Applied Multivariate Statistical Analysis Solutions

Nathan Crouse

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1 Chapter 1

2 Chapter 2

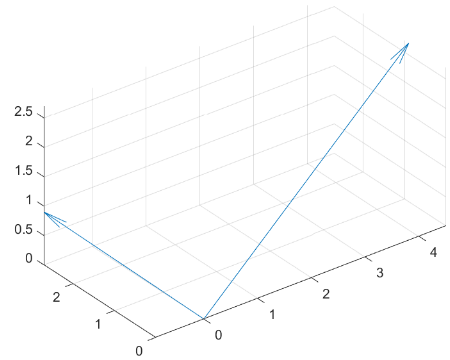
2.1

Let $\mathbf{x}' = [5, 1, 3]$ and $\mathbf{y}' = [-1, 3, 1]$.

(a) Graph the two vectors.

```
x = [5,1,3]'; y = [-1,3,1]';
starts = zeros(2,3);  % Starts at the origin.
ends = [x'; y'];  % Ends at the point.

% quiver3 args are x,y,z,u,v,w. x,y,z are the start positions and u,v,w are the end positions.
a = quiver3(starts(:,1), starts(:,2), starts(:,3), ends(:,1), ends(:,2), ends(:,3));
axis equal
saveas(a, '.\applied-multivariate-statistics\solutions\chapter-2\sol2.1a.png', 'png')
```



(b) Find (i) the length of $\mathbf{x}$, (ii) the angle between $\mathbf{x}$ and $\mathbf{y}$, and (iii) the projection of $\mathbf{y}$ onto $\mathbf{x}$.

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}} = \sqrt{5^2 + 1^2 + 3^2} = 5.9161$$

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right) = \cos^{-1}\left(\frac{1}{5.9161 \times 3.3166}\right) = 87.0787^\circ$$

MATLAB code `acosd((x'*y)/(norm(x)*norm(y)))` returns the angle in degrees.

$$\mathrm{comp}_{\mathbf{x}}\mathbf{y} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|}$$

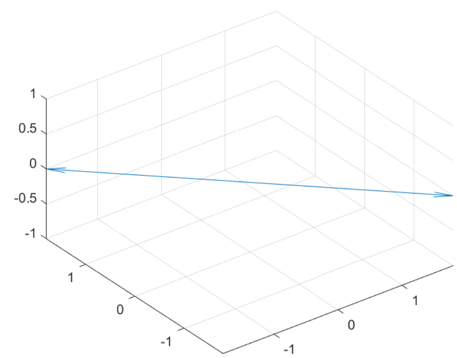
$$\mathrm{proj}_{\mathbf{x}}\mathbf{y} = \mathrm{comp}_{\mathbf{y}}\mathbf{x}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|}\right)\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\right)\mathbf{x} = \left(\frac{1}{35}\right)\begin{bmatrix}5\\1\\3\end{bmatrix} = \begin{bmatrix}5/35\\1/35\\3/35\end{bmatrix}$$

(c) Since $\bar{x} = 3$ and $\bar{y} = 1$, graph $[5-3, 1-3, 3-3] = [2, -2, 0]$ and $[-1-1, 3-1, 1-1] = [-2, 2, 0]$.

```
starts = zeros(2,3);  % Starts at the origin.
ends = [(x-mean(x))'; (y-mean(y))'];  % Ends at the point. Subtract the mean values.

% quiver3 args are x,y,z,u,v,w. x,y,z are the start positions and u,v,w are
% the end positions.
b = quiver3(starts(:,1), starts(:,2), starts(:,3), ends(:,1), ends(:,2), ends(:,3));
```

```
    axis equal
    saveas(b, '.\applied-multivariate-statistics\
       solutions\chapter-2\sol2.1c.png', 'png')
```



After subtracting off the respective means from both $\mathbf{x}$ and $\mathbf{y}$ (centering), the results of both vectors exist on the same line through the origin, but point in different directions.

2.2

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$$

(a) $5\mathbf{A}$

$$5\mathbf{A} = 5 \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$$

(b) $\mathbf{BA}$

$$\mathbf{BA} = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$$

(c) $\mathbf{A}'\mathbf{B}'$

$$\mathbf{A}'\mathbf{B}' = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ -3 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$$

(d) $\mathbf{C}'\mathbf{B}$

$$\mathbf{C}'\mathbf{B} = \begin{bmatrix} 5 & -4 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 12 & -7 \end{bmatrix}$$

(e) Is **AB** defined?

No, **A** is a 2×2 matrix and **B** is a 3×2 matrix, the number of columns in $\mathbf{A} = 2$ is not the same as the number of rows in $\mathbf{B} = 3$, so the two matrices are not conformable.

2.3

Verify the following properties of transpose when

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix},$$

(a) $(\mathbf{A}')' = \mathbf{A}$

$$(\mathbf{A}')' = \left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}' \right)' = \left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right)' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \mathbf{A}$$

Matrix **A** is symmetric since $\mathbf{A} = \mathbf{A}'$.

(b) $(\mathbf{C}')^{-1} = \left(\mathbf{C}^{-1}\right)'$

$$(\mathbf{C}')^{-1} = \left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}' \right)^{-1} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}^{-1} = \frac{1}{2 - 12} \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -2/10 & 3/10 \\ 4/10 & -1/10 \end{bmatrix}$$

$$\left(\mathbf{C}^{-1}\right)' = \left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}^{-1} \right)' = \frac{1}{2 - 12} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}' = \begin{bmatrix} -2/10 & 3/10 \\ 4/10 & -1/10 \end{bmatrix}$$

So $(\mathbf{C}')^{-1} = \left(\mathbf{C}^{-1}\right)'$.

(c) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

$$(\mathbf{AB})' = \left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix} \right)' = \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$$

$$\mathbf{B}'\mathbf{A}' = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}' \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$$

So $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

(d) For general $\underset{(m\times k)}{\mathbf{A}}$ and $\underset{(k\times l)}{\mathbf{B}}$, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

$$(\mathbf{AB})' = \left(\begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{k1} & \dots & b_{k\ell} \end{bmatrix}\right)' = \left(\begin{bmatrix} \sum_{i=1}^k a_{1i}b_{i1} & \dots & \sum_{i=1}^k a_{1i}b_{i\ell} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^k a_{mi}b_{i1} & \dots & \sum_{i=1}^k a_{mi}b_{i\ell} \end{bmatrix}\right)' =$$

$$= \begin{bmatrix} \sum_{i=1}^k a_{1i}b_{i1} & \dots & \sum_{i=1}^k a_{mi}b_{i1} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^k a_{1i}b_{i\ell} & \dots & \sum_{i=1}^k a_{mi}b_{i\ell} \end{bmatrix}$$

$$\mathbf{B}'\mathbf{A}' = \begin{bmatrix} b_{11} & \dots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{k1} & \dots & b_{k\ell} \end{bmatrix}' \begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mk} \end{bmatrix}' = \begin{bmatrix} b_{11} & \dots & b_{k1} \\ \vdots & \ddots & \vdots \\ b_{1\ell} & \dots & b_{k\ell} \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1k} & \dots & a_{mk} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^k b_{i1}a_{1i} & \dots & \sum_{i=1}^k b_{i1}a_{mi} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^k b_{i\ell}a_{1i} & \dots & \sum_{i=1}^k b_{i\ell}a_{mi} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^k a_{1i}b_{i1} & \dots & \sum_{i=1}^k a_{mi}b_{i1} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^k a_{1i}b_{i\ell} & \dots & \sum_{i=1}^k a_{mi}b_{i\ell} \end{bmatrix}$$

So $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

2.4

When $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist, prove each of the following.

(a) $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$

$$(\mathbf{A}^{-1})' = (\mathbf{A}^{-1})'\mathbf{I} = (\mathbf{A}^{-1})'\mathbf{A}'(\mathbf{A}')^{-1} = (\mathbf{AA}^{-1})'(\mathbf{A}')^{-1} = \mathbf{I}(\mathbf{A}')^{-1} = (\mathbf{A}')^{-1}$$

(b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

If we define $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, and if **Def 2A.27** is satisifed ($\mathbf{CD} = \mathbf{DC} = \mathbf{I}$), then $\mathbf{D}$ is the inverse of $\mathbf{C}$.

$$\mathbf{CD} = (\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$\mathbf{DC} = \left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) = \mathbf{B}\left(\mathbf{AA}^{-1}\right)\mathbf{B}^{-1} = \mathbf{BIB}^{-1} = \mathbf{BB}^{-1} = \mathbf{I}$$

Now, **Def 2A.27** is satisifed ($\mathbf{CD} = \mathbf{DC} = \mathbf{I}$), so the inverse of $\mathbf{AB}$ is $\mathbf{B}^{-1}\mathbf{A}^{-1}$. That is, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Hint: Part a can be proven noting that $\mathbf{AA}^{-1} = \mathbf{I}$, $\mathbf{I} = \mathbf{I}'$, and $\left(\mathbf{AA}^{-1}\right)' = \left(\mathbf{A}^{-1}\right)'\mathbf{A}'$. Part b follows from $\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)\mathbf{AB} = \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$.

2.5

Check that

$$\mathbf{Q} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{bmatrix}$$

is an orthogonal matrix.

If the conditions of **Result 2A.13**, ($\mathbf{AA}' = \mathbf{A}'\mathbf{A} = \mathbf{I}$) are true, then we have an orthogonal matrix.

$$\mathbf{Q}'\mathbf{Q} = \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{bmatrix} = \begin{bmatrix} \frac{25}{169} + \frac{144}{169} & \frac{60}{169} - \frac{60}{169} \\ \frac{60}{169} - \frac{60}{169} & \frac{144}{169} + \frac{25}{169} \end{bmatrix} = \begin{bmatrix} \frac{169}{169} & \frac{0}{169} \\ \frac{0}{169} & \frac{169}{169} \end{bmatrix} = \mathbf{I}$$

$$\mathbf{Q}\mathbf{Q}' = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} = \begin{bmatrix} \frac{25}{169} + \frac{144}{169} & -\frac{60}{169} + \frac{60}{169} \\ -\frac{60}{169} + \frac{60}{169} & \frac{144}{169} + \frac{25}{169} \end{bmatrix} = \begin{bmatrix} \frac{169}{169} & \frac{0}{169} \\ \frac{0}{169} & \frac{169}{169} \end{bmatrix} = \mathbf{I}$$

Because $\mathbf{QQ}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}$, $\mathbf{Q}$ is an orthofonal matrix.

2.6

Let

$$\mathbf{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

(a) Is $\mathbf{A}$ symmetric?

Yes, $\mathbf{A}' = \mathbf{A}$, so $\mathbf{A}$ is symmetric.

(b) Show that $\mathbf{A}$ is positive definite.

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = (9-\lambda)(6-\lambda) - 4 = \lambda^2 - 15\lambda + 54 - 4 = (\lambda - 5)(\lambda - 10)$$

The two eigenvalues of 5 and 10 are both positive, so from what's in **2.3** on page 63, $\mathbf{A}$ is positive definite.

2.7

Let $\mathbf{A}$ be as given in Exercise 2.6.

(a) Determine the eigenvalues and eigenvectors of $\mathbf{A}$.

From problem 2.7, the eigenvalues are 5 and 10.To get the eigenvectors,
$\underline{\lambda_1 = 5}$:

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$9x_1 - 2x_2 = 5x_1 \Rightarrow 4x_1 = 2x_2 \Rightarrow 2x_1 = x_2$$

and

$$-2x_1 + 6x_2 = 5x_1 \Rightarrow x_2 = 2x_1$$

So $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and normalizing, $\mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$.

$\underline{\lambda_2 = 10}$:

$$\mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 10 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$9x_1 - 2x_2 = 10x_1 \Rightarrow x_1 = -2x_2$$

and

$$-2x_1 + 6x_2 = 10x_1 \Rightarrow 12x_1 = -6x_2 \Rightarrow x_1 = -2x_2$$

So $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and normalizing, $\mathbf{e}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

(b) Write the spectral decomposition of $\mathbf{A}$.

The spectral decomposition would be,

$$\mathbf{A} = \sum_{k=1}^{2} \lambda_k \mathbf{e}_k \mathbf{e}_k' = 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}' + 10 \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}' =$$

$$= 5 \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} + 10 \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

(c) Find $\mathbf{A}^{-1}$.

Using the spectral decomposition for the inverse in **(2-21)** on page 66,

$$\mathbf{A}^{-1} = (\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}')^{-1} = \left(\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \boldsymbol{\Lambda} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}' \right)^{-1} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \boldsymbol{\Lambda}^{-1} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}' =$$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1/5 & -2/10 \\ 2/5 & 1/10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} =$$

$$= \frac{1}{5} \begin{bmatrix} 30/50 & 10/50 \\ 10/50 & 45/50 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$$

By direct computation,

$$\mathbf{A}^{-1} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}^{-1} = \frac{1}{54-4} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$$

(d) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

Using **2-21** on page 66 again, the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocal of the eigenvalues of $\mathbf{A}$ and the eigenvectors are the same as those for $\mathbf{A}$.

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

If we did it by-hand, setting $\mathbf{B} = \mathbf{A}^{-1}$,

$$0 = |\mathbf{B} - \lambda \mathbf{I}| = \frac{1}{50} \begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = \frac{1}{50} \left[(6-\lambda)(9-\lambda) - 54 + 4\right] = \frac{1}{50} \left[(6-\lambda)(9-\lambda) - 50\right] =$$

$$\frac{1}{50} \left(\lambda^2 - 15\lambda + 50\right) = \left(\lambda - \frac{5}{50}\right) \left(\lambda - \frac{10}{50}\right) = \left(\lambda - \frac{1}{10}\right) \left(\lambda - \frac{1}{5}\right)$$

These eigenvalues are the same as the reciprocal of the eigenvalues for $\mathbf{A}$.

2.8

Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

find the eigenvalues λ_1 and λ_2 and the associated eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Determine the spectral decomposition **(2-16)** of $\mathbf{A}$.

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = -2 + 2\lambda - \lambda + \lambda^2 - 4 =$$

$$= \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$$

The eigenvalues are -3 and 2.

<u>$\lambda_1 = -3$</u>:

$$\mathbf{A}\mathbf{x}_1 = \lambda \mathbf{x}_1 \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 \\ -3x_2 \end{bmatrix}$$

$$x_1 + 2x_2 = -3x_1 \Rightarrow -4x_1 = 2x_2 \Rightarrow -2x_1 = x_2$$

and

$$2x_1 - 2x_2 = -3x_1 \Rightarrow 2x_1 = -x_2 \Rightarrow -2x_1 = x_2$$

So $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and normalizing $\mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$.

<u>$\lambda_2 = 2$</u>:

$$\mathbf{A}\mathbf{x}_2 = \lambda \mathbf{x}_2 \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$x_1 + 2x_2 = 2x_1 \Rightarrow 2x_2 = x_1$$

and

$$2x_1 - 2x_2 = 2x_1 \Rightarrow x_2 = 4x_1 \Rightarrow x_1 = 2x_2$$

So $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and normalizing $\mathbf{e}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

$$\mathbf{A} = \sum_{k=1}^{2} \lambda_k \mathbf{e}_k \mathbf{e}_k' = -3 \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} + 2 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} =$$

$$= \frac{1}{5}\left(-3 \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix} + 2 \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = \frac{1}{5}\left(- \begin{bmatrix} 3 & -6 \\ -6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\right) =$$

$$\frac{1}{5} \begin{bmatrix} 5 & 10 \\ 10 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

2.9

Let $\mathbf{A}$ be as in Exercise 2.8.

(a) Find $\mathbf{A}^{-1}$.

Using **(2-21)**,

$$\mathbf{A}^{-1} = \mathbf{P}\mathbf{\Lambda}^{-1}\mathbf{P}' = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} =$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 1 & 1/2 \end{bmatrix} =$$

$$= \frac{1}{5} \begin{bmatrix} 5/3 & 5/3 \\ 5/3 & -5/6 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{bmatrix}$$

Using direct computation,

$$\mathbf{A}^{-1} = \frac{1}{-2-4} \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{bmatrix}$$

(b) Compute the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

Also from **(2-21)** on page 66, we can see that the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocal of the eigenvalues of $\mathbf{A}$, so

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix} = \begin{bmatrix} -1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

and the eigenvectors for $\mathbf{A}^{-1}$ are the same as those for $\mathbf{A}$,

$$\mathbf{P} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

(c) Write the spectral decomposition of $\mathbf{A}^{-1}$, and compare it with that of $\mathbf{A}$ from Exercise 2.8.

$$\mathbf{A}^{-1} = \sum_{k=1}^{2} \frac{1}{\lambda_k}\mathbf{e}_k\mathbf{e}_k' = -\frac{1}{3}\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} =$$

$$= \frac{1}{5}\left(-\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = \frac{1}{30}\left(-\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}\right) =$$

$$= \frac{1}{30}\begin{bmatrix} 10 & 10 \\ 10 & -5 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{bmatrix}$$

In the spectral decomposition of both $\mathbf{A}$ and $\mathbf{A}^{-1}$ the matrices created for all of the $\mathbf{e}_k\mathbf{e}_k'$ components are the same. The difference is in the eigenvalues. The eigenvalues for $\mathbf{A}$ are λ_k and the eigenvalues for $\mathbf{A}^{-1}$ are $1/\lambda_k$.

2.10

Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the (2, 2) position. Moreover, the columns of $\mathbf{A}$ (and $\mathbf{B}$) are nearly dependent. Show that $\mathbf{A}^{-1} \doteq (-3)\mathbf{B}^{-1}$. Consequently, small changes — perhaps caused by rounding — can give substantially different inverses.

$$\mathbf{A}^{-1} = \frac{1}{16.008 - 16.008001}\begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix} = \frac{1}{-0.000001}\begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix} =$$

$$= -1000000\begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix} = \begin{bmatrix} -4002000 & 4001000 \\ 4001000 & -4000000 \end{bmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{16.008004 - 16.008001}\begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix} = \frac{1}{0.000003}\begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix} =$$

$$= 333333.\bar{3}\begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix} = \begin{bmatrix} 1334000.3333331999333 & -1333666.6666665333 \\ -1333666.6666665333 & 1333333.3333332 \end{bmatrix}$$

$$(-3)\mathbf{B}^{-1} = -3\begin{bmatrix} 1334000.3333331999333 & -1333666.6666665333 \\ -1333666.6666665333 & 1333333.3333332 \end{bmatrix} =$$

$$= \begin{bmatrix} -4002000.9999995997999 & 4000999.9999995999 \\ 4000999.9999995999 & -3999999.9999996 \end{bmatrix} \doteq \mathbf{A}^{-1}$$

Used $\frac{1}{|B|} = \frac{1}{0.000003} = 333333.3333333$ for computation.

2.11

Show that the determinant of the $p \times p$ diagonal matrix $\mathbf{A} = a_{ij}$ with $a_{ij} = 0$, $i \neq j$, is given by the product of the diagonal elements; thus, $|\mathbf{A}| = a_{11}a_{22}, \ldots a_{pp}$. *Hint:* By **Definition 2A.24**, $|\mathbf{A}| = a_{11}\mathbf{A}_{11} + 0 + \cdots + 0$. Repeat for the submatrix $\mathbf{A}_{11}$ obtained by deleting the first row and first column of $\mathbf{A}$.

$$|\mathbf{A}| = \sum_{j=1}^{p} a_{1j} \left| A_{1j} \right| (-1)^{1+j} =$$

$$= a_{11} \left| \mathbf{A}_{11} \right| + 0 + \cdots + 0 =$$

$$= a_{11} \left(\sum_{j=2}^{p} a_{2j} \left| A_{2j} \right| (-1)^{2+j} \right) =$$

$$= a_{11} \left(a_{22} \left| \mathbf{A}_{22} \right| + 0 + \cdots + 0 \right) =$$

$$\vdots$$

$$= a_{11}a_{22} \ldots a_{p-2,p-2} \left| \mathbf{A}_{p-2,p-2} \right|$$

$$= a_{11}a_{22} \ldots a_{p-2,p-2} \left(a_{p-1,p-1}a_{pp} - 0 \right)$$

$$= a_{11}a_{22} \ldots a_{p-2,p-2}a_{p-1,p-1}a_{pp} = \prod_{i=1}^{p} a_{ii}$$

2.12

Show that the determinant of a square symmetric $p \times p$ matrix $\mathbf{A}$ can be expressed as the product of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$; that is, $|\mathbf{A}| = \prod_{i=1}^{p} \lambda_i$. *Hint:* From **(2-16)** and **(2-20)**, $\mathbf{A} = \mathbf{P \Lambda P'}$ with $\mathbf{P'P} = \mathbf{I}$. From **Result 2A.ll(e)**, $|A| = |\mathbf{P\Lambda P'}| = |\mathbf{P}|\,|\mathbf{\Lambda P'}| = |\mathbf{P}|\,|\mathbf{\Lambda}|\,|\mathbf{P'}| = |\mathbf{\Lambda}|\,|\mathbf{I}|$,since $|\mathbf{I}| = |\mathbf{P'P}| = |\mathbf{P'}|\,|\mathbf{P}|$. Apply **Exercise 2.11**.

$$|\mathbf{A}| = |\mathbf{P\Lambda P'}| \overset{\text{2A.11(e)}}{=} |\mathbf{P}|\,|\mathbf{\Lambda P'}| \overset{\text{2A.11(e)}}{=} |\mathbf{P}|\,|\mathbf{\Lambda}|\,|\mathbf{P'}| = |\mathbf{\Lambda}|\,|\mathbf{P}|\,|\mathbf{P'}| =$$

$$= |\mathbf{\Lambda}|\,|\mathbf{PP'}| = |\mathbf{\Lambda}|\,|\mathbf{I}| = |\mathbf{\Lambda}|\,(1) = |\mathbf{\Lambda}| \overset{\text{Exercise 2.11}}{=} \prod_{i=1}^{p} \lambda_i$$

2.13

Show that $|\mathbf{Q}| = +1$ or -1 if $\mathbf{Q}$ is a $p \times p$ orthogonal matrix.
Hint: $|\mathbf{Q}\mathbf{Q}'| = |\mathbf{I}|$. Also, from **Result 2A.ll**, $|\mathbf{Q}|\,|\mathbf{Q}'| = |\mathbf{Q}|^2$. Thus, $|\mathbf{Q}|^2 = |\mathbf{I}|$. Now use **Exercise 2.11**.

We know $\mathbf{Q}$ is orthogonal iff $\mathbf{Q}\mathbf{Q}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}$ from **Definition 2A.13**.

$$|\mathbf{Q}'\mathbf{Q}| = |\mathbf{Q}'|\,|\mathbf{Q}| = |\mathbf{Q}|\,|\mathbf{Q}| = |\mathbf{Q}|^2 = |\mathbf{I}| \overset{\text{Exercise 2.11}}{=} 1$$

So

$$|\mathbf{Q}| = \left(|\mathbf{Q}|^2\right)^{1/2} = (|\mathbf{I}|)^{1/2} = \pm\sqrt{1} = \pm 1$$

2.14

Show that $\underset{(p\times p)}{\mathbf{Q}'}\underset{(p\times p)}{\mathbf{A}}\underset{(p\times p)}{\mathbf{Q}}$ and $\underset{p\times p}{\mathbf{A}}$ have the same eigenvalues if $\mathbf{Q}$ is orthogonal.
Hint: Let λ be an eigenvalue of $\mathbf{A}$. Then $0 = |\mathbf{A} - \lambda\mathbf{I}|$. By **Exercise 2.13** and **Result 2A.11(e)**, we can write $0 = |\mathbf{Q}'|\,|\mathbf{A} - \lambda\mathbf{I}|\,|\mathbf{Q}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{I}|$, since $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$ from **Exercise 2.11**.

Answer is pretty much in the hint. We already know $|\mathbf{Q}'\mathbf{Q}| = |\mathbf{I}| = 1$. We also already know $|\mathbf{Q}'\mathbf{Q}| = |\mathbf{Q}'|\,|\mathbf{Q}|$ from **Result 2A.11(e)**. Not mentioned in the book, but the determinant is a scalar output, so it's also commutative, so $|\mathbf{Q}'\mathbf{Q}| = |\mathbf{Q}'|\,|\mathbf{Q}| = |\mathbf{Q}|\,|\mathbf{Q}'|$.

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{Q}'\mathbf{Q}|\,|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{Q}'|\,|\mathbf{Q}|\,|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{Q}'|\,|\mathbf{A} - \lambda\mathbf{I}|\,|\mathbf{Q}| =$$
$$= |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{Q}'\mathbf{I}\mathbf{Q}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{Q}'\mathbf{Q}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{I}|$$

There it is, $0 = |\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{I}|$, so the eigenvalues of $\mathbf{A}$ are the same as those for $\mathbf{Q}'\mathbf{A}\mathbf{Q}$.

2.15

A quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is said to be positive definite if the matrix $\mathbf{A}$ is positive definite. Is the quadratic form $3x_1^2 + 3x_2^2 - 2x_1x_2$ positive definite?

Converting $3x_1^2 + 3x_2^2 - 2x_1x_2$ to a matrix, we'd have,

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\,(3x_1 - x_2) + x_2\,(-x_1 + 3x_2) = 3x_1^2 + 3x_2^2 - 2x_1x_2$$

So now,

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Now check if $\mathbf{A}$ has positive eigenvalues,

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 9 - 1 = (\lambda - 2)(\lambda - 4)$$

The eigenvalues are $\{\lambda_1, \lambda_2\} = \{2, 4\}$, so since $\lambda_i > 0$, i.e., the eigenvalues are all positive, the matrix $\mathbf{A}$ is positive definite.

2.16

Consider an arbitrary $n \times p$ matrix $\mathbf{A}$. Then $\mathbf{A}'\mathbf{A}$ is a symmetric $p \times p$ matrix. Show that $\mathbf{A}'\mathbf{A}$ is necessarily nonnegative definite.
Hint: Set $\mathbf{y} = \mathbf{A}\mathbf{x}$ so that $\mathbf{y}'\mathbf{y} = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x}$.

Ignoring the hint, could use the Singular-Value Decomposition from **Result 2A.15** for nonsquare matrices,

$$\mathbf{A}'\mathbf{A} = \left(\mathbf{U}\boldsymbol{\Lambda}\mathbf{V}'\right)'\left(\mathbf{U}\boldsymbol{\Lambda}\mathbf{V}'\right) = \mathbf{V}\boldsymbol{\Lambda}\mathbf{U}'\mathbf{U}\boldsymbol{\Lambda}\mathbf{V}' = \mathbf{V}\boldsymbol{\Lambda}\mathbf{I}\boldsymbol{\Lambda}\mathbf{V}' = \mathbf{V}\boldsymbol{\Lambda}\boldsymbol{\Lambda}\mathbf{V}' = \mathbf{V}\boldsymbol{\Lambda}^2\mathbf{V}'$$

The eigenvalues in $\boldsymbol{\Lambda}^2$ are all squared values of the eigenvalues of $\boldsymbol{\Lambda}$, so they are either zero or a positive value. Thus, $\mathbf{A}'\mathbf{A}$ is nonnegative definite.

Using the hint, $\mathbf{y} = \mathbf{A}\mathbf{x}$,

$$\mathbf{y}'\mathbf{y} = \left(\mathbf{A}\mathbf{x}\right)'\left(\mathbf{A}\mathbf{x}\right) = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{x}$$

As explained on page 62 **(2-17)**, this is in quadratic form. The matrix, $\mathbf{B} = \mathbf{A}'\mathbf{A}$, is $p \times p$ and is nonnegative definite. This could also be explained as, $\mathbf{y}'\mathbf{y} = y_1^2 + \cdots + y_p^2 = \|\mathbf{y}\|^2$, the sum of squared values, and so the sum cannot be negative, so must be at least zero (nonnegative definite).

2.17

Prove that every eigenvalue of a $k \times k$ positive definite matrix $\mathbf{A}$ is positive.
Hint: Consider the definition of an eigenvalue, where $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$. Multiply on the left by $\mathbf{e}'$ so that $\mathbf{e}'\mathbf{A}\mathbf{e} = \lambda\mathbf{e}'\mathbf{e}$.

Using the hint,

$$\begin{aligned}
\mathbf{A}\mathbf{e} &= \lambda\mathbf{e} \\
\Rightarrow \mathbf{e}'\mathbf{A}\mathbf{e} &= \mathbf{e}'\lambda\mathbf{e} \\
\Rightarrow \mathbf{e}'\mathbf{A}\mathbf{e} &= \lambda\mathbf{e}'\mathbf{e} \\
\Rightarrow \mathbf{e}'\mathbf{A}\mathbf{e} &= \lambda(1) \\
\Rightarrow \mathbf{e}'\mathbf{A}\mathbf{e} &= \lambda
\end{aligned}$$

We now have a positive definite form, like in the definition of positive definiteness **(2-18)**, $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$, so all of the eigenvalues nust be positive, that is,

$$\Rightarrow \mathbf{e}'\mathbf{A}\mathbf{e} = \lambda > 0$$

2.18

Consider the sets of points (x_1, x_2) whose 'distances' from the origin are given by

$$c^2 = 4x_1^2 + 3x_2^2 - 2\sqrt{2}x_1x_2$$

for $c^2 = 1$ and for $c^2 = 4$. Determine the major and minor axes of the ellipses of constant distances and their associated lengths. Sketch the ellipses of constant distances and comment on their positions. What will happen as c^2 increases?

Converting the quadratic polynomial to a matrix,

$$c^2 = 4x_1^2 + 3x_2^2 - 2\sqrt{2}x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}'\mathbf{A}\mathbf{x}$$

Finding the eigenvalues and eigenvectors,

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & -\sqrt{2} \\ -\sqrt{2} & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 12 - 2 =$$

$$= (\lambda - 2)(\lambda - 5)$$

The eigenvalues are $\{\lambda_1, \lambda_2\} = \{2, 5\}$. Finding the eigenvectors,

<u>For $\lambda_1 = 2$:</u>

$$\mathbf{A}\mathbf{x}_1 - \lambda_1\mathbf{x}_1 =$$

$$\begin{bmatrix} 4 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \xrightarrow{\text{Row } 2 + (\sqrt{2}/2)\text{Row } 1} \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

So $2x_1 - \sqrt{2}x_2 = 0 \Rightarrow 2x_1 = \sqrt{2}x_2 \Rightarrow x_1 = \frac{\sqrt{2}}{2}x_2$. Pick,

$$\mathbf{x}_1 = \begin{bmatrix} \sqrt{2}/2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}\|} = \frac{1}{\sqrt{3/2}} \begin{bmatrix} \sqrt{2}/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}$$

<u>For $\lambda_1 = 5$:</u>

$$\mathbf{A}\mathbf{x}_2 - \lambda_2\mathbf{x}_2 =$$

$$\begin{bmatrix} 4 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 5x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \xrightarrow{\text{Row } 2 - (\sqrt{2})\text{Row } 1} \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

So $x_1 + \sqrt{2}x_2 = 0 \Rightarrow x_1 = -\sqrt{2}x_2 \to x_2 = -\frac{\sqrt{2}}{2}x_1$. Pick,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -\sqrt{2}/2 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}\|} = \frac{1}{\sqrt{3/2}} \begin{bmatrix} 1 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

We now have all the parts,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

and

$$\mathbf{P} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

<u>As an aside:</u>

$$\mathbf{A} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \lambda_2 \mathbf{e}_2 \mathbf{e}_2' \Rightarrow$$

$$\Rightarrow \mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}' \left(\lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \lambda_2 \mathbf{e}_2 \mathbf{e}_2'\right) \mathbf{x} = \lambda_1 \mathbf{x}'\mathbf{e}_1 \mathbf{e}_1' \mathbf{x} + \lambda_2 \mathbf{x}'\mathbf{e}_2 \mathbf{e}_2' \mathbf{x} =$$

$$= \lambda_1 \mathbf{x}'\mathbf{e}_1 (\mathbf{x}'\mathbf{e}_1)' + \lambda_2 \mathbf{x}'\mathbf{e}_2 (\mathbf{x}'\mathbf{e}_2)' = \lambda_1 (\mathbf{x}'\mathbf{e}_1)^2 + \lambda_2 (\mathbf{x}'\mathbf{e}_2)^2 = \lambda_1 y_1^2 + \lambda_2 y_2^2 =$$

$$\Rightarrow c^2 = \mathbf{x}'\mathbf{A}\mathbf{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$\Rightarrow \frac{\lambda_1 y_1^2}{c^2} + \frac{\lambda_2 y_2^2}{c^2} = 1$$

$$\Rightarrow \frac{y_1^2}{\left(c/\sqrt{\lambda_1}\right)^2} + \frac{y_2^2}{\left(c/\sqrt{\lambda_2}\right)^2} = 1$$

The formula for a horizontal ellipse is,

$$\frac{(y_1 - h)^2}{b^2} + \frac{(y_2 - k)^2}{a^2} = 1$$

Where the ellipse is centered at $(y_1, y_2) = (h, k)$. The major axis are at $\pm b$ and the minor axis are at $\pm a$. Here, for quadratic form we are centered at the origin, so $h = k = 0$ the major axis are in the direction of $\mathbf{e}_1$ with length $b = \pm c/\sqrt{\lambda_1}$, and the minor axis are in the direction of $\mathbf{e}_2$ with length $\pm a = c/\sqrt{\lambda_2}$. Note that $0 \leq \lambda_1 \leq \lambda_2$.

For $c^2 = 1$ with our data, the <u>major axis</u> are in the direction of

$$\mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}$$

with length $\pm 1/\sqrt{2}$. These are the red circles in figure 1.
The <u>minor axis</u> are in the direction of

$$\mathbf{e}_2 = \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

with length $\pm 1/\sqrt{5}$. These are the green circles in figure 1. When $c^2 = 4$, the direction vectors stay the same, but the lengths change to $\pm 2/\sqrt{2}$ in the $\mathbf{e}_1$ direction and $\pm 2/\sqrt{5}$ in the $\mathbf{e}_2$ direction. The major and minor axis are represented in figure 2, where the red circles are the major axis and the green circles are the minor axis. The plot of the $c^2 = 1$ ellipse is in figure 1, and the plot of the ellipse for $c^2 = 4$ is in figure 2. As c^2 increases the ellipse grows larger maintaining the same shape.

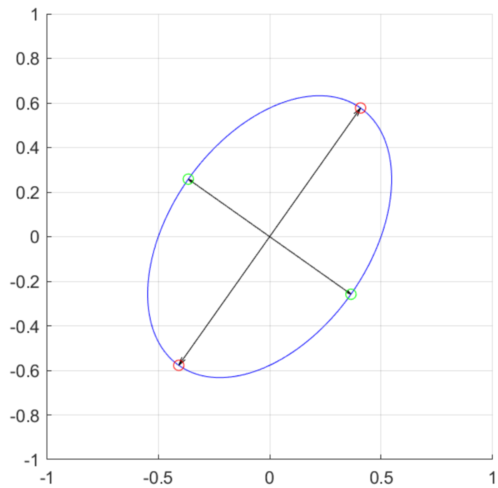


Figure 1: When $c^2 = 1$

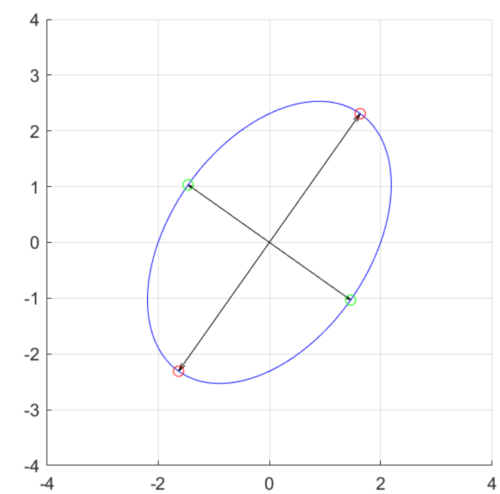


Figure 2: When $c^2 = 4$

MATLAB code:

```
A = [4 -sqrt(2); -sqrt(2) 3];
[V,D] = eig(A);
rref(A - D(1,1)*eye(width(A)))
rref(A - D(2,2)*eye(width(A)))
MyPlotEllipse(V,D,1,'sol2.18.c1')
MyPlotEllipse(V,D,4,'sol2.18.c4')
```

```
        function [] = MyPlotEllipse(V, D, c, fName)
% Compute points corresponding to axis-oriented ellipse.
% Where to center the ellipse.
xc = 0;
yc = 0;
% The length in the major and minor axis.
b = c/sqrt(D(1,1));
a = c/sqrt(D(2,2));
theta = acos(-V(:,1)'*[1 ; 0]); % acos(1/sqrt(3));

t = linspace(0, 2*pi, 200);
xt = b * cos(t) + xc;
yt = a * sin(t) + yc;

% Apply rotation by angle theta (in radians).
cot = cos(theta); sit = sin(theta);
x = xt * cot - yt * sit;
y = xt * sit + yt * cot;

hold on
    % Plot the ellipse.
    p=plot(x, y, '-', 'Color', 'blue');

    % Plot the vector for the major axis.
    quiver(0, 0, V(1,1), V(2,1), c, 'color', 'k');
    quiver(0, 0, -V(1,1), -V(2,1), c, 'color', 'k');

    % Plot the vector for the minor axis.
    quiver(0, 0, V(1,2), V(2,2), c, 'color', 'k');
    quiver(0, 0, -V(1,2), -V(2,2), c, 'color', 'k');

    % Plot red point for major axis.
    plot((c/sqrt(D(1,1)))*V(1,1), (c/sqrt(D(1,1)))*V(2,1), 'o', 'Color', 'red');
```

```
        plot(-(c/sqrt(D(1,1)))*V(1,1), -(c/sqrt(D(1,1)))*V(2,1), 'o', 'Color', 'red');

        % Plot green point for minor axis.
        plot((c/sqrt(D(2,2)))*V(1,2), (c/sqrt(D(2,2)))*V(2,2), 'o', 'Color', 'green');
        plot(-(c/sqrt(D(2,2)))*V(1,2), -(c/sqrt(D(2,2)))*V(2,2), 'o', 'Color', 'green');
        title(append('c^2 = ', num2str(c)))
        grid on
        pbaspect([1 1 1])
    hold off
    saved_file = append('.\applied-multivariate-statistics\solutions\chapter-2\', fName, '.png');
    saveas(p, saved_file, 'png')
end
```

2.19

Let $\underset{(m\times m)}{\mathbf{A}^{1/2}} = \sum_{i=1}^{m}\sqrt{\lambda_i}\mathbf{e}_i\mathbf{e}_i' = \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'$, where $\mathbf{P}\mathbf{P}' = \mathbf{P}'\mathbf{P} = \mathbf{I}$. (The λ_i's and the $\mathbf{e}$'s are the eigenvalues and associated normalized eigenvectors of the matrix $\mathbf{A}$.) Show Properties **(1)-(4)** of the square-root matrix in **(2-22)**.

(1) $\left(\mathbf{A}^{1/2}\right)' = \mathbf{A}^{1/2}$ (that is, $\mathbf{A}$ is symmetric).

$$\left(\mathbf{A}^{1/2}\right)' = \left(\mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'\right)' \overset{\text{Exercise 2.3(c)}}{=} (\mathbf{P}')'\left(\mathbf{\Lambda}^{1/2}\right)'(\mathbf{P})' =$$
$$= \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}' = \mathbf{A}^{1/2}$$

(2) $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$.

$$\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \left(\mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'\right)\left(\mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'\right) = \mathbf{P}\mathbf{\Lambda}^{1/2}\left(\mathbf{P}'\mathbf{P}\right)\mathbf{\Lambda}^{1/2}\mathbf{P}' =$$
$$= \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{I}\mathbf{\Lambda}^{1/2}\mathbf{P}' = \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{P}' =$$
$$= \mathbf{P}\begin{bmatrix}\sqrt{\lambda_1} & 0 & \cdots & 0\\ 0 & \sqrt{\lambda_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sqrt{\lambda_m}\end{bmatrix}\begin{bmatrix}\sqrt{\lambda_1} & 0 & \cdots & 0\\ 0 & \sqrt{\lambda_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sqrt{\lambda_m}\end{bmatrix}\mathbf{P}' =$$
$$\mathbf{P}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_m\end{bmatrix}\mathbf{P}' = \mathbf{P}\mathbf{\Lambda}\mathbf{P}' = \mathbf{A}$$

(3) $\left(\mathbf{A}^{1/2}\right)^{-1} = \sum_{i=1}^{k} \frac{1}{\sqrt{\lambda_i}} \mathbf{e}_i \mathbf{e}_i' = \mathbf{P}\mathbf{\Lambda}^{-1/2}\mathbf{P}'$, where $\mathbf{\Lambda}^{-1/2}$ is a diagonal matrix with $1/\sqrt{\lambda_i}$ as the ith diagonal element.

First off, something useful,

$$\mathbf{\Lambda}^{-1/2}\mathbf{\Lambda}^{1/2} = \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_m} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_m} \end{bmatrix} = \mathbf{I}$$

$$\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{-1/2} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_m} \end{bmatrix} \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_m} \end{bmatrix} = \mathbf{I}$$

We have $\mathbf{\Lambda}^{-1/2}\mathbf{\Lambda}^{1/2} = \mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{-1/2} = \mathbf{I}$, so $\mathbf{\Lambda}^{-1/2} = \left(\mathbf{\Lambda}^{1/2}\right)^{-1}$ is the inverse of $\mathbf{\Lambda}^{1/2}$ by **Definition 2A.27**.

$$\left(\mathbf{A}^{1/2}\right)^{-1} = \left(\mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{P}'\right)^{-1} \overset{Exercise 2.4(b)}{=} (\mathbf{P}')^{-1}\left(\mathbf{\Lambda}^{1/2}\right)^{-1}\mathbf{P}^{-1} =$$

$$\overset{\mathbf{P}'=\mathbf{P}^{-1}}{=} (\mathbf{P}')'\left(\mathbf{\Lambda}^{1/2}\right)^{-1}\mathbf{P}' = \mathbf{P}\left(\mathbf{\Lambda}^{1/2}\right)^{-1}\mathbf{P}'$$

$$= \mathbf{P}\mathbf{\Lambda}^{-1/2}\mathbf{P}' = \sum_{i=1}^{m} \frac{1}{\sqrt{\lambda_i}} \mathbf{e}_i \mathbf{e}_i'$$

(4) $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$, and $\mathbf{A}^{-1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1}$, where $\mathbf{A}^{-1/2} = \left(\mathbf{A}^{1/2}\right)^{-1}$.

Above in **(3)** it was shown that $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$, so $\mathbf{A}^{-1/2} = \left(\mathbf{A}^{1/2}\right)^{-1}$.

$$\mathbf{A}^{-1/2}\mathbf{A}^{-1/2} = \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_m} \end{bmatrix} \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_m} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_m \end{bmatrix} = \mathbf{A}^{-1}$$

2.20

Determine the square-root matrix $\mathbf{A}^{1/2}$, using the matrix $\mathbf{A}$ in **Exercise 2.3**. Also, determine $\mathbf{A}^{-1/2}$, and show that $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 = \lambda^2 - 5\lambda + 6 - 1 =$$

$$= \left(\lambda - \frac{5-\sqrt{5}}{2}\right)\left(\lambda - \frac{5+\sqrt{5}}{2}\right)$$

The eigenvalues are $\{\lambda_1, \lambda_2\} = \left\{\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right\}$.
To sumplify things, define variables a and b,

$$a = (5 - \sqrt{(}5))^{1/2}$$

$$b = (5 + \sqrt{(}5))^{1/2}$$

Using a few useful facts,

$$a^{1/2}(1 + 5^{1/2}) = 2b^{1/2}$$

and

$$b^{1/2}(1 - 5^{1/2}) = -2a^{1/2}$$

$\underline{\lambda_1 = \frac{5-\sqrt{5}}{2}:}$

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{5-\sqrt{5}}{2}\right)x_1 \\ \left(\frac{5-\sqrt{5}}{2}\right)x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \left(\frac{-1+\sqrt{5}}{2}\right) & 1 \\ 1 & \left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{-1+\sqrt{5}}{2}\right) & 1 \\ 1 & \left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} \xrightarrow{\text{Row } 2 - \left(\frac{2}{\sqrt{5}-1}\right)\text{Row } 1} \begin{bmatrix} \left(\frac{-1+\sqrt{5}}{2}\right) & 1 \\ 0 & 0 \end{bmatrix}$$

So $\left(\frac{-1+5^{1/2}}{2}\right)x_1 + x_2 = 0 \Rightarrow x_2 = -\left(\frac{-1+5^{1/2}}{2}\right)x_1 = \left(\frac{1-5^{1/2}}{2}\right)x_1$
We have $x_2 = \left(\frac{1-5^{1/2}}{2}\right)x_1$, so when $x = -1$, $x_2 = -\left(\frac{1-5^{1/2}}{2}\right) > 0$.

$$\|\mathbf{x}_1\| = \frac{\left(5 - 5^{1/2}\right)^{1/2}}{2^{1/2}} = \frac{a^{1/2}}{2^{1/2}}$$

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -\left(\frac{1-5^{1/2}}{2}\right) \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{2^{1/2}}{a^{1/2}} \begin{bmatrix} -1 \\ -\left(\frac{1-5^{1/2}}{2}\right) \end{bmatrix} = \begin{bmatrix} -\frac{2^{1/2}}{a^{1/2}} \\ -\left(\frac{1-5^{1/2}}{2^{1/2}a^{1/2}}\right)\left(\frac{1+5^{1/2}}{1+5^{1/2}}\right) \end{bmatrix} = \begin{bmatrix} -(2/a)^{1/2} \\ (2/b)^{1/2} \end{bmatrix}$$

$\underline{\lambda_2 = \frac{5+\sqrt{5}}{2}:}$

$$\mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{5+\sqrt{5}}{2}\right) x_1 \\ \left(\frac{5+\sqrt{5}}{2}\right) x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{-1-\sqrt{5}}{2}\right) & 1 \\ 1 & \left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{-1-\sqrt{5}}{2}\right) & 1 \\ 1 & \left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix} \xrightarrow{\text{Row } 2+\left(\frac{2}{1+\sqrt{5}}\right)\text{Row } 1} \begin{bmatrix} \left(\frac{-1-\sqrt{5}}{2}\right) & 1 \\ 0 & 0 \end{bmatrix}$$

So $\left(\frac{-1-\sqrt{5}}{2}\right) x_1 + x_2 = 0 \Rightarrow x_2 = \left(\frac{1+\sqrt{5}}{2}\right) x_1$

We have $x_2 = \left(\frac{1+5^{1/2}}{2}\right) x_1$, so when $x = 1$, $x_2 = \left(\frac{1+5^{1/2}}{2}\right) > 0$.

$$\|\mathbf{x}_2\| = \frac{\left(5 + 5^{1/2}\right)^{1/2}}{2^{1/2}} = \frac{b^{1/2}}{2^{1/2}}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ \left(\frac{1+5^{1/2}}{2}\right) \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{2^{1/2}}{b^{1/2}} \begin{bmatrix} 1 \\ \left(\frac{1+5^{1/2}}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{2^{1/2}}{b^{1/2}} \\ \left(\frac{1+5^{1/2}}{2^{1/2}b^{1/2}}\right)\left(\frac{1-5^{1/2}}{1-5^{1/2}}\right) \end{bmatrix} = \begin{bmatrix} (2/b)^{1/2} \\ (2/a)^{1/2} \end{bmatrix}$$

We finally have all of the eigenvalues and eigenvectors,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a/2 & 0 \\ 0 & b/2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} (2/a)^{1/2} & (2/b)^{1/2} \\ -(2/b)^{1/2} & (2/a)^{1/2} \end{bmatrix}$$

Now to find $\mathbf{A}^{1/2}$. Rewrite a few things,

$$\mathbf{A}^{1/2} = 2^{1/2} \begin{bmatrix} \left(b/a^2\right)^{1/2} & \left(a/b^2\right)^{1/2} \\ \left(a/b^2\right)^{1/2} & \left(2^2/a\right)^{1/2} \end{bmatrix}$$

2.21

(See **Result 2A.15**) Using the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

(a) Calculate $\mathbf{A}'\mathbf{A}$ and obtain the eigenvalues and eigenvectors.

$$\mathbf{A}'\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

$$0 = |\mathbf{A}'\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 9-\lambda & 1 \\ 1 & 9-\lambda \end{vmatrix} = (9-\lambda)^2 - 1 = \lambda^2 - 18\lambda + 81 - 1 = (\lambda - 8)(\lambda - 10)$$

$$\left(\lambda_1^2, \lambda_1^2\right) = (8, 10)$$

$\underline{\lambda_1^2 = 8:}$

$$\mathbf{A}'\mathbf{A}\mathbf{x}_1 = \lambda_1^2 \mathbf{x}_1 \Rightarrow \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8x_1 \\ 8x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Row 2 - Row 1}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$. Pick,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\underline{\lambda_2^2 = 10:}$

$$\mathbf{A}'\mathbf{A}\mathbf{x}_1 = \lambda_1^2 \mathbf{x}_1 \Rightarrow \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Row 2 + Row 1}} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $-x_1 + x_2 = 0 \Rightarrow x_1 = x_2$. Pick,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_2 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

The eigenvectors are,

$$\mathbf{V} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(b) Calculate $\mathbf{A}\mathbf{A}'$ and obtain the eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part a.

$$\mathbf{A}\mathbf{A}' = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

$$0 = |\mathbf{AA}' - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 8-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{vmatrix} = (2-\lambda)(8-\lambda)^2 - 4(4(8-\lambda)) =$$

$$= (8-\lambda)[(2-\lambda)(8-\lambda) - 16] = (8-\lambda)\left(\lambda^2 - 10\lambda\right) = \lambda(8-\lambda)(\lambda - 10)$$

$$\left(\lambda_1^2, \lambda_2^2, \lambda_2^2\right) = (0, 8, 10)$$

Yes, these nonzero eigenvales are the same as those in part a.

$\underline{\lambda_1^2 = 0}$:

$$\mathbf{AA}'\mathbf{x}_1 = \lambda_1^2 \mathbf{x}_1 \Rightarrow$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Row 3 - 2Row 1}} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $2x_1 + 4x_3 = 0 \Rightarrow x_1 = -2x_3$ and $8x_2 = 0 \Rightarrow x_2 = 0$. Pick,

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

$\underline{\lambda_2^2 = 8}$:

$$\mathbf{AA}'\mathbf{x}_2 = \lambda_2^2 \mathbf{x}_2 \Rightarrow$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $-6x_1 + 4x_3 = 0 \Rightarrow x_1 = 2/3x_3$ and $4x_1 = 0 \Rightarrow x_1 = 0$. Pick,

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_3^2 = 10}$:

$$\mathbf{AA}'\mathbf{x}_2 = \lambda_2^2 \mathbf{x}_2 \Rightarrow$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \\ 10x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -8 & 0 & 4 \\ 0 & -2 & 0 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Row 3 + 1/2 Row 1}} \begin{bmatrix} -8 & 0 & 4 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $-8x_1 + 4x_3 = 0 \Rightarrow x_3 = 2x_1$ and $-2x_2 = 0 \Rightarrow x_3 = 0$. Pick,

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$$

The eigenvectors are,

$$\mathbf{U} = \begin{bmatrix} 0 & 1/\sqrt{5} \\ 1 & 0 \\ 0 & 2/\sqrt{5} \end{bmatrix}$$

(c) Obtain the sigular-value decomposition of $\mathbf{A}$.

$$\mathbf{A} = \mathbf{U\Lambda V}' = \begin{bmatrix} 0 & 1/\sqrt{5} \\ 1 & 0 \\ 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \mathbf{A}$$

$$\mathbf{A} = \sum_{k=1}^{2} \lambda_k \mathbf{e}_k \mathbf{e}_k' = \sqrt{8} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + \sqrt{10} \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} =$$

$$= \sqrt{8} \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \end{bmatrix} + \sqrt{10} \begin{bmatrix} 1/\sqrt{10} & 1/\sqrt{10} \\ 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \mathbf{A}$$

2.22

(See **Result 2A.15**) Using the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

(a) Calculate $\mathbf{AA}'$ and obtain its eigenvalues and eigenvectors.

$$\mathbf{AA}' = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix} = 6 \times \begin{bmatrix} 24 & -2 \\ -2 & 21 \end{bmatrix}$$

$$0 = \left|\mathbf{AA}' - \lambda^2 \mathbf{I}\right| = |\mathbf{AA}' - \gamma \mathbf{I}| = \begin{vmatrix} 24-\gamma & -2 \\ -2 & 21-\gamma \end{vmatrix} =$$

$$= (24-\gamma)(21-\gamma) - 4 = 504 - 45\gamma + \gamma^2 - 4 = \gamma^2 - 45\gamma + 500 = (\gamma - 25)(\gamma - 20)$$

The two eigenvalues are:
$\gamma_1 = (1/6)\lambda_1^2 = 20 \Rightarrow \lambda_1 = \sqrt{6 \times 20} = \sqrt{120}$

$\gamma_2 = (1/6)\lambda_2^2 = 25 \Rightarrow \lambda_2 = \sqrt{6 \times 25} = \sqrt{150}$

<u>For $\gamma_1 = 20$:</u>

$$\mathbf{AA'x}_1 = \gamma_1 \mathbf{x} \Rightarrow \begin{bmatrix} 24 & -2 \\ -2 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20x_1 \\ 20x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Row 2 + (1/2) Row 1}} \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $4x_1 - 2x_2 = 0 \Rightarrow x_2 = 2x_1$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

<u>For $\gamma_1 = 25$:</u>

$$\mathbf{AA'x}_2 = \gamma_2 \mathbf{x} \Rightarrow \begin{bmatrix} 24 & -2 \\ -2 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 25x_1 \\ 25x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Row 2 - 2 Row 1}} \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $-x_1 - 2x_2 = 0 \Rightarrow x_1 = -2x_2$

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \sqrt{120} & 0 \\ 0 & \sqrt{150} \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

(b) Calculate $\mathbf{A'A}$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part a.

$$\mathbf{A'A} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} = 5 \times \begin{bmatrix} 5 & 10 & 1 \\ 10 & 20 & 2 \\ 1 & 2 & 29 \end{bmatrix}$$

$$0 = \left|\mathbf{A'A} - \lambda^2 \mathbf{I}\right| = |\mathbf{A'A} - \gamma \mathbf{I}| = \begin{vmatrix} 5-\gamma & 10 & 1 \\ 10 & 20-\gamma & 2 \\ 1 & 2 & 29-\gamma \end{vmatrix} =$$

$$= (5-\gamma)[(20-\gamma)(29-\gamma) - 4] - 10[10(29-\gamma) - 2] + [20 - (20-\gamma)] =$$

$$= (29\gamma^2 - 725\gamma + 2900) + (-\gamma^3 + 25\gamma^2 - 100\gamma) - 40 + 5\gamma - 2900 + 100\gamma + 40 =$$

$$= -\gamma(\gamma - 24)(\gamma - 30)$$

The two nonzero eigenvalues are:
$\gamma_1 = (1/5)\lambda_1^2 = 24 \Rightarrow \lambda_1 = \sqrt{5 \times 24} = \sqrt{120}$
$\gamma_2 = (1/5)\lambda_2^2 = 30 \Rightarrow \lambda_2 = \sqrt{5 \times 30} = \sqrt{150}$

These are the same two eigenvalues as in part (a).

For $\gamma_1 = 24$:

$$\mathbf{A}'\mathbf{A}\mathbf{x}_1 = \gamma_1\mathbf{x} \Rightarrow \begin{bmatrix} 5 & 10 & 1 \\ 10 & 20 & 2 \\ 1 & 2 & 29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24x_1 \\ 24x_2 \\ 24x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -19 & 10 & 1 \\ 10 & -4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -19 & 10 & 1 \\ 10 & -4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \xrightarrow{\text{Swap Row 1 with Row 3}} \begin{bmatrix} 1 & 2 & 5 \\ 10 & -4 & 2 \\ -19 & 10 & 1 \end{bmatrix} \xrightarrow{\text{Row 3 + 19 Row 1}} \begin{bmatrix} 1 & 2 & 5 \\ 10 & -4 & 2 \\ 0 & 48 & 96 \end{bmatrix}$$

$$\xrightarrow{\text{Row 2 - 10 Row 1}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -24 & -48 \\ 0 & 48 & 96 \end{bmatrix} \xrightarrow{\text{Simplify rows}} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Row 3 - Row 2}}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1 - 2 Row 2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$ and $x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$.

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

For $\gamma_2 = 30$:

$$\mathbf{A}'\mathbf{A}\mathbf{x}_1 = \gamma_1\mathbf{x} \Rightarrow \begin{bmatrix} 5 & 10 & 1 \\ 10 & 20 & 2 \\ 1 & 2 & 29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30x_1 \\ 30x_2 \\ 30x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -25 & 10 & 1 \\ 10 & -10 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -25 & 10 & 1 \\ 10 & -10 & 2 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{Swap Row 1 with Row 3}} \begin{bmatrix} 1 & 2 & -1 \\ 10 & -10 & 2 \\ -25 & 10 & 1 \end{bmatrix} \xrightarrow{\text{Row 3 + 25 Row 1}} \begin{bmatrix} 1 & 2 & -1 \\ 10 & -10 & 2 \\ 0 & 60 & -24 \end{bmatrix}$$

$$\xrightarrow{\text{Row 2 - 10 Row 1}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -30 & 12 \\ 0 & 48 & 96 \end{bmatrix} \xrightarrow{\text{Simplify rows}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 5 & -2 \end{bmatrix} \xrightarrow{\text{Row 3 + Row 2}}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1 - (2/5) Row 2}} \begin{bmatrix} 1 & 0 & -1/5 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 - (1/5)x_3 = 0 \Rightarrow x_3 = 5x_1$ and $-5x_2 + 2x_3 = 0 \Rightarrow x_2 = (2/5)x_3$.

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \sqrt{120} & 0 \\ 0 & \sqrt{150} \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{30} \\ -2/\sqrt{6} & 2/\sqrt{30} \\ 2/\sqrt{6} & 5/\sqrt{30} \end{bmatrix}$$

(c) Obtain the Singular value decomposition of $\mathbf{A}$.

Need to multiply SVD by -1, so we get the right result. Back when computing the eigenvalues notice the negative in front.

$$\mathbf{A} = \mathbf{U\Lambda V}' = (-1) \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{120} & 0 \\ 0 & \sqrt{150} \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & -2/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{30} & 5/\sqrt{30} \end{bmatrix} =$$

$$= \left(\frac{1}{\sqrt{5}}\right) \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \left(\sqrt{5}\sqrt{6}\right) \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \left(\frac{1}{\sqrt{6}}\right) \begin{bmatrix} -1 & -2 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 5/\sqrt{5} \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -2 & -4 & 2 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \mathbf{A}$$

2.23

Verify the relationships $\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \mathbf{\Sigma}$ and $\boldsymbol{\rho} = \left(\mathbf{V}^{1/2}\right)^{-1}\mathbf{\Sigma}\left(\mathbf{V}^{1/2}\right)^{-1}$, where $\mathbf{\Sigma}$ is the $p \times p$ population covariance matrix [**Equation (2-32)**], $\boldsymbol{\rho}$ is the $p \times p$ population correlation matrix [**Equation (2-34)**], and $\mathbf{V}^{1/2}$ is the population standard deviation matrix [**Equation (2-35)**].

$$\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\sigma_{pp}} \end{bmatrix} \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{bmatrix} \begin{bmatrix} \sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\sigma_{pp}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{pp}}} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}} \end{bmatrix} \begin{bmatrix} \sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\sigma_{pp}} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix} = \mathbf{\Sigma}$$

$$\left(\mathbf{V}^{1/2}\right)^{-1}\mathbf{\Sigma}\left(\mathbf{V}^{1/2}\right)^{-1} =$$

$$= \left(\begin{bmatrix} \sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\sigma_{pp}} \end{bmatrix}\right)^{-1} \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix} \left(\begin{bmatrix} \sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\sigma_{pp}} \end{bmatrix}\right)^{-1} =$$

$$= \begin{bmatrix} 1/\sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & 1/\sqrt{\sigma_{pp}} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix} \begin{bmatrix} 1/\sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & 1/\sqrt{\sigma_{pp}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/\sqrt{\sigma_{11}} & & 0 \\ & \ddots & \\ 0 & & 1/\sqrt{\sigma_{pp}} \end{bmatrix} \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{pp}}} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{bmatrix} = \boldsymbol{\rho}$$

2.24

Let $\mathbf{X}$ have covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find

(a) $\mathbf{\Sigma}^{-1}$

Using what's on page 59, the inverse of a diagonal matrix is the reciprocal of the elements.

$$\mathbf{\Sigma}^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) The eigenvalues and eigenvectors of $\mathbf{\Sigma}$.

$$0 = |\mathbf{\Sigma} - \lambda \mathbf{I}| = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (4-\lambda)(9-\lambda)(1-\lambda)$$

The eigenvalues are simply the diagonal elements of $\mathbf{\Sigma}$, $(\lambda_1, \lambda_2, \lambda_3) = (1, 4, 9)$.

$\underline{\lambda_1 = 1}$:

$$\mathbf{\Sigma}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_1 = x_2 = 0$ and x_3 is free.

$$\mathbf{x}_1 = \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 4}$:

$$\mathbf{\Sigma}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_2 = x_3 = 0$ and x_1 is free.

$$\mathbf{x}_2 = \mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda_3 = 9}$:

$$\mathbf{\Sigma}\mathbf{x}_3 = \lambda_3\mathbf{x}_3 \Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9x_1 \\ 9x_2 \\ 9x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_1 = x_3 = 0$ and x_2 is free.

$$\mathbf{x}_3 = \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Putting it all together,

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The eigenvalues are the elements in the diagonal matrix and the eigenvectors are the identity matrix.

(c) The eigenvalues and eigenvectors of $\mathbf{\Sigma}^{-1}$.

$$0 = |\mathbf{\Sigma} - \lambda\mathbf{I}| = \begin{bmatrix} 1/4 - \lambda & 0 & 0 \\ 0 & 1/9 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix} = (1/4 - \lambda)\,(1/9 - \lambda)\,(1/9 - \lambda)$$

The eigenvalues are simply the diagonal elements of $\mathbf{\Sigma}$, $(\lambda_1, \lambda_2, \lambda_3) = (1/9, 1/4, 1)$.

$\underline{\lambda_1 = 1/9}$:

$$\mathbf{\Sigma}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \Rightarrow \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/9)x_1 \\ (1/9)x_2 \\ (1/9)x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5/36 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_1 = x_3 = 0$ and x_2 is free.

$$\mathbf{x}_1 = \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_2 = 1/4}$:

$$\mathbf{\Sigma}\mathbf{x}_2 = \lambda_1\mathbf{x}_2 \Rightarrow \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/4)x_1 \\ (1/4)x_2 \\ (1/4)x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5/36 & 0 \\ 0 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_2 = x_3 = 0$ and x_1 is free.

$$\mathbf{x}_2 = \mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\lambda_3 = 1}$:

$$\mathbf{\Sigma}\mathbf{x}_3 = \lambda_3\mathbf{x}_3 \Rightarrow \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3/4 & 0 & 0 \\ 0 & -8/9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_1 = x_2 = 0$ and x_3 is free.

$$\mathbf{x}_3 = \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Putting it all together,

$$\mathbf{\Lambda} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues are the elements in the diagonal matrix and the eigenvectors are the identity matrix.

2.25

Let $\mathbf{X}$ have covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

(a) Determine $\boldsymbol{\rho}$ and $\mathbf{V}^{1/2}$.

$$\boldsymbol{\rho} = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \frac{\sigma_{13}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{33}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \frac{\sigma_{23}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{33}}} \\ \frac{\sigma_{31}}{\sqrt{\sigma_{33}}\sqrt{\sigma_{11}}} & \frac{\sigma_{32}}{\sqrt{\sigma_{33}}\sqrt{\sigma_{22}}} & \frac{\sigma_{33}}{\sqrt{\sigma_{33}}\sqrt{\sigma_{33}}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{25}{\sqrt{25}\sqrt{25}} & \frac{-2}{\sqrt{25}\sqrt{4}} & \frac{4}{\sqrt{25}\sqrt{9}} \\ \frac{-2}{\sqrt{4}\sqrt{25}} & \frac{4}{\sqrt{4}\sqrt{4}} & \frac{1}{\sqrt{4}\sqrt{9}} \\ \frac{4}{\sqrt{9}\sqrt{25}} & \frac{1}{\sqrt{9}\sqrt{4}} & \frac{9}{\sqrt{9}\sqrt{9}} \end{bmatrix} = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix}$$

(b) Multiply your matrices to check the relation $\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \mathbf{\Sigma}$.

$$\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

2.26

Use $\mathbf{\Sigma}$ as given in Exercise 2.25.

(a) Find ρ_{13}.

We can pick ρ_{13} off the result from Exercise 2.25 (a),

$$\rho_{13} = 4/15$$

(b) Find the correlation between X_1 and $\frac{1}{2}X_2 + \frac{1}{2}X_3$.

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

$$\mathbf{X}^{(1)} = \begin{bmatrix} X_1 \end{bmatrix} \quad \text{and} \quad \mathbf{X}^{(2)} = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$$

$$Y = \frac{1}{2}X_1 + \frac{1}{2}X_2 = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathbf{c}'\mathbf{X}^{(2)}$$

So $\mathbf{c} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

$$\boldsymbol{\Sigma} = \left[\begin{array}{c:cc} 25 & -2 & 4 \\ \hdashline -2 & 4 & 1 \\ 4 & 1 & 9 \end{array}\right] = \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right] =$$

$$\operatorname{Cov}\left(\mathbf{X}^{(1)}, Y\right) = \operatorname{Cov}\left(\mathbf{X}^{(1)}, \mathbf{c}'\mathbf{X}^{(2)}\right) =$$

$$= E\left[\left(\mathbf{X}^{(1)} - E\left[\mathbf{X}^{(1)}\right]\right)\left(\mathbf{c}'\mathbf{X}^{(2)} - E\left[\mathbf{c}'\mathbf{X}^{(2)}\right]\right)'\right] =$$

$$= E\left[\left(\mathbf{X}^{(1)} - E\left[\mathbf{X}^{(1)}\right]\right)\left(\mathbf{c}'\mathbf{X}^{(2)} - \mathbf{c}'E\left[\mathbf{X}^{(2)}\right]\right)'\right] =$$

$$= E\left[\left(\mathbf{X}^{(1)} - E\left[\mathbf{X}^{(1)}\right]\right)\left\{\mathbf{c}'\left(\mathbf{X}^{(2)} - E\left[\mathbf{X}^{(2)}\right]\right)\right\}'\right] =$$

$$= E\left[\left(\mathbf{X}^{(1)} - E\left[\mathbf{X}^{(1)}\right]\right)\left\{\left(\mathbf{X}^{(2)} - E\left[\mathbf{X}^{(2)}\right]\right)\right\}'\mathbf{c}\right] =$$

$$= E\left[\left(\mathbf{X}^{(1)} - E\left[\mathbf{X}^{(1)}\right]\right)\left\{\left(\mathbf{X}^{(2)} - E\left[\mathbf{X}^{(2)}\right]\right)\right\}'\right]\mathbf{c} =$$

$$= \operatorname{Cov}\left(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right)\mathbf{c} = \begin{bmatrix} -2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1$$

2.27

Derive expressions for the mean and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.

(a) $X_1 - 2X_2$

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$E\left[\mathbf{c}'\mathbf{X}\right] = E\left[\begin{bmatrix} 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right] = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} E\left[\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right] =$$

$$= \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} E\left[X_1\right] \\ E\left[X_2\right] \\ E\left[X_3\right] \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = 1\times\mu_1 - 2\times\mu_2 + 0\times\mu_3 = \mu_1 - 2\mu_2$$

$$\operatorname{V}\left(\mathbf{c}'\mathbf{X}\right) = \operatorname{Cov}\left(\mathbf{c}'\mathbf{X}, \mathbf{c}'\mathbf{X}\right) = E\left[\left(\mathbf{c}'\mathbf{X} - E\left[\mathbf{c}'\mathbf{X}\right]\right)\left(\mathbf{c}'\mathbf{X} - E\left[\mathbf{c}'\mathbf{X}\right]\right)'\right] =$$

$$= \mathbf{c}'E\left[(\mathbf{X}-E[\mathbf{X}])(\mathbf{X}-E[\mathbf{X}])'\right]\mathbf{c} = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} =$$

$$\begin{bmatrix}1 & -2 & 0\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix} = \begin{bmatrix}(\sigma_{11}+\sigma_{21}) & (\sigma_{12}+\sigma_{22}) & (\sigma_{13}+\sigma_{23})\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix} =$$

$$= 1\times(\sigma_{11}+\sigma_{21}) - 2\times(\sigma_{12}+\sigma_{22}) + 0\times(\sigma_{13}+\sigma_{23}) = (\sigma_{11}-2\sigma_{12}) + (\sigma_{21}-2\sigma_{22})$$

(b) $-X_1 + 3X_2$

$$\mathbf{X} = \begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix}-1\\3\\0\end{bmatrix}$$

$$E[\mathbf{c}'\mathbf{X}] = E\left[\begin{bmatrix}-1 & 3 & 0\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] = \begin{bmatrix}-1 & 3 & 0\end{bmatrix}E\left[\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] =$$

$$= \begin{bmatrix}-1 & 3 & 0\end{bmatrix}\begin{bmatrix}E[X_1]\\E[X_2]\\E[X_3]\end{bmatrix} = \begin{bmatrix}-1 & 3 & 0\end{bmatrix}\begin{bmatrix}\mu_1\\\mu_2\\\mu_3\end{bmatrix} = -1\times\mu_1 3\times\mu_2 + 0\times\mu_3 = -\mu_1 + 3\mu_2$$

$$\mathrm{V}(\mathbf{c}'\mathbf{X}) = \mathrm{Cov}(\mathbf{c}'\mathbf{X}, \mathbf{c}'\mathbf{X}) = E\left[(\mathbf{c}'\mathbf{X}-E[\mathbf{c}'\mathbf{X}])(\mathbf{c}'\mathbf{X}-E[\mathbf{c}'\mathbf{X}])'\right] =$$

$$= \mathbf{c}'E\left[(\mathbf{X}-E[\mathbf{X}])(\mathbf{X}-E[\mathbf{X}])'\right]\mathbf{c} = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} =$$

$$\begin{bmatrix}-1 & 3 & 0\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}\begin{bmatrix}-1\\3\\0\end{bmatrix} = \begin{bmatrix}(-\sigma_{11}+3\sigma_{21}) & (-\sigma_{12}+3\sigma_{22}) & (-\sigma_{13}+3\sigma_{23})\end{bmatrix}\begin{bmatrix}-1\\3\\0\end{bmatrix} =$$

$$= -1\times(-\sigma_{11}+3\sigma_{21}) + 3\times(-\sigma_{12}+3\sigma_{22}) + 0\times(-\sigma_{13}+3\sigma_{23}) = \sigma_{11} + 3(\sigma_{21}-\sigma_{12}) + 6\sigma_{22}$$

(c) $X_1 + X_2 + X_3$

$$\mathbf{X} = \begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$E[\mathbf{c}'\mathbf{X}] = E\left[\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] = \begin{bmatrix}1 & 1 & 1\end{bmatrix}E\left[\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] =$$

$$= \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}E[X_1]\\E[X_2]\\E[X_3]\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}\mu_1\\\mu_2\\\mu_3\end{bmatrix} = 1\times\mu_1 + 1\times\mu_2 + 1\times\mu_3 = \mu_1 + \mu_2 + \mu_3$$

$$\mathrm{V}(\mathbf{c}'\mathbf{X}) = \mathrm{Cov}(\mathbf{c}'\mathbf{X}, \mathbf{c}'\mathbf{X}) = E\left[(\mathbf{c}'\mathbf{X}-E[\mathbf{c}'\mathbf{X}])(\mathbf{c}'\mathbf{X}-E[\mathbf{c}'\mathbf{X}])'\right] =$$

$$= \mathbf{c}'E\left[(\mathbf{X}-E[\mathbf{X}])(\mathbf{X}-E[\mathbf{X}])'\right]\mathbf{c} = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} =$$

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}(\sigma_{11}+\sigma_{21}+\sigma_{31}) & (\sigma_{12}+\sigma_{22}+\sigma_{32}) & (\sigma_{13}+\sigma_{23}+\sigma_{33})\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix} =$$

$$= 1\times(\sigma_{11}+\sigma_{21}+\sigma_{31}) + 1\times(\sigma_{12}+\sigma_{22}+\sigma_{32}) + 1\times(\sigma_{13}+\sigma_{23}+\sigma_{33}) =$$

$$= \sum_{i=1}^{3}\sigma_{i1} + \sum_{i=1}^{3}\sigma_{i3} + \sum_{i=1}^{3}\sigma_{i3} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}$$

(d) $X_1 + 2X_2 - X_3$

$$\mathbf{X} = \begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$E[\mathbf{c}'\mathbf{X}] = E\left[\begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] = \begin{bmatrix}1 & 2 & -1\end{bmatrix} E\left[\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] =$$

$$= \begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}E[X_1]\\E[X_2]\\E[X_3]\end{bmatrix} = \begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}\mu_1\\\mu_2\\\mu_3\end{bmatrix} = 1\times\mu_1+2\times\mu_2-1\times\mu_3 = \mu_1+2\mu_2-\mu_3$$

$$\mathrm{V}(\mathbf{c}'\mathbf{X}) = \mathrm{Cov}(\mathbf{c}'\mathbf{X}, \mathbf{c}'\mathbf{X}) = E\left[(\mathbf{c}'\mathbf{X} - E[\mathbf{c}'\mathbf{X}])(\mathbf{c}'\mathbf{X} - E[\mathbf{c}'\mathbf{X}])'\right] =$$

$$= \mathbf{c}'E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])'\right]\mathbf{c} = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} =$$

$$\begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\\sigma_{21} & \sigma_{22} & \sigma_{23}\\\sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix} =$$

$$= \begin{bmatrix}(\sigma_{11}+2\sigma_{21}-\sigma_{31}) & (\sigma_{12}+2\sigma_{22}-\sigma_{32}) & (\sigma_{13}+2\sigma_{23}-\sigma_{33})\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix} =$$

$$= 1\times(\sigma_{11}+2\sigma_{21}-\sigma_{31}) + 2\times(\sigma_{12}+2\sigma_{22}-\sigma_{32}) - 1\times(\sigma_{13}+2\sigma_{23}-\sigma_{33}) =$$

$$= \sigma_{11}+2\sigma_{12}-\sigma_{13}+2(\sigma_{21}+\sigma_{22}-\sigma_{23}) - (\sigma_{31}+\sigma_{32}-\sigma_{33})$$

(e) $3X_1 - 4X_2$ if X_1 and X_2 are independent random variables.

First off, if $X_1 \perp\!\!\!\perp X_2$, then $\sigma_{12} = \sigma_{21} = 0$.

$$\mathbf{X} = \begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix}3\\-4\\0\end{bmatrix}$$

$$E[\mathbf{c}'\mathbf{X}] = E\left[\begin{bmatrix}3 & -4 & 0\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] = \begin{bmatrix}3 & -4 & 0\end{bmatrix} E\left[\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\right] =$$

$$= \begin{bmatrix} 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} E[X_1] \\ E[X_2] \\ E[X_3] \end{bmatrix} = \begin{bmatrix} 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = 3 \times \mu_1 - 4 \times \mu_2 + 0 \times \mu_3 = 3\mu_1 - 4\mu_2$$

$$\mathrm{V}(\mathbf{c}'\mathbf{X}) = \mathrm{Cov}(\mathbf{c}'\mathbf{X}, \mathbf{c}'\mathbf{X}) = E\left[(\mathbf{c}'\mathbf{X} - E[\mathbf{c}'\mathbf{X}])(\mathbf{c}'\mathbf{X} - E[\mathbf{c}'\mathbf{X}])'\right] =$$

$$= \mathbf{c}'E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])'\right]\mathbf{c} = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} =$$

$$\begin{bmatrix} 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} (3\sigma_{11} - 4\sigma_{21}) & (3\sigma_{12} - 4\sigma_{22}) & (3\sigma_{13} - 4\sigma_{23}) \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} =$$

$$= 3 \times (3\sigma_{11} - 4\sigma_{21}) - 4 \times (3\sigma_{12} - 4\sigma_{22}) + 0 \times (3\sigma_{13} - 4\sigma_{23}) =$$

$$= 9\sigma_{11} - 12\sigma_{21} - 12\sigma_{12} - 16\sigma_{22} = 9\sigma_{11} - 0 - 0 - 16\sigma_{22} = 9\sigma_{11} - 16\sigma_{22}$$

2.28

Show that

$$\mathrm{Cov}(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p) = \mathbf{c}_1'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{c}_2$$

where $\mathbf{c}_1' = \left[c_{11}, c_{12}, \ldots, c_{1p}\right]$ and $\mathbf{c}_2' = \left[c_{21}, c_{22}, \ldots, c_{2p}\right]$. This verifies the off-diagonal elements $\mathbf{C}\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{C}'$ in (2-45) or diagonal elements if $\mathbf{c}_1 = \mathbf{c}_2$.
Hint: By **(2-45)**, $Z_1 - E(Z_1) = c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p)$ and $Z_2 - E(Z_2) = c_{21}(X_1 - \mu_1) + \cdots + c_{2p}(X_p - \mu_p)$. So $\mathrm{Cov}(Z_1, Z_2) = E[(Z_1 - E(Z_1))(Z_2 - E(Z_2))] = E[(c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p))(c_{21}(X_1 - \mu_1) + \cdots + c_{2p}(X_p - \mu_p))]$. The product

$$(c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p))(c_{21}(X_1 - \mu_1) + \cdots + c_{2p}(X_p - \mu_p)) =$$

$$= \left(\sum_{\ell=1}^{p} c_{1\ell}(X_\ell - \mu_\ell)\right)\left(\sum_{m=1}^{p} c_{2m}(X_m - \mu_m)\right)$$

$$= \sum_{\ell=1}^{p}\sum_{\ell=1}^{p} c_{1\ell}c_{2m}(X_\ell - \mu_\ell)(X_m - \mu_m)$$

has expected value

$$= \sum_{\ell=1}^{p}\sum_{\ell=1}^{p} c_{1\ell}c_{2m}\sigma_{\ell m} = [c_{11}, \ldots, c_{1p}]\,\boldsymbol{\Sigma}\,[c_{21}, \ldots, c_{2p}]'.$$

Verify the last step by the definition of matrix multiplication. The same steps should hold for all elements.

Ignoring the hint, using the definition,

$$Z_1 = \mathbf{c}_1'\mathbf{X} = \sum_{i=1}^{p} c_{1i} X_i$$

$$Z_2 = \mathbf{c}_2'\mathbf{X} = \sum_{i=1}^{p} c_{2i} X_i$$

$$\begin{aligned}
\mathrm{Cov}\,(Z_1, Z_2) &= \\
&= E\left[(Z_1 - E\left[Z_1\right])\left(Z_2 - E\left[Z_2\right]\right)'\right] = \\
&= E\left[(\mathbf{c}_1'\mathbf{X} - E\left[\mathbf{c}_1'\mathbf{X}\right])\left(\mathbf{c}_2'\mathbf{X} - E\left[\mathbf{c}_2'\mathbf{X}\right]\right)'\right] = \\
&= E\left[\mathbf{c}_1'\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)\left(\mathbf{c}_2'\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)\right)'\right] = \\
&= E\left[\mathbf{c}_1'\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)'\left(\mathbf{c}_2'\right)'\right] = \\
&= E\left[\mathbf{c}_1'\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)'\mathbf{c}_2\right] = \\
&= \mathbf{c}_1' E\left[\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)\left(\mathbf{X} - E\left[\mathbf{X}\right]\right)'\right]\mathbf{c}_2 = \\
&= \mathbf{c}_1'\mathrm{Cov}\,(\mathbf{X})\,\mathbf{c}_2 = \\
&= \mathbf{c}_1'\boldsymbol{\Sigma}\mathbf{c}_2
\end{aligned}$$

Another way, using the hint and multiplying everything out,

$$\mathrm{Cov}\,(Z_1, Z_2) = E\left[(Z_1 - E[Z_2])\left(Z_1 - E[Z_2]\right)'\right] =$$

$$E\left[\left\{\sum_{\ell=1}^{p} c_{1\ell}\left(X_\ell = \mu_\ell\right)\right\}\left\{\sum_{m=1}^{p} c_{2m}\left(X_m = \mu_m\right)\right\}\right] =$$

$$= E[c_{11}c_{21}(X_1-\mu_1)(X_1-\mu_1)+c_{11}c_{22}(X_1-\mu_1)(X_2-\mu_2)+\cdots+c_{11}c_{2p}(X_1-\mu_1)(X_p-\mu_p)+$$
$$+c_{12}c_{21}(X_2-\mu_2)(X_1-\mu_1)+c_{12}c_{22}(X_2-\mu_2)(X_2-\mu_2)+\cdots+c_{12}c_{2p}(X_2-\mu_2)(X_p-\mu_p)+$$
$$\cdots$$
$$+c_{1p}c_{21}(X_p-\mu_p)(X_1-\mu_1)+c_{1p}c_{22}(X_p-\mu_p)(X_2-\mu_2)+\cdots+c_{1p}c_{2p}(X_p-\mu_p)(X_p-\mu_p)] =$$

$$E\left[\sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}\left(X_\ell - \mu_\ell\right)\left(X_m - \mu_m\right)\right] = \sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m} E\left[\left(X_\ell - \mu_\ell\right)\left(X_m - \mu_m\right)\right] =$$

$$= \sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}\sigma_{\ell m}$$

$$\mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} = \begin{bmatrix} c_{11}, & \cdots, & c_{1p} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix} \begin{bmatrix} c_{21} \\ \vdots \\ c_{2p} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{\ell=1}^{p} c_{1\ell}\sigma_{\ell 1}, & \cdots, & \sum_{\ell=1}^{p} c_{1\ell}\sigma_{\ell p} \end{bmatrix} \begin{bmatrix} c_{21} \\ \vdots \\ c_{2p} \end{bmatrix} =$$

$$= c_{21}\sum_{\ell=1}^{p} c_{1\ell}\sigma_{\ell 1} + \cdots + c_{2p}\sum_{\ell=1}^{p} c_{1\ell}\sigma_{\ell p} = \sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}\sigma_{\ell m}$$

Now we finally have,

$$\text{Cov}\,(Z_1, Z_2) = \sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}\sigma_{\ell m} = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c}$$

2.29

Consider the arbitrary random vector $\mathbf{X}' = \begin{bmatrix} X_1, & X_2, & X_3, & X_4, & X_5 \end{bmatrix}$ with mean vector $\boldsymbol{\mu}' = \begin{bmatrix} \mu_1, & \mu_2, & \mu_3, & \mu_4, & \mu_5 \end{bmatrix}$. Partition $\mathbf{X}$ into

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

where

$$\mathbf{X}^{(1)} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{and} \quad \mathbf{X}^{(2)} = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

Let $\boldsymbol{\Sigma}$ be the covariance matrix of $\mathbf{X}$ with general element σ_{ik}. Partition $\boldsymbol{\Sigma}$ into the covariance matrices of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ and the covariance matrix of an element of $\mathbf{X}^{(1)}$ and an element of $\mathbf{X}^{(2)}$.

$$\boldsymbol{\Sigma} = \left[\begin{array}{cc:ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \hdashline \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{array}\right] = \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]$$

$$\boldsymbol{\Sigma}_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \quad \boldsymbol{\Sigma}_{12} = \begin{bmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{23} & \sigma_{24} & \sigma_{25} \end{bmatrix}, \quad \boldsymbol{\Sigma}_{21} = \begin{bmatrix} \sigma_{31} & \sigma_{32} \\ \sigma_{41} & \sigma_{42} \\ \sigma_{51} & \sigma_{52} \end{bmatrix}, \quad \boldsymbol{\Sigma}_{22} \begin{bmatrix} \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{53} & \sigma_{54} & \sigma_{55} \end{bmatrix}$$

where $\boldsymbol{\Sigma}_{12} = \boldsymbol{\Sigma}'_{21}$.

2.30

You are given the random vector $\mathbf{X}' = \begin{bmatrix} X_1, & X_2, & X_3, & X_4 \end{bmatrix}$ with mean vector $\boldsymbol{\mu}'_{\mathbf{x}} = \begin{bmatrix} 4, & 3, & 2, & 1 \end{bmatrix}$ and the variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{AX}^{(2)}$. Find

(a) $E\left(\mathbf{X^{(1)}}\right)$

$$E\left[\mathbf{X}^{(1)}\right] = E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} E[X_1] \\ E[X_2] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(b) $E\left(\mathbf{AX^{(1)}}\right)$

$$E\left[\mathbf{AX}^{(1)}\right] = E\left[\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} 1 & 2 \end{bmatrix} E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4{+}6 = 10$$

(c) $\text{Cov}\left(\mathbf{X^{(1)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(1)}}\right) = \mathbf{\Sigma}_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) $\text{Cov}\left(\mathbf{AX^{(1)}}\right)$

$$\text{Cov}\left(\mathbf{AX^{(1)}}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X^{(1)}}\right)\mathbf{A}' = \mathbf{A\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 7$$

(e) $E\left(\mathbf{X^{(2)}}\right)$

$$E\left[\mathbf{X}^{(2)}\right] = E\left[\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\right] = \begin{bmatrix} E[X_3] \\ E[X_4] \end{bmatrix} = \begin{bmatrix} \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(f) $E\left(\mathbf{BX^{(2)}}\right)$

$$E\left[\mathbf{BX}^{(2)}\right] = E\left[\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\right] = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} E\left[\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\right] = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(g) $\text{Cov}\left(\mathbf{X^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(2)}}\right) = \mathbf{\Sigma}_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$$

(h) Cov $\left(\mathbf{BX^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{BX^{(2)}}\right) = \mathbf{B}\text{Cov}\left(\mathbf{X^{(2)}}\right)\mathbf{B}' = \mathbf{B\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 13 & -10 \\ 20 & -8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 33 & 36 \\ 36 & 48 \end{bmatrix}$$

(i) Cov $\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right) = \mathbf{\Sigma}_{12} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(j) Cov $\left(\mathbf{AX^{(1)}}, \mathbf{BX^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{AX^{(1)}}, \mathbf{BX^{(2)}}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right)\mathbf{B}' = \mathbf{A\Sigma}_{12}\mathbf{B}' =$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \end{bmatrix}$$

2.31

Repeat **Exercise 2.30**, but with **A** and **B** replaced by

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

(a) $E\left(\mathbf{X^{(1)}}\right)$

$$E\left[\mathbf{X}^{(1)}\right] = E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} E[X_1] \\ E[X_2] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(b) $E\left(\mathbf{AX^{(1)}}\right)$

$$E\left[\mathbf{AX}^{(1)}\right] = E\left[\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} 1 & -1 \end{bmatrix} E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4 - 3 = 1$$

(c) Cov $\left(\mathbf{X^{(1)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(1)}}\right) = \mathbf{\Sigma}_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Cov $\left(\mathbf{AX^{(1)}}\right)$

$$\text{Cov}\left(\mathbf{AX^{(1)}}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X^{(1)}}\right)\mathbf{A}' = \mathbf{A\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$$

(e) $E\left(\mathbf{X^{(2)}}\right)$

$$E\left[\mathbf{X}^{(2)}\right] = E\left[\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\right] = \begin{bmatrix} E[X_3] \\ E[X_4] \end{bmatrix} = \begin{bmatrix} \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(f) $E\left(\mathbf{BX^{(2)}}\right)$

$$E\left[\mathbf{BX}^{(2)}\right] = E\left[\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\right] = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} E\left[\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\right] = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(g) $\text{Cov}\left(\mathbf{X^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(2)}}\right) = \mathbf{\Sigma}_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$$

(h) $\text{Cov}\left(\mathbf{BX^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{BX^{(2)}}\right) = \mathbf{B}\text{Cov}\left(\mathbf{X^{(2)}}\right)\mathbf{B}' = \mathbf{B\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 20 & -8 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 48 & -8 \\ -8 & 4 \end{bmatrix}$$

(i) $\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right) = \mathbf{\Sigma}_{12} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(j) $\text{Cov}\left(\mathbf{AX^{(1)}}, \mathbf{BX^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{AX^{(1)}}, \mathbf{BX^{(2)}}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right)\mathbf{B}' = \mathbf{A\Sigma}_{12}\mathbf{B}' =$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

2.32

You are given the random vector $\mathbf{X}' = \begin{bmatrix} X_1, & X_2, & \cdots, & X_5 \end{bmatrix}$ with the mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = \begin{bmatrix} 2, & 4, & -1, & 3, & 0 \end{bmatrix}$ and the variance-covariance matrix

$$\mathbf{\Sigma_X} = \begin{bmatrix} 4 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 3 & 1 & -1 & 0 \\ \frac{1}{2} & 1 & 6 & 1 & -1 \\ -\frac{1}{2} & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(a) $E\left(\mathbf{X^{(1)}}\right)$

$$E\left[\mathbf{X}^{(1)}\right] = E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} E[X_1] \\ E[X_2] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(b) $E\left(\mathbf{AX^{(1)}}\right)$

$$E\left[\mathbf{AX}^{(1)}\right] = E\left[\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(c) $\text{Cov}\left(\mathbf{X^{(1)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(1)}}\right) = \mathbf{\Sigma}_{11} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

(d) $\text{Cov}\left(\mathbf{AX^{(1)}}\right)$

$$\text{Cov}\left(\mathbf{AX^{(1)}}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X^{(1)}}\right)\mathbf{A}' = \mathbf{A\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

(e) $E\left(\mathbf{X^{(2)}}\right)$

$$E\left[\mathbf{X}^{(2)}\right] = E\left[\begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}\right] = \begin{bmatrix} E[X_3] \\ E[X_4] \\ E[X_5] \end{bmatrix} = \begin{bmatrix} \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

(f) $E\left(\mathbf{BX^{(2)}}\right)$

$$E\left[\mathbf{BX}^{(2)}\right] = E\left[\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} E\left[\begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}\right] =$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(g) Cov $\left(\mathbf{X^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(2)}}\right) = \mathbf{\Sigma}_{22} = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(h) Cov $\left(\mathbf{BX^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{BX^{(2)}}\right) = \mathbf{B}\text{Cov}\left(\mathbf{X^{(2)}}\right)\mathbf{B'} = \mathbf{B\Sigma}_{22}\mathbf{B'} =$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 1 \\ 9 & 5 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix}$$

(i) Cov $\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right) = \mathbf{\Sigma}_{12} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(j) Cov $\left(\mathbf{AX^{(1)}}, \mathbf{BX^{(2)}}\right)$

$$\text{Cov}\left(\mathbf{AX^{(1)}}, \mathbf{BX^{(2)}}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X^{(1)}}, \mathbf{X^{(2)}}\right)\mathbf{B'} = \mathbf{A\Sigma}_{12}\mathbf{B'} =$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

2.33

Repeat Exercise 2.32, but with $\mathbf{X}$ partitioned as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \hdashline X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

and with $\mathbf{A}$ and $\mathbf{B}$ replaced by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

(a) $E\left(\mathbf{X}^{(\mathbf{1})}\right)$

$$E\left[\mathbf{X}^{(1)}\right] = E\left[\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ E[X_3] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

(b) $E\left(\mathbf{A}\mathbf{X}^{(\mathbf{1})}\right)$

$$E\left[\mathbf{A}\mathbf{X}^{(1)}\right] = E\left[\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right] = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} E\left[\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right] =$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(c) $\text{Cov}\left(\mathbf{X}^{(\mathbf{1})}\right)$

$$\text{Cov}\left(\mathbf{X}^{(\mathbf{1})}\right) = \mathbf{\Sigma}_{11} = \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix}$$

(d) $\text{Cov}\left(\mathbf{A}\mathbf{X}^{(\mathbf{1})}\right)$

$$\text{Cov}\left(\mathbf{A}\mathbf{X}^{(\mathbf{1})}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X}^{(\mathbf{1})}\right)\mathbf{A}' = \mathbf{A}\mathbf{\Sigma}_{11}\mathbf{A}' =$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 0 \\ \frac{9}{2} & 5 & \frac{39}{2} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 4 \\ 4 & 68 \end{bmatrix}$$

(e) $E\left(\mathbf{X}^{(\mathbf{2})}\right)$

$$E\left[\mathbf{X}^{(2)}\right] = E\left[\begin{bmatrix} X_4 \\ X_5 \end{bmatrix}\right] = \begin{bmatrix} E[X_4] \\ E[X_5] \end{bmatrix} = \begin{bmatrix} \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

(f) $E\left(\mathbf{B}\mathbf{X}^{(\mathbf{2})}\right)$

$$E\left[\mathbf{B}\mathbf{X}^{(2)}\right] = E\left[\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_4 \\ X_5 \end{bmatrix}\right] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} E\left[\begin{bmatrix} X_4 \\ X_5 \end{bmatrix}\right] =$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

(g) $\text{Cov}\left(\mathbf{X}^{(\mathbf{2})}\right)$

$$\text{Cov}\left(\mathbf{X}^{(\mathbf{2})}\right) = \mathbf{\Sigma}_{22} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(h) Cov $\left(\mathbf{B}\mathbf{X}^{(\mathbf{2})}\right)$

$$\text{Cov}\left(\mathbf{B}\mathbf{X}^{(\mathbf{2})}\right) = \mathbf{B}\text{Cov}\left(\mathbf{X}^{(\mathbf{2})}\right)\mathbf{B}' = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' =$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$$

(i) Cov $\left(\mathbf{X}^{(\mathbf{1})}, \mathbf{X}^{(\mathbf{2})}\right)$

$$\text{Cov}\left(\mathbf{X}^{(\mathbf{1})}, \mathbf{X}^{(\mathbf{2})}\right) = \boldsymbol{\Sigma}_{12} = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(j) Cov $\left(\mathbf{A}\mathbf{X}^{(\mathbf{1})}, \mathbf{B}\mathbf{X}^{(\mathbf{2})}\right)$

$$\text{Cov}\left(\mathbf{A}\mathbf{X}^{(\mathbf{1})}, \mathbf{B}\mathbf{X}^{(\mathbf{2})}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X}^{(\mathbf{1})}, \mathbf{X}^{(\mathbf{2})}\right)\mathbf{B}' = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}' =$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{3}{2} & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ -\frac{9}{2} & \frac{9}{2} \end{bmatrix}$$

2.34

Consider the vectors $\mathbf{b}' = \begin{bmatrix} 2, & -1, & 4, & 0 \end{bmatrix}$ and $\mathbf{d}' = \begin{bmatrix} -1, & 3, & -2, & 1 \end{bmatrix}$. Verify the Cauchy-schwarz inequality $\left(\mathbf{b}'\mathbf{d}\right)^2 \leq \left(\mathbf{b}'\mathbf{b}\right)\left(\mathbf{d}'\mathbf{d}\right)$.

$$\left(\mathbf{b}'\mathbf{d}\right)^2 = \left(\begin{bmatrix} 2 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}\right)^2 = (-13)^2 = 169$$

$$\left(\mathbf{b}'\mathbf{b}\right) = \|\mathbf{b}\|^2 = \begin{bmatrix} 2 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix} = 21$$

$$\left(\mathbf{d}'\mathbf{d}\right) = \|\mathbf{d}\|^2 = \begin{bmatrix} -1 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix} = 15$$

$$\|\mathbf{b}\|^2\|\mathbf{d}\|^2 = (21)(15) = 315$$

$$169 = \left(\mathbf{b}'\mathbf{d}\right)^2 < \|\mathbf{b}\|^2\|\mathbf{d}\|^2 = 315$$

2.35

Using the vector $\mathbf{b}' = \begin{bmatrix}-4, & 3\end{bmatrix}$ and $\mathbf{d}' = \begin{bmatrix}1, & 1\end{bmatrix}$, verify the extended Cauchy-schwarz inequality $\left(\mathbf{b}'\mathbf{d}\right)^2 \leq \left(\mathbf{b}'\mathbf{B}\mathbf{b}\right)\left(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d}\right)$ if

$$\mathbf{B} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$$\left(\mathbf{b}'\mathbf{d}\right)^2 = \left(\begin{bmatrix}-4 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)^2 = (-1)^2 = 1$$

$$\left(\mathbf{b}'\mathbf{B}\mathbf{b}\right) = \begin{bmatrix}-4 & 3\end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix} = \begin{bmatrix}-14 & 23\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix} = 125$$

$$\mathbf{B}^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\left(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d}\right) = \begin{bmatrix}1 & 1\end{bmatrix}\left(\frac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\right)\begin{bmatrix}1\\1\end{bmatrix} = \left(\frac{1}{6}\right)\begin{bmatrix}7 & 4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{11}{6}$$

$$\left(\mathbf{b}'\mathbf{B}\mathbf{b}\right)\left(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d}\right) = 125 \times \frac{11}{6} = \frac{1375}{6} = 229.16\bar{6}$$

$$1 = \left(\mathbf{b}'\mathbf{d}\right)^2 < \left(\mathbf{b}'\mathbf{B}\mathbf{b}\right)\left(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d}\right) = 229.16\bar{6}$$

2.36

Find the maximum and minimum values of the quadratic form $4x_1^2+4x_2^2+6x_1x_2$ for all points $\mathbf{x}' = \begin{bmatrix}x_1, & x_2\end{bmatrix}$ such that $\mathbf{x}'\mathbf{x}' = 1$.

Convert into matrix form

$$4x_1^2 + 4x_2^2 + 6x_1x_2 = \begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{x}'\mathbf{A}\mathbf{x}$$

so we have that

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Find the eigenvalues of $\mathbf{A}$,

$$0 = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 9 = \lambda^2 - 8\lambda + 16 - 9 =$$

$$= \lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1)$$

The eigenvalues are $(\lambda_1, \lambda_2) = (7, 1)$. The matrix $\mathbf{A}$ is positive definite, since $\lambda_1 > \lambda_2 > 0$. By **(2-51)** on page 80, $\lambda_1 = 7$ is the maximum and $\lambda_2 = 1$ is the minimum.

2.37

With $\mathbf{A}$ as given in Exercise 2.6, find the maximum value of $\mathbf{x}'\mathbf{A}\mathbf{x}$ for $\mathbf{x}'\mathbf{x} = 1$.

From Exercise 2.6, the eigenvalues are $(\lambda_1, \lambda_2) = (10, 5)$. The matrix $\mathbf{A}$ is positive definite, since $\lambda_1 > \lambda_2 > 0$. By **(2-51)** on page 80, $\lambda_1 = 10$ is the maximum and $\lambda_2 = 5$ is the minimum.

2.38

Find the maximum and minimum values of the ratio $\mathbf{x}'\mathbf{A}\mathbf{x}/\mathbf{x}'\mathbf{x}$ for any nonzero vector $\mathbf{x}' = \begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix}$ if

$$\mathbf{A} = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}$$

Finding the eigenvalues of $\mathbf{A}$,

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{vmatrix} =$$

$$= (13-\lambda)\left[(13-\lambda)(10-\lambda) - 4\right] + 4\left[-4(10-\lambda) + 4\right] + 2\left[8 - 2(13-\lambda)\right] =$$

$$= (13-\lambda)^2 (10-\lambda) - 4(13-\lambda) - 16(10-\lambda) + 16 + 16 - 4(13-\lambda)$$

$$= \left(\lambda^2 - 26\lambda + 169\right)(10-\lambda) - 52 + 4\lambda - 160 + 16\lambda + 32 - 52 + 4\lambda =$$

$$= \left(10\lambda^2 - 260\lambda + 1690\right) + \left(-\lambda^3 + 26\lambda^2 - 169\lambda\right) 24\lambda - 232 =$$

$$= 1458 - 405\lambda + 36\lambda^2 - \lambda^3 =$$

$$= -\left(\lambda^3 - 36\lambda^2 + 405\lambda - 1458\right) =$$

$$= -\left(\lambda^2(\lambda-9) - 27\lambda(\lambda-9) + 162(\lambda-9)\right) =$$

$$= -(\lambda-9)\left(\lambda^2 - 27\lambda + 162\right) =$$

$$= -(\lambda-9)(\lambda-18)(\lambda-9)$$

The eigenvalues are $(\lambda_1, \lambda_2, \lambda_3) = (18, 9, 9)$. The matrix $\mathbf{A}$ is positive definite, since $\lambda_1 > \lambda_2 > \lambda_3 > 0$. By **(2-51)** on page 80, $\lambda_1 = 18$ is the maximum and $\lambda_2 = \lambda_3 = 9$ is the minimum.

2.39

Show that

$$\underset{(r\times s)}{\mathbf{A}}\underset{(s\times t))}{\mathbf{B}}\underset{(t\times v)}{C} \text{ has } (i,j)\text{th entry } \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{i\ell}b_{\ell k}c_{kj}$$

Hint: $\mathbf{BC}$ has (ℓ, j)th entry $\sum_{k=1}^{t} b_{\ell k}c_{kj} = d_{\ell j}$. So $\mathbf{A}(\mathbf{BC})$ has (i,j)th element

$$a_{i1}d_{1j} + a_{i2}d_{2j} + \cdots + a_{is}d_{sj} = \sum_{\ell=1}^{s} a_{i\ell}\left(\sum_{k=1}^{t} b_{\ell k}c_{kj}\right) = \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{i\ell}b_{\ell k}c_{kj}$$

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = \mathbf{A}\left(\begin{bmatrix} b_{11} & \cdots & b_{1t} \\ \vdots & \ddots & \vdots \\ b_{s1} & \cdots & b_{st} \end{bmatrix}\begin{bmatrix} c_{11} & \cdots & c_{1v} \\ \vdots & \ddots & \vdots \\ c_{t1} & \cdots & c_{tv} \end{bmatrix}\right) =$$

$$= \mathbf{A}\begin{bmatrix} \sum_{k=1}^{t} b_{1k}c_{k1} & \cdots & \sum_{k=1}^{t} b_{1k}c_{kv} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{t} b_{sk}c_{k1} & \cdots & \sum_{k=1}^{t} b_{sk}c_{kv} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rs} \end{bmatrix}\begin{bmatrix} \sum_{k=1}^{t} b_{1k}c_{k1} & \cdots & \sum_{k=1}^{t} b_{1k}c_{kv} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{t} b_{sk}c_{k1} & \cdots & \sum_{k=1}^{t} b_{sk}c_{kv} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{\ell=1}^{s} a_{1\ell}\left(\sum_{k=1}^{t} b_{\ell k}c_{k1}\right) & \cdots & \sum_{\ell=1}^{s} a_{1\ell}\left(\sum_{k=1}^{t} b_{\ell k}c_{kv}\right) \\ \vdots & \ddots & \vdots \\ \sum_{\ell=1}^{s} a_{r\ell}\left(\sum_{k=1}^{t} b_{\ell k}c_{k1}\right) & \cdots & \sum_{\ell=1}^{s} a_{r\ell}\left(\sum_{k=1}^{t} b_{\ell k}c_{kv}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{1\ell}b_{\ell k}c_{k1} & \cdots & \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{1\ell}b_{\ell k}c_{kv} \\ \vdots & \ddots & \vdots \\ \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{r\ell}b_{\ell k}c_{k1} & \cdots & \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{r\ell}b_{\ell k}c_{kv} \end{bmatrix}$$

The output of $\mathbf{ABC}$ is a $r \times v$ matrix, whose elements are

$$\sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{i\ell}b_{\ell k}c_{kj}$$

where $i = 1, \ldots, r$ and $j = 1, \ldots, v$.

2.40

Verify **(2-24)**: $E(\mathbf{X}+\mathbf{Y}) = E(\mathbf{X}) + E(\mathbf{Y})$ and $E(\mathbf{AXB}) = \mathbf{A}E(\mathbf{X})\mathbf{B}$.
Hint: $\mathbf{X}+\mathbf{Y}$ has $X_{ij}+Y_{ij}$ as its (i,j)th element. Now, $E(X_{ij}+Y_{ij}) = E(X_{ij}) + E(Y_{ij})$ by the univariate property of expectation, and this last quantity is the (i,j)th element of $E(\mathbf{X}) + E(\mathbf{Y})$. Next (see Exercise 2.39), $\mathbf{AXB}$ has (i,j)th entry $\sum_\ell \sum_k a_{i\ell} X_{\ell k} b_{kj}$, and by the additive property of expectation,

$$E\left(\sum_\ell \sum_k a_{i\ell} X_{\ell k} b_{kj}\right) = \sum_\ell \sum_k a_{i\ell} E(X_{\ell k}) b_{kj}$$

which is the (i,j)th element of $\mathbf{A}E(\mathbf{X})\mathbf{B}$.

Let both $\mathbf{X}$ and $\mathbf{Y}$ be $n \times p$ matrices, so

$$E[\mathbf{X}+\mathbf{Y}] = E\left[\begin{bmatrix} X_{11} & \cdots & X_{1p} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{np} \end{bmatrix} + \begin{bmatrix} Y_{11} & \cdots & Y_{1p} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{np} \end{bmatrix}\right] =$$

$$= E\left[\begin{bmatrix} X_{11}+Y_{11} & \cdots & X_{1p}+Y_{1p} \\ \vdots & \ddots & \vdots \\ Y_{n1}+X_{n1} & \cdots & X_{np}+Y_{np} \end{bmatrix}\right] =$$

$$= \begin{bmatrix} E[X_{11}+Y_{11}] & \cdots & E[X_{1p}+Y_{1p}] \\ \vdots & \ddots & \vdots \\ E[X_{n1}+Y_{n1}] & \cdots & E[X_{np}+Y_{np}] \end{bmatrix} =$$

$$= \begin{bmatrix} E[X_{11}]+E[Y_{11}] & \cdots & E[X_{1p}]+E[Y_{1p}] \\ \vdots & \ddots & \vdots \\ E[X_{n1}]+E[Y_{n1}] & \cdots & E[X_{np}]+E[Y_{np}] \end{bmatrix} =$$

$$= \begin{bmatrix} E[X_{11}] & \cdots & E[X_{1p}] \\ \vdots & \ddots & \vdots \\ E[X_{n1}] & \cdots & E[X_{np}] \end{bmatrix} + \begin{bmatrix} E[Y_{11}] & \cdots & E[Y_{1p}] \\ \vdots & \ddots & \vdots \\ E[Y_{n1}] & \cdots & E[Y_{np}] \end{bmatrix} =$$

$$= E\left[\begin{bmatrix} X_{11} & \cdots & X_{1p} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{np} \end{bmatrix}\right] + E\left[\begin{bmatrix} Y_{11} & \cdots & Y_{1p} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{np} \end{bmatrix}\right] =$$

$$= E[\mathbf{X}] + E[\mathbf{Y}]$$

Using Exercise 2.39

$$E(\mathbf{AXB}) =$$

$$= E\left[\begin{bmatrix} \sum_{\ell=1}^s \sum_{k=1}^t a_{1\ell} X_{\ell k} c_{k1} & \cdots & \sum_{\ell=1}^s \sum_{k=1}^t a_{1\ell} X_{\ell k} c_{kv} \\ \vdots & \ddots & \vdots \\ \sum_{\ell=1}^s \sum_{k=1}^t a_{r\ell} X_{\ell k} c_{k1} & \cdots & \sum_{\ell=1}^s \sum_{k=1}^t a_{r\ell} X_{\ell k} c_{kv} \end{bmatrix}\right] =$$

$$= \begin{bmatrix} E\left[\sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{1\ell}X_{\ell k}c_{k1}\right] & \cdots & E\left[\sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{1\ell}X_{\ell k}c_{kv}\right] \\ \vdots & \ddots & \vdots \\ E\left[\sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{r\ell}X_{\ell k}c_{k1}\right] & \cdots & E\left[\sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{r\ell}X_{\ell k}c_{kv}\right] \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{\ell=1}^{s}\sum_{k=1}^{t} E\left[a_{1\ell}X_{\ell k}c_{k1}\right] & \cdots & \sum_{\ell=1}^{s}\sum_{k=1}^{t} E\left[a_{1\ell}X_{\ell k}c_{kv}\right] \\ \vdots & \ddots & \vdots \\ \sum_{\ell=1}^{s}\sum_{k=1}^{t} E\left[a_{r\ell}X_{\ell k}c_{k1}\right] & \cdots & \sum_{\ell=1}^{s}\sum_{k=1}^{t} E\left[a_{r\ell}X_{\ell k}c_{kv}\right] \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{1\ell}E\left[X_{\ell k}\right]c_{k1} & \cdots & \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{1\ell}E\left[X_{\ell k}\right]c_{kv} \\ \vdots & \ddots & \vdots \\ \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{r\ell}E\left[X_{\ell k}\right]c_{k1} & \cdots & \sum_{\ell=1}^{s}\sum_{k=1}^{t} a_{r\ell}E\left[X_{\ell k}\right]c_{kv} \end{bmatrix} =$$

$$= \mathbf{A}E(\mathbf{X})\mathbf{B}$$

2.41

You are given the random vector $\mathbf{X}' = \begin{bmatrix} X_1, & X_2, & X_3, & X_4 \end{bmatrix}$ with the mean vector $\boldsymbol{\mu}_{\mathbf{X}}' = \begin{bmatrix} 3, & 2, & -2, & 0 \end{bmatrix}$ and variance-covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

(a) Find $E(\mathbf{AX})$, the mean of $\mathbf{AX}$.

$$E\left[\mathbf{AX}\right] = \mathbf{A}E\left[\mathbf{X}\right] = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

(b) Find $\text{Cov}\left(\mathbf{AX}\right)$, the variances and covariances of $\mathbf{AX}$.

$$\text{Cov}\left(\mathbf{AX}\right) = \mathbf{A}\text{Cov}\left(\mathbf{X}\right)\mathbf{A}' = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}' =$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} =$$

$$= 3\begin{bmatrix}1 & -1 & 0 & 0\\ 1 & 1 & -2 & 0\\ 1 & 1 & 1 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix} =$$

$$= 3\begin{bmatrix}2 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 12\end{bmatrix} = \begin{bmatrix}6 & 0 & 0\\ 0 & 18 & 0\\ 0 & 0 & 36\end{bmatrix}$$

(c) Which pairs of linear combinations have zero covariances?

$$\mathrm{Cov}\,(X_1, X_2) = \mathrm{Cov}\,(X_1, X_3) = \mathrm{Cov}\,(X_2, X_3) = 0$$

2.42

Repeat Exercise 2.41, but with

$$\mathbf{\Sigma_X} = \begin{bmatrix}3 & 1 & 1 & 1\\ 1 & 3 & 1 & 1\\ 1 & 1 & 3 & 1\\ 1 & 1 & 1 & 3\end{bmatrix}$$

(a) Find $E(\mathbf{AX})$, the mean of $\mathbf{AX}$.

$$E\,[\mathbf{AX}] = \mathbf{A}E\,[\mathbf{X}] = \mathbf{A}\boldsymbol{\mu}_\mathbf{X} = \begin{bmatrix}1 & -1 & 0 & 0\\ 1 & 1 & -2 & 0\\ 1 & 1 & 1 & -3\end{bmatrix}\begin{bmatrix}3\\ 2\\ -2\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 9\\ 3\end{bmatrix}$$

(b) Find $\mathrm{Cov}\,(\mathbf{AX})$, the variances and covariances of $\mathbf{AX}$.

$$\mathrm{Cov}\,(\mathbf{AX}) = \mathbf{A}\mathrm{Cov}\,(\mathbf{X})\,\mathbf{A}' = \mathbf{A\Sigma_X A}' =$$

$$= \begin{bmatrix}1 & -1 & 0 & 0\\ 1 & 1 & -2 & 0\\ 1 & 1 & 1 & -3\end{bmatrix}\begin{bmatrix}3 & 1 & 1 & 1\\ 1 & 3 & 1 & 1\\ 1 & 1 & 3 & 1\\ 1 & 1 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix} =$$

$$= \begin{bmatrix}2 & -2 & 0 & 0\\ 2 & 2 & -4 & 0\\ 2 & 2 & 2 & -6\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix} =$$

$$= \begin{bmatrix}4 & 0 & 0\\ 0 & 12 & 0\\ 0 & 0 & 24\end{bmatrix}$$

(c) Which pairs of linear combinations have zero covariances?

$$\mathrm{Cov}\,(X_1, X_2) = \mathrm{Cov}\,(X_1, X_3) = \mathrm{Cov}\,(X_2, X_3) = 0$$

3 Chapter 3

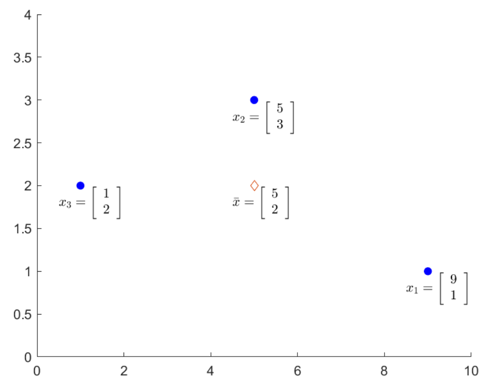
3.1

Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

(a) Graph the scatter plot in $p = 2$ dimensions. Locate the sample mean on the diagram.

```
X = [9 1; 5 3; 1 2];
mean_pt = mean(X);
hold on
    % Plot data.
    scatter(X(:,1),X(:,2),'blue','filled')
    % Plot mean.
    plot(mean_pt(1),mean_pt(2),'d')
    % Text for mean.
    text(mean_pt(1)-0.5,mean_pt(2)-0.2, ...
        join(["$$\bar{x}=\left[\begin{array}{c}",mean_pt(1),"\\",mean_pt(2),"\end{array}\right]$$"],''), ...
        'interpreter','latex')
    % Text for data.
    for r = 1:height(X)
        anno = join(["$$x_",r," = \left[\begin{array}{c}",X(r,1),"\\",X(r,2),"\end{array}\right]$$"],'');
        text(X(r,1)-0.5,X(r,2)-0.2,anno,'interpreter','latex');
    end
    xlim([0 10])
    ylim([0 4])
    saveas(gcf,'sol3.1a.png')
hold off
```



(b) Sketch the $n = 3$-dimensional representation of the data, and plot the deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$.

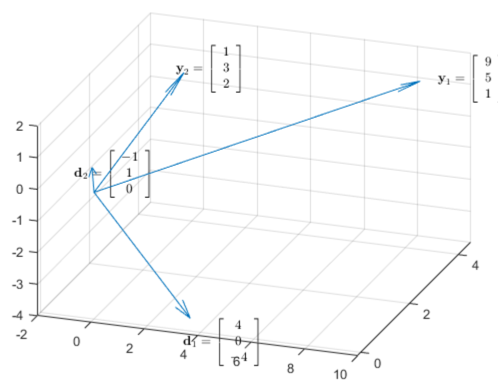
$$\mathbf{y}_1 - \bar{x}_1\mathbf{1} = \begin{bmatrix}9\\5\\1\end{bmatrix} - \begin{bmatrix}5\\5\\5\end{bmatrix} = \begin{bmatrix}4\\0\\-4\end{bmatrix}$$

$$\mathbf{y}_2 - \bar{x}_2\mathbf{1} = \begin{bmatrix}1\\3\\2\end{bmatrix} - \begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

```
% Continuing from part (a)...
% Compute the deviation vectors.
d1 = X(:,1) - mean_pt(1)*ones([3,1]);
d2 = X(:,2) - mean_pt(2)*ones([3,1]);
% Combine the data with the deviation vectors. First two rows are data,
% second two are the deviation vectors.
D = [X d1 d2]';
start = zeros(size(D));

% Plot the y_1 and y_2 vectors and the d_1, d_2 deviation vectors.
quiver3(start(:,1), start(:,2), start(:,3), D(:,1), D(:,2), D(:,3));
% Text for data.
for r = 1:height(D)
    if r < 3
        % Labels for the data, y_1 and y_2.
        anno = join(["$$\textbf{y}_",r," = \left[\begin{array}{c}",D(r,1)
```

```
                    ,"\\",D(r,2),"\\",D(r,3),"\end{
                    array}\right]$$"],'');
                text(D(r,1)-0.5,D(r,2)-0.2,D(r,3)
                    -0.2,anno,'interpreter','latex'
                    );
            else
                % Labels for the deviation vectors
                    , d_1 and d_2.
                anno = join(["$$\textbf{d}_",r-2,"
                     = \left[\begin{array}{c}",D(r
                    ,1),"\\",D(r,2),"\\",D(r,3),"\
                    end{array}\right]$$"],'');
                text(D(r,1)-0.5,D(r,2)-0.2,D(r,3)
                    -0.2,anno,'interpreter','latex'
                    );
            end
        end
```



(c) Sketch the deviation vectors in (b) emanating from the origin. Calculate the lengths of these vectors and the cosine of the angle between them. Relate the quantities to $\mathbf{S}_n$ and $\mathbf{R}$.

The sketch of $\mathbf{d}_1$ and $\mathbf{d}_2$ are in part (b). The lengths are

$$\|\mathbf{d}_1\| = \sqrt{(4)^2 + (0)^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32}$$

$$\|\mathbf{d}_2\| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = \begin{bmatrix} 4 & 0 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -4$$

$$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\|\mathbf{d}_1\|\,\|\mathbf{d}_2\|} = \frac{-4}{\sqrt{32}\sqrt{2}}$$

$$\Rightarrow \theta = \arccos\left(\frac{-4}{\sqrt{32}\sqrt{2}}\right) = \arccos\left(\frac{-4}{\sqrt{64}}\right) = \arccos\left(\frac{-4}{8}\right) = \arccos\left(\frac{-1}{2}\right) = 120^{\circ}$$

For $\mathbf{S}_n$, element $s_{12} = \frac{1}{n}(\mathbf{y}_1 - \bar{x}_1\mathbf{1})'(\mathbf{y}_2 - \bar{x}_2\mathbf{1}) = \frac{1}{n}\mathbf{d}_1 \cdot \mathbf{d}_2$, so what we computed for $\mathbf{d}_1\mathbf{d}_2 = n \times s_{12}$. For $\mathbf{R}$, when using $\mathbf{S}_n$, element $r_{12} = \frac{s_{12}}{\sqrt{s_{22}}\sqrt{s_{22}}} = \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2/n)}{\sqrt{\mathbf{d}_1 \cdot \mathbf{d}_1/n}\sqrt{\mathbf{d}_2 \cdot \mathbf{d}_2/n}} = \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2)}{\sqrt{\mathbf{d}_1 \cdot \mathbf{d}_1}\sqrt{\mathbf{d}_2 \cdot \mathbf{d}_2}}$

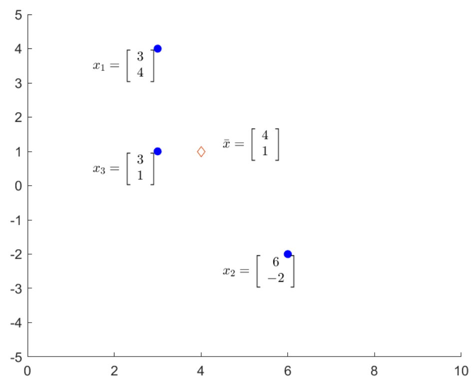
3.2

Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 3 & 4 \\ 6 & -2 \\ 3 & 1 \end{bmatrix}$$

(a) Graph the scatter plot in $p = 2$ dimensions. Locate the sample mean on the diagram.

```
        X = [3 4; 6 -2; 3 1];
        mean_pt = mean(X);
        hold on
            % Plot data.
            scatter(X(:,1),X(:,2),'blue','filled')
            % Plot mean.
            plot(mean_pt(1),mean_pt(2),'d')
            % Text for mean.
            text(mean_pt(1)+0.5,mean_pt(2)+0.2, ...
                join(["$$\bar{x}=\left[\begin{array}{c}",mean_pt(1),"\\",mean_pt(2),"\end{array}\right]$$"],''), ...
                'interpreter','latex')
            % Text for data.
            for r = 1:height(X)
                anno = join(["$$x_",r," = \left[\begin{array}{c}",X(r,1),"\\",X(r,2),"\end{array}\right]$$"],'');
                text(X(r,1)-1.5,X(r,2)-0.5,anno,'interpreter','latex');
            end
            xlim([0 10])
            ylim([-5 5])
            saveas(gcf,'sol3.2a.png')
        hold off
```



(b) Sketch the $n = 3$-dimensional representation of the data, and plot the deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$.

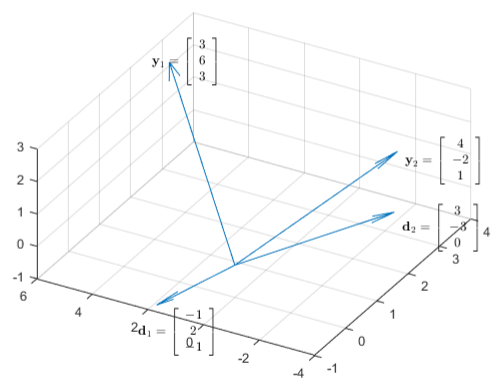
$$\mathbf{y}_1 - \bar{x}_1\mathbf{1} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{y}_2 - \bar{x}_2\mathbf{1} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

```
% Continuing from part (a)...
% Compute the deviation vectors.
d1 = X(:,1) - mean_pt(1)*ones([3,1]);
d2 = X(:,2) - mean_pt(2)*ones([3,1]);
% Combine the data with the deviation
    vectors. First two rows are data,
% second two are the deviation vectors.
D = [X d1 d2]';
start = zeros(size(D));

% Plot the y_1 and y_2 vectors and the d_1
    , d_2 deviation vectors.
quiver3(start(:,1), start(:,2), start(:,3)
    , D(:,1), D(:,2), D(:,3));
% Text for data.
for r = 1:height(D)
    if r < 3
        % Labels for the data, y_1 and y_2
            .
        anno = join(["$$\textbf{y}_",r," =
            \left[\begin{array}{c}",D(r,1)
```

```
                    ,"\\",D(r,2),"\\",D(r,3),"\end{
                    array}\right]$$"],'');
                text(D(r,1)-0.5,D(r,2)-0.2,D(r,3)
                    -0.2,anno,'interpreter','latex'
                    );
            else
                % Labels for the deviation vectors
                    , d_1 and d_2.
                anno = join(["$$\textbf{d}_",r-2,"
                     = \left[\begin{array}{c}",D(r
                    ,1),"\\",D(r,2),"\\",D(r,3),"\
                    end{array}\right]$$"],'');
                text(D(r,1)-0.5,D(r,2)-0.2,D(r,3)
                    -0.2,anno,'interpreter','latex'
                    );
            end
        end
```



(c) Sketch the deviation vectors in (b) emanating from the origin. Calculate the lengths of these vectors and the cosine of the angle between them. Relate the quantities to $\mathbf{S}_n$ and $\mathbf{R}$.

The sketch of $\mathbf{d}_1$ and $\mathbf{d}_2$ are in part (b). The lengths are

$$\|\mathbf{d}_1\| = \sqrt{(-1)^2 + (2)^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$\|\mathbf{d}_2\| = \sqrt{(3)^2 + (-3)^2 + (0)^2} = \sqrt{18}$$

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = \begin{bmatrix}-1 & 2 & -1\end{bmatrix} \begin{bmatrix}3\\-3\\0\end{bmatrix} = -9$$

$$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\|\mathbf{d}_1\| \, \|\mathbf{d}_2\|} = \frac{-9}{\sqrt{6}\sqrt{18}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-9}{\sqrt{32}\sqrt{2}}\right) = \cos^{-1}\left(\frac{-9}{\sqrt{108}}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right) = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = 150^\circ$$

For $\mathbf{S}_n$, element $s_{12} = \frac{1}{n}(\mathbf{y}_1 - \bar{x}_1\mathbf{1})'(\mathbf{y}_2 - \bar{x}_2\mathbf{1}) = \frac{1}{n}\mathbf{d}_1 \cdot \mathbf{d}_2$, so what we computed for $\mathbf{d}_1\mathbf{d}_2 = n \times s_{12}$. For $\mathbf{R}$, when using $\mathbf{S}_n$, element $r_{12} = \frac{s_{12}}{\sqrt{s_{22}}\sqrt{s_{22}}} = \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2/n)}{\sqrt{\mathbf{d}_1 \cdot \mathbf{d}_1/n}\sqrt{\mathbf{d}_2 \cdot \mathbf{d}_2/n}} = \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2)}{\sqrt{\mathbf{d}_1 \cdot \mathbf{d}_1}\sqrt{\mathbf{d}_2 \cdot \mathbf{d}_2}}$

3.3

Perform the decomposition of $\mathbf{y}_1$ into $\bar{x}_1\mathbf{1}$ and $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ using the first column of the data matrix in Example 3.9.

$$X = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 1 & 6 \\ 4 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{bmatrix}$$

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} (1+4+4)/3 \\ (2+1+0)/3 \\ (5+6+4)/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \qquad \bar{x}_1\mathbf{1}_3 = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{y}_1 - \bar{x}_1\mathbf{1}_3 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

3.4

Use the six observations on the variable X_1 in units of millions, from Table 1.1.

(a) Find the projection on $\mathbf{1}' = [1, 1, 1, 1, 1, 1]$.

Convert X_1 to units of millions

$$\mathbf{y}_1 = \left(\frac{1}{1000000}\right)\begin{bmatrix} 3497900 \\ 2485475 \\ 1782875 \\ 1725450 \\ 1645575 \\ 1469800 \end{bmatrix} = \begin{bmatrix} 3.4979 \\ 2.4855 \\ 1.7829 \\ 1.7254 \\ 1.6456 \\ 1.4698 \end{bmatrix}$$

$$\text{Proj}_{\mathbf{1}_6}\mathbf{y}_1 = \left(\frac{\mathbf{1}_6 \cdot \mathbf{y}_1}{\|\mathbf{1}_6\|}\right)\frac{\mathbf{1}_6}{\|\mathbf{1}_6\|} = \bar{x}_1\mathbf{1}_6 = 2.1012\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2.1012\\2.1012\\2.1012\\2.1012\\2.1012\\2.1012\end{bmatrix}$$

(b) Calculate the deviation vectot $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$. Relate its length to the sample standard deviation.

$$\mathbf{y}_1 - \bar{x}_1\mathbf{1} = \begin{bmatrix}3.4979\\2.4855\\1.7829\\1.7254\\1.6456\\1.4698\end{bmatrix} - \begin{bmatrix}2.1012\\2.1012\\2.1012\\2.1012\\2.1012\\2.1012\end{bmatrix} = \begin{bmatrix}1.3967\\0.3843\\-0.3183\\-0.3757\\-0.4556\\-0.6314\end{bmatrix}$$

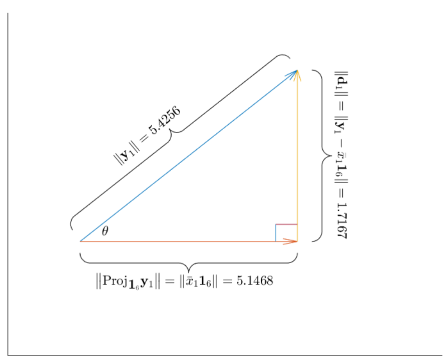
The sample variance for the first variable (feature, predictor, or whatever) is

$$\sqrt{s_{11}} = \sqrt{\frac{\sum_{i=1}^{6}(x_{i1} - \bar{x}_1)^2}{n-1}} = \sqrt{\frac{(\mathbf{y}_1 - \bar{x}_1\mathbf{1})'(\mathbf{y}_1 - \bar{x}_1\mathbf{1})}{n-1}} = \frac{\|\mathbf{y}_1 - \bar{x}_1\mathbf{1}\|}{\sqrt{n-1}}$$

$$\Rightarrow \|\mathbf{y}_1 - \bar{x}_1\mathbf{1}\| = \sqrt{s_{11}}\sqrt{n-1}$$

The length of $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ is the sample standard deviation times the square root of $n-1$. For our data, $\|\mathbf{y}_1 - \bar{x}_1\mathbf{1}\| = 1.7167$.

(c) Graph (to scale) the triangle formed by $\mathbf{y}_1$, $\bar{x}_1\mathbf{1}$, and $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$. Identify the length of each component on your graph.




```
        % Divide original units by 1,000,000 so we
          're in units of millions.
```

```
y1 = [3497900 2485475 1782875 1725450 1645575 1469800]'/1000000;
a1 = ones(height(y1),1);
% The projection of y1 onto a1.
x1bar1 = ((y1'*a1)/(norm(a1)*norm(a1)))*a1;

% Exercise 3.4 (b)
d1 = y1 - x1bar1;

% Exercise 3.4 (c)
clear clf
hold on
start = zeros(1,2);
quiver(start(:,1), start(:,2), 1, 1, 3);
quiver(start(:,1), start(:,2), 1, 0, 3);
quiver(3, 0, 0, 1, 3);
xlim([-1,5])
ylim([-2,4])
% Use the DRAWBRACE created by Pal Naverlid Savik
drawbrace([3.2, -0], [3.2, -3],10,1,'Color','k') % Draws a curly brace for deviance vector.
% Text for the norm of deviance vector d1, norm(d1).
anno = join(["$$\left\|\textbf{d}_1\right\|"," = \left\|\textbf{y}_1 - \bar{x}_1 \textbf{1}_6 \right\| = 1.7167$$"],'');
text(3.2+0.4,1.5+1.5,anno,'Rotation',270,'interpreter','latex');

drawbrace([0, 0.2], [3, 0.2],10,1,'Color','k') % Draws a curly brace for projection vector
% % Text for the norm of the projection vector, norm(x1bar1).
anno = join(["$$\left\|\rm{Proj}_{\textbf{1}_6}\textbf{y}_1 \right\| = \left\|\bar{x}_1\textbf{1}_6\right\|"," = 5.1468$$"],'');
text(0.2,-0.7,anno,'interpreter','latex');

drawbrace([-0.1, 0.2], [2.9, 3.2],10,0,'Color','k') % Draws a curly brace for y vector.
% Text for the norm of y2, norm(y1).
anno = join(["$$\left\|\textbf{y}_1 \right\| = "," 5.4256$$"],'');
text(0.5,1.4,anno,'Rotation',40,'interpreter','latex');

```

```
% Include text for angle theta.
text(0.3,0.2,'\theta')

% Include the right-angle symbol on plot.
plot([3,2.7],[0.3,0.3])
plot([2.7,2.7],[0,0.3])
set(gca,'xtick',[],'ytick',[]);
hold off
saveas(gcf,'sol3.4c.png')
```

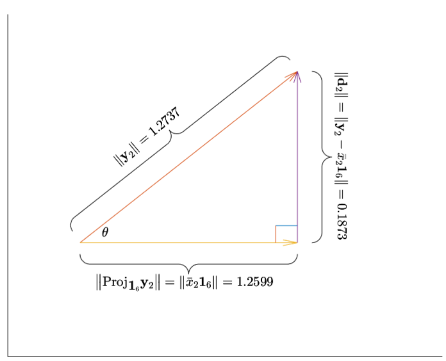
(d) Repeat Parts a-c for the variable X_2 in Table 1.1.

$$\mathbf{y}_2 = \begin{bmatrix} 0.623 \\ 0.593 \\ 0.512 \\ 0.500 \\ 0.463 \\ 0.395 \end{bmatrix}$$

$$\text{Proj}_{\mathbf{1}_6}\mathbf{y}_2 = \left(\frac{\mathbf{1}_6 \cdot \mathbf{y}_2}{\|\mathbf{1}_6\|}\right)\frac{\mathbf{1}_6}{\|\mathbf{1}_6\|} = \bar{x}_2\mathbf{1}_6 = 0.5143\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5143 \\ 0.5143 \\ 0.5143 \\ 0.5143 \\ 0.5143 \\ 0.5143 \end{bmatrix}$$

$$\mathbf{y}_2 - \bar{x}_2\mathbf{1} = \begin{bmatrix} 0.623 \\ 0.593 \\ 0.512 \\ 0.500 \\ 0.463 \\ 0.395 \end{bmatrix} - \begin{bmatrix} 0.5143 \\ 0.5143 \\ 0.5143 \\ 0.5143 \\ 0.5143 \\ 0.5143 \end{bmatrix} = \begin{bmatrix} 1.3967 \\ 0.3843 \\ -0.3183 \\ -0.3757 \\ -0.4556 \\ -0.6314 \end{bmatrix}$$

For our data, $\|\mathbf{y}_2 - \bar{x}_2\mathbf{1}\| = 0.1873$.



```
        y2 = [0.623 0.593 0.512 0.500 0.463 0.395]';
        % The projection of y2 onto a1.
        x2bar1 = ((y2'*a1)/(norm(a1)*norm(a1)))*a1;

        % Compute the deviance vector for y2.
        d2 = y2 - x2bar1;

        clear clf
        hold on
        start = zeros(1,2);
        quiver(start(:,1), start(:,2), 1, 1, 3);
        quiver(start(:,1), start(:,2), 1, 0, 3);
        quiver(3, 0, 0, 1, 3);
        xlim([-1,5])
        ylim([-2,4])
        % Use the DRAWBRACE created by Pal Naverlid Savik
        drawbrace([3.2, -0], [3.2, -3],10,1,'Color','k') % Draws a curly brace for deviance vector.
        % Text for the norm of deviance vector d2, norm(d2).
        anno = join(["$$\left\|\textbf{d}_2\right\|"," = \left\|\textbf{y}_2 - \bar{x}_2 \textbf{1}_6 \right\| = 0.1873$$"],'');
        text(3.2+0.4,1.5+1.5,anno,'Rotation',270,'interpreter','latex');

        drawbrace([0, 0.2], [3, 0.2],10,1,'Color','k') % Draws a curly brace for projection vector.
        % Text for the norm of the projection vector, norm(x2bar1).
        anno = join(["$$\left\|\rm{Proj}_{\textbf{1}_6}\textbf{y}_2 \right\| = \left\|\bar{x}_2\textbf{1}_6\right\|"," = 1.2599$$"],'');
        text(0.2,-0.7,anno,'interpreter','latex');

        drawbrace([-0.1, 0.2], [2.9, 3.2],10,0,'Color','k') % Draws a curly brace for y vector.
```

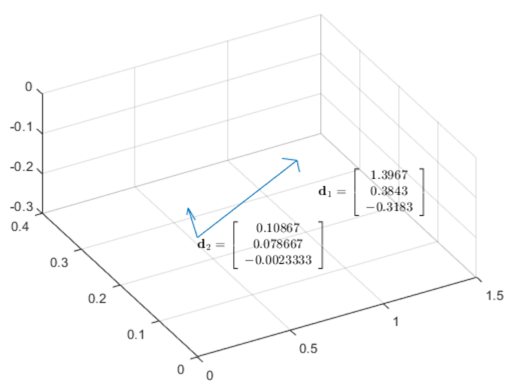
```
        % Text for the norm of y2, norm(y2).
        anno = join(["$$\left\|\textbf{y}_2 \right
           \| = "," 1.2737$$"],'');
        text(0.5,1.4,anno,'Rotation',40,'
           interpreter','latex');

        % Include text for angle theta.
        text(0.3,0.2,'\theta')

        % Include the right-angle symbol on plot.
        plot([3,2.7],[0.3,0.3])
        plot([2.7,2.7],[0,0.3])
        set(gca,'xtick',[],'ytick',[]);
        hold off
        saveas(gcf,'sol3.4d.png')
```

(e) Graph (to scale) the two deviation vectors $\mathbf{y}_1 - \bar{x}_1\mathbf{1}$ and $\mathbf{y}_2 - \bar{x}_2\mathbf{1}$. Calculate the value of the angle between them.

The deviation vectors are in $\mathbb{R}^6$, so plotting that isn't feasible so here's the plot of the first 3 dimensions.



$$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\|\mathbf{d}_1\|\,\|\mathbf{d}_2\|} = \frac{1}{1.7167 \times 0.1873} \begin{bmatrix} 1.3967 \\ 0.3843 \\ -0.3183 \\ -0.3757 \\ -0.4556 \\ -0.6314 \end{bmatrix}' \begin{bmatrix} 1.3967 \\ 0.3843 \\ -0.3183 \\ -0.3757 \\ -0.4556 \\ -0.6314 \end{bmatrix} = 0.8921$$

$$\Rightarrow \theta = \cos^{-1}(0.8921) = 26.8583^{\circ}$$

Also could do

$$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\|\mathbf{d}_1\|\,\|\mathbf{d}_2\|} = \frac{(n-1)s_{12}}{\sqrt{(n-1)s_{11}}\sqrt{(n-1)s_{22}}} = \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}} = r_{12}$$

$$\Rightarrow \theta = \cos^{-1} r_{12} = \cos^{-1}(0.8921) = 26.8583^\circ$$

```
% Combine the deviation vectors.
D = [d1 d2]';
start = zeros(size(D));

clear clf
% Plot the d_1, d_2 deviation vectors.
quiver3(start(:,1), start(:,2), start(:,3), D(:,1), D(:,2), D(:,3));
% Text for data.
for r = 1:height(D)
    % Labels for the deviation vectors, d_1 and d_2.
    anno = join(["$$\textbf{d}_",r," = \left[\begin{array}{c}",D(r,1),"\\",D(r,2),"\\",D(r,3),"\end{array}\right]$$"],'');
    text(D(r,1)-0.05,D(r,2)-0.05,D(r,3)-0.05,anno,'interpreter','latex');
end

% Compute the angle between d1 and d2.
acosd(d1'*d2/(norm(d1)*norm(d2)))
% Same answer using (2-36) on page 72, R = V^{-1/2} \Sigma V^{-1/2}.
acosd(diag(sqrt(inv(diag(diag(cov(y1,y2)))))*cov(y1,y2)*sqrt(inv(diag(diag(cov(y1,y2))))),1))
```

3.5

Calculate the generalized sample variance $|\mathbf{S}|$ for (a) the data matrix $\mathbf{X}$ in Exercise 3.1 and (b) the data matrix $\mathbf{X}$ in Exercise 3.2.

(a)

$$\mathbf{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 \end{bmatrix}, \qquad \bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\mathbf{d}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1}_3 = \begin{bmatrix} 9 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

$$\mathbf{d}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}$$

$$\mathbf{S} = \left(\frac{1}{n-1}\right) \mathbf{D}'\mathbf{D} = \left(\frac{1}{3-1}\right) \begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$$

$$|\mathbf{S}| = \begin{vmatrix} 16 & -2 \\ -2 & 1 \end{vmatrix} = 16 - 4 = 12$$

(b)

$$\mathbf{X} = \begin{bmatrix} 3 & 4 \\ 6 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 \end{bmatrix}, \qquad \bar{\mathbf{x}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\mathbf{d}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1}_3 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{d}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1}_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$$

$$\mathbf{S} = \left(\frac{1}{n-1}\right) \mathbf{D}'\mathbf{D} = \left(\frac{1}{3-1}\right) \begin{bmatrix} -1 & 2 & -1 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} 6 & -9 \\ -9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & -(9/2) \\ -(9/2) & 9 \end{bmatrix}$$

$$|\mathbf{S}| = \begin{vmatrix} 3 & -(9/2) \\ -(9/2) & 9 \end{vmatrix} = 27 - (81/4) = \frac{27}{4}$$

3.6

Consider the data matrix

$$\mathbf{X} = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

(a) Calculate the matrix of deviations (residuals), $\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}'$. Is this matrix of full rank? Explain.

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}' = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

No, the residual matrix is not full rank. The third column is column one plus column two, so there's a linear dependency. To be full rank the three columns in the square matrix must be linearly independent.

(b) Determine $\mathbf{S}$ and calculate the generalized sample variance $|\mathbf{S}|$. Interpret the latter geometrically.

$$\mathbf{S} = \left(\frac{1}{n-1}\right)\mathbf{D}^{-1}\mathbf{D} = \left(\frac{1}{2}\right) \begin{bmatrix} -3 & 0 & 3 \\ 0 & 1 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix} =$$

$$= \left(\frac{1}{2}\right) \begin{bmatrix} 18 & -3 & 15 \\ -3 & 2 & -1 \\ 15 & -1 & 14 \end{bmatrix} = \begin{bmatrix} 9 & -(3/2) & (15/2) \\ -(3/2) & 1 & -(1/2) \\ (15/2) & -(1/2) & 7 \end{bmatrix}$$

$$|\mathbf{S}| = \begin{vmatrix} 9 & -(3/2) & (15/2) \\ -(3/2) & 1 & -(1/2) \\ (15/2) & -(1/2) & 7 \end{vmatrix} = 0$$

The matrix isn't full rank, so the determinant is 0. From result 3.2 on page 130, when at least one of the deviation vectors lies in the hyperplane formed by the linear combinations from the others, the generalized variance is zero.

(c) Using the results in (b), calculate the total sample variance. [See (3-23).]

The total samle variance is the trace of $\mathbf{S}$

$$tr\left\{\mathbf{S}\right\} = 9 + 1 + 7 = 17$$

3.7

Sketch the solid ellipsoids $(\mathbf{x}-\bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}-\bar{\mathbf{x}})$ [see (3-16)] for the three matrices

$$\mathbf{S}=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix},\qquad \mathbf{S}=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix},\qquad \mathbf{S}=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

(Note that these matrices have the *same* generalized variance $|\mathbf{S}|$.)

$$\mathbf{S}=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$$

$$0=|\mathbf{S}-\lambda\mathbf{I}|=\begin{vmatrix}5-\lambda & 4\\ 4 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-16=\lambda^2-10\lambda+9=(\lambda-9)(\lambda-1)$$

The eigenvalues are $\{\lambda_1,\lambda_2\}=\{1,9\}$.
$\underline{\lambda_1=1:}$

$$\mathbf{S}\mathbf{x}_1=\lambda_1\mathbf{x}_1\Rightarrow\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}\Rightarrow\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}\Rightarrow\mathbf{e}_1=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$$

$\underline{\lambda_2=9:}$

$$\mathbf{S}\mathbf{x}_2=\lambda_2\mathbf{x}_2\Rightarrow\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}9x_1\\ 9x_2\end{bmatrix}\Rightarrow\begin{bmatrix}-4 & 4\\ 4 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}\Rightarrow\mathbf{e}_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$$

$$\mathbf{S}=\lambda_1\mathbf{e}_1\mathbf{e}_1'+\lambda_2\mathbf{e}_2\mathbf{e}_2'$$

$$\Rightarrow\mathbf{x}'\mathbf{S}\mathbf{x}=\mathbf{x}'\left(\lambda_1\mathbf{e}_1\mathbf{e}_1'+\lambda_2\mathbf{e}_2\mathbf{e}_2'\right)\mathbf{x}=\lambda_1\mathbf{x}'\mathbf{e}_1\mathbf{e}_1'\mathbf{x}+\lambda_2\mathbf{x}'\mathbf{e}_2\mathbf{e}_2'\mathbf{x}=$$

$$=\left(\frac{\mathbf{x}'\mathbf{e}_1}{(1/\sqrt{\lambda_1})}\right)\left(\frac{\mathbf{x}'\mathbf{e}_1}{(1/\sqrt{\lambda_1})}\right)'+\left(\frac{\mathbf{x}'\mathbf{e}_2}{(1/\sqrt{\lambda_2})}\right)\left(\frac{\mathbf{x}'\mathbf{e}_2}{(1/\sqrt{\lambda_2})}\right)'=$$

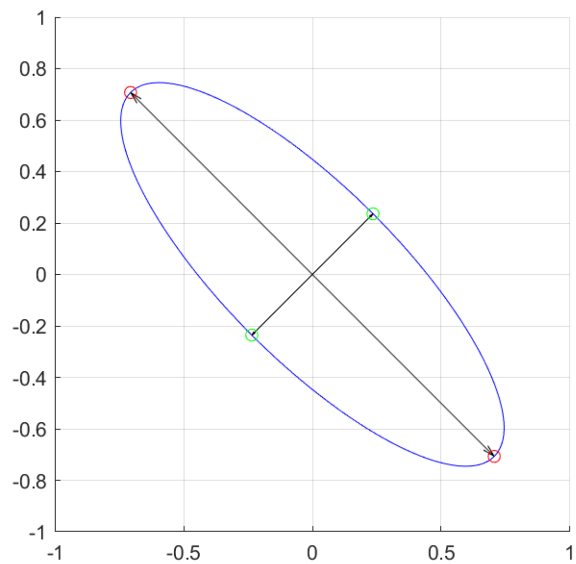
$$=\frac{(\mathbf{x}'\mathbf{e}_1)^2}{(1/\sqrt{\lambda_1})^2}+\frac{(\mathbf{x}'\mathbf{e}_2)^2}{(1/\sqrt{\lambda_2})^2}$$

Set equal to c^2 to have the equation of an ellipse centered at the origin.

$$\Rightarrow\mathbf{x}'\mathbf{S}\mathbf{x}=c^2=\frac{(\mathbf{x}'\mathbf{e}_1)^2}{(c/\sqrt{\lambda_1})^2}+\frac{(\mathbf{x}'\mathbf{e}_2)^2}{(c/\sqrt{\lambda_2})^2}$$

$$=\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}$$

The major axis (related to the smallest eigenvalue) is in the direction of $\mathbf{e}_1 = [(1/\sqrt{2}), -(1/\sqrt{2})]'$ with length $a = c/\sqrt{\lambda_1} = 1/1 = 1$. The minor axis (related to the largest eigenvalue) is in the direction of $\mathbf{e}_2 = [(1/\sqrt{2}), (1/\sqrt{2})]'$, with length $a = c/\sqrt{\lambda_2} = 1/3$. Here, $c = 1$.




```
S = [5 4; 4 5];
[V,D] = eig(S);
MyPlotEllipse(V,D,1,output_path,'sol3.7.1')
```

$$\mathbf{S} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$0 = |\mathbf{S} - \lambda\mathbf{I}| = \begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1)$$

The eigenvalues are $\{\lambda_1, \lambda_2\} = \{1, 9\}$.
$\underline{\lambda_1 = 1:}$

$$\mathbf{S}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \Rightarrow \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\underline{\lambda_2 = 9:}$

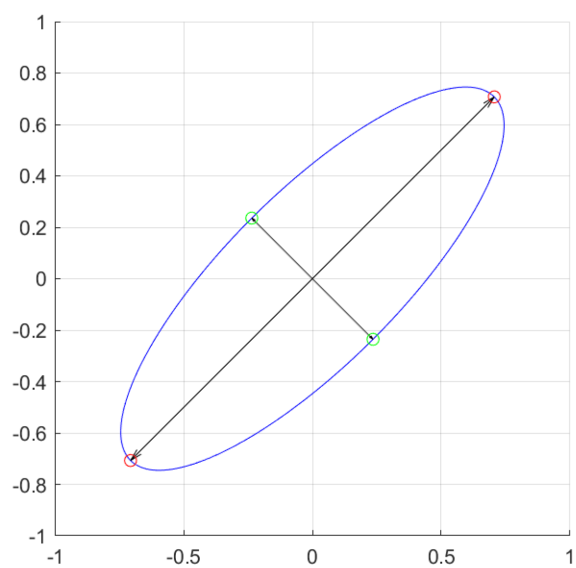
$$\mathbf{S}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \Rightarrow \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9x_1 \\ 9x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Set equal to c^2 to have the equation of an ellipse centered at the origin.

$$\Rightarrow \mathbf{x}'\mathbf{S}\mathbf{x} = c^2 = \frac{(\mathbf{x}'\mathbf{e}_1)^2}{(c/\sqrt{\lambda_1})^2} + \frac{(\mathbf{x}'\mathbf{e}_2)^2}{(c/\sqrt{\lambda_2})^2}$$
$$= \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}$$

The major axis (related to the smallest eigenvalue) is in the direction of $\mathbf{e}_1 = [(1/\sqrt{2}), (1/\sqrt{2})]'$ with length $a = c/\sqrt{\lambda_1} = 1/1 = 1$. The minor axis (related to the largest eigenvalue) is in the direction of $\mathbf{e}_2 = [(1/\sqrt{2}), -(1/\sqrt{2})]'$, with length $a = c/\sqrt{\lambda_2} = 1/3$. Here, $c = 1$. This sample covariance matrix, $\mathbf{S}$, has the same eigenvalues as the one above, but the eigenvectors here are switched so the major axis is also switched.




```
S = [5 -4; -4 5];
[V,D] = eig(S);
MyPlotEllipse(V,D,1,output_path,'sol3.7.2')
```

$$\mathbf{S} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$0 = |\mathbf{S} - \lambda \mathbf{I}| = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 0 = (\lambda - 3)(\lambda - 3)$$

The eigenvalues are both 3, $\{\lambda_1, \lambda_2\} = \{3, 3\}$.
$\underline{\lambda_1 = \lambda_2 = 3:}$

$$\mathbf{S}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Both eigenvalues are the same, so need to pick two vectors. They can be anything, so why not the stand basis vectors for $\mathbb{R}^2$.

$$\mathbf{x}_1 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

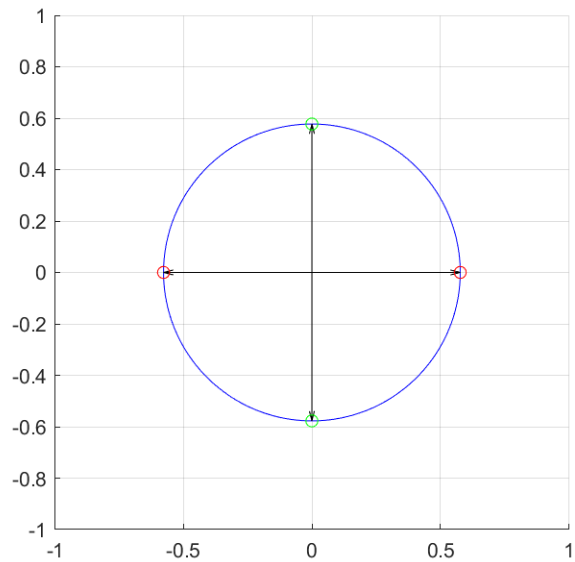
$$\mathbf{x}_2 = \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Set equal to c^2 to have the equation of an ellipse centered at the origin.

$$\Rightarrow \mathbf{x}'\mathbf{S}\mathbf{x} = c^2 = \frac{(\mathbf{x}'\mathbf{e}_1)^2}{(c/\sqrt{\lambda_1})^2} + \frac{(\mathbf{x}'\mathbf{e}_2)^2}{(c/\sqrt{\lambda_2})^2}$$

$$= \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}$$

Here, we have two eigenvalues that are the same, so the lengths in the major and minor axis are also the same. The eigenvectors are the standard basis, so we have a circle centered at the origin with radius $a = b = c/\sqrt{\lambda_1} = 1/\sqrt{3}$. Here, $c = 1$.




```
S = [3 0; 0 3];
[V,D] = eig(S);
MyPlotEllipse(V,D,1,output_path,'sol3.7.3')
```

3.8

Given

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

(a) Calculate the total sample variance for each $\mathbf{S}$. Compare the results.

$$tr\{\mathbf{S}\} = tr\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = 3$$

$$tr\{\mathbf{S}\} = tr\left\{ \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \right\} = 3$$

Both sample covariance matrices have the same total sample variance values, since both have the same sample variance values of 1 on the diagonal. The total sample variance metric doesn't account for any the covariance structure for $i \neq j$ (off-diagonal values).

(b) Calculate the generalized sample variance for each $\mathbf{S}$, and compare the results. Comment on the discrepancies, if any, found between Parts a and b.

$$|\mathbf{S}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 + 0 = 1(1-0) = 1$$

$$|\mathbf{S}| = \begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{vmatrix} =$$

$$= 1\left(1 - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\right) = \frac{3}{4} - \frac{3}{8} - \frac{3}{8} = 0$$

The generalized sample variance of the diagonal matrix is 1. The sample covariance matrix form the standard basis for $\mathbb{R}^3$, so each element is one unit from zero and form a cube of length 1 on all sides. The volumn of this cube determined by the determinant for the generalized sample variance is of course 1. The second covariance matrix has $\text{Cov}(x_i, x_j) = -1/2 \; \forall \; i \neq j$. For this sample covariance matrix there is a linear dependence where the third column is -1 times the first column plus -1 times the second column. By result 3.2 on page 130, if at least one deviation vector lies in the (hyper) plane formed by all linear combos of the others then we have a linear dependence. If there's a linear dependence the parallelepiped will have volumn 0 (using (2) on page 133). The generalized sample variance (GSV) accounts for the off-diagonal covariance values, not simply the diagonal (variance) values, like the total sample variance does, so its result is more representative of the data when we have nonzero covariance. If the vectors are closely related the GSV is small, or zero if some vectors lie in the same (hyper) plane. If vectors are far from each other the GSV will be large.

3.9

The following data matrix contains data on test scores, with x_1 = score on first test, x_2 = score on second test, and x_3 = total score on the two tests:

$$\mathbf{X} = \begin{bmatrix} 12 & 17 & 29 \\ 18 & 20 & 38 \\ 14 & 16 & 30 \\ 20 & 18 & 38 \\ 16 & 19 & 35 \end{bmatrix}$$

(a) Obtain the mean corrected data matrix, and verify that the columns are linearly dependent. Specify an $\mathbf{a}' = [a_1, a_2, a_3]$ vector that establishes the linear dependence.

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 80/5 \\ 90/5 \\ 170/5 \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \\ 34 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}}' = \begin{bmatrix} 12 & 17 & 29 \\ 18 & 20 & 38 \\ 14 & 16 & 30 \\ 20 & 18 & 38 \\ 16 & 19 & 35 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 16 & 18 & 34 \end{bmatrix} =$$

$$= \begin{bmatrix} 12 & 17 & 29 \\ 18 & 20 & 38 \\ 14 & 16 & 30 \\ 20 & 18 & 38 \\ 16 & 19 & 35 \end{bmatrix} - \begin{bmatrix} 16 & 18 & 34 \\ 16 & 18 & 34 \\ 16 & 18 & 34 \\ 16 & 18 & 34 \\ 16 & 18 & 34 \end{bmatrix} = \begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

We have a linear dependence for column 3, whose the sum of the first two columns.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{Xa} = \begin{bmatrix} 12 & 17 & 29 \\ 18 & 20 & 38 \\ 14 & 16 & 30 \\ 20 & 18 & 38 \\ 16 & 19 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

or

$$(\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}}')\,\mathbf{a} = \begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

(b) Obtain the sample covariance matrix $\mathbf{S}$, and verify that the generalized variance is zero. Also, show that $\mathbf{Sa} = \mathbf{0}$, so a can be rescaled to be an eigenvector corresponding to eigenvalue zero.

$$\mathbf{D} = (\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}})$$

$$\mathbf{S} = \left(\frac{1}{n-1}\right)\mathbf{D}'\mathbf{D} = \left(\frac{1}{4}\right) \begin{bmatrix} -4 & 2 & -2 & 4 & 0 \\ -1 & 2 & -2 & 0 & 1 \\ -5 & 4 & -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \left(\frac{1}{4}\right) \begin{bmatrix} 40 & 12 & 52 \\ 12 & 10 & 22 \\ 52 & 22 & 74 \end{bmatrix} = \begin{bmatrix} 10 & 3 & 13 \\ 3 & (5/2) & (11/2) \\ 13 & (11/2) & (37/2) \end{bmatrix}$$

Another way, using (3-27) on page 139,

$$\mathbf{S} = \left(\frac{1}{n-1}\right)\mathbf{X}'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_5\mathbf{1}_5'\right)\mathbf{X} = \left(\frac{1}{n-1}\right)\mathbf{X}'\left(\mathbf{X} - \frac{1}{n}\mathbf{1}_5\mathbf{1}_5'\mathbf{X}\right) =$$

$$= \left(\frac{1}{n-1}\right)\mathbf{X}'\left(\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}}'\right) = \left(\frac{1}{4}\right)\begin{bmatrix} 12 & 18 & 14 & 20 & 16 \\ 17 & 20 & 16 & 18 & 19 \\ 29 & 38 & 30 & 38 & 35 \end{bmatrix}\begin{bmatrix} -4 & -1 & -5 \\ 2 & 2 & 4 \\ -2 & -2 & -4 \\ 4 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \left(\frac{1}{4}\right)\begin{bmatrix} 40 & 12 & 52 \\ 12 & 10 & 22 \\ 52 & 22 & 74 \end{bmatrix} = \begin{bmatrix} 10 & 3 & 13 \\ 3 & (5/2) & (11/2) \\ 13 & (11/2) & (37/2) \end{bmatrix}$$

Computing the generalized sample variance

$$|\mathbf{S}| = \begin{vmatrix} 10 & 3 & 13 \\ 3 & (5/2) & (11/2) \\ 13 & (11/2) & (37/2) \end{vmatrix} =$$

$$= 10\begin{vmatrix} (5/2) & (11/2) \\ (11/2) & (37/2) \end{vmatrix} - 3\begin{vmatrix} 3 & (11/2) \\ 13 & (37/2) \end{vmatrix} + 13\begin{vmatrix} 3 & (5/2) \\ 13 & (11/2) \end{vmatrix} =$$

$$= \frac{10}{4}(185 - 121) - \frac{3}{2}(111 - 143) + \frac{13}{2}(33 - 65) =$$

$$= \frac{640}{4} + \frac{96}{2} - \frac{416}{2} =$$

$$= \frac{416}{2} - \frac{416}{2} = 0$$

In part (a) we could see that the third column is the sum of the first two, so we defined a vector $\mathbf{a}$ as

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

and using that same vector to compute $\mathbf{Sa}$,

$$\mathbf{Sa} = \begin{bmatrix} 10 & 3 & 13 \\ 3 & (5/2) & (11/2) \\ 13 & (11/2) & (37/2) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 - 13 \\ 11/2 - 11/2 \\ 37/2 - 37/2 \end{bmatrix} = \mathbf{0}$$

(c) Verify that the third column of the data matrix is the sum of the first two columns. That is, show that there is linear dependence, with $a_1 = 1$, $a_2 = 1$, and $a_3 = -1$.

In part (a) this was shown for the computation of $\mathbf{Xa} = \mathbf{0}$. Here's another way using column vectors in $\mathbf{X}$

$$\mathbf{Xa} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 =$$

$$= \begin{bmatrix} 12 \\ 18 \\ 14 \\ 20 \\ 16 \end{bmatrix} + \begin{bmatrix} 17 \\ 20 \\ 16 \\ 18 \\ 19 \end{bmatrix} - \begin{bmatrix} 29 \\ 38 \\ 30 \\ 38 \\ 35 \end{bmatrix} = \begin{bmatrix} 29 \\ 38 \\ 30 \\ 38 \\ 35 \end{bmatrix} - \begin{bmatrix} 29 \\ 38 \\ 30 \\ 38 \\ 35 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

3.10

When the generalized variance is zero, it is the columns of the mean corrected data matrix $\mathbf{X}_c = \mathbf{X} - \mathbf{1}\bar{\mathbf{x}}'$ that are linearly dependent, not necessarily those of the data matrix itself Given the data

$$\mathbf{X} = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 6 \\ 4 & 2 & 2 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$$

(a) Obtain the mean corrected data matrix, and verify that the columns are linearly dependent. Specify an $\mathbf{a}' = [a_1, a_2, a_3]$ vector that establishes the linear dependence.

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 25/5 \\ 10/5 \\ 15/5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}}' = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 6 \\ 4 & 2 & 2 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 6 \\ 4 & 2 & 2 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 2 & 3 \\ 5 & 2 & 3 \\ 5 & 2 & 3 \\ 5 & 2 & 3 \\ 5 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

For $\mathbf{X}_c$ we have a linear dependence for column 3, whose the sum of the first two columns.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

In Exercise 3.9. both $\mathbf{X}_c$a $= \mathbf{0}$ and $\mathbf{Xa} = \mathbf{0}$ for the same $\mathbf{a}$, but that isn't the case here. Here, $r(\mathbf{X}) = p = 3$ so the columns of $\mathbf{X}$ are linearly independent, and the columns of $\mathbf{X}_c$ are linearly dependent ($r(\mathbf{X}_c) = 2 <$

$p = 3$).

$$(\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}}')\,\mathbf{a} = \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

(b) Obtain the sample covariance matrix $\mathbf{S}$, and verify that the generalized variance is zero.

$$\mathbf{D} = (\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}})$$

$$\mathbf{S} = \left(\frac{1}{n-1}\right)\mathbf{D}'\mathbf{D} = \left(\frac{1}{4}\right) \begin{bmatrix} -2 & 1 & -1 & 2 & 0 \\ -1 & 2 & 0 & -2 & 1 \\ -3 & 3 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \left(\frac{1}{4}\right) \begin{bmatrix} 10 & 0 & 10 \\ 0 & 10 & 10 \\ 10 & 10 & 20 \end{bmatrix} = \begin{bmatrix} (5/2) & 0 & (5/2) \\ 0 & (5/2) & (5/2) \\ (5/2) & (5/2) & 5 \end{bmatrix}$$

Another way, using (3-27) on page 139,

$$\mathbf{S} = \left(\frac{1}{n-1}\right)\mathbf{X}'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_5\mathbf{1}_5'\right)\mathbf{X} = \left(\frac{1}{n-1}\right)\mathbf{X}'\left(\mathbf{X} - \frac{1}{n}\mathbf{1}_5\mathbf{1}_5'\mathbf{X}\right) =$$

$$= \left(\frac{1}{n-1}\right)\mathbf{X}'\,(\mathbf{X} - \mathbf{1}_5\bar{\mathbf{x}}') = \left(\frac{1}{4}\right) \begin{bmatrix} 3 & 6 & 4 & 7 & 5 \\ 1 & 4 & 2 & 0 & 3 \\ 0 & 6 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \left(\frac{1}{4}\right) \begin{bmatrix} 10 & 0 & 10 \\ 0 & 10 & 10 \\ 10 & 10 & 20 \end{bmatrix} = \begin{bmatrix} (5/2) & 0 & (5/2) \\ 0 & (5/2) & (5/2) \\ (5/2) & (5/2) & 5 \end{bmatrix}$$

Computing the generalized sample variance

$$|\mathbf{S}| = \begin{vmatrix} (5/2) & 0 & (5/2) \\ 0 & (5/2) & (5/2) \\ (5/2) & (5/2) & 5 \end{vmatrix} =$$

$$= \left(\frac{5}{2}\right) \begin{vmatrix} (5/2) & (5/2) \\ (5/2) & 5 \end{vmatrix} - 0 + \left(\frac{5}{2}\right) \begin{vmatrix} 0 & (5/2) \\ (5/2) & (5/2) \end{vmatrix} =$$

$$= \frac{5}{8}(50 - 25) + \frac{5}{8}(0 - 25) =$$

$$= \frac{125}{8} - \frac{125}{8} = 0$$

(c) Show that the columns of the data matrix are linearly independent in this case.

To show this, work $\mathbf{X}$ into reduced row echelon form and count the pivot columns.

$$\mathbf{X} = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 6 \\ 4 & 2 & 2 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \xrightarrow{\text{Simplify rows}} \begin{bmatrix} 3 & 1 & 0 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \\ 7 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \xrightarrow{\text{Row 5}-\frac{5}{3}\text{Row 1}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \\ 7 & 0 & 3 \\ 0 & (4/3) & 4 \end{bmatrix} \xrightarrow{\text{Row 4}-\frac{7}{3}\text{Row 1}} \begin{bmatrix} 3 & 1 & 0 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & -(7/3) & 3 \\ 0 & (4/3) & 4 \end{bmatrix} \xrightarrow{\text{Row 3}-\frac{2}{3}\text{Row 1}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 3 & 2 & 3 \\ 0 & (1/3) & 1 \\ 0 & -(7/3) & 3 \\ 0 & (4/3) & 4 \end{bmatrix} \xrightarrow{\text{Row 3}-\text{Row 1}} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & (1/3) & 1 \\ 0 & -(7/3) & 3 \\ 0 & (4/3) & 4 \end{bmatrix} \xrightarrow{\text{Simplify rows}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & -7 & 9 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{Row 5}-\text{Row 2}} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & -7 & 9 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 4}+7\text{Row 2}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 30 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 3}-\text{Row 2}} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 30 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Swap rows}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 30 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Simplify}} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 2}-3\text{Row 3}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1}-\text{Row 2}} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Simplify}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that in reduced row echelon form that there are 3 pivot columns (nonzero rows), so the rank of $\mathbf{X}$ is 3 and is of full rank ($r(\mathbf{X}) = 3 = p$).

3.11

Use the sample covariance obtained in Example 3.7 to verify (3-29) and (3-30), which state that $\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2}$ and $\mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2} = \mathbf{S}$.

$$\mathbf{S} = \begin{bmatrix} 252.04 & -68.43 \\ -68.43 & 123.67 \end{bmatrix}$$

From (3-28) on Page 139

$$\mathbf{D}^{1/2} = \begin{bmatrix} \sqrt{s_{11}} & 0 \\ 0 & \sqrt{s_{22}} \end{bmatrix} = \begin{bmatrix} \sqrt{252.04} & 0 \\ 0 & \sqrt{123.67} \end{bmatrix} = \begin{bmatrix} 15.87577 & 0 \\ 0 & 11.1207 \end{bmatrix}$$

$$\mathbf{D}^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{s_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{s_{22}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{252.04}} & 0 \\ 0 & \frac{1}{\sqrt{123.67}} \end{bmatrix} = \begin{bmatrix} 0.06298908 & 0 \\ 0 & 0.08992239 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2} =$$

$$= \begin{bmatrix} 0.06298908 & 0 \\ 0 & 0.08992239 \end{bmatrix} \begin{bmatrix} 252.04 & -68.43 \\ -68.43 & 123.67 \end{bmatrix} \begin{bmatrix} 0.06298908 & 0 \\ 0 & 0.08992239 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -0.3876 \\ -0.3876 & 1 \end{bmatrix}$$

$$\mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2} =$$

$$= \begin{bmatrix} 15.87577 & 0 \\ 0 & 11.1207 \end{bmatrix} \begin{bmatrix} 1 & -0.3876 \\ -0.3876 & 1 \end{bmatrix} \begin{bmatrix} 15.87577 & 0 \\ 0 & 11.1207 \end{bmatrix} =$$

$$\begin{bmatrix} 252.04 & -68.43 \\ -68.43 & 123.67 \end{bmatrix}$$

3.12

Show that $|\mathbf{S}| = (s_{11}s_{22}\cdots s_{pp})\,|\mathbf{R}|$.

The output of the determinant is scalar value, so we can rearrange them (commutative property), i.e., $|\mathbf{A}|\,|\mathbf{B}| = |\mathbf{B}|\,|\mathbf{A}|$.

$$|\mathbf{S}| \overset{\text{(3-30)}}{=} \left|\mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}\right| \overset{\text{Result 2A.11(e)}}{=} \left|\mathbf{D}^{1/2}\right| \left|\mathbf{R}\right| \left|\mathbf{D}^{1/2}\right| = \left|\mathbf{D}^{1/2}\right| \left|\mathbf{D}^{1/2}\right| \left|\mathbf{R}\right| =$$

$$= \left|\mathbf{D}^{1/2}\mathbf{D}^{1/2}\right|\left|\mathbf{R}\right| = |\mathbf{D}|\,|\mathbf{R}| = \prod_{i=1}^{p} s_{ii}\,|\mathbf{R}|$$

3.13

Given a data matrix $\mathbf{X}$ and the resulting sample correlation matrix $\mathbf{R}$, consider the standardized observations $(x_{ik} - \bar{x}_k)/\sqrt{s_{kk}}$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$. Show that these standardized quantities have sample covariance matrix $\mathbf{R}$.

Here are two solutions. One, using matrices and a second using the summations and algebra.

Solution 1:

$$\left[(x_{ik} - \bar{x}_k)/\sqrt{s_{kk}}\right]_{ik} = (\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2} = \mathbf{X}_\star$$

Use the formula for $\mathbf{S}$ from (3-27),

$$\mathbf{S} = \frac{1}{n-1}\mathbf{X}_\star'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'\right)\mathbf{X}_\star =$$

$$= \frac{1}{n-1}\left((\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right)'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'\right)\left((\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\left(\mathbf{D}^{-1/2}\right)'(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'\right)\left((\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'\right)(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2} =$$

$$= \frac{1}{n-1}\left(\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\,(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2} - \frac{1}{n}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\mathbf{1}_n\mathbf{1}_n'\,(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\left(\mathbf{D}^{-1/2}((n-1)\mathbf{S})\mathbf{D}^{-1/2} - \frac{1}{n}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\,(\mathbf{1}_n\mathbf{1}_n'\mathbf{X} - \mathbf{1}_n\mathbf{1}_n'\mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\left((n-1)\mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2} - \frac{1}{n}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\,(\mathbf{1}_n\,(\mathbf{1}_n'\mathbf{X}) - \mathbf{1}_n\,(\mathbf{1}_n'\mathbf{1}_n)\,\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\left((n-1)\mathbf{R} - \frac{1}{n}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\,(\mathbf{1}_n\,(n\bar{\mathbf{x}}') - \mathbf{1}_n\,(n)\,\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\left((n-1)\mathbf{R} - \frac{1}{n}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'\,(n\mathbf{1}_n\bar{\mathbf{x}}' - n\mathbf{1}_n\bar{\mathbf{x}}')\,\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}\left((n-1)\mathbf{R} - \frac{1}{n}\mathbf{D}^{-1/2}(\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')'0\mathbf{D}^{-1/2}\right) =$$

$$= \frac{1}{n-1}((n-1)\mathbf{R} - 0) =$$

$$= \frac{n-1}{n-1}\mathbf{R} =$$

$$= \mathbf{R}$$

Solution 2:
For an observation j variable k, define the standardized values as

$$y_{jk} = \frac{x_{jk} - \bar{x}_k}{\sqrt{s_{kk}}}$$

A mean value for variable k from the p by 1 mean vector would be

$$\bar{y}_k = \sum_{j=1}^{n} \frac{y_{jk}}{n} = \sum_{j=1}^{n} \frac{(x_{jk} - \bar{x}_k)}{n\sqrt{s_{kk}}}$$

An element ik from the p by p covariance matrix would be

$$s_{ik} = \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} (y_{ji} - \bar{y}_i)(y_{jk} - \bar{y}_k) =$$

$$= \left(\frac{1}{n-1}\right) \sum_{j=1}^{n} \left(\left(\frac{x_{ji} - \bar{x}_i}{\sqrt{s_{ii}}}\right) - \left(\sum_{\ell=1}^{n} \frac{(x_{\ell i} - \bar{x}_i)}{n\sqrt{s_{ii}}}\right)\right) \left(\left(\frac{x_{jk} - \bar{x}_k}{\sqrt{s_{kk}}}\right) - \left(\sum_{\ell=1}^{n} \frac{(x_{\ell k} - \bar{x}_k)}{n\sqrt{s_{kk}}}\right)\right) =$$

$$= \left(\frac{1}{n-1}\right) \left(\frac{1}{\sqrt{s_{ii}}\sqrt{s_{kk}}}\right) \sum_{j=1}^{n} \left(x_{ji} - \bar{x}_i - \left(\sum_{\ell=1}^{n} \frac{x_{\ell i} - \bar{x}_i}{n}\right)\right) \left(x_{jk} - \bar{x}_k - \left(\sum_{\ell=1}^{n} \frac{x_{\ell k} - \bar{x}_k}{n}\right)\right) =$$

$$= \left(\frac{1}{n-1}\right) \left(\frac{1}{\sqrt{s_{ii}}\sqrt{s_{kk}}}\right) \sum_{j=1}^{n} \left(x_{ji} - \bar{x}_i - \left(\sum_{\ell=1}^{n} \frac{x_{\ell i}}{n}\right) + n\frac{\bar{x}_i}{n}\right) \left(x_{jk} - \bar{x}_k - \left(\sum_{\ell=1}^{n} \frac{x_{\ell k}}{n}\right) - n\frac{\bar{x}_k}{n}\right) =$$

$$= \left(\frac{1}{n-1}\right) \left(\frac{1}{\sqrt{s_{ii}}\sqrt{s_{kk}}}\right) \sum_{j=1}^{n} (x_{ji} - \bar{x}_i - \bar{x}_i + \bar{x}_i)(x_{jk} - \bar{x}_k - \bar{x}_k + \bar{x}_k) =$$

$$= \left(\frac{1}{n-1}\right) \left(\frac{1}{\sqrt{s_{ii}}\sqrt{s_{kk}}}\right) \sum_{j=1}^{n} (x_{ji} - \bar{x}_i)(x_{jk} - \bar{x}_k) =$$

$$= \left(\frac{1}{n-1}\right) \left(\frac{1}{\sqrt{s_{ii}}\sqrt{s_{kk}}}\right) (n-1)s_{ik} =$$

$$= \frac{s_{ik}}{\sqrt{s_{ii}}\sqrt{s_{kk}}} = r_{ik}$$

This is element ik from the p by p sample correlation matrix.

3.14

Consider the data matrix $\mathbf{X}$ in Exercise 3.1. We have $n = 3$ observations on $p = 2$ variables X_1 and X_2. Form the linear combinations

$$\mathbf{c}'\mathbf{X} = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -X_1 + 2X_2$$

$$\mathbf{b}'\mathbf{X} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 2X_1 + 3X_2$$

(a) Evaluate the sample means, variances, and covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ from first principles. That is, calculate the observed values of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ and then use the sample mean, variance, and covariance formulas.

$$\mathbf{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{c}'\mathbf{x}_1 &= -1x_{11} + 2x_{12} = -1(9) + 2(1) = -7 \\ \mathbf{c}'\mathbf{x}_2 &= -1x_{21} + 2x_{22} = -1(5) + 2(3) = 1 \\ \mathbf{c}'\mathbf{x}_3 &= -1x_{31} + 2x_{32} = -1(1) + 2(2) = 3 \end{aligned}$$

$$\text{Sample mean} = \frac{(-7+1+3)}{3} = -1$$

$$\text{Sample variance} = \frac{(-7+1)^2 + (1+1)^2 + (3+1)^2}{3-1} = 28$$

$$\begin{aligned} \mathbf{b}'\mathbf{x}_1 &= 2x_{11} + 3x_{12} = 2(9) + 3(1) = 21 \\ \mathbf{b}'\mathbf{x}_2 &= 2x_{21} + 3x_{22} = 2(5) + 3(3) = 19 \\ \mathbf{b}'\mathbf{x}_3 &= 2x_{31} + 3x_{32} = 2(1) + 3(2) = 8 \end{aligned}$$

$$\text{Sample mean} = \frac{(21+19+8)}{3} = 16$$

$$\text{Sample variance} = \frac{(21-16)^2 + (19-16)^2 + (8-16)^2}{3-1} = 49$$

The covariance between $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$

$$\text{Sample covariance} =$$

$$= \frac{(-7+1)(21-16) + (1+1)(19-16) + (3+1)(8-16)}{3-1} =$$
$$= -28$$

(b) Calculate the sample means, variances, covariance of $\mathbf{B}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ using (3-36). Compare the results in (a) and (b).

First, compute $\bar{\mathbf{x}}$ and $\mathbf{S}_{\mathbf{X}}$

$$\bar{\mathbf{x}} = \frac{1}{n}\mathbf{X}'\mathbf{1}_n = \frac{1}{3}\mathbf{X}'\mathbf{1}_3 = \frac{1}{3}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 15 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The deviance is

$$\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}' = \mathbf{X} - \mathbf{1}_3\bar{\mathbf{x}}' = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 5 & 2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix}$$

Using the deviance to get $\mathbf{S_X}$

$$\mathbf{S_X} = \left(\frac{1}{n-1}\right) (\mathbf{X} - \mathbf{1}_n \bar{\mathbf{x}}')' (\mathbf{X} - \mathbf{1}_n \bar{\mathbf{x}}') =$$

$$= \left(\frac{1}{3-1}\right) \begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix} =$$

$$= \left(\frac{1}{3-1}\right) \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\text{sample mean} = E[\mathbf{c}'\mathbf{X}] = \mathbf{c}'E[\mathbf{X}] = \mathbf{c}'\bar{\mathbf{x}} = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = -5 + 4 = -1$$

$$\text{sample variance} = V[\mathbf{c}'\mathbf{X}] = \mathbf{c}'V[\mathbf{X}]\mathbf{c} = \mathbf{c}'\mathbf{S_X}\mathbf{c} =$$

$$= \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -20 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 28$$

$$\text{sample mean} E[\mathbf{b}'\mathbf{X}] = \mathbf{b}'E[\mathbf{X}] = \mathbf{b}'\bar{\mathbf{x}} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 10 + 6 = 16$$

$$\text{sample variance} = V[\mathbf{b}'\mathbf{X}] = \mathbf{b}'V[\mathbf{X}]\mathbf{c} = \mathbf{b}'\mathbf{S_X}\mathbf{c} =$$

$$= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 26 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 49$$

The sample covariance between $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$

$$\text{Sample covariance} = \text{Cov}(\mathbf{c}'\mathbf{X}, \mathbf{b}'\mathbf{X}) = \mathbf{c}'\text{Cov}(\mathbf{X}, \mathbf{X})\mathbf{b} = \mathbf{c}'V[\mathbf{X}]\mathbf{b} = \mathbf{c}'\mathbf{S_X}\mathbf{b} =$$

$$= \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -20 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -40 + 12 = -28$$

Everything computed here in part (b) agrees with the corresponding values computed in part (a) using first principles.

3.15

Repeat Exercise 3.14 using the data matrix

$$\begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

and the linear combinations

$$\mathbf{b}'\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\mathbf{c}'\mathbf{X} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

(a) Evaluate the sample means, variances, and covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ from first principles. That is, calculate the observed values of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ and then use the sample mean, variance, and covariance formulas.

$$\begin{aligned} \mathbf{b}'\mathbf{x}_1 &= 1x_{11} + 1x_{12} + 1x_{13} = 1(1) + 1(4) + 1(3) = 8 \\ \mathbf{b}'\mathbf{x}_2 &= 1x_{21} + 1x_{22} + 1x_{23} = 1(6) + 1(2) + 1(6) = 14 \\ \mathbf{b}'\mathbf{x}_3 &= 1x_{31} + 1x_{32} + 1x_{33} = 1(8) + 1(3) + 1(3) = 14 \end{aligned}$$

$$\text{Sample mean} = \frac{(8 + 14 + 14)}{3} = 12$$

$$\text{Sample variance} = \frac{(8-12)^2 + (14-12)^2 + (14-12)^2}{3-1} = 12$$

$$\begin{aligned} \mathbf{c}'\mathbf{x}_1 &= 1x_{11} + 2x_{12} - 3x_{13} = 1(1) + 2(4) - 3(3) = 0 \\ \mathbf{c}'\mathbf{x}_2 &= 1x_{21} + 2x_{22} - 3x_{23} = 1(6) + 2(2) - 3(6) = -8 \\ \mathbf{c}'\mathbf{x}_3 &= 1x_{31} + 2x_{32} - 3x_{33} = 1(8) + 2(3) - 3(3) = 5 \end{aligned}$$

$$\text{Sample mean} = \frac{(0 - 8 + 5)}{3} = -1$$

$$\text{Sample variance} = \frac{(0+1)^2 + (-8+1)^2 + (5+1)^2}{3-1} = 43$$

The covariance between $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$

$$\text{Sample covariance} =$$

$$= \frac{(8-12)(0+1) + (14-12)(-8+1) + (14-12)(5+1)}{3-1} =$$
$$= -3$$

(b) Calculate the sample means, variances, covariance of $\mathbf{B}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$ using (3-36). Compare the results in (a) and (b).

First, compute $\bar{\mathbf{x}}$ and $\mathbf{S}_{\mathbf{X}}$

$$\bar{\mathbf{x}} = \frac{1}{n}\mathbf{X}'\mathbf{1}_n = \frac{1}{3}\mathbf{X}'\mathbf{1}_3 = \frac{1}{3}\begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 3 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 15 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

The deviance is

$$\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}' = \mathbf{X} - \mathbf{1}_3\bar{\mathbf{x}}' = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 4 \\ 5 & 3 & 4 \\ 5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

Using the deviance to get $\mathbf{S_X}$

$$\mathbf{S_X} = \left(\frac{1}{n-1}\right) (\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}')' (\mathbf{X} - \mathbf{1}_n\bar{\mathbf{x}}') =$$

$$= \left(\frac{1}{3-1}\right) \begin{bmatrix} -4 & 1 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -4 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 0 & -1 \end{bmatrix} =$$

$$= \left(\frac{1}{3-1}\right) \begin{bmatrix} 26 & -5 & 3 \\ -5 & 2 & -3 \\ 3 & -3 & 6 \end{bmatrix} = \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}$$

$$\text{sample mean} = E[\mathbf{b}'\mathbf{X}] = \mathbf{b}'E[\mathbf{X}] = \mathbf{b}'\bar{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} = 5{+}3{+}4 = 12$$

$$\text{sample variance} = V[\mathbf{b}'\mathbf{X}] = \mathbf{b}'V[\mathbf{X}]\mathbf{c} = \mathbf{b}'\mathbf{S_X}\mathbf{c} =$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 12$$

$$\text{sample mean} = E[\mathbf{c}'\mathbf{X}] = \mathbf{c}'E[\mathbf{X}] = \mathbf{c}'\bar{\mathbf{x}} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} =$$

$$= 5 + 6 - 12 = -1$$

$$\text{sample variance} = V[\mathbf{c}'\mathbf{X}] = \mathbf{c}'V[\mathbf{X}]\mathbf{c} = \mathbf{c}'\mathbf{S_X}\mathbf{c} =$$

$$= \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7/2 & 4 & 21/5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 43$$

The sample covariance between $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$

$$\text{Sample covariance} = \mathrm{Cov}(\mathbf{c}'\mathbf{X}, \mathbf{b}'\mathbf{X}) = \mathbf{c}'\mathrm{Cov}(\mathbf{X}, \mathbf{X})\mathbf{b} = \mathbf{c}'V[\mathbf{X}]\mathbf{b} =$$

$$= \mathbf{c}'\mathbf{S_X}\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} =$$

$$= \begin{bmatrix} 12 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 12 - 6 - 9 = -3$$

Everything computed here in part (b) agrees with the corresponding values computed in part (a) using first principles.

3.16

Let $\mathbf{V}$ be a vector random variable with mean vector $E(\mathbf{V}) = \boldsymbol{\mu}_{\mathbf{V}}$ and covariance matrix $E(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})' = \boldsymbol{\Sigma}_{\mathbf{V}}$. Show that $E(\mathbf{V}\mathbf{V}') = \boldsymbol{\Sigma}_{\mathbf{V}} + \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}'$.

$$\begin{aligned}
\boldsymbol{\Sigma}_{\mathbf{V}} = E(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})' = E(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})(\mathbf{V}' - \boldsymbol{\mu}_{\mathbf{V}}') = \\
= E\left[\mathbf{V}\mathbf{V}' - \mathbf{V}\boldsymbol{\mu}_{\mathbf{V}}' - \boldsymbol{\mu}_{\mathbf{V}}\mathbf{V}' + \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}'\right] = \\
= E\left[\mathbf{V}\mathbf{V}'\right] - E\left[\mathbf{V}\boldsymbol{\mu}_{\mathbf{V}}'\right] - E\left[\boldsymbol{\mu}_{\mathbf{V}}\mathbf{V}'\right] + E\left[\boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}'\right] = \\
= E\left[\mathbf{V}\mathbf{V}'\right] - E\left[\mathbf{V}\right]\boldsymbol{\mu}_{\mathbf{V}}' - \boldsymbol{\mu}_{\mathbf{V}}E\left[\mathbf{V}'\right] + \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}' = \\
= E\left[\mathbf{V}\mathbf{V}'\right] - \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}' - \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}' + \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}' = \\
= E\left[\mathbf{V}\mathbf{V}'\right] - \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}' = \boldsymbol{\Sigma}_{\mathbf{V}} \\
\Rightarrow E\left[\mathbf{V}\mathbf{V}'\right] = \boldsymbol{\Sigma}_{\mathbf{V}} + \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}_{\mathbf{V}}'
\end{aligned}$$

3.17

Show that if $\underset{(p\times 1)}{\mathbf{X}}$ and $\underset{(q\times 1)}{\mathbf{Z}}$ are independent, then each component of $\mathbf{X}$ is independent of each component of $\mathbf{Z}$.
Hint:

$$\begin{aligned}
&P[X_1 \le x_1, X_2 \le x_2, \ldots, X_p \le x_p \text{ and } Z_1 \le z_1, Z_2 \le z_2, \ldots, Z_q \le z_q] \\
&\qquad P[X_1 \le x_1, X_2 \le x_2, \ldots, X_p \le x_p] \cdot P[Z_1 \le z_1, Z_2 \le z_2, \ldots, Z_q \le z_q]
\end{aligned}$$

by independence. Let $x_2, \ldots, x_p$ and $z_2, \ldots, z_q$ tend to infinity, to obtain

$$P[X_1 \le x_1 \text{ and } Z_1 \le Z_1] = P[X_1 \le x_1] \cdot P[Z_1 \le Z_1]$$

for all x_1, z_1. So X_1 and Z_1 are independent. Repeat for all other pairs.

We know $\mathbf{X}$ and $\mathbf{Z}$ are independent, so first we can break the joint multivariate CDF's into two. If we're looking at some pair of components i and j, where i is in $\mathbf{X}$ and j is in $\mathbf{Z}$, and taking the limit to ∞ for everything other than the i, j pair the limit will go to 1 for the non i, j components and we're left with the marginal CDF.

$$\lim_{\substack{i'\to\infty \\ i'\neq i}} \lim_{\substack{j'\to\infty \\ j'\neq j}} P[X_1 \le x_1, X_2 \le x_2, \ldots, X_p \le x_p \text{ and } Z_1 \le z_1, Z_2 \le z_2, \ldots, Z_q \le z_q] =$$

$$= \lim_{\substack{i'\to\infty \\ i'\neq i}} P[X_1 \le x_1, X_2 \le x_2, \ldots, X_p \le x_p] \cdot \lim_{\substack{j'\to\infty \\ j'\neq j}} P[Z_1 \le z_1, Z_2 \le z_2, \ldots, Z_q \le z_q] =$$

$$= P[X_i \le x_i \text{ and } Z_j \le z_j] = P[X_i \le x_i] \cdot P[Z_j \le z_j]$$

3.18

Energy consumption in 2001, by state, from the major resources

$$x_1 = \text{petroleum} \qquad x_2 = \text{natural gas}$$
$$x_3 = \text{hydroelectric power} \qquad x_4 = \text{nuclear electric power}$$

is recorded in quadrillions (10^{15}) of BTUs (Source: *Statistical Abstract of the United States 2006*)
The resulting mean and covariance matrix are

$$\bar{\mathbf{x}} = \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix}$$

(a) Using the summary statistics, determine the sample mean and variance of a state's total energy consumption for these major sources.

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{b}'\mathbf{x} = X_1 + X_2 + X_3 + X_4$$

$$E[\mathbf{b}'\mathbf{x}] = \mathbf{b}'E[\mathbf{x}] = \mathbf{b}'\bar{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix} =$$

$$= 0.766 + 0.508 + 0.438 + 0.161 = 1.8730$$

$$\text{Cov}(\mathbf{b}'\mathbf{x}) = \mathbf{b}'\text{Cov}(\mathbf{x})\mathbf{b} = \mathbf{b}'\mathbf{S}\mathbf{b} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1.76 & 1.3970 & 0.5110 & 0.2450 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3.9130$$

(b) Determine the sample mean and variance of the excess of petroleum consumption over natural gas consumption. Also find the sample covariance of this variable with the total variable in part a.

$$\mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{c}'\mathbf{x} = X_1 - X_2$$

$$E[\mathbf{c}'\mathbf{x}] = \mathbf{c}'E[\mathbf{x}] = \mathbf{c}'\bar{\mathbf{x}} = \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0.766\\0.508\\0.438\\0.161\end{bmatrix} =$$

$$= 0.766 - 0.508 + 0 + 0 = 0.2580$$

$$\mathrm{Cov}(\mathbf{b}'\mathbf{x}) = \mathbf{c}'\mathrm{Cov}(\mathbf{x})\mathbf{c} = \mathbf{c}'\mathbf{S}\mathbf{c} =$$

$$= \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0.856 & 0.635 & 0.173 & 0.096\\0.635 & 0.568 & 0.128 & 0.067\\0.173 & 0.127 & 0.171 & 0.039\\0.096 & 0.067 & 0.039 & 0.043\end{bmatrix}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} =$$

$$= \begin{bmatrix}0.2210 & 0.0670 & 0.0450 & 0.0290\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 0.1540$$

The sample covariance between excess petroleum consumption over gas consumption with the total energy consumption is

$$\mathrm{Cov}(\mathbf{c}'\mathbf{x}, \mathbf{b}'\mathbf{x}) = \mathbf{c}'\mathrm{Cov}(\mathbf{x})\mathbf{b} = \mathbf{c}'\mathbf{S}\mathbf{b} =$$

$$= \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0.856 & 0.635 & 0.173 & 0.096\\0.635 & 0.568 & 0.128 & 0.067\\0.173 & 0.127 & 0.171 & 0.039\\0.096 & 0.067 & 0.039 & 0.043\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}0.2210 & 0.0670 & 0.0450 & 0.0290\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 0.3620$$

3.19

Using the summary statistics for the first three variables in Exercise 3.18, verify the relation

$$|\mathbf{S}| = (s_{11}s_{22}s_{33})|\mathbf{R}|$$

$$|\mathbf{S}| = \begin{vmatrix}0.856 & 0.635 & 0.173\\0.635 & 0.568 & 0.128\\0.173 & 0.127 & 0.171\end{vmatrix} =$$

$$= 0.856\begin{vmatrix}0.568 & 0.128\\0.127 & 0.171\end{vmatrix} - 0.635\begin{vmatrix}0.635 & 0.128\\0.173 & 0.171\end{vmatrix} + 0.173\begin{vmatrix}0.635 & 0.568\\0.173 & 0.127\end{vmatrix} =$$

$$= 0.856(0.568 * 0.171 - 0.128 * 0.127) -$$
$$0.635(0.635 * 0.171 - 0.128 * 0.173) +$$
$$0.173(0.635 * 0.127 - 0.568 * 0.173) =$$

$$= 0.01128831$$

$$\mathbf{D}^{-1/2} = \begin{bmatrix} 1/\sqrt{s_{11}} & 0 & 0 \\ 0 & 1/\sqrt{s_{22}} & 0 \\ 0 & 0 & 1/\sqrt{s_{33}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/\sqrt{0.856} & 0 & 0 \\ 0 & 1/\sqrt{0.568} & 0 \\ 0 & 0 & 1/\sqrt{0.171} \end{bmatrix} =$$

$$= \begin{bmatrix} 1.0808 & 0 & 0 \\ 0 & 1.3269 & 0 \\ 0 & 0 & 2.4183 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2} =$$

$$= \begin{bmatrix} 1.0808 & 0 & 0 \\ 0 & 1.3269 & 0 \\ 0 & 0 & 2.4183 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 \\ 0.635 & 0.568 & 0.128 \\ 0.173 & 0.127 & 0.171 \end{bmatrix} \begin{bmatrix} 1.0808 & 0 & 0 \\ 0 & 1.3269 & 0 \\ 0 & 0 & 2.4183 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.9252 & 0.6863 & 0.1870 \\ 0.8426 & 0.7537 & 0.1698 \\ 0.4184 & 0.3071 & 0.4135 \end{bmatrix} \begin{bmatrix} 1.0808 & 0 & 0 \\ 0 & 1.3269 & 0 \\ 0 & 0 & 2.4183 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0.9107 & 0.4522 \\ 0.9107 & 1 & 0.4107 \\ 0.4522 & 0.4075 & 1 \end{bmatrix}$$

$$|\mathbf{R}| = \begin{vmatrix} 1 & 0.9107 & 0.4522 \\ 0.9107 & 1 & 0.4107 \\ 0.4522 & 0.4075 & 1 \end{vmatrix} =$$

$$= 1 \begin{vmatrix} 1 & 0.4107 \\ 0.4075 & 1 \end{vmatrix} - 0.9107 \begin{vmatrix} 0.9107 & 0.4107 \\ 0.4522 & 1 \end{vmatrix} + 0.4522 \begin{vmatrix} 0.9107 & 1 \\ 0.4522 & 0.4075 \end{vmatrix} =$$

$$= 1(1 * 1 - 0.4107 * 0.4075) -$$
$$0.9107(0.9107 * 1 - 0.4107 * 0.4522) +$$
$$0.4522(0.9107 * 0.4075 - 1 * 0.4522) =$$

$$= 0.13577$$

$(s_{11}s_{22}s_{33})|\mathbf{R}| = (0.856 * 0.568 * 0.171)0.135770.083141568 * 0.13577 = 0.011288$

Okay, so we have that $|\mathbf{S}| = (s_{11}s_{22}s_{33})|\mathbf{R}|$.

3.20

In northern climates, roads must be cleared of snow quickly following a storm. One measure of storm severity is x_1 = its duration in hours, while the effectiveness of snow removal can be quantified by x_2 = the number of hours crews, men, and machine, spend to clear snow. Here are the results for 25 incidents in Wisconsin.

Table 3.2 Snow Data

x_1	x_2	x_3	x_4	x_5	x_6
12.5	13.7	9.0	24.4	3.5	26.1
14.5	16.5	6.5	18.2	8.0	14.5
8.0	17.4	10.5	22.0	17.5	42.3
9.0	11.0	10.0	32.5	10.5	17.5
19.5	23.6	4.5	18.7	12.0	21.8
8.0	13.2	7.0	15.8	6.0	10.4
9.0	32.1	8.5	15.6	13.0	25.6
7.0	12.3	6.5	12.0		
7.0	11.8	8.0	12.8		

(a) Find the sample mean and variance of the difference $x_2 - x_1$ by first obtaining the summary statistics.

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 9.42 \\ 19.272 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 14.139167 & 13.472667 \\ 13.472667 & 62.238767 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{b}'\mathbf{X} = X_1 - X_2$$

$$E[\mathbf{b}'\mathbf{X}] = \mathbf{b}'E[\mathbf{X}] = \mathbf{b}'\bar{\mathbf{x}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 9.42 \\ 19.272 \end{bmatrix} = -9.852$$

$$\mathrm{Cov}(\mathbf{b}'\mathbf{X}) = \mathbf{b}'\mathrm{Cov}(\mathbf{X})\mathbf{b} =$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 14.139167 & 13.472667 \\ 13.472667 & 62.238767 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.6665 & -84.7661 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 49.43260$$

(b) Obtain the mean and variance by first obtaining the individual values $x_{j2} - x_{j1}$ for $j = 1, 2, \ldots, 25$ and then calculating the mean and variance.

Compare these values with those obtained in part a.

$$\mathbf{Y} = \mathbf{X}\mathbf{b} = \begin{bmatrix} 12.5 & 13.7 \\ \vdots & \vdots \\ 13.0 & 25.6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.2 \\ \vdots \\ -12.6 \end{bmatrix}$$

$$E[\mathbf{Y}] = \frac{-1.2 + \cdots - 12.6}{25} = -9.852$$

$$\text{Cov}(\mathbf{Y}) = V[\mathbf{Y}] = \frac{(-1.2 + 9.852)^2 + \cdots + (-12.6 + 9.852)^2}{25 - 1} = 49.43260$$

These values are the same as those computed in part (a).

4 Chapter 4

4.1

Consider a bivariate normal distribution with $\mu_1 = 1$, $\mu_2 = 3$, $\sigma_{11} = 2$, $\sigma_{22} = 1$ and $\rho_{12} = -0.8$.

(a) Write out the bivariate normal density. For

$$\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\rho_{12} \\ \sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\rho_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 2 & -0.80\sqrt{2} \\ -0.80\sqrt{2} & 1 \end{bmatrix}$$

$$f(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} e^{\frac{-1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}(\mathbf{x}-\boldsymbol{\mu})} =$$

$$= \frac{1}{(2\pi)^{2/2}\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}} \exp\left\{ \frac{-1}{2(1-\rho_{12}^2)} \left[\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right)^2 + \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right)^2 \right.\right.$$
$$\left.\left. - 2\rho_{12} \left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right) \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right) \right] \right\} =$$

$$= \frac{1}{(2\pi)^{2/2}\sqrt{2(1-0.64)}} \exp\left\{ \frac{-1}{2(1-0.64)} \left[\left(\frac{x_1 - 1}{\sqrt{2}} \right)^2 + \left(\frac{x_2 - 3}{\sqrt{1}} \right)^2 \right.\right.$$
$$\left.\left. + 2(0.80) \left(\frac{x_1 - 1}{\sqrt{2}} \right) \left(\frac{x_2 - 3}{\sqrt{1}} \right) \right] \right\} =$$

$$= \frac{1}{1.2\pi\sqrt{2}} \exp\left\{ \frac{-1}{0.72} \left[\left(\frac{x_1 - 1}{\sqrt{2}} \right)^2 + (x_2 - 3)^2 \right.\right.$$
$$\left.\left. + 1.6 \left(\frac{x_1 - 1}{\sqrt{2}} \right) (x_2 - 3) \right] \right\}$$

(b) Write out the squared statistical distance expression $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as a quadratic function of x_1 and x_2.

This is most of what's inside the exponent.

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) =$$

$$= \frac{1}{1-\rho_{12}^2} \left[\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right)^2 + \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right)^2 - 2\rho_{12} \left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right) \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right) \right] =$$

$$= \frac{1}{0.36}\left[\left(\frac{x_1-1}{\sqrt{2}}\right)^2 + \left(\frac{x_2-3}{1}\right)^2 + 1.6\left(\frac{x_1-1}{\sqrt{2}}\right)(x_2-3)\right] =$$

$$= \frac{1}{0.36}\left[\frac{1}{2}({x_1}^2-2x_1+1)+({x_2}^2-6x_2+9)+\frac{1.6\sqrt{2}}{2}(x_1x_2-3x_1-x_2+3)\right] =$$

$$= \frac{1}{0.36}\left[\frac{1}{2}({x_1}^2-2x_1+1)+({x_2}^2-6x_2+9)+\frac{1.6\sqrt{2}}{2}(x_1x_2-3x_1-x_2+3)\right] =$$

$$= \frac{25}{18}{x_1}^2+\frac{50}{18}{x_2}^2-\frac{5(5+12\sqrt{2})}{9}x_1-x_2\frac{10(15+2\sqrt{2})}{9}+\frac{20\sqrt{2}}{9}x_1x_2+\frac{5(95+24\sqrt{2})}{18} =$$

$$= 1.3889x_1^2 + 2.7778x_2^2 - 12.2059x_1 - 19.8094x_2 + 3.1427x_1x_2 + 35.8170$$

4.2

Consider a bivariate normal distribution with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$ and $\rho_{12} = 0.5$.

(a) Write out the bivariate normal density. For

$$\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\rho_{12} \\ \sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\rho_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0.50\sqrt{2} \\ 0.50\sqrt{2} & 1 \end{bmatrix}$$

$$f(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} e^{\frac{-1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}(\mathbf{x}-\boldsymbol{\mu})} =$$

$$= \frac{1}{(2\pi)^{2/2}\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}} \exp\left\{\frac{-1}{2(1-\rho_{12}^2)}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 \right.\right.$$
$$\left.\left. - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]\right\} =$$

$$= \frac{1}{(2\pi)^{2/2}\sqrt{2(1-0.25)}} \exp\left\{\frac{-1}{2(1-0.25)}\left[\left(\frac{x_1-0}{\sqrt{2}}\right)^2 + \left(\frac{x_2-2}{\sqrt{1}}\right)^2 \right.\right.$$
$$\left.\left. - 2(0.50)\left(\frac{x_1-0}{\sqrt{2}}\right)\left(\frac{x_2-2}{\sqrt{1}}\right)\right]\right\} =$$

$$= \frac{\sqrt{6}}{6\pi} \exp\left\{\frac{-2}{3}\left[\frac{x_1^2}{2} + (x_2-2)^2 - \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)\right]\right\}$$

(b) Write out the squared statistical distance expression $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ as a quadratic function of x_1 and x_2.

This is most of what's inside the exponent.

$$(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) =$$

$$= \frac{1}{1-\rho_{12}^2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right] =$$

$$= \frac{4}{3}\left[\frac{1}{2}x_1^2 + (x_2-2)^2 + \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)\right] =$$

$$= \frac{4}{3}\left[\frac{1}{2}x_1^2 + \left(x_2^2 - 4x_2 + 4\right) + \frac{\sqrt{2}}{2}(x_1x_2 - 2x_1)\right] =$$

$$= \frac{2}{3}x_1^2 + \frac{4}{3}x_2^2 - \frac{4\sqrt{2}}{3}x_1 - \frac{16}{3}x_2 + \frac{2\sqrt{2}}{3}x_1x_2 - \frac{16}{3} =$$

$$= 0.6667x_1^2 + 1.3333x_2^2 - 1.8856x_1 - 5.3333x_2 + 0.9428x_1x_2 + 5.3333$$

(c) Determine (and sketch) the constant-density contour that contains 50% of the probability.

First, find the eigenvalues.

$$0 = |\boldsymbol{\Sigma} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & \sqrt{2}/2 \\ \sqrt{2}/2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - \frac{2}{4} =$$

$$= \lambda^2 - 3\lambda + 2 - \frac{1}{2} = \lambda^2 - 3\lambda + \frac{3}{2} = (\lambda - (3-\sqrt{3})/2)(\lambda - (3+\sqrt{3})/2)$$

The eigenvalues are $(3\pm\sqrt{3})/2$.

$\underline{\lambda_1 = \frac{3-\sqrt{3}}{2}:}$

$$\boldsymbol{\Sigma}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \Rightarrow \begin{bmatrix} 2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3-\sqrt{3})/2x_1 \\ (3-\sqrt{3})/2x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1+\sqrt{3})/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & (-1+\sqrt{3})/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1+\sqrt{3})/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & (-1+\sqrt{3})/2 \end{bmatrix} \xrightarrow{\text{Row 2}-\left(\frac{2}{1+\sqrt{3}}\right)\frac{\sqrt{2}}{2}\text{Row 1}} \begin{bmatrix} (1+\sqrt{3})/2 & \\ 0 & 0 \end{bmatrix}$$

$$\frac{(1+\sqrt{3})}{2}x_1 = -\frac{\sqrt{2}}{2}x_2 \Rightarrow x_1 = -\frac{\sqrt{2}}{(1+\sqrt{3})}x_2$$

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{\sqrt{2}}{(1+\sqrt{3})} \\ 1 \end{bmatrix}$$

$$\|\mathbf{x}_1\| = \sqrt{\left(-\frac{\sqrt{2}}{(1+\sqrt{3})}\right)^2 + 1^2} = \sqrt{\frac{2+(4+2\sqrt{3})}{(1+\sqrt{3})^2}} = \frac{\sqrt{2(3+\sqrt{3})}}{(1+\sqrt{3})}$$

$$\mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{(1+\sqrt{3})}{\sqrt{2(3+\sqrt{3})}} \begin{bmatrix} -\frac{\sqrt{2}}{(1+\sqrt{3})} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3+\sqrt{3}}}\frac{\sqrt{3-\sqrt{3}}}{\sqrt{3-\sqrt{3}}} \\ \frac{1+\sqrt{3}}{\sqrt{2(3+\sqrt{3})}}\frac{\sqrt{3-\sqrt{3}}}{\sqrt{3-\sqrt{3}}} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{\sqrt{3-\sqrt{3}}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}} \\ \frac{\sqrt{2(3+\sqrt{3})}}{\sqrt{12}}\frac{\sqrt{6}}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{6(3-\sqrt{3})}}{6} \\ \frac{\sqrt{6(3+\sqrt{3})}}{6} \end{bmatrix}$$

$\underline{\lambda_2 = \frac{3+\sqrt{3}}{2}:}$

$$\mathbf{\Sigma}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \Rightarrow \begin{bmatrix} 2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3+\sqrt{3})/2x_1 \\ (3+\sqrt{3})/2x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1-\sqrt{3})/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & (-1-\sqrt{3})/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1-\sqrt{3})/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & (-1-\sqrt{3})/2 \end{bmatrix} \xrightarrow{\text{Row } 2-\left(\frac{2}{1-\sqrt{3}}\right)\frac{\sqrt{2}}{2}\text{Row } 1} \begin{bmatrix} (1-\sqrt{3})/2 & \\ 0 & 0 \end{bmatrix}$$

$$\frac{(1-\sqrt{3})}{2}x_1 = -\frac{\sqrt{2}}{2}x_2 \Rightarrow x_1 = -\frac{\sqrt{2}}{(1-\sqrt{3})}x_2$$

$$\mathbf{x}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{(1-\sqrt{3})} \\ 1 \end{bmatrix}$$

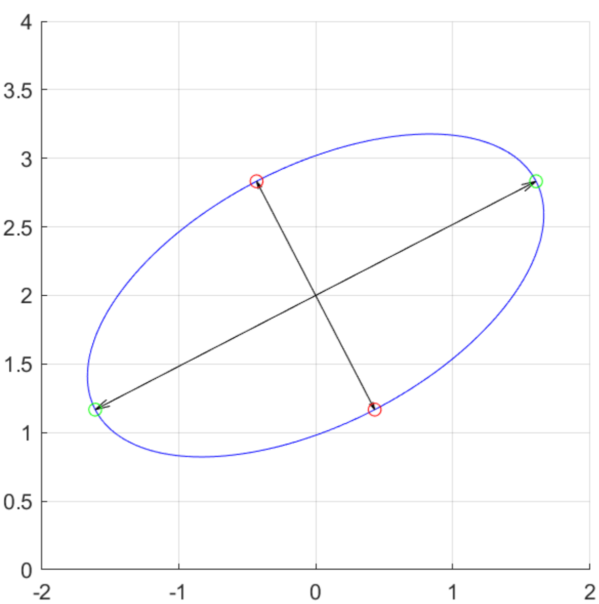
$$\|\mathbf{x}_2\| = \sqrt{\left(-\frac{\sqrt{2}}{(1-\sqrt{3})}\right)^2 + 1^2} = \sqrt{\frac{2+(4-2\sqrt{3})}{(1-\sqrt{3})^2}} = \frac{\sqrt{2(3-\sqrt{3})}}{(1-\sqrt{3})}$$

$$\mathbf{e}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{(1-\sqrt{3})}{\sqrt{2(3-\sqrt{3})}} \begin{bmatrix} -\frac{\sqrt{2}}{(1-\sqrt{3})} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3-\sqrt{3}}}\frac{\sqrt{3+\sqrt{3}}}{\sqrt{3+\sqrt{3}}} \\ \frac{1-\sqrt{3}}{\sqrt{2(3-\sqrt{3})}}\frac{\sqrt{3+\sqrt{3}}}{\sqrt{3-\sqrt{3}}} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{\sqrt{3+\sqrt{3}}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}} \\ -\frac{\sqrt{2(3-\sqrt{3})}}{\sqrt{12}}\frac{\sqrt{6}}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{6(3+\sqrt{3})}}{6} \\ -\frac{\sqrt{6(3-\sqrt{3})}}{6} \end{bmatrix}$$

(d) Determine (and sketch) the constant-density contour that contains 50% of the probability.

The $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution assigns probability 0.5 to the solid ellipsoid $\{\mathbf{x} : (\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \leq \chi^2(0.50)\} = F_2^{-1}(0.50)$, which happens at 1.3863, so $c = \sqrt{1.3863}$. The major axis has length $c\sqrt{\lambda_2} = \sqrt{1.3863 * \frac{3+\sqrt{3}}{2}} = 1.8111$ and the minor axis has length $c\sqrt{\lambda_1} = \sqrt{1.3863 * \frac{3-\sqrt{3}}{2}} = 0.9375$.



4.3

Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [-3, 1, 4]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) X_1 and X_2

$\text{Cov}(X_1, X_2) = -2 \neq 0$, so no, X_1 and X_2 are not $\perp\!\!\!\perp$.

(b) X_2 and X_3

$\text{Cov}(X_2, X_3) = 0$, so yes, X_2 and X_3 are $\perp\!\!\!\perp$.

(c) (X_1, X_2) and X_3

If we partition the matrix so that column 3 with rows 1 and 2 make up $\mathbf{\Sigma}_{12}$,

$$\mathbf{\Sigma} = \left[\begin{array}{rr:r} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}$$

$$\mathbf{\Sigma}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because $\mathbf{\Sigma}_{12} = \text{Cov}((X_1, X_2), X_2) = \mathbf{0}$, yes, (X_1, X_2) and X_2 are $\perp\!\!\!\perp$.

(d) $\frac{X_1+X_2}{2}$ and X_3

If we define $Y_1 = \frac{X_1+X_2}{2}$ and $Y_2 = X_3$ we could then setup $\mathbf{A}$ to be

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{AX} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{X_1+X_2}{2} \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\text{Cov}(\mathbf{AX}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}' = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

Because $\text{Cov}(Y_1, Y_2) = 0$, yes, $\frac{X_1+X_2}{2}$ and X_3 are $\perp\!\!\!\perp$. Another way would be to partition the matrix, like in chapter 3

$$\mathbf{\Sigma} = \left[\begin{array}{rr:r} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right] = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}$$

Then apply a linear combination to X_1 and X_2, $Y = \mathbf{b}'\mathbf{X}^\star = \frac{X_1+X_2}{2}$, where $\mathbf{X}^\star = [X_1, X_2]'$ then

$$\mathbf{\Sigma}_{11} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\text{Cov}(Y, X_3) = \text{Cov}(\mathbf{b}'\mathbf{X}^\star, X_3) = \mathbf{b}'\text{Cov}(\mathbf{X}^\star, X_3) = \mathbf{b}'\mathbf{\Sigma}_{12} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Because $\text{Cov}(Y, X_3) = 0$, yes, $Y = \frac{X_1+X_2}{2}$ and X_3 are $\perp\!\!\!\perp$.

(e) X_2 and $X_2 - \frac{5}{2}X_1 - X_3$

If we define $Y_1 = X_2 - \frac{5}{2}X_1 - X_3$ and $Y_2 = X_2$ we could then setup $\mathbf{A}$ to be

$$\mathbf{A} = \begin{bmatrix} -\frac{5}{2} & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{AX} = \begin{bmatrix} -\frac{5}{2} & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_2 - \frac{5}{2}X_1 - X_3 \\ X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\text{Cov}(\mathbf{AX}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}' = \begin{bmatrix} -\frac{5}{2} & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{5}{2} & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{9}{2} & 10 & -2 \\ -2 & 5 & 0 \end{bmatrix} \begin{bmatrix} -\frac{5}{2} & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 93/4 & 10 \\ 10 & 5 \end{bmatrix}$$

Because $\text{Cov}(Y_1, Y_2) = 10$, no, X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not $\perp\!\!\!\perp$.

4.4

Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \mathbf{\Sigma})$ with $\boldsymbol{\mu} = [2, -3, 1]$ and

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Find the distribution of $3X_1 - 2X_2 + X_3$.

$$\mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \qquad \mathbf{b}'\mathbf{X} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 3X_1 - 2X_2 + X_3$$

$$E[\mathbf{b}'\mathbf{X}] = \mathbf{b}'E[\mathbf{X}] = \mathbf{b}'\boldsymbol{\mu} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 13$$

$$\text{Cov}(\mathbf{b}'\mathbf{X}) = \mathbf{b}'\text{Cov}(\mathbf{X})\mathbf{b} = \mathbf{b}'\mathbf{\Sigma}\mathbf{b} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 9$$

Now we have that

$$\mathbf{b}'\mathbf{X} \sim N(13, 9)$$

(b) Relabel the variables if necessary, and find a 2×1 vector $\mathbf{a}$ such that X_2 and $X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ are independent.

I don't see X_3 in $X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, so if we first partition $\mathbf{\Sigma}$ and focus on $\mathbf{\Sigma}_{11}$, that way we're working with only X_1 and X_2, also defining $\mathbf{X}^\star = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

$$\mathbf{\Sigma} = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 3 & 2 \\ \hdashline 1 & 2 & 2 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \hdashline \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{array}\right]$$

$$X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_2 - a_1X_1 - a_2X_2 = -a_1X_1 + (1-a_2)X_2 = \begin{bmatrix} -a_1 & (1-a_2) \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

We're interested in the covariance of X_2 with this, so we can setup $\mathbf{A}$ to be

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & (1-a_2) \end{bmatrix}$$

$$\text{Cov}(\mathbf{A}\mathbf{X}^\star) = \mathbf{A}\text{Cov}(\mathbf{X}^\star)\mathbf{A}' = \mathbf{A}\mathbf{\Sigma}_{11}\mathbf{A}' =$$

$$= \begin{bmatrix} 0 & 1 \\ -a_1 & (1-a_2) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & -a_1 \\ 1 & (1-a_2) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3 \\ -a_1 & (3-a_1-3a_2) \end{bmatrix}\begin{bmatrix} 0 & -a_1 \\ 1 & (1-a_2) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & (3-a_1-3a_2) \\ (3-a_1-3a_2) & (a_1^2 + (1-a_2)(3-a_1-3a_2)) \end{bmatrix}$$

We want to know when X_2 and $X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ are independent. This is when the off-diagonal elements of this covariance matrix are zero, that is, when

$$3 - a_1 - 3a_2 = 0 \Rightarrow a_1 + 3a_2 = 3$$

If we pick $a_1 = 1$, then $a_2 = \frac{2}{3}$ and $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$. Now to check that this is correct.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & (1-\frac{2}{3}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{3} \end{bmatrix}$$

$$\text{Cov}(\mathbf{A}\mathbf{X}^\star) = \mathbf{A}\text{Cov}(\mathbf{X}^\star)\mathbf{A}' = \mathbf{A}\mathbf{\Sigma}_{11}\mathbf{A}' =$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -\frac{2}{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

The off-diagonal values are zero, so $\text{Cov}(X_2, X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}) = 0$ and are $\perp\!\!\!\perp$.

4.5

Specify the following.

(a) The conditional distribution of X_1, given that $X_2 = x_2$ for the joint distribution in Exercise 4.2.

$$\boldsymbol{\mu} = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} = \begin{bmatrix}0\\ 2\end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}\sigma_{11} & \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}\\ \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}2 & \sqrt{2}/2\\ \sqrt{2}/2 & 1\end{bmatrix}$$

Using **Result 4.6** on page 160, the conditional distribution is normally distributed with

$$\text{Mean} = \mu_1 + \sigma_{12}\sigma_{22}^{-1}(x_2 - \mu_2) = 0 + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{1}\right)(x_2 - 2) = \frac{\sqrt{2}}{2}(x_2 - 2)$$

$$\text{Covariance} = \sigma_{11} - \sigma_{12}\sigma_{22}^{-1}\sigma_{21} = 2 - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{1}\right)\left(\frac{\sqrt{2}}{2}\right) = 2 - \frac{2}{4} = \frac{3}{2}$$

$$X_1 \Big| X_2 = x_2 \sim N\left(\frac{\sqrt{2}}{2}(x_2 - 2), \frac{3}{2}\right)$$

(b) The conditional distribution of X_2, given that $X_1 = x_1$ and $X_3 = x_3$ for the joint distribution in Exercise 4.3.

$$\boldsymbol{\mu} = \begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix} = \begin{bmatrix}-3\\ 1\\ 4\end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix}1 & -2 & 0\\ -2 & 5 & 0\\ 0 & 0 & 2\end{bmatrix}$$

This time we need to rearrange the rows and columns so that X_2 is the first row and the first column. Then, from **Result 4.4** on page 158, the subsets of a normal are also normal, and again, using **Result 4.6** to find the mean and variance of the normal for the conditional distribution.

$$\boldsymbol{\mu} = \begin{bmatrix}\mu_2\\ \hdashline \mu_1\\ \mu_3\end{bmatrix} = \begin{bmatrix}1\\ \hdashline -3\\ 4\end{bmatrix} = \begin{bmatrix}\boldsymbol{\mu}_1\\ \hdashline \boldsymbol{\mu}_2\end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix}1 & -2 & 0\\ -2 & 5 & 0\\ 0 & 0 & 2\end{bmatrix} \xrightarrow[\text{Swap column 1 and column 2}]{\text{Swap row 1 and row 2}} \left[\begin{array}{c:cc}5 & -2 & 0\\ \hdashline -2 & 1 & 0\\ 0 & 0 & 2\end{array}\right] = \left[\begin{array}{c:c}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{array}\right]$$

The conditional distribution is normally distributed with

$$\text{Mean} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) = 1 + \begin{bmatrix}-2 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\left(\begin{bmatrix}x_1\\ x_3\end{bmatrix} - \begin{bmatrix}1\\ 4\end{bmatrix}\right) =$$

$$= 1 + \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} (x_1 - 1) \\ (x_3 - 4) \end{bmatrix} = 1 - 2\,(x_1 - 1) = 3 - 2x_1$$

$$\text{Covariance} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} = 5 - \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} =$$

$$= 5 - \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 5 - 4 = 1$$

$$X_2 \Big| X_1 = x_1, X_3 = x_3 \sim N\Big((3 - 2x_1), 1\Big)$$

(c) The conditional distribution of X_3, given that $X_1 = x_1$ and $X_2 = x_2$ for the joint distribution in Exercise 4.4.

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \qquad \mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

This time, again, we need to rearrange the rows and columns of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$. For this conditional distribution we need to partition so that X_3 is the first row and the first column. From **Result 4.4**, the subsets of the normal are also normal, and using **Result 4.6** to get the conditional distribution.

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_3 \\ \hdashline \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \hdashline 2 \\ -3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \xrightarrow[\text{Move column 3 to column 1}]{\text{Move row 3 and row 1}} \begin{bmatrix} \sigma_{33} & \sigma_{31} & \sigma_{32} \\ \sigma_{13} & \sigma_{11} & \sigma_{12} \\ \sigma_{23} & \sigma_{21} & \sigma_{22} \end{bmatrix} =$$

$$= \left[\begin{array}{c:cc} 2 & 1 & 2 \\ \hdashline 1 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \hdashline \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{array}\right]$$

The conditional distribution is normally distributed with

$$\text{Mean} = \boldsymbol{\mu}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) = 1 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) =$$

$$= 1 + \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} (x_1 - 2) \\ (x_2 + 3) \end{bmatrix} = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{3}{2}$$

$$\text{Covariance} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} = 2 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$= 2 - \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$X_3 \Big| X_1 = x_1, X_2 = x_2 \sim N\left(\frac{1}{2}(x_1 + x_2 + 3), \frac{1}{2}\right)$$

4.6

Let $\mathbf{X}$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}' = [1, -1, 2]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) X_1 and X_2

Picking out the value of $\text{Cov}(X_1, X_2) = \sigma_{12} = \sigma_{21}$ from $\boldsymbol{\Sigma}$, we can see that $\text{Cov}(X_1, X_2) = 0$, so by part **(a)** of **Result 4.5** on page 159 X_1 and X_2 are $\perp\!\!\!\perp$.

(b) X_1 and X_3

Picking out the value of $\text{Cov}(X_1, X_3) = \sigma_{13} = \sigma_{31}$ from $\boldsymbol{\Sigma}$, we can see that $\text{Cov}(X_1, X_3) = -1$, so by part **(a)** of **Result 4.5** on page 159 X_1 and X_3 are not $\perp\!\!\!\perp$.

(c) X_2 and X_3

Picking out the value of $\text{Cov}(X_2, X_3) = \sigma_{23} = \sigma_{32}$ from $\boldsymbol{\Sigma}$, we can see that $\text{Cov}(X_2, X_3) = 0$, so by part **(a)** of **Result 4.5** on page 159 X_2 and X_3 are $\perp\!\!\!\perp$.

(d) (X_1, X_3) and X_2

This time partition $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ so that X_2 is the last row of $\boldsymbol{\mu}$ and the last row/column of $\boldsymbol{\Sigma}$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_3 \\ \hdashline \mu_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \hdashline -1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \xrightarrow[\text{Swap row 2 and row 3}]{\text{Swap column 2 and column 3}} \left[\begin{array}{cc:c} \sigma_{11} & \sigma_{13} & \sigma_{12} \\ \sigma_{31} & \sigma_{33} & \sigma_{32} \\ \hdashline \sigma_{21} & \sigma_{23} & \sigma_{22} \end{array}\right] =$$

$$= \left[\begin{array}{cc:c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hdashline 0 & 0 & 5 \end{array}\right] = \left[\begin{array}{c:c} \boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} \\ \hdashline \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}} \end{array}\right]$$

Here, $\mathbf{X}_1 = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ and $\mathbf{X}_2 = \begin{bmatrix} X_2 \end{bmatrix}$. Picking out the value of $\text{Cov}(\mathbf{X}_1, \mathbf{X}_2) = \boldsymbol{\Sigma}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$ from the partitioned $\boldsymbol{\Sigma}$, we can see that $\boldsymbol{\Sigma}_{12} = \mathbf{0}$, so by part **(b)** of **Result 4.5** on page 159 (X_1, X_3) and X_2 are $\perp\!\!\!\perp$.

(e) X_1 and $X_1 + 3X_2 - 2X_3$

For this, we have $q = 2$ linear combinations, so

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

From **Result 4.3** on page 157, this would be distributed as $N_q(, (\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$. Computing $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$ we have

$$\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ 6 & 15 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 6 \\ 6 & 61 \end{bmatrix}$$

This is the variance-covariance matrix of X_1 and $X_1+3X_2-2X_3$. Picking out the off-diagonal term, $\text{Cov}(X_1, X_1 + 3X_2 - 2X_3) = 6$. This value is not zero, so by **Result 4.5** part **(a)**, X_1 and $X_1 + 3X_2 - 2X_3$ are not $\perp\!\!\!\perp$.

4.7

Refer to Exercise 4.6 and specify each of the following.

(a) The conditional distribution of X_1, given that $X_3 = x_3$.

We only care about X_1 and X_3, so to remove X_2 we could first apply the linear combinations

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so by **Result 4.3**, this would be distributed as $N_2(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$, where

$$\mathbf{A}\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix}$$

From here we can use **Result 4.6** to get the conditional distribution

$$\text{Mean} = \mu_1 + \sigma_{13}\sigma_{33}^{-1}(x_3 - \mu_3) = 1 + (-1)(1/2)(x_3 - 2) = 2 - \frac{1}{2}x_3$$

$$\text{Covariance} = \sigma_{11} + \sigma_{13}\sigma_{33}^{-1}\sigma_{31} = 4 - (-1)^2(1/2) - 4 - 1/2 = \frac{7}{2}$$

and putting it together we have

$$X_1 \Big| X_3 = x_3 \sim N\left(2 - \frac{1}{2}x_3, \frac{7}{2}\right)$$

(b) The conditional distribution of X_1, given that $X_2 = x_2$ and $X_3 = x_3$.

This time partition things so that one group contains X_1 and another group contains X_2 and X_3.

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \left[\begin{array}{r} 1 \\ \hline -1 \\ 2 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{\mu}_1 \\ \hline \boldsymbol{\mu}_2 \end{array}\right]$$

$$\boldsymbol{\mu} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \left[\begin{array}{r|rr} 4 & 0 & -1 \\ \hline 0 & 5 & 0 \\ -1 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]$$

Again, apply **Result 4.6**,

$$\text{Mean} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) = 1 + \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 2/10 & 0 \\ 0 & 5/10 \end{bmatrix} \left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) =$$

$$= 1 + \begin{bmatrix} 0 & -1/2 \end{bmatrix} \begin{bmatrix} x_2 + 1 \\ x_3 - 2 \end{bmatrix} = 1 - \frac{1}{2}(x_3 - 2) = 2 - \frac{1}{2}x_3$$

$$\text{Covariance} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{21} = 4 - \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 2/10 & 0 \\ 0 & 5/10 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} =$$

$$= 4 - \begin{bmatrix} 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 4 - \frac{1}{2} = \frac{7}{2}$$

Putting this together we have

$$X_1 \Big| X_2, X_3 \sim \left(2 - \frac{1}{2}x_3, \frac{7}{2}\right)$$

4.8

(Example of a nonnormal bivariate distribution with normal marginals.) Let X_1 be $N(0,1)$ and let

$$X_2 = \begin{cases} -X_1 & \text{if } -1 \le X_1 \le 1 \\ \ \ X_1 & \text{otherwise} \end{cases}$$

Show each of the following.

(a) X_2 also has an $N(0,1)$ distribution.

The hint is basically the answer. Using symmetry we can start with

$$P(-1 < X_1 \le x) = P(-x \le X_1 < 1)$$

Using the definition of the CDF for X_2

$$\begin{aligned}
F_{X_2}(x_2) &= \\
= P(X_2 \leq x_2) &= \\
= P(X_2 \leq -1) + P(-1 < X_2 \leq x_2) &= \\
= P(X_1 \leq -1) + P(-1 < -X_1 \leq x_2) &= && \text{Using the definition of } X_2. \\
= P(X_1 \leq -1) + P(-x_2 \leq X_1 < 1) &= \\
= P(X_1 \leq -1) + P(-1 < -X_1 \leq x_2) &= && \text{Using symmetry argument.} \\
P(X_1 \leq x_2) &= && \text{CDF definition for } X_1. \\
F_{X_1}(x_2)
\end{aligned}$$

The CDF for X_1 and X_2 are the same, so since $X_1 \sim N(0,1)$, then X_2 is also distributed as $N(0,1)$.

(b) X_1 and X_2 do *not* have a bivariate normal distribution.

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad Y = \mathbf{a}'\mathbf{X} = X_1 - X_2$$

$$\mathbf{a}'\mathbf{X} = X_1 - X_2 = \begin{cases} X_1 - (-X_1) & \text{if } -1 \leq X_1 \leq 1 \\ X_1 - X_1 & \text{otherwise} \end{cases} =$$

$$= \begin{cases} 2X_1 & \text{if } -1 \leq X_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If X_1 is in the interval $-1 \leq X_1 \leq 1$ then it's distribution as $Y = (X_1 - X_2) \sim N(0,4)$, since $E[\mathbf{a}'\mathbf{X}] = E[2X_1] = 2E[X_1] = 2\mu_1 = 2*0 = 0$ and $V[\mathbf{a}'\mathbf{X}] = V[2X_1] = 4V[X_1] = 4\sigma_1^2 = 4*1 = 4$. But if X_1 is not in the interval $-1 \leq X_1 \leq 1$, then $X_1 - X_2 = 0$. This happens with point probability

$$P(\mathbf{a}'\mathbf{X}) = P(X_1 - X_2) = P(0) = P(X_1 < -1 \text{ and } X_1 > 1) =$$

$$= P(X_1 < -1) + P(X_1 > 1) = 2P(X_1 > 1) = P(|X_1| > 1) = 0.3173105$$

So since $P(0) \neq 0$, X_1 and X_2 don't have a bivariate normal. Basically, a Normal distribution is either entirely continuous or entirely discrete, so here we have a discrete point mass at 0 that doesn't have a probability of zero and a continuous $N(0,4)$ distribution from -2 to 2, so this is not consistent with a normal distribution.

The R code for this probability calculation is `2*pnorm(1, lower.tail=FALSE)`. Could have also done the probability calculation as

$$P(|X_1| > 1) = 1 - P(|X_1| \leq 1) = 1 - P(-1 \leq X_1 \leq 1) =$$

$$= 1 - (\Phi(1) - \Phi(-1)) = 1 - 0.6826895 = 0.3173105$$

Where Φ is the CDF of a standard normal distribution. The R code is `1 - (pnorm(1)- pnorm(-1))`.
Hint:

(a) Since X_1 is $N(0,1)$, $P[-1 < X_1 \leq x] = P[-x \leq X_1 < 1]$ for any x. When $-1 < x_2 < 1$, $P[X_2 \leq x_2] = P[X_2 \leq -1]+P[-1 < X_2 \leq x_2] = P[X_1 \leq -1]+P[-1 < -X_1 \leq x_2] = P[X_1 \leq -1]+P[-x_2 \leq X_1 < 1]$. But $P[-x_2 \leq X_1 < 1] = P[-1 < X_1 \leq x_2]$ from the symmetry argument in the first line of this hint. Thus $P[X_2 \leq x_2] = P[X_1 \leq -1] + P[-1 < X_1 \leq x_2] = P[X_1 \leq x_2]$, which is a standard normal probability.

(b) Consider the linear combination $X_1 - X_2$, which equals zero with probability $P[|X_1| > 1] = 0.3174$.

4.9

Refer to Exercise 4.8, but modify the construction by replacing the break point 1 by c so that

$$X_2 = \begin{cases} -X_1 & \text{if } -c \leq X_1 \leq c \\ X_1 & \text{elsewhere} \end{cases}$$

Show that c can be chosen so that $\text{Cov}(X_1, X_2) = 0$, but that the two random variables are not independent.
Hint:
For $c = 0$, evaluate $\text{Cov}(X_1, X_2) = E[X_1(X_1)]$
For c very large, evaluate $\text{Cov}(X_1, X_2) \doteq E[X_1(-X_1)]$

We already know X_1 and X_2 are not $\perp\!\!\!\perp$, since X_2 is a function of X_1, so we just need to show it's possible for c to be chosen so the covariance of X_1 and X_2 will be zero. Conveniently, the hint suggests trying out both $c = 0$ and c very big, these are two extreme values that c could take on. If the covariance for the two extreme c values form an interval that contains 0 we can use the interemediate value theorem to say the covariance of X_1 and X_2 will pass through zero for some c.
First off, the covariance is

$$\text{Cov}(X_1, X_2) = E\left[(X_1 - E[X_1])\,(X_2 - E[X_2])\right] =$$

$$= E\left[X_1X_2 - X_1E[X_2] - E[X_1]X_2 + E[X_1]E[X_2]\right] =$$

$$= E[X_1X_2] - E[X_1]E[X_2] - E[X_1]E[X_2] + E[X_1]E[X_2] =$$

$$= E[X_1X_2] - E[X_1]E[X_2]$$

When $c = 0$, the probability that $X_2 = -X_1$ is

$$P(-c \leq X_1 \leq c) = P(0 \leq X_1 \leq 0) = \Phi(0) - \Phi(0) = 0$$

and so from the definition of X_2, the probability of $X_2 = -X_1$ is zero, and the probability of $X_2 = X_1$ is 1. Now to compute the covariance for $c = 0$,

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2] = E[X_1(X_1)] - E[X_1]E[-X_1] =$$

$$= E[X_1^2] - (E[X_1])^2 = \mathrm{Var}[X_1] = 1$$

When c is very big, say ∞, the probability that $X_2 = -X_1$ is

$$P(-c \leq X_1 \leq c) = P(-\infty \leq X_1 \leq \infty) = \Phi(\infty) - \Phi(-\infty) = 1 - 0 = 1$$

and so from the definition of X_2, the probability of $X_2 = -X_1$ is 1, and the probability of $X_2 = X_1$ is 0. This is opposite what we got when c was 0. Computing the covariance,

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2] = E[X_1(-X_1)] - E[X_1]E[-X_1] =$$

$$= -\left(E[X_1^2] - (E[X_1])^2\right) = -\mathrm{Var}[X_1] = -1$$

We now have that the covariance for extreme values of c, 1 for $c = 0$ and -1 for c at ∞. Since the covariance is a smooth function of c, then by the intermediate value theorem, $\mathrm{Cov}(X_1, X_2) = 0$ for some value of c. We're done, we showed that there exists some value of c that causes the covariance to be zero when X_1 and X_2 are not independent.

4.10

Show the following

(a)

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}|\,|\mathbf{B}|$$

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \left| \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{bmatrix} \right| = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} =$$

$$= |\mathbf{AI} - \mathbf{00}'|\,|\mathbf{IB} - \mathbf{00}'| = |\mathbf{AI}|\,|\mathbf{IB}| = |\mathbf{A}|\,|\mathbf{B}|$$

(b)

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}|\,|\mathbf{B}| \quad \text{for} \quad |\mathbf{A}| \neq 0$$

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \left| \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \right| = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} =$$

$$= |\mathbf{A}|\,|\mathbf{B}|\,\left|\mathbf{II} - \mathbf{A}^{-1}\mathbf{C0}'\right| = |\mathbf{A}|\,|\mathbf{B}|\,|\mathbf{I}| = |\mathbf{A}|\,|\mathbf{B}|$$

Hint:
(a) $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$. Expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$ by the first row (see definition 2A.24) gives 1 times a determinant of the same form, with the order of $\mathbf{I}$ reduced by one. This procedure is repeated until $1 \times |\mathbf{B}|$ is obtained. Similarly, expanding the determinant $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}|$.
(b) $\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$. Expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = 1$. Now use the results in Part a.

4.11

Show that, if $\mathbf{A}$ is square,

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{A}_{22}| \left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| \quad \text{for} \quad |\mathbf{A}_{22}| \neq 0 \\ &= |\mathbf{A}_{11}| \left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right| \quad \text{for} \quad |\mathbf{A}_{11}| \neq 0 \end{aligned}$$

First, using the hint

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{A}_{12} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{22} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{A}_{12} - \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{A}_{21} - \mathbf{A}_{22}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{A}_{21} - \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} \end{bmatrix} =$$

Now, for the determinant

$$\left|\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}\right| = \left|\begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} \end{bmatrix}\right|$$

$$\Rightarrow \begin{vmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{vmatrix} = \begin{vmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} \end{vmatrix}$$

$$\Rightarrow \left|\mathbf{II} - -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{0}'\right| |\mathbf{A}| \left|\mathbf{II} - -\mathbf{0}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| = \left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| |\mathbf{A}_{22}|$$

$$\Rightarrow (1)\,|\mathbf{A}|\,(1) = |\mathbf{A}_{22}| \left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right|$$

$$\Rightarrow |\mathbf{A}| = |\mathbf{A}_{22}| \left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right|$$

Verifying the second part of the hint

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{11} & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} - \mathbf{A}_{21} & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} - \mathbf{A}_{11}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} - \mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} =$$

$$\left| \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \right| = \left| \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} \begin{vmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = \left| \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix} \right|$$

$$\Rightarrow \left|\mathbf{II} - -\mathbf{0}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\right| |\mathbf{A}| \left|\mathbf{II} - -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{0}'\right| = |\mathbf{A}_{11}| \left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right|$$

$$\Rightarrow (1)\,|\mathbf{A}|\,(1) = |\mathbf{A}_{11}| \left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right|$$

$$\Rightarrow |\mathbf{A}| = |\mathbf{A}_{11}| \left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right|$$

Finally, we have what we want, that

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{A}_{22}| \left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| \quad \text{for} \quad |\mathbf{A}_{22}| \neq 0 \\ &= |\mathbf{A}_{11}| \left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right| \quad \text{for} \quad |\mathbf{A}_{11}| \neq 0 \end{aligned}$$

Hint: Partition $\mathbf{A}$ and verify that

$$\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}\end{bmatrix}$$

Take the determinants on both sides of this inequality. Use Exercise 4.10 for the first and third determinants on the left and for the determinant on the right. The second inequality for $|\mathbf{A}|$ follows by considering

$$\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\end{bmatrix}$$

4.12

Show that, for $\mathbf{A}$ symmetric,

$$\mathbf{A}^{-1}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\left(\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}$$

Thus, $\left(\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1}$ is the upper left-hand block of $\mathbf{A}^{-1}$.
From Exercise 4.11 we know

$$\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}\end{bmatrix}$$

$$\begin{aligned}\Rightarrow &\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\\ &\times\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}^{-1}=\\ &\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}^{-1}\end{aligned}$$

$$\Rightarrow \mathbf{A}=\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}^{-1}$$

$$\Rightarrow \mathbf{A}^{-1}=\left(\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}^{-1}\right)^{-1}$$

$$\Rightarrow \mathbf{A}^{-1}=\begin{bmatrix}\mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\\ \mathbf{0}' & \mathbf{I}\end{bmatrix}\left(\begin{bmatrix}\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0}\\ \mathbf{0}' & \mathbf{A}_{22}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I}\end{bmatrix}$$

$$\Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \left(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}$$

Hint: Premultiply the expression in the hint to Exercise 4.11 by $\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}$ and postmultiply by $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}$. Take inverses of the resulting expression.

4.13

Show the following if $\mathbf{X}$ is $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| \neq 0$.

(a) Check that $|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{22}| \left|\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right|$. (Note that $|\boldsymbol{\Sigma}|$ can be factored into the product of contributions from the marginal and conditional distributions.)

This follows directly fom Exercise 4.11 since $\boldsymbol{\Sigma}$ is square.

$$\boldsymbol{\Sigma} = \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]$$

$$|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{22}| \left|\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right|$$

(b) Check that

$$\begin{aligned}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = {} & [\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)]' \\ & \times \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1}[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{\mathbf{12}}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)] \\ & + (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)\end{aligned}$$

(Thus the joint density exponent can be written as the sum of two terms corresponding to contributions from the conditional and marginal distributions.)

$$\begin{aligned}&(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \\ &\quad \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}' \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \\ &\qquad \times \begin{bmatrix} \mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix} =\end{aligned}$$

$$\begin{aligned}&= \left[(\mathbf{x}_1 - \boldsymbol{\mu}_1)' - (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}, \quad (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\right] \\ &\qquad \times \begin{bmatrix} \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \\ &\qquad \times \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix} =\end{aligned}$$

$$= \left[\left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right]', \quad (\mathbf{x}_2 - \boldsymbol{\mu}_2)' \right]$$
$$\times \begin{bmatrix} \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix}$$
$$\times \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix} =$$

$$= \left[\left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right]', \quad (\mathbf{x}_2 - \boldsymbol{\mu}_2)' \right]$$
$$\times \begin{bmatrix} \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1} \left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right] \\ \boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \end{bmatrix} =$$

$$= \left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right]'$$
$$\times \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1}$$
$$\times \left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right]$$
$$+ (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$$

(c) Given the results in Parts a and b, identify the marginal distribution of $\mathbf{X}_2$ and the conditional distribution of $\mathbf{X}_1 \big| \mathbf{X}_2 = \mathbf{x}_2$.

From part (a)
$$|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{22}| \left| \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} \right|$$
The value $\boldsymbol{\Sigma}_{22}$ is the covariance of the marginal (normal) distribution of $\mathbf{X}_2$. The determinant, $|\boldsymbol{\Sigma}_{22}|$, is the scaling factor found in the first part of the normal density equation for the normal marginal distribution of $\mathbf{X}_2$.

The value of $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$ is the covariance of the normal conditional distribution of $\mathbf{X}_1 \big| \mathbf{X}_2 = \mathbf{x}_2$. The determinant, $\left| \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} \right|$, is the scaling factor found in the first part of the normal density equation for the normal conditional distribution of $\mathbf{X}_1 \big| \mathbf{X}_2 = \mathbf{x}_2$.

From part (b)

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) = \left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right]'$$
$$\times \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)^{-1} \left[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{\mathbf{12}}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right]$$
$$+ (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$$

The left and right parts of the first half of the equation can be written as

$$\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) = \mathbf{x}_1 - \left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \right).$$

From there, we can see that,

$$[\mathbf{x}_1 - (\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2))]'(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1} \times [\mathbf{x}_1 - (\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{\mathbf{12}}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2))]$$

, is the exponent for the normal conditional distribution of $\mathbf{X}_1 \big| \mathbf{X}_2 = x_2$, where $\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$ is the mean from page 160 and $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$ is the covariance from page 161.
The second part of the addition, $(\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$, is the exponent of the normal marginal for $\mathbf{X}_2$, where the mean is $\boldsymbol{\mu}_2$ and the covariance is $\boldsymbol{\Sigma}_{22}$.

Hint:
(a) Apply Exercise 4.11
(b) Note from Exercise 4.12 that we can write $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as

$$\begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}' \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}$$

If we group the product so that

$$\begin{bmatrix} \mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}$$

the result follows.

4.14

If $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| \neq 0$, show that the joint density can be written as the product of marginal densities for

$$\underset{(q\times 1)}{\mathbf{X}_1} \quad \text{and} \quad \underset{((p-q)\times 1)}{\mathbf{X}_2} \quad \text{if} \quad \boldsymbol{\Sigma}_{12} = \underset{(q\times(p-q))}{\mathbf{0}}$$

First, parition $\mathbf{X}$ and $\boldsymbol{\Sigma}$,

$$\underset{p\times 1}{\mathbf{X}} = \begin{bmatrix} \underset{(q\times 1)}{\mathbf{X}_1} \\ \hdashline \underset{((p-q)\times 1)}{\mathbf{X}_2} \end{bmatrix}$$

$$\underset{p\times 1}{\boldsymbol{\mu}} = \begin{bmatrix} \underset{(q\times 1)}{\boldsymbol{\mu}_1} \\ \hdashline \underset{((p-q)\times 1)}{\boldsymbol{\mu}_2} \end{bmatrix}$$

$$\underset{p\times p}{\boldsymbol{\Sigma}} = \left[\begin{array}{c:c} \underset{(q\times q)}{\boldsymbol{\Sigma}_{11}} & \underset{(q\times(p-q))}{\boldsymbol{\Sigma}_{12}} \\ \hdashline \underset{((p-q)\times q)}{\boldsymbol{\Sigma}_{21}} & \underset{((p-q)\times(p-q))}{\boldsymbol{\Sigma}_{22}} \end{array}\right] = \left[\begin{array}{c:c} \underset{(q\times q)}{\boldsymbol{\Sigma}_{11}} & \underset{(q\times(p-q))}{\mathbf{0}} \\ \hdashline \underset{((p-q)\times q)}{\mathbf{0}'} & \underset{((p-q)\times(p-q))}{\boldsymbol{\Sigma}_{22}} \end{array}\right]$$

We can pick $\boldsymbol{\Sigma}^{-1}$ and use the definition of an inverse on page 58 to show it is the inverse of $\boldsymbol{\Sigma}$, (if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{B}$ is an inverse of $\mathbf{A}$.) So using the hint to pick $\boldsymbol{\Sigma}^{-1}$

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix}$$

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} = \mathbf{I}$$

$$\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}\boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} = \mathbf{I}$$

Because $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = \mathbf{I}$, our pick of $\boldsymbol{\Sigma}^{-1}$ is the inverse of $\boldsymbol{\Sigma}$.
Using the second part of the hint

$$\begin{aligned}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) &= [(\mathbf{x}_1-\boldsymbol{\mu}_1)', (\mathbf{x}_2-\boldsymbol{\mu}_2)']\begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix} \\ &= (\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1) + (\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\end{aligned}$$

Using the third part of the hint

$$|\boldsymbol{\Sigma}| = \begin{vmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22} \end{vmatrix} = |\boldsymbol{\Sigma}_{11}|\,|\boldsymbol{\Sigma}_{22}| \text{ (from Exercise 4.10 (a))}$$

Now, using the formula for the normal density (4–4)

$$\begin{aligned} f(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) &= \frac{1}{(1/2)^{p/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \\ &= \frac{1}{(1/2)^{q/2}(1/2)^{(p-q)/2}(|\boldsymbol{\Sigma}_{11}|\,|\boldsymbol{\Sigma}_{22}|)^{1/2}} \\ &\times \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1) + (\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\right\} \\ &= \left(\frac{1}{(1/2)^{q/2}|\boldsymbol{\Sigma}_{11}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\}\right) \\ &\times \left(\frac{1}{(1/2)^{(p-q)/2}|\boldsymbol{\Sigma}_{22}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\right\}\right) \\ &= f(\mathbf{x}_1|\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)f(\mathbf{x}_2|\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2)\end{aligned}$$

so we've shown that the joint density can be written as the product of marginal densities

$$f(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = f(\mathbf{x}_1|\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)f(\mathbf{x}_2|\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_2)$$

Hint: Show by block multiplication that

$$\begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \text{ is the inverse of } \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

Then write

$$\begin{aligned}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) &= [(\mathbf{x}_1-\boldsymbol{\mu}_1)', (\mathbf{x}_2-\boldsymbol{\mu}_2)'] \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix} \\ &= (\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1) + (\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\end{aligned}$$

Note that $|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{11}|\,|\boldsymbol{\Sigma}_{22}|$ from Exercise 4.10(a). Now factor the joint density.

4.15

Show that $\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})'$ and $\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})'$ are both $p \times p$ matrices of zeros. Here $\mathbf{x}_j' = [x_{j1}, x_{j2}, \ldots, x_{jp}]$, $j = 1, 2, \ldots, n$ and

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j$$

Here are both a short answer (using distribution and vector-vector multiplication) and long ugly answer (multiply matrix out).
<u>Short answer:</u>

$$\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}'-\boldsymbol{\mu}') =$$

$$= \sum_{j=1}^{n}\left(\underset{(p\times1)}{\mathbf{x}_j}\underset{(1\times p)}{\bar{\mathbf{x}}'} - \underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\bar{\mathbf{x}}'} - \underset{(p\times1)}{\mathbf{x}_j}\underset{(1\times p)}{\boldsymbol{\mu}'} + \underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\boldsymbol{\mu}'}\right) =$$

$$= \sum_{j=1}^{n}\left(\underset{(p\times1)}{\mathbf{x}_j}\underset{(1\times p)}{\bar{\mathbf{x}}'}\right) - n\underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\bar{\mathbf{x}}'} - \sum_{j=1}^{n}\left(\underset{(p\times1)}{\mathbf{x}_j}\underset{(1\times p)}{\boldsymbol{\mu}'}\right) + n\underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\boldsymbol{\mu}'} =$$

$$= n\underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\bar{\mathbf{x}}'} - n\underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\bar{\mathbf{x}}'} - n\underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\boldsymbol{\mu}'} + n\underset{(p\times1)}{\bar{\mathbf{x}}}\underset{(1\times p)}{\boldsymbol{\mu}'} =$$

$$= \underset{p\times p}{\mathbf{0}}$$

and

$$\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})' =$$

$$= \sum_{j=1}^{n} (\bar{\mathbf{x}} - \boldsymbol{\mu})(\mathbf{x}_j' - \bar{\mathbf{x}}') =$$

$$= \sum_{i=1}^{n} \left(\underset{(p\times 1)}{\bar{\mathbf{x}}} \underset{(1\times p)}{\mathbf{x}_j'} - \underset{(p\times 1)}{\boldsymbol{\mu}} \underset{(1\times p)}{\mathbf{x}_j'} - \underset{(p\times 1)}{\bar{\mathbf{x}}} \underset{(1\times p)}{\bar{\mathbf{x}}'} + \underset{(p\times 1)}{\boldsymbol{\mu}} \underset{(1\times p)}{\bar{\mathbf{x}}'} \right) =$$

$$= \sum_{i=1}^{n} \left(\underset{(p\times 1)}{\bar{\mathbf{x}}} \underset{(1\times p)}{\mathbf{x}_j'} \right) - \sum_{i=1}^{n} \left(\underset{(p\times 1)}{\boldsymbol{\mu}} \underset{(1\times p)}{\mathbf{x}_j'} \right) - n \underset{(p\times 1)}{\bar{\mathbf{x}}} \underset{(1\times p)}{\bar{\mathbf{x}}'} + n \underset{(p\times 1)}{\boldsymbol{\mu}} \underset{(1\times p)}{\bar{\mathbf{x}}'} =$$

$$= n \underset{(p\times 1)}{\bar{\mathbf{x}}} \underset{(1\times p)}{\bar{\mathbf{x}}'} - n \underset{(p\times 1)}{\boldsymbol{\mu}} \underset{(1\times p)}{\bar{\mathbf{x}}'} - n \underset{(p\times 1)}{\bar{\mathbf{x}}} \underset{(1\times p)}{\bar{\mathbf{x}}'} + n \underset{(p\times 1)}{\boldsymbol{\mu}} \underset{(1\times p)}{\bar{\mathbf{x}}'} =$$

$$= \underset{p\times p}{\mathbf{0}}$$

Long answer:

$$\sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\bar{\mathbf{x}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n} \left(\begin{bmatrix} x_{j1} \\ \vdots \\ x_{jp} \end{bmatrix} - \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix} \right) \left(\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} \right)' =$$

$$= \sum_{j=1}^{n} \left(\begin{bmatrix} x_{j1} - \bar{x}_1 \\ \vdots \\ x_{jp} - \bar{x}_p \end{bmatrix} \right) \left(\begin{bmatrix} \bar{x}_1 - \mu_1 \\ \vdots \\ \bar{x}_p - \mu_p \end{bmatrix} \right)' =$$

$$= \sum_{j=1}^{n} \left(\begin{bmatrix} (x_{j1} - \bar{x}_1)(\bar{x}_1 - \mu_1) & \cdots & (x_{j1} - \bar{x}_1)(\bar{x}_p - \mu_p) \\ \vdots & \ddots & \vdots \\ (x_{jp} - \bar{x}_p)(\bar{x}_1 - \mu_1) & \cdots & (x_{jp} - \bar{x}_p)(\bar{x}_p - \mu_p) \end{bmatrix} \right) =$$

$$= \sum_{j=1}^{n} \left(\begin{bmatrix} (x_{j1}\bar{x}_1 - \bar{x}_1\bar{x}_1 - x_{j1}\mu_1 + \bar{x}_1\mu_1) & \cdots & (x_{j1}\bar{x}_p - \bar{x}_1\bar{x}_p - x_{j1}\mu_p + \bar{x}_1\mu_p) \\ \vdots & \ddots & \vdots \\ (x_{jp}\bar{x}_1 - \bar{x}_p\bar{x}_1 - x_{jp}\mu_1 + \bar{x}_p\mu_1) & \cdots & (x_{jp}\bar{x}_p - \bar{x}_p\bar{x}_p - x_{jp}\mu_p + \bar{x}_p\mu_p) \end{bmatrix} \right) =$$

$$= \begin{bmatrix} \sum_{j=1}^{n} (x_{j1}\bar{x}_1 - \bar{x}_1\bar{x}_1 - x_{j1}\mu_1 + \bar{x}_1\mu_1) & \cdots & \sum_{j=1}^{n} (x_{j1}\bar{x}_p - \bar{x}_1\bar{x}_p - x_{j1}\mu_p + \bar{x}_1\mu_p) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} (x_{jp}\bar{x}_1 - \bar{x}_p\bar{x}_1 - x_{jp}\mu_1 + \bar{x}_p\mu_1) & \cdots & \sum_{j=1}^{n} (x_{jp}\bar{x}_p - \bar{x}_p\bar{x}_p - x_{jp}\mu_p + \bar{x}_p\mu_p) \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{j=1}^{n} (x_{j1}\bar{x}_1) - n\bar{x}_1\bar{x}_1 - \sum_{j=1}^{n} (x_{j1}\mu_1) + n\bar{x}_1\mu_1 & \cdots & \sum_{j=1}^{n} (x_{j1}\bar{x}_p) - n\bar{x}_1\bar{x}_p - \sum_{j=1}^{n} (x_{j1}\mu_p) + n\bar{x}_1\mu_p \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} (x_{jp}\bar{x}_1) - n\bar{x}_p\bar{x}_1 - \sum_{j=1}^{n} (x_{jp}\mu_1) + n\bar{x}_p\mu_1 & \cdots & \sum_{j=1}^{n} (x_{jp}\bar{x}_p) - n\bar{x}_p\bar{x}_p - \sum_{j=1}^{n} (x_{jp}\mu_p) + n\bar{x}_p\mu_p \end{bmatrix} =$$

$$= \begin{bmatrix} n\bar{x}_1\bar{x}_1 - n\bar{x}_1\bar{x}_1 - n\bar{x}_1\mu_1 + n\bar{x}_1\mu_1 & \cdots & n\bar{x}_1\bar{x}_p - n\bar{x}_1\bar{x}_p - n\bar{x}_1\mu_p + n\bar{x}_1\mu_p \\ \vdots & \ddots & \vdots \\ n\bar{x}_p\bar{x}_1 - n\bar{x}_p\bar{x}_1 - n\bar{x}_p\mu_1 + n\bar{x}_p\mu_1 & \cdots & n\bar{x}_p\bar{x}_p - n\bar{x}_p\bar{x}_p - n\bar{x}_p\mu_p + n\bar{x}_p\mu_p \end{bmatrix} =$$

$$
= \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \underset{(p \times p)}{\mathbf{0}}
$$

and

$$
\sum_{j=1}^{n} (\bar{\mathbf{x}} - \boldsymbol{\mu})(\mathbf{x}_j - \bar{\mathbf{x}})' =
$$

$$
= \sum_{j=1}^{n} \left(\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} \right) \left(\begin{bmatrix} x_{j1} \\ \vdots \\ x_{jp} \end{bmatrix} - \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix} \right)' =
$$

$$
= \sum_{j=1}^{n} \left(\begin{bmatrix} \bar{x}_1 - \mu_1 \\ \vdots \\ \bar{x}_p - \mu_p \end{bmatrix} \right) \left(\begin{bmatrix} x_{j1} - \bar{x}_1 \\ \vdots \\ x_{jp} - \bar{x}_p \end{bmatrix} \right)' =
$$

$$
= \sum_{j=1}^{n} \left(\begin{bmatrix} (\bar{x}_1 - \mu_1)(x_{j1} - \bar{x}_1) & \cdots & (\bar{x}_1 - \mu_1)(x_{jp} - \bar{x}_p) \\ \vdots & \ddots & \vdots \\ (\bar{x}_p - \mu_p)(x_{j1} - \bar{x}_1) & \cdots & (\bar{x}_p - \mu_p)(x_{jp} - \bar{x}_p) \end{bmatrix} \right) =
$$

$$
= \sum_{j=1}^{n} \left(\begin{bmatrix} (\bar{x}_1 x_{j1} - \mu_1 x_{j1} - \bar{x}_1 \bar{x}_1 + \mu_1 \bar{x}_1) & \cdots & (\bar{x}_1 x_{jp} - \mu_1 x_{jp} - \bar{x}_1 \bar{x}_p + \mu_1 \bar{x}_p) \\ \vdots & \ddots & \vdots \\ (\bar{x}_p x_{j1} - \mu_p x_{j1} - \bar{x}_p \bar{x}_1 - \mu_p \bar{x}_1) & \cdots & (\bar{x}_p x_{jp} - \mu_p x_{jp} - \bar{x}_p \bar{x}_p + \mu_p \bar{x}_p) \end{bmatrix} \right) =
$$

$$
= \begin{bmatrix} \sum_{j=1}^{n} (\bar{x}_1 x_{j1} - \mu_1 x_{j1} - \bar{x}_1 \bar{x}_1 + \mu_1 \bar{x}_1) & \cdots & \sum_{j=1}^{n} (\bar{x}_1 x_{jp} - \mu_1 x_{jp} - \bar{x}_1 \bar{x}_p + \mu_1 \bar{x}_p) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} (\bar{x}_p x_{j1} - \mu_p x_{j1} - \bar{x}_p \bar{x}_1 - \mu_p \bar{x}_1) & \cdots & \sum_{j=1}^{n} (\bar{x}_p x_{jp} - \mu_p x_{jp} - \bar{x}_p \bar{x}_p + \mu_p \bar{x}_p) \end{bmatrix} =
$$

$$
= \begin{bmatrix} \sum_{j=1}^{n} (\bar{x}_1 x_{j1}) - \sum_{j=1}^{n} (\mu_1 x_{j1}) - n\bar{x}_1 \bar{x}_1 + n\mu_1 \bar{x}_1 & \cdots & \sum_{j=1}^{n} (\bar{x}_1 x_{jp}) - \sum_{j=1}^{n} (\mu_1 x_{jp}) - n\bar{x}_1 \bar{x}_p + n\mu_1 \bar{x}_p \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} (\bar{x}_p x_{j1}) - \sum_{j=1}^{n} (\mu_p x_{j1}) - n\bar{x}_p \bar{x}_1 + n\mu_p \bar{x}_1 & \cdots & \sum_{j=1}^{n} (\bar{x}_p x_{jp}) - \sum_{j=1}^{n} (\mu_p x_{jp}) - n\bar{x}_p \bar{x}_p + n\mu_p \bar{x}_p \end{bmatrix} =
$$

$$
= \begin{bmatrix} n\bar{x}_1 \bar{x}_1 - n\mu_1 \bar{x}_1 - n\bar{x}_1 \bar{x}_1 + n\mu_1 \bar{x}_1 & \cdots & n\bar{x}_1 \bar{x}_p - n\mu_1 \bar{x}_p - n\bar{x}_1 \bar{x}_p + n\mu_1 \bar{x}_p \\ \vdots & \ddots & \vdots \\ n\bar{x}_p \bar{x}_1 - n\mu_p \bar{x}_1 - n\bar{x}_p \bar{x}_1 + n\mu_p \bar{x}_1 & \cdots & n\bar{x}_p \bar{x}_p - n\mu_p \bar{x}_p - n\bar{x}_p \bar{x}_p + n\mu_p \bar{x}_p \end{bmatrix} =
$$

$$
= \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \underset{(p \times p)}{\mathbf{0}}
$$

4.16

Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, and $\mathbf{X}_4$ be independent $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ random vectors.

(a) Find the marginal distributions for each of the random vectors

$$\mathbf{V}_1 = \frac{1}{4}\mathbf{X}_1 - \frac{1}{4}\mathbf{X}_2 + \frac{1}{4}\mathbf{X}_3 - \frac{1}{4}\mathbf{X}_4$$

and

$$\mathbf{V}_2 = \frac{1}{4}\mathbf{X}_1 + \frac{1}{4}\mathbf{X}_2 - \frac{1}{4}\mathbf{X}_3 - \frac{1}{4}\mathbf{X}_4$$

First, note that we have $n = 4$ random vectors. We're looking at the linear combination of random vectors, not a linear combination of the components of a random vector, so we use Result 4.8 on page 158, $\mathbf{V}_1 = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$ is distributed as $N_p\left(\sum_{j=1}^n c_j\boldsymbol{\mu}_j, \left(\sum_{j=1}^n c_j^2\right)\boldsymbol{\Sigma}\right)$. For $\mathbf{V}_1$ we have

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

$$\text{mean} = \sum_{j=1}^n c_j \underset{(p\times 1)}{\boldsymbol{\mu}_j} = \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_1} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_2} + \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_3} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_4} =$$

$$= \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} + \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} = (\frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4})\underset{(p\times 1)}{\boldsymbol{\mu}} = \underset{(p\times 1)}{\mathbf{0}}$$

$$\text{Covariance} = \left(\sum_{j=1}^n c_j^2\right)\boldsymbol{\Sigma} = \left(\left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2\right)\boldsymbol{\Sigma} =$$

$$= \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right)\boldsymbol{\Sigma} = \frac{4}{16}\boldsymbol{\Sigma} = \frac{1}{4}\boldsymbol{\Sigma}$$

so that $\mathbf{V}_1 \sim N_p\left(\underset{(p\times 1)}{\mathbf{0}}, \underset{(p\times p)}{\left(\tfrac{1}{4}\right)\boldsymbol{\Sigma}}\right)$.

For $\mathbf{V}_2$ we have

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

$$\text{mean} = \sum_{j=1}^n d_j \underset{(p\times 1)}{\boldsymbol{\mu}_j} = \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_1} + \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_2} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_3} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}_4} =$$

$$= \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} + \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} - \frac{1}{4}\underset{(p\times 1)}{\boldsymbol{\mu}} = (\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4})\underset{(p\times 1)}{\boldsymbol{\mu}} = \underset{(p\times 1)}{\mathbf{0}}$$

$$\text{Covariance} = \left(\sum_{j=1}^{n} d_j^2\right)\mathbf{\Sigma} = \left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2\right)\mathbf{\Sigma} =$$

$$= \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right)\mathbf{\Sigma} = \frac{4}{16}\mathbf{\Sigma} = \frac{1}{4}\mathbf{\Sigma}$$

so that $\mathbf{V}_2 \sim N_p\left(\underset{(p\times 1)}{\mathbf{0}}, \underset{(p\times p)}{\left(\tfrac{1}{4}\right)\mathbf{\Sigma}}\right)$.

(b) Find the joint density of the random vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ defined in (a).

Again, from Result 4.8 on page 165, the joint distribution of $\mathbf{V}_1$ and $\mathbf{V}_2$ is multivariate normal. To compute the mean vector using what's on page 166, we know $n = 4$, but p can be anything, for $\underset{(p\times p)}{\mathbf{I}}$,

$$\underset{(2p\times np)}{\mathbf{A}} = \begin{bmatrix} c_1\mathbf{I} & c_2\mathbf{I} & c_3\mathbf{I} & c_4\mathbf{I} \\ d_1\mathbf{I} & d_2\mathbf{I} & d_3\mathbf{I} & d_4\mathbf{I} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\mathbf{I} & -\frac{1}{4}\mathbf{I} & \frac{1}{4}\mathbf{I} & -\frac{1}{4}\mathbf{I} \\ \frac{1}{4}\mathbf{I} & \frac{1}{4}\mathbf{I} & -\frac{1}{4}\mathbf{I} & -\frac{1}{4}\mathbf{I} \end{bmatrix}$$

$$\text{Vec}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \mathbf{X}_4 \end{bmatrix} = \underset{(np\times 1)}{\begin{bmatrix} X_{11} \\ \vdots \\ X_{1p} \\ X_{21} \\ \vdots \\ X_{2p} \\ X_{31} \\ \vdots \\ X_{3p} \\ X_{41} \\ \vdots \\ X_{4p} \end{bmatrix}}$$

$$\mathbf{A}\mathrm{Vec}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4) = \begin{bmatrix} c_1\mathbf{I} & c_2\mathbf{I} & c_3\mathbf{I} & c_4\mathbf{I} \\ d_1\mathbf{I} & d_2\mathbf{I} & d_3\mathbf{I} & d_4\mathbf{I} \end{bmatrix} \begin{bmatrix} X_{11} \\ \vdots \\ X_{1p} \\ X_{21} \\ \vdots \\ X_{2p} \\ X_{31} \\ \vdots \\ X_{3p} \\ X_{41} \\ \vdots \\ X_{4p} \end{bmatrix} =$$

$$\begin{bmatrix} c_1X_{11}+c_2X_{21}+c_3X_{31}+c_4X_{41} \\ c_1X_{12}+c_2X_{22}+c_3X_{32}+c_4X_{42} \\ \vdots \\ c_1X_{1p}+c_2X_{2p}+c_3X_{3p}+c_4X_{4p} \\ d_1X_{11}+d_2X_{21}+d_3X_{31}+d_4X_{41} \\ d_1X_{12}+d_2X_{22}+d_3X_{32}+d_4X_{42} \\ \vdots \\ d_1X_{1p}+d_2X_{2p}+d_3X_{3p}+d_4X_{4p} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^4 c_jX_{j1} \\ \sum_{j=1}^4 c_jX_{j2} \\ \vdots \\ \sum_{j=1}^4 c_jX_{jp} \\ \sum_{j=1}^4 d_jX_{j1} \\ \sum_{j=1}^4 d_jX_{j2} \\ \vdots \\ \sum_{j=1}^4 d_jX_{jp} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^4 c_j\mathbf{X}_j \\ \sum_{j=1}^4 d_j\mathbf{X}_j \end{bmatrix}$$

The expected value of this is (using part (a))

$$\mathrm{E}\left\{\begin{bmatrix} \sum_{j=1}^4 c_j\mathbf{X}_j \\ \sum_{j=1}^4 d_j\mathbf{X}_j \end{bmatrix}\right\} = \begin{bmatrix} \sum_{j=1}^4 c_j\mathrm{E}\left[\mathbf{X}_j\right] \\ \sum_{j=1}^4 d_j\mathrm{E}\left[\mathbf{X}_j\right] \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^4 c_j\boldsymbol{\mu}_j \\ \sum_{j=1}^4 d_j\boldsymbol{\mu}_j \end{bmatrix} = \begin{bmatrix} \underset{p\times 1}{\mathbf{0}} \\ \underset{p\times 1}{\mathbf{0}} \end{bmatrix} = \underset{2p\times 1}{\mathbf{0}}$$

The covariance matrix from Result 4.8 is on page 165,

$$\mathbf{d}'\mathbf{c} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} = \frac{1}{16} - \frac{1}{16} - \frac{1}{16} + \frac{1}{16} = 0$$

$$\begin{bmatrix} \left(\sum_{i=1}^n c_j^2\right)\boldsymbol{\Sigma} & (\mathbf{d}'\mathbf{c})\,\boldsymbol{\Sigma} \\ (\mathbf{d}'\mathbf{c})\,\boldsymbol{\Sigma} & \left(\sum_{i=1}^n d_j^2\right)\boldsymbol{\Sigma} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{4}\right)\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \left(\frac{1}{4}\right)\boldsymbol{\Sigma} \end{bmatrix} = \left(\frac{1}{4}\right)\begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma} \end{bmatrix}$$

and mean vector

$$\mathrm{Vec}(\mathbf{V}_1,\mathbf{V}_2) = \begin{bmatrix} \underset{p\times 1}{\mathbf{0}} \\ \underset{p\times 1}{\mathbf{0}} \end{bmatrix} = \underset{2p\times 1}{\mathbf{0}}$$

Also, $\mathbf{V}_1$ and $\mathbf{V}_2$ are independent, since $\mathbf{d}'\mathbf{c} = 0$.

4.17

Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, and $\mathbf{X}_4$ be independent and identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Find the mean vector and covariance matrices for each of the two linear combinations of random vectors

$$\frac{1}{5}\mathbf{X}_1 + \frac{1}{5}\mathbf{X}_2 + \frac{1}{5}\mathbf{X}_3 + \frac{1}{5}\mathbf{X}_4 + \frac{1}{5}\mathbf{X}_5$$

and

$$\mathbf{X}_1 - \mathbf{X}_2 + \mathbf{X}_3 - \mathbf{X}_4 + \mathbf{X}_5$$

in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Also, obtain the covariance betweeen the two linear combinations of random vectors.

First some detail for my own education. The vector $\mathbf{c} = [1/5, 1/5, 1/5, 1/5, 1/5]'$ is (5×1), but the vector $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5]'$ is $(np \times 1)$, so computing $\mathbf{c}'\mathbf{X}$ is not conformable (can't multiply $(1 \times p)$ by a $(np \times 1)$). We can get the job done by multiplying $\mathbf{c}'$ by the kronecker product of the identity matrix, $\mathbf{c}' \otimes \mathbf{I}_p$ first. Each of the $\mathbf{X}_j$ is $p \times 1$ and $\mathbf{I}_p$ is $p \times p$.

$$(\mathbf{c}' \otimes \mathbf{I}_p)\,\mathbf{X} = \left(\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix} \otimes \mathbf{I}_p\right) \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \mathbf{X}_4 \\ \mathbf{X}_5 \end{bmatrix} =$$

$$= \begin{bmatrix} c_1\mathbf{I}_p & c_2\mathbf{I}_p & c_3\mathbf{I}_p & c_4\mathbf{I}_p & c_5\mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \mathbf{X}_4 \\ \mathbf{X}_5 \end{bmatrix} =$$

$$= c_1\mathbf{I}_p\mathbf{X}_1 + c_2\mathbf{I}_p\mathbf{X}_2 + c_3\mathbf{I}_p\mathbf{X}_3 + c_4\mathbf{I}_p\mathbf{X}_4 + c_5\mathbf{I}_p\mathbf{X}_5 =$$

$$= c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3 + c_4\mathbf{X}_4 + c_5\mathbf{X}_5 =$$

$$= \frac{1}{5}\mathbf{X}_1 + \frac{1}{5}\mathbf{X}_2 + \frac{1}{5}\mathbf{X}_3 + \frac{1}{5}\mathbf{X}_4 + \frac{1}{5}\mathbf{X}_5$$

<u>Mean:</u>

$$\mathrm{E}\left[(\mathbf{c}' \otimes \mathbf{I}_p)\,\mathbf{X}\right] =$$

$$= \mathrm{E}\left[\frac{1}{5}\mathbf{X}_1 + \frac{1}{5}\mathbf{X}_2 + \frac{1}{5}\mathbf{X}_3 + \frac{1}{5}\mathbf{X}_4 + \frac{1}{5}\mathbf{X}_5\right] =$$

$$= \mathrm{E}\left[\frac{1}{5}\mathbf{X}_1\right] + \mathrm{E}\left[\frac{1}{5}\mathbf{X}_2\right] + \mathrm{E}\left[\frac{1}{5}\mathbf{X}_3\right] + \mathrm{E}\left[\frac{1}{5}\mathbf{X}_4\right] + \mathrm{E}\left[\frac{1}{5}\mathbf{X}_5\right] =$$

$$= \frac{1}{5}\mathrm{E}\left[\mathbf{X}_1\right] + \frac{1}{5}\mathrm{E}\left[\mathbf{X}_2\right] + \frac{1}{5}\mathrm{E}\left[\mathbf{X}_3\right] + \frac{1}{5}\mathrm{E}\left[\mathbf{X}_4\right] + \frac{1}{5}\mathrm{E}\left[\mathbf{X}_5\right] =$$

$$= \frac{1}{5}\boldsymbol{\mu} + \frac{1}{5}\boldsymbol{\mu} + \frac{1}{5}\boldsymbol{\mu} + \frac{1}{5}\boldsymbol{\mu} + \frac{1}{5}\boldsymbol{\mu}$$
$$= \left(\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}\right)\boldsymbol{\mu} = \boldsymbol{\mu}$$

<u>Covariance:</u>

$$\text{Covar}\left[\left(\mathbf{c}' \otimes \mathbf{I}_p\right)\mathbf{X}\right] =$$
$$= \text{Covar}\left[\frac{1}{5}\mathbf{X}_1 + \frac{1}{5}\mathbf{X}_2 + \frac{1}{5}\mathbf{X}_3 + \frac{1}{5}\mathbf{X}_4 + \frac{1}{5}\mathbf{X}_5\right] =$$
$$= \text{Covar}\left[\frac{1}{5}\mathbf{X}_1\right] + \text{Covar}\left[\frac{1}{5}\mathbf{X}_2\right] + \text{Covar}\left[\frac{1}{5}\mathbf{X}_3\right] + \text{Covar}\left[\frac{1}{5}\mathbf{X}_4\right] + \text{Covar}\left[\frac{1}{5}\mathbf{X}_5\right] =$$
$$= \left(\frac{1}{5}\right)^2 \text{Covar}\left[\mathbf{X}_1\right] + \left(\frac{1}{5}\right)^2 \text{Covar}\left[\mathbf{X}_2\right] + \left(\frac{1}{5}\right)^2 \text{Covar}\left[\mathbf{X}_3\right] + \left(\frac{1}{5}\right)^2 \text{Covar}\left[\mathbf{X}_4\right] + \left(\frac{1}{5}\right)^2 \text{Covar}\left[\mathbf{X}_5\right] =$$
$$= \left(\frac{1}{5}\right)^2 \boldsymbol{\Sigma} + \left(\frac{1}{5}\right)^2 \boldsymbol{\Sigma} + \left(\frac{1}{5}\right)^2 \boldsymbol{\Sigma} + \left(\frac{1}{5}\right)^2 \boldsymbol{\Sigma} + \left(\frac{1}{5}\right)^2 \boldsymbol{\Sigma} =$$
$$= 5\left(\frac{1}{5}\right)^2 \boldsymbol{\Sigma} = \frac{1}{5}\boldsymbol{\Sigma}$$

and for

$$\mathbf{X}_1 - \mathbf{X}_2 + \mathbf{X}_3 - \mathbf{X}_4 + \mathbf{X}_5$$
$$\text{Here, we have} \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

<u>Mean:</u>

$$\text{E}\left[\left(\mathbf{d}' \otimes \mathbf{I}_p\right)\mathbf{X}\right] = \sum_{j=1}^{5} d_j \boldsymbol{\mu}_j = \sum_{j=1}^{5} d_j \boldsymbol{\mu} = (1 - 1 + 1 - 1 + 1)\,\boldsymbol{\mu} = \boldsymbol{\mu}$$

<u>Covariance:</u>

$$\text{Covar}\left[\left(\mathbf{d}' \otimes \mathbf{I}_p\right)\mathbf{X}\right] = \left(\sum_{j=1}^{5} d_j^2\right)\boldsymbol{\Sigma} = \left(1^2 - 1^2 + 1^2 - 1^2 + 1^2\right)\boldsymbol{\Sigma} = 5\boldsymbol{\Sigma}$$

Before finding the covariance betweeen the two linear combinations we need to compute

$$\mathbf{d}'\mathbf{c} = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} = \frac{1}{5}$$

Now, the covariance betweeen the two linear combinations is

$$\begin{bmatrix} \left(\sum_{j=1}^{5} c_j^2\right) \boldsymbol{\Sigma} & \left(\mathbf{d}'\mathbf{c}\right) \boldsymbol{\Sigma} \\ \left(\mathbf{d}'\mathbf{c}\right) \boldsymbol{\Sigma} & \left(\sum_{j=1}^{5} d_j^2\right) \boldsymbol{\Sigma} \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\boldsymbol{\Sigma} & \frac{1}{5}\boldsymbol{\Sigma} \\ \frac{1}{5}\boldsymbol{\Sigma} & 5\boldsymbol{\Sigma} \end{bmatrix}$$

4.18

Find the maximum likelihood estimates of the 2×1 mean vector $\boldsymbol{\mu}$ and the 2×2 covariance matrix $\boldsymbol{\Sigma}$ based on the random sample

$$\begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

from a bivariate normal population.

Using Result 4.11 on page 171

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \frac{(3+4+5+4)}{4} \\ \frac{(6+4+7+7)}{4} \end{bmatrix} = \begin{bmatrix} \frac{16}{4} \\ \frac{24}{4} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\hat{\Sigma} = \left(\frac{(n-1)}{n}\right) \mathbf{S} = \left(\frac{1}{4}\right) \sum_{j=1}^{4} \left(\mathbf{X}_j - \bar{\mathbf{X}}\right) \left(\mathbf{X}_j - \bar{\mathbf{X}}\right)' =$$

$$= \left(\frac{1}{4}\right) \left\{ \left(\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) \left(\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)' + \left(\begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) \left(\begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)' + \right.$$

$$\left. \left(\begin{bmatrix} 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) \left(\begin{bmatrix} 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)' + \left(\begin{bmatrix} 4 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) \left(\begin{bmatrix} 4 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)' \right\} =$$

$$= \left(\frac{1}{4}\right) \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}' + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}' + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}' + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}' \right\} =$$

$$= \left(\frac{1}{4}\right) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} =$$

$$= \left(\frac{1}{4}\right) \begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} (1/2) & -(1/4) \\ -(1/4) & (3/2) \end{bmatrix}$$

4.19

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{20}$ be a random sample of size $n = 20$ from a $N_6(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ population. Specify each of the following completely.

(a) The distribution of $(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu})$
Can use Result 4.7 (a) on page 163 for this

$$(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu}) \sim \chi^2_6$$

(b) The distribution of $\bar{\mathbf{X}}$ and $\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})$

$$\mathbf{Y}_1 = \bar{\mathbf{X}} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \vdots \\ \bar{\mathbf{x}}_6 \end{bmatrix} = (\mathbf{X}_1 + \cdots + \mathbf{X}_{20})/20 = \frac{\sum_{j=1}^{20} \mathbf{X}_j}{20}$$

$$\mathrm{E}[\mathbf{Y}_1] = \mathrm{E}\left[\frac{\sum_{j=1}^{20} \mathbf{X}_j}{20}\right] = \frac{\sum_{j=1}^{20} \mathrm{E}[\mathbf{X}_j]}{20} = \frac{\sum_{j=1}^{20} \boldsymbol{\mu}}{20} = \frac{20\boldsymbol{\mu}}{20} = \boldsymbol{\mu}$$

$$\mathrm{Covar}[\mathbf{Y}_1] = \mathrm{Covar}\left[\frac{\sum_{j=1}^{20} \mathbf{X}_j}{20}\right] = \left(\frac{1}{20}\right)^2 \sum_{j=1}^{20} \mathrm{Covar}[\mathbf{X}_j] =$$

$$\left(\frac{1}{20}\right)^2 \left(\sum_{j=1}^{20} \boldsymbol{\Sigma}\right) = \left(\frac{1}{20}\right)^2 (20\boldsymbol{\Sigma}) = \frac{1}{20}\boldsymbol{\Sigma}$$

$$\mathbf{Y}_1 = \bar{\mathbf{X}} \sim N_6\left(\boldsymbol{\mu}, \frac{1}{20}\boldsymbol{\Sigma}\right)$$

This is the same as (4–23) in section 4.4 on page 174.
If $\mathbf{Y}_2 = \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})$, then

$$\mathrm{E}[\mathbf{Y}] = \mathrm{E}\left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] = \mathrm{E}\left[\sqrt{n}\bar{\mathbf{X}}\right] - \mathrm{E}\left[\sqrt{n}\boldsymbol{\mu}\right] = \mathrm{E}\left[\sqrt{n}\bar{\mathbf{X}}\right] - \mathrm{E}\left[\sqrt{n}\boldsymbol{\mu}\right] =$$

$$= \sqrt{n}\mathrm{E}\left[\bar{\mathbf{X}}\right] - \sqrt{n}\boldsymbol{\mu} = \sqrt{n}\boldsymbol{\mu} - \sqrt{n}\boldsymbol{\mu} = \mathbf{0}$$

$$\mathrm{Covar}[\mathbf{Y}_2] = \mathrm{Covar}\left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] = \mathrm{Covar}\left[\sqrt{n}\bar{\mathbf{X}}\right] - \mathrm{Covar}\left[\sqrt{n}\boldsymbol{\mu}\right] =$$

$$= \left(\sqrt{n}\right)^2 \mathrm{Covar}\left[\bar{\mathbf{X}}\right] - \mathbf{0} = n\left(\frac{1}{n}\boldsymbol{\Sigma}\right) = \boldsymbol{\Sigma}$$

$$\mathbf{Y}_2 = \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N_6(\mathbf{0}, \boldsymbol{\Sigma})$$

This is the same as the approximation in (4–28) in section 4.5 on page 176.

(c) The distribution of (n-1)$\mathbf{S}$

Answers in (4–23) on page 174.

$$(n-1)\mathbf{S} \sim W_{m=20-1}\left((n-1)\mathbf{S} \middle| \boldsymbol{\Sigma}\right)$$

4.20

For the random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{20}$ in Exercise 4.19, specify the distribution of $\mathbf{B}(19\mathbf{S})\mathbf{B}'$ in each case

(a) $\mathbf{B} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$

Use the properties of the Wishart distribution from page 174 along with (4–23)

$$\mathbf{B}(19\mathbf{S})\mathbf{B}' \sim W_{19}\left(\mathbf{B}(19\mathbf{S})\mathbf{B}' \middle| \mathbf{B}(19\boldsymbol{\Sigma})\mathbf{B}'\right)$$

where

$$\mathbf{B}(19\mathbf{S})\mathbf{B}' = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} (s_{11} - \frac{1}{2}s_{21} - \frac{1}{2}s_{31}) & (s_{12} - \frac{1}{2}s_{22} - \frac{1}{2}s_{32}) & \cdots & (s_{16} - \frac{1}{2}s_{26} - \frac{1}{2}s_{36}) \\ (-\frac{1}{2}s_{41} - \frac{1}{2}s_{51} + s_{61}) & (-\frac{1}{2}s_{42} - \frac{1}{2}s_{52} + s_{62}) & \cdots & (-\frac{1}{2}s_{46} - \frac{1}{2}s_{56} + s_{66}) \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \text{element}_{11} & \text{element}_{12} \\ \text{element}_{21} & \text{element}_{22} \end{bmatrix}$$

where
$\text{element}_{11} = (s_{11} - \frac{1}{2}s_{21} - \frac{1}{2}s_{31}) - \frac{1}{2}(s_{12} - \frac{1}{2}s_{22} - \frac{1}{2}s_{32}) - \frac{1}{2}(s_{13} - \frac{1}{2}s_{23} - \frac{1}{2}s_{33})$
$\text{element}_{12} = -\frac{1}{2}(s_{14} - \frac{1}{2}s_{24} - \frac{1}{2}s_{34}) - \frac{1}{2}(s_{15} - \frac{1}{2}s_{25} - \frac{1}{2}s_{35}) + (s_{16} - \frac{1}{2}s_{26} - \frac{1}{2}s_{36})$
$\text{element}_{21} = (-\frac{1}{2}s_{41} - \frac{1}{2}s_{51} + s_{61}) - \frac{1}{2}(-\frac{1}{2}s_{42} - \frac{1}{2}s_{52} + s_{62}) - \frac{1}{2}(-\frac{1}{2}s_{43} - \frac{1}{2}s_{53} + s_{63})$
$\text{element}_{22} = -\frac{1}{2}(-\frac{1}{2}s_{44} - \frac{1}{2}s_{54} + s_{64}) - \frac{1}{2}(-\frac{1}{2}s_{45} - \frac{1}{2}s_{55} + s_{65}) + (-\frac{1}{2}s_{46} - \frac{1}{2}s_{56} + s_{66})$ and

$$\mathbf{B}(19\boldsymbol{\Sigma})\mathbf{B}' = \begin{bmatrix} \text{element}_{11} & \text{element}_{12} \\ \text{element}_{21} & \text{element}_{22} \end{bmatrix}$$

where
element$_{11}$ $= (\sigma_{11}-\frac{1}{2}\sigma_{21}-\frac{1}{2}\sigma_{31})-\frac{1}{2}(\sigma_{12}-\frac{1}{2}\sigma_{22}-\frac{1}{2}\sigma_{32})-\frac{1}{2}(\sigma_{13}-\frac{1}{2}\sigma_{23}-\frac{1}{2}\sigma_{33})$
element$_{12}$ $= -\frac{1}{2}(\sigma_{14}-\frac{1}{2}\sigma_{24}-\frac{1}{2}\sigma_{34})-\frac{1}{2}(\sigma_{15}-\frac{1}{2}\sigma_{25}-\frac{1}{2}\sigma_{35})+(\sigma_{16}-\frac{1}{2}\sigma_{26}-\frac{1}{2}\sigma_{36})$
element$_{21}$ $= (-\frac{1}{2}\sigma_{41}-\frac{1}{2}\sigma_{51}+\sigma_{61})-\frac{1}{2}(-\frac{1}{2}\sigma_{42}-\frac{1}{2}\sigma_{52}+\sigma_{62})-\frac{1}{2}(-\frac{1}{2}\sigma_{43}-\frac{1}{2}\sigma_{53}+\sigma_{63})$
element$_{22}$ $= -\frac{1}{2}(-\frac{1}{2}\sigma_{44}-\frac{1}{2}\sigma_{54}+\sigma_{64})-\frac{1}{2}(-\frac{1}{2}\sigma_{45}-\frac{1}{2}\sigma_{55}+\sigma_{65})+(-\frac{1}{2}\sigma_{46}-\frac{1}{2}\sigma_{56}+\sigma_{66})$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

Again, use the properties of the Wishart distribution from page 174 along with (4–23)

$$\mathbf{B}(19\mathbf{S})\mathbf{B}' \sim W_{19}\left(\mathbf{B}(19\mathbf{S})\mathbf{B}' \middle| \mathbf{B}(19\mathbf{\Sigma})\mathbf{B}'\right)$$

where

$$\mathbf{B}(19\mathbf{S})\mathbf{B}' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{13} \\ s_{31} & s_{33} \end{bmatrix}$$

and

$$\mathbf{B}(19\mathbf{\Sigma})\mathbf{B}' = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix}$$

4.21

Let $\mathbf{X}_1, \ldots, \mathbf{X}_{60}$ be a random sample of size 60 from a four-variate normal distribution having mean $\boldsymbol{\mu}$ and covariance $\mathbf{\Sigma}$. Specify each of the following completely.

(a) The distribution of $\bar{\mathbf{X}}$

Using 1. from (4–23) on page 174

$$\bar{\mathbf{X}} \sim N_4\left(\underset{(4\times1)}{\mathbf{0}}, \frac{1}{60}\underset{(4\times4)}{\mathbf{\Sigma}}\right)$$

(b) The distribution of $(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu})$

Using Result 4.7 (a) in section 4.2 on page 163.

$$(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu}) \sim \chi^2_4$$

(c) The distribution of $n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu})$

Similar to part (b), but I'll do the work.

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) =$$
$$= \sqrt{n}\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2}(\bar{\mathbf{X}} - \boldsymbol{\mu}) =$$
$$= \left[(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1/2}\sqrt{n}\right]\left[\sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] =$$
$$= \left[\sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right]'\left[\sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] =$$
$$= \mathbf{Z}'\mathbf{Z}$$
$$\mathrm{E}[\mathbf{Z}] = \mathrm{E}\left[\sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}\mathrm{E}\left[(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] =$$
$$= \sqrt{n}\boldsymbol{\Sigma}^{-1/2}\left(\mathrm{E}\left[\bar{\mathbf{X}}\right] - \mathrm{E}[\boldsymbol{\mu}]\right) = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\mu} - \boldsymbol{\mu}) = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}\mathbf{0} = \mathbf{0}$$
$$\mathrm{Covar}[\mathbf{Z}] = \mathrm{Covar}\left[\sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] =$$
$$= \left(\sqrt{n}\right)^2\boldsymbol{\Sigma}^{-1/2}\mathrm{Covar}\left[(\bar{\mathbf{X}} - \boldsymbol{\mu})\right]\left(\boldsymbol{\Sigma}^{-1/2}\right)' =$$
$$= n\boldsymbol{\Sigma}^{-1/2}\left(\mathrm{Covar}\left[\bar{\mathbf{X}}\right] + \mathrm{Covar}[\boldsymbol{\mu}]\right)\boldsymbol{\Sigma}^{-1/2} =$$
$$= n\boldsymbol{\Sigma}^{-1/2}\left(\frac{1}{n}\boldsymbol{\Sigma} + 0\right)\boldsymbol{\Sigma}^{-1/2} =$$
$$= n\boldsymbol{\Sigma}^{-1/2}\left(\frac{1}{n}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}\right)\boldsymbol{\Sigma}^{-1/2} =$$
$$= n\frac{1}{n}\left(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\right)\left(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\right) =$$
$$= (\mathbf{I}_p)(\mathbf{I}_p) = \mathbf{I}_p$$

We now have that

$$\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{I}_p)$$

Now $\mathbf{Z}'\mathbf{Z} = \sum_{j=1}^{p} Z_j^2$, the sum of squared standard normal random variables is a chi-squared distribution with p degrees of freedom, so

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim \chi^2_4$$

(d) The approximate distribution of $n\big(\bar{\mathbf{X}}-\boldsymbol{\mu}\big)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right)$

Well, if we know that $\bar{\mathbf{X}} \sim N(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma})$ and that $(n-1)\mathbf{S}$ is a $p=4$-variate Wishart distribution with $(60-1)$ degrees of freedom, then $n\big(\bar{\mathbf{X}}-\boldsymbol{\mu}\big)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right)$ is a Hotelling's $T^2_{p=4,n-1=60-1}$ distribution. The T^2 is related to an F-distribution as $T^2_{4,59} = \frac{4(60-1)}{60-4}F_{p=4,n-p=60-4}$. Here, they're asking for an approximation though, so instead we can use the chi-squared approximation in (4–28) on page 176, where they aren't making assumtions about the distribution of the random sample (same as the answer to Exercise 4.22 (b)).

$$n\big(\bar{\mathbf{X}}-\boldsymbol{\mu}\big)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right) \dot{\sim} \chi^2_4$$

4.22

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{75}$ be a random sample from a population distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. What is the approximate distribution of each of the following?

(a) $\bar{\mathbf{X}}$

We have 75 observations, which is considered fairly large, so the asymptotic results from the section 4.5 apply. As n increases, $\bar{\mathbf{X}}$ converges in probability to $\boldsymbol{\mu}$ and $\mathbf{S}$ converges in probability to $\boldsymbol{\Sigma}$. Result 4.13 basically gives our answer. In the result, they have $\sqrt{n}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right) \dot{\sim} N_p\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$. If we leave off the $\sqrt{n}$ and the subtraction of $\boldsymbol{\mu}$ we have

$$\bar{\mathbf{X}} \dot{\sim} N_p\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right)$$

(b) $n\big(\bar{\mathbf{X}}-\boldsymbol{\mu}\big)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right)$

Again, we only know that we have a population mean vector $\boldsymbol{\mu}$ and covaraince matrix $\boldsymbol{\Sigma}$. We're not making any statements about what distribution they belong to. Using the central limit theorem from Result 4.13 from page 176, where $\sqrt{n}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right) \dot{\sim} N_p\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$ and then $n\big(\bar{\mathbf{X}}-\boldsymbol{\mu}\big)'\boldsymbol{\Sigma}^{-1}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right) \dot{\sim} \chi^2_p$ so replacing the true population covaraince, $\boldsymbol{\Sigma}$, with the sample covariance, $\mathbf{S}$, won't cause massive errors as long as $n >> p$, and we have

$$n\big(\bar{\mathbf{X}}-\boldsymbol{\mu}\big)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}}-\boldsymbol{\mu}\right) \dot{\sim} \chi^2_p$$