Homework Assignment #4

1. I. Base case:

Let $m=0$

n must equal 1 since $n=0$ is not a tree and $n=2$ has one edge.

$m = n-1 = 0$

II. Let $n \geq 1$ and assume $\forall k$ in $1 \leq k < n$ $P(k)$ is true. We must prove: $P(n)$ is true.

Let T be a tree having n vertices and remove any edge e from T. T splits into 2 subtrees with both having fewer vertices than n. Suppose the 2 subtrees have k_1 and k_2 vertices. Since no vertices were removed, $k_1 + k_2 = n$. By induction hypothesis, we have $k_1 - 1$ and $k_2 - 1$ edges. Thus:

$$(k_1 - 1) + (k_2 - 1) + 1 = k_1 + k_2 - 1 = n - 1$$

2. Let the connected components (Trees) be $T_1, T_2, T_3 \dots T_k$

Suppose T_j has m_j and n_j edges/vertices.

By 1 we proved $m = n-1$. So $m_j = n_j - 1$

$$\text{So: } m = \sum_{j=1}^{k} m_j = \sum_{j=1}^{k} n_j - \sum_{j=1}^{k} 1 = n - k$$

3. $T(n) = 2T(\lfloor \frac{n}{3} \rfloor) + 5$ $\quad k = 0$

$= 2(2T(\lfloor \frac{\lfloor n/3 \rfloor}{3} \rfloor) + 5) + 5$ $\quad k = 1$

$= 15 + 4T(\lfloor \frac{n}{9} \rfloor)$

$= 15 + 4T(2T(\lfloor \frac{\lfloor n/3 \rfloor}{9} \rfloor) + 5)$ $\quad k = 2$

$= 35 + 8T(\lfloor \frac{n}{27} \rfloor)$

$\sum_{i=0}^{k-1} 5 \cdot 2^i + 2^k T(\lfloor \frac{n}{3^k} \rfloor)$

$1 \leq \lfloor \frac{n}{3^k} \rfloor < 3$ $\quad k = \lfloor \log_3(n) \rfloor$

$\sum_{i=0}^{k-1} 5 \cdot 2^i \cdot 2^{\lfloor \log_3(n) \rfloor} T(\lfloor \frac{\lfloor \log_3(n) \rfloor}{3^k} \rfloor)$

$= 5 \cdot (2^{\lfloor \log_3(n) \rfloor} - 1) + 2^{\lfloor \log_3(n) \rfloor}$

$= 12^{\lfloor \log_3(n) \rfloor} - 5$

4. $T(n) = 4T(\lfloor \frac{n}{5} \rfloor) + n$

$= 4(4T(\lfloor \frac{\lfloor n/5 \rfloor}{5} \rfloor) + n) + n$

$= 16T(\lfloor \frac{n}{5^2} \rfloor) + 5n$

$= 16(4T(\lfloor \frac{\lfloor n/5^2 \rfloor}{5} \rfloor) + n) + 5n$

$= 4^3 T(\lfloor \frac{n}{5^3} \rfloor) + 21n$

$T(n) = \sum_{i=0}^{k-1} 4^i (\frac{n}{5^i}) + 4^k T(\lfloor \frac{n}{5^k} \rfloor)$

$1 \le \lfloor n/5^k \rfloor < 5 \quad k = \lfloor \log_5(n) \rfloor$

$T(n) = \sum_{i=0}^{\lfloor \log_5(n) \rfloor - 1} 4^i (\frac{n}{5^i}) + 4^{\lfloor \log_5(n) \rfloor} T(\lfloor \frac{n}{5^{\log_5(n)}} \rfloor)$

$= T(n) = \sum_{i=0}^{\lfloor \log_5(n) - 1 \rfloor} 4^i (\frac{n}{5^i}) + 3 \cdot 4^{\lfloor \log_5(n) \rfloor}$ ✓

Show that $T(n) = \Theta(n)$

$T(n) \le \sum_{i=0}^{\infty} 4^i (\frac{n}{5^i}) + 3 \cdot 4^{\log_5(n)}$

$= \sum_{i=0}^{\infty} \frac{4^i n}{5^i} + 3 \cdot 4^{\log_5(n)}$

$= n \sum_{i=0}^{\infty} \frac{4^i}{5^i} + 3n^{\log_5 4}$

$= n(\frac{1}{1-4/5}) + 3n^{\log_5 4}$

$= n(\frac{1}{1/5}) + 3n^{\log_5 4}$

$= 5n + 3n^{\log_5 4}$

$\therefore \; = O(n)$

Also $T(n) = 4T\lfloor \frac{n}{5} \rfloor + n \ge n = \Omega(n)$

Since $T(n) = O(n)$ and $\Omega(n)$

$\therefore T(n) = \Theta(n)$

5. a. $T(n) = 2T(\lfloor \frac{n}{3} \rfloor) + n$

$= 2(2T(\lfloor \frac{n}{3^2} \rfloor) + n) + n$

$= 2^3 T(\lfloor \frac{n}{3^3} \rfloor) + 2^2 n + n$

$T(n) = \sum_{i=0}^{k-1} 2^k T(\lfloor \frac{n}{3^k} \rfloor) + 2^i \lfloor \frac{n}{3^i} \rfloor$

$1 \le \lfloor n/3^k \rfloor < 3 \quad \log_3(n)$

$T(n) = \sum_{i=0}^{\log_3(n)-1} 12^{\log_3(n)} + 2^i \lfloor \frac{n}{3^i} \rfloor$

b. $T(n) = \sum_{i=0}^{\infty} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 6 \cdot 2^{\log_3(n)}$

$= n \sum_{i=0}^{\infty} \frac{2^i}{3^i} + 6 \cdot 2^{\log_3(n)}$

$= n\left(\frac{1}{1-2/3}\right) + 6 \cdot n^{\log_3 2}$

$= 3n + 6n^{\log_3 2}$

$= O(n)$

c. Compare: n to $n^{\log_3 2}$

Let $\varepsilon = 1 - \log_3 2$. Then $\varepsilon > 0$

Also $\log_3 2 + \varepsilon = 1$

$n = n^{\log_3 2 + \varepsilon} = \Omega(n^{\log_3 2 + \varepsilon})$

Regularity condition $af\left(\frac{n}{b}\right) \leq cf(n)$ $a=2, b=3$

$2\left(\frac{n}{3}\right) \leq cn$

$2/3\, n < cn$

$2/3 \leq c < 1$

c is in range $(0,1)$, therefore regularity condition holds so by case (3)

$\therefore T(n) = \Theta(n)$