A Long and a Short Leg Make For a Wobbly Equilibrium

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Abstract

We provide evidence that the online discussion on the WSB subreddit had a substantial negative impact on the profitability of shorting strategies across a number of stocks — even those that were neither heavily discussed on the subreddit, nor experienced an unusual increase in retail buying volume. We provide a model to explain how fears among short sellers can become self-fulfilling. In the model, the market for shares and the lending market clear jointly. Despite standard assumptions, the model features multiple equilibria and "run-type" behavior by shorting agents. More broadly, the model provides a tractable framework to interpret several empirical observations on the relation between short interest, lending fees, and expected returns.

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In January and February of 2021 the press paid a lot of attention to the spectacular rise of Gamestop's stock and the role played by the contributors to the WallStreetBets (WSB) subreddit online forum. Somewhat less noticed was the broader "ripple effect" of these events on short-selling strategies. We show that, as the price of Gamestop skyrocketed, short sellers withdrew from several other stocks. These stocks were substantially larger than Gamestop, were not particularly discussed on the WSB subreddit, and did not experience an extraordinary rise in retail purchases (as Gamestop did). The result was an unprecedentedly bad month for shorting strategies. This "contagion" effect occurred despite Gamestop being a relatively small stock. (The dollar value of shares shorted as of December 31, 2020 was $1.3 billion, 0.1% percent of the total value of all shares shorted in the US stock market.)

A natural explanation is that the Gamestop episode spread fear among short sellers across the board. In this paper we develop a model to explain why short selling can be particularly disrupted by short seller fears, which become self-fulfilling. In particular, the paper provides a novel reason why rational investors may choose to retreat from a market even as prices deviate further from fundamentals. We show that a simple, unified model of the spot and lending market can feature multiple equilibria: an equilibrium with relatively high short interest can coexist with two other equilibria featuring smaller, respectively zero, short interest. This multiplicity is a time-honored device to illustrate the amplification effects of fears and beliefs.

We next provide more detail on our empirical findings, outline the ingredients of the model, and summarize the basic intuitions for our findings.

January 2021 was an unprecedented month for a strategy that goes long stocks with high short interest and shorts the broad market portfolio (a bet against the "shorts"). During that month, the online discussion on the WSB subreddit saw explosive growth, largely centered around Gamestop. Consistent with articles in the popular press linking this online discussion with retail purchases, we show that high-frequency fluctuations of Gamestop mentions on the WSB subreddit exhibited a very high correlation (above 90 percent at hourly intervals) with retail purchases of Gamestop. This strong high-frequency correlation suggests that the WSB subreddit was an effective vehicle in coordinating retail purchases for this particular stock. Remarkably, several other high-short-interest stocks experienced dramatic declines in short interest over the same period, even though they were barely mentioned on the WSB subreddit, did not experience any unusual increase in retail purchases, and were much larger than Gamestop. Indeed, the performance of shorting strategies continues to be strikingly poor even when we exclude stocks that attracted some attention on the WSB subreddit, and even if we only consider large market-capitalization stocks. For stocks with these properties, there appears to be no evidence of a coordinated "short squeeze" by the WSB subreddit participants, yet short interest nonetheless declined. An illustration of the gloomy mood among short sellers at that time was the emblematic decision of Citron Research to stop publishing short selling research after over 20 years.[1]

Using the behavior of short sellers in early 2021 as a backdrop, we develop a model that explains why short selling can be particularly prone to disruptions and runs. Short sellers could be either active or idle for the same market fundamentals, i.e., our model features multiple equilibria. For our purposes, multiple equilibria are primarily a convenient device to illustrate the feedback loop between the Sharpe ratio and short interest that we wish to highlight.

The model features investors with heterogeneous beliefs about the expected return of a positive-supply, risky stock: one group is optimistic, while the other holds rational beliefs.[2] This difference of opinion between investors prompts them to trade with each other, with the rational investors having an incentive to short the stock whenever the expected excess return

[1] "Citron to stop publishing short-selling research, Andrew Left says" (`https://financialpost.com/investing/citron-research-andrew-left-stop-short-selling-research-publishing`).

[2] Motivated by the empirical fact that stocks with high short interest tend to have low subsequent returns, we assume that the comparatively more pessimistic investors are actually rational, but this is not an essential assumption for our results.

becomes negative. Shorting stock requires borrowing it, for a fee determined endogenously in the lending market as a result of bargaining.

As is recognized, the presence of lending fees modifies the returns experienced by both long and short investors. The equilibrium risk compensation (the "Sharpe ratio") is impacted both by the magnitude of the lending fee and the fraction of a representative lender's shares that are shorted (which we refer to as short interest).[3] All else equal, a higher short interest acts as an increased subsidy for long positions. This basic property of the model is responsible for equilibrium multiplicity. To see this, consider two equilibria. In equilibrium A the Sharpe ratio is higher than in equilibrium B, while short interest is lower. In equilibrium B, the relatively low Sharpe ratio entices short sellers to pay the lending fee, which in turn provides lending income to the long investors and makes them content to hold long positions, despite the low Sharpe ratio. By an analogous argument, in equilibrium A the relatively higher Sharpe ratio deters short sellers, who find the post-lending fee Sharpe ratio to be unattractive; at the same time, the relatively high Sharpe ratio in equilibrium A makes long investors content to hold a long position in the stock despite the lack of lending income.

The discussion in the above paragraph takes the wealth share of short sellers at a given point in time as fixed. Phrased differently, for a *given* wealth share of short sellers, the economy could be in either a high or low shorting equilibrium. One advantage of our dynamic setup is that we can study the evolution of the wealth shares depending on whether investors coordinate on a high or a low shorting equilibrium. We show that the wealth growth of short sellers is higher in the high shorting equilibrium (A in the above paragraph) than in the no shorting equilibrium (B). An implication is that the (stochastic) steady-state fraction of wealth controlled by short sellers is lower in the equilibrium without shorting. Since the Sharpe ratio is increasing in the wealth share of these investors, the *steady state* Sharpe

[3]Short interest commonly refers to the number of shares shorted as a fraction of the float. This is a monotonic transformation of the quantity that we refer to as short interest.

ratio may well be lower if investors coordinate on the low-shorting equilibrium than if they coordinate on the high-shorting equilibrium.

To fully explore the implications of this dynamic effect in a more realistic setup, we extend the model to allow for multiple stocks. After showing that our conclusions from the single-stock economy extend to the multi-stock economy, we focus on the case where there is a large and a small stock, with disagreement affecting only the small stock.[4] For realism, we assume that only a small fraction of investors pay attention to the small stock and incur a small participation cost in doing so. In that extended version of the model, we show that the shift to a low shorting equilibrium causes rational investors to exit the market for the small stock, since remaining in a market without a trading opportunity is not worth paying that participation cost. The exit of rational short-sellers causes a rise in the price of the stock — consistent with the empirical observation that bad returns of shorting strategies coincide with drops in short interest.[5]

To summarize, the goal of this paper is to show that short selling can be fickle: In the model, short sellers may suddenly abandon their short positions, even if there is no change in any fundamental quantity (such as dividends, the technology of matching borrowers and lenders, or retail trading volume). While the specific motivation for the paper is the surprisingly far-reaching effects that a small stock such as Gamestop had on short selling across the entire market, the conclusions of the model are broader. For instance, variants of this model could help explain the somewhat puzzling phenomenon that short selling may well decline during episodes in which the stock market appears overvalued (Brunnermeier and Nagel (2004), Lamont and Stein (2004)).

[4]With this assumption, the interest rate becomes essentially fixed and therefore fluctuations in the Sharpe ratio are mirrored in the price-dividend ratio of the small stock. By contrast, in the baseline model the assumption of log utility and i.i.d. dividend growth imply that fluctuations in the Sharpe ratio are exactly offset by fluctuations in the interest rate, leaving the price-dividend ratio unaffected.

[5]This was the case, for example, in January 2021.

Related Literature

Our work relates to several strands of the asset-pricing literature. The most closely related one considers the joint determination of lending fees, short interest, and returns. In particular, D'Avolio (2002), Duffie et al. (2002), Vayanos and Weill (2008), Banerjee and Graveline (2013), and Atmaz and Basak (2018) consider explicit frictions to lending and borrowing shares, which translate into non-zero lending fees that in turn impact expected returns.[6] Our model is closer in spirit to D'Avolio (2002),[7] Banerjee and Graveline (2013), and Atmaz and Basak (2018), which also feature instantaneous clearing of lending and spot markets — rather than explicit time-consuming search. For our purposes, a key feature of all these papers is the fact that the feedback loop between the Sharpe ratio and short interest is severed by the hard constraint on the proportion of shares a given agent may lend. As a consequence, the equilibrium is unique. In addition, our model allows for a different class of dynamics, driven by the endogenous fluctuations in wealth of the different types of agents.

An even larger number of papers assume that shorting is prohibited and analyze implications for returns. Indicative papers here include Harrison and Kreps (1978), Miller (1977), Diamond and Verrecchia (1987), Hong and Stein (2003), and Detemple and Murthy (1997). As in Harrison and Kreps (1978) and Miller (1977), we model the motive for trade in our paper in the convenient form of (dogmatic) differences of opinions among agents.

A large body of work studies the empirical relation between short interest and stock returns. Seneca (1967), Senchack and Starks (1993), Desai et al. (2002), Diether et al. (2009), Asquith et al. (2005), Blocher et al. (2013), Beneish et al. (2015), and Dechow et al. (2001) study the cross-sectional relation and find that stocks with higher short interest under-perform those with lower short interest. Later work by Cohen et al. (2007) and

[6]Such frictions also motivated the empirical studies of Geczy et al. (2002), Lamont (2012), Jones and Lamont (2002), Kaplan et al. (2013), Porras Prado et al. (2016), and Asquith et al. (2005) among others.

[7]More precisely, to a working-paper version of this study, which contains a theoretical model that did not make it in the published article. Unfortunately, we have been unable to find a copy.

Boehmer et al. (2008) uses proprietary data on quantities lent as well as shorting fees and find consistent results. Duong et al. (2017) studies the empirical relation between lending fees and stock returns and find that high lending fees predict lower future returns. Drechsler and Drechsler (2014) documents that asset pricing anomalies concentrate in stocks with high shorting fees, while Lamont and Stein (2004) studies the information content in aggregate short interest and finds that short interest declined as stock market valuations rose in the late 90's. Rapach et al. (2016) shows that predictive power of aggregate short interest stems predominantly from a cash flow channel. Engelberg et al. (2012) characterizes short sellers as skilled information processors rather than insiders.

Our paper also relates to a sizable theoretical literature analyzing multiple equilibria in asset pricing and macroeconomics. Multiple equilibria can arise through a number of mechanisms, chief among them a) bubbles (or money) in OLG economies, b) increasing returns to scale and production externalities, and c) portfolio constraints.[8] To our knowledge, ours is the first paper in which multiple equilibria are due to shorting fees that can make a long position sufficiently attractive to sustain a higher level of the short interest.

We highlight the ability of the feedback loop between returns and short interest to generate contagion across shorted securities through equilibrium coordination. Another, well studied, contagion mechanism is based on balance-sheet effects. See, for instance, Kyle and Xiong (2001). For a model targeting specifically the set of events involving Gamestop and the role of social communication, see Pedersen (2021).

Finally, as in canonical asset pricing models with heterogeneous agents — see, e.g., Dumas (1989) and Gârleanu and Panageas (2015) — the relative wealth shares of agents is an important state variable in our model. Rather than attempting to summarize the large body of work in this area, we refer readers to Panageas (2020), which discusses both theoretical

[8]We refer the reader to the survey by Benhabib and Farmer (1999), which lists and discusses the different mechanisms that lead to multiple equilibria and indeterminacies. Gârleanu and Panageas (2021) discusses the asset-pricing implications of models with multiple equilibria and "sunspot shocks".

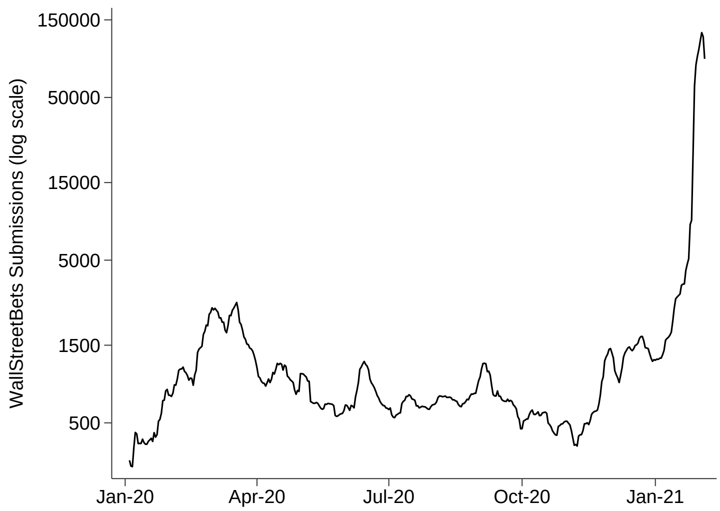

Figure 1: Seven-day moving average of daily submissions to the WallstreetBets subreddit, (January 1, 2020 – February 7, 2021). The vertical axis is on a logarithmic scale.

and empirical contributions in this area of asset pricing research.

1 Empirical Motivation

We motivate our theoretical model by first documenting some empirical facts. Specifically, we show that: a) January 2021 was the worst month for shorting strategies across all years for which data are available (48 years); b) these remarkably bad returns coincided with an exponential growth in discussion on the WSB subreddit, focused primarily on Gamestop; c) the events of January 2021 impacted shorting strategies across the board — they even impacted stocks that were substantially larger than Gamestop, did not experience a notable change in retail purchase volume patterns, and that were not particularly discussed on the WSB subreddit.

In terms of data, we combine standard academic data-sets with social media posts collected from the WallstreetBets subreddit (WSB), a subdomain of the Reddit website. (For a detailed description of the data collection process, see Appendix D.) Reddit is a large online

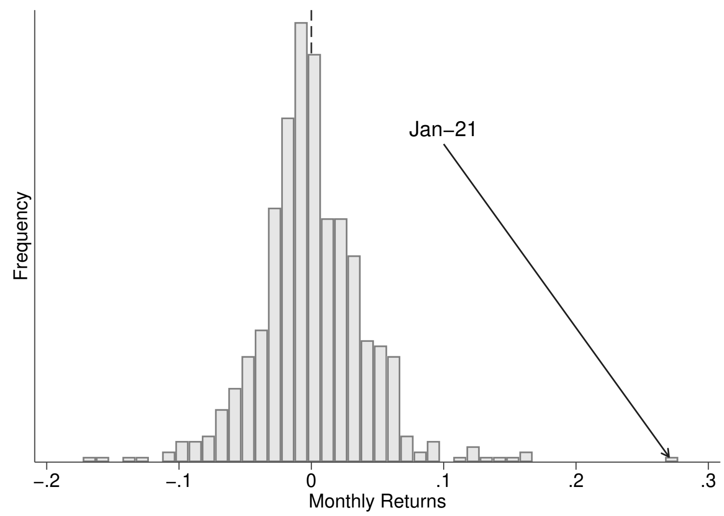

Figure 2: Histogram of monthly returns, 1973–2021. Equal weighted returns on a portfolio long highly shorted stocks and short the market. The arrow indicates the portfolio return in the month of January 2021.

website featuring specialized communities in which users post messages and other users can comment on these posts in message-board fashion. Users on the WSB subreddit actively discuss financial news, investments, and individual securities with one another. We plot the daily submissions to WSB in Figure 1 on a logarithmic scale. Though the subreddit has existed for a number of years (it was created in 2012), daily activity on WSB grew exponentially in late 2020, starting around November and continuing to increase through January 2021.

Concurrently with the rise in this online discussion, the returns of shorting strategies collapsed. In Figure 2, we plot the returns to an equal-weighted portfolio that "bets against the shorts." The portfolio is long the top decile of Russell 3000 stocks, ranked by short interest, and short the broad market. To put the recent returns of shorting strategies in historical perspective, we plot a histogram of the monthly returns of this strategy for as long as data are readily available (since 1973). Stock return data are from CRSP and short interest data are from the SEC. Figure 2 depicts these returns and shows that the January

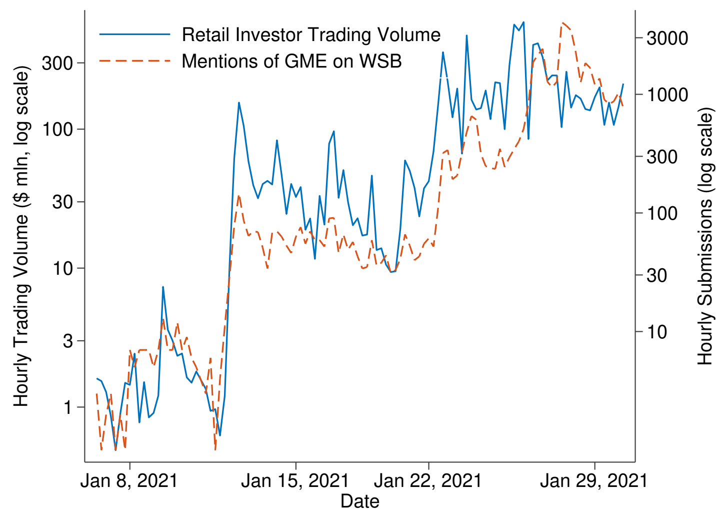

Figure 3: Retail trading volume in GME (January 7 – January 25, 2021). Hourly trading volume in GME, measured using the methodology of Boehmer et al. (2020), plotted together with hourly mentions of the GME ticker on the WallStreetBets subreddit. Both vertical axes are on logarithmic scales.

2021 return is unprecedented historically (approximately six standard deviations larger than the historical mean). For details of the portfolio construction, see Appendix D.

The popular press focused primarily on one stock, namely GameStop (GME). The strong returns of GameStop were linked to the belligerent posts on the WSB subreddit targeting the hedge funds that were shorting this stock. There is high-frequency evidence that this was indeed the case. One advantage of using high-frequency data on WSB mentions is that we are able to identify a strong "real-time" link between mentions of Gamestop on WSB and retail trading volume, which we measure in TAQ data with the methodology of Boehmer et al. (2020). Figure 3 plots the two series at hourly frequency. Over the three trading weeks shown, Gamestop mentions on WSB and retail trading purchases exhibit strikingly strong comovement (0.93 rankcorrelation).

It is important to note that the vast majority of the discussion on the WSB subreddit focused on a very limited number of stocks. Our textual analysis enables the creation of high-frequency time series of ticker-mentions by aggregating mentions within a time interval. As

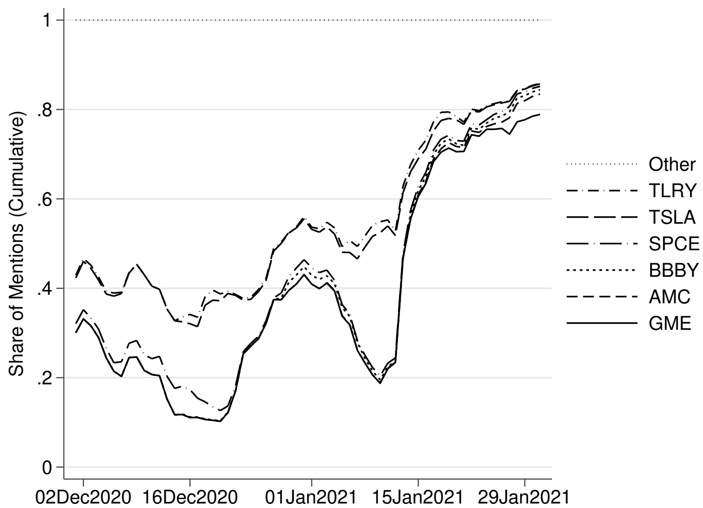

Figure 4: Cumulative fraction of discussion (December 1, 2020 – January 31, 2021). Relative shares for ticker s are computed as $m_{st} = \frac{\text{Mentions}_{st}}{\sum_{s' \in S} \text{Mentions}_{s't}}$. "Other" consists of all other tickers mentioned on WSB.

can be seen in Figure 4, the "lion's share" of mentions centered around GME. From December 1, 2020 to February 1, 2021, six stocks account for about 80% of the discussion. That number peaks to over 80% in the week of January 18–25, primarily driven by the rise in Gamestop mentions.

Despite the focus of the discussion on only a few stocks, the events surrounding Gamestop had a substantial impact on short positions — even for stocks that were not particularly discussed on the WSB subreddit. We next show that a) shorting activity exhibited a broad retreat over the two weeks between January 15 and January 29, b) losses on short positions extended to the larger universe of highly shorted stocks, not just those discussed on WSB, and c) unlike Gamestop, the decline in short interest for other stocks does not coincide with an unusual increase in retail purchase volume.

In Figure 5, we show a bin scatter of short interest on January 15, 2021 against the subsequent decline in short interest, as well as a quadratic curve of best fit. The decline in short interest is concentrated among those stocks with high short interest on January 15,

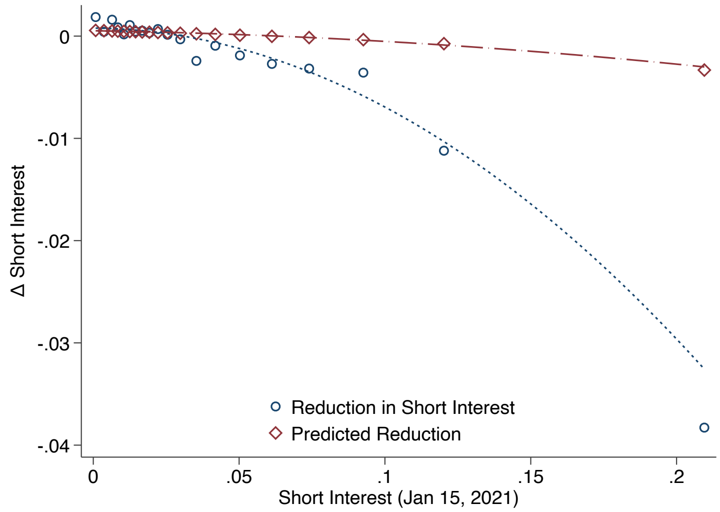

Figure 5: Change in short interest, January 15–January 29, 2021. Stocks are binned into one of twenty groups based on short interest as of January 15, 2021. The average percentage point reduction in short interest from January 15 to January 29, 2021 for each bin is plotted on the y-axis. Predicted reduction is computed based on an AR(1) fitted on five-percent bins of short interest fitted on historical changes in shorting data. The dashed and dotted lines represent best-fit quadratic curves.

2021. Figure 5 also shows that the decline in short interest amongst stocks with high short interest is not just a manifestation of mean reversion. The change in short interest that would be predicted by a simple AR1 model would be much smaller than the one observed in the data. In Figure 6 we put the January 2021 reduction in aggregate short interest into historical context. We compute the monthly reduction in short interest for the top decile of Russell 3000 stocks, ranked by short interest (i.e., the stocks in our betting-against-the-shorts trading strategy in Figure 2) and show that the steep decline in January 2021 was a significant negative outlier.

The historically unparalleled losses on short selling in January 2021 extended to a much broader set of stocks than GME. The returns we show in Figure 2 correspond to monthly returns on an equal-weighted portfolio, of which GME and other reddit stocks comprise only a small fraction. To further illustrate this point, in the first column of Figure 7, we

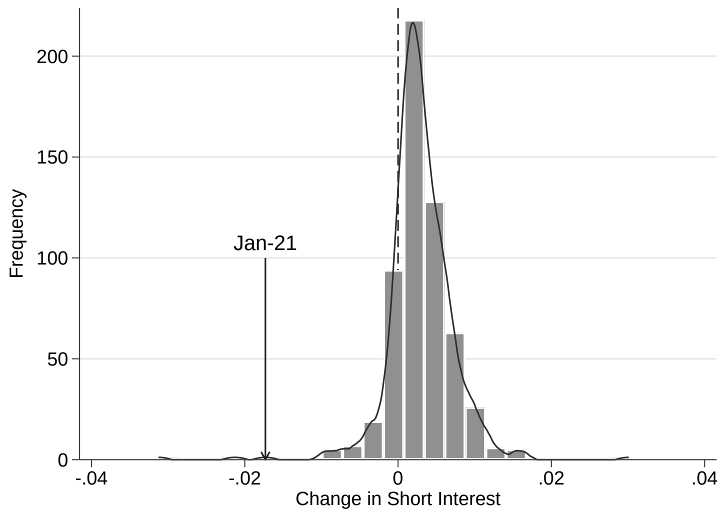

Figure 6: Monthly change in short interest, January 1973–January 2021. Histogram of equal weighted reduction in short interest for stocks in the betting against the shorts long portfolio. Kernel density in black.

repeat the histogram of figure 2, but also show that the unusual returns hold for: a) a value-weighted portfolio; b) the residuals of a regression of the equal-weighted "bet against the shorts" strategy regressed on the Fama-French factors and c) the residuals of a regression of the value-weighted "bet against the shorts" strategy regressed on the Fama-French factors. The second column repeats this exercise while explicitly excluding popular reddit tickers (including GME) from the set of portfolio holdings. Finally, in the third column we restrict our sample to S&P 500 constituents (i.e., larger stocks) in which there is high short interest. Regardless of the exact choices underlying the construction of the long-short portfolio, January 2021 was among the worst, if not the worst, months for shorting strategies (i.e., the portfolio that bets against the shorts had an unprecedentedly strong performance). Table 1 formalizes this discussion in a regression framework, in which we show the magnitude of the portfolio return in January 2021 and test whether it is statistically different from the average return on the strategy over the full 48-year sample.

The systematic reduction in short interest in January 2021 cannot be attributed to retail

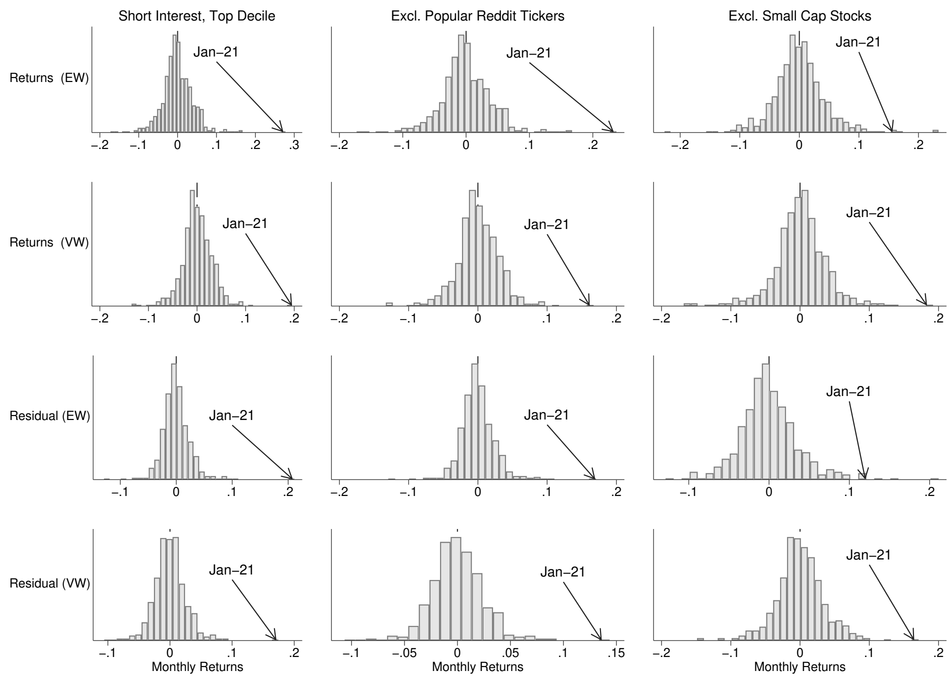

Figure 7: Monthly returns, 1973–2021. Histograms show monthly returns to a trading strategy long highly shorted stocks and short the market index. The first column shows returns to holding the top decile of stocks, sorted on short interest. The second column excludes popular stocks discussed on Reddit. The third column excludes small market capitalization stocks. Returns during the month of January 2021 are indicated with an arrow. The first row shows a histogram of unadjusted returns to the long-short strategy that equal weights stocks in the portfolio. The second row shows the same for a value-weighted portfolio. The third row shows the monthly returns to the equal-weighted portfolio, net of exposures to the 3 Fama-French factors. The fourth row shows the adjusted returns for the value-weighted portfolio.

	Highly Shorted Stocks	Excl. Popular Reddit Stocks	Excl. Small Stocks
r^{EW}	0.270	0.232	0.156
	(6.728)	(5.773)	(3.491)
r^{VW}	0.194	0.161	0.183
	(6.287)	(5.246)	(4.716)
α^{EW}	0.207	0.169	0.120
	(8.458)	(6.894)	(3.218)
α^{VW}	0.170	0.136	0.164
	(6.686)	(5.431)	(4.781)

Table 1: Portfolio returns, January 2021. Monthly return to a long-short portfolio buying highly shorted stocks and shorting the market index. Here, r denotes raw returns, and α residuals after controlling for Fama-French 3-factor exposure. EW denotes equal weighted portfolio returns and VW denotes value weighted portfolio returns. t-statistics on a January 2021 indicator variable are shown in parentheses. Standard errors are the larger of OLS standard errors and White standard errors.

traders' stock purchases. In Figure 8, we plot changes in retail purchase volume against the change in short interest for all stocks in the top decile of short interest in January 2021. Changes in both retail purchase volume and change in short interest are reported as standardized z-scores using TAQ and SEC data from January 2015 through January 2021. We can see that, while there is a negative slope coefficient, it is economically small and statistically indistinguishable from zero, suggesting no relation between changes in short interest and changes in retail purchase volume. Indeed, the x-axis values of most observations are between ± 2 indicating that January 2021 was not an unusual month for retail purchase volume for these highly shorted stocks. This is in sharp contrast to the y-axis values which are overwhelmingly negative, indicating a systematic decline in short interest in that month.

In summary, the month of January 2021 was associated with historically bad returns for short sellers even for stocks that were not the focus of discussion on the WSB subreddit, were much larger than GME and did not experience a notable change in retail purchasing volume. This suggests that the events surrounding Gamestop stoked fears among short

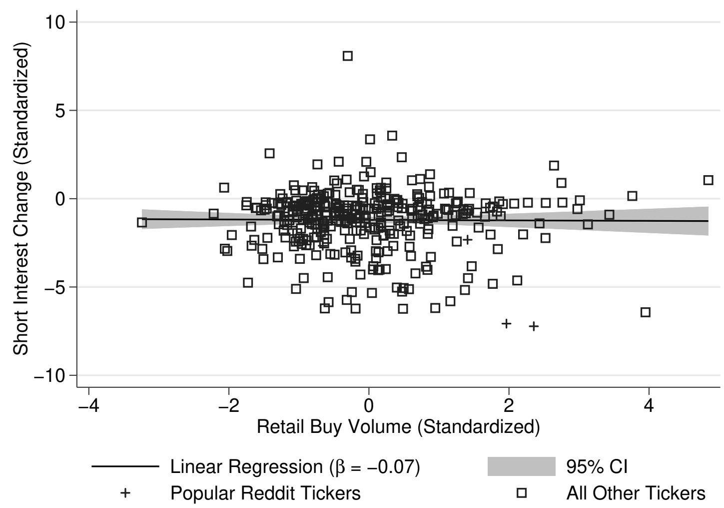

Figure 8: Cross-sectional relationship between retail volume and short interest, January 2021. Changes in both retail purchase volume and change in short interest were calculated as standardized z-scores using TAQ and SEC data from January 2015 through January 2021. Each month, we calculate retail purchase volume and short interest; standardized values are demeaned and divided by the sample standard deviation on this period. Tickers which were popular discussion topics on WSB and which are also in the top decile of short interest are indicated with "+", while all other tickers are indicated with "□".

sellers that made them retreat from short selling. In the next section we develop a model to better understand why short selling is a strategy that is likely to be particularly exposed to changes in market fears and sentiments – to the point where any event that makes short sellers fearful could make the shorting market unravel.

2 Model

2.1 Agents: life-cycle and preferences

Time is continuous and infinite for tractability. To obtain a stationary wealth distribution, we follow Gârleanu and Panageas (2015) and assume that investors continuously arrive ("births") and depart ("deaths") from the economy. Per unit of time a mass π of investors arrives, and a mass π departs. By the law of large numbers, the population of agents who were born at time $s \leq t$ still remaining at time t is $\pi e^{-\pi(t-s)}$, while the total population is constant and equal to $\int_{-\infty}^{t} \pi e^{-\pi(t-s)} ds = 1$. "Births" and "deaths" should be understood as arrivals and departures from the stock market, a point that will become clearer in section 5, when we introduce multiple stocks.

To introduce trade in equities, we assume that investors have heterogeneous beliefs. For simplicity, a fraction $\nu \in (0,1)$ of investors perceive the correct data-generating process. We refer to them as rational investors ("R" investors). The remaining fraction are overly optimistic (we model this optimism shortly), and we refer to these investors as "I" investors.

For tractability, both investors have logarithmic utilities and their expected discounted utility from consumption is

$$V_t^i \equiv E_t^i \int_t^{\infty} e^{-(\rho+\pi)(u-t)} \log\left(c_{u,t}^i\right) du \tag{1}$$

for $i \in \{I, R\}$, with ρ a discount factor and $c_{u,t}^i$ the time-u consumption of an agent of type i born at time $t \leq u$. The notation E_t^i reflects the different investor beliefs. Because of death, the effective discount rate is $\rho + \pi$.

Before proceeding, we note that while we require heterogeneous beliefs to introduce a motivation for trading, the assumption that one group has correct beliefs helps mostly to save notation and can be easily relaxed. Similarly, the overlapping-generations structure is just a technical device to ensure that no investor type disappears in the long run.[9]

2.2 Endowments

In order to support their consumption over their lives, we assume that the arriving investors at time t are equally endowed with shares of new "trees," which are created at time t. Letting

[9]In particular, the lack of inter-generational risk sharing, which is a feature of some of these models, is not driving any of the results in this paper.

$s \leq t$ denote the time of creation of a tree, we specify its time-t dividends as

$$D_{t,s} = \delta e^{-\delta(t-s)} D_t, \tag{2}$$

where $\delta > 0$ captures depreciation and D_t follows a geometric Brownian motion with mean μ_D and volatility $\sigma_D > 0$,

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t, \tag{3}$$

with B_t a standard Brownian motion. Accordingly, the time-t total endowment of this economy is the sum of the endowment produced by all trees born up to to time t,

$$\int_{-\infty}^{t} D_{t,s} ds = \left(\int_{-\infty}^{t} \delta e^{-\delta(t-s)} ds \right) \times D_t = D_t.$$

The assumption that investors are endowed with shares of newly arriving trees follows Gârleanu et al. (2012) and Panageas (2020). We adopt this assumption rather than introducing labor income (as in Gârleanu and Panageas, 2015 or Gârleanu and Panageas, 2020), because — for the purposes of this paper — labor income would just complicate matters without providing any novel insights.

We note that since the returns of all stocks (across all vintages) are perfectly correlated, in effect there is a single, "representative" stock, which is convenient to model. In the real world, shorting frictions are more relevant for a small fraction of stocks rather than the broad stock market. In section 5 we extend the model to allow for multiple stocks and study the special case in which the shorting frictions are relevant for small stocks only.

2.3 Beliefs

The irrational investors are optimistic and believe that the aggregate endowment grows at the rate $\mu^I > \mu_D$. Irrational investors hold this optimistic view over their life-time and do not learn ("dogmatic beliefs"). Introducing learning would be a distraction for the purposes of this paper and therefore we omit it.

For future reference, we define

$$\eta \equiv \frac{\mu^I - \mu_D}{\sigma_D}.$$

2.4 Dynamic budget constraint and short-selling frictions

This section embeds stock lending fees (and lending income) into an investor's dynamic budget constraint, which is the novel aspect of our setup. In preparation, we start by defining the return of the market portfolio and introduce a standard assumption on annuitization.

Specifically, as in Gârleanu et al. (2012) and Panageas (2020), the arriving investors support their life-time consumption by selling their firms into the stock market. These firms become part of the market index (the "market portfolio"). Given our assumptions, the market value of arriving companies, $P_{t,t}$, over total market capitalization P_t is $\frac{P_{t,t}}{P_t} = \delta$.

Letting P_t denote aggregate stock market capitalization, the instantaneous return of the market portfolio is

$$\begin{aligned} dR_t = & \underbrace{\frac{dP_t}{P_t}}_{\substack{\text{Change in} \\ \text{aggregate market cap.}}} - \underbrace{\frac{P_{t,t}}{P_t}dt}_{\substack{\text{cost to purchase} \\ \text{the new firms}}} + \underbrace{\frac{D_t}{P_t}dt}_{\text{dividend yield}} \\ = & \frac{dP_t + D_t dt}{P_t} - \delta dt. \end{aligned}$$

The definition of dR_t reflects the fact that existing investors need to pay arriving investors to purchase the new firms, and hence the increase in stock market capitalization, $\frac{dP_t}{P_t}$, has to be reduced by the payments that existing investors need to make to new investors, $\frac{P_{t,t}}{P_t}$.[10]

Aside from investing in shares of the market portfolio and (zero net supply) risk-free assets, we follow Blanchard (1985) in assuming that each investor annuitizes her entire wealth (since there are no bequest motives) by pledging it to a competitive insurance company upon death in exchange for receiving an income stream while alive. This income stream is equal to the hazard rate of death, π, times the wealth of the investor, so that the insurance company breaks even.[11]

The main departure from a frictionless market is that if investors want to short stocks, they have to pay a lending fee, f_t. Specifically, letting $W^i_{t,s}$ denote the time-t wealth of an investor of type i who was born at time $s \leq t$ and $w^i_{t,s}$ the fraction of wealth invested in stocks, the dynamic budget constraint is

$$dW^i_{t,s} = W^i_{t,s}\left(r_t + \pi + n_t + w^i_{t,s}\left(\mu_t - r_t + \lambda^i_{t,s}\right) - \frac{c^i_{t,s}}{W^i_{t,s}}\right)dt + w^i_{t,s}W^i_{t,s}\sigma_t dB_t, \tag{4}$$

where μ_t and σ_t are the equilibrium expected return and volatility (respectively) of a stock investment and r_t is the equilibrium interest rate. The non-standard terms in equation (4) are the terms $\lambda^i_{t,s}$ and n_t, which we describe next.

[10]For a more detailed derivation, start from $P_t = \int_{-\infty}^{t} P_{t,s}ds$. Time-differentiating $\frac{dP_t}{P_t}$, using Leibniz's rule, and adding $\frac{D_t}{P_t} = \frac{\int D_{t,s}ds}{P_t}$ we obtain

$$\begin{aligned}\frac{dP_t}{P_t} + \frac{D_t}{P_t}dt &= \frac{\int_{-\infty}^{t}\left(dP_{t,s} + D_{t,s}dt\right)ds}{P_t} + \frac{P_{t,t}}{P_t}dt = \int_{-\infty}^{t}\left(\frac{P_{t,s}}{P_t}\right)\left(\frac{dP_{t,s} + D_{t,s}dt}{P_{t,\,s}}\right)ds + \frac{P_{t,t}}{P_t}dt \\ &= \int_{-\infty}^{t} w_{t,s}\left(\frac{dP_{t,s} + D_{t,s}dt}{P_{t,\,s}}\right)ds + \frac{P_{t,t}}{P_t}dt = dR_t + \frac{P_{t,t}}{P_t}dt = dR_t + \delta dt,\end{aligned}$$

where $w_{t,s}$ are market-capitalization weights and the equality $dR_t = \int_{-\infty}^{t} w_{t,s}\left(\frac{dP_{t,s}+D_{t,s}dt}{P_{t,\,s}}\right)ds$ constitutes the definition of a portfolio's return.

[11]This is an implication of the Law of Large Numbers.

The term $\lambda^i_{t,s}$ captures the presence of lending fees. The term $\lambda^i_{t,s}$ is defined as

$$\lambda^i_{t,s}(w^i_{t,s}) \equiv f_t \times \left(1_{\{w^i_{t,s}<0\}} + \tau y_t 1_{\{w^i_{t,s}\geq 0\}}\right), \tag{5}$$

where y_t is the fraction of a long portfolio that is lent out by the representative "brokerage house" and τ is the fraction of the lending fees that accrues to the investor. (We discuss the determination of y_t and τ shortly.) Equation (5) reflects that an investor with a short position $w^i_{t,s} < 0$ has to pay a proportion f_t of the value of her entire short position, $|w^i_{t,s}|W^i_{t,s}$, so that the net-of-fee rate of return per dollar shorted is $-(\mu_t - r_t + f_t)\,dt - \sigma_t dB_t$. Similarly, an investor holding a positive position, $w^i_{t,s} > 0$, obtains a rate of return equal to $(\mu_t - r_t + \tau y_t f_t)\,dt + \sigma_t dB_t$ on her stock investments.

Market clearing in the lending market requires

$$y_t W^+_t = W^-_t, \tag{6}$$

where W^-_t is the value of the aggregate short interest and W^+_t that of the aggregate long position,

$$W^-_t \equiv \sum_{i\in\{I,R\}} \int_{-\infty}^{t} |w^i_{t,s}| W^i_{t,s} 1_{w^i_{t,s}<0} ds \tag{7}$$

$$W^+_t \equiv \sum_{i\in\{I,R\}} \int_{-\infty}^{t} w^i_{t,s} W^i_{t,s} 1_{w^i_{t,s}>0} ds. \tag{8}$$

To close the model, we need to specify n_t and also a "supply curve" for lending shares, that is, we need to provide a relation between y_t and the fee f_t. To that end, in Appendix A we model the supply curve for lendable shares by introducing competitive firms specializing in servicing either borrowers ("brokers") or lenders ("dealers"). Brokers are faced with a demand from would-be short sellers, while dealers obtain investors' long portfolios. Brokers and dealers are matched pairwise subject to a "labor cost" and engage in bilateral negotiations that result in a lending fee f_t. In equilibrium, the fee is the same for all shares that are lent, and therefore the total revenue from lending shares equals the fee multiplied by the value of all shares lent. This revenue is shared between the stock lenders (a fraction τ of the lending revenue) and the households as compensation for their labor cost (the remaining $1-\tau$ fraction of lending revenue) .

Appendix A derives τ as a function of parameters governing the search-and-bargaining protocol. In addition, we establish that the lending fee, f_t, is a non-decreasing function $f_t = l\,(y_t)$ with $l'\,(\cdot) \geq 0$, where $l\,(\cdot)$ depends on the (exogenous) "technology" of finding a match. From now on, we refer to $l(y_t)$ as the supply curve for lendable shares.

The term n_t in equation (4) is the fraction $(1-\tau)$ of the lending revenue that is paid to the households as compensation for their labor cost in operating the matching technology. Denoting aggregate wealth at time t by W_t, we define

$$n_t \equiv \frac{(1-\tau)\, f_t W_t^-}{W_t}. \tag{9}$$

To better understand (9), use (6) and (9) and aggregate across all households to obtain

$$\begin{aligned} f_t W_t^- &= (1-\tau)\, f_t W_t^- + \tau f_t W_t^- \\ &= n_t W_t + \tau f_t y_t W_t^+. \end{aligned} \tag{10}$$

The left hand side of (10) reflects the aggregate lending fees $f_t W_t^-$. The right-hand side reflects the ultimate division of lending income between the households (who obtain a fraction $1-\tau$ of lending, irrespective of their portfolio) and long investors, who obtain a fraction τ of the lending income.

Equation (10) shows that share lending does not result in any loss of aggregate resources: All payments made by investors with short positions are received either by investors with long positions or by brokerage firms, who rebate them to the household sector as labor compensation. This shows the dual role played by brokerage firms. On the one hand, their optimizing choices provide a micro-foundation for an upward sloping supply curve $f_t = l(y_t)$. On the other hand, they help capture the notion that only a fraction τ of lending fees is received by long investors, with the remainder of the lending fees being rebated to the households in a manner that does not depend on their portfolio choice.

2.5 Equilibrium

Equilibrium in the lending market requires that the lending fee be such that the supply of lendable shares $y_t W_t^+ = l^{-1}(f_t) W_t^+$ is equal to the demanded short interest, W_t^- (equation (6)).

The rest of the equilibrium definition is standard. We require that investors I and R maximize (1) over $c_{t,s}^i$ and $w_{t,s}^i$ subject to the budget constraint (4), and μ_t, r_t, and σ_t are such that the bond market clears, $\sum_{i\in\{I,R\}} \int_{-\infty}^{t} \nu^i \left(1 - w_{t,s}^i\right) W_{t,s}^i ds = 0$, the stock market clears, $\sum_{i\in\{I,R\}} \int_{-\infty}^{t} \nu^i w_{t,s}^i W_{t,s}^i ds = P_t$, and the goods market clears, $\sum_{i\in\{I,R\}} \int_{-\infty}^{t} \nu^i c_{t,s}^i ds = D_t$. By Walras' Law, market clearing of the bond market implies stock market clearing and vice versa, and accordingly the asset-market clearing requirements can be written equivalently as $W_t = \sum_{i\in\{I,R\}} \int_{-\infty}^{t} \nu^i W_{t,s}^i ds = P_t$.

For future reference, we note that the clearing of the stock market requires that $y_t < 1$:

$$y_t = \frac{W_t^-}{W_t^+} = \frac{W_t^-}{P_t + W_t^-} < 1.$$

3 Analysis

We analyze the model in two steps. First, we consider a special parametric case that allows us to characterize all equilibrium quantities in closed form. The special case we analyze is the "elastic supply" case, that is, the limiting case where the supply of lendable shares is horizontal at some level $l(y_t) = \varphi$. (As we explain in Appendix A, this special case corresponds to a linear specification for the cost of lending out shares.) In Section 6 we repeat the analysis for increasing functions $l(y_t)$ and show how the key results readily extend to this more general case.

3.1 Optimal portfolio and consumption

For a log investor the wealth-to-consumption ratio is constant and equal to

$$\frac{c^i_{t,s}}{W^i_{t,s}} = \rho + \pi. \tag{11}$$

Another convenient property of logarithmic utility is that the portfolio is myopic and maximizes the logarithmic growth rate of an investor's wealth, under the investor's beliefs,

$$w^i_{t,s} = \arg\max_w \left\{ w \left(\mu_t + \eta\sigma_t 1_{\{i=I\}} - r_t + \lambda^i_{t,s}(w) \right) - \frac{1}{2} \left(w\sigma_t \right)^2 \right\}, \tag{12}$$

where $1_{\{i=I\}}$ is an indicator function taking the value one when $i = I$ and zero otherwise.

Letting

$$\widehat{\mu}^i_t \equiv \mu_t + \eta\sigma_t 1_{\{i=I\}}$$

denote the expected return on the stock as perceived by investor $i \in \{I, R\}$, the optimal portfolio is

$$w^i_{t,s} = \begin{cases} \frac{\widehat{\mu}^i_t - r_t + f_t}{\sigma^2_t} & \text{if } \widehat{\mu}^i_t - r_t + f_t < 0 \\ \frac{\widehat{\mu}^i_t - r_t + \tau f_t y_t}{\sigma^2_t} & \text{if } \widehat{\mu}^i_t - r_t + f_t y_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

By inspection, the optimal portfolios do not depend on the cohort s, only on the type of investor $i \in \{I, R\}$. Therefore, from now on we drop the subscript s and write w_t^R and w_t^I.

One straightforward implication of equation (13) is that if investor R is actively shorting ($w_t^R < 0$) then it must be the case that the excess rate of return per dollar shorted is positive, even after netting out the fee f_t. Indeed, evaluating (13) with $i = R$, assuming that $w_t^R < 0$ and re-arranging leads to $-(\mu_t - r - f_t) = -(\widehat{\mu}_t^R - r - f_t) = -w_t^R \sigma_t^2 > 0$. The term $-w_t^R \sigma_t^2$, which corresponds to the absolute value of the covariance of the stock's return with the short seller's portfolio, is the risk compensation to the short seller for taking a short position.

3.2 Equilibrium

It is useful to start by defining the the wealth-weight ω_t^i

$$\omega_t^i \equiv \frac{\nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} W_{t,s}^i ds}{W_t} \tag{14}$$

where W_t is aggregate wealth. The goods market and stock market clearing requirements imply

$$\begin{aligned} D_t &= \sum_{i \in \{I,R\}} \int_{-\infty}^t \nu^i \pi e^{-\pi(t-s)} c_{t,s}^i ds = (\rho + \pi) \sum_{i \in \{I,R\}} \int_{-\infty}^t \nu^i \pi e^{-\pi(t-s)} W_{t,s}^i ds \\ &= (\rho + \pi) W_t = (\rho + \pi) P_t. \end{aligned} \tag{15}$$

Taking logarithms gives $d \log D_t = d \log P_t$ and therefore the stock market volatility equals $\sigma = \sigma_D$. The implication of a constant stock volatility is convenient for obtaining closed-form solutions. In Section 5 we discuss extensions of the model that allow for a time-varying price-dividend ratio and volatility by introducing multiple stocks.

Applying Itô's Lemma to (14) and using (4) and (15) yields

$$d\omega_t^i = \mu_t^i dt + \sigma_t^i dB_t,$$

where

$$\sigma_t^i = \omega_t^i \left(w_t^i - 1\right) \sigma_D, \tag{16}$$

$$\mu_t^i = \omega_t^i \left(-\mu_D + \sigma_D^2 - \pi + r_t - \rho + w_t^i \left(\mu_t - r_t + s_t^i\right) - w_t^i \sigma_D^2\right) + \nu^i \delta. \tag{17}$$

The market clearing requirement $\sum_{i\in\{I,R\}} \omega_t^i = 1$ implies $\sum_{i\in\{I,R\}} d\omega_t^i = 0$ and therefore $\sum_{i\in\{I,R\}} \sigma_t^i = 0$ and $\sum_{i\in\{I,R\}} \mu_t^i = 0$. To simplify notation, we let $\omega_t \equiv \omega_t^R$. As mentioned earlier, in an effort to obtain a tractable solution, we assume that the supply of lendable shares is perfectly elastic at the rate φ:

Assumption 1 $l(y) = \varphi > 0$.

We maintain this assumption until Section 6 in order to develop intuition. In Section 6 we generalize the results to an upward-sloping supply function $l(\cdot)$, so that the lending fee is increasing with short interest.

In preparation for the description of the equilibrium, we start with the following definition and assumptions on the parameters.

Definition 1 *Let*

$$\omega_1^* \equiv 1 - \frac{\sigma_D}{\eta - \frac{\varphi}{\sigma_D}}. \tag{18}$$

and

$$F(\omega) \equiv \left(\sigma_D - \omega\left((1+\tau)\frac{\varphi}{\sigma_D} - \eta\right)\right)^2 - 4\tau\frac{\omega^2}{1-\omega}\frac{\varphi}{\sigma_D}\left(\sigma_D + (1-\omega)\left(\frac{\varphi}{\sigma_D} - \eta\right)\right). \tag{19}$$

Assumption 2 *Assume that η, φ, σ_D, and τ are such that*

$$(1+\tau)\frac{\varphi}{\sigma_D} > \eta > \frac{\varphi}{\sigma_D}, \tag{20}$$

$$\omega_1^* > \frac{\sigma_D}{(1+\tau)\frac{\varphi}{\sigma_D} - \eta} > 0, \tag{21}$$

and $F(\omega)$ has a unique root in the interval $(0,1)$, denoted by ω_2^*.

The following proposition asserts that the set of parameters η, φ, σ_D, and τ that satisfy Assumption 2 is non-empty.

Proposition 1 *There exists an open set of positive values η, φ, σ_D, and τ that jointly satisfy Assumption 2.*

The next proposition describes the equilibria in our economy.

Proposition 2 *Suppose that Assumption 2 holds. Then $\omega_2^* > \omega_1^*$ and the equilibria in this economy can be described as follows.*

i) If $\omega_t \in (\omega_2^*, 1]$ there is no short-selling in equilibrium. The equilibrium is unique and the Sharpe ratio $\kappa_t \equiv \frac{\mu_t - r_t}{\sigma_D}$ is given by

$$\kappa_t = \begin{cases} \sigma_D - (1-\omega_t)\,\eta & \text{if } \omega_t > 1 - \frac{\sigma_D}{\eta} \\ \frac{\sigma_D}{1-\omega_t} - \eta & \text{if } \omega_t \in (\omega_2^*, 1 - \frac{\sigma_D}{\eta}] \end{cases}. \tag{22}$$

ii) If $\omega_t \in [\omega_1^*, \omega_2^*]$, then there are three equilibria. The first equilibrium continues to be given by (22) and involves no short-selling. The second and third equilibria involve shorting and the ratio of shorted-to-lendable shares y_t corresponds to the two roots y^+ and y^- of the quadratic equation

$$y\left(\eta + \frac{\sigma_D}{\omega_t} - \frac{\varphi}{\sigma_D}(1-\tau y)\right) - \left(\eta - \frac{\sigma_D}{1-\omega_t} - \frac{\varphi}{\sigma_D}(1-\tau y)\right) = 0, \tag{23}$$

which has two real roots y^+ and y^- in the interval $(0,1)$. The Sharpe ratio in the equilibria associated with y^+ and y^-) are

$$\kappa_t^{\pm} = \sigma_D - (1-\omega_t)\,\eta - \frac{\varphi}{\sigma_D}\left(\omega_t + \tau y^{\pm}(1-\omega_t)\right). \tag{24}$$

iii) If $\omega_t \in [0, \omega_1^*)$, then the equilibrium is unique and involves shorting. In this case only the larger of the two roots (y^+) of equation (23) lies in the interval $(0,1)$, and the unique equilibrium Sharpe ratio is given by κ^+.

In all three cases the interest rate is given by

$$r_t = \rho + \pi + \mu_D - \delta - \kappa_t \sigma_D. \tag{25}$$

Additionally, because κ_t, r_t, and y_t are functions of ω_t, equations (13), (16), and (17) imply that μ_t^R and σ_t^R are functions of ω_t and hence the equilibrium is Markov in ω_t.

Figure 9 illustrates Proposition 2. The figure plots $\kappa(\omega_t)$, the Sharpe ratio, as a function of the wealth share of rational agents.

As a benchmark, the line labeled "Costless shorting eqlm" depicts $\sigma_D - (1-\omega_t)\,\eta$, i.e., the Sharpe ratio that would obtain in this economy in the absence of any shorting frictions ($\varphi = 0$). The curve "No shorting equil." depicts the Sharpe ratio in the equilibrium that involves no shorting for the values of ω_t that such an equilibrium exists. Similarly for the curves "Medium shorting" and "High shorting," which depict equilibria with shorting, assuming that the value of ω_t permits such equilibria.

The figure shows that when ω_t is larger than $1 - \frac{\sigma_D}{\eta}$ the lines "Zero shorting cost" and "No shorting equil." coincide, reflecting that all investors invest strictly positive amounts in the stock market in this region.

When ω_t becomes smaller than $1 - \frac{\sigma_D}{\eta}$ (but larger than ω_2^*), the rational investor puts

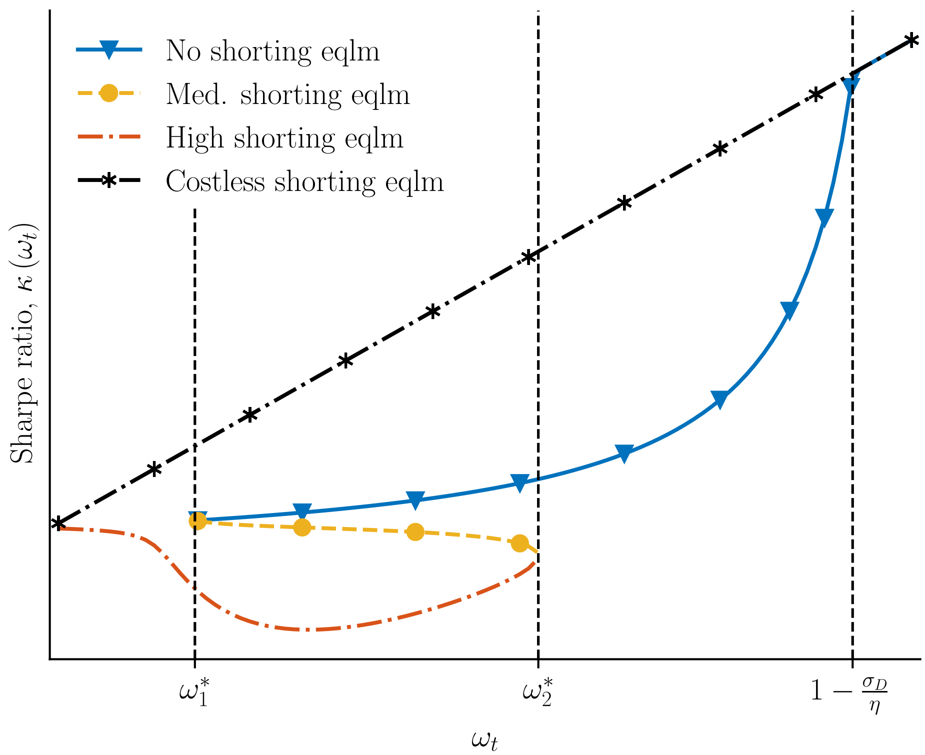

Figure 9: An illustration of Proposition 2.

zero weight on stocks, but the shorting fee φ deters her from actively short-selling. Since only the irrational investor is marginal in financial markets, the lines "Zero shorting cost" and "No shorting equil." deviate from each other when $\omega_t < 1 - \frac{\sigma_D}{\eta}$. In this region the magnitude of the lending fee, φ, does not impact the Sharpe ratio directly (only by deterring the R investors from actively shorting).

If ω_t becomes smaller than ω_2^* (but larger than ω_1^*) the economy exhibits three equilibria. In the first equilibrium, there is still no shorting. In the second and third, there is active shorting by the rational investor. Across these three equilibria, the higher the extent of shorting, the lower the Sharpe ratio. This is illustrated in Figure 10. If ω_t becomes smaller

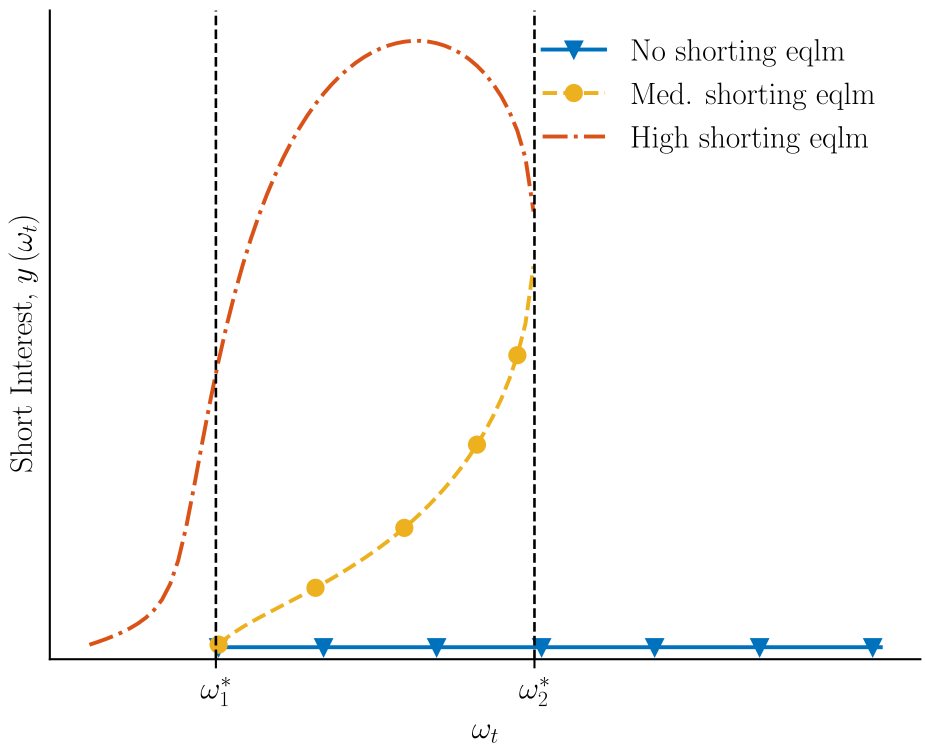

Figure 10: The ratio of shorted-to-lendable shares, y_t in the equilibria of model as a function of ω_t.

than ω_1^*, then the equilibrium becomes unique and involves shorting.[12]

Several features of Figure 9 are noteworthy. First, the Sharpe ratio is always lower than it would be in the absence of lending fees, even if investor R is not actively shorting shares, but is only investing in bonds.

Second, the presence of a region where multiple equilibria co-exist is not a very common

[12]To see why an equilibrium without shorting can no longer be an equilibrium when $\omega_t < \omega_1^*$, assume otherwise. Indeed assume that the R investor holds zero stocks and is not marginal in the stock market $\left(w_t^R = 0\right)$. The market clearing requirement, $\omega_t w_t^R + (1-\omega_t) w_t^I = 1$, along with $w_t^I = \frac{\kappa_t + \eta}{\sigma_D}$ implies that the Sharpe ratio would be $\kappa_t = \frac{\sigma_D}{1-\omega_t} - \eta$. Under this supposition, it would therefore be the case that $\mu_t - r + \varphi = \sigma_D \left(\kappa_t + \frac{\varphi}{\sigma^D}\right) = \sigma_D \left(\frac{\sigma_D}{1-\omega_t} - \eta + \frac{\varphi}{\sigma^D}\right) < 0$, where the inequality follows from $\omega_t < \omega_1^*$. Because $\mu_t - r + \varphi < 0$, equation (13) implies that the R investor would want to short the market, contradicting the assumption that she is optimally holding zero stocks.

feature of asset pricing models, especially when there is only one good and one positive-supply asset. To better understand the source of this multiplicity, it is useful to provide a concise derivation of the key statements in Proposition 1.

Specifically, suppose that we consider equilibria that involve active shorting ($w_t^R < 0$). In such equilibria, the optimal portfolio holdings can be expressed as

$$w_t^R = \frac{\kappa_t + \frac{\varphi}{\sigma_D}}{\sigma_D} \tag{26}$$

$$w_t^I = \frac{\kappa_t + \eta + \frac{\varphi}{\sigma_D}\tau y_t}{\sigma_D}, \tag{27}$$

while asset market clearing requires

$$\omega_t w_t^R + (1-\omega_t)\, w_t^I = 1. \tag{28}$$

Combining equations (26)–(28) leads to

$$\kappa_t = \sigma_D - (1-\omega_t)\,\eta - \frac{\omega_t}{\sigma_D}\varphi\left(1 + \tau y_t \frac{1-\omega_t}{\omega_t}\right), \tag{29}$$

which is equation (24) of Proposition 1. Note that the partial derivative of κ_t with respect to y_t is negative. This is intuitive: A higher value of y_t increases the effective rate of return to (long-portfolio) stock holders (I investors). The increased appetite by I investors to hold long positions lowers the Sharpe ratio. (Phrased differently, the absolute value of the Sharpe ratio increases, since the Sharpe ratio is negative when $w_t^R < 0$.)

This lowering of the Sharpe ratio strengthens the short-sellers' appetite to borrow the stock and short it. In equilibrium, the increased shorting demand raises the ratio of shorted-to-lendable shares, y_t, increasing the effective return to I investors, which further reduces the Sharpe ratio, etc.

These self-reinforcing effects are the root cause of the multiple equilibria. The easiest way to see this is by completing the computation of the Sharpe ratio, which requires us to determine the value of y_t that clears the lending market. Indeed, in any equilibrium involving $w_t^R < 0$ and $w_t^I > 0$ we must have

$$y_t = \frac{W_t^-}{W_t^+} = \frac{-w_t^R W_t^R}{w_t^I W_t^I} = -\frac{w_t^R}{w_t^I} \times \frac{\omega_t}{1-\omega_t}. \tag{30}$$

Using (26) to compute the ratio $\frac{w_t^R}{w_t^I}$ gives

$$\begin{aligned} y_t &= -\frac{\kappa_t + \frac{\varphi}{\sigma_D}}{\kappa_t + \eta^I + \frac{\varphi}{\sigma_D}\tau y_t} \times \frac{\omega_t}{1-\omega_t} \\ &= \frac{\eta - \frac{\sigma_D}{1-\omega_t} - \frac{\varphi}{\sigma_D}\tau\left(1-y_t\right)}{\eta + \frac{\sigma_D}{\omega_t} - \frac{\varphi}{\sigma_D}\tau\left(1-y_t\right)}, \end{aligned} \tag{31}$$

where the last line follows from (29) after collecting terms and simplifying. Rearranging (31) gives the quadratic equation (23), which is the key equation of Proposition 1. The rest of the proposition is devoted to studying this quadratic equation and confirming that its roots correspond to valid equilibria.

While equation (31) is particularly simple to analyze, the multiplicity of equilibria does not hinge on assuming that the supply curve $l\left(y_t\right)$ is constant at the level φ, as we show in Section 6.

The intuition behind the multiplicity of equilibria is contained in equation (31). For a given wealth distribution and belief discrepancy, a higher y_t makes long investors content with holding the same positive position at a lower equilibrium Sharpe ratio. This negative relation between the Sharpe ratio and y_t is responsible for equilibrium multiplicity: For instance, if something prompts rational investors to abandon their short positions, the resulting reduction in lending income requires a higher Sharpe ratio to compensate the long investors and clear the market. But this rise in the Sharpe ratio reinforces the incentive of short sellers to abandon their positions, which further lowers lending income, and further raises the Sharpe ratio, etc., until the market settles on a new equilibrium with (possibly zero) short interest.

4 Properties of the Equilibria

In Section 4.1 we show that equilibria with high shorting are beneficial for R investors. This implies that the worst possible outcome for R investors is for markets to coordinate on the equilibrium that deters them from short selling. Sections 4.2 and 4.3 discuss some broader implications of the model that are unrelated to coordination, but help further illustrate the model's key intuitions. Specifically, we perform comparative statics with respect to changes in φ, which capture shifts in the supply of lendable shares. Section 4.2 shows how marginal changes in the supply curve can lead to discontinuous drops in short interest. Section 4.3 shows that exogenous shifts in the supply of lendable shares may impact lending fees and short interest but have a muted impact on equilibrium expected returns.

4.1 Dynamics of the wealth shares

The three equilibria we identified above have different implications for the dynamics of the wealth shares of R investors. The next proposition shows that both the drift rate $\mu_t^R(\omega_t)$ of the wealth share of type R investors and the expected logarithmic growth rate of their wealth are higher in equilibria that feature higher short interest y_t.

Proposition 3 *For a fixed wealth share of the R-agents, ω_t, consider two equilibria A and B with the following properties:*

1. $w_t^R \leq 0$ *in both equilibria A and B.*
2. $y_t^B > y_t^A$ *(and accordingly $\kappa_t^B < \kappa_t^A$).*

Then the drift of investor R's wealth share in equilibrium $i \in \{A, B\}$, $\mu_t^{R,i}(\omega_t)$, satisfies

$$\mu_t^{R,B}(\omega_t) > \mu_t^{R,A}(\omega_t).$$

In addition, the drift of the logarithmic growth rate of investor R, defined as

$$g_t^R \equiv r_t + \max_{w^R \leq 0} \left\{ w_t^R (\kappa_t \sigma_D + \varphi) - \frac{1}{2} \left(w_t^R \sigma_D \right)^2 \right\} - (\rho + \pi), \tag{32}$$

is higher in equilibrium B than in equilibrium A, i.e., $g_t^{R,B}(\omega_t) > g_t^{R,A}(\omega_t)$.

Figure 11 provides an illustration of Proposition 3. The figure shows the stationary distribution of ω_t in the equilibrium associated with no shorting for values $\omega_t \in (\omega_1^*, \omega_2^*)$ and in the equilibrium associated with the highest shorting, $y^+(\omega_t)$, for $\omega_t \in (\omega_1^*, \omega_2^*)$. The figure shows that the distribution of ω_t has a higher stationary mean in the second equilibrium rather than in the first equilibrium. This is consistent with Proposition 3, which asserts a higher (logarithmic) growth rate for the wealth of R investors in the second equilibrium.

The comparatively higher probability mass of larger values of ω_t in the equilibrium that features shorting implies that there are two competing effects on the stationary mean of the Sharpe ratio κ_t. On the one hand — for a fixed ω_t — the Sharpe ratio is lower in equilibria featuring comparatively higher short selling. On the other hand, low values of ω_t become less likely in equilibria with comparatively more shorting activity. The first effect tends to lower the stationary mean of the Sharpe ratio in equilibria with comparatively higher shorting, the second effect tends to raise it. The overall effect on the stationary value of the Sharpe ratio is ambiguous. We revisit this issue in Section 5, when we discuss an extension of the model that allows for multiple stocks, a time-varying price-dividend ratio, and endogenous exit of R investors. Specifically, that section shows that R investors tend to exit the market for a stock when it is not profitable to short, thus resulting in low values of ω_t in the no-

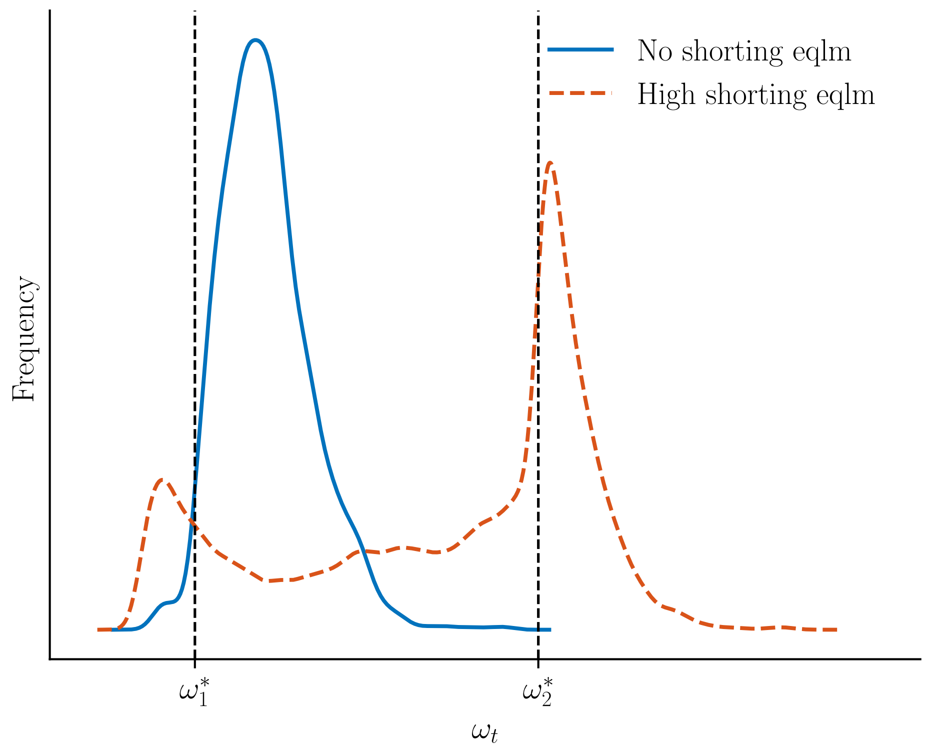

Figure 11: An illustration of Proposition 3. Simulating the model for the case in which market participants coordinate on the "high shorting" (respectively, "no shorting") equilibrium, the figure depicts the stationary distribution of the wealth share of the rational investor, ω_t, for the economy of Figure 9.

shorting equilibrium (and therefore low values of the Sharpe ratio and high values of the price-dividend ratio).

4.2 The instability of short interest

Besides the sensitivity that emanates from demand-side coordination, our model also implies that small shifts in the supply of lendable shares can lead to discontinuous changes in equilibrium short interest.

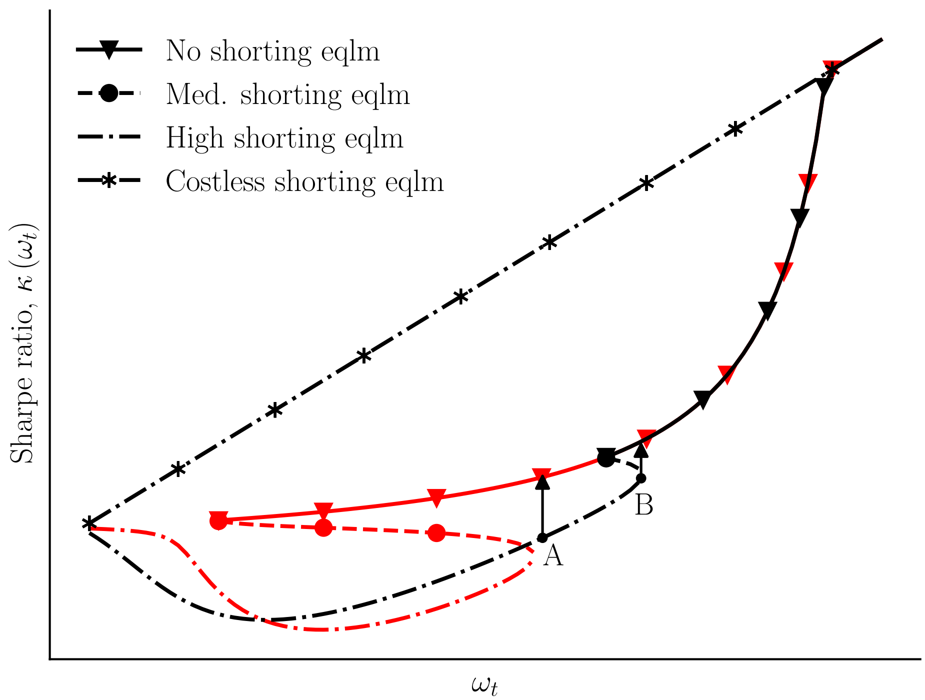

Figure 12: The red curve corresponds to a higher fee φ. Points A and B, which are equilibrium points for the low value of φ, would entail a discontinuous rise in the Sharpe ratio when φ increases.

Lemma 1

$$\frac{d\omega_2^*}{d\varphi} < 0.$$

Lemma 1 states that an increase in φ lowers the range of values ω_t that are associated with multiple equilibria. By implication, if, say, a given company can take some action to to induce a shift to the left in the supply curve for its lendable shares (i.e., an increase in φ), this can lead to a discontinuous change of short interest from a positive value to 0 if ω_t is close to ω_2^*.

Figure 12 illustrates the effects of an increase in φ (an upward shift in the supply for lendable shares) on the equilibrium values ω_t. The black curves correspond to the original value of φ, while the red curves to the higher one. The figure shows that points such as A and B, which represent equilibria with positive short interest for the original value of φ stop being equilibria if φ increases.

4.3 The ambiguous relation between Sharpe ratio and short interest

It would seem natural to expect that an increase in the supply of lendable shares (a downward shift of the lending supply curve) raises the Sharpe ratio, as it incentivizes short sellers to short the stock and thus lowers the absolute value of the (negative) Sharpe ratio. Remarkably, this outcome need not obtain in this model. Depending on the equilibrium, there is no unambiguous relation between the Sharpe ratio and shorting costs. This may be one of the reasons why the empirical literature finds that randomized increases in lendable shares affect short interest and lending rates but not excess returns.

The following proposition illustrates the novel implications of the model by focusing on the case of small ω_t.

Proposition 4 *Assume that the equilibrium involves a positive short interest. In the equilibrium associated with y^+ (which is the unique equilibrium if $\omega_t < \omega_1^*$), it holds that, for sufficiently small ω_t,*

$$\frac{d\kappa}{d\varphi} > 0. \tag{33}$$

In the equilibrium associated with y^-, for any value of ω_t,

$$\frac{d\kappa}{d\varphi} < 0. \tag{34}$$

Equation (33) in Proposition 4 appears counterintuitive. The explanation is that decreasing φ has two opposing effects. Inspection of equation (29) shows that a decline in φ has the direct effect of raising κ_t; however, since y_t is endogenous, the decline in φ also increases y_t, which — for a given φ — has the effect of lowering κ_t. Therefore, it is possible that a decline in φ (say, because of an exogenous change in the cost of supplying shares) lowers the fee f_t and increases the short interest y_t, but leaves the expected return on the stock unchanged. This is consistent with the empirical findings of Kaplan et al. (2013).

Figure 12 illustrates that an increase in φ could either raise or lower $\kappa_t\,(\omega_t)$, depending on the equilibrium and on whether ω_t is large or small.

5 Multiple Risky Assets and Time-Varying Price-Dividend Ratio

In the baseline model, the price-dividend ratio and the volatility of the stock market are both constant. This is an implication of a) logarithmic utility over intermediate consumption (which implies a constant wealth-to-consumption ratio) and b) a single asset in positive net supply. As is typical of models with similar setups, fluctuations in the interest rate offset the fluctuations of the risk premium, thus rendering the overall discount rate — and by implication the price-dividend ratio[13] — constant.

We next present a version of the model that features multiple risky assets and, by implication, a time-varying price-dividend ratio. After extending Proposition 2 to allow for multiple risky assets — a result of independent interest — we consider a limiting case of the multi-asset model that permits simple computations. Specifically, we study the limit in which there is a "small" stock subject to shorting costs and a "large" stock that can be shorted costlessly. In that limit, only the endowment of the large stock matters for the

[13]Note also that the expected dividend growth is constant.

interest rate and thus the price-dividend ratio of the small stock is time-varying and reflects variations in its risk premium. As a byproduct, we can study how the price of the asset changes when the nature of the equilibrium shifts, a phenomenon that lies at the core of the paper. In particular, we show that moving to an equilibrium with low shorting can be accompanied by a price increase, driven by the shift in the composition of the investors for the asset: would-be shorters are driven away, and the price is set predominantly by optimists.

5.1 Multiple risky assets

In this section we introduce an additional Lucas tree (stock 2) to our baseline model, which is not subject to any trading frictions, and comprises a potentially large part of the total market capitalization. We continue to assume that borrowing stock 1, which now comprises only (a possibly small) part of the market capitalization, requires lending fees, as in the baseline model.

We make one more convenient and realistic assumption. Specifically, while all investors participate in the markets for stock 2 and the risk-free asset, only a fraction of investors pays any "attention" to stock 1. The remaining fraction of investors simply optimize their portfolio over the risk-free asset and stock 2 and assign zero weight to stock 1. This assumption is in the spirit of Robert Merton's "limited recognition hypothesis," the idea that only a fraction of investors actively trade in some smaller stocks. .

Because stock 1 is no longer the only positive-supply asset, consumption-market clearing no longer implies a constant price-to-dividend ratio for stock 1, and a full analytical solution of the model is no longer available. However, we can still provide an analytic "CAPM-style" formula,[14] which constitutes a natural extension of the results of Proposition 2.

To start, we assume that in equilibrium the returns on stocks 1 and 2 follow a (possibly

[14] By CAPM-style formula we mean that the formula provides a connection between expected excess returns and the covariance matrix of returns.

correlated) vector diffusion process of the form

$$dR_{1,t} = \mu_{1,t}dt + \sigma_{1,t}dB_{1,t} + b_t\sigma_{2,t}dB_{2,t} \tag{35}$$

$$dR_{2,t} = \mu_{2,t}dt + \sigma_{2,t}dB_{2,t}, \tag{36}$$

where $B_{1,t}$ and $B_{2,t}$ are independent Brownian motions, $\mu_{1,t}$ and $\mu_{2,t}$ are the expected excess returns of the two stocks, and

$$\sigma_t \equiv \begin{bmatrix} \sigma_{1,t} & b_t\sigma_{2,t} \\ 0 & \sigma_{2,t} \end{bmatrix}$$

is a matrix capturing the loadings of the two stocks on the two Brownian motions. We assume that investors I believe that Brownian motion 1 follows the dynamics[15] $dB_{1,t} + \eta dt$, while no investor has any belief distortions pertaining to Brownian motion 2. We let $\overrightarrow{m}_t$ denote the vector of market-capitalization weights of the two stocks, and $m_{j,t}$, $j \in \{1, 2\}$, its entries.

From now on we use W_t^i to denote the wealth of all agents of type i that participate in market 1 and define $\omega_t^i \equiv W_t^i/(W_t^R + W_t^I)$. Letting $\widehat{\omega}_t$ denote the wealth share of the investors who actively participate in the market for stock 1, the market clearing condition is

$$\widehat{\omega}_t \sum_{i\in\{I,R\}} \omega_t \overrightarrow{w}_t^i + (1 - \widehat{\omega}_t) \begin{bmatrix} 0 \\ \widehat{w}_{2,t} \end{bmatrix} = \overrightarrow{m}_t, \tag{37}$$

[15] More formally, the Radon-Nikodym derivative of the true probability measure with respect to the subjective one is given by

$$Z_t^I \equiv e^{-\frac{\eta^2}{2}t+\eta B_{1,t}}.$$

where $\widehat{w}_{2,t} = \frac{\mu_{2,t}-r_t}{\sigma_{2,t}^2}$ is the optimal portfolio holding of stock 2 by investors who don't participate in stock 1, and $\overrightarrow{w}_t^i$ is the vector of portfolio holdings of an investor $i \in \{I, R\}$ that is active in the market for stock 1. The market clearing condition (37) leads to the following result.

Proposition 5 *Define* $\kappa_{1,t} = \frac{(\mu_{1,t}-r)-b_t(\mu_{2,t}-r)}{\sigma_{1,t}}$ *as the Sharpe ratio of a portfolio that invests 1 unit in asset* 1 *and shorts* b_t *units of asset 2. Let* $\widetilde{m}_{1,t} \equiv \frac{m_{1,t}}{\widehat{\omega}_t}$.

In an equilibrium with shorting in asset 1 ($y_t > 0$), y_t *is given by the root(s) of the quadratic equation*

$$0 = y\left(\eta + \frac{\widetilde{m}_{1,t}}{\omega_t}\sigma_{1,t} - \frac{\varphi}{\sigma_{1,t}}(1-\tau y)\right) - \left(\eta - \frac{\widetilde{m}_{1,t}}{1-\omega_t}\sigma_{1,t} - \frac{\varphi}{\sigma_{1,t}}(1-\tau y)\right) \tag{38}$$

that lie(s) in the interval $[0, 1)$*, and the Sharpe ratio is given by*

$$\kappa_{1,t} = \widetilde{m}_{1,t}\sigma_{1,t} - (1-\omega_t)\,\eta - \frac{\varphi}{\sigma_{1,t}}\left(\omega_t + (1-\omega_t)\,\tau y_t\right). \tag{39}$$

Similarly, in an equilibrium without shorting in asset 1 we have $\kappa_{1,t} = \sigma_{1,t}\widetilde{m}_{1,t} - (1-\omega_t)\,\eta$ *if investor R holds an interior position in asset* 1 *and* $\kappa_{1,t} = \frac{\sigma_{1,t}\widetilde{m}_{1,t}}{1-\omega_t} - \eta$ *otherwise.*

The excess return to asset 2 is given by the conventional CAPM relationship

$$\mu_{2,t} - r_t = [0,1]\,\sigma_t\sigma_t'\overrightarrow{m}_t. \tag{40}$$

Equations (39) and (38) specialize to (24) and (23), respectively, when $\widetilde{m}_{1,t} = 1$ and $\sigma_{1,t} = \sigma_D$. In this sense, Proposition 5 is a natural extension of Proposition 2, except that the Sharpe ratio in Proposition 5 pertains to a portfolio that invests one dollar in asset 1 and shorts b_t units of asset 2 (so as to "hedge out" the exposure of the portfolio to the second Brownian shock).

As in Proposition 2, the excess return on asset 1 can be decomposed into a risk premium, a (wealth-weighted) belief distortion, and a component that reflects the impact of shorting costs. Specifically, equation (39) implies that in an equilibrium with active shorting, the expected return of stock 1 is

$$\mu_{1,t} - r_t = \underbrace{b_t\left(\mu_{2,t} - r_t\right) + \widetilde{m}_{1,t}\sigma_{1,t}^2}_{\text{risk compensation}} - \underbrace{\left(1-\omega_t\right)\eta\sigma_{1,t}}_{\text{wealth-weighted optimism}} - \underbrace{\varphi\left(\omega_t + \left(1-\omega_t\right)\tau y_t\right)}_{\text{impact of shorting costs}}. \qquad (41)$$

All else equal, a higher level of y_t lowers $\mu_{1,t} - r$ — consistent with the empirical finding that short interest negatively predicts returns — and higher values of the lending fee φ lower equilibrium expected excess returns.

5.2 A limiting economy with a small and a large stock

The CAPM-style formulas provide equilibrium returns conditional on the equilibrium covariance matrix and the investor wealth shares. To fully solve the model, we consider a limiting, multi-stock economy, in which trees of type 1 are small compared to trees of type 2 and also the fraction of investors that pay attention to trees of type 1 is small. Since this section involves some detailed modeling assumptions, we relegate the full presentation to Appendix B. In the text we simply summarize the setup and the main findings.

Specifically, suppose that there are two types of trees, namely "small" trees (type-1 trees) and "large" trees (type-2 trees). Type-2 trees have dividends similar to the baseline model, namely $D_{2,t,s} = \phi_2\delta_2 D_{2,t} e^{-\delta_2(t-s)}$, where $\phi_2 > 0$, $\delta_2 > 0$, and $D_{2,t}$ follows a geometric Brownian motion, $\frac{dD_{2,t}}{D_{2,t}} = \mu_{2,D}dt + \sigma_{2,D}dB_{2,t}$. Type-1 trees produce dividends

$$D_{1,t,s} = \phi_1\delta_1 D_{2,s} e^{-\delta_1(t-s)+\sigma_{1,D}(B_{1,t}-B_{1,s})},$$

with $\phi_1 > 0$ and $\delta_1 > 0$. With the above dividend specifications, the dividend ratio of type-1 to type-2 trees is

$$\frac{D_{1,t}}{D_{2,t}} = \frac{\int_{-\infty}^{t} D_{1,t,s} ds}{\int_{-\infty}^{t} D_{2,t,s} ds} = \frac{\phi_1}{\phi_2} \int_{-\infty}^{t} \frac{D_{2,s}}{D_{2,t}} \delta_1 e^{-\delta_1 (t-s) + \sigma_{1,D} (B_{1,t} - B_{1,s})} ds, \tag{42}$$

which is a stationary process.

The above assumptions imply that the dividend shares of type-1 and type-2 trees at an arbitrary time t are stationary fractions of aggregate consumption $D_{1,t} + D_{2,t}$, while the dividends of the tree with which a fixed cohort s is endowed upon entering the economy follow a geometric Brownian motion.[16] Moreover, when type-1 trees are small compared to type-2 trees $\left(\frac{\phi_1}{\phi_2} \approx 0\right)$, aggregate consumption is approximately equal to the aggregate dividends of the large, type-2 trees, and therefore aggregate consumption follows a geometric Brownian motion. The implication is that the interest rate and the risk premium for type-2 trees both converge to constants as the ratio $\frac{\phi_1}{\phi_2} \to 0$ goes to zero.

In the baseline model, entry and exit of investors into the single stock market was tied to the arrival and departure of agents in the economy and was essentially exogenous. The extension to two risky-asset markets requires that we model the entry and exit into the market for stock 1. Specifically, we assume that investors of both types (R and I) gain and lose interest in stock 1 at the rate χ per unit of time dt. Phrased differently, a measure χ of investors becomes interested in market 1 per unit of time and a measure χ of investors loses interest for exogenous reasons. Of the arriving investors a fraction ν is of type R, as in the baseline model.

We go further and make an additional assumption that captures the shifts in investor-base composition that can accompany shifts in equilibrium. Specifically, we assume that

[16] Ito's Lemma implies that

$$\frac{dD_{1,t,s}}{D_{1,t,s}} = \left(\frac{\sigma_{1,D}^2}{2} - \delta_1\right) dt + \sigma_{1,D} dB_{1,t} \text{ and } \frac{dD_{2,t,s}}{D_{2,t,s}} = (\mu_{2,D} - \delta_2)\, dt + \sigma_{2,D} dB_{2,t}.$$

investors incur a small, non-pecuniary, disutility flow ε from paying attention to stock 1. An investor of type $i \in (I, R)$ will consequently choose to keep staying in the market if and only if her expected discounted utility from remaining attentive to stock 1 is above the present value of the (small) disutility cost of attention. To anticipate the results, this participation cost will drive some type-R agents out of the market when no-shorting equilibria prevail, capturing the notion that short sellers lose interest in stock 1 in such times.

For more detail, observe first that, as long as this disutility is small enough, it is irrelevant for investors of type I, since they always choose a strictly positive portfolio in stock 1. For investors of type R, however, there are regions of ω_t where their optimal holding of stock 1 is zero. Even a small disutility, therefore, can lead them to exit the market. Formally, an investor of type R finds it optimal to remain in the market for stock 1 if and only if

$$V^R(\omega_t) \equiv \mathrm{E}_t \left[\max_{w^R_{1,u}} \int_t^T e^{-\rho(u-t)} \left(w^R_{1,u} \left(\mu_u - r_u + \lambda^R_u \right) - \frac{1}{2} \left(w^R_{1,u} \sigma_u \right)^2 - \varepsilon \right) du \right] \geq 0, \quad (43)$$

where T is the stochastic time of exit from the market for stock 1 (be it endogenous or exogenous). Equation (43) uses the assumption of logarithmic preferences — along with the simplifying assumption that stocks 1 and 2 are independent — to express the net expected utility gain from continued presence in market 1 as the increase in investor R's logarithmic growth rate of wealth, $w^R_{1,u} \left(\mu_u - r_u + \lambda^R_u \right) - \frac{1}{2} \left(w^R_{1,u} \sigma_u \right)^2$, net of the flow disutility of presence in the market ε. The requirement that this net gain stay positive at all times implies that, for given equilibrium functions $\kappa(\omega_t)$ and $y(\omega_t)$, there is a critical boundary $\bar{\omega}$, typically lying in the region of ω_t where $w^R_{1,u}(\omega_t) = 0$, that acts as a "reflecting barrier" for ω_t. Specifically, if the process ω_t were to ever exceed $\bar{\omega}$, there would be enough exit to restore ω_t back to $\bar{\omega}$.[17]

Some further technical assumptions on investor entry and exit are detailed in Appendix B. Here we simply state the main result, which provides an ordinary differential equation

[17] This behavior is reminiscent of models of industry equilibrium with endogenous entry and exit (e.g., Leahy (1993), Baldursson and Karatzas (1996).)

(ODE) for the price-dividend ratio. For simplicity, we assume that the Brownian motions $B_{1,t}$ and $B_{2,t}$ are independent.

Proposition 6 *Using the expressions for w_t^i, $\kappa_{1,t}$ (with $b = 0$), and y_t from Proposition 5, the wealth share ω_t follows the diffusion process*

$$d\omega_t = \mu_t^R dt + \sigma_t^R dB_{1,t} - dF_t, \tag{44}$$

where F_t is an increasing (singular) process that reflects ω_t to remain below the value $\bar{\omega}$ that makes equation (43) hold as an equality and μ_t^R and σ_t^R are given by

$$\mu_t^R = \omega_t \left(\left(w_{1,t}^R - \widetilde{m} \right) \sigma_{1,t} \left(\kappa_t - \sigma_{1,t} \widetilde{m} \right) + w_{1,t}^R \varphi + \frac{y_t \widetilde{m}}{1 - y_t} \varphi \left(1 - \tau \right) \right) + \chi \left(\nu - \omega_t \right), \tag{45}$$

$$\sigma_t^R = \omega_t \left(w_{1,t}^R - \widetilde{m} \right) \sigma_{1,t}, \tag{46}$$

where $\sigma_{1,t} = \frac{p'(\omega_t)}{p(\omega_t)} \sigma_t^R + \sigma_{1,D}$ is the volatility of stock 1 and the price-dividend ratio $p_t = p(\omega_t)$ solves the ordinary differential equation

$$\frac{1}{2} \frac{\partial^2 p}{\partial \omega_t^2} \left(\sigma_t^R \right)^2 + \frac{\partial p}{\partial \omega_t} \left(\mu_t^R + \left(\sigma_{1,D} - \kappa_{1,t} \right) \sigma_t^R \right) - p \left(r + \delta_1 + \kappa_{1,t} \sigma_{1,D} \right) + 1 = 0 \tag{47}$$

in the region $0 \leq \omega_t \leq \bar{\omega}$.

Remark 1 *Since there are multiple equilibrium values for w_t^i, $\kappa_{1,t}$, and y_t in Proposition 6, there are a continuum of solutions for $p(\cdot)$ and $\bar{\omega}$, depending on the equilibria on which agents coordinate at each value of ω_t.*

The expressions for μ_t^i and σ_t^i in Proposition 6 coincide with (17) and (16) when $\widetilde{m} = 1$, $\chi = \pi$, and $\sigma_{1,t} = \sigma_D$.[18] Moreover, with the dividend growths of stocks 1 and 2 being independent, so are their stock-price processes (in the limit where stock 1 becomes small)

[18] To see this, substitute the expression for the equilibrium interest rate (25) to (17).

and the expressions for y_t, $w^i_{1,t}$, and $\kappa_{1,t}$ in Proposition 6 (when $\widetilde{m} = 1$ and $\sigma_{1,t} = \sigma_D$) coincide with the respective expressions in the baseline model. Finally, if $\varepsilon = 0$, then $\bar{\omega} = 1$, as in the baseline model. In short, if one dropped the goods-market clearing requirement from the baseline model, the resulting expression for the price-to-dividend ratio would be given by (47) (with $\widetilde{m} = 1$ and $\varepsilon = 0$).

The main complications with solving (47) are that a) it is a non-linear ODE[19] and b) if $\varepsilon > 0$, this ODE is to be solved over a domain of values of ω_t so that (43) holds, together with the boundary $\bar{\omega}$ implied by Equation (43). The appendix describes an iterative numerical procedure to solve this problem conditional on the equilibrium that investors coordinate on at each value of ω_t.

To illustrate the results, Figure 13 presents the solution for the price-dividend ratio. We are interested in situations where the disagreement is large ($\eta = 0.9$), and the speed of investor churn in market 1 is quite large ($\chi = 2$), to capture short-termism. The idiosyncratic dividend volatility is not too large, $\sigma_{1,D} = 7\%$, and the shorting fee is at the high levels that one encounters for stocks that are "on special" ($\varphi = 5.7\%$). A proportion $\nu = 0.7$ of new investors are of type R. In equilibrium, this value of ν ensures that the endogenous exit decision is meaningful, that is, under any equilibrium there would a possibility that ω_t "spends time" in a region where a zero holding of asset 1 is optimal for investor R. Finally, we assume that the sum of interest rate and depreciation $r + \delta_1$ for stock 1 is 0.1. We choose a value of $\tau = 0.8$ based on the industry practice of rebating about 80% to the mutual funds or ETFs that provide their shares for lending.[20] Finally, for the disutility ε we intentionally choose a very small amount (2.5 basis points).

Figure 13 shows the price-dividend ratio under two different assumptions on the equilib-

[19]The reason why (47) is non-linear is that μ^i_t and σ^i_t depend on $\sigma_{1,t}$, which in turn depends on $p(\omega_t)$ and $p'(\omega_t)$.

[20]Source: "Unlocking the potential of your portfolios: iShares Security Lending." Blackrock, 2021. Available at `https://www.ishares.com`.

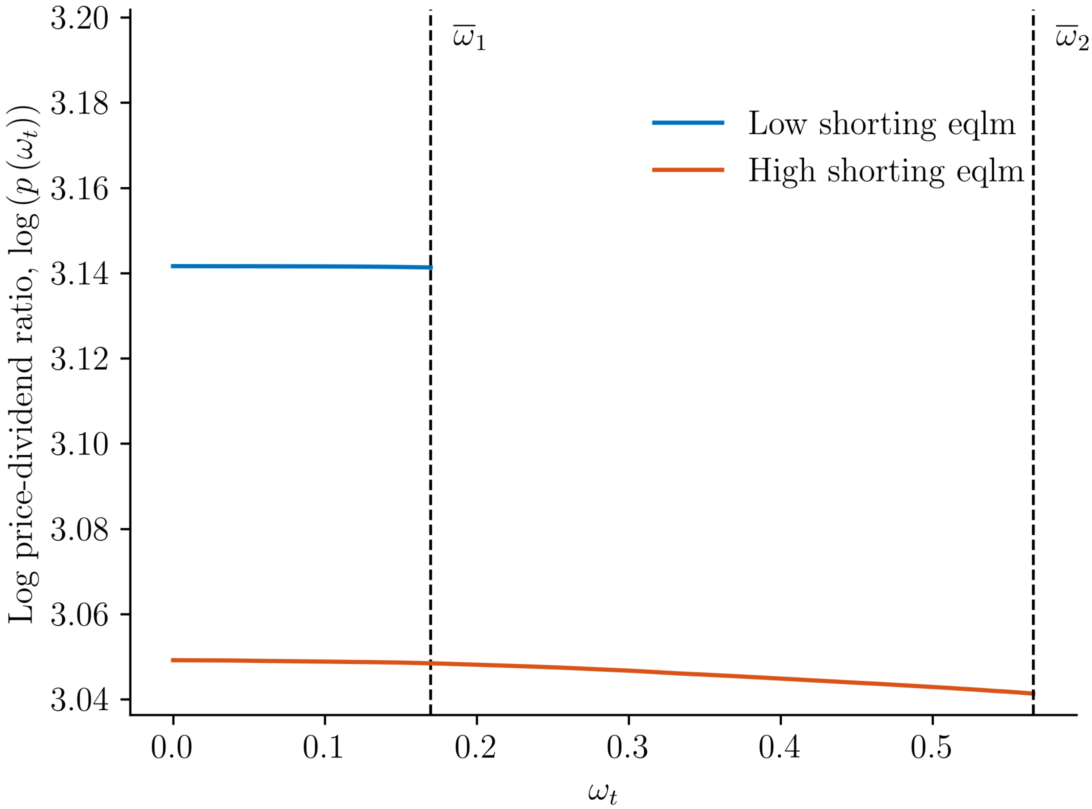

Figure 13: The price-dividend ratio in the equilibria involving the highest and the lowest extent of shorting.

rium that investors coordinate on (when multiple equilibria are possible). Specifically, the line "low shorting" assumes that investors always coordinate on the equilibrium with zero shorting, if it exists. By contrast, the line "high shorting" assumes that investors always coordinate on the equilibrium with the highest possible shorting. Note that both lines extend only until the levels $\bar{\omega}_1$ and $\bar{\omega}_2$, respectively, which are the levels of ω_t at which R investors exit in the two equilibria.

There are several noteworthy features of Figure 13. First, the price-dividend ratio for the low shorting equilibrium is higher than the price-dividend ratio for the high shorting equilibrium. This may seem counterintuitive in light of the fact that the high shorting equilibrium implies a lower Sharpe ratio *for a fixed* ω_t. The reason is that in the low shorting equilibrium many R investors do not maintain a presence in the market for stock 1. In that equilibrium shorting occurs only for low values of ω_t (below 0.3 in this numerical example), which realize infrequently. Consequently, a significant fraction of R investors leave the market, meaning that the typical steady-state values of ω_t are low. The few remaining R investors maintain a presence in the hope that, over the course of their stay in market 1, the process for ω_t may drift even lower, thus activating shorting — a low probability event in this equilibrium. Since the (stochastic) steady-state wealth share of rational investors, ω_t, is quite low, so is the Sharpe ratio, and thus the price-dividend ratio is high.

By contrast, the high shorting equilibrium attracts more R investors, who rationally anticipate that they will be actively shorting (with high probability) over the course of their stay in the market for stock 1. Therefore, the typical values of ω_t are higher, the Sharpe ratio is higher, and the price-dividend ratio is lower than in the low-shorting equilibrium.

We also note that our assumption of a high χ ("short-termism") is reflected in the fact that the curves for the price-dividend ratio are essentially flat lines, since the process ω_t mean-reverts quickly to its stochastic steady state under either equilibrium. Because of this feature, the price-dividend ratio in either equilibrium is roughly equal to its steady-state value irrespective of the current value of ω_t.

In terms of quantities, Figure 13 implies that an unanticipated shift in equilibrium (from the "high shorting" to the "low shorting") will make the price-dividend ratio jump upward by about 10%.

6 General Supply Curve for Shorting

The baseline version of the model assumes an elastic supply of lendable shares, so that the lending fee is constant. The results generalize readily to the case in which the supply of lendable shares is increasing in y_t so that $f_t = l\left(y_t\right)$, where $l'\left(y_t\right) > 0$.

We obtain the following proposition.

Proposition 7 *Consider the model of Section 2, but without Assumption 1. Define*

$$z(y_t) \equiv f_t(1-\tau y_t) = l(y_t)(1-\tau y_t). \tag{48}$$

Assume that $\frac{\sigma_D}{1-\omega_t} - \eta < 0$ *and that there exists* $y_t^* \in (0,1)$ *such that*

$$y_t^* = \frac{\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(y_t^*)}{\eta + \frac{\sigma_D}{\omega_t} - \frac{1}{\sigma_D} z(y_t^*)}, \tag{49}$$

and $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(y^*) > 0$. *Moreover, if* $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(0) < 0$, *then there exist at least two values of* $y_t^{(j)}$, $j = \{1,2\}$ *satisfying both (49) and* $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z\left(y^{(j)}\right) > 0$ *and three equilibria co-exist. In the first equilibrium, the R investor holds a zero portfolio, and the Sharpe ratio is* $\kappa_t = \frac{\sigma_D}{1-\omega_t} - \eta$. *There also exist two other equilibria, with R investors holding negative portfolios and the Sharpe ratio given by*

$$\kappa_t = \sigma_D - (1-\omega_t)\eta - \frac{\omega_t}{\sigma_D} l\left(y_t^{(j)}\right)\left[1 + \tau y_t^{(j)} \frac{1-\omega_t}{\omega_t}\right]. \tag{50}$$

In all equilibria the interest rate is given by (25), and the lending fee is given by $l\left(y_t^{(j)}\right)$.

Remark 2 *In the special case* $l(y_t) = \varphi$, *equations (49) and (50) become identical to (23) and (24), respectively.*

Figure 14 illustrates Proposition 7. For this particular numerical example we choose $l(y_t) = \varphi(1 + 2y_t^3)$, $\eta = 0.9$, $\sigma_D = 7\%$, $\tau = 0.8$, $\omega = 0.5$, and $\varphi = 0.055$. The figure plots the left hand side (dotted line) of equation (49) and the right hand side (solid line). Points B and C correspond to the two fixed points. Point A in the figure illustrates the assumption $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(0) < 0$. This inequality implies that there is a third equilibrium in which R investors choose to not go short. The fees in the three equilibria differ, with the lending

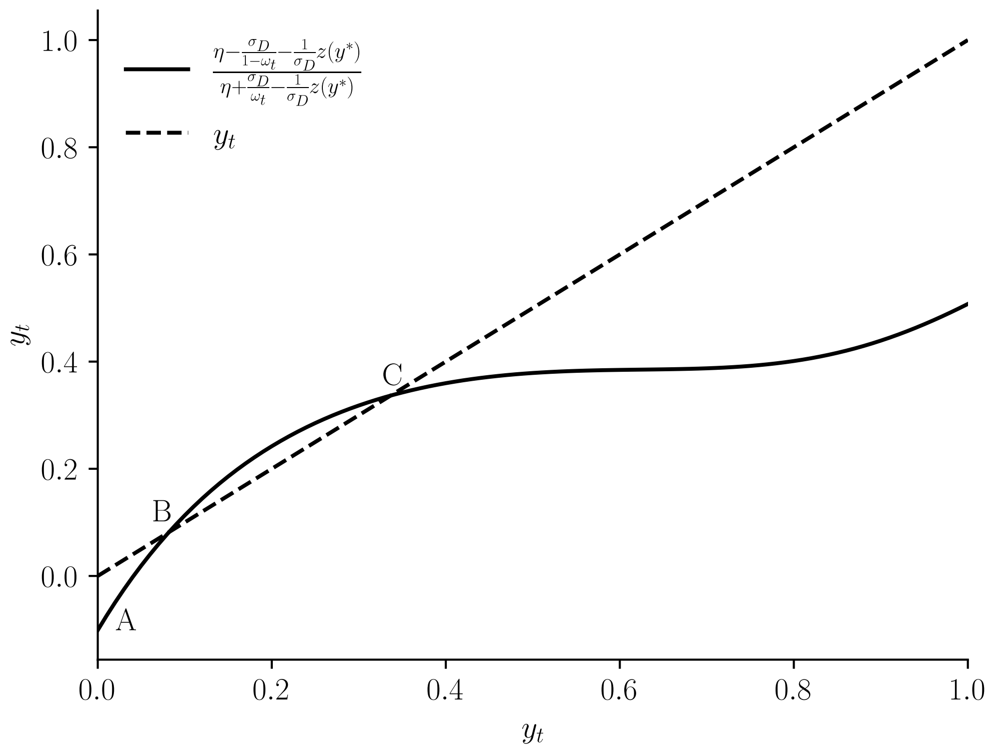

Figure 14: An illustration of Proposition 7.

fee being lowest ($l\,(0) = \varphi$) in equilibrium A, in which the shorting market is inactive, and highest in equilibrium C, in which y is highest.

7 Conclusion

The stratospheric rise in the price of Gamestop in January of 2021 appears to have affected shorting strategies across many other stocks. Short sellers retreated across the board. This retreat affected even larger stocks (S&P 500 constituents), stocks that were not particularly discussed in the WSB subreddit, and stocks whose retail purchase volume did not change abnormally.

The magnitude of these spillover effects is remarkable, given the very small capitalization of Gamestop. Moreover, it does not appear that, for stocks other than Gamestop, there was any obvious change in the supply behavior of lendable shares in January of 2021 (such as a regulatory change, or some increased technological difficulty in identifying stock lenders). It is more plausible that Gamestop stoked fears among the investors shorting other stocks.

In this paper we propose a theory for why shorting can be fickle and subject to sudden reversals, not attributable to a change in the fundamentals or the stock lending technology. To illustrate the notion of sudden reversals, we use a time-honored device in economic theory, namely the existence of multiple equilibria — a modeling device to illustrate the feedback effects between the spot market and the lending market.

In the model, shorting can exhibit "run-type" patterns. An event that prompts some short sellers to abandon their short positions can ignite a self-propagating cycle: Less shorting also implies less lending income for investors with long positions, who now need to be compensated with a higher Sharpe ratio, which in turn further prompts short sellers to abandon their strategies. (Going in the opposite direction, a high level of shorting activity further subsidizes long positions, thus lowering the equilibrium Sharpe ratio and attracting further short selling). Thus, for the same fundamentals, there can be multiple equilibria — a manifestation of the self-reinforcing nature of shorting decisions.

The paper can consequently provide an explanation of how attention-capturing events (such as a short squeeze in a stock of small market capitalization) can spiral quickly across many shorting strategies — if such events are interpreted by market participants as indicative of a change in equilibrium play by other market participants.

The model can also help explain a simultaneous decline in short selling and rise in the price of a stock (such as the one that we document in our empirical analysis) even when neither lending fees, nor fundamentals, nor the supply curve for lendable shares change. At first sight, it would appear that a rise in the stock price (absent a change in fundamentals or lending fees) should attract, rather than repel short sellers. We show, though, that the likelihood of a switch to a no-shorting equilibrium can drive the incentive of participating in the market low enough to prompt would-be short sellers to “abandon” the asset to the optimists.

While motivated by recent events, the analysis of the paper has broader implications for the empirical relationship between shorting and stock returns. One aspect we wish to highlight is that this model helps explain why shifts in the supply of lendable shares do not have a clear impact on the Sharpe ratio.

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Appendix

A The determination of the Lending Fee

In the text we assume a “flat” supply curve for lending shares. That is, we assume $f_t = l(y_t) = \varphi$. We provide here the simplest model that supports this assumption. We also discuss how to extend the model to allow for an increasing $l(\cdot)$.

All interactions considered in this section happen anew every period, where the length of the period is idealized to be “dt,” that is, infinitesimal. (We could formalize this assumption by considering a discrete-time model where the length Δ of a period is taken to go to zero, and focusing on the limit of resultant equilibria.)

We start by considering the long investors, who wish to lend their shares. Each investor lends all her shares to any one of a competitive fringe of risk-neutral “lender’s dealers” in exchange for an income stream that is proportional to the dollar value of shares the investor lends. This income stream is determined as follows. In equilibrium, each broker receives a fee f_l per dollar of shares it lends out, which constitute only a proportion y_t of the shares it borrows from investors. (We omit time subscripts from now on.) Therefore, competition between the brokers drives the income stream of investors to yf_l per dollar of shares they lend.

At the other end of the lending transaction, desirous short sellers interact with a competitive set of “borrower’s brokers.” Specifically, for every borrowing fee f_b the would-be short sellers provide the dollar amount that they would like to short, and the brokers take the value f_b as a given when they attempt to fill the investor’s borrowing orders.

All of the frictions in this model pertain to the interaction between lender’s dealers and borrower’s brokers. Specifically, to initiate a stock loan the representative broker must pay a cost ξ per dollar value of share “located” with a dealer, per unit of time. This cost is construed as labor cost that compensates brokers for their disutility of labor.

The interaction between the broker and the dealer takes the form of bilateral Nash bargaining in which the broker has bargaining power $1/(1+z)$ for a parameter $z \in (0, \infty)$. Given our assumption that all interactions (between investors and brokers or dealers and between brokers and dealers) happen anew every period, the outside option for both brokers and dealers is the failure to transact during the period. This means that the gains from trade to the dealer equal the lending fee f_l, while to the broker the borrowing fee net of the lending one $f_b - f_l$ — the searching and matching cost ξ has been sunk at this point. The total gains from trade equal f_b, the foregone revenue from the would-be short seller. Given the bargaining protocol, it follows that

$$f_l = \frac{z}{1+z}(f_l + f_b - f_l) = \frac{z}{1+z} f_b. \tag{51}$$

Since brokers are competitive, they break even on net, meaning that

$$f_b = f_l + \xi, \tag{52}$$

so that

$$f_l = z\xi, \tag{53}$$

$$f_b = (1+z)\xi. \tag{54}$$

To keep the model transparent and tractable, assume that all brokers are members of the representative household, and therefore the fees that compensate them for their effort are rebated to each households as an income stream proportional to the household's wealth and independent of the composition of the household's portfolio.

Setting $\beta = (1+z)\xi$ and $\tau = z/(1+z)$, this extended model is equivalent to the model we assumed in the text. To generalize to upward-sloping supply curves, one would simply assume an increasing cost $\xi(y)$.

B The Price-Dividend Ratio of a Small Stock

This section provides the details of the entry-and-exit process for the model of Section 5.2 and proves Proposition 6.

The entry and exit into market 1 happens either for endogenous or exogenous reasons. By "endogenous" we mean that investors conduct a cost-benefit analysis before deciding whether to keep paying attention to the market for stock 1. In addition to this optimizing choice, we assume that investors enter and exit the market for exogenous reasons. This exogenous flux of investors is modeled with the sole purpose of making the model solution more tractable and transparent.

Specifically, recalling that W_t^i denotes the (aggregate) wealth of type-i investors that participate in market 1, we assume

$$dW_t^i = dW_t^{i,\text{part}} + \chi\left(\nu^i(W_t^I + W_t^R) - W_t^i\right)dt - 1_{i=R} \times \frac{W_t^I + W_t^R}{1-\omega_t} dF_t + \omega_t^i\left(dL_t - dN_t\right), \quad (55)$$

where $dW_t^{i,\text{part}}$ is the wealth growth of an investor of type $i \in \{I, R)$ who is already participating in the market for stock 1.[21] The term $\chi\left(\nu_i(W_t^I + W_t^R) - W_t^i\right)dt$ reflects entirely exogenous, non-optimizing entry, which happens at some rate χ.

As in the baseline model, we assume that this exogenous entry-and-exit process affects

[21]For completeness, $dW_t^{i,\text{part}} = W_t^{i,\text{part}}\mu_W^i dt + W_t^{i,\text{part}}\overrightarrow{\sigma}_W^i dW_t$, where

$$\mu_W^i = r_t + \pi + n_t + \overrightarrow{w}_{t,s}^i\left(\overrightarrow{\mu}_t - r_t 1_{\{2\times1\}} + \lambda_{t,s}^i \begin{bmatrix}1\\0\end{bmatrix}\right) - \frac{c_{t,s}^i}{W_{t,s}^i}$$

for any $s \le t$ and $\overrightarrow{\sigma}_W^i = \overrightarrow{w}^{i\prime}\sigma_t$.

the composition, but not the sum, of $W_t^I + W_t^R$, since

$$\sum_{i \in \{I,R\}} \chi \left(\nu^i \left(W_t^I + W_t^R \right) - W_t^i \right) = 0.$$

The term $-1_{i=R} \times \frac{W_t^I + W_t^R}{1-\omega_t} dF_t$ captures the endogenous exit of R investors. As in the text, the (singular) process dF_t is constructed so that ω_t stays below the critical value of ω_t, which ensures that (43) holds.

Mostly for technical tractability reasons, we assume another source of exogenous entry and exit, which is reflected in the term $\omega_t^i \left(dL_t - dN_t \right)$ on the right-hand side of (55). This entry and exit process leaves the composition of wealth in the market (between R and I investors) unaffected, but ensures that the wealth of the investors who pay attention to the market "1" stays proportional to the "size" of market 1. Specifically, we define dL_t and dN_t as the two singular, increasing processes that "control" $W_t^I + W_t^R$ so that the ratio of stock market capitalization of asset 1 to the total wealth of investors participating in market 1, $\widetilde{m}_t = \frac{M_{1,t}}{W_t^I + W_t^R}$, stays constant across time ($\widetilde{m}_t = \widetilde{m}$).[22] Because $(dL_t - dN_t)$ is multiplied by ω_t^i, this exogenous entry-and-exit process does not impact the composition of wealth between R and I investors. The purpose of this exogenous entry-and-exit term is transparency and tractability: By ensuring a constant $\widetilde{m}_t$, if there were no differences of opinion ($\eta = 0$), the excess return, the price-dividend ratio, and the volatility of stock 1 would all be constant. Thus, we can eliminate a state variable from the problem, namely the ratio of market capitalization to the total wealth of investors in market 1). Economically, this means that we can abstract from the economic effects of limited participation (that have been studied extensively in the literature) and isolate the impact of shorting frictions. It is also worth highlighting that the term $\omega_t^i \left(dL_t - dN_t \right)$ would endogenously approach zero as

[22] These processes can be uniquely constructed from the running maximum and minimum of the difference between $(W_t^R + W_t^I) - M_{1,t}$. For details see Karatzas and Shreve (2012, p. 210) on the Skorohod equation.

δ_1 and χ approach infinity.[23] Thus, our computations would be approximately valid, if we eliminated the term $\omega_t^i\left(dL_t - dN_t\right)$, as long as the analysis focuses on cases where investors are short-termist (χ is large) and the ratio of the dividends of a typical tree 1 to tree 2 mean reverts fast.

Having described the entry and exit of investors into the market for stock 1, we now proceed to derive the differential equation in Proposition 6. Using the market clearing condition $\sum_{i\in\{I,R\}} \omega_t^i w_t^{i,1} = \widetilde{m}$, and applying Ito's Lemma to $\omega_t^i = \frac{W_t^i}{W_t^I + W_t^R}$ leads to

$$d\omega_t^i = \mu_t^i dt + \sigma_t^i dB_{1,t}, \tag{56}$$

where

$$\begin{aligned}
\mu_t^i &= \omega_t^i\left[\left(w_{1,t}^i - \widetilde{m}\right)\sigma_{1,t}\left(\kappa_t - \sigma_{1,t}\widetilde{m}\right) + w_{1,t}^i f_t + \widetilde{n}_t\right] + \chi\left(\nu_t^i - \omega_t^i\right),\\
\sigma_t^i &= \omega_t^i\left(w_{1,t}^i - \widetilde{m}\right)\sigma_{1,t},
\end{aligned}$$

and[24]

$$\widetilde{n}_t \equiv -\sum_{i\in\{I,R\}} w_{1,t}^i {\omega_t}^i \lambda_t^i = \frac{y_t \widetilde{m}}{1-y_t} f_t \left(1-\tau\right).$$

Since $\frac{\phi_1}{\phi_2} \approx 0$, the aggregate endowment follows a geometric Brownian motion in the limit, and the interest rate is constant $r_t = r$. Accordingly, the price of a stock of type 1 follows

[23] The reason is that the price-dividend ratio and the ratio of the dividend processes for the two trees (given in (42)) approach constants, thus implying that $\widetilde{m}_t = \widetilde{m}$ approaches a constant.

[24] Using $\sum_{i\in\{I,R\}} w_{1,t}^i \omega_t^i = \widetilde{m}_t$, the definition $y_t = -\frac{w_{1,t}^R \omega_t 1_{\{w_{1,t}^R<0\}}}{w_{1,t}^I \omega_t^I}$ and the definition of λ_t^i leads to

$$-\sum_{i\in\{I,R\}} w_{1,t}^i \omega_t^i \lambda_t^i = \frac{y_t \widetilde{m}}{1-y_t} f_t \left(1-\tau\right).$$

the dynamics

$$\frac{dP_{1,t,s}+D_{1,t,s}dt}{P_{1,t,s}}=(r+\kappa_{1,t}\sigma_{1,t})dt+\sigma_t dB_{1,t}. \tag{57}$$

Applying Ito's Lemma to the product $P_{1,t,s}=p\left(\omega_t\right)D_{1,t,s}$ also implies that

$$\frac{dP_{1,t,s}}{P_{1,t,s}}=\frac{dp_t}{p_t}+\frac{dD_{1,t,s}}{D_{1,t,s}}+\frac{p'\left(\omega_t\right)}{p\left(\omega_t\right)}\sigma_t^R\sigma_{1,D}dt. \tag{58}$$

Combining (57) with (58) and using $\sigma_{1,t}=\frac{p'(\omega_t)}{p(\omega_t)}\sigma_t^R+\sigma_{1,D}$ and Ito's Lemma to compute the drift of $\frac{dp_t}{p_t}$ leads to

$$\frac{1}{2}\frac{\partial^2 p}{\partial\omega_t^2}\left(\sigma_t^R\right)^2+\frac{\partial p}{\partial\omega_t}\left(\mu_t^R+\sigma_t^R\sigma_{1,D}\right)-p\times\left(r+\delta_1+\kappa_{1,t}\sigma_{1,t}\right)+1=0, \tag{59}$$

which in turn leads to (47) after substituting $\sigma_{1,t}=\frac{p'(\omega_t)}{p(\omega_t)}\sigma_t^R+\sigma_{1,D}$.

We solve (47) with iterated Monte Carlo. We start with the initial guess $\sigma_{1,t}=\sigma_{1,D}$ and some guess for the cutoff $\bar{\omega}$. With that guess we use a Monte carlo simulation to evaluate $V_t^R(\omega_t)$ on a grid of ω_t values. We find the value that implies $V_t^R(\omega_t)=0$ and update our guess for $\bar{\omega}$ until $V_t^R(\bar{\omega})=0$. With this guess for $\bar{\omega}$ we draw paths of ω_t for different initial values and solve for the price-dividend ratio by using the Feynman-Kac theorem to express (47) as an expectation, which we evaluate with Monte Carlo simulation. After obtaining the price-dividend ratio on a fine grid of values, we evaluate $\frac{p'(\omega_t)}{p(\omega_t)}$, and compute $\sigma_{1,t}=\frac{p'(\omega_t)}{p(\omega_t)}\sigma_t^R+\sigma_{1,D}$. Using this new guess for $\sigma_{1,t}$ we repeat the above procedure until convergence.

C Proofs

Proof of Proposition 1. Fix parameters $\eta > 0$ and $\psi > 1$ and define φ according to

$$\varphi = \sigma_D \left(\eta - \psi \sigma_D\right) \tag{60}$$

for any value of σ_D. Note that when σ_D is sufficiently small, φ is guaranteed to be positive.

We show next that, as σ_D gets close to zero, Assumption 2 is satisfied. Rearranging (60) gives

$$\frac{\eta}{\frac{\varphi}{\sigma_D}} = \frac{1}{1 - \psi \frac{\sigma_D}{\eta}}. \tag{61}$$

For sufficiently small σ_D we obtain

$$1 + \tau > \frac{1}{1 - \psi \frac{\sigma_D}{\eta}} > 1. \tag{62}$$

Combining (61) and (62) yields (20).

Turning to (21), we note that the definition of ω_1^* along with (60) implies

$$\omega_1^* = 1 - \frac{\sigma_D}{\psi \sigma_D} = \frac{\psi - 1}{\psi} > 0,$$

while also

$$\lim_{\sigma_D \to 0} \frac{\sigma_D}{\left(1 + \tau\right) \frac{\varphi}{\sigma_D} - \eta} = \lim_{\sigma_D \to 0} \frac{\sigma_D}{\left(1 + \tau\right) \left(\eta - \psi \sigma_D\right) - \eta} = 0.$$

Therefore, for sufficiently small σ_D, the left-hand side of (21) converges to $\frac{\psi - 1}{\psi} > 0$, while the right-hand side converges to zero, and therefore the inequality holds.

We conclude the proof by showing that $F(\omega)$ has a unique root in the interval $(\omega_1^*, 1)$.

To this end, it is useful to introduce the definitions

$$A(\omega) \equiv \tau \frac{\omega}{\sigma_D} \varphi, \tag{63}$$

$$B(\omega) \equiv \sigma_D - \omega \left((1+\tau) \frac{\varphi}{\sigma_D} - \eta \right), \tag{64}$$

$$C(\omega) \equiv \frac{\omega}{1-\omega} \left(\sigma_D + (1-\omega) \left(\frac{\varphi}{\sigma_D} - \eta \right) \right). \tag{65}$$

With these definitions, $F(\omega)$ can be written as $F(\omega) = B^2(\omega) - 4A(\omega)C(\omega)$. We start by observing that $C(\omega_1^*) = 0$ for any parametric choice (since the definition of ω_1^* in equation (18) implies $\sigma_D + (1-\omega_1^*) \left(\frac{\varphi}{\sigma_D} - \eta \right) = 0$). Also, inequality (21) implies that $B(\omega_1^*) \neq 0$, and thus $B^2(\omega_1^*) > 0$. Accordingly, $F(\omega_1^*) > 0$. Also $B(1) < \infty$, while $C(1) = \infty$. By continuity, there exists at least one value $\omega_2^* \in (\omega_1^*, 1)$ such that $F(\omega_2^*) = 0$.

To show that this value is unique, consider any value $\omega_2^* \in (\omega_1^*, 1)$ such that $F(\omega_2^*) = 0$. We next show that $F'(\omega_2^*) < 0$.

To this end, note that

$$\begin{aligned} F'(\omega) &= 2B(\omega)B'(\omega) - 4\left[A'(\omega)C(\omega) + A(\omega)C'(\omega)\right] \\ &= 2B^2(\omega)\frac{B'(\omega)}{B(\omega)} - 4A(\omega)C(\omega)\left(\frac{A'(\omega)}{A(\omega)} + \frac{C'(\omega)}{C(\omega)}\right). \end{aligned}$$

Since ω_2^* is a root of $F(\omega)$ it follows that $B^2(\omega_2^*) = 4A\left(\omega^{(2)}\right) C(\omega_2^*)$. Therefore,

$$F'(\omega_2^*) = B^2(\omega_2^*) \left(2\frac{B'(\omega_2^*)}{B(\omega_2^*)} - \frac{A'(\omega_2^*)}{A(\omega_2^*)} - \frac{C'(\omega_2^*)}{C(\omega_2^*)} \right). \tag{66}$$

We have

$$\frac{A'(\omega_2^*)}{A(\omega_2^*)} = \frac{1}{\omega_2^*}$$

$$\frac{B'(\omega_2^*)}{B(\omega_2^*)} = -\frac{(1+\tau)\frac{\varphi}{\sigma_D} - \eta}{\sigma_D - \omega_2^* \left((1+\tau)\frac{\varphi}{\sigma_D} - \eta \right)}$$

and

$$\frac{C'\left(\omega_2^*\right)}{C\left(\omega_2^*\right)}=\frac{1}{\omega_2^*\left(1-\omega_2^*\right)}+\frac{\eta-\frac{\varphi}{\sigma_D}}{\sigma_D+\left(1-\omega_2^*\right)\left(\frac{\varphi}{\sigma_D}-\eta\right)}.$$

Combining terms gives

$$\begin{aligned}&2\frac{B'\left(\omega_2^*\right)}{B\left(\omega_2^*\right)}-\frac{A'\left(\omega_2^*\right)}{A\left(\omega_2^*\right)}-\frac{C'\left(\omega_2^*\right)}{C\left(\omega_2^*\right)} \\ &=-\frac{2\left(\left(1+\tau\right)\frac{\varphi}{\sigma_D}-\eta\right)}{\sigma_D-\omega_2^*\left(\left(1+\tau\right)\frac{\varphi}{\sigma_D}-\eta\right)}-\frac{1}{\omega_2^*}-\frac{1}{\omega_2^*\left(1-\omega_2^*\right)}-\frac{\eta-\frac{\varphi}{\sigma_D}}{\sigma_D+\left(1-\omega_2^*\right)\left(\frac{\varphi}{\sigma_D}-\eta\right)}.\end{aligned} \tag{67}$$

For future reference, we note that using $\omega_2^*>\omega_1^*$ along with (20) and the definition of ω_1^* implies that

$$\sigma_D+\left(1-\omega_2^*\right)\left(\frac{\varphi}{\sigma_D}-\eta\right)>\sigma_D+\left(1-\omega_1^*\right)\left(\frac{\varphi}{\sigma_D}-\eta\right)=0. \tag{68}$$

Using (60) we can write the right-hand side of (67) as

$$-\frac{2\left(\left(1+\tau\right)\left(\eta-\psi\sigma_D\right)-\eta\right)}{\sigma_D-\omega_2^*\left(\left(1+\tau\right)\left(\eta-\psi\sigma_D\right)-\eta\right)}-\frac{1}{\omega_2^*}-\frac{1}{\omega_2^*\left(1-\omega_2^*\right)}-\frac{\psi}{1-\psi\left(1-\omega_2^*\right)}. \tag{69}$$

Taking the limit as σ_D approaches zero, the expression (69) converges to

$$-\frac{1}{1-\omega_2^*}-\frac{\psi}{1-\psi\left(1-\omega_2^*\right)}<0,$$

where the inequality follows from (68) along with (60).[25]

The fact that the derivative $F'\left(\omega_2^*\right)<0$ for any root of the equation $F\left(\omega_2^*\right)=0$ in the interval $(\omega_1^*,1)$ implies that the root ω_2^* must be unique. ■

[25] Equation (60) implies $\frac{\varphi}{\sigma_D}-\eta=-\psi\sigma_D$, and therefore $0<\sigma_D+\left(1-\omega_2^*\right)\left(\frac{\varphi}{\sigma_D}-\eta\right)=\sigma_D\left(1-\left(1-\omega_2^*\right)\psi\right)$, where the inequality follows from (68).

Proof of Proposition 2. In preparation for the proof, we state and prove an auxiliary result.

Lemma 2 *The following statements hold for the quadratic equation (23).*

1. $\omega_1^* < \omega_2^*$ *and the discriminant of (23) is non-negative for all* $\omega_t \leq \omega_2^*$.
2. *When* $\omega_1^* \leq \omega_t \leq \omega_2^*$, *the two roots of the equation are both in the interval* $[0, 1)$.
3. *For* $\omega_t \in [0, \omega_1^*)$, *only the larger root of (23) is in the interval* $(0, 1)$.
4. *If* y *is a root of (23), then* $(1 - \omega_t)\,\eta - \sigma_D - \frac{1-\omega_t}{\sigma_D}\varphi\,(1 - \tau y) > 0$.

Proof of Lemma 2. We start with part 1. Using the definitions (63)–(65), equation (23) can be written in the familiar form

$$A\,(\omega_t)\,y^2 + B\,(\omega_t)\,y + C\,(\omega_t) = 0,$$

and the discriminant of this quadratic equation is given by $F\,(\omega_t)$ as defined in equation (19).

For $\omega_t \leq \omega_1^*$, $C\,(\omega_t) < 0$ and the discriminant, $B^2\,(\omega_t) - 4A\,(\omega_t)\,C\,(\omega_t)$, is positive. The assumption that ω_2^* is the unique root of $F\,(\omega)$ along with the facts that $F\,(\omega_1^*) = B^2\,(\omega_1^*) > 0$ and $F\,(1) = -\infty$ imply that $\omega_1^* < \omega_2^*$.[26] The uniqueness of the root ω_2^* also implies that $F\,(\omega_t) = B^2\,(\omega_t) - 4A\,(\omega_t)\,C\,(\omega_t) \geq 0$ for all $\omega_t \leq \omega_2^*$.

We now turn to part 2. To economize on notation we write A rather $A\,(\omega_t)$ and similarly for B and C. Fix a given ω_t and let $g\,(y) = Ay^2 + By + C$. We have $g\,(1) = A + B + C = \frac{\sigma_D}{1-\omega_t} > 0$ and $g'\,(1) = 2A + B = \sigma_D + \omega_t\left(\eta - (1 - \tau)\,\frac{\varphi}{\sigma_D}\right) > 0$, where the inequality follows from (20). Since $A > 0$, it follows that all roots of $g\,(y)$ must be smaller than one. Also, the

[26] Assumption (21) implies that $B\,(\omega_1^*) \neq 0$ and therefore $B^2\,(\omega_1^*) > 0$.

fact that $\omega_t \geq \omega_1^*$ implies that $g(0) = C > 0$, while assumptions (20) and (21) together with the fact that $\omega_t \geq \omega_1^*$ imply that $g'(0) = B < 0$.

The facts that i) $g(y)$ is a convex, quadratic function of y, ii) $g(1) > 0$, $g(0) > 0$, $g'(1) > 0$, and $g'(0) < 0$ and iii) $B^2 - 4AC > 0$ for $\omega_t \in [\omega_1^*, \omega_2^*)$ imply that there are two roots in $(0,1)$.

For part 3, we note that, when $\omega_t < \omega_1^*$, $g(0) = C < 0$, while $g(1) = A+B+C = \frac{\sigma_D}{1-\omega_t} > 0$. Therefore there exists one and only one root in $(0,1)$.

Finally, let $y \in (0,1)$ denote a root of the quadratic equation (23). Accordingly,

$$\begin{aligned}
(1-\omega_t)\eta - \sigma_D - (1-\omega_t)\frac{\varphi}{\sigma_D}(1-\tau y) &= \frac{1-\omega_t}{\omega_t} y \left(\sigma_D + \omega_t\eta - \omega_t\frac{\varphi}{\sigma_D}(1-\tau y)\right) \\
&= \frac{1-\omega_t}{\omega_t} y \left[\sigma_D + \omega_t\left(\eta - \frac{\varphi}{\sigma_D}\right) + \omega_t\frac{\varphi}{\sigma_D}\tau y\right] \\
&> 0
\end{aligned}$$

where the last inequality follows from (20). This proves property 4. ■

We now continue with the proof of the proposition. We provide expressions for r_t and κ_t that apply in any equilibrium in which $w_t^R \neq 0$. Since $\sum\limits_i \omega_t^i = 1$, it follows that $\sum_i \sigma_t^i = 0$ and $\sum_i \mu_t^i = 0$. Using (16) and $\sum_i \sigma_t^i = 0$ implies that $\sum_i \omega_t^i w_t^i = 1$. Combining $\sum_i \omega_t^i w_t^i = 1$ with (13) along with the definition $y_t = \frac{W_t^-}{W_t^+}$ gives

$$\kappa_t + (1-\omega_t)\eta + \left(\omega_t\frac{1}{\sigma_D}\varphi + (1-\omega_t)\tau y_t\frac{1}{\sigma_D}\varphi\right)1_{\{w_t^R<0\}} = \sigma^D. \tag{70}$$

Similarly, using (17) along with $\sum_i \mu_t^i = 0$ and $\sum_i \omega_t^i (n_t + w_t^i s_t^i) = 0$ gives (25).

We next describe the equilibria for the three intervals of ω_t described in the statement of the proposition.

i) In this case, $\omega_t > \omega_2^*$. The equilibrium prescribes non-negative portfolios for both investors. If $\omega_t > 1 - \frac{\sigma_D}{\eta}$, equation (70) implies that $\kappa_t > 0$ and (13) implies that both investors hold positive portfolios and the shorting market is inactive. If $\omega_t \in [\omega_1^*, 1 - \frac{\sigma_D}{\eta})$, then there exists an equilibrium that involves no shorting and a zero portfolio for investor R. We check this assertion by osberving that the associated market clearing requirement becomes $(1 - \omega_t) w_t^I = 1$, which together with $y_t = 0$ leads to (22). We then note that

$$\begin{aligned} \kappa_t + \frac{\varphi}{\sigma_D} &= \frac{\sigma_D}{1 - \omega_t} - \eta + \frac{\varphi}{\sigma_D} \\ &> \frac{\sigma_D}{1 - \omega_1^*} - \eta + \frac{\varphi}{\sigma_D} \\ &= 0. \end{aligned} \tag{71}$$

The first line follows from (22), the second line follows from $\omega_t > \omega_1^*$ and the third line follows from the definition of ω_1^*. Since $\kappa_t + \frac{\varphi}{\sigma_D} > 0$, investor R does not choose a negative portfolio. And since $\kappa_t < 0$ for $\omega_t \in [\omega_1^*, 1 - \frac{\sigma_D}{\eta})$, the investor chooses a zero portfolio.

ii) In this case, $\omega_1^* < \omega_t < \omega_2^*$. Since $\omega_t > \omega_1^*$, equation (71) implies that the no-shorting equilibrium continues to be an equilibrium. There exist, however, two more equilibria. To compute them, we guess (and verify shortly) that $w_t^R < 0$. Using (13) and (70) gives

$$\begin{aligned} y_t &= \frac{W_t^-}{W_t^+} = \frac{-\omega_t w_{t,s}^R}{(1 - \omega_t) w_{t,s}^I} = \frac{\omega_t}{1 - \omega_t} \frac{-\left(\kappa_t + \frac{1}{\sigma_D}\varphi\right)}{\kappa_t + \eta_t + \frac{1}{\sigma_D}\varphi\tau y_t} \\ &= \frac{\omega_t}{1 - \omega_t} \frac{(1 - \omega_t)\eta - \sigma_D - \frac{1 - \omega_t}{\sigma_D}\varphi(1 - \tau y_t)}{\sigma_D + \omega_t \eta - \frac{\omega_t}{\sigma_D}\varphi(1 - \tau y_t)}. \end{aligned}$$

Rearranging leads to (23). Statement 1 of Lemma 2 implies that, when $\omega_t \in (\omega_1^*, \omega_2^*)$, equation (23) has two roots in $(0, 1)$. Under the supposition that $w_t^R < 0$, Equation

(70) leads to (24). In turn

$$\begin{aligned}
\kappa_t^{\pm} + \frac{\varphi}{\sigma_D} &= \sigma_D - (1-\omega_t)\,\eta - \frac{\omega_t}{\sigma_D}\varphi\left(1 + \tau y^{\pm}\frac{1-\omega_t}{\omega_t}\right) + \frac{\varphi}{\sigma_D} \\
&= \sigma_D - (1-\omega_t)\left(\eta + \frac{\varphi}{\sigma_D}\left(1-\tau y_t^{\pm}\right)\right) < 0, \qquad (72)
\end{aligned}$$

where the last inequality follows from statement 4 of Lemma 2. Combining this observation with (13) confirms that $w_t^R < 0$. Note that in the second and third equilibria we have that

$$\kappa_t^{\pm} + \eta_t + \frac{1}{\sigma_D}\varphi\tau y_t^{\pm} = \sigma_D + \omega_t\eta - \frac{\varphi\omega_t}{\sigma_D}\left(1-\tau y_t^{\pm}\right) > 0,$$

where the last inequality follows from (72) along with the fact that $y^{\pm}$ satisfy the equation (23). This implies that $w_t^I > 0$.

iii) In this case, $\omega_t < \omega_1^*$. Statement 3 of Lemma 2 implies that the quadratic equation (23) has only one solution in $(0,1)$. This shows that there can only be one equilibrium with shorting. Moreover, this is the unique equilibrium. If w_t^R were zero and the Sharpe ratio were $\frac{\sigma_D}{1-\omega_t} - \eta$, then the inequality in (71) reverses, i.e., $\frac{\sigma_D}{1-\omega_t} - \eta + \frac{\varphi}{\sigma_D} < 0$ and investor R would want to deviate from the equilibrium prescription and choose a negative portfolio.

■

Proof of Proposition 3. We distinguish two cases according to whether investor R holds an interior positions in both equilbria.

Case i: Suppose that $w_t^{R,A} = 0$ in equilibrium A and $w_t^{R,B} < 0$ in equilibrium B. We have

$$\begin{aligned} g_t^{R,B} - g_t^{R,A} &= -\left(\kappa^B - \kappa^A\right)\sigma_D + \max_{w_t \leq 0}\left\{w_t\left(\kappa^B\sigma_D + \varphi\right) - \frac{1}{2}w_t^2\sigma_t^2\right\} \\ &> \left(\kappa^A - \kappa^B\right)\sigma_D \geq 0, \end{aligned}$$

where the first inequality follows from the fact that $w_t = 0$ is suboptimal for investor R in equilibrium B (by assumption). Similarly, using (17) gives

$$\begin{aligned} \mu_t^{R,B} - \mu_t^{R,A} &= \omega_t\left(\left(\kappa^A - \kappa^B\right)\sigma_D + w_t^{R,B}\sigma_D\left(\kappa^B + \frac{\varphi}{\sigma_D} - \sigma_D\right)\right) \\ &= \omega_t\left[\left(\kappa^A - \kappa^B\right)\sigma_D + (1-\omega_t)\, w_t^{R,B}\sigma_D\left(\frac{\varphi}{\sigma_D}(1-y) - \eta\right)\right] \\ &= \omega_t\left[\left(\kappa^A - \kappa^B\right)\sigma_D + (1-\omega_t)\left|w_t^{R,B}\right|\sigma_D\left(\eta - \frac{\varphi}{\sigma_D}(1-y)\right)\right] \\ &> 0, \end{aligned}$$

where the first equality follows from (24), the second equality from $w_t^{R,B} < 0$ and the inequality from assumption (20) along with $y < 1$.

Case ii: In this case the portfolio choice of investor R is interior in both equilibria. Using the fact that in any interior equilibrium the optimal value of w_t satisfies

$$w_t\left(\kappa^B\sigma_D + \varphi\right) - \frac{1}{2}w_t^2\sigma_D^2 = \frac{1}{2}w_t^2\sigma_D^2,$$

we obtain

$$\begin{aligned}
g_t^{R,B} - g_t^{R,A} &= -\left(\kappa^B - \kappa^A\right)\sigma_D + \frac{\sigma_D^2}{2}\left[\left(w_t^{R,B}\right)^2 - \left(w_t^{R,A}\right)^2\right] \\
&= \left(\kappa^A - \kappa^B\right)\sigma_D + \frac{\sigma_D^2}{2}\left(w_t^{R,B} + w_t^{R,A}\right)\left(w_t^{R,B} - w_t^{R,A}\right) \\
&= \left(\kappa^A - \kappa^B\right)\sigma_D + \frac{\sigma_D}{2}\left(w_t^{R,B} + w_t^{R,A}\right)\left(\kappa^B - \kappa^A\right) \\
&= \left(\kappa^A - \kappa^B\right)\sigma_D\left(1 + \left|w_t^{R,B} + w_t^{R,A}\right|\right) \\
&> 0.
\end{aligned}$$

Using (17) gives

$$\begin{aligned}
\mu_t^{R,B} - \mu_t^{R,A} &= \omega_t\left(\left(\kappa^A - \kappa^B\right)\sigma_D + w_t^{R,B}\sigma_D\left(\kappa^B + \frac{\varphi}{\sigma_D} - \sigma_D\right) - w_t^{R,A}\sigma_D\left(\kappa^A + \frac{\varphi}{\sigma_D} - \sigma_D\right)\right) \\
&= \omega_t\left(\left(\kappa^A - \kappa^B\right)\sigma_D + \sigma_D^2\left[w_t^{R,B}\left(w_t^{R,B} - 1\right) - w_t^{R,A}\left(w_t^{R,A} - 1\right)\right]\right) \\
&= \omega_t\left(\left(\kappa^A - \kappa^B\right)\sigma_D + \sigma_D^2\left[\left(w_t^{R,B} - \frac{1}{2}\right)^2 - \left(w_t^{R,A} - \frac{1}{2}\right)^2\right]\right) \\
&= \omega_t\left(\left(\kappa^A - \kappa^B\right)\sigma_D + \sigma_D^2\left[\left(\left|w_t^{R,B}\right| + \frac{1}{2}\right)^2 - \left(\left|w_t^{R,A}\right| + \frac{1}{2}\right)^2\right]\right) \\
&> 0,
\end{aligned}$$

where the last inequality follows from $w_t^{R,B} < w_t^{R,A} < 0$ (since $\kappa^B < \kappa^A < 0$) and therefore $\left|w_t^{R,B}\right| > \left|w_t^{R,A}\right|$. ■

Proof of Lemma 1. By the implicit function theorem,

$$\frac{d\omega_2^*}{d\varphi} = -\frac{F_\varphi}{F_\omega}.$$

Since $\lim_{\omega\to\infty} F(\omega) = -\infty$ and the root $F(\omega_2^*) = 0$ is unique (by assumption), it follows that $F_\omega(\omega_2^*) < 0$. So it suffices to prove that $F_\varphi(\omega_2^*) < 0$.

Differentiating F with respect to φ, multiplying the resulting expression by φ and evaluating at ω_2^* (recall $F(\omega_2^*) = 0$) gives

$$\varphi F_\varphi = -2\omega_2^* \frac{\varphi}{\sigma_D}(1+\tau)\left(\sigma_D - \omega_2^*\left((1+\tau)\frac{\varphi}{\sigma_D} - \eta\right)\right) - \left(\sigma_D - \omega_2^*\left((1+\tau)\frac{\varphi}{\sigma_D} - \eta\right)\right)^2 - 4\tau(\omega_2^*)^2\frac{\varphi^2}{\sigma_D^2}. \tag{73}$$

Completing the square gives

$$\begin{aligned}
\varphi F_\varphi &= -\left(\sigma_D - \omega_2^*\left((1+\tau)\frac{\varphi}{\sigma_D} - \eta\right) + \omega_2^*\frac{\varphi}{\sigma_D}(1+\tau)\right)^2 + (\omega_2^*)^2\frac{\varphi^2}{\sigma_D^2}(1-\tau)^2 \\
&= -(\sigma_D + \omega_2^*\eta)^2 + (\omega_2^*)^2\frac{\varphi^2}{\sigma_D^2}(1-\tau)^2 \\
&= -\left(\sigma_D + \omega_2^*\eta + \omega_2^*\frac{\varphi}{\sigma_D}(1-\tau)\right)\left(\sigma_D + \omega_2^*\eta - \omega_2^*\frac{\varphi}{\sigma_D}(1-\tau)\right) \\
&< 0,
\end{aligned}$$

where the last inequality follows from the assumption $\eta \geq \frac{\varphi}{\sigma_D}$. ■

Proof of Proposition 4. Differentiating κ_t with respect to φ (in an equilibrium where $y > 0$), we obtain

$$\frac{d\kappa_t}{d\varphi} = -\frac{\omega_t}{\sigma_D}\left(1 + \frac{1-\omega_t}{\omega_t}\tau y_t\left(1 + \frac{\varphi}{y_t}\frac{dy_t}{d\varphi}\right)\right). \tag{74}$$

In turn, the implicit function theorem applied to (23) gives

$$\frac{dy_t}{d\varphi} = -\frac{\frac{\omega_t}{\sigma_D}(1-\tau y)(1-y)}{\sigma_D + \omega_t\eta - \frac{\omega_t}{\sigma_D}\varphi(1+\tau-2\tau y)} = -\frac{\frac{\omega_t}{\sigma_D}(1-\tau y)(1-y)}{g'(y)},$$

where $g(y)$ is defined in Proposition 1. Since $g'(y^-) < 0$ and $g'(y^+) > 0$, we have $\frac{dy^-}{d\varphi} > 0$ and $\frac{dy^+}{d\varphi} < 0$. Combining $\frac{dy^-}{d\varphi} > 0$ with (74) implies $\frac{d\kappa_t}{d\varphi} < 0$ in the equilibrium associated with y^{-}. For the equilibrium associated with y^{+} we have

$$1+\frac{\varphi}{y^{+}}\frac{dy^{+}}{d\varphi}=\frac{\left(\sigma_D+\omega_t\eta-\frac{\omega_t}{\sigma_D}\varphi\left(1+\tau-2\tau y^{+}\right)\right)y^{+}-\frac{\varphi\omega_t}{\sigma_D}\left(1-\tau y^{+}\right)\left(1-y^{+}\right)}{\left(\sigma_D+\omega_t\eta-\frac{\omega_t}{\sigma_D}\varphi\left(1+\tau-2\tau y^{+}\right)\right)y^{+}}. \tag{75}$$

We are interested in the behavior of (75) as ω_t approaches zero. Letting $x\equiv\frac{y^{+}}{\omega_t}$, dividing both sides of (23) by ω_t and re-arranging terms yields

$$x\left(\sigma_D+\omega_t\eta\right)+\frac{\varphi}{\sigma_D}\left(1-\omega_t x\right)\left(1-\tau\omega_t x\right)=\frac{1}{1-\omega_t}\left(\left(1-\omega_t\right)\eta^I-\sigma_D\right).$$

Taking limits as ω_t approaches zero, implies

$$\lim_{\omega_t\to 0}x=\frac{\eta-\sigma_D-\frac{\varphi}{\sigma_D}}{\sigma_D}. \tag{76}$$

Using (76), as ω_t approaches zero we obtain

$$\begin{aligned}\lim_{\omega_t\to 0}\left(1+\frac{1-\omega_t}{\omega_t}\tau y_t\left(1+\frac{\varphi}{y_t}\frac{dy_t}{d\varphi}\right)\right)&=1+\tau\lim_{\omega_t\to 0}x\times\lim_{\omega_t\to 0}\left(1+\frac{\varphi}{y_t}\frac{dy_t}{d\varphi}\right)\\&=\tau\lim_{\omega_t\to 0}x\times\left(1-\frac{\frac{\varphi}{\sigma_D}}{\sigma_D\lim_{\omega_t\to 0}x_t}\right)\\&=\tau\left(\frac{\eta-\sigma_D-2\frac{\varphi}{\sigma_D}}{\sigma_D}\right)\\&<0,\end{aligned} \tag{77}$$

where we used (20) to derive the last inequality. Combining (77) with (74) implies that, for small ω_t, $\frac{d\kappa\left(y^{+}\right)}{d\varphi}>0$. ■

Proof of Proposition 5. The proof essentially repeats the steps from the one-risky asset case, so we provide only a sketch, focusing on the elements that differ. We define

$$\overrightarrow{\varphi} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix}, \overrightarrow{\eta} = \begin{bmatrix} \eta \\ 0 \end{bmatrix}.$$

We consider first an equilibrium with $y_t > 0$. Investor R's and I's optimal portfolios are given by

$$\overrightarrow{w}_t^R = (\sigma_t \sigma_t')^{-1} (\overrightarrow{\mu}_t - r_t 1_{2\times 1} + \overrightarrow{\varphi}), \tag{78}$$

$$\overrightarrow{w}_t^I = (\sigma_t \sigma_t')^{-1} (\overrightarrow{\mu}_t - r_t 1_{2\times 1} + \sigma_{1,t} \overrightarrow{\eta} + \tau y_t \overrightarrow{\varphi}). \tag{79}$$

Using (78) inside (37) yields

$$\begin{aligned}(\sigma_t \sigma_t') \overrightarrow{m}_t = \widehat{\omega}_t [\omega_t (\overrightarrow{\mu}_t - r 1_N + \overrightarrow{\varphi}) + (1 - \omega_t) (\overrightarrow{\mu}_t - r 1_N + \sigma_1 \overrightarrow{\eta} + \tau y_t \overrightarrow{\varphi})] \\ + (1 - \widehat{\omega}_t) (\sigma_t \sigma_t') \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix}.\end{aligned} \tag{80}$$

Next we use the row selection vector $[0, 1]$ to pre-multiply both sides of (80). Noting that $[0, 1] \overrightarrow{\varphi} = [0, 1] \overrightarrow{\eta} = 0$, and also

$$(\sigma_t \sigma_t') \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix} = \begin{bmatrix} b_t (\mu_{2,t} - r) \\ \mu_{2,t} - r \end{bmatrix}, \tag{81}$$

leads to (40). We next note that

$$\begin{aligned}[1, -b_t] \, \sigma_t \sigma_t' \begin{bmatrix} m_{1,t} \\ m_{2,t} \end{bmatrix} &= [\sigma_{1,t}, 0] \begin{bmatrix} \sigma_{1,t} & 0 \\ b_t \sigma_{2,t} & \sigma_{2,t} \end{bmatrix} \begin{bmatrix} m_{1,t} \\ m_{2,t} \end{bmatrix} \\ &= \sigma_{1,t}^2 m_{1,t}.\end{aligned} \tag{82}$$

Pre-multiplying both sides of (80) with the row vector $[1, -b_t]$, using (81), (82), and the definition of $\kappa_{1,t}$, and re-arranging yields

$$\kappa_{1,t} = \widetilde{m}_{1,t}\sigma_{1,t} - (1-\omega_t)\,\eta - \frac{\varphi}{\sigma_{1,t}}\left(\omega_t + (1-\omega_t)\tau y_t\right). \tag{83}$$

Using the definition of $\kappa_{1,t}$ inside (78) gives

$$w_{1,t}^R = \frac{\kappa_{1,t}}{\sigma_{1,t}} + \frac{\varphi}{\sigma_{1,t}^2} \tag{84}$$

$$w_{1,t}^I = \frac{\kappa_{1,t}+\eta}{\sigma_{1,t}} + \frac{\tau y_t \varphi}{\sigma_{1,t}^2}, \tag{85}$$

where we used the notation $w_t^{1,i}$, $i \in \{R, I\}$, to denote the first element of w_t^i.

Using the market clearing condition $y_t = -\frac{\omega_t^R w_{1,t}^R}{\omega_t^I w_{1,t}^I} = -\frac{\omega_t w_{1,t}^R}{(1-\omega_t) w_{1,t}^I}$ leads to (38).

If agent R chooses not to short then the market clearing condition becomes

$$\widehat{\omega}_t(1-\omega_t)w_t^I + (1-\widehat{\omega}_t)\begin{bmatrix} 0 \\ \widehat{w}_{2,t} \end{bmatrix} = \overrightarrow{m}_t. \tag{86}$$

Substituting (79), pre-mutiplying by $(\sigma_t\sigma_t')$ gives

$$(\sigma_t\sigma_t')\,\overrightarrow{m}_t = \widehat{\omega}_t(1-\omega_t)\left(\overrightarrow{\mu}_t - r1_N + \sigma_1\overrightarrow{\eta}\right) + (1-\widehat{\omega}_t)\,(\sigma_t\sigma_t')\begin{bmatrix} 0 \\ \frac{\mu_{2,t}-r}{\sigma_{2,t}^2} \end{bmatrix}. \tag{87}$$

Premultiplying (87) by the row $[1, -b_t]$ and using (81) and (82) gives

$$\sigma_{1,t}^2\widetilde{m}_{1,t} = (1-\omega_t)\,\sigma_{1,t}\left(\kappa_{1,t}+\eta\right),$$

and therefore

$$\kappa_{1,t} = \sigma_{1,t} \frac{\widetilde{m}_{1,t}}{1-\omega_t} - \eta. \tag{88}$$

Finally, when both agents hold positive portfolios, the optimal portfolios are $\overrightarrow{w}_t^R = (\sigma_t \sigma_t')^{-1} (\overrightarrow{\mu}_t - r_t 1_{2\times 1})$, $\overrightarrow{w}_t^I = (\sigma_t \sigma_t')^{-1} (\overrightarrow{\mu}_t - r_t 1_{2\times 1} + \sigma_{1,t} \overrightarrow{\eta})$. Repeating the arguments in equations (78)–(83), we obtain $\kappa_{1,t} = \widetilde{m}_{1,t} \sigma_{1,t} - (1-\omega_t)\eta$. ■

Proof of Proposition 6. The proof of this Proposition is contained in Appendix B. ■

Proof of Proposition 7. Since this proof is essentially identical to the proof of Proposition 2, we only provide a sketch. Combining (13) with $\sum_i \omega_t^i w_t^i = 1$ implies that in any equilibrium with $w_t^R < 0$ and $w_t^I > 0$ the Sharpe ratio is

$$\kappa_t + (1-\omega_t)\eta + \omega_t \frac{1}{\sigma_D} f_t + (1-\omega_t)\tau y_t \frac{1}{\sigma_D} f_t = \sigma_D. \tag{89}$$

Re-arranging (89) and using $f_t = l(y_t)$ gives (50). Substituting (50) back into the investors' optimal portfolios (13) and the fact that $y_t = -\frac{\omega_t w_t^R}{(1-\omega_t) w_t^I}$ leads to (49).

We next study the roots of (49). Let $Z(y) \equiv \frac{\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(y)}{\eta + \frac{\sigma_D}{\omega_t} - \frac{1}{\sigma_D} z(y)}$, so that equation (49) can be expressed as $y = Z(y)$. The assumption of the proposition is that there exists at least one y such that $y = Z(y)$. Let $\bar{y}^*$ be the largest root of (49) that satisfies $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(y) > 0$. We consider two cases: i) $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(y) > 0$ for all $y \in [\bar{y}^*, 1]$ and ii) $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(\overline{y}) = 0$ for some $\overline{y} \in [\bar{y}^*, 1]$. In case i) it must be that $Z'(\bar{y}^*) \leq 1$, since $\bar{y}^* = Z(\bar{y}^*)$ and $1 > Z(1)$. In case ii) it must also be that $Z'(\bar{y}^*) \leq 1$ since $\overline{y} > Z(\overline{y}) = 0$.

Furthermore, by continuity values of y lower than $\bar{y}^*$ exist such that $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(y) = 0$; let $\underline{y}$ be the highest such value and note $0 < \underline{y} < \bar{y}^*$. Since the numerator of Z is positive on $(\underline{y}, \bar{y}^*)$, so is the denominator. Moreover, we have $Z(\underline{y}) = 0 < \underline{y}$. A solution $Z(y) = y$ therefore exists in $(\underline{y}, \bar{y}^*)$, and it defines a second equilibrium with strictly positive short interest.

Finally, to confirm that a no-shorting equilibrium is also an equilibrium, $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} z(0) < 0$ is equivalent to $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} l(0) < 0$. If the Sharpe ratio is given by $\kappa_t = \frac{\sigma_D}{1-\omega_t} - \eta < 0$, the assumption $\eta - \frac{\sigma_D}{1-\omega_t} - \frac{1}{\sigma_D} l(0) < 0$ implies $\kappa_t + \frac{1}{\sigma_D} l(0) > 0$. Accordingly, investor R does not wish to short when the fee is $f_t = l(0)$ and the lending market clears with $y = 0$ at the lending fee $l(0)$. Moreover, $w_t^I = \frac{\kappa_t + \eta}{\sigma_D} = \frac{1}{1-\omega_t}$. Therefore $\omega_t \times 0 + (1-\omega_t) \times w_t^I = 1$ and the stock market clears. ■

D Additional Data Discussion

D.1 Methodology

D.1.1 Measuring ticker discussion on WallstreetBets

Our measure of ticker mentions on WallstreetBets is constructed as follows. We use the PushshiftAPI to collect all submissions posted on WallstreetBets subreddit from January 1, 2020 through February 7, 2021 (Baumgartner et al., 2020). For each submission, we observe the title text, the body of the submission, the author of the submission, and the time of the submission.

In order to identify which tickers are discussed in the submission, we take advantage of the fact that users often tag tickers with a leading $ (i.e. $TSLA or $AAPL). This practice is entirely voluntary and is therefore insufficient for identifying all mentions of a ticker. We use regular expressions to identify all words tagged in this way and match those words to CRSP tickers that were traded on the NYSE, AMEX, and NASDAQ exchanges in 2020. This gives us a set S of roughly 4,000 tickers that are mentioned on WSB between January 2020 and February 2021.

We then identify all cases in which these tickers are mentioned in submissions, irrespective of whether they are prefixed with a dollar sign. To address the possibility of falsely identifying tickers, we require that, if the ticker is a common word in the written English language, it must be prefaced by a dollar sign. For example, AT&T's ticker T is also a common word in written English, and thus we require that the text "$T" appear in a submission for it to be considered as mentioned AT&T. We consider a ticker as being mentioned in a submission if it appears in either the title or the body of the submission. We identify common word-stems based on the Google Trillion Word Corpus (Michel et al., 2011). In a robustness check, we account for the downward bias this restriction introduces by scaling common-word tickers by an in-sample estimated adjustment factor. This adjustment leaves the relative ranking of ticker mentions largely unchanged. We estimate the adjustment factor by comparing the frequency of tagged ticker mentions versus untagged ticker mentions for the set of tickers which do not commonly appear in written English.

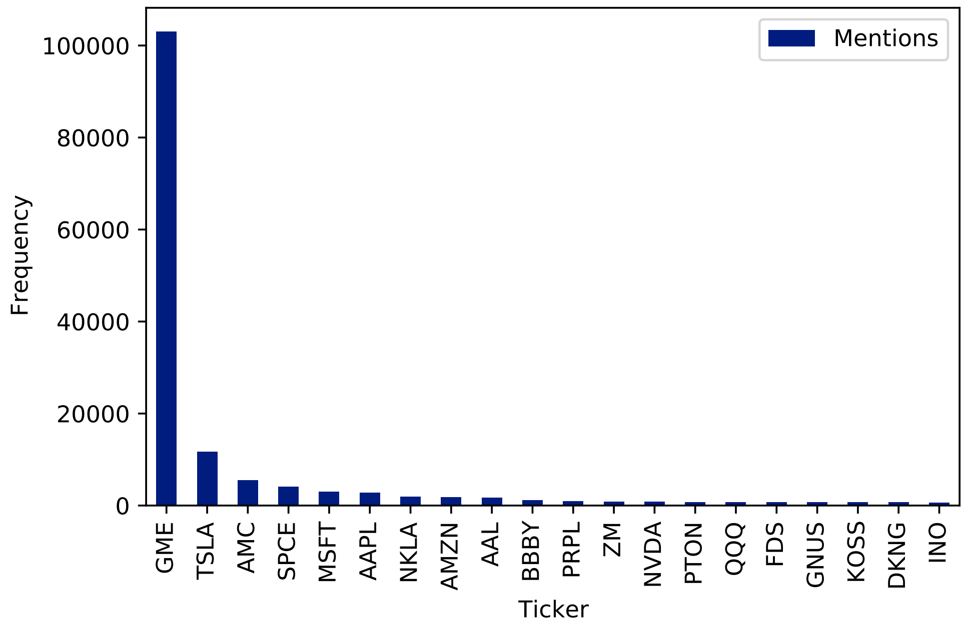
Revised submissions and comments. Authors of Reddit comments have the ability to edit their comments even after the comment has been posted. The PushshiftAPI records the comment text as of a certain day, and does not update to reflect potential revised comments. The same constraint applies to the content body of submissions. Titles of submissions cannot be revised and thus do not have this measurement problem.

Missed tickers Tickers that, for whatever reason, are never tagged with a leading dollar sign will be omitted from our dataset. Similarly, we under-count the occurrences of tickers that are common words, owing to requiring they appear with a leading "$" We attempt to correct for this by scaling the observed counts for common word tickers. For AAPL and GME, which are not common word tickers, the ticker appears with the leading "$" roughly 20% of the time. We can thus simply multiply our observed frequencies by a factor of five to adjust for the more stringent matching procedure. As can be seen in Figures 15a and 15b, the adjustment does not have a significant impact on the relative popularity of the top tickers.

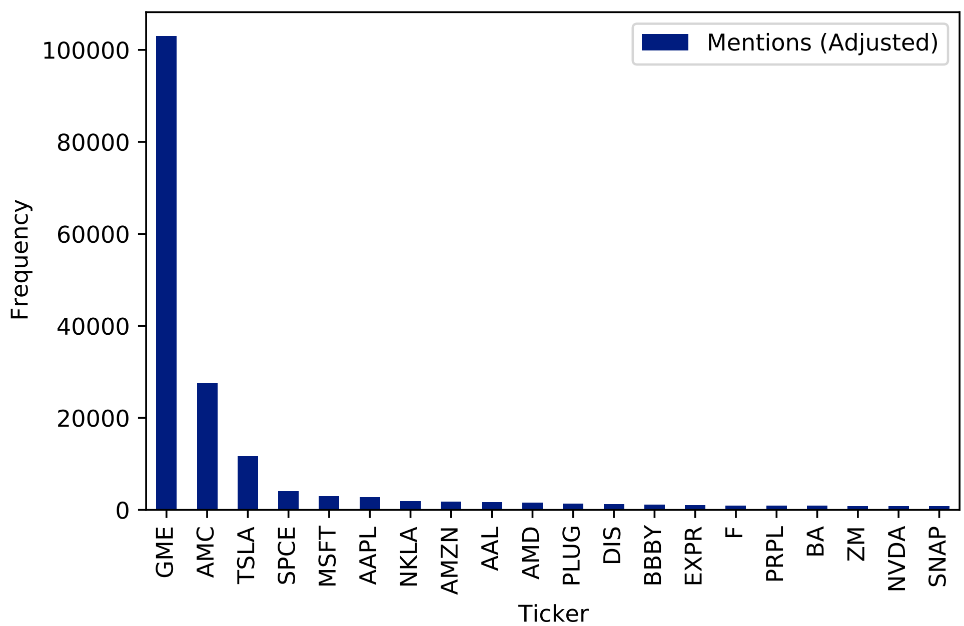
In some cases, users may choose to refer to the company by its name, rather than by its ticker. We do not attempt to identify mentions of companies by name.

D.2 Measuring retail trading

We adopt the methodology of Boehmer et al. (2020) to identify retail trades in the TAQ data. We briefly summarize the methodology here and refer readers to the paper for details.



(a) Submissions mentioning each Ticker



(b) Submissions mentioning each Ticker, adjusted for word-ticker overlap

Figure 15: Popular Tickers on WallstreetBets (January 1, 2020 - February 7, 2021).

The intuition behind the methodology is the knowledge that retail trades are often executed by wholesalers or via broker internalization, rather than on the major trading exchanges. These trades appear in the TAQ consolidated tape data under the exchange code "D". These trades are given a small price improvement on the order of tenths of a penny as a means to induce brokers to route orders to the wholesaler. Similarly, brokers which internalize retail trades offer a subpenny price improvement in order to comply with Regulation 606T. Importantly, institutional trades are rarely, if ever, internalized or directed to wholesalers and their trades are usually in round penny prices, with the notable exception of midpoint trades.

The methodology of Boehmer et al. (2020) uses these institutional details to identify retail trades in the TAQ consolidated tape data. Trades flagged with exchange code "D" and with a subpenny amount in the set $(0, 0.40) \bigcup (0.60, 1.00)$ are identified as retail trades. Splitting these trades further, retail trades with subpenny amounts between zero- and forty-hundredths of a penny are labeled as "sell orders", whereas subpenny amounts between sixty- and one hundred-hundredths are considered "buy orders". The midpoint trades are excluded to avoid mis-classifying institutional trades executed at midpoints as retail trades.

D.2.1 Challenges

Derivatives The TAQ data only contains trades of equities. Options offer another way to benefit for investors to benefit from increases in the price of stock. As an added advantage for retail investors, options offer embedded leverage greater than what might otherwise be available through their broker. The Boehmer et al. (2020) methodology relies on institutional details to identify off-exchange retail trades, and thus cannot reliably identify replication trades by market makers.

D.3 Betting against the shorts portfolio

As is standard in the literature, we restrict attention to common shares of COMPUSTAT firms which trade on the NYSE, AMEX, and NASDAQ exchanges. We further exclude companies for whom no share class has a price exceeding $1. The strategy equally weights each firm in the top decile, shorts the market index, and reconstitutes 8 trading days following the disclosure date, which is the first opportunity following the public dissemination of the short interest data.