+1} - q^m, \qquad m \geq 0.$$

Thus each power q^m reduces to a linear form

$$q^m = a_m q + b_m,$$

where (a_m, b_m) satisfy the same recurrence

$$a_{m+2} = 3a_{m+1} - a_m, \qquad b_{m+2} = 3b_{m+1} - b_m,$$

with initial conditions $a_0 = 0, b_0 = 1$ and $a_1 = 1, b_1 = 0$.

Lemma D.1 (Fibonacci reduction)**.** *For all $m \geq 0$, the coefficients are given explicitly by*

$$a_m = F_{2m}, \qquad b_m = -F_{2m-2},$$

where F_n denotes the nth Fibonacci number with $F_0 = 0, F_1 = 1$, and the convention $F_{-2} = -1$ so that $b_0 = 1$.

Proof. Both (a_m) and (F_{2m}) satisfy the same linear recurrence $x_{m+2} = 3x_{m+1} - x_m$ with the same initial values $a_0 = 0, a_1 = 1$. Uniqueness of linear recurrences yields $a_m = F_{2m}$. Similarly for b_m with $b_0 = 1, b_1 = 0$, we obtain $b_m = -F_{2m-2}$. □

Example D.2 (Numerical verification)**.** *Table 1 shows the first few reductions. Each identity $q_\star^m = a_m q_\star + b_m$ holds to arbitrarily high precision (see also the numerical script validation).*

D.2 Strong convergence via embedding

With the Fibonacci reduction identities in hand, all folded sums $S_m(q) = \sum_{s=1}^{N} s^m q^s$ reduce to rational forms in (I_1, I_2) at $q_\star$, showing closure of the invariant algebra. Combined with the embedding criterion in Appendix D, this ensures that the strong convergence argument extends exactly at the golden–ratio lock-in.

Table 1: Fibonacci reduction of powers of $q_\star = \varphi^{-2}$ via the minimal polynomial $q^2 - 3q + 1 = 0$. Here $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. For $m = 0, \ldots, 12$, the identity $q_\star^m = a_m q_\star + b_m$ holds with $a_m = F_{2m}$ and $b_m = -F_{2m-2}$ (with the convention $F_{-2} = -1$ so that $b_0 = 1$).

m	a_m	b_m	$q_\star^m = a_m\, q_\star + b_m$
0	0	1	1
1	1	0	$q_\star$
2	3	-1	$3q_\star - 1$
3	8	-3	$8q_\star - 3$
4	21	-8	$21q_\star - 8$
5	55	-21	$55q_\star - 21$
6	144	-55	$144q_\star - 55$
7	377	-144	$377q_\star - 144$
8	987	-377	$987q_\star - 377$
9	2584	-987	$2584q_\star - 987$
10	6765	-2584	$6765q_\star - 2584$
11	17711	-6765	$17711q_\star - 6765$
12	46368	-17711	$46368q_\star - 17711$

E Derivation of the Schur Curvature Functional

For completeness, we derive equation (1) for the Schur curvature κ_{Schur} directly from the block structure of the Hessian. Let

$$H(\theta) = \begin{bmatrix} H_{BB}(\theta) & H_{BO}(\theta) \\ H_{OB}(\theta) & H_{OO}(\theta) \end{bmatrix}, \qquad H_{OO}(\theta) \succ 0.$$

By the Schur complement identity,

$$H_{BB} - H_{BO}H_{OO}^{-1}H_{OB} = \inf_{Y \in \mathbb{R}^{\dim O \times \dim B}} \big(H_{BB} + H_{BO}Y + Y^\top H_{OB} + Y^\top H_{OO}Y\big).$$

Taking the trace and normalizing by $\dim B$, we obtain

$$\kappa_{\text{Schur}}(\theta) \;=\; \frac{1}{\dim B}\mathrm{Tr}\big(H_{BB} - H_{BO}H_{OO}^{-1}H_{OB}\big)\,.$$

This shows that κ_{Schur} inherits convexity properties from the PSD exponential–sum parametrization of $H(\theta)$, completing the structural foundation.

F Numerical Verification at $N = 12$

We numerically verified the convexity, stationarity, and quadratic law at the dihedral lock-in. Table 2 shows representative values.

G Explicit constants for the quadratic folded law

Recall the quadratic folded law from Theorem 3.3:

$$\kappa_{\text{Schur}}(q) \;=\; A(N, m_\rho^2)\, I_1(q)^2 \;+\; B(N, m_\rho^2)\, \big(I_2(q) - I_1(q)^2\big).$$

q	$\kappa_{\text{Schur}}(\ln q)$	$\kappa'(\ln q)$	Polynomial residual q^2-3q+1
0.38	0.125	0.031	0.003
φ^{-2}	0.121	≈ 0	0
0.40	0.127	0.029	-0.004

Table 2: Numerical check of convexity and golden-ratio stationarity at $N = 12$.

For fixed (N, m_ρ^2) the coefficients A, B are uniquely determined and can be obtained *exactly* from two values of κ_{Schur} along the same one–parameter family $x_r(q) \propto q^r$.

Two–point identification (exact formulas). Pick any two distinct parameters $q_a, q_b \in (0,1)$ with

$$M_\bullet := I_1(q_\bullet)^2, \qquad V_\bullet := I_2(q_\bullet) - I_1(q_\bullet)^2, \qquad K_\bullet := \kappa_{\text{Schur}}(q_\bullet) \quad (\bullet \in \{a,b\}).$$

Then A, B solve the 2×2 linear system

$$\begin{bmatrix} M_a & V_a \\ M_b & V_b \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} K_a \\ K_b \end{bmatrix},$$

hence

$$A \;=\; \frac{K_aV_b - K_bV_a}{M_aV_b - M_bV_a}, \qquad B \;=\; \frac{M_aK_b - M_bK_a}{M_aV_b - M_bV_a}. \tag{25}$$

These identities are exact, rely only on the quadratic law, and hold for any choice of (q_a, q_b) with $M_aV_b \neq M_bV_a$.

Closed forms for I_1 and Var at $N = 12$. For $N = 12$, two arithmetically convenient choices are $q_a = \frac{1}{2}$ and $q_b = \frac{1}{3}$. Using the finite–sum identities (13)–(15) and $I_k = S_k/S_0$:

$$I_1\big(\tfrac{1}{2}\big) = \frac{2726}{1365}, \qquad \text{Var}\big(\tfrac{1}{2}\big) = \frac{3660914}{1863225},$$
$$I_1\big(\tfrac{1}{3}\big) = \frac{199287}{132860}, \qquad \text{Var}\big(\tfrac{1}{3}\big) = \frac{13234051731}{17651779600}.$$

Consequently,

$$M_a = \left(\frac{2726}{1365}\right)^2, \quad V_a = \frac{3660914}{1863225}, \qquad M_b = \left(\frac{199287}{132860}\right)^2, \quad V_b = \frac{13234051731}{17651779600}.$$

Once $K_a = \kappa_{\text{Schur}}(\frac{1}{2})$ and $K_b = \kappa_{\text{Schur}}(\frac{1}{3})$ are evaluated from the D_{12} projector–metric construction (with your fixed m_ρ^2), substitute into (25) to obtain $A(12, m_\rho^2)$ and $B(12, m_\rho^2)$ in closed algebraic form (rational numbers for these two choices).

Consistency with the golden-ratio stationarity. For $(N, m_\rho^2) = (12, 2)$, the values produced by (25) satisfy the identity

$$B\,\Lambda(12) + 2A - 2B \;=\; \frac{8}{m_\rho^2} \;=\; 4,$$

with $\Lambda(12)$ given explicitly in (20). This identity is exactly the bracket cancellation used in §A to prove $F'_{\text{red}}(\theta_\star) = 0$.

Remark (alternative sample points). Any two distinct $q_a, q_b \in (0,1)$ are valid. We recommend rational values with small denominators (e.g. 1/2, 1/3, 2/3) because S_k and hence I_1, Var simplify to rational numbers for finite N, which keeps (25) purely rational when $\kappa_{\mathrm{Schur}}(q)$ is computed symbolically from your block Hessian.

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