Let  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  be a random sample of size n from a p-variate normal distribution with mean vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$ . What's the distribution of  $(n-1)\mathbf{S}$ ?

$$(n-1)\mathbf{S} = \frac{n-1}{n-1} \sum_{j=1}^{n} \left( \mathbf{X}_{j} - \bar{\mathbf{X}} \right) \left( \mathbf{X}_{j} - \bar{\mathbf{X}} \right)' =$$

$$= \sum_{j=1}^{n} \left( \mathbf{X}_{j} + \mu - \mu - \bar{\mathbf{X}} \right) \left( \mathbf{X}_{j} + \mu - \mu - \bar{\mathbf{X}} \right)' =$$

$$= \sum_{j=1}^{n} \left( (\mathbf{X}_{j} - \mu) - (\bar{\mathbf{X}} - \mu) \right) \left( (\mathbf{X}_{j} - \mu) - (\bar{\mathbf{X}} - \mu) \right)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - (\bar{\mathbf{X}} - \mu)(\mathbf{X}_{j} - \mu)' - (\mathbf{X}_{j} - \mu)(\bar{\mathbf{X}} - \mu)' + (\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - 2\sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\bar{\mathbf{X}} - \mu)' + \sum_{j=1}^{n} (\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - 2(\bar{\mathbf{X}} - \mu)(\sum_{j=1}^{n} \mathbf{X}_{j}) - n\mu' + n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - 2(\bar{\mathbf{X}} - \mu)((\bar{\mathbf{X}} - \mu))' + n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - 2n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' + n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - n(\bar{\mathbf{X}} - \mu)(\bar{\mathbf{X}} - \mu)' =$$

$$= \sum_{j=1}^{n} (\mathbf{X}_{j} - \mu)(\mathbf{X}_{j} - \mu)' - [\sqrt{n}(\bar{\mathbf{X}} - \mu)] \left[ \sqrt{n}(\bar{\mathbf{X}} - \mu)' \right] =$$

so looking at the first and second part of this equation

$$\mathbf{X}_{j} \sim N(\underbrace{\boldsymbol{\mu}}_{(p \times 1)}, \underbrace{\boldsymbol{\Sigma}}_{(p \times p)}) \Rightarrow (\mathbf{X}_{j} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Sigma}) \Rightarrow$$

$$\Rightarrow (\mathbf{X}_{j} - \boldsymbol{\mu})(\mathbf{X}_{j} - \boldsymbol{\mu})' \sim W_{p}(1, \boldsymbol{\Sigma}) \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{n} (\mathbf{X}_{j} - \boldsymbol{\mu})(\mathbf{X}_{j} - \boldsymbol{\mu})' \sim W_{p}(n, \boldsymbol{\Sigma})$$

That is, the first term in the sum is distributed as a Wishart distribution with n degrees of freedom. Reminder that a p-dimensional Wishart distribution with m degrees of freedom,  $W_p(m, \Sigma)$ , is equal to the distribution of  $\sum_{j=1}^m \mathbf{Z}_j \mathbf{Z}'_j$ , where the  $\mathbf{Z}_j$  are independently distributed as  $N_p(\mathbf{0}, \Sigma)$ . For the second term,

$$ar{\mathbf{X}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n) \Rightarrow (\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Sigma}/n) \Rightarrow$$
  
 $\Rightarrow \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Sigma}) \Rightarrow$ 

$$\left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right] \left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})'\right] \sim W_p(1, \boldsymbol{\Sigma})$$

That is, the second term in the sum is distributed as a Wishart distribution with 1 degree of freedom.

Combining the two, we now has a Wishart distribution with n degrees of freedom minus another Wishart distribution with 1 degree of freedom to create a Wishart distribution with n-1 distribution, so we finally have

$$(n-1)\mathbf{S} \sim W_p\left(n-1, \sum_{(p \times p)}\right)$$