May 26, 2020

Notes on Parameter Distributions for Monte Carlo simulations of an SIR model

David Kaiser

Center for Theoretical Physics, MIT

We want to sample both the per-capita contact rate β and the average duration of infection γ^{-1} from (semi-)realistic distributions while also keeping the relative width of each distribution comparable, so that the effects on the basic reproductive number $\mathcal{R}_0$ are comparable. We want to keep the average value of $(\mathcal{R}_0)_{\rm avg} = 2.31$ to reflect a global average for analyses of COVID-19. We also want to keep the average duration of infection at $(\gamma^{-1})_{\rm avg} = 7.0$ days. Hence we want to keep the average per-capita rate $\beta_{\rm avg} = (\gamma)_{\rm avg} \times (\mathcal{R}_0)_{\rm avg} = 0.33$.

We will sample the independent parameters β and γ^{-1} from Gamma distributions. The probability density function for the Gamma distribution is given, in general, in the form

$$f_X(x; \mu, \kappa, s) = \frac{1}{\kappa \Gamma(s)} \left(\frac{x - \mu}{\kappa} \right)^{s-1} \exp\left[-\frac{(x-\mu)}{\kappa} \right]. \tag{1}$$

The parameter s is known as the "shape" parameter, and κ is known as the "scale" parameter, while μ sets the minimum value that the parameter X may assume. For the Gamma distribution, we take $\mu, s, \kappa > 0$. According to Wikipedia (!), given the Gamma distribution, the mean value is given by

$$(\bar{x} - \mu) = s\kappa\,, \tag{2}$$

and the variance is given by

$$\text{Var} = s\kappa^2\,. \tag{3}$$

To define the (dimensionless) relative width, I suggest we consider the quantity

$$W_X \equiv \frac{\sqrt{\text{Var}}}{\bar{x}} = \frac{\sqrt{s}\,\kappa}{\mu + s\kappa}. \tag{4}$$

Our challenge is therefore to find values of μ, s, and κ for distributions both for β and γ^{-1} such that the mean values obey $\bar{\beta} = 0.330$ and $\overline{\gamma^{-1}} = 7.0$ days, while also keeping $W_\beta \simeq W_{1/\gamma}$. I suggest the following selections:

$$\begin{aligned} \mu_\beta &= 0.21\,,\ \kappa_\beta = 0.01\,,\ s_\beta = 12, \\ \mu_{1/\gamma} &= 4.5\,,\ \kappa_{1/\gamma} = 0.25\,,\ s_{1/\gamma} = 10. \end{aligned} \tag{5}$$

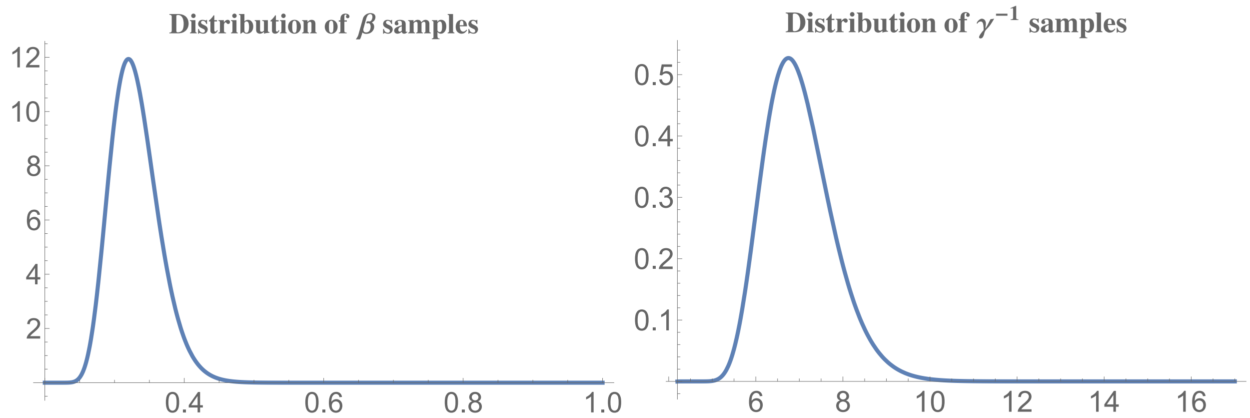

Figure 1: Proposed distributions from which to draw β and γ^{-1} for our Monte Carlo simulations, consistent with the parameter selections in Eq. (5).

With those selections, we find

$$\begin{aligned} \bar{\beta} &= \mu_\beta + s_\beta\, \kappa_\beta = 0.21 + 12 \times 0.01 = 0.330\,, \\ \overline{\gamma^{-1}} &= \mu_{1/\gamma} + s_{1/\gamma}\, \kappa_{1/\gamma} = 4.5 + 10 \times 0.25 = 7.0\,\text{days}\,, \end{aligned} \tag{6}$$

exactly as desired. Moreover, we find

$$\begin{aligned} W_\beta &= \frac{\sqrt{s_\beta}\, \kappa_\beta}{\mu_\beta + s_\beta \kappa_\beta} = 0.105 \simeq 0.11\,, \\ W_{1/\gamma} &= \frac{\sqrt{s_{1/\gamma}}\, \kappa_{1/\gamma}}{\mu_{1/\gamma} + s_{1/\gamma}\, \kappa_{1/\gamma}} = 0.113 \simeq 0.11\,. \end{aligned} \tag{7}$$

The two distributions are shown in Fig. 1.

As a quick reality check, we may calculate the average values of β and γ^{-1} given these selections for the relevant Gamma distributions:

$$\begin{aligned} (\beta)_{\text{avg}} &= \int_{\mu_\beta}^{\infty} x f_\beta(x; \mu_\beta, \kappa_\beta, s_\beta) = 0.330\,, \\ \left(\gamma^{-1}\right)_{\text{avg}} &= \int_{\mu_{1/\gamma}}^{\infty} x f_{1/\gamma}(x; \mu_{1/\gamma}, \kappa_{1/\gamma}, s_{1/\gamma}) = 7.00\,\text{days}\,, \end{aligned} \tag{8}$$

exactly as desired.