Homework Assignment 3

1. $\binom{2n}{n} = \dfrac{2n!}{n!(2n-n)!}$

$= \dfrac{2n!}{(n!)^2}$

$2n! = \sqrt{2\pi 2n} \cdot (\frac{2n}{e})^{2n} \cdot (1+\Theta(\frac{1}{2n}))$

$(n!)^2 = (\sqrt{2\pi n} \cdot (\frac{n}{e})^n \cdot (1+\Theta(\frac{1}{2n})))^2$

$= \frac{1}{\sqrt{\pi n}} \cdot 4^n \cdot \frac{1+\Theta(\frac{1}{2n})}{(1+\Theta(\frac{1}{2n}))^2}$

$= \frac{1}{\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \cdot \frac{1+\Theta(\frac{1}{2n})}{(1+\Theta(\frac{1}{2n}))^2}$

$\frac{4^n}{\sqrt{n}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1+\Theta(\frac{1}{2n})}{(1+\Theta(\frac{1}{2n}))^2} \leftarrow \lim_{n\to\infty}$

$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{1}$

$= \frac{1}{\sqrt{\pi}}$

$\frac{1}{\sqrt{\pi}} \in (0, \infty)$

2. a. Prove $\forall n \geq n_0 : P(n) \to P(n+1)$

Let $n_0 = 1 \to n \geq 1$

$\sum_{i=1}^{1} i^3 = 1 = (\frac{1(1+1)}{2})^2$ ✓

Assume $P(n)$ is true, prove $P(n+1)$

$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n+1} i^3 + (n+1)^3$

$= (\frac{n(n+1)}{2})^2 + (n+1)^3$ * Inductive step

$= \frac{n^2(n+1)^2}{4} + (n+1)^3$

$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$

$= \frac{n^4+6n^3+13n^2+12n+4}{4}$

$= \frac{(n+1)^2(n+2)^2}{4}$

$= (\frac{(n+1)(n+2)}{2})^2$ ✓

b. Prove $\forall n > n_0 : P(n-1) \to P(n)$

Let $n_0 = 1 \to n > 1$

$n=2$ $\sum_{i=1}^{1} i^3 = 1 = (\frac{(2-1)\cdot 2}{2})^2$ ✓

Assume $P(n-1)$ is true, prove $P(n)$

$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n-1} i^3 + n^3$

$= (\frac{(n-1)n}{2})^2 + n^3$ * Inductive Step

$= \frac{(n-1)^2 n^2 + 4n^3}{4}$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \frac{n^2(n+1)^2}{4}$$

$$= \left(\frac{n(n+1)}{2}\right)^2 \checkmark$$

3. I. $S(1)$ says

$$S(1) \geq \lg(1)$$

$$0 \geq 0 \checkmark$$

II d. Let $n \geq 1$: Assume $\forall k \quad 1 \leq k < n$: $S(k) \geq \lg(k)$

prove : $S(n) \geq \lg(n)$

$$S(n) \geq S(\lceil \tfrac{n}{2} \rceil) + 1$$

$$\geq \lg(\lceil \tfrac{n}{2} \rceil) + 1 \quad \text{* induction step}$$

$$\geq \lg(\tfrac{n}{2}) + 1$$

$$\geq \lg(n) - \lg(2) + 1$$

$$\geq \lg(n) - 1 + 1$$

$$\geq \lg(n)$$

So $S(n) \geq \lg(n)$

4. I. Base case $S(1)$

$$T(1) \leq \tfrac{4}{3} n^2$$

$$1 \leq \tfrac{4}{3} \checkmark$$

II d. Let $n > 1$ be arbitrary

Assume $\forall k$ in the range $1 \leq k < n$

$$T(k) \leq \tfrac{4}{3} k^2$$

We must show : $T(n) \leq \tfrac{4}{3} k^2$

$$T(n) = T(\lfloor \tfrac{n}{2} \rfloor) + n^2$$

$$\leq \tfrac{4}{3}(\lfloor \tfrac{n}{2} \rfloor)^2 + n^2 \quad \text{* inductive step}$$

$$\leq \tfrac{4}{3}(\tfrac{n}{2})^2 + n^2$$

$$\leq \frac{n^2}{3} + n^2$$

$$\leq \frac{n^2 + 3n^2}{3}$$

$$\leq \tfrac{4}{3} n^2$$

So $T(n) \leq \tfrac{4}{3} k^2$ hence $T(n) = O(n^2)$

5. I. Base case $T(1)$, $T(2)$

$T(1) \le 3n^2-1$

$2 \le 2$ ✓

$T(2) \le 3n^2-1$

$2 \le 11$ ✓

II. Let $n>2$ be arbitrary

Assume $\forall k$ $1 \le k < n$

$T(k) \le 3k^2-1$

We must show $T(n) \le 3n^2-1$

$T(n) = 9T(\lfloor \frac{n}{3} \rfloor)+1$

$\le 9(3\lfloor \frac{n}{3} \rfloor^2-1)+1$ * inductive step

$\le 9 \cdot 3(\frac{n}{3})^2-9+1$

$\le 9 \cdot \frac{3n^2}{27} -8$

$\le 3n^2-8$

$\le 3n^2-1$ * ($-8<-1$ So we can replace the constant)

So: $T(n) \le 3n^2-1$ hence $T(n)=O(n^2)$

6. I. Base case $T(1)$, $T(2)$

$T(1) \le 6n$

$6 \le 6$ ✓

$T(2) \le 6n$

$6 \le 12$ ✓

II. Let $n>2$ be arbitrary

Assume $\forall k$ $1 \le k < n$

$T(k) \le 6n$

We must show: $T(n) \le 6n$

$T(n) = 2T(\lfloor \frac{n}{3} \rfloor)+n$

$\le 2 \cdot 6(\lfloor \frac{n}{3} \rfloor)+n$ * inductive step

$\le 12(\frac{n}{3})+n$

$\le 4n+n$

$\le 5n$

$\le 6n$ * $(5<6)$

So $T(n) \le 6n$ hence $T(n) = O(n)$