Some ideas about Tomography

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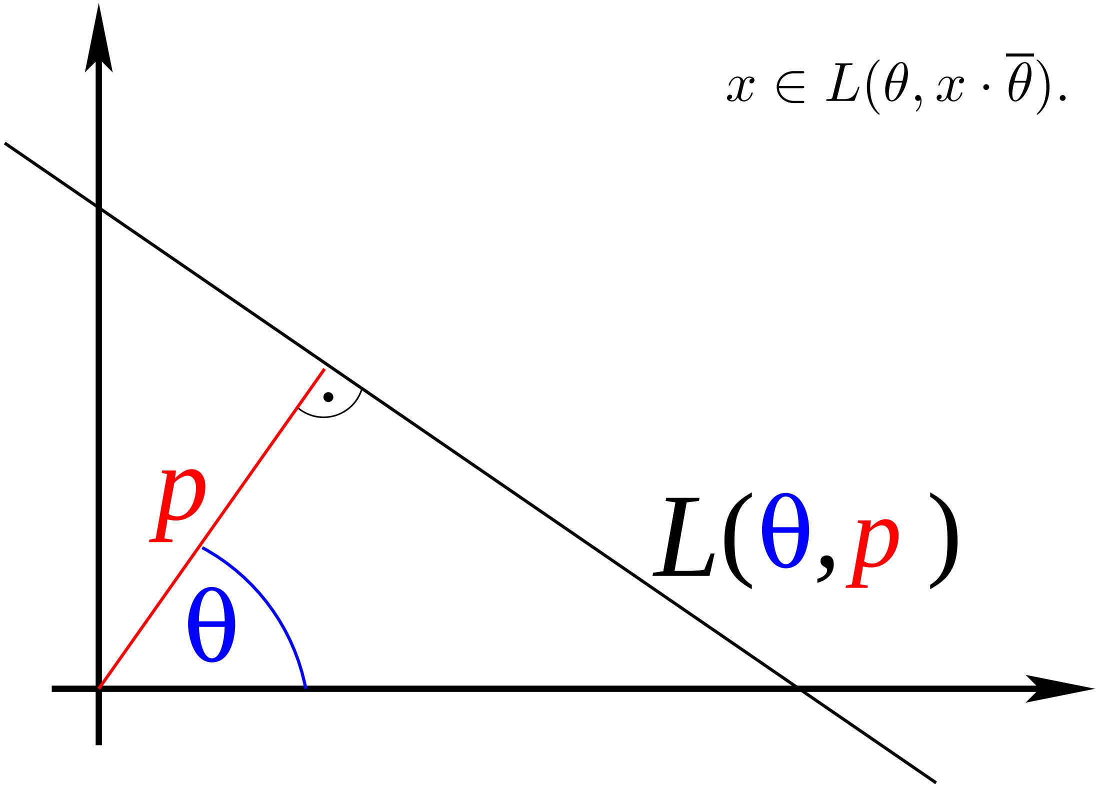
2. *Microlocal analysis (pure math) can show what features are stably visible from tomographic data.*
   - It provides the relation between singularities of an object and those of its tomographic data ([Q 1993], [NC]).

Radon Transform:

$$L(\theta, p) = \{x \in \mathbb{R}^2 \,|\, x \cdot \overline{\theta} = p\} \qquad \overline{\theta} = (\cos\theta, \sin\theta).$$

$$x \in L(\theta, x \cdot \overline{\theta}).$$



$$Rf(\theta, p) = \int_{x \in L(\theta,p)} f(x) ds$$

Notation: $f = f(x), \quad g = g(\theta, p)$

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Fourier Transform: $\mathcal{F} f(y) = \dfrac{1}{2\pi} \displaystyle\int_{x \in \mathbb{R}^2} e^{-ix \cdot y} f(x)\, dx$

1-D Fourier Transform: $\mathcal{F}_p g(\theta, \tau) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} e^{-ip\tau} g(\theta, p)\, dp$

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Lambda Operators:

$$\Lambda_p g(\theta, p) = \sqrt{-\frac{d^2}{dp^2}}\, g = \frac{1}{\sqrt{2\pi}} \int_{\tau=-\infty}^{\infty} e^{ip\tau}\, |\tau|\, \big(\mathcal{F}_p g\big)(\theta, \tau) d\tau,$$

$$\Lambda_x f = \sqrt{-\Delta} f = \frac{1}{2\pi} \int_{\xi\in\mathbb{R}^2} e^{ix\cdot\xi} |\xi| \widehat{f}(\xi) d\xi.$$

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$$f = \frac{1}{4\pi} R^* \Lambda_p R f \approx \frac{1}{4\pi} R^* \big(\varphi(p) *_p R f\big)$$

- FBP reconstructs f.
- Reconstruction **is not** local as the approximate to Λ_p, $\phi(p)$, is nonzero "everywhere."

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- **Key point:** Λ_x emphasizes singularities of f

(Λ_x is an elliptic ΨDO.)