What's the distribution of (n-1)$\mathbf{S}$?

$$(n-1)\mathbf{S} = \frac{n-1}{n-1}\sum_{j=1}^{n}\left(\mathbf{X}_j - \bar{\mathbf{X}}\right)\left(\mathbf{X}_j - \bar{\mathbf{X}}\right)' =$$

$$= \sum_{j=1}^{n}\left(\mathbf{X}_j + \boldsymbol{\mu} - \boldsymbol{\mu} - \bar{\mathbf{X}}\right)\left(\mathbf{X}_j + \boldsymbol{\mu} - \boldsymbol{\mu} - \bar{\mathbf{X}}\right)' =$$

$$= \sum_{j=1}^{n}\left((\mathbf{X}_j - \boldsymbol{\mu}) - (\bar{\mathbf{X}} - \boldsymbol{\mu})\right)\left((\mathbf{X}_j - \boldsymbol{\mu}) - (\bar{\mathbf{X}} - \boldsymbol{\mu})\right)' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - (\bar{\mathbf{X}} - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - (\mathbf{X}_j - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' + (\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - 2\sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' + \sum_{j=1}^{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - 2(\bar{\mathbf{X}} - \boldsymbol{\mu})\sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})' + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - 2(\bar{\mathbf{X}} - \boldsymbol{\mu})((\sum_{j=1}^{n}\mathbf{X}_j) - n\boldsymbol{\mu})' + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - 2(\bar{\mathbf{X}} - \boldsymbol{\mu})(n\bar{\mathbf{X}} - n\boldsymbol{\mu})' + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - 2n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' + n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - n(\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})' =$$

$$= \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' - \left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right]\left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})'\right] =$$

so looking at the first and second part of this equation

$$\mathbf{X}_j \sim N(\underset{(p\times 1)}{\boldsymbol{\mu}}, \underset{(p\times p)}{\boldsymbol{\Sigma}}) \;\Rightarrow\; (\mathbf{X}_j - \boldsymbol{\mu}) \sim N(\mathbf{0}, \Sigma) \Rightarrow$$

$$\Rightarrow (\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' \sim \chi_1^2 \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{n}(\mathbf{X}_j - \boldsymbol{\mu})(\mathbf{X}_j - \boldsymbol{\mu})' \sim W_p(n, \Sigma)$$

That is, the first term in the sum is distributed as a Wishart distribution with n degrees of freedom. For the second term,

$$\bar{\mathbf{X}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n) \Rightarrow (\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Sigma}/n) \Rightarrow$$

$$\Rightarrow \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Sigma}) \Rightarrow$$

$$\left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})\right]\left[\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})'\right] \sim W_p(1, \boldsymbol{\Sigma})$$

That is, the second term in the sum is distributed as a Wishart distribution with 1 degree of freedom.

Combining the two, we now has a Wishart distribution with n degrees of freedom minus another Wishart distribution with 1 degree of freedom to create a Wishart distribution with $n-1$ distribution, so we finally have

$$(n-1)\mathbf{S} \sim W_p\left(n-1, \underset{(p\times p)}{\boldsymbol{\Sigma}}\right)$$