Toroidal Compactification Proof

This appendix provides a concise constructive proof that identifies the Euclidean 24-torus obtained by quotienting $\mathbb{R}^{24}$ by the Niemeier lattice Λ as a compact symmetric manifold preserving the universal symmetry group.

1. **Quotient Construction**:
   Define the map
   $$\pi : \mathbb{R}^{24} \to T^{24} = \mathbb{R}^{24}/\Lambda,$$
   identifying each point x with $x + v$ for all lattice vectors $v \in \Lambda$.
2. **Compactness**:
   The fundamental domain of Λ is bounded and closed, and the quotient by a discrete group yields a compact manifold.
3. **Symmetry Preservation**:
   Any automorphism of Λ descends to a diffeomorphism of T^{24}, ensuring the lattice's Weyl group actions are well-defined on the torus.
4. **Modular Form Well-Definedness**:
   Theta functions $\vartheta_\Lambda(\tau)$ are invariant under translations by Λ, hence define modular forms on T^{24}.

Figure: Fundamental domain cut-and-glue operation.


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![Fundamental domain illustration]({"id":"generated_image:1","caption":"Square fundamental domain cut and glued to form a torus","file_name":"torus_cut.png"})
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