Stability of Lagrange Points

Prereqs

- Taylor series expansions
- Linear stability analysis

Prep

$$\ddot{x}_s - \dot{\theta}^2 x_s - 2\dot{y}_s\dot{\theta} = -\frac{Gm_1}{|r_{13}|^3}(x_s + x_1) - \frac{Gm_2}{|r_{23}|^3}(x_s - x_2)$$

$$\ddot{y}_s - \dot{\theta}^2 y_s + 2\dot{x}_s\dot{\theta} = -\frac{Gm_1}{|r_{13}|^3} y_s - \frac{Gm_2}{|r_{23}|^3} y_s$$

$$\ddot{z}_s = -\frac{Gm_1}{|r_{13}|^3} z_s - \frac{Gm_2}{|r_{23}|^3} z_s$$

- nondimensionalize
- coordinate change
- $\dot{x} = f(x)$

nondimensionalization

$\ell = 1\,m^* = 384400\,km$ $\quad x_{nd} = \frac{x_s}{\ell}$ $\quad x_s = \ell x_{nd}$ $\quad x_{1nd} = \frac{x_1}{\ell}$

$$\cancel{\ell}\ddot{x}_{nd} - \dot{\theta}^2 \cancel{\ell} x_{nd} - 2\cancel{\ell}\dot{y}_{nd}\dot{\theta} = -\frac{Gm_1}{\ell^3|r_{13}|^3}\cancel{\ell}(x_{nd} + x_{1nd}) - \frac{Gm_2}{\ell^3|r_{23}|^3}\cancel{\ell}(x_{nd} - x_{2nd})$$

$$\dot{\theta} = \sqrt{\frac{G(m_1+m_2)}{\ell^3}}\ \frac{rad}{s} \cdot \frac{\cancel{\sqrt{\ell^3/G(m_1+m_2)}}\ s}{1\ s^*} = 1\ \frac{rad}{s^*}$$

$$G = \cancel{6.67e^{-11}}\ \frac{m^3}{kg\,s^2} \cdot \frac{\cancel{\frac{\ell^3}{G(m_1+m_2)}}\ s^2}{1\ s^{*2}} = \frac{\ell^3}{m_1+m_2}\ \frac{m^3}{kg\,s^{*2}} \qquad \frac{G}{\ell^3} = \frac{1}{m_1+m_2}\ \frac{1}{kg\,s^{*2}}$$

$$\ddot{x}_{nd} - x_{nd} - 2\dot{y}_{nd} = \frac{-m_1}{m_1+m_2}\frac{1}{|r_{13}|^3}(x_{nd}+x_{1nd}) - \frac{m_2}{m_1+m_2}\frac{1}{|r_{23}|^3}(x_{nd}-x_{2nd})$$

$$\mu = \frac{m_2}{m_1+m_2} \qquad 1-\mu = \frac{m_1}{m_1+m_2} \qquad x_{1nd} = \frac{x_1}{l} = \mu \qquad x_{2nd} = 1-\mu$$

$$\ddot{x}_{nd} - x_{nd} - 2\dot{y}_{nd} = \frac{-(1-\mu)}{|r_{13}|^3}(x_{nd}+\mu) - \frac{\mu}{|r_{23}|^3}(x_{nd}-(1-\mu))$$

Coordinate change

$$x = x_{nd} - L_x \qquad y = y_{nd} - L_y \qquad z = z_{nd}$$

$$\dot{x} = \dot{x}_{nd}$$

$$\ddot{x} - (x+L_x) - 2\dot{y} = \frac{-(1-\mu)}{|r_{13}|^3}(x+L_x+\mu) - \frac{\mu}{|r_{23}|^3}(x+L_x-(1-\mu))$$

$$|r_{13}| = \sqrt{(x+L_x+\mu)^2+(y+L_y)^2+z^2} \qquad |r_{23}| = \sqrt{(x+L_x-(1-\mu))^2+(y+L_y)^2+z^2}$$

$$\dot{x} = f(x)$$

$x \to [x \; y \; z \; \dot{x} \; \dot{y} \; \dot{z}]^T$

$\dot{x} \to [\dot{x} \; \dot{y} \; \dot{z} \; \ddot{x} \; \ddot{y} \; \ddot{z}]^T$

$$\dot{x} = \dot{x}$$

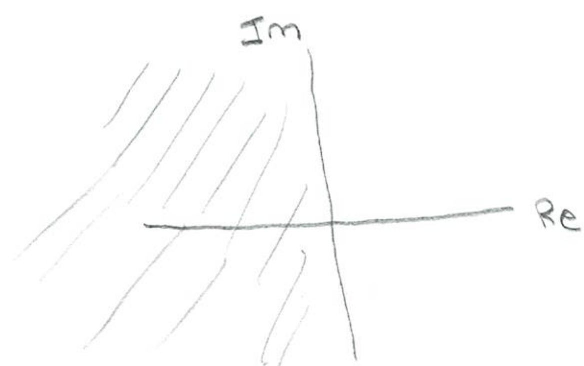
$$\dot{y} = \dot{y}$$

$$\dot{z} = \dot{z}$$

$$\ddot{x} = x + L_x + 2\dot{y} - \frac{(1-\mu)}{|r_{13}|^3}(x+L_x+\mu) - \frac{\mu}{|r_{23}|^3}(x+L_x-(1-\mu))$$

$$\ddot{y} = y + L_y - 2\dot{x} - \left(\frac{(1-\mu)}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3}\right)(y+L_y)$$

$$\ddot{z} = -\left(\frac{(1-\mu)}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3}\right)z$$



Taylor series expansions

$$\ddot{x} = f_x(x, y, z, \dot{x}, \dot{y}, \dot{z})$$

$$\ddot{y} = f_y(x, y, z, \dot{x}, \dot{y}, \dot{z})$$

$$\ddot{z} = f_z(x, y, z, \dot{x}, \dot{y}, \dot{z})$$

$$T(f_x, 0) = \cancelto{0}{f_x(0)} + \frac{\partial f_x}{\partial x}(0)\cdot x + \frac{\partial f_x}{\partial y}(0)\cdot y + \cancelto{0}{\frac{\partial f_x}{\partial z}(0)\cdot z} +$$

$$\cancelto{0}{\frac{\partial f_x}{\partial \dot{x}}}(0)\cdot \dot{x} + \underbrace{\frac{\partial f_x}{\partial \dot{y}}}_{2}(0)\,\dot{y} + \cancelto{0}{\frac{\partial f_x}{\partial \dot{z}}(0)\cdot \dot{z}} + \cancel{\frac{\partial^2 f_x}{\partial x^2}(0)\cdot\frac{x^2}{2!} + \frac{\partial f_x}{\partial x \partial y}(0)\frac{y^2}{2!}}$$

ignore H.O.T

$$\frac{\partial f_x}{\partial x} = 1 - \left(\frac{(1-\mu)}{|r_{13}|^3} + (x+L_x+\mu)(1-\mu)\frac{\partial}{\partial x}\frac{1}{|r_{13}|^3}\right) - \left(\frac{\mu}{|r_{23}|^3} + (x+L_x-(1-\mu))\mu\frac{\partial}{\partial x}\frac{1}{|r_{23}|^3}\right)$$

$$\frac{\partial}{\partial x}\frac{1}{|r_{13}|^3} = \frac{1}{\sqrt{(x+L_x+\mu)^2+(y+L_y)^2+z^2}^{\,3}} = \left((x+L_x+\mu)^2+(y+L_y)^2+z^2\right)^{-3/2}$$

$$-\frac{3}{\cancel{2}}\left((x+L_x+\mu)^2+(y+L_y)^2+z^2\right)^{-5/2}\underbrace{\frac{\partial}{\partial x}\left((x+L_x+\mu)^2+(y+L_y)^2+z^2\right)}_{\cancel{2}(x+L_x+\mu)}$$

$$\frac{-3(x+L_x+\mu)}{|r_{13}|^5}$$

$$\frac{\partial}{\partial x}\frac{1}{|r_{23}|^3} = \frac{-3(x+L_x-(1-\mu))}{|r_{23}|^5}$$

$$\frac{\partial f}{\partial x} = 1 - \left(\frac{1-\mu}{|r_{13}|^3} + \frac{(x+L_x+\mu)^2\,(1-\mu)\cdot -3}{|r_{13}|^5}\right) - \left(\frac{\mu}{|r_{23}|^3} - \frac{3(x+L_x-(1-\mu))^2\,\mu}{|r_{23}|^5}\right)$$

$$\frac{\partial f}{\partial x}(0) = 1 - \left(\frac{1-\mu}{\sqrt{(L_x+\mu)^2+L_y^2}^{\,3}} - \frac{3(L_x+\mu)^2(1-\mu)}{\sqrt{(L_x+\mu)^2+L_y^2}^{\,5}}\right) \qquad R_{13} = \sqrt{(L_x+\mu)^2+L_y^2}$$

$$- \left(\frac{\mu}{\sqrt{(L_x-(1-\mu))^2+L_y^2}^{\,3}} - \frac{3(L_x-(1-\mu))^2\,\mu}{\sqrt{(L_x-(1-\mu))^2+L_y^2}^{\,5}}\right) \qquad R_{23} = \sqrt{(L_x-(1-\mu))^2+L_y^2}$$

$$\frac{\partial f_x}{\partial x}(0) = 1 - \left(\frac{1-\mu}{R_{13}^3} - \frac{3(L_x+\mu)^2(1-\mu)}{R_{13}^5}\right) - \left(\frac{\mu}{R_{23}^3} - \frac{3(L_x-(1-\mu))^2\,\mu}{R_{23}^5}\right)$$

$$\frac{\partial f_x}{\partial y} = -(1-\mu)(x+L_x+\mu)\frac{\partial}{\partial y}\frac{1}{|r_{13}|^3} - \mu(x+L_x-(1-\mu))\frac{\partial}{\partial y}\frac{1}{|r_{23}|^3}$$

$$\frac{\partial}{\partial y}\frac{1}{|r_{13}|^3} = -\frac{3}{\cancel{2}}\left((x+L_x+\mu)^2+(y+L_y)^2+z^2\right)^{-5/2}\underbrace{\frac{\partial}{\partial y}\left((x+L_x+\mu)^2+(y+L_y)^2+z^2\right)}_{\cancel{2}(y+L_y)}$$

$$= \frac{-3(y+L_y)}{|r_{13}|^5} \qquad \frac{\partial}{\partial y}\frac{1}{|r_{23}|^3} = \frac{-3(y+L_y)}{|r_{23}|^5}$$

$$\frac{\partial f_x}{\partial y} = \frac{3(1-\mu)(x+L_x+\mu)(y+L_y)}{|r_{13}|^5} + \frac{3\mu(x+L_x-(1-\mu))(y+L_y)}{|r_{23}|^5}$$

$$\frac{\partial f_x}{\partial y}(0) = \frac{3(1-\mu)(L_x+\mu)L_y}{R_{13}^5} + \frac{3\mu(L_x-(1-\mu))L_y}{R_{23}^5}$$

$$\frac{\partial f_x}{\partial z} = \frac{3(1-\mu)(x+L_x+\mu)z}{|r_{13}|^5} + \frac{3\mu(x+L_x-(1-\mu))z}{|r_{23}|^5}$$

$$\frac{\partial f_x}{\partial z}(0) = 0$$

$$\boxed{\begin{aligned} T(f_x, 0) = {} & \overbrace{\left(1 - \left(\frac{1-\mu}{R_{13}^3} - \frac{3(L_x+\mu)^2(1-\mu)}{R_{13}^5}\right) - \left(\frac{\mu}{R_{23}^3} - \frac{3(L_x-(1-\mu))^2\mu}{R_{23}^5}\right)\right)}^{f_{xx}} x \\ & + \overbrace{3\left(\frac{(1-\mu)(L_x+\mu)}{R_{13}^5} + \frac{\mu(L_x-(1-\mu))}{R_{23}^5}\right) L_y}^{f_{xy}} \cdot y + 2\dot{y} \end{aligned}}$$

$$\begin{aligned} T(f_y, 0) = {} & \cancel{f_y(0)} \to 0 + \frac{\partial f_y}{\partial x}(0)\cdot x + \frac{\partial f_y}{\partial y}(0)\cdot y + \cancel{\frac{\partial f_y}{\partial z}(0)\cdot z} \\ & + \underbrace{\frac{\partial f_y}{\partial \dot{x}}(0)}_{-2}\cdot \dot{x} + \cancel{\frac{\partial f_y}{\partial \dot{y}}(0)} \to 0 \cdot \dot{y} + \cancel{\frac{\partial f_y}{\partial \dot{z}}(0)} \to 0 \cdot \dot{z} + \cancel{H.O.T} \end{aligned}$$

$$\frac{\partial f_y}{\partial x} = -(y+L_y)\left(\frac{-3(x+L_x+\mu)(1-\mu)}{|r_{13}|^5} - \frac{3(x+L_x-(1-\mu))\mu}{|r_{23}|^5}\right)$$

$$\frac{\partial f_y}{\partial x}(0) = 3L_y\left(\frac{(L_x+\mu)(1-\mu)}{R_{13}^5} + \frac{(L_x-(1-\mu))\mu}{R_{23}^5}\right)$$

$$\frac{\partial f_y}{\partial y} = 1 - \left(\left(\frac{1-\mu}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3}\right) + (y+L_y)\frac{\partial}{\partial y}\left(\frac{1-\mu}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3}\right)\right)$$

$$= 1 - \left(\frac{1-\mu}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3} - \frac{3(1-\mu)(y+L_y)^2}{|r_{13}|^5} - \frac{3\mu(y+L_y)^2}{|r_{23}|^5}\right)$$

$$\frac{\partial f_y}{\partial y}(0) = 1 - \frac{1-\mu}{R_{13}^3} - \frac{\mu}{R_{23}^5} + \frac{3(1-\mu)L_y^2}{R_{13}^5} + \frac{3\mu L_y^2}{R_{23}^5}$$

$$\frac{\partial f_y}{\partial z} = -(y+L_y)\left((1-\mu)\frac{\partial}{\partial z}\frac{1}{|r_{13}|^3} + \mu\frac{\partial}{\partial z}\frac{1}{|r_{23}|^3}\right)$$

$$= -(y+L_y)\left(\frac{-3(1-\mu)z}{|r_{13}|^5} - \frac{3\mu z}{|r_{23}|^5}\right)$$

$$\frac{\partial f_y}{\partial z}(0) = 0$$

$$\boxed{T(f_y, 0) = 3L_y\left(\frac{(L_x+\mu)(1-\mu)}{R_{13}^5} + \frac{(L_x-(1-\mu))\mu}{R_{23}^5}\right)x + \left(1 - \frac{(1-\mu)}{R_{13}^3} - \frac{\mu}{R_{23}^3} + \frac{3(1-\mu)L_y^2}{R_{13}^5} + \frac{3\mu L_y^2}{R_{23}^5}\right)y - 2\dot{x}}$$

$$T(f_z, 0) = \cancelto{0}{\frac{\partial f_z}{\partial x}(0)}\cdot x + \cancelto{0}{\frac{\partial f_z}{\partial y}(0)}\cdot y + \frac{\partial f_z}{\partial z}(0)\cdot z$$

$$\frac{\partial f_z}{\partial x} = -z\frac{\partial}{\partial x}\left(\frac{(1-\mu)}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3}\right) \qquad \frac{\partial f_z}{\partial x}(0) = 0$$

$$\frac{\partial f_z}{\partial z} = -\left(\frac{1-\mu}{|r_{13}|^3} + \frac{\mu}{|r_{23}|^3}\right) - z\frac{\partial}{\partial z}(\cdots)$$

$$\frac{\partial f_z}{\partial z}(0) = -\frac{1-\mu}{R_{13}^3} - \frac{\mu}{R_{23}^3}$$

$$\boxed{T(f_z, 0) = -\left(\frac{1-\mu}{R_{13}^3} + \frac{\mu}{R_{23}^3}\right)z}$$

$$\ddot{x} \approx f_{xx} x + f_{xy} y + 2\dot{y}$$

$$\ddot{y} \approx f_{xy} x + f_{yy} y - 2\dot{x}$$

$$\ddot{z} \approx f_{zz} z$$

$$\ddot{z} - f_{zz} z \approx 0$$

$$r^2 - f_{zz} \qquad r^2 = \pm\sqrt{f_{zz}} \rightarrow \pm ki$$

$$X = [x \;\; y \;\; \dot{x} \;\; \dot{y}]^T$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_{xx} & f_{xy} & 0 & 2 \\ f_{xy} & f_{yy} & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\dot{X} = A \quad X$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ f_{xx} & f_{xy} & -\lambda & 2 \\ f_{xy} & f_{yy} & -2 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ f_{xy} & -\lambda & 2 \\ f_{yy} & -2 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & -\lambda & 1 \\ f_{xx} & f_{xy} & 2 \\ f_{xy} & f_{yy} & -\lambda \end{vmatrix}$$

$$= -\lambda\left(-\lambda(\lambda^2+4) + 1\cdot(-2f_{xy} + \lambda f_{yy})\right) + 1\cdot\left(\lambda(-\lambda f_{xx} - 2f_{xy}) + 1\cdot(f_{xx}f_{yy} - f_{xy}^2)\right)$$

$$= (\lambda^2(\lambda^2+4) - \lambda(\lambda f_{yy} - 2f_{xy}) + (-\lambda^2 f_{xx} - 2\lambda f_{xy} + f_{xx}f_{yy} - f_{xy}^2$$

$$= \lambda^4 + 4\lambda^2 - \lambda^2 f_{yy} \cancel{+2\lambda f_{xy}} - \lambda^2 f_{xx} \cancel{-2\lambda f_{xy}} + f_{xx}f_{yy} - f_{xy}^2$$

$$\lambda^4 + \lambda^2(4 - f_{xx} - f_{yy}) + f_{xx} f_{yy} - f_{xy}^2 = 0$$

$$\Lambda = \lambda^2 \qquad \lambda = \pm\sqrt{\Lambda}$$

$$\Lambda^2 + \Lambda(4 - f_{xx} - f_{yy}) + f_{xx} f_{yy} - f_{xy}^2 = 0$$

$$\Lambda = \frac{-(4 - f_{xx} - f_{yy}) \pm \sqrt{(4 - f_{xx} - f_{yy})^2 - 4(f_{xx} f_{yy} - f_{xy}^2)}}{2}$$

Collinear points $(L_x \neq 0, L_y = 0)$ $\qquad R_{13} = |L_x + \mu|$

$\qquad R_{23} = |L_x - (1-\mu)|$

$f_{xy} = 0$

$$f_{xx} = 1 - \frac{1-\mu}{|L_x+\mu|^3} + \frac{3\cancel{(L_x+\mu)^2}(1-\mu)}{|L_x+\mu|^{\cancel{5}\,3}} - \frac{\mu}{|L_x-(1-\mu)|^3} + \frac{3\cancel{(L_x-(1-\mu))^2}\mu}{|L_x-(1-\mu)|^{\cancel{5}\,3}}$$

$$= 1 + \frac{2(1-\mu)}{|L_x+\mu|^3} + \frac{2\mu}{|L_x-(1-\mu)|^3} > 0$$

$$f_{yy} = 1 - \frac{(1-\mu)}{|L_x+\mu|^3} - \frac{\mu}{|L_x-(1-\mu)|^3} < 0$$

$$\Lambda \quad -b \pm \sqrt{b^2+k} \qquad k > 0$$

$\Lambda_1 > 0 \qquad \lambda_{12} = \pm\sqrt{\Lambda_1} \qquad \lambda_1 > 0$

$\qquad \lambda_2 < 0$

$\Lambda_2 < 0 \qquad \lambda_{34} = \pm\sqrt{\Lambda_2} \qquad \lambda_{34}$ imaginary

Equilateral points $L_x = \frac{x_2 - x_1}{2} = \frac{\mu - (1-\mu)}{2} = \frac{1}{2} - \mu \quad L_y = \pm\sqrt{3}/2$

$$R_{13} = \sqrt{\left(\frac{1}{2} - \mu + \mu\right)^2 + \frac{3}{4}} = \sqrt{\frac{1}{2}^2 + 3/4} = 1$$

$$R_{23} = \sqrt{\left(\frac{1}{2} - \mu - (1-\mu)\right)^2 + \frac{3}{4}} = \sqrt{\left(\frac{1}{2} \cancel{-\mu} - 1 \cancel{+\mu}\right)^2 + 3/4} = 1$$

$$f_{xx} = 1 - \left(1 - \mu - 3\left(\tfrac{1}{2}\right)^2(1-\mu)\right) - \left(\mu - 3\left(\tfrac{1}{2}\right)^2 \mu\right)$$

$$= \cancel{1 - 1 + \mu} + \frac{3}{4}(1-\mu) \cancel{- \mu} + \frac{3}{4}\mu = \frac{3}{4} \cancel{- \frac{3}{4}\mu + \frac{3}{4}\mu} = 3/4$$

$$f_{xy} = 3\left((1-\mu)\left(\tfrac{1}{2}\right) + \mu\left(-\tfrac{1}{2}\right)\right) \pm \frac{\sqrt{3}}{2} = \pm\frac{3\sqrt{3}}{2}\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\mu\right) = \pm\frac{3\sqrt{3}}{2}\left(\frac{1}{2} - \mu\right)$$

$$f_{yy} = 1 - (1-\mu) - \mu + 3(1-\mu)\frac{3}{4} + 3\mu\frac{3}{4}$$

$$= 1 - 1 + \mu - \mu \Longrightarrow 9/4$$

$$\lambda = \frac{-\left(4 - \frac{3}{4} - \frac{9}{4}\right) \pm \sqrt{\left(4 - \frac{3}{4} - \frac{9}{4}\right)^2 - 4\left(\frac{3}{4}\cdot\frac{9}{4} - \frac{27}{4}\left(\frac{1}{2} - \mu\right)^2\right)}}{2}$$

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \left(3\cdot\frac{9}{4} - 27\left(\frac{1}{2} - \mu\right)^2\right)}$$

$\left(\frac{1}{2} - \mu\right)^2 = \frac{1}{4} - \mu + \mu^2$

$\cdot 27$

$\frac{27}{4} - 27\mu - 27\mu^2$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - (27\mu - 27\mu^2)}$$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{\underbrace{1 - 27\mu(1-\mu)}_{s(\mu)}}$$

$s(\mu) = 1 - 27\mu(1-\mu) \qquad 0 \le \mu \le 1/2$

$$s(0) = 1 \qquad s\left(\tfrac{1}{2}\right) = 1 - 27\cdot\tfrac{1}{2}\left(1 - \tfrac{1}{2}\right) = 1 - \frac{27}{4} = -\frac{23}{4}$$

$$s(\mu) = 0 = 1 - 27\mu(1-\mu) \qquad 1 = 27\mu(1-\mu) \Longrightarrow 27\mu^2 - 27\mu + 1 = 0$$

$\mu = 0.961479\ldots$ X

$\mu = 0.03852\ldots$

$\mu_{sun\,jupiter} = 0.001\ldots \quad \mu_{earth\,moon} = 0.012$

$\mu_{sun\,earth} = 0.000003002\ldots$

$$0 = \frac{-Gm_1}{(x_s+x_1)|x_s+x_1|} - \frac{Gm_2}{(x_s-x_2)|x_s-x_2|} + \dot{\theta}^2 x_s$$

$$= \frac{-Gm_1}{l^2(x+\mu)|x+\mu|} - \frac{Gm_2}{l^2(x-(1-\mu))|(x-(1-\mu))|} + lx$$

$$= -\frac{G}{l^3}\frac{m_1}{(x+\mu)|(x+\mu)|} - \frac{G}{l^3}\frac{m_2}{(x-(1-\mu))|x-(1-\mu)|} + x$$

$$= \frac{-(1-\mu)}{(x+\mu)|(x+\mu)|} - \frac{\mu}{(x-(1-\mu))|x-(1-\mu)|} + x$$