- Office hours friday moved to 2 to 3
- HW #1 due tomoooorroooooow
- HW2 posted tomorrow too, due in one weeeek
- And now for the important stuff
- Is it possible for there to be a uniformly best estimator? Not really, no. Unbiased vs smallest MSE matters. Uniformly best would pick the parameter with probability 1 but that is rather pointless
- Defn:

$$\text{Let } T \text{ be an unbiased estimator of } \theta. \text{ If, for every unbiased estimator } S \text{ of } \theta,$$
$$V(T) \leq V(S), \text{ and } V(T) < V(S) \text{ for some } \theta,$$
$$\text{then } T \text{ is the uniformly minimum variance unbiased estimator.}$$

- UMVUEs exist, but some biased estimators have a smaller MSE, and there is no good algorithm to find them
- So, what happens to estimator as sample size $n \to \infty$?
- Defn:

$$\text{Let } X_1, \ldots, X_n \text{ be IID. For each } n, \text{ let } \hat{\theta}_n(X_1, \ldots, X_n)$$
$$\text{be an estimator of } \theta. \text{ Then, } \hat{\theta}_1, \hat{\theta}_2, \ldots,$$
$$\text{is a sequence of estimators for } \theta.$$

- Ex for $\bar{x}$: $X_1, \frac{X_1+X_2}{2}, \ldots, \frac{X_1+\cdots+X_n}{n}$
- Defn:

$$\text{Let } X_1, \ldots, X_n \text{ be IID. Let } \hat{\theta}_n \text{ be a sequence of estimators.}$$
$$\text{The estimator sequence } \hat{\theta}_n \text{ is consistent for } \theta \text{ if}$$
$$\forall_\theta \text{ in the parameter space }, \hat{\theta}_n \to \theta.$$

- Imagine there's a p over that arrow, that is there sometimes
- Another way to put it:

$$\forall \epsilon > 0, \lim_{n\to\infty} P\left(\left|\hat{\theta}_n\right| > \epsilon\right) = 0$$
$$\lim_{n\to\infty} P\left(\left|\hat{\theta}_n\right| < \epsilon\right) = 1$$

- In english: For any number ϵ, no matter how small, if you go far enough down the $\hat{\theta}_n$ sequence you'll find some $\hat{\theta}_n$s that are guaranteed to be within ϵ-distance of θ. Even if ϵ is $\frac{1}{TREE(G(64))}$ or something crazy

- Can biased estimators be consistent? Yep! But most consistent estimators we will use are unbiased

- Some examples!!!!!!!!!!!!!!

$$\text{Let } Z_1, \ldots, \text{ be IID, } U(0,1). \text{ For each } n, y_{(n)} = \max(Z_1, \ldots, Z_n).$$

$$\text{Show } y_{(n)} \text{ is consistent for 1.}$$

Proof: Show:

$$\lim_{n \to \infty} P\left(\left|y_{(n)} - 1\right|\right) = 1$$

Let $\epsilon = 0$, fix n.

$$P\left(\left|y_{(n)} - 1\right| < \epsilon\right) = P\left(1 - y_{(n)} < \epsilon\right)$$

$$= P(y_{(n)} > 1 - \epsilon)$$

$$= 1 - P(y_{(n)} < 1 - \epsilon)$$

$$= 1 - \prod_{i=1}^{n} P(Z_i < 1 - \epsilon)$$

$$= 1 - \prod_{i=1}^{n} (1 - \epsilon) = 1 - (1 - \epsilon)^n$$

Now let's do the limit.

$$\lim_{n \to \infty} 1 - (1 - \epsilon)^n = 1 - 0 = 1$$

That's what was wanted! Yay! Happy check mark! (how do i write a check mark in LaTeX? lol)

- That's brute force consistency. But there's better ways

- Theorem:

The Weak Law of Large Numbers (WLLNS):

$$\text{Let } X_1, \ldots, \text{ be IID, with } E(X_i) = \mu.$$

$$\text{For each positive integer } n, \text{ let } \bar{x} = \frac{X_1 + \cdots + X_n}{n}$$

$$\text{Then } \bar{x} \to \mu, \text{ that is, } \lim_{n \to \infty} P(|\bar{x}_n - \mu| > \epsilon) = 0$$

- Another one:

Markov's Inequality:

If X is a nonnegative RV, then

$$\forall A > 0, P(X \geq A) \leq \frac{E(x)}{a}.$$

- That one puts a bound on our expectation value. We learned that one a while ago, I've got the proof in here somewhere

- Proof of WLLNs:

$$\forall \epsilon > 0, P(|\bar{x}_n - \mu| > \epsilon)$$

We shall now drop the n from $\bar{x}_n$ for expediency but assume it's there

$$= P((\bar{x} - \mu)^2 > \epsilon^2)$$

which, by Markov's Inequality,

$$\leq \frac{E((\bar{x} - \mu)^2)}{\epsilon^2}$$

Hey the numerator is $V(\bar{x})$! And $V(\bar{x}) = \frac{\sigma^2}{n}$ So we have:

$$\frac{\sigma^2}{n\epsilon^2}$$

And now evaluate the limit:

$$\lim_{n \to \infty} \frac{\sigma^2}{n\epsilon^2} = 0$$

Therefore, $\bar{x}$ is consistent for $\mu.\square$

- But what about mean squared error?

- A theorem!!!!

Let $\hat{\theta}_n$ be a sequence of estimators of parameter θ.

If $\forall_\theta \lim_{n \to \infty} MSE(\hat{\theta}_n) = 0,$ then $\hat{\theta}_n$ is consistent for θ.

Proof:

$$\text{Let } \epsilon > 0. P\left(\left|\hat{\theta}_n - \theta\right| > \epsilon\right)$$

$$= P((\hat{\theta}_n - \theta)^2 > \epsilon^2)$$

Dropping the ns from that $\hat{\theta}$s now

$$\leq \frac{E((\hat{\theta} - \theta)^2)}{\epsilon^2} = \frac{MSE(\hat{\theta})}{\epsilon^2}$$

Now take limit:

$$\lim_{n \to \infty} \frac{MSE(\hat{\theta})}{\epsilon^2} = \frac{0}{\epsilon^2} = 0$$

- Use the above for consistency proofs most of the time. It's easier
- Now another theorem

  Let $\hat{\theta}$ be a consistent estimtator of θ.

  Let g be a continuous function. Then, $g(\hat{\theta})$ is consistent for $g(\theta)$.

  Proof is in the class notes
- Another one!

  Let $\hat{\theta}$ be consistent for $\theta, \hat{\tau}$ be consistent for τ.

  Let g be a continuous function of two variables. Then, $g(\hat{\theta}, \hat{\tau})$ is consistent for $g(\theta, \tau)$.

  Special case: $\hat{\theta} + \hat{\tau}$ is consistent for $\theta + \tau$

  Also $\hat{\theta} \times \hat{\tau}$ consistent for $\theta \times \tau, \dfrac{\hat{\theta}}{\hat{\tau}}$ consistent for $\dfrac{\theta}{\tau}$
- Mead! Another!

  Let $x_1, \ldots,$ be IID, mean $\mu,$ variance σ^2.

  $$\text{Let } S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{x})^2$$

  Then S^2 is consistent for σ^2.

  Proof:

  $$S^2 = \frac{\sum_{i=1}^{n} X_i^2}{n-1} - \frac{n\bar{x}^2}{n-1}$$
  $$= \frac{n}{n-1} \frac{\sum_{i=1}^{n} x_i^2}{n} - \frac{n}{n-1}\bar{x}^2$$

  Since $\bar{x} \to \mu$ by WLLN, $\bar{x}^2 \to \mu^2$. So:

  $$\lim_{n\to\infty} \frac{n}{n-1}\bar{x}^2 = \mu^2 \text{ (solve the ns and it goes to 1)}$$

  Now: $\dfrac{\sum_{i=1}^{n} X_i^2}{n}$, which is an ugly sample mean. It's not $\bar{x}$.

  But it is an average of the squares. Like Silence of the Lambs, but Average of the Squares

  It converges to $E(X^2)$

  $$= V(X) + E(X)^2$$
  $$= \sigma^2 + \mu^2$$
  $$\Rightarrow \lim_{n\to\infty} \frac{n}{n-1} \frac{\sum_{i=1}^{n} X_i^2}{n} = \sigma^2 + \mu^2$$

  Thus we have:

  $$\sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

  This concludes the proof. We now have $S^2 \to \sigma^2$.$\square$