REPORT ENGLISH

1. GENERAL INFORMATION AND PFEFACE

Here is a report on the internship that Xu Wentao carried out at the LPSC between 9 and 30 January 2023. This report will be handed over to PROF. INGO SCHEIBEN. It will therefore be mainly academic in nature.

My work (and that of FILIP) is to a large extent along the lines of Miss DERYA's report. However, from my personal point of view (and allow me to speak from the point of view of a student who does not know much about the field), DERYA's report is imperfect and unorganised. It caused a great deal of frustration in our work.

I have therefore decided to rewrite a report of my own, or rather reading notes.

Obviously, there are already many excellent introductory books on the Internet about Lie Group. For exampel, Timothy Budd's liealgebras.pdf (ru.nl)(*Lie Algebras in Particle Physics*)is a very suitable document for beginners.

Similarly, my BINOME has written an introductory document. If you find it still too abstract, start with Nathan Carter's *Visual Group Theory*, which doesn't even contain many formulas.

Again, as the author (me) has only less than half a year of training in physics, and his previous education has been more in the field of engineering. As a result, much of this document will be unproven, i.e. it is a very strong physics (engineering) style document. There will be some "black boxes" in which I have not understood, but have used directly to draw conclusions. I may or may not be able to point out where specific explanations are given (sometimes I don't even point out where the black boxes are). It is therefore more like a user manual.

Befor Lie groups

The definition of a group

omitted. See the FILIP documentation

Group representation theory

(In general, we deal with physically encountered group representation problems on linear spaces. Therefore, I am simplifying the problem. However the more important reason is that I do not have any grounding in abstract algebra)

(It is sufficient that the quantum mechanics problem can be solved on linear space. Therefore there is no need to introduce more concepts from abstract algebra (*linear algebra is enough*), such as morphism or homomorphism)

There are three players in group representation theory, which are

1. the group G
2. a general linear group GL on a representation space (V).
3. a homomorphic mapping D.

For these three players, we have certain requirements.

1. for the representation space V. Typically, it is either $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$ (the latter is more common in physics)
2. for a homomorphic mapping D
    a. It serves to map the group G onto the general linear group GL(V) on V. i.e.

    $$D : G \to \mathrm{GL}(V)$$

    b. for a homomorphic mapping D
    it should have the following properties.

    $$D(g)D(h) = D(g \cdot h) \quad \text{for all} \quad g, h \in G$$

Group representation theory allows us to use the tools of linear algebra to study the otherwise more abstract group theory. We can now study groups in terms of matrices over representation spaces.

Equivalent representations

(I don't know if this part can be omitted)

(Budd, page 3&4, around the *definiton 1.11*)

> Having introduced abstract groups and their representations, we disentangle two aspects of the symmetry of a physical system: the abstract group captures how symmetry transformations com- pose, while the representation describes how the symmetry transformations act on the system.
>
> This separation is very useful in practice, thanks to the fact that there are far fewer abstract groups than conceivable physical systems with symmetries.
>
> Understanding the properties of some abstract group teaches us something about all possible systems that share that symmetry group. As we will see later in the setting of Lie groups, one can to a certain extent classify all possible abstract groups with certain properties. Furthermore, for a given abstract group one can then try to classify all its (inequivalent) representations, i.e. all the ways in which the group can be realized as a symmetry group in the system.
>
> --original text

Regular representation

(Budd, page 5, *definition 1.12*)

> Higher-dimentional representations can be constructed naturally for any finite group is the regular representation.

For a finite group G, typically, we use a |G|-dimensional complex linear representation space: $V_G \equiv \mathbb{C}^{|G|}$, where $|G|$ is the order of the group.

Since the dimension is equal to the group order, it is possible to put the group element in one-to-one correspondence with the basis, i.e. the basis vector is written as $\{e^g \mid g \in G\}$. (It should be noted that, as in the Einstein convention, the superscript here does not denote a multiplicative power, but a mere token.)

We can think of V_G as a space tensed by $\{e^g \mid g \in G\}$. Again, it meets the requirements described above.

> $$D(g)D(h) = D(g \cdot h) \quad \text{for all} \quad g, h \in G$$

A very natural idea is to define $D(g), D(h)$ here as vectors in linear algebra, and the meaning of the dot product - in vector multiplication is fully preserved.

Typically, we use column vectors to form the basis for this group.

> The dimension of the regular representation thus equals the order of the group, $\dim\left(D_{\text{reg}}\right) = |G|$. Note that in the basis $\left|g_1\right\rangle, \ldots, \left|g_n\right\rangle$ these vectors correspond to the column vectors
>
> $$\left|g_1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad, \left|g_n\right\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$
>
> *-Budd, page 5*

From column vectors to matrix

There is a very clear textual description in *Group theory in a nutshell for physicists,* A.ZEE, chp II: Representing Group Elements by Matrices.(around page 90)

> Consider S_4, the permutation group of four objects. Think of the four objects as the four vectors* $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Then we can represent the element (2413), which takes $2 \rightarrow 4, 4 \rightarrow 1, 1 \rightarrow 3$, and $3 \rightarrow 2$, by the 4-by-4 matrix $D(2413) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. By construction, $D(2413)v_2 = v_4, D(2413)v_4 = v_1$, and so on. The action of the matrix D (2413) on the four vectors mirrors precisely the action of the permutation (2413) on the four objects labeled 1, 2, 3, and 4. Similarly, we have, for example, $D(34) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

> According to what we have learned in chapter I.2, we have $(34)(2413)=(23)(14)$. Here, let us multiply the two matrices $D(34)$ and $D(2413)$ together. (Go ahead, do it!) We find $D(34) D(2413)=\left(\begin{array}{llll}0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0\end{array}\right)$, which is precisely $D((23)(14))$, as expected. This verifies (1), at least in this particular instance. Exercise! Write down a few more matrices in this 4-dimensional representation of S_4 and multiply them.
>
> I presume that you are not surprised that we have found a 4-dimensional representation of S_4. In summary, the group S_4 can be represented by 24 distinct $4 \otimes 4$ matrices. Note that these are very special matrices, with 0 almost everywhere except for four 1 s, with one single 1 in each column (and in each row). All this should be fairly self-evident: what is the difference between four vectors labeled v_1, v_2, v_3, v_4 and four balls labeled $1,2,3,4$?

It is important to note that the concept of group elements is derived from transformations. Very naturally, we again apply the concept of matrices in linear algebra. For example, in the above description we have successfully represented the transformation of two group elements in the form of a matrix.

Since then, we have completed the definition of group elements and group multiplication, from conceptual to (most commonly used) representations. Namely, this step as follows：

$$
\begin{array}{ccccc}
g_1 & \cdot & g_2 & = & g_1 \cdot g_2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
D\left(g_1\right) & \cdot & D\left(g_2\right) & = & D\left(g_1 \cdot g_2\right)
\end{array}
$$

It's also says:

> In fact, physicists often confound group elements with the matrices that represent them.

But it's still quite important to know the difference of the group elements and their matrix representations.

Other things not clearly definied

I'm not sure if I want to include the *direct sum* and *tensor product* here.

Block-diagonal form

-*Budd,* page 6, definition 1.14

irreducible representations

Schur Lemma

Group center, Kernel

Every representation of a finite group is completely reducible.

Every representation of a finite group is equivalent to a unitary representation. (unitary representation)

Lie groups and Lie algebras

Before start

1. Lie group and Lie algebra are not an equivalent concept (although for a long time I got them confused)
2. The expression of group theory in physics is the theory of representation of groups.
3. Inevitably, therefore, there will be a whole host of questions and missing bits. Considering that this is the result of an internship for two undergraduates (one of whom has only been on a formal physics course for four months), forgive me.

Let's begin from some basic definitions

In physics, a group is an algebraic structure used to study the 'action' on a physical system. Further, physicists want to apply tools that are "smooth" (for the parameters), i.e. derivable to infinite order. In this case, the group that meets the corresponding requirements constitutes the Lie group. Its most distinctive feature is that it can be derived to infinite order. (i.e. C^n, which remains continuous after an infinite number of derivations.)

> Recall that, informally speaking, a set G is an n-dimensional smooth manifold if the neighbourhoud of any element can be smoothly parametrized by n parameters $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$。

The definition of a Lie group is "both a differential manifold and a group". Similarly, in such an algebraic structure, operations such as group multiplication and taking the inverse must be compatible with smooth structures. (i.e. they are all infinite-order derivable operations.)

> The Lie group is very interested, but unfortunately a direct study of it is more difficult. This difficulty obviously arises from two things: its propriety, as a differential manifold, which is often a high-dimensional, topologically non-trivial, curved space, and the fact that its algebraic operations are *non-linear operations*.

An example

An example that we come across from time to time and that fits the definition of a Lie group is the rotation group $(SO(2)\&SO(3))$

Take the SO(2) for example:

$$\mathrm{SO}(2)=\left\{g \in \mathbb{R}^{2 \times 2}: g^T g=1, \det g=1\right\}$$

The rotationnal group of the two-dimensional plane, consisting of orthogonal 2×2 matrices with determinant 1 .

The group elements $g \in \mathrm{SO}(2)$ can be parametrized by a single real parameter, the (counterclockwise) rotation angle α

$$g(\alpha)=\left(\begin{array}{cc}\cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha\end{array}\right)$$

Definition(Lie group)

A formal definition:

A Lie group of dimension n is a group $(G, \cdot)$ whose set G possesses the additional structure of a real n-dimensional smooth manifold, such that the group structures, composition $(g, h) \rightarrow g \cdot h$ and inversion $g \rightarrow g^{-1}$, are smooth maps.

Generators and Lie algebra

As we said above, Lie group is interesting and powerful, but a direct study of him has great difficulties. Therefore, as we have done in mathematical analysis and differential geometry, we have to straighten the *curved object and approximate the curved object with a straight line.*

> On differential manifolds, we can study **tangent vectors** at any point along the direction of the differential manifold, and then observe the behaviour of these vectors as they move along the differential manifold. The subtext of this statement is, of course, that "there are vectors that do not follow the direction of the differential manifold", which means that the classical differential geometry perspective of "putting the differential manifold into a higher dimensional space" is taken. This is certainly a viable perspective, but modern mathematicians and physicists prefer to study differential manifolds from an 'internal Bing' point of view, where the 'tangent vector' is defined as 'the derivative of a function on a differential manifold operation".
>
> In either view, the introduction of tangent vectors brought a **new algebraic structure** to what had been a very "non-linear, curved space": at each point we have a linear space of tangent vectors that can be added and multiplied. This is a great leap forward from before, when nothing algebraic could be done on a smooth differential manifold. This is often referred to as "linearization".

The Lie group is therefore a differential manifold, consisting of an infinite number of points. We wish to study it in a linearised way.

How do we begin? Let us turn our attention to *group multiplication*: any point $g \in G$ can be moved back to the unit element e by its inverse g^{-1} left action, i.e. $L_{g^{-1}} g=e$, while also moving all the tangent vectors at g to e.

What this means in effect is that the points on the Lie group G are equal, and there is no difference between all points (group elements) at the level of differential geometry, so we can just pick the simplest point, say e, and study its (all) tangent vectors.

reference(optionnal):

Naive Lie theory, John Stillwell, CHP4&5, (*The exponential map&The tangent space*)

(It is much more detailed and well-written than my notes. And, of course, much thicker. So I chose not to read it)

> Let us parametrize the elements $g(\alpha)$ in a neighbourhood of the identity of the Lie group with the convention that $\alpha_a=0$ corresponds to the identity, $g(0)=e=1$. It turns out that a lot of information on the Lie group can be recovered by studying the infinitesimal vicinity of the identity. The generators play an important role in this.
>
> *-Budd*

Definition(Lie algebra)

In summary, we obtain a (rough) induced definition of the Lie algebra $\mathfrak{g}$: the tangent space $\mathfrak{g} \equiv T_e G$ of a Lie group G at the unit element is called the Lie algebra of the Lie group G, also noted as $\mathfrak{g} \equiv \operatorname{Lie} G$.

The exponential map and tagent space, and some of their properties

More precisely, for any curve $g(t)$ in G passing through $e \in G$, we can use "derivatives" to extract its tangent vector at e:

$$X \equiv \left.\frac{d}{dt}\right|_{t=0} g(t)$$

When we have finished exhausting all possible curves passing through e, we can exhaust all the tangent vectors at e and obtain a tangent space $T_e G$.

In turn, using the tools of differential geometry, we can in turn define an "exponential map" $\exp : \mathfrak{g} \rightarrow G$ (or simply e^X) which satisfies some simple properties:

- For any $X \in \mathfrak{g}$, $\exp\left(t_1 X\right) \exp\left(t_2 X\right) = \exp\left(\left(t_1 + t_2\right) X\right)$
- For any $X \in \mathfrak{g}$, $\exp(X) \exp(-X) = e$
- For any $X \in \mathfrak{g}$, $\exp(0X) = e$
- $d/\left.dt\right|_{t=0}$ and $\exp$ "Mutually Inverse":

  $$\left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X,$$

- We can linearise the group multiplication into vector addition

  $$\left.\frac{d}{dt}\right|_{t=0} \exp(tX) \exp(tY) = X + Y.$$

  This relationship can of course be extended to any two curves $g_1(t), g_2(t)$ that pass through e.

  $$\left.\frac{d}{dt}\right|_{t=0} g_i(t) = X_i \quad \Rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} g_1(t) g_2(t) = X_1 + X_2.$$

In the language of differential geometry, given $X \in \mathfrak{g}$, the curve $\gamma_X(t) \equiv \exp(tX)$ defined by the exponential map forms a "one-parameter subgroup" in G satisfying the condition:

$$\gamma_X(0) = e, \quad \gamma_X\left(t_1 + t_2\right) = \gamma_X\left(t_1\right) \gamma_X\left(t_2\right), \quad \left.\frac{d}{dt}\right|_{t=0} \gamma_X(t) = X$$