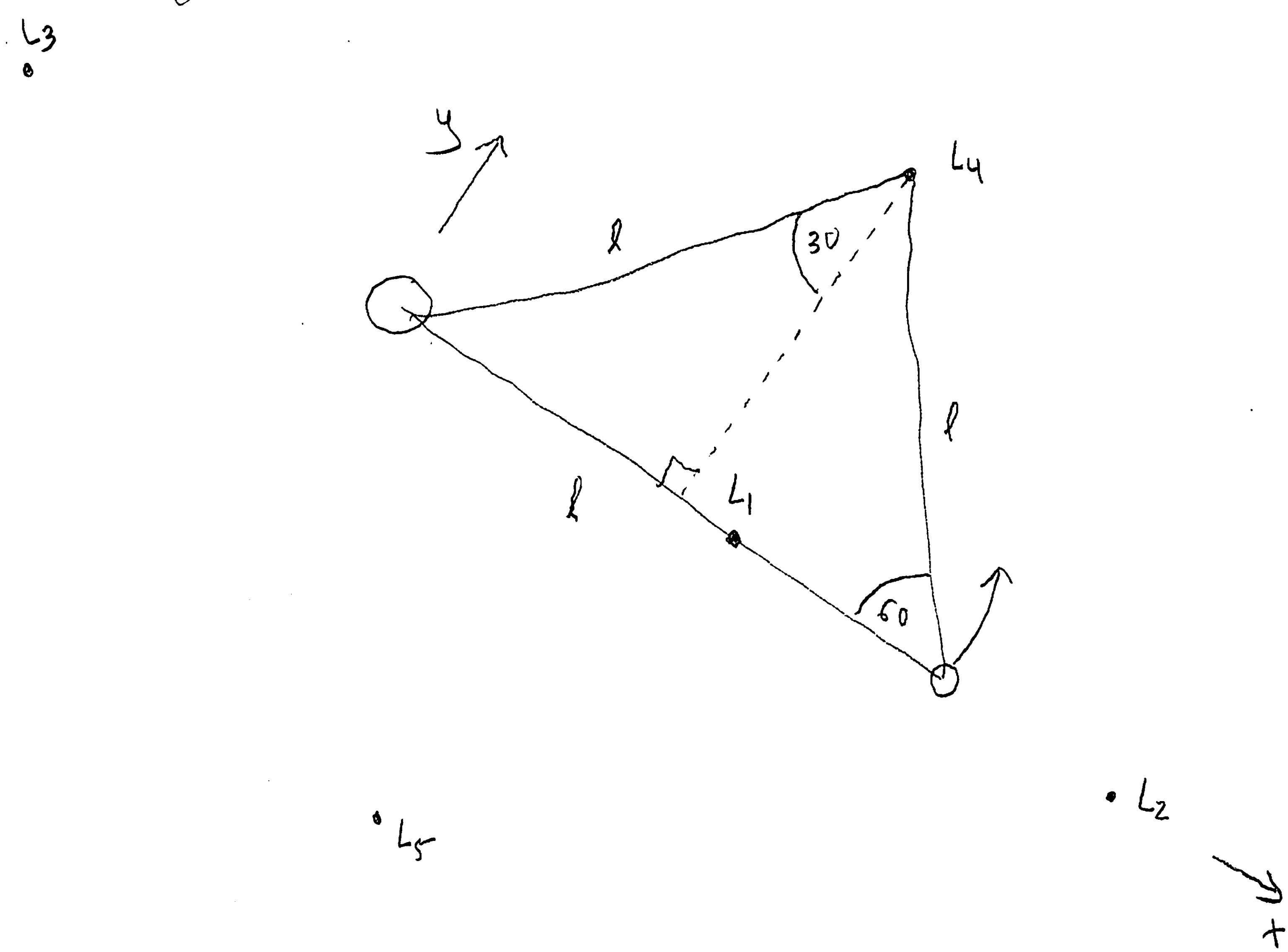
Lagrange Points

Eqn Soln Diff EOMs   CR3BP



$$\sin 30 = \frac{x}{\ell} = \frac{1}{2}$$

$$x = \ell/2$$

$$\cos 30 = \frac{y}{\ell} = \frac{\sqrt{3}}{2}$$

$$y = \pm\frac{\sqrt{3}}{2}\ell$$

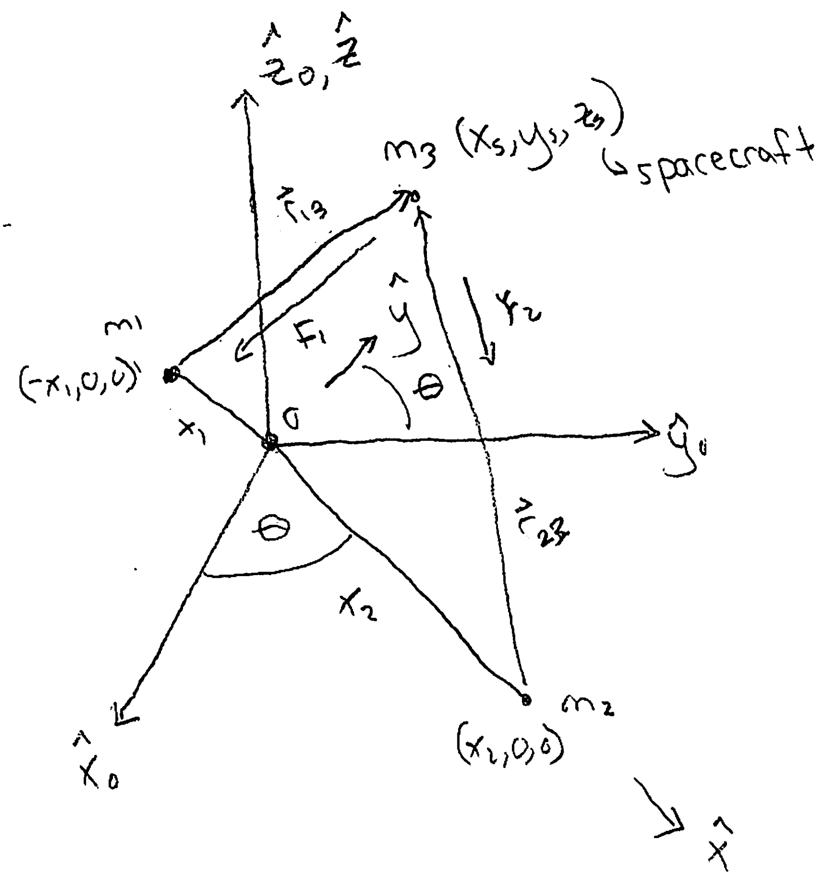
$\ell = 384400$ km

$L_1 = 326390$ km

$L_2 = 448900$ km

$L_3 = -381680$ km

Coordinate System



$$\frac{\sum m_i \cdot \vec{r}_i}{\sum m_i}$$

$$-m_1 x_1 + m_2 x_2 = 0$$

$$x_1 + x_2 = l$$

$$m_1 x_1 = m_2 x_2$$

$$x_1 = l - x_2$$

$$m_1 (l - x_2) = m_2 x_2$$

$$m_1 l - m_1 x_2 = m_2 x_2$$

$$m_1 l = m_2 x_2 + m_1 x_2$$

$$= (m_1 + m_2) x_2$$

$$x_2 = \frac{m_1 l}{m_1 + m_2}$$

$$x_1 = \frac{m_2 l}{m_1 + m_2}$$

$$\hat{x} = \cos\theta\, \hat{x}_0 + \sin\theta\, \hat{y}_0$$

$$\hat{y} = -\sin\theta\, \hat{x}_0 + \cos\theta\, \hat{y}_0$$

$$\hat{z} = \hat{z}_0$$

$$\dot{\hat{x}} = -\sin\theta\, \dot{\theta}\, \hat{x}_0 + \cos\theta\, \dot{\theta}\, \hat{y}_0 = \dot{\theta}(-\sin\theta\, \hat{x}_0 + \cos\theta\, \hat{y}_0) = \dot{\theta}\hat{y}$$

$$\dot{\hat{y}} = -\cos\theta\, \dot{\theta}\, \hat{x}_0 - \sin\theta\, \dot{\theta}\, \hat{y}_0 = -\dot{\theta}(\cos\theta\, \hat{x}_0 + \sin\theta\, \hat{y}_0) = -\dot{\theta}\hat{x}$$

$$\dot{\hat{z}} = 0$$

$$\ddot{\hat{x}} = \ddot{\theta}\hat{y} + \dot{\theta}\dot{\hat{y}} = \ddot{\theta}\hat{y} - \dot{\theta}^2 \hat{x}$$

$$\ddot{\hat{y}} = -\ddot{\theta}\hat{x} - \dot{\theta}\dot{\hat{x}} = -\ddot{\theta}\hat{x} - \dot{\theta}^2 \hat{y}$$

Assumptions

- Bodies in an elliptical orbit
- $m_3 \ll m_2$, $m_2 < m_1$

EOMs

$$\vec{F} = \cancel{m_3}\vec{a}$$

Basic Kinematic Equation
Transport theorem
Time derivatives in a rotating frame

$$\vec{r} = x_s\hat{x} + y_s\hat{y} + z_s\hat{z}$$

$$\vec{v} = \dot{\vec{r}} = \dot{x}_s\hat{x} + x_s\dot{\hat{x}} + \dot{y}_s\hat{y} + y_s\dot{\hat{y}} + \dot{z}_s\hat{z} + \cancel{z_s}\dot{\hat{z}}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{x}_s\hat{x} + \dot{x}_s\dot{\hat{x}} + \dot{x}_s\dot{\hat{x}} + x_s\ddot{\hat{x}} + \ddot{y}_s\hat{y} + 2\dot{y}_s\dot{\hat{y}} + y_s\ddot{\hat{y}} + \ddot{z}_s\hat{z}$$

$$\vec{a} = \ddot{x}_s\hat{x} + 2\dot{x}_s\dot{\theta}\hat{y} + x_s\left(\ddot{\theta}\hat{y} - \dot{\theta}^2\hat{x}\right) + \ddot{y}_s\hat{y} + 2\dot{y}_s\left(-\dot{\theta}\hat{x}\right) + y_s\left(-\ddot{\theta}\hat{x} - \dot{\theta}^2\hat{y}\right) + \ddot{z}_s\hat{z}$$

$$\vec{a} = \left(\ddot{x}_s - \dot{\theta}^2 x_s - 2\dot{y}_s\dot{\theta} - y_s\ddot{\theta}\right)\hat{x} + \left(\ddot{y}_s - \dot{\theta}^2 y_s + 2\dot{x}_s\dot{\theta} + x_s\ddot{\theta}\right)\hat{y} + \ddot{z}_s\hat{z}$$

$$F_1 = -\frac{Gm_1m_3}{|r_{13}|^2}\cdot\hat{r}_{13} \qquad \hat{r}_{13} = \frac{\vec{r}_{13}}{|r_{13}|} \qquad F_1 = \frac{-Gm_1m_3}{|r_{13}|^3}\vec{r}_{13}$$

$$\vec{r}_{13} = (x_s, y_s, z_s) - (-x_1, 0, 0) = (x_s + x_1, y_s, z_s) = (x_s + x_1)\hat{x} + y_s\hat{y} + z_s\hat{z}$$

$$F_2 = -\frac{Gm_2m_3}{|r_{23}|^2}\cdot\hat{r}_{23} = -\frac{Gm_2m_3}{|r_{23}|^3}\vec{r}_{23} \qquad |r_{13}| = \sqrt{(x_s+x_1)^2 + y_s^2 + z_s^2}$$

$$|r_{23}| = \sqrt{(x_s - x_2)^2 + y_s^2 + z_s^2}$$

$$\vec{r}_{23} = (x_s, y_s, z_s) - (x_2, 0, 0) = (x_s - x_2, y_s, z_s) = (x_s - x_2)\hat{x} + y_s\hat{y} + z_s\hat{z}$$

$$\sum F = F_1 + F_2 = -\frac{Gm_1m_3}{|r_{13}|^3}\vec{r}_{13} - \frac{Gm_2m_3}{|r_{23}|^3}\vec{r}_{23} = \cancel{m_3}\left(-\frac{Gm_1}{|r_{13}|^3}\vec{r}_{13} - \frac{Gm_2}{|r_{23}|^3}\vec{r}_{23}\right)$$

$$-\frac{Gm_1}{|r_{13}|^3}\vec{r}_{13} - \frac{Gm_2}{|r_{23}|^3}\vec{r}_{23} = \left(\ddot{x}_s - \dot{\theta}^2 x_s - 2\dot{y}_s\dot{\theta} - y_s\ddot{\theta}\right)\hat{x} + \left(\ddot{y}_s - \dot{\theta}^2 y_s + 2\dot{x}_s\dot{\theta} + x_s\ddot{\theta}\right)\hat{y} + \ddot{z}_s\hat{z}$$

$$-\frac{Gm_1}{|r_{13}|^3}\left((x_s+x_1)\hat{x}+y_s\hat{y}+z_s\hat{z}\right)-\frac{Gm_2}{|r_{23}|^3}\left((x_s-x_2)\hat{x}+y_s\hat{y}+z_s\hat{z}\right)=$$

$$(\ddot{x}_s-\dot{\theta}^2x_s-2\dot{y}_s\dot{\theta}-y_s\ddot{\theta})\hat{x}+(\ddot{y}_s-\dot{\theta}^2y_s+2\dot{x}_s\dot{\theta}+x_s\ddot{\theta})\hat{y}+\ddot{z}_s\hat{z}$$

$$\left(\frac{-Gm_1(x_s+x_1)}{|r_{13}|^3}-\frac{Gm_2}{|r_{23}|^3}(x_s-x_2)\right)\hat{x}+\left(\frac{-Gm_1y_s}{|r_{13}|^3}-\frac{Gm_2y_s}{|r_{23}|^3}\right)\hat{y}+\left(\frac{-Gm_1z_s}{|r_{13}|^3}-\frac{Gm_2z_s}{|r_{23}|^3}\right)\hat{z}$$

$$\hat{x}:\ \ddot{x}_s-\dot{\theta}^2x_s-2\dot{y}_s\dot{\theta}-y_s\ddot{\theta}=\frac{-Gm_1}{|r_{13}|^3}(x_s+x_1)-\frac{Gm_2}{|r_{23}|^3}(x_s-x_2)$$

$$\hat{y}:\ \ddot{y}_s-\dot{\theta}^2y_s+2\dot{x}_s\dot{\theta}+x_s\ddot{\theta}=-\frac{Gm_1}{|r_{13}|^3}y_s-\frac{Gm_2}{|r_{23}|^3}y_s$$

$$\hat{z}:\ \ddot{z}_s=\frac{-Gm_1}{|r_{13}|^3}z_s-\frac{Gm_2}{|r_{23}|^3}z_s$$

– Assume circular ($\ddot{\theta}=0$)

$$\boxed{\begin{aligned}
&\hat{x}:\ \ddot{x}_s-\dot{\theta}^2x_s-2\dot{y}_s\dot{\theta}=-\frac{Gm_1}{|r_{13}|^3}(x_s+x_1)-\frac{Gm_2}{|r_{23}|^3}(x_s-x_2)\\
&\hat{y}:\ \ddot{y}_s-\dot{\theta}^2y_s+2\dot{x}_s\dot{\theta}=-\frac{Gm_1}{|r_{13}|^3}y_s-\frac{Gm_2}{|r_{23}|^3}y_s\\
&\hat{z}:\ \ddot{z}_s=-\frac{Gm_1}{|r_{13}|^3}z_s-\frac{Gm_2}{|r_{23}|^3}z_s
\end{aligned}}$$

Equilibrium points (x_s, y_s, z_s are constant, $\dot{x}_s = \dot{y}_s = \dot{z}_s = \ddot{x}_s = \ddot{y}_s = \ddot{z}_s = 0$)

$$-\dot{\Theta}^2 x_s = -\frac{Gm_1}{|r_{13}|^3}(x_s + x_1) - \frac{Gm_2}{|r_{23}|^3}(x_s - x_2)$$

$$-\dot{\Theta}^2 y_s = -\frac{Gm_1}{|r_{13}|^3} y_s - \frac{Gm_2}{|r_{23}|^3} y_s$$

$$0 = -\frac{Gm_1}{|r_{13}|^3} z_s - \frac{Gm_2}{|r_{23}|^3} z_s$$

z-equation

$$0 = -G\left(\frac{m_1}{|r_{13}|^3} + \frac{m_2}{|r_{23}|^3}\right) z_s$$

$z_s = 0$

$z_s \neq 0$ impossible

y-equation

$$0 = \left(-\frac{Gm_1}{|r_{13}|^3} - \frac{Gm_2}{|r_{23}|^3} + \dot{\Theta}^2\right) y_s$$

$y_s = 0$

$y_s \neq 0$?

Case $y_s \neq 0$

$$-\frac{Gm_1}{|r_{13}|^3} - \frac{Gm_2}{|r_{23}|^3} + \dot{\Theta}^2 = 0$$

$$\dot{\Theta}^2 = \frac{Gm_1}{|r_{13}|^3} + \frac{Gm_2}{|r_{23}|^3} = G\left(\frac{m_1}{|r_{13}|^3} + \frac{m_2}{|r_{23}|^3}\right)$$

$$\frac{\dot{\Theta}^2}{G} = \frac{m_1}{|r_{13}|^3} + \frac{m_2}{|r_{23}|^3}$$

$$-\left(\frac{Gm_1}{|r_{13}|^3} + \frac{Gm_2}{|r_{23}|^3}\right)x_5 = -\frac{Gm_1}{|r_{13}|^3}(x_5+x_1) - \frac{Gm_2}{|r_{23}|^3}(x_5-x_2)$$

$$\cancel{-\frac{Gm_1}{|r_{13}|^3}x_5} - \cancel{\frac{Gm_2}{|r_{23}|^3}x_5} = -\cancel{\frac{Gm_1}{|r_{13}|^3}x_5} - \frac{Gm_1}{|r_{13}|^3}x_1 - \cancel{\frac{Gm_2}{|r_{23}|^3}x_5} + \frac{Gm_2}{|r_{23}|^3}x_2$$

$$\frac{\cancel{G}m_1}{|r_{13}|^3}x_1 = \frac{\cancel{G}m_2}{|r_{23}|^3}x_2$$

$$\cancel{\frac{m_1}{|r_{13}|^3}\cdot\frac{m_2 l}{m_1+m_2} = \frac{m_2}{|r_{23}|^3}\cdot\frac{m_1 l}{m_1+m_2}} \qquad \frac{1}{|r_{13}|^3} = \frac{1}{|r_{23}|^3}$$

$$|r_{13}|^3 = |r_{23}|^3 \qquad |r_{13}| = |r_{23}|$$

$$\sqrt{(x_5+x_1)^2+y_5^2+z_5^2} = \sqrt{(x_5-x_2)^2+y_5^2+z_5^2}$$

$$(x_5+x_1)^2 + \cancel{y_5^2} + \cancel{z_5^2} = (x_5-x_2)^2 + \cancel{y_5^2} + \cancel{z_5^2}$$

$$|x_5+x_1| = |x_5-x_2|$$

Cases	$(x_5-x_2)<0$	$(x_5-x_2)>0$
$(x_5+x_1)<0$	I	II
$(x_5+x_1)>0$	III	IV

Case I

$$-(x_5+x_1) = -(x_5-x_2)$$

$$\cancel{x_5}+x_1 = \cancel{x_5}-x_2$$

$$x_1 = -x_2 \text{ ??}$$

Case II

DNE

Case III

$$x_5+x_1 = -(x_5-x_2)$$

$$x_5+x_1 = -x_5+x_2$$

$$2x_5 = x_2-x_1$$

$$x_5 = \frac{x_2-x_1}{2}$$

$+x_1$

$$x_5+x_1 = \frac{x_2-x_1}{2}+x_1 \quad = x_5+x_1 = \frac{x_2+x_1}{2} = \frac{l}{2}$$

Case IV

$$(x_5+x_1) = (x_5-x_2)$$

$$x_1 = -x_2 \quad \times$$

(Case $y_5 \neq 0$, found that $x_5 = \frac{x_2 - x_1}{2}$)

$$\frac{\dot{\Theta}^2}{G} = \frac{m_1}{|r_{13}|^3} + \frac{m_2}{|r_{23}|^3}$$ know that $z_5 = 0$

$$\frac{\dot{\Theta}^2}{G} = \frac{m_1}{\sqrt{\left(\frac{x_2-x_1}{2}+x_1\right)^2+y_5^2}^{\,3}} + \frac{m_2}{\sqrt{\left(\frac{x_2-x_1}{2}-x_2\right)^2+y_5^2}^{\,3}}$$

$$\frac{\dot{\Theta}^2}{G} = \frac{m_1}{\sqrt{\ell^2/4+y_5^2}^{\,3}} + \frac{m_2}{\sqrt{\ell^2/4+y_5^2}^{\,3}} = \frac{m_1+m_2}{\sqrt{\ell^2/4+y_5^2}^{\,3}}$$

$$\sqrt{\ell^2/4+y_5^2}^{\,3} = \frac{(m_1+m_2)G}{\dot{\Theta}^2} \Longrightarrow \sqrt{\ell^2/4+y_5^2} = \sqrt[3]{\frac{(m_1+m_2)G}{\dot{\Theta}^2}}$$

$$\frac{\ell^2}{4}+y_5^2 = \left(\frac{(m_1+m_2)G}{\dot{\Theta}^2}\right)^{2/3}$$

$$y_5^2 = \left(\frac{(m_1+m_2)G}{\dot{\Theta}^2}\right)^{2/3} - \frac{\ell^2}{4} \qquad y_5 = \pm\sqrt{\left(\frac{(m_1+m_2)G}{\dot{\Theta}^2}\right)^{2/3} - \frac{\ell^2}{4}}$$

$$\dot{\Theta} = \frac{2\pi}{T} \qquad T = \frac{2\pi\ell}{\sqrt{4/\ell}} = \frac{2\pi\ell}{\sqrt{\frac{G(m_1+m_2)}{\ell}}}$$

$$\dot{\Theta} = \cancel{2\pi}\cdot\frac{\sqrt{\frac{G(m_1+m_2)}{\ell}}}{\cancel{2\pi}\ell} = \sqrt{\frac{G(m_1+m_2)}{\ell}}\cdot\frac{1}{\ell} = \sqrt{\frac{G(m_1+m_2)}{\ell^3}}$$

$$\dot{\Theta}^2 = \frac{G(m_1+m_2)}{\ell^3}$$

$$y_5 = \pm\sqrt{\left(\cancel{G(m_1+m_2)}\cdot\frac{\ell^3}{\cancel{G(m_1+m_2)}}\right)^{2/3} - \frac{\ell^2}{4}} = \pm\sqrt{\ell^2-\frac{\ell^2}{4}} = \pm\ell\frac{\sqrt{3}}{2}$$

Case $y_s = 0$ (and $z_s = 0$)

$$|r_{13}| = \sqrt{(x_s + x_1)^2 + \cancel{y_s^2} + \cancel{z_s^2}} = |x_s + x_1|$$

$$|r_{23}| = |x_s - x_2|$$

$$|x|^3 = x^2 |x|$$

$$-\dot{\theta}^2 x_s = \frac{-Gm_1}{|x_s + x_1|^3}(x_s + x_1) - \frac{Gm_2}{|x_s - x_2|^3}(x_s - x_2)$$

$$0 = \frac{-Gm_1}{(x_s + x_1)^{\cancel{2}} |x_s + x_1|}\cancel{(x_s + x_1)} - \frac{Gm_2}{(x_s - x_2)^{\cancel{2}} |x_s - x_2|}\cancel{(x_s - x_2)} + \dot{\theta}^2 x_s$$

$$0 = \frac{-Gm_1}{(x_s + x_1)|x_s + x_1|} - \frac{Gm_2}{(x_s - x_2)|x_s - x_2|} + \dot{\theta}^2 x_s$$