$$x(t) = \cancel{c_1 \eta_1 e^{\lambda_1 t}} + \cancel{c_2 \eta_2} e^{\lambda_2 t} + c_3 \eta_3 e^{\lambda_3 t} + c_4 \eta_4 e^{\lambda_4 t}$$

$$x(0) = c_3 \eta_3 + c_4 \eta_4$$

$4 \times 2 \qquad 2 \times 1$

$$\begin{bmatrix} x_0 \\ y_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix} = c_3 \eta_3 + c_4 \eta_4 = \begin{bmatrix} \eta_3 & \eta_4 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}$$

$$\left[\begin{aligned} x_0 &= c_3 \eta_{31} + c_4 \eta_{41} \\ y_0 &= c_3 \eta_{32} + c_4 \eta_{42} \end{aligned}\right. \Rightarrow \begin{bmatrix} \eta_{31} & \eta_{41} \\ \eta_{32} & \eta_{42} \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{aligned} \dot{x}_0 &= c_3 \eta_{33} + c_4 \eta_{43} \\ \dot{y}_0 &= c_3 \eta_{34} + c_4 \eta_{43} \end{aligned} \Longrightarrow \begin{bmatrix} \eta_{33} & \eta_{43} \\ \eta_{34} & \eta_{43} \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \end{bmatrix}$$

$$\to \frac{1}{\eta_{31}\eta_{42} - \eta_{32}\eta_{41}} \begin{bmatrix} \eta_{42} & -\eta_{41} \\ -\eta_{32} & \eta_{31} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}$$

$\Rightarrow$ go to page 2

$$\frac{1}{\eta_{42} - \eta_{32}} \begin{bmatrix} \eta_{42} & -1 \\ -\eta_{32} & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} \xRightarrow{\eta_{42} = -\eta_{32}} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2\eta_{42}} \\ \frac{1}{2} & \frac{1}{2\eta_{42}} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} \lambda_3 & -\lambda_3 \\ \eta_{34} & \eta_{34} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\eta_{42}} \\ 1 & \frac{1}{\eta_{42}} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \end{bmatrix} \Rightarrow \cancel{\tfrac{1}{2}} \begin{bmatrix} 0 & -\cancel{2}\lambda_3/\eta_{42} \\ \cancel{2}\eta_{34} & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \end{bmatrix}$$

$$\dot{x}_0 = \frac{-\lambda_3}{\eta_{42}} y_0 = \frac{+2\lambda_2}{\lambda_2 - f_{xx}} y_0$$

$$\dot{y}_0 = \eta_{34} x_0 = \frac{\lambda_2 - f_{xx}}{2} x_0$$

$$A\eta = \lambda\eta$$

$$(A-\lambda I)\eta = 0$$

$$\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ f_{xx} & f_{xy} & -\lambda & 2 \\ f_{xy} & f_{yy} & -2 & -\lambda \end{bmatrix} \begin{bmatrix} \eta_{x1} \\ \eta_{x2} \\ \eta_{x3} \\ \eta_{x4} \end{bmatrix} = 0$$

$$-\lambda\eta_{x1} + \eta_{x3} = 0$$

$$-\lambda\eta_{x2} + \eta_{x4} = 0$$

$\eta_{x4} = \lambda\eta_{x2}$

$\eta_{x3} = \lambda\eta_{x1}$

$$f_{xx}\eta_{x1} + f_{xy}\eta_{x2} - \lambda\eta_{x3} + 2\eta_{x4} = 0 \Rightarrow (f_{xx} - \lambda^2)\eta_{x1} + (\cancel{f_{xy}} + 2\lambda)\eta_{x2} = 0$$

$$f_{xy}\eta_{x1} + f_{yy}\eta_{x2} - 2\eta_{x3} - \lambda\eta_{x4} = 0 \Rightarrow (\cancel{f_{xy}} - 2\lambda)\eta_{x1} + (f_{yy} - \lambda^2)\eta_{x2} = 0$$

$$\Rightarrow 2\lambda\eta_{x2} = (\lambda^2 - f_{xx})\eta_{x1}$$

$$\eta_{x2} = \frac{(\lambda^2 - f_{xx})\eta_{x1}}{2\lambda}$$

$$\begin{bmatrix} \eta_{x1} \\ \eta_{x2} \\ \eta_{x3} \\ \eta_{x4} \end{bmatrix} = \begin{bmatrix} \eta_{x1} \\ \frac{(\lambda^2 - f_{xx})\eta_{x1}}{2\lambda} \\ \lambda\eta_{x1} \\ \frac{(\lambda^2 - f_{xx})\eta_{x1}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\lambda^2 - f_{xx}}{2\lambda} \\ \lambda \\ \frac{\lambda^2 - f_{xx}}{2} \end{bmatrix} \quad \eta_{x1} \to 1$$

$$\eta_3 = \begin{bmatrix} 1 \\ \frac{\Lambda_2 - f_{xx}}{2\lambda_3} \\ \lambda_3 \\ \frac{\Lambda_2 - f_{xx}}{2} \end{bmatrix}$$

$\eta_4 =$ (note that $\lambda_4 = -\lambda_3$)

$$\begin{bmatrix} 1 \\ \frac{\Lambda_2 - f_{xx}}{-2\lambda_3} \\ -\lambda_3 \\ \frac{\Lambda_2 - f_{xx}}{2} \end{bmatrix}$$

$\Rightarrow$ return to page 1