- Homework #1 is up! Do it
- Let $X_1, \ldots, X_n$ have joint distribution $F(\vec{x};\theta)$
- θ is the unknown parameter
- The unknown parameter generally goes on the right
- Statistic used to estimate θ is the estimator of θ. Name checks out
- Let's say $N(\mu, \sigma^2)$, σ^2 is known
- Estimators for μ: $\bar{x}$, median
- What if $X \sim U(0,\theta)$?
- So it has height $\frac{1}{\theta}$ everywhere that it isn't 0
- How to estimate θ after sampling it a bunch? We could do double the median or double the mean (both of which should be right about in the middle so twice that should get the endpoint), or the largest value we get from all our samples
- If we get the biggest value we will be able to underestimate θ but never overestimate it. That means selecting the biggest value is biased! OOOOH NOOOO
- So that means that that one is possibly less good
- Bias: Let T be an estimator of θ. The **bias** of T is $E(T) - \theta$, where the bias is a function of θ.
- BIAS IS A FUNCTION OF θ! But we don't actually know the true value of θ so that's lame
- If $E(T) = \theta$, then T is unbiased
- Unbiased estimators are the cool kids on the street
- Example time! $X_1, \ldots, X_n$, IID with $E(X_i) = \mu$. Show $\bar{x}$ is unbiased for μ.
  Remember: Expectation of sum is sum of expecations

$$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n}E\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} \mu\right)$$

$$= \frac{n\mu}{\mu}$$
$$= \mu$$

Therefore $\bar{x}$ is unbiased for μ

- Now let's think about the variance of $\bar{x}$!
- The x guys are independent so the variance of the sum are the sum of the variances!

$$V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}V\left(\sum_{i=1}^{n} X_i\right)$$
$$= \frac{n}{n^2}V(X_i)$$
$$= \frac{V(X_i)}{n}$$
$$= \frac{\sigma^2}{n}$$

- Recall: $V(x) = E\left((x-\mu)^2\right)$
- Therefore $V(\bar{x}) = E((\bar{x}-\mu)^2)$
- Recall from waaaaaay back in teh olden dayz: Our sample variance is:

$$S^2 =: \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- A theorem! Let $X_1, \ldots, X_n$ be IID, $E(X_i) = \mu, V(X_i) = \sigma^2$. Also:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

- Then S^2 is an unbiased estimator for σ^2.
- Proooooof:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i - \mu + \mu - \bar{x})^2$$
$$= \sum_{i=1}^{n}((x_i - \mu) - (\bar{x} + \mu))^2$$
$$= \sum_{i=1}^{n}((x_i - \mu)^2 - 2(\bar{x} - \mu)(x_i - \mu) + (\bar{x} - \mu)^2)$$

$$= \sum_{i=1}^{n} (x_i - \mu)^2 - 2(\bar{x} - \mu) \sum_{i=1}^{n} (x_i - \mu) + n(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n} (x_i - \mu)^2 - 2(\bar{x} - \mu) n(\bar{x} - \mu) + n(\bar{x} - \mu)^2$$

$$\sum_{i=1}^{n} (x_i - \mu)^2 - n(\bar{x} - \mu)^2$$

Now let's take the expectation of that giant boi

$$E\left(\sum_{i=1}^{n} (x_i - \mu)^2) - E(n(\bar{x} - \mu)^2\right)$$

$$= \sum_{i=1}^{n} E\left((x_i - \mu)^2) - nE((\bar{x} - \mu)^2\right)$$

Hey that first one is the variance

$$= \sum_{i=1}^{n} \sigma^2 - n\frac{\sigma^2}{n}$$

$$= n\sigma^2 - \sigma^2$$

And so:

$$E(S^2) = \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2. \ \square$$

- That is our first fixed sample sized metric for how good our estimator is. In general, unbiased is good
- But the only way to know for sure what the bias is, is to know the parameter, but you can get pretty good
- Bias can be negative or positive
- Note: Bias $\approx$ accuracy
- Standard deviation $\approx$ precision
- A good estimator is both accurate and precise
- Mean squared error!

Let $X_1, \ldots, X_n$ have joint distribution$f(\vec{x}, \theta)$. Let $T(X_1, \ldots, X_n) = T(\vec{x})$ be an estimator of θ.

The mean squared error of T is $E((T - \theta)^2)$.

$$E((T - \theta)^2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (T(X) - \theta)^2 f(\vec{x}; \theta) \ dx_1 \ldots dx_\theta$$

- But that's horrible so don't do that

  Let T be an estimator of $\theta,$ where $\mu_T = E(T)$. Then $MSE(T) = V(T) + (Bias(T))^2$

  Proof:

$$
\begin{aligned}
E((T-\theta)^2) &= E((T-\mu_T+\mu_T-\theta)^2) \\
&= E(((T-\mu_T)+(\mu_T-\theta))^2) \\
&= E((T-\mu_T)^2) + 2E((T-\mu_T)(\mu_T-\theta)) + E((\mu_T-\theta)^2) \\
&= V(T) + 2(\mu_T + Bias(T)) + 2E((T-\mu_T)(\mu_T-\theta))
\end{aligned}
$$

  Consider that last term:

$$2(\mu_T-\theta)(E(T)-\mu_T)$$

$$2(\mu_T-\theta)(E(T)-E(T)) = 0$$

  This concludes the proof. $\square$

- All right, an example!

  Let $X_1, \ldots, X_n$ be IID with $X_i \sim N(\mu, \sigma^2)$. $\mu \approx \bar{x}$. Find MSE of $\bar{x}$.

- Well $\bar{x}$ is unbiased so that goes to 0. In the formula the MSE is the bias + the variance so really it should just be the variance right? So the MSE is σ^2 I think.

$$
\begin{aligned}
MSE(\bar{x}) &= Bias(\bar{x})^2 + V(\bar{x}) \\
&= V(\bar{x}) \\
&= \frac{\sigma^2}{n}
\end{aligned}
$$