

# ALGEBRA QUALIFYING EXAM PROBLEMS RING THEORY

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# Contents

<b>1</b>	<b>Ring Theory</b>	<b>3</b>
1.1	General Ring Theory . . . . .	3
1.2	Prime, Maximal, and Primary Ideals . . . . .	7
1.3	Commutative Rings . . . . .	9
1.4	Domains . . . . .	10
1.5	Polynomial Rings . . . . .	12
1.6	Non-commutative Rings . . . . .	12
1.7	Local Rings, Localization, Rings of Fractions . . . . .	13
1.8	Chains and Chain Conditions . . . . .	14

# 1 Ring Theory

## 1.1 General Ring Theory

1. Give an example of each of the following

- (a) An irreducible polynomial of degree 3 in  $\mathbb{Z}_3[x]$
- (b) A polynomial in  $\mathbb{Z}[x]$  that is not irreducible in  $\mathbb{Z}[x]$  but is irreducible in  $\mathbb{Q}[x]$
- (c) A non-commutative ring of characteristic  $p$ ,  $p$  a prime
- (d) A ring with exactly 6 invertible elements
- (e) An infinite non-commutative ring with only finitely many ideals.
- (f) An infinite non-commutative ring with non-zero characteristic.
- (g) An integral domain which is not a unique factorization domain.
- (h) A unique factorization domain that is not a principal ideal domain
- (i) A principal ideal domain that is not a Euclidean domain.
- (j) A Euclidean domain other than the ring of integers of a field.
- (k) A finite non-commutative ring.
- (l) A commutative ring with a sequence  $\{P_n\}_{n=1}^{\infty}$  of prime ideals such that  $P_n$  is properly contained in  $P_{n+1}$  for all  $n$ .
- (m) A non-zero prime ideal of a commutative ring that is not a maximal ideal.
- (n) An irreducible element of a commutative ring that is not a prime element.
- (o) An irreducible element of an integral domain that is not a prime element.
- (p) A commutative ring that has exactly one maximal ideal and is not a field.
- (q) A non-commutative ring with exactly two maximal ideals.

2. (a) How many units does the ring  $\mathbb{Z}/60\mathbb{Z}$  have? Explain your answer.  
(b) How many ideals does the ring  $\mathbb{Z}/60\mathbb{Z}$  have? Explain your answer.

*Proof.* For (a), we can use the fact that  $a$  is a unit if  $\gcd(a, n) = 1$ . Thus  $|U_{60}| = |\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 57, 59\}| = 16$ .

For (b) use the fact that  $\mathbb{Z}/d\mathbb{Z}$  is an ideal of  $\mathbb{Z}/n\mathbb{Z}$  when  $d \mid n$ . So let  $D = \{d \in \mathbb{Z}/60\mathbb{Z} : d \mid 60\} = \{2^x \cdot 3^y \cdot 5^z : 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ . Thus the number of ideals is equal to  $|D| = 3 \cdot 2 \cdot 2 = 12$   $\square$

3. How many ideals does the ring  $\mathbb{Z}/90\mathbb{Z}$  have? Explain your answer.

*Proof.* Use the fact that  $\mathbb{Z}/d\mathbb{Z}$  is an ideal of  $\mathbb{Z}/n\mathbb{Z}$  when  $d \mid n$ . Let  $D = \{d \in \mathbb{Z}/90\mathbb{Z} : d \mid 90\} = \{2^x \cdot 3^y \cdot 5^z : 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1\}$ . Then the number of ideals is equal to  $|D| = 2 \cdot 3 \cdot 2 = 12$ .  $\square$

4. Denote the set of invertible elements of the ring  $\mathbb{Z}_n$  by  $U_n$ .

- (a) List all the elements of  $U_{18}$ .
- (b) Is  $U_{18}$  a cyclic group under multiplication? Justify your answer.

*Proof.* Use the fact that  $a$  is a unit if  $\gcd(a, 18) = 1$ . So the invertible elements of  $\mathbb{Z}_{18}$  are  $U_{18} = \{1, 5, 7, 11, 13, 17\}$ . Note that  $\langle 5 \rangle = \{5, 7, 17, 13, 11, 1\} = U_{18}$ . Therefore  $U_{18}$  is cyclic.  $\square$

5. Denote the set of invertible elements of the ring  $\mathbb{Z}_n$  by  $U_n$ .

- (a) List all the elements of  $U_{24}$ .
- (b) Is  $U_{24}$  a cyclic group under multiplication? Justify your answer.

*Proof.* Use the fact that  $a$  is a unit if  $\gcd(a, 24) = 1$ . So the invertible elements of  $\mathbb{Z}_{18}$  are  $U_{24} = \{1, 5, 7, 11, 13, 17, 19, 23\}$ . Note that  $x^2 = 1$  for all  $x \in U_{24}$ . Therefore  $U_{24}$  is not cyclic.  $\square$

6. Find all positive integers  $n$  having the property that the group of units of  $\mathbb{Z}/n\mathbb{Z}$  is an elementary abelian 2-group.
7. Let  $U(R)$  denote the group of units of a ring  $R$ . Prove that if  $m$  divides  $n$ , then the natural ring homomorphism  $\mathbb{Z}_n \rightarrow \mathbb{Z}_m$  maps  $U(\mathbb{Z}_n)$  onto  $U(\mathbb{Z}_m)$ . Give an example that shows that  $U(R)$  does not have to map onto  $U(S)$  under a surjective ring homomorphism  $R \rightarrow S$ .
8. If  $p$  is a prime satisfying  $p \equiv 1 \pmod{4}$ , then  $p$  is a sum of two squares.
9. If  $(\div)$  denotes the Legendre symbol, prove Euler's Critereon: if  $p$  is a prime and  $a$  is any integer relatively prime to  $p$ , then  $a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \pmod{p}$ .
10. Let  $R_1$  and  $R_2$  be commutative rings with identities and let  $R = R_1 \times R_2$ . Show that every ideal  $I$  of  $R$  is of the form  $I = I_1 \times I_2$  with  $I_i$  an ideal of  $R_i$  for  $i = 1, 2$ .

*Proof.* Let  $I$  be an ideal fo  $R = R_1 \times R_2$ . Define  $I_1 = \{r \in R_1 : \exists s \in R_2 \text{ where } (r, s) \in I\}$  and  $I_2 = \{s \in R_2 : \exists r \in R_1 \text{ where } (r, s) \in I\}$ .

Note that if  $r \in I_1$ , then there exists  $s \in R_2$  such that  $(r, s) \in I$ . Thus  $(a, 1_{R_2})(r, s) = (ar, s) \in I$ , and so  $ar \in I_1$ . Thus  $I_1$  is an ideal of  $R_1$ . By a similar argument, we can see that  $I_2$  is an ideal of  $R_2$ . Now it is clear that  $I \subseteq I_1 \times I_2$ .

To show the reverse inclusion, suppose that  $r \in I_1$ , and  $s \in I_2$  such that  $(r, s) \in I_1 \times I_2$ . Since  $r \in I_1$ , then there is an  $s' \in I_2$  such that  $(r, s') \in I$ . Also since  $s \in I_2$ , then there exists an  $r' \in I_1$  such that  $(r', s) \in I$ . Now we see that  $(1_R, s)(r, s') = (r, ss') \in I$ , and  $(1_R, s')(r', s) = (r', ss') \in I$ . Thus  $(r, ss') - (r', ss') + (r', s) = (r, s) \in I$ . Therefore  $I_1 \times I_2 \subseteq I$  so  $I = I_1 \times I_2$ , as desired.  $\square$

11. Show that a non-zero ring  $R$  in which  $x^2 = x$  for all  $x \in R$  is of characteristic 2 and is commutative.

*Proof.* Let  $x \in R$  and note that  $x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$ . Thus  $x + x = 0$ , so  $R$  has characteristic zero.

Now let  $x, y \in R$ . Then  $x + y = (x + y)^2 = x^2 + xy + yx + y^2$ . Rearranging terms gives  $0 = xy + yx$ . Now  $xy = xy + xy + yx = (xy + xy) + yx = 0 + yx = yx$ . Thus we conclude that  $R$  is commutative.  $\square$

12. Let  $R$  be a finite commutative ring with more than one element and no zero-divisors. Show that  $R$  is a field.

*Proof.* Define  $\phi : R \rightarrow R$  by  $\phi(r) = xr$  for all  $r \in R$  and a fixed nonzero element  $x \in R$ . Suppose that for two elements  $r, s \in R$ ,  $\phi(r) = \phi(s)$ . Then we have  $xr = xs$  or  $xr - xs = 0$  or  $x(r - s) = 0$ . Since  $x$  is nonzero and  $R$  has no zero divisors, we see that  $r - s = 0$ , and so  $r = s$ . Thus the map  $\phi$  is injective. Since we are mapping from  $R \rightarrow R$ , then  $\phi$  must also be surjective and thus bijective.

First we show that  $R$  has an identity element. By the surjectivity of  $\phi$  there must be an element  $e \in R$  such that  $x = \phi(e) = ex$ . Now let  $b \in R$ . Then by the surjectivity of  $\phi$  there is an element  $y_b \in R$  such that  $b = \phi(y_b) = y_b x$ . Now we have  $b = y_b x = y_b ex = (y_b x)e = be$ . Thus  $e = 1$  since it fixes every element in  $R$ .

Now again by the surjectivity of  $\phi$ , we must have some  $r \in R$  such that  $\phi(r) = rx = 1$ . Thus  $x$  is a unit and since the choice of  $x$  was arbitrary (except for the restriction of  $x$  being nonzero), we conclude that  $R$  must be a field.  $\square$

13. Determine for which integers  $n$  the ring  $\mathbb{Z}/n\mathbb{Z}$  is a direct sum of fields. Prove your answer.
14. Let  $R$  be a subring of a field  $F$  such that for each  $x$  in  $F$  either  $x \in R$  or  $x^{-1} \in R$ . Prove that if  $I$  and  $J$  are ideals of  $R$ , then either  $I \subseteq J$  or  $J \subseteq I$ .

*Proof.* Suppose  $I \not\subseteq J$ . It suffices to show that  $J \subseteq I$ . Since  $I \not\subseteq J$ , then there is a nonzero  $a \in I \setminus J$ . If  $b \in J$ , we want to show that  $b \in I$ .

If  $b = 0$ , then we are done since  $I$  is an ideal, and must contain zero. Now suppose  $b \neq 0$ . Then there is an inverse  $b^{-1} \in F$  and so  $ab^{-1} \in F$ . Thus either  $ab^{-1} \in R$  or  $ba^{-1} \in R$ . If  $ab^{-1} \in R$ , then  $a = ab^{-1}b \in J$ , a contradiction.

Thus it must be the case that  $ba^{-1} \in R$ , and so  $b = ba^{-1}a \in I$ . Therefore  $b \in I$  and  $J \subseteq I$ , as desired.  $\square$

15. The *Jacobson Radical*  $J(R)$  of a ring  $R$  is defined to be the intersection of all maximal ideals of  $R$ . Let  $R$  be a commutative ring with 1 and let  $x \in R$ . Show that  $x \in J(R)$  if and only if  $1 - xy$  is a unit for all  $y$  in  $R$ .

*Proof.* Suppose that  $x \in J(R)$ . Then  $x \in I_i$  where  $I_i$  is a maximal ideal. Then  $xy \in I_i$  for all  $i$ , and so  $xy \in J(R)$ . Thus  $xy$  is quasiregular and so  $1 - xy$  is a unit for all  $y$  in  $R$ .

Now suppose that  $1 - xy$  is a unit for all  $y \in R$  and  $x \notin J(R)$ . Then there exists a maximal ideal  $M$  such that  $x \notin M$ . Thus  $\langle x \rangle + M = R$ . So there exists a  $y \in R$  and  $m \in M$  such that  $xy + m = 1$ . Thus  $m = 1 - xy \in M$  is a unit. So we must have  $M = R$ , but this is a contradiction to the maximality of  $M$ .  $\square$

16. Let  $R$  be any ring with identity, and  $n$  any positive integer. If  $M_n(R)$  denotes the ring of  $n \times n$  matrices with entries in  $R$ , prove that  $M_n(I)$  is an ideal of  $M_n(R)$  whenever  $I$  is an ideal of  $R$ , and that every ideal of  $M_n(R)$  has this form.
17. Let  $m, n$  be positive integers such that  $m$  divides  $n$ . Then the natural map  $\phi : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$  given by  $a + (n) \mapsto a + (m)$  is a surjective ring homomorphism. If  $U_n, U_m$  are the units of  $\mathbb{Z}_n$  and  $\mathbb{Z}_m$ , respectively, show that  $\phi : U_n \rightarrow U_m$  is a surjective group homomorphism.
18. Let  $R$  be a ring with the ideals  $A$  and  $B$ . Let  $R/A \times R/B$  be the ring with coordinate-wise addition and multiplication. Show the following.
- (a) The map  $R \rightarrow R/A \times R/B$  given by  $r \mapsto (r + A, r + B)$  is a ring homomorphism.
  - (b) The homomorphism in part (a) is surjective if and only if  $A + B = R$ .
19. Let  $m$  and  $n$  be relatively prime integers.
- (a) Show that if  $c$  and  $d$  are any integers, then there is an integer  $x$  such that  $x \equiv c \pmod{m}$  and  $x \equiv d \pmod{n}$ .
  - (b) Show that  $\mathbb{Z}_{mn}$  and  $\mathbb{Z}_m \times \mathbb{Z}_n$  are isomorphic as rings.
20. Let  $R$  be a commutative ring with 1 and let  $I$  and  $J$  be ideals of  $R$  such that  $I + J = R$ . Show that  $R/(I \cap J) \cong R/I \oplus R/J$ .
21. Let  $R$  be a commutative ring with identity and let  $I_1, I_2, \dots, I_n$  be pairwise co-maximal ideals of  $R$  (i.e.,  $I_i + I_j = R$  if  $i \neq j$ ). Show that  $I_i + \bigcap_{j \neq i} I_j = R$  for all  $i$ .
22. Let  $R$  be a commutative ring, not necessarily with identity, and assume there is some fixed positive integer  $n$  such that  $nr = 0$  for all  $r \in R$ . Prove that  $R$  embeds in a ring  $S$  with identity so that  $R$  is an ideal of  $S$  and  $S/R \cong \mathbb{Z}/n\mathbb{Z}$ .
23. Let  $R$  be a ring with identity 1 and  $a, b \in R$  such that  $ab = 1$ . Denote  $X = \{x \in R \mid ax = 1\}$ . Show the following.
- (a) If  $x \in X$ , then  $b + (1 - x) \in X$ .
  - (b) If  $\phi : X \rightarrow X$  is the mapping given by  $\phi(x) = b + (1 - xa)$ , then  $\phi$  is one-to-one.
  - (c) If  $X$  has more than one element, then  $X$  is an infinite set.
24. Let  $R$  be a commutative ring with identity and define  $U_2(R) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in R \right\}$ . Prove that every  $R$ -automorphism of  $U_2(R)$  is inner.

25. Let  $\mathbb{R}$  be the field of real numbers and let  $F$  be the set of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ -3b & a \end{bmatrix}$ , where  $a, b \in \mathbb{R}$ . Show that  $F$  is a field under the usual matrix operations.
26. Let  $R$  be the ring of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$  where  $a$  and  $b$  are real numbers. Prove that  $R$  is isomorphic to  $\mathbb{C}$ , the field of complex numbers.
27. Let  $p$  be a prime and let  $R$  be the ring of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ pb & a \end{bmatrix}$ , where  $a, b \in \mathbb{Z}$ . Prove that  $R$  is isomorphic to  $\mathbb{Z}[\sqrt{p}]$ .
28. Let  $p$  be a prime and  $F_p$  the set of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ , where  $a, b \in \mathbb{Z}_p$ .
- Show that  $F_p$  is a commutative ring with identity.
  - Show that  $F_7$  is a field.
  - Show that  $F_{13}$  is not a field.
29. Let  $I \subseteq J$  be right ideals of a ring  $R$  such that  $J/I \cong R$  as right  $R$ -modules. Prove that there exists a right ideal  $K$  such that  $I \cap K = (0)$  and  $I + K = J$ .
30. A ring  $R$  is called simple if  $R^2 \neq 0$  and  $0$  and  $R$  are its only ideals. Show that the center of a simple ring is  $0$  or a field.
31. Give an example of a field  $F$  and a one-to-one ring homomorphism  $\phi : F \rightarrow F$  which is not onto. Verify your example.
32. Let  $D$  be an integral domain and let  $D[x_1, x_2, \dots, x_n]$  be the polynomial ring over  $D$  in the  $n$  indeterminates  $x_1, x_2, \dots, x_n$ . Let

$$V = \begin{bmatrix} x_1^{n-1} & \dots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & \dots & x_2^2 & x_2 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ x_n^{n-1} & \dots & x_n^2 & x_n & 1 \end{bmatrix}.$$

Prove that the determinant of  $V$  is  $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ .

33. Let  $R = C[0, 1]$  be the set of all continuous real-valued functions on  $[0, 1]$ . Define addition and multiplication on  $R$  as follows. For  $f, g \in R$  and  $x \in [0, 1]$ ,
- $$(f + g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x).$$
- Show that  $R$  with these operations is a commutative ring with identity.
  - Find the units of  $R$ .
  - If  $f \in R$  and  $f^2 = f$ , then  $f = 0_R$  or  $f = 1_R$ .
  - If  $n$  is a positive integer and  $f \in R$  is such that  $f^n = 0_R$ , then  $f = 0_R$ .
34. Let  $S$  be the ring of all bounded, continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  is the set of real numbers. Let  $I$  be the set of functions  $f$  in  $S$  such that  $f(t) \rightarrow 0$  as  $|t| \rightarrow \infty$ .
- Show that  $I$  is an ideal of  $S$ .
  - Suppose  $x \in S$  is such that there is an  $i \in I$  with  $ix = x$ . Show that  $x(t) = 0$  for all sufficiently large  $|t|$ .
35. Let  $\mathbb{Q}$  be the field of rational numbers and  $D = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ .
- Show that  $D$  is a subring of the field of real numbers.
  - Show that  $D$  is a principal ideal domain.
  - Show that  $\sqrt{3}$  is not an element of  $D$ .
36. Show that if  $p$  is a prime such that  $p \equiv 1 \pmod{4}$ , then  $x^2 + 1$  is not irreducible in  $\mathbb{Z}_p[x]$ .
37. Show that if  $p$  is a prime such that  $p \equiv 4 \pmod{4}$ , then  $x^2 + 1$  is irreducible in  $\mathbb{Z}_p[x]$ .
38. Show that if  $p$  is a prime such that  $p \equiv 1 \pmod{6}$ , then  $x^3 + 1$  splits in  $\mathbb{Z}_p[x]$ .

## 1.2 Prime, Maximal, and Primary Ideals

39. Let  $R$  be a non-zero commutative ring with 1. Show that an ideal  $M$  of  $R$  is maximal if and only if  $R/M$  is a field.

*Proof.* By the correspondence theorem,  $M$  is a maximal ideal of  $R$  if and only if  $0$  is a maximal ideal of  $R/M$ . Also, it is known that a commutative ring is a field if and only if  $0$  is a maximal ideal. Thus the result follows.  $\square$

40. Let  $R$  be a commutative ring with 1. Show that an ideal  $P$  of  $R$  is prime if and only if  $R/P$  is an integral domain.

*Proof.* First let  $P$  be a prime ideal of  $R$ . Then  $R/P$  is a prime ring. Let  $a, b \in R/P$  such that  $ab = 0$ . Then we can say that  $(a)(b) = 0$  where  $(a)$  and  $(b)$  are ideals generated by  $a$  and  $b$  respectively in  $R/P$ . But then we know that either  $(a) = 0$  or  $(b) = 0$  since  $R/P$  is a prime ring. Therefore, we must have  $a = 0$  or  $b = 0$  and so  $R/P$  is an integral domain.

Now suppose that  $R/P$  is an integral domain. Suppose  $A$  and  $B$  are ideals such that  $AB = 0$ . Then pick  $a_1 \in A$ . Then, since  $R/P$  is an integral domain, we have  $a_1 b_i = 0$  for all  $b_i \in B$  or  $a_1 = 0$ . Continuing in this way, we must have either  $A = 0$  or  $B = 0$ . Thus  $R/P$  is prime so  $P$  is a prime ideal  $\square$

41. (a) Let  $R$  be a commutative ring with 1. Show that if  $M$  is a maximal ideal of  $R$  then  $M$  is a prime ideal of  $R$ .  
(b) Give an example of a non-zero prime ideal in a ring  $R$  that is not a maximal ideal.

*Proof.* For (a) since  $M$  is a maximal ideal then  $R/M$  is a field and thus an integral domain. Since  $R/M$  is an integral domain, then  $M$  is a prime ideal of  $R$ .

For (b) note that  $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$  so  $(x)$  is a prime ideal of  $\mathbb{Z}[x]$  but  $(x)$  is not maximal in  $\mathbb{Z}[x]$  since  $\mathbb{Z}$  is not a field.  $\square$

42. Let  $R$  be a non-zero ring with identity. Show that every proper ideal of  $R$  is contained in a maximal ideal.

*Proof.* Let  $I \subsetneq R$  be a proper ideal. Define the set  $P = \{A \mid I \subseteq A \subsetneq R \text{ is an ideal of } R\}$ . Let  $\mathcal{C}$  be a chain of ideals in  $P$ . Then  $U_{\mathcal{C}} = \bigcup_{A \in \mathcal{C}} A$  is an ideal of  $R$  containing  $I$  and is an upper bound for  $\mathcal{C}$ . Notice that  $1 \notin U_{\mathcal{C}}$  since  $1$  was not in any of the ideals from  $\mathcal{C}$  as the ideals in  $\mathcal{C}$  are proper in  $R$ . Thus we have that  $U_{\mathcal{C}} \in P$ . Now we can apply Zorn's Lemma. So there exists a maximal element  $\mathcal{A}$  of the set  $P$ . It follows that  $I \subseteq \mathcal{A} \subsetneq R$  and so  $I$  is contained in the maximal ideal  $\mathcal{A}$ .  $\square$

43. Let  $R$  be a commutative ring with 1 and  $P$  a prime ideal of  $R$ . Show that if  $I$  and  $J$  are ideals of  $R$  such that  $I \cap J \subseteq P$  and  $J \not\subseteq P$ , then  $I \subseteq P$ .  
44. Let  $M_1 \neq M_2$  be two maximal ideals in the commutative ring  $R$  and let  $I = M_1 \cap M_2$ . Prove that  $R/I$  is isomorphic to the direct sum of two fields.  
45. Let  $R$  be a non-zero commutative ring with 1. Show that if  $I$  is an ideal of  $R$  such that  $1 + a$  is a unit in  $R$  for all  $a \in I$ , then  $I$  is contained in every maximal ideal of  $R$ .  
46. Let  $R$  be a commutative ring with identity. Suppose  $R$  contains an idempotent element  $a$  other than 0 or 1. Show that every prime ideal of  $R$  contains an idempotent element other than 0 or 1. (An element  $a \in R$  is idempotent if  $a^2 = a$ )  
47. Let  $R$  be a commutative ring with 1.

- (a) Prove that  $(x)$  is a prime ideal in  $R[x]$  if and only if  $R$  is an integral domain.  
(b) Prove that  $(x)$  is a maximal ideal in  $R[x]$  if and only if  $R$  is a field.

48. Find all values of  $a$  in  $\mathbb{Z}_3$  such that the quotient ring

$$\mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1)$$

is a field. Justify your answer.

49. Find all values of  $a$  in  $\mathbb{Z}_5$  such that the quotient ring

$$\mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3)$$

is a field. Justify your answer.

50. Let  $R$  be a commutative ring with identity and let  $U$  be maximal among non-finitely generated ideals of  $R$ . Prove  $U$  is a prime ideal.
51. Let  $R$  be a commutative ring with identity and let  $U$  be a maximal among non-principal ideals of  $R$ . Prove  $U$  is a prime ideal.
52. Let  $R$  be a non-zero commutative ring with 1 and  $S$  a multiplicative subset of  $R$  not containing 0. Show that if  $P$  is maximal in the set of ideals of  $R$  not intersecting  $S$ , then  $P$  is a prime ideal.
53. Let  $R$  be a non-zero commutative ring with 1.

(a) Let  $S$  be a multiplicative subset of  $R$  not containing 0 and let  $P$  be maximal in the set of ideals of  $R$  not intersecting  $S$ . Show that  $P$  is a prime ideal.

(b) Show that the set of nilpotent elements of  $R$  is the intersection of all prime ideals.

54. Let  $R$  be a commutative ring with identity and let  $x \in R$  be a non-nilpotent element. Prove that there exists a prime ideal  $P$  of  $R$  such that  $x \notin P$ .
55. Let  $R$  be a commutative ring with identity and let  $S$  be the set of all elements of  $R$  that are *not* zero-divisors. Show that there is a prime ideal  $P$  such that  $P \cap S$  is empty. (Hint: Use Zorn's Lemma)
56. Let  $R$  be a commutative ring with identity and let  $\mathcal{C}$  be a chain of prime ideals of  $R$ . Show that  $\bigcup_{P \in \mathcal{C}} P$  and  $\bigcap_{P \in \mathcal{C}} P$  are prime ideals of  $R$ .
57. Let  $R$  be a commutative ring and  $P$  be a prime ideal of  $R$ . Show that there is a prime ideal  $P_0 \subseteq P$  that does not properly contain any prime ideal.
58. Let  $R$  be a commutative ring with 1 such that every  $x$  in  $R$  there is an integer  $n > 1$  (depending on  $x$ ) such that  $x^n = x$ . Show that every prime ideal of  $R$  is maximal.
59. Let  $R$  be a commutative ring with 1 in which every ideal is a prime ideal. Prove that  $R$  is a field. (Hint: For  $a \neq 0$  consider the ideals  $(a)$  and  $(a^2)$ .)
60. Let  $D$  be a principal ideal domain. Prove that every nonzero prime ideal of  $D$  is a maximal ideal.
61. Show that if  $R$  is a finite commutative ring with identity, then every prime ideal of  $R$  is a maximal ideal.
62. Let  $R = C[0, 1]$  be the ring of all continuous real-valued functions on  $[0, 1]$ , with addition and multiplication defined as follows. For  $f, g \in R$  and  $x \in [0, 1]$ ,

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x).$$

Prove that if  $M$  is a maximal ideal of  $R$ , then there is a real number  $x_0 \in [0, 1]$  such that  $M = \{f \in R \mid f(x_0) = 0\}$ .

63. Let  $R$  be a commutative ring with identity, and let  $P \subset Q$  be prime ideals of  $R$ . Prove there exists prime ideals  $P^*, Q^*$  satisfying  $P \subseteq P^* \subset Q^* \subseteq Q$ , such that there are no prime ideals strictly between  $P^*$  and  $Q^*$ . (Hint: Fix  $x \in Q \setminus P$  and show that there exists a prime ideal  $P^*$  containing  $P$ , contained in  $Q$  and maximal with respect to not containing  $x$ .)



64. Let  $R$  be a commutative ring with 1. An ideal  $I$  of  $R$  is called a *primary* ideal if  $I \neq R$  and for all  $x, y \in R$  with  $xy \in I$ , either  $x \in I$  or  $y^n \in I$  for some integer  $n \geq 1$ .
- (a) Show that an ideal  $I$  of  $R$  is primary if and only if  $R/I \neq 0$  and every zero-divisor in  $R/I$  is nilpotent.
  - (b) Show that if  $I$  is a primary ideal of  $R$  then the radical  $\text{Rad}(I)$  of  $I$  is a prime ideal. (Recall that  $\text{Rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n\}$ .)

### 1.3 Commutative Rings

65. Let  $R$  be a commutative ring with identity. Show that  $R$  is an integral domain if and only if  $R$  is a subring of a field.
66. Let  $R$  be a commutative ring with identity. Show that if  $x$  and  $y$  are nilpotent elements of  $R$  then  $x + y$  is nilpotent and the set of all nilpotent elements is an ideal in  $R$ .
67. Let  $R$  be a commutative ring with identity. An ideal  $I$  of  $R$  is *irreducible* if it cannot be expressed as the intersection of two ideals of  $R$  neither of which is contained in the other. Show the following.
- (a) If  $P$  is a prime ideal then  $P$  is irreducible.
  - (b) If  $x$  is a non-zero element of  $R$ , then there is an ideal  $I_x$ , maximal with respect to the property that  $x \notin I_x$ , and  $I_x$  is irreducible.
  - (c) If every irreducible ideal of  $R$  is a prime ideal, then 0 is the only nilpotent element of  $R$ .
68. Let  $R$  be a commutative ring with 1 and let  $I$  be an ideal of  $R$  satisfying  $I^2 = \{0\}$ . Show that if  $a + I \in R/I$  is an idempotent element of  $R/I$ , then the coset  $a + I$  contains an idempotent element of  $R$ .
69. Let  $R$  be a commutative ring with identity that has exactly one prime ideal  $P$ . Prove the following
- (a)  $R/P$  is a field.
  - (b)  $R$  is isomorphic to  $R_P$ , the ring of quotients of  $R$  with respect to the multiplicative set  $R \setminus P = \{s \in R \mid s \notin P\}$
70. Let  $R$  be a commutative ring with identity and  $\sigma : R \rightarrow R$  a ring automorphism.
- (a) Show that  $F = \{r \in R \mid \sigma(r) = r\}$  is a subring of  $R$  and the identity of  $R$  is in  $F$ .
  - (b) Show that if  $\sigma^2$  is the identity map on  $R$ , then each element of  $R$  is the root of a monic polynomial of degree two in  $F[x]$ .
71. Let  $R$  be a commutative ring with identity that has exactly three ideals  $\{0\}$ ,  $I$ , and  $R$ .
- (a) Show that if  $a \notin I$ , then  $a$  is a unit of  $R$ .
  - (b) Show that if  $a, b \in I$  then  $ab = 0$
72. Let  $R$  be a commutative ring with 1. Show that if  $u$  is a unit in  $R$  and  $n$  is nilpotent, then  $u + n$  is a unit.
73. Let  $R$  be a commutative ring with identity. Suppose that for every  $a \in R$ , either  $a$  or  $1 - a$  is invertible. Prove that  $N = \{a \in R \mid a \text{ is not invertible}\}$  is an ideal of  $R$ .
74. Let  $R$  be a commutative ring with 1. Show that the sum of any two principal ideals of  $R$  is principal if and only if every finitely generated ideal of  $R$  is principal.
75. Let  $R$  be a commutative ring with identity such that not every ideal is a principal ideal.
- (a) Show that there is an ideal  $I$  maximal with respect to the property that  $I$  is not a principal ideal.
  - (b) If  $I$  is the ideal of part (a), show that  $R/I$  is a principal ideal ring.

76. Recall that if  $R \subseteq S$  is an inclusion of commutative rings (with the same identity) then an element  $s \in S$  is *integral over  $R$*  if  $s$  satisfies some monic polynomial with coefficients in  $R$ . Prove the equivalence of the following statements.
- (a)  $s$  is integral over  $R$ .
  - (b)  $R[s]$  is finitely generated as an  $R$ -module.
  - (c) There exists a faithful  $R[s]$  module which is finitely generated as an  $R$ -module.
77. Recall that if  $R \subseteq S$  is an inclusion of commutative rings (with the same identity) then  $S$  is an *integral extension* of  $R$  if every element of  $S$  satisfies some monic polynomial with coefficients in  $R$ . Prove that if  $R \subseteq S \subseteq T$  are commutative rings with the same identity, then  $S$  is integral over  $R$  and  $T$  if and only if  $T$  is integral over  $R$ .
78. Let  $R \subseteq S$  be commutative domains with the same identity, and assume that  $S$  is an integral Extension of  $R$ . Let  $I$  be a nonzero ideal of  $S$ . Prove the  $I \cap R$  is a nonzero ideal of  $R$ .

## 1.4 Domains

79. Suppose  $R$  is a domain and  $I$  and  $J$  are ideals of  $R$  such that  $IJ$  is principal. Show that  $I$  (and by symmetry  $J$ ) is finitely generated.  
[Hint: If  $IJ = (a)$ , then  $\sum_{i=1}^n x_i y_i$  for some  $x_i \in I$  and  $y_i \in J$ . Show the  $x_i$  generate  $I$ .]
80. Prove that if  $D$  is a Euclidean Domain, then  $D$  is a Principal Ideal Domain.
81. Show that if  $p$  is a prime such that there is an integer  $b$  with  $p = b^2 + 4$ , then  $\mathbb{Z}[\sqrt{p}]$  is not a unique factorization domain.
82. Show that if  $p$  is a prime such that  $p \equiv 1 \pmod{4}$ , then  $\mathbb{Z}[\sqrt{p}]$  is not a unique factorization domain.
83. Let  $D = \mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} \mid m, n \in \mathbb{Z}\}$  - a subring of the field of real numbers and necessarily an integral domain (you need not show this) - and  $F = \mathbb{Q}(\sqrt{5})$  its field of fractions. Show the following:
- (a)  $x^2 + x - 1$  is irreducible in  $D[x]$  but not in  $F[x]$ .
  - (b)  $D$  is not a unique factorization domain.
84. Let  $D = \mathbb{Z}(\sqrt{21}) = \{m + n\sqrt{21} \mid m, n \in \mathbb{Z}\}$  and  $F = \mathbb{Q}(\sqrt{21})$ , the field of fractions. Show the following:
- (a)  $x^2 - x - 5$  is irreducible in  $D[x]$  but not in  $F[x]$ .
  - (b)  $D$  is not a unique factorization domain.
85. Let  $D = \mathbb{Z}(\sqrt{-11}) = \{m + n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$  and  $F = \mathbb{Q}(\sqrt{-11})$ , the field of fractions. Show the following:
- (a)  $x^2 - x + 3$  is irreducible in  $D[x]$  but not in  $F[x]$ .
  - (b)  $D$  is not a unique factorization domain.
86. Let  $D = \mathbb{Z}(\sqrt{13}) = \{m + n\sqrt{13} \mid m, n \in \mathbb{Z}\}$  and  $F = \mathbb{Q}(\sqrt{13})$ , the field of fractions. Show the following:
- (a)  $x^2 + 3x - 1$  is irreducible in  $D[x]$  but not in  $F[x]$ .
  - (b)  $D$  is not a unique factorization domain.
87. Let  $D$  be an integral domain and  $F$  a subring of  $D$  that is a field. Show that if each element of  $D$  is algebraic over  $F$ , then  $D$  is a field.
88. Let  $R$  be an integral domain containing the subfield  $F$  and assume that  $R$  is finite dimensional over  $F$  when viewed as a vector space over  $F$ . Prove that  $R$  is a field.

89. Let  $D$  be an integral domain.
- (a) For  $a, b \in D$  define a *greatest common divisor* of  $a$  and  $b$ .
  - (b) For  $x \in D$  denote  $(x) = \{dx \mid d \in D\}$ . Prove that if  $(a) + (b) = (d)$ , then  $d$  is a greatest common divisor of  $a$  and  $b$ .
90. Let  $D$  be a principal ideal domain.
- (a) For  $a, b \in D$ , define a *least common multiple* of  $a$  and  $b$ .
  - (b) Show that  $d \in D$  is a least common multiple of  $a$  and  $b$  if and only if  $(a) \cap (b) = (d)$ .
91. Let  $D$  be a principal ideal domain and let  $a, b \in D$ .
- (a) Show that there is an element  $d \in D$  that satisfies the properties
    - i.  $d \mid a$  and  $d \mid b$  and
    - ii. if  $e \mid a$  and  $e \mid b$  then  $e \mid d$
  - (b) Show that there is an element  $m \in D$  that satisfies the properties
    - i.  $a \mid m$  and  $b \mid m$  and
    - ii. if  $a \mid e$  and  $b \mid e$  then  $m \mid e$ .
92. Let  $R$  be a principal ideal domain. Show that if  $(a)$  is a nonzero ideal in  $R$ , then there are only finitely many ideals in  $R$  containing  $(a)$ .
93. Let  $D$  be a unique factorization domain and  $F$  its field of fractions. Prove that if  $d$  is an irreducible element in  $D$ , then there are no  $x \in F$  such that  $x^2 = d$ .
94. Let  $D$  be a Euclidean domain. Prove that every non-zero prime ideal is a maximal ideal.
95. Let  $\pi$  be an irreducible element of a principal ideal domain  $R$ . Prove that  $\pi$  is a prime element (that is,  $\pi \mid ab$  implies  $\pi \mid a$  or  $\pi \mid b$ ).
96. Let  $D$  with  $\phi : D \setminus \{0\} \rightarrow \mathbb{N}$  be a Euclidean domain. Suppose  $\phi(a + b) \leq \max\{\phi(a), \phi(b)\}$  for all  $a, b \in D$ . Prove that  $D$  is either a field or isomorphic to a polynomial ring over a field.
97. Let  $D$  be an integral domain and  $F$  its field of fractions. Show that if  $g$  is an isomorphism of  $D$  onto itself, then there is a unique isomorphism  $h$  of  $F$  onto  $F$  such that  $h(d) = g(d)$  for all  $d \in D$ . ( $h|_D = g$ ).
98. Let  $D$  be a unique factorization domain such that if  $p$  and  $q$  are irreducible elements of  $D$ , then  $p$  and  $q$  are associates. Show that if  $A$  and  $B$  are ideals of  $D$ , then either  $A \subseteq B$  or  $B \subseteq A$ .
99. Let  $D$  be a unique factorization domain and  $p$  a fixed irreducible element of  $D$  such that if  $q$  is any irreducible element of  $D$ , then  $q$  is an associate of  $p$ . Show the following.
- (a) If  $d$  is a nonzero element of  $D$ , then  $d$  is uniquely expressible in the form  $up^n$ , where  $u$  is a unit of  $D$  and  $n$  is a non-negative integer.
  - (b)  $D$  is a Euclidean domain.
100. Prove that  $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$  is a Euclidean domain.
101. Show that the ring  $\mathbb{Z}[i]$  of Gaussian integers is a Euclidean ring and compute the greatest common divisor of  $5 + i$  and  $13$  using the Euclidean algorithm.

## 1.5 Polynomial Rings

102. Show that the polynomial  $f(x) = x^4 + 5x^2 + 3x + 2$  is irreducible over the field of rational numbers.
103. Let  $D$  be an integral domain and  $D[x]$  the polynomial ring over  $D$ . Suppose  $\phi : D[x] \rightarrow D$  is an isomorphism such that  $\phi(d) = d$  for all  $d \in D$ . Show that  $\phi(x) = ax + b$  for some  $a, b \in D$  and that  $a$  is a unit of  $D$ .
104. Let  $f(x) = a_0 + a_1x + \dots + a_kx^k + \dots + a_nx^n \in \mathbb{Z}[x]$  and  $p$  a prime such that  $p \mid a_i$  for  $i = 1, \dots, k-1$ ,  $p \nmid a_k$ ,  $p \nmid a_n$ , and  $p^2 \nmid a_0$ . Show that  $f(x)$  has an irreducible factor in  $\mathbb{Z}[x]$  of degree at least  $k$ .
105. Let  $D$  be an integral domain and  $D[x]$  the polynomial ring over  $D$  in the indeterminate  $x$ . Show that if every nonzero prime ideal of  $D[x]$  is a maximal ideal, then  $D$  is a field.
106. Let  $R$  be a commutative ring with 1 and let  $f(x) \in R[x]$  be nilpotent. Show that the coefficients of  $f$  are nilpotent.
107. Show that if  $R$  is an integral domain and  $f(x)$  is a unit in the polynomial ring  $R[x]$ , then  $f(x)$  is in  $R$ .
108. Let  $D$  be a unique factorization domain and  $F$  its field of fractions. Prove that if  $f(x)$  is a monic polynomial in  $D[x]$  and  $\alpha \in F$  is a root of  $f$ , then  $\alpha \in D$ .
109. (a) Show that  $x^4 + x^3 + x^2 + x + 1$  is irreducible in  $\mathbb{Z}_3[x]$ .  
(b) Show that  $x^4 + 1$  is not irreducible in  $\mathbb{Z}_3[x]$ .
110. Let  $F[x, y]$  be the polynomial ring over a field  $F$  in two indeterminates  $x, y$ . Show that the ideal generated by  $\{x, y\}$  is not a principal ideal.
111. Let  $F$  be a field. Prove that the polynomial ring  $F[x]$  is a PID and that  $F[x, y]$  is not a PID.
112. Let  $D$  be an integral domain and let  $c$  be an irreducible element in  $D$ . Show that the ideal  $(x, c)$  generated by  $x$  and  $c$  in the polynomial ring  $D[x]$  is not a principal ideal.
113. Show that if  $R$  is a commutative ring with 1 that is not a field, then  $R[x]$  is not a principal ideal domain.
114. (a) Let  $\mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^n} \mid a, n \in \mathbb{Z}, n \geq 0\}$ , the smallest subring of  $\mathbb{Q}$  containing  $\mathbb{Z}$  and  $\frac{1}{2}$ . Let  $(2x - 1)$  be the ideal of  $\mathbb{Z}[x]$  generated by the polynomial  $2x - 1$ . Show that  $\mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}[\frac{1}{2}]$ .  
(b) Find an ideal  $I$  of  $\mathbb{Z}[x]$  such that  $(2x - 1) \subsetneq I \subsetneq \mathbb{Z}[x]$ .

## 1.6 Non-commutative Rings

115. Let  $R$  be a ring with identity such that the identity map is the only ring automorphism of  $R$ . Prove that the set  $N$  of all nilpotent elements of  $R$  is an ideal of  $R$ .
116. Let  $p$  be a prime. A ring  $S$  is called a  $p$ -ring if the characteristic of  $S$  is a power of  $p$ . Show that if  $R$  is a ring with identity of finite characteristic, then  $R$  is isomorphic to a finite direct product of  $p$ -rings for distinct primes.
117. If  $R$  is any ring with identity, let  $J(R)$  denote the Jacobson radical of  $R$ . Show that if  $e$  is any idempotent of  $R$ , then  $J(eRe) = eJ(R)e$ .
118. If  $n$  is a positive integer and  $F$  is any field, let  $M_n(F)$  denote the ring of  $n \times n$  matrices with entries in  $F$ . Prove that  $M_n(F)$  is a simple ring. Equivalently,  $\text{End}_F(V)$  is a simple ring if  $V$  is a finite dimensional vector space over  $F$ .
119. Let  $R$  be a ring.
- (a) Show that there is a unique smallest (with respect to inclusion) ideal  $A$  such that  $R/A$  is a commutative ring.
- (b) Give an example of a ring  $R$  such that for every proper ideal  $I$ ,  $R/I$  is not commutative. Verify your example.

- (c) For the ring  $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$  with the usual matrix operations, find the ideal of  $A$  of part (a).

120. A ring  $R$  is *nilpotent-free* if  $a^n = 0$  for  $a \in R$  and some positive integer  $n$  implies  $a = 0$ .
- (a) Suppose there is an ideal  $I$  such that  $R/I$  is nilpotent-free. Show there is a unique smallest (with respect to inclusion) ideal  $A$  such that  $R/A$  is nilpotent-free.
  - (b) Give an example of a ring  $R$  such that for every proper ideal  $I$ ,  $R/I$  is not nilpotent-free. Verify your example.
  - (c) Show that if  $R$  is a commutative ring with identity, then there is a proper ideal  $I$  of  $R$  such that  $R/I$  is nilpotent-free, and find the ideal  $A$  of part (a).

## 1.7 Local Rings, Localization, Rings of Fractions

121. Let  $R$  be an integral domain. Construct the field of fractions  $F$  of  $R$  by defining the set  $F$  and the two binary operations, and show that the two operations are well-defined. Show that  $F$  has a multiplicative identity element and that every nonzero element of  $F$  has a multiplicative inverse.
122. A *local* ring is a commutative ring with 1 that has a unique maximal ideal. Show that a ring  $R$  is local if and only if the set of non-units in  $R$  is an ideal.
123. Let  $R$  be a commutative ring with  $1 \neq 0$  in which the set of nonunits is closed under addition. Prove that  $R$  is local, i.e., has a unique maximal ideal.
124. Let  $D$  be an integral domain and  $F$  its field of fractions. Let  $P$  be a prime ideal in  $D$  and  $D_P = \{ab^{-1} \mid a, b \in D, b \notin P\} \subseteq F$ . Show that  $D_P$  has a unique maximal ideal.
125. Let  $R$  be a commutative ring with identity and  $M$  a maximal ideal of  $R$ . Let  $R_M$  be the ring of quotients of  $R$  with respect to the multiplicative set  $R \setminus M = \{s \in R \mid s \notin M\}$ . Show the following.
- (a)  $M_M = \{\frac{a}{s} \mid a \in M, s \notin M\}$  is the unique maximal ideal of  $R_M$ .
  - (b) The fields  $R/M$  and  $R_M/M_M$  are isomorphic.
126. Let  $R$  be an integral domain,  $S$  a multiplicative set, and let  $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$  (contained in the field of fractions of  $R$ ). Show that if  $P$  is a prime ideal of  $R$ , then  $S^{-1}P$  is either a prime ideal of  $S^{-1}R$  or else equals  $S^{-1}R$ .
127. Let  $R$  be a commutative ring with identity and  $P$  a prime ideal of  $R$ . Let  $R_P$  be the ring of quotients of  $R$  with respect to the set  $R \setminus P = \{s \in R \mid s \notin P\}$ . Show that  $R_P/P_P$  is the field of fractions of the integral domain  $R/P$ .
128. Let  $D$  be an integral domain and  $F$  its field of fractions. Denote by  $\mathcal{M}$  the set of all maximal ideals of  $D$ . For  $M \in \mathcal{M}$ , let  $D_M = \{\frac{a}{s} \mid a, s \in D, s \notin M\} \subset F$ . Show that  $\bigcap_{M \in \mathcal{M}} D_M = D$ .
129. Let  $R$  be a commutative ring with 1 and  $D$  a multiplicative subset of  $R$  containing 1. Let  $J$  be an ideal in the ring of fractions  $D^{-1}R$  and let

$$I = \left\{ a \in R \mid \frac{a}{d} \in J \text{ for some } d \in D \right\}.$$

Show that  $I$  is an ideal of  $R$ .

130. Let  $D$  be a principal ideal domain and let  $P$  be a non-zero prime ideal. Show that  $D_P$ , the localization of  $D$  at  $P$ , is a principal ideal domain and has a unique irreducible element, up to associates.

## 1.8 Chains and Chain Conditions

131. Let  $R$  be a commutative ring with identity. Prove that any non-empty set of prime ideals of  $R$  contains maximal *and* minimal elements.
132. Let  $R$  be a commutative ring with 1. We say  $R$  satisfies the *ascending chain condition* if whenever  $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$  is an ascending chain of ideals, there is an integer  $N$  such that  $I_k = I_N$  for all  $k \geq N$ . Show that  $R$  satisfies the ascending chain condition if and only if every ideal of  $R$  is finitely generated.
133. Define *Noetherian ring* and prove that if  $R$  is Noetherian, then  $R[x]$  is Noetherian.
134. Let  $R$  be a commutative Noetherian ring with identity. Prove that there are only finitely many *minimal* prime ideals of  $R$ .
135. Let  $R$  be a commutative Noetherian ring in which every 2-generated ideal is principal. Prove that  $R$  is a Principal Ideal Domain.
136. Let  $R$  be a commutative Noetherian ring with identity and let  $I$  be an ideal in  $R$ . Let  $J = \text{Rad}(I)$ . Prove that there exists a positive integer  $n$  such that  $j^n \in I$  for all  $j \in J$ .
137. Let  $R$  be a commutative Noetherian domain with identity. Prove that every nonzero ideal of  $R$  contains a product of nonzero *prime* ideals of  $R$ .
138. Let  $R$  be a ring satisfying the *descending chain condition* on right ideals. If  $J(R)$  denotes the Jacobson radical of  $R$ , prove that  $J(R)$  is nilpotent.
139. Show that if  $R$  is a commutative Noetherian ring with identity, then the polynomial ring  $R[x]$  is also Noetherian.
140. Let  $P$  be a nonzero prime ideal of the commutative Noetherian domain  $R$ . Assume  $P$  is principal. Prove that there does not exist a prime ideal  $Q$  satisfying  $(0) < Q < P$ .
141. Let  $R$  be a commutative Noetherian ring. Prove that every nonzero ideal  $A$  of  $R$  contains a product of prime ideals (not necessarily distinct) each of which contains  $A$ .
142. Let  $R$  be a commutative ring with 1 and let  $M$  be an  $R$ -module that is not Artinian (Noetherian, of finite composition length). Let  $\mathcal{I}$  be the set of ideals  $I$  of  $R$  such that there exists an  $R$ -submodule  $N$  of  $M$  with the property that  $N/NI$  is not Artinian (Noetherian, of finite composition length, respectively). Show that if  $A \in \mathcal{I}$  is a maximal element of  $\mathcal{I}$ , then  $A$  is a prime ideal of  $R$ .