Homework Assignment 5

1. a. $T(n) = 7T(n/4) + n$

Compare: $n$ to $n^{\log_4 7}$

let $\varepsilon = \log_4 7 - 1$ Then $\varepsilon > 0$

Since $\log_4 7 > \log_4 4 = 1$

$1 = \log_4 7 - \varepsilon$ and thus $n = O(n^1) = O(n^{\log_4 7 - \varepsilon})$

$\therefore$ By case 1 we have $T(n) = \Theta(n^{\log_4 7})$

b. $T(n) = 9T(n/3) + n^2$

Compare: $n^2$ to $n^{\log_3 9}$

$n^{\log_3 9} = n^2$

$n^2 = n^2 = \Theta(n^{\log_3 9})$

$\therefore$ By case 2 we have $T(n) = \Theta(n^2 \log n)$

c. $T(n) = 6T(n/5) + n^2$

Compare: $n^2$ to $n^{\log_5 6}$

Let $\varepsilon = 2 - \log_5 6$ Then $\varepsilon > 0$

Since $6 < 25 \Rightarrow \log_5 6 < \log_5 25 = 2$

Then $\log_5 6 + \varepsilon = 2$ hence

$n^2 = \Omega(n^2) = \Omega(n^{\log_5 6 + \varepsilon})$

Reg cond.

$6\left(\frac{n}{5}\right)^2 \le cn^2$

$= \frac{6}{25} \le c$

Pick any $c$: $\frac{6}{25} \le c < 1$

$\therefore$ By case 3 we have $T(n) = \Theta(n^2)$

d. $T(n) = 6T(n/5) + n\log n$

Compare: $n\log n$ to $n^{\log_5 6}$

let $\varepsilon = \frac{1}{2}(\log_5 6 - 1)$ Then $\varepsilon > 0$

$2\varepsilon = \log_5 6 - 1$

$\therefore 1 + \varepsilon = \log_5 6 - 2$

so $\frac{n\log n}{n^{\log_5 6 - \varepsilon}} = \frac{n\log n}{n^{1+\varepsilon}} = \frac{\log n}{n^\varepsilon} = 0$

$\therefore n\log(n) = o(n^{\log_5 6 - \varepsilon}) \subseteq O(n^{\log_5 6 - \varepsilon})$

$\therefore$ By case 1 we have $T(n) = \Theta(n^{\log_5 6})$

e. $T(n) = 7T(n/2) + n^2$

Compare: $n^2$ to $n^{\log_2 7}$

let $\varepsilon = \log_2 7 - 2$ Then $\varepsilon > 0$

$n^2 = n^{\log_2 7 - \varepsilon} = O(n^{\log_2 7 - \varepsilon})$

$\therefore$ By case 1 we have $T(n) = \Theta(n^{\log_2 7})$

f. $S(n) = aS(n/4) + n^2$

compare: $n^2$ to $n^{\log_4 a}$

1. $a > 16$ let $\varepsilon = \log_4 a - 2$, $n^2 = O(n^{\log_4 a - \varepsilon})$

$\therefore S(n) = \Theta(n^{\log_4 a})$

2. $a = 16$ $\log_4 16 = 2$, $n^2 = n^2 = \Theta(n^{\log_4 a})$

$\therefore S(n) = \Theta(n^2 \log n)$

3. $1 \le a < 16$ let $\varepsilon = 2 - \log_4 a$, $n^2 = \Omega(n^{\log_4 a + \varepsilon})$

reg condition: $\frac{a}{16} \le c < 1$

$\therefore S(n) = \Theta(n^2)$

2. Compare: $n^{\deg(f)}$ and $n^{\log_b(a)}$

Let $\varepsilon = \deg(f) - \log_b a$

$\deg(f) = \log_b a + \varepsilon$

Hence $n^{\deg(f)} = \Omega(n^{\deg(f)}) = \Omega(n^{\log_b a + \varepsilon})$

Reg. Condition

Pick any $c$ in $\frac{a}{b^{\deg(f)}} \le c < 1$.

We have $(a/b^{\deg(f)})n^{\deg(f)} \le cn^{\deg(f)}$

3. Use e and f from Prob. 1

From case 1:

$n^2 = o(n^{\log_2 7})$

so $n^{\log_4 a} = o(n^{\log_2 7})$

$\log_4 a < \log_2 7$

$= a < 4^{\log_2 7}$

$= a < 7^{\log_2 4}$

$= a < 7^2$

$= a < 49$

$a = 48$ is the max integer

4. Handshake lemma

$$\sum_{x \in V(G)} \deg(x) = 2 \cdot |E(G)|$$

Two cases
A $\{x \in V(G) \mid \deg(x)$ is even
B $\{y \in V(G) \mid \deg(x)$ is odd

$$\sum_{x \in A} \deg(x) + \sum_{y \in B} \deg(y) = 2 \cdot |E(G)|$$

$2 \cdot |E(G)|$ is even due to the 2. A is even given the 1st case A. Thus $\sum_{y \in B} \deg(y)$ must be even or the equality does not hold

5. I. Base case:
Let $|E(G)| = 0$. G can only have one vertex, thus $|V(G)| = 1$. $0 \geq 1-1 = 0 \geq 0$ ✓
II. Let $|E(G)| > 0$
Assume $|E(G')| \geq |V(G')| - 1$
We must show $|E(G)| \geq |V(G)| - 1$
remove edge $e \in E(G)$.
Case 1: $G-e$ is connected
$G-e$ has $m-1$ edges / $n$ vertices

$m-1 \geq n-1$
$m \geq n-1$ ✓

Case 2: $G-e$ is disconnected
Two components $C_1$ and $C_2$
$C_1$ has $m_1$ edges and $n_1$ vertices and $C_2$ has $m_2$ edges / $n_2$ vertices
$n = n_1 + n_2$
$m \geq m_1 + m_2 + 1 \geq (n_1 - 1) + (n_2 - 1) + 1$
$\geq n_1 + n_2 - 1$
$= n - 1$ ✓
$m \geq n - 1$