Bias, MSE, Consistency

- Squishing things together into the mean squared error

$$\text{Bias: For estimator } \theta, Bias = E(\hat{\theta}) - \theta$$

$$\text{If } Bias = 0, \hat{\theta} \text{ is unbiased for } \theta$$

- IMPORTANT: Use θ whatever that is. On HW that's $np(1-p)$. Keep that in mind for work! Wooo

$$MSE(\hat{\theta}) = V(\hat{\theta}) + (Bias(\hat{\theta}))^2$$

$$\text{Recall: } V(x) = E(x^2) - (E(x))^2$$

- Note: If you need $E(x^2)$ and have $E(x), V(x)$, use the above!
- Defn:

  Let $X_1, \ldots, X_n$ be a sample. The ordered sample is written $X_{(1)}, \ldots, X_{(n)}$

  Known as order statistics

- If you estimate you gotta put a haaaaaaaat on it — Beyonce, probably
- Ex:

  Let $X_1, \ldots, X_n$ be IID, $U(0, \theta)$. Previously we picked estimators for θ to be either:

  - $\max_(x_1, \ldots, x_n) = X_n$
  - $2\bar{x}$

  1. Find Bias, MSE for $X_{(n)}$.

     Find CDF, PDF of $X_{(n)}$.

$$CDF : P(X_{(n)} \leq x) = P(X_1 \leq x \cap X_2 \leq x \cap \ldots \cap X_n \leq x)$$

$$\prod_{i=1}^{n} P(X_i \leq x)$$

$$= \prod_{i=1}^{n} \frac{x}{\theta}$$

$$= \frac{x^n}{\theta^n}$$

Therefore, $\frac{x^n}{\theta^n}$ is the CDF of $X_{(n)}$. Now for PDF via derivative:

$$fx_{(n)}(x) = \frac{nx^{n-1}}{\theta^n} I_{(0,\theta)}(x)$$

Now we can find the bias!

$$Bias(X_{(n)}) = E(X_{(n)}) - \theta$$

$$\begin{aligned} E(X_{(n)}) &= \int_0^\theta x \frac{nx^{n-1}}{\theta^n}\, dx \\ &= \int_0^\theta \frac{nx^n}{\theta^n}\, dx \\ &= \frac{nx^{n+1}}{(n+1)\theta^n}\Big|_0^\theta \\ &= \frac{n\theta}{n+1} \end{aligned}$$

$$\Rightarrow Bias = \frac{n}{n+1}\theta - \theta = \theta\left(\frac{n}{n+1} - \frac{n+1}{n+1}\right)$$

$$\Rightarrow Bias = -\frac{\theta}{n+1}$$

Does that make sense? Yeah. We expected underestimation $\Rightarrow$ negative bias. That seems about right so yay.

Now for MSE:

$$V(X_{(n)}) = E\left(X_{(n)}^2\right) - E\big(X_{(n)}\big)^2$$

$$E(X_{(n)}^2) = \int_0^\theta x^2 \frac{nx^{n-1}}{\theta^n}\, dx$$

Some calculus later:

$$E(X_{(n)}^2) = \frac{n}{n+2}\theta^2$$

And the rest is algebra So:

$$V(X_{(n)}) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2$$

$$\Rightarrow MSE(X_{(n)}) = \left(\frac{-\theta}{n+1}\right)^2 + \frac{n}{n+2}\theta^2 - \frac{n^2\theta^2}{(n+1)^2}$$

That's bias squared - variance
... Algebra occurs ...

$$= \frac{2\theta^2}{(n+1)(n+2)}$$

2. Example yall: Find MSE for $2\bar{x}$ estimating θ for $X_1, \ldots, X_n$, IID $U(0, \theta)$ (done by hand)

   You get $\frac{\theta^2}{3n}$

   So what's better? max or double $\bar{x}$? The max actually, because it has n^2 in the denominator. Even though $\bar{x}$ is unbiased.

- So what makes one estimator dood uniformly better than another?
- Defn:

$$\text{Let } T_1, T_2 \text{ be estimators for } \theta. \text{ If } \forall_\theta, MSE(T_1) \leq MSE(T_2),$$

$$\text{and } \exists_\theta, MSE(T_1) \leq MSE(T_2), T_1 \text{ is uniformly better than } T_2.$$

- if T is uniformly best estimator of θ then $\forall_\theta MSE(T) = 0$.
- Ex: Uniform between $\theta, \theta + 1$
- Next time: Uniformly minimum variance unbiased estimators