Dynamic Portfolio Choice with Frictions[*]

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PRELIMINARY AND INCOMPLETE

Abstract

We show that the optimal portfolio can be derived explicitly in a large class of models with transitory and persistent transaction costs, multiple signals predicting returns, multiple assets, general correlation structure, time-varying volatility, and general dynamics. Our tractable continuous-time model is shown to be the limit of discrete-time models with endogenous transaction costs due to optimal dealer behavior. Depending on the dealers' inventory dynamics, we show that transitory transaction costs survive, respectively vanish, in the limit, corresponding to an optimal portfolio with bounded, respectively quadratic, variation. Finally, we provide equilibrium implications and illustrate the model's broader applicability to economics.

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A fundamental question in financial economics is how to choose an optimal portfolio. Investors must consider the risks, correlations, expected returns, and transaction costs of all their available assets and their portfolio choice is a dynamic problem for several reasons. First, expected returns are driven by several economic factors that vary over time, leading to variation in the optimal portfolio.[1] Second, some return-driving factors mean-revert more slowly than others, leading their information to be relevant for longer. Third, transaction costs imply that an investor must consider the portfolio's optimality both currently and in the future. Fourth, investors face these trade offs continually.

We provide a general and tractable framework to address these issues, deriving a simple expression for the optimal portfolio choice in light of all these dynamic considerations. Further, we show how the continuous-time solution obtains as the limit of discrete-time models in which transaction costs are modeled endogenously. We provide several additional applications of the framework and derive implications for equilibrium expected returns.

Our framework's innovation is to consider a continuous-time model in which transaction costs are quadratic in the number of securities traded. The natural interpretation of a quadratic cost is that the price impact is linear in the trade size, resulting in a quadratic cost. This assumption makes our framework highly tractable, allowing us to provide a closed-form optimal portfolio choice with multiple assets, multiple return-predicting factors, and general correlation structure. The tractability of our framework contrasts that of standard models in the literature. Indeed, standard models using proportional transaction costs are complex and rely on numerical solutions even in the case of a *single* asset with *i.i.d.* returns (i.e., no return predicting factors).[2]

In discrete time, quadratic costs have been shown to provide tractability and we rely in particular on Gârleanu and Pedersen (2013).[3] However, it has been questioned whether

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[1] See, e.g, Campbell and Viceira (2002) and Cochrane (2011) and references therein.

[2] There is an extensive literature on proportional transaction costs following Constantinides (1986). Davis and Norman (1990) provide a more formal analysis and Liu (2004) determines the optimal trading strategy for an investor with constant absolute risk aversion (CARA) and many independent securities with both fixed and proportional costs (without predictability). The assumptions of CARA and independence across securities imply that the optimal position for each security is independent of the positions in the other securities.

[3] See also Heaton and Lucas (1996) and Grinold (2006) who also assume quadratic costs. Further, Glasser-

market impact costs apply in continuous time or vanish in the limit. For instance, in the model of Cetin, Jarrow, Protter, and Warachka (2006), transaction costs are irrelevant in continuous time. To see why quadratic costs might be irrelevant in continuous time, consider what happens when one splits a trade into two equal parts. The quadratic transaction costs of each part of the trade is $(\frac{1}{2})^2 = \frac{1}{4}$ of the cost of the original trade, leading to a total cost that is half (two times $\frac{1}{4}$) what it was before. This insight leads to two apparent conclusions, the latter of which we wish to dispel: (i) Splitting orders up and trading gradually over time is optimal, as is the case in our optimal strategy and in real-world electronic markets. (ii) One can continue to halve one's cost by splitting the trade up further, so the cost goes to zero as trading approaches continuous time. We refute this argument under certain conditions, as it relies on an implicit assumption that, when the trading frequency increases, the parameter of the quadratic cost function remains unchanged. This assumption does not hold in general when trading costs are micro-founded.

We provide an economic foundation for quadratic transaction costs and show that they *do* matter in continuous time, under natural conditions.[4] This is important for several reasons. First, most trading today occurs in electronic markets in which traders can trade almost arbitrarily fast. Hence, the continuous-time model is arguably at least as realistic as a discrete-time model, so it is important to understand continuous-time trading costs. Further, if transaction costs did not matter in continuous time, then it would imply that the discrete-time models either rely heavily on the sufficient length of the time period or that transaction costs also have a small effect in these models. Second, it is important to understand how models of different period length are connected and how parameters should be scaled as a function of the period length. Third, our continuous-time model is more tractable than its discrete-time counterpart and the continuous-time framework opens the door for further applications with all the usual benefits of continuous time. Our micro foundation justifies the use of such a cost specification.

To provide an economic foundation for a continuous-time model with transaction costs, we discretize the model and let transaction costs arise endogenously due to dealers' inventory considerations à la Grossman and Miller (1988).[5] We consider both persistent and transitory costs, corresponding to dealers who can lay off their inventory gradually or in one shot. We show that the discrete-time persistent market-impact costs converge to a continuous-time model with the same persistent market-impact parameter and a resiliency parameter that depends on the length of the time periods to the first order.

There are two ways to model the dependence of the transitory costs on the trading frequency: (a) If dealers can always lay off their inventory in one time period, then shorter time periods imply that dealers need only hold inventories for a shorter time and, in this case, transitory costs vanish in the limit. (b) If, instead, the time it takes dealers to unload inventories does not go to zero even as the trading frequency increases, then transitory costs survive in the limit. In this case, the limit transaction costs are quadratic in the *trading intensity*, i.e., the number of shares traded per time unit.

We show that both trading costs and the optimal portfolio converge to their continuous-time counterparts as trading frequencies increase. In the case with vanishing transitory costs, the optimal continuous-time portfolio has quadratic variation. With transitory costs, however, our optimal continuous-time strategy is smooth and has a finite turnover. Our optimal strategy is qualitatively different from the strategy with proportional or fixed transaction costs, which exhibits long periods of no trading. Our strategy resembles the method used by many real-world traders in electronic markets, namely to continually post limit orders close to the mid-quote. The trading speed is the limit orders' "fill rate," which naturally depends on the price-aggressiveness of the limit orders, i.e., on the cost that the trader is willing to accept — just as in our model. Our strategy has several advantages in the real world according to discussions with people who design trading systems: Trading continuously minimizes the order sizes at each point in time and exploits the liquidity that is available throughout the day/week/month, rather than submitting large infrequent orders when limited liquid-

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[5]Inventory models with multiple correlated assets include Greenwood (2005) and Gârleanu, Pedersen, and Poteshman (2009).

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ity may be available. Consistently, the empirical literature generally finds transaction costs to be convex (e.g., Engle, Ferstenberg, and Russell (2008), Lillo, Farmer, and Mantegna (2003)), with some researchers estimating quadratic trading costs (e.g., Breen, Hodrick, and Korajczyk (2002) and Kyle and Obizhaeva (2011)).

The tractability of our framework makes it a potentially powerful "workhorse" for other applications involving transactions costs. As one such application, we embed the continuous-time model in an equilibrium setting. Rational investors facing transaction costs trade with several groups of noise traders who provide a time-varying excess supply or demand of assets. We show that, in order for the market to clear, the investors must be offered return premia depending on the properties of the noise-traders' positions. In particular, the noise trader positions that mean revert more quickly generate larger alphas in equilibrium, as the rational investors must be compensated for incurring higher transaction costs per time unit. Long-lived supply fluctuations, on the other hand, give rise to smaller and more persistent alphas. This can help explain the short-term return reversals documented by Lehman (1990) and Lo and MacKinlay (1990), and their relation to transaction costs documented by Nagel (2011).

Finally, our work relates to several strands of literature in addition to the research cited above. One strand of literature studies equilibrium asset pricing with trading costs (Amihud and Mendelson (1986), Vayanos (1998), Vayanos and Vila (1999), Lo, Mamaysky, and Wang (2004), Jang, Koo, Liu, and Loewenstein (2007), and Gârleanu (2009)) and time-varying trading costs (Acharya and Pedersen (2005), Lynch and Tan (2011)). Second, a strand of literature derives the optimal trade execution, treating the asset and quantity to trade as given exogenously (see, e.g., Perold (1988), Bertsimas and Lo (1998), Almgren and Chriss (2000), Obizhaeva and Wang (2006), and Engle and Ferstenberg (2007)). Finally, quadratic programming techniques are also used in macroeconomics and other fields, and, usually, the solution comes down to algebraic matrix Riccati equations (see, e.g., Ljungqvist and Sargent (2004) and references therein). We solve our model explicitly, including the Riccati equations.

This paper contributes to the literature by proposing a continuous time model with a far

more general framework — including general return dynamics (e.g., time-varying volatility) and multiple return-predicting signals with general dynamics — providing a micro foundation for the trading costs, linking transaction costs to the time horizon and deriving their continuous-time limit, and considering equilibrium implications of transaction costs.

At a high level, our model shows how to act optimality in light of frictions and several signals with varying mean-reversion rates. Our framework and insights may therefore also have implications for other areas of social science as we discuss in the concluding section of the paper. For instance, a politician may face varying signals from several constituents and incur costs from political changes. A firm may receive several signals about consumer preferences and face costs to adjusting its products. A central bank may receive several signals about inflation (e.g., in several regions such as the European member states) and face costs of changing monetary policy. In each of these cases, our framework could be used to see how to optimally weight the signals in light of their dynamics and costs. Our model shows that the optimal policy moves gradually in the direction of an aim, which incorporates an average of current and future expected signals, thus giving must weight to persistent signals, and how the answer depends on the time horizon.

The rest of the paper is organized as follows. Section 1 lays out our continuous-time framework and solves the model with transitory and persistent transaction costs. Section 2 provides a discrete-time foundation for the model, providing endogenous transaction costs and deriving the limit as the length of the time periods becomes small. Section 3 shows how to extend the framework to accommodate time variation in such quantities as volatility, risk aversion, or transaction costs, while Section 4 derives equilibrium implications of the framework. Section 5 concludes with broader implications. All proofs are in appendix.

1   Continuous-Time Model

We start by introducing our tractable continuous-time framework and illustrating its solution. We first consider the case of purely transitory transaction costs, then introduce persistent transaction costs, and finally consider the case of purely persistent costs.

1.1 Purely Temporary Transaction Costs

An investor must choose an optimal portfolio among S risky securities and a risk-free asset. The risky securities have prices p with dynamics

$$dp_t = \left(r^f p_t + B f_t\right) dt + du_t, \tag{1}$$

Here, f_t is a $K \times 1$ vector which contains the factors that predict excess returns, B is an $S \times K$ matrix of factor loadings, and u is an unpredictable "noise term," i.e., a martingale (e.g., a Brownian motion) with instantaneous variance-covariance matrix $\mathrm{var}_t(du_t) = \Sigma dt$. The return-predicting factors follow a general Markovian jump diffusion:

$$df_t = \mu_f(f_t)dt + \sigma_f(f_t)dw_t + dN_t^f, \tag{2}$$

where w is a Wiener process and N^f a purely-discontinuous martingale. We impose on the dynamics of f conditions sufficient to ensure that it is stationary. Occasionally, we also make use of the following simplifying assumption.

Assumption A1. The drift of f_t is given by $\mu_f(f) = -\Phi f$.

The agent chooses his trading intensity $\tau_t \in \mathbb{R}^S$, which determines the rate of change[6] of his position x_t:

$$dx_t = \tau_t dt. \tag{3}$$

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[6]We only consider smooth portfolio policies here because discrete jumps in positions or quadratic variation would be associated with infinite trading costs in this setting. Further, when we consider the discrete-time foundation for temporary transaction costs in Section 2.2, we see that such non-smooth strategies would have infinite transaction costs when the length of the trading periods approaches zero. E.g., if the agent trades n shares over a time period of Δt, then the cost according to (4) is $\int_0^{\Delta t} TC(\frac{n}{\Delta t})dt = \frac{1}{2}\Lambda \frac{n^2}{\Delta t}$, which approaches infinity as Δt approaches 0.

The transaction cost TC per time unit of trading with intensity τ_t is

$$TC(\tau_t) = \frac{1}{2}\tau_t^\top \Lambda \tau_t. \tag{4}$$

Here, Λ is a symmetric positive-definite matrix measuring the level of trading costs.[7] This quadratic transaction cost arises as a trade Δx_t shares moves the price by $\frac{1}{2}\Lambda \Delta x_t$, and this results in a total trading cost of Δx_t times the price move. This is a multi-dimensional version of Kyle's lambda. Most of our results hold with this general transaction cost function, but some of the resulting expressions are simpler in the following special case.

Assumption A2. Transaction costs are proportional to the amount of risk: $\Lambda = \lambda \Sigma$ for a scalar $\lambda > 0$.

This assumption is natural and, in fact, implied by the micro-foundation that we provide in Section 2.2. To understand this, suppose that a dealer takes the other side of the trade Δx_t, holds this position for a period of time dt, and "lays it off" at the end of the period. Then the dealer's risk is $\Delta x_t^\top \Sigma \Delta x_t\, dt$ and the trading cost is the dealer's compensation for risk, depending on the dealer's risk aversion reflected by λ. Section 2.2 further analyzes the conditions under which the compensation for risk is strictly positive.

The investor chooses his optimal trading strategy to maximize the present value of the future stream of expected excess returns, penalized for risk and trading costs:

$$\max_{(\tau_s)_{s\geq t}} E_t \int_t^\infty e^{-\rho(s-t)} \left(x_s^\top B f_s - \frac{\gamma}{2} x_s^\top \Sigma x_s - \frac{1}{2}\tau_s^\top \Lambda \tau_s \right) ds. \tag{5}$$

This objective function means that the investor has mean-variance preferences over the change in his wealth W_t each time period.

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[7]The assumption that Λ is symmetric is without loss of generality. To see this, suppose that $TC(\Delta x_t) = \frac{1}{2}\Delta x_t^\top \hat{\Lambda} \Delta x_t$, where $\hat{\Lambda}$ is not symmetric. Then, letting Λ be the symmetric part of $\hat{\Lambda}$, i.e., $\Lambda = (\hat{\Lambda} + \hat{\Lambda}^\top)/2$, generates the same trading costs as $\hat{\Lambda}$.

We conjecture and verify that the value function is quadratic in x:

$$V(x, f) = -\frac{1}{2} x^\top A_{xx} x + x^\top A_x(f) + A(f). \tag{6}$$

We solve the model explicitly, as the following proposition states. It is helpful to compare our result with the optimal portfolio under the classical no-friction assumption, for which we use the notation *Markowitz* as a reference to the classical findings of Markowitz (1952):

$$Markowitz_t = (\gamma\Sigma)^{-1} Bf_t. \tag{7}$$

The Markowitz portfolio has an optimal trade-off between risk and expected excess return, leveraged to suit the agent's risk aversion γ. We show that the optimal portfolio in light of transaction costs moves gradually towards an "aim portfolio," which incorporates current and future expected Markowitz portfolios.

Proposition 1 *(i) There exists a unique optimal portfolio strategy.*
(ii) The optimal portfolio x_t tracks a moving "aim portfolio" $\bar{M}^{aim}(f_t)$ with a tracking speed of $\bar{M}^{rate}$. That is, the optimal trading intensity $\tau_t = \frac{dx_t}{dt}$ is

$$\tau_t = \bar{M}^{rate} \left(\bar{M}^{aim}(f_t) - x_t \right), \tag{8}$$

where the coefficient matrix $\bar{M}^{rate}$ is given by

$$\bar{M}^{rate} = \Lambda^{-1} A_{xx} \tag{9}$$

$$A_{xx} = -\frac{\rho}{2}\Lambda + \Lambda^{\frac{1}{2}} \left(\gamma\Lambda^{-\frac{1}{2}}\Sigma\Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4}I \right)^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \tag{10}$$

and the aim portfolio by

$$\bar{M}^{aim}(f_t) = A_{xx}^{-1} A_x(f_t), \tag{11}$$

and $A_x(f)$ satisfies a second-order ODE given in the Appendix.

(iii) The aim portfolio $M^{aim}(f)$ has the intuitive representation

$$M^{aim}(f) = b \int_0^\infty e^{-bt} \, \mathrm{E}\,[Markowitz_t | f_0 = f] \, dt \tag{12}$$

with $b = \gamma A_{xx}^{-1} \Sigma$.

(iv) Under Assumption A1, the solution simplifies: $A_{xx} = a\Sigma$, $b > 0$ is a scalar, and

$$\begin{aligned}
\bar{M}^{rate} &= a/\lambda = \frac{1}{2}(\sqrt{\rho^2 + 4\gamma/\lambda} - \rho) \tag{13} \\
\bar{M}^{aim} &= \gamma^{-1}\Sigma^{-1}B\,(I + a/\gamma\Phi)^{-1}\,, \tag{14}
\end{aligned}$$

where the last equation also requires Assumption A2, $\mu(f) = -\Phi f$.

This proposition provides an intuitive method of portfolio choice. The optimal portfolio can be written in a simple closed-form expression. In light of the literature on portfolio choice with proportional transaction costs (Constantinides (1986)), which relies on numerical results even for a single asset with i.i.d. returns, our framework offers remarkable tractability even with correlated multiple assets and multiple signals.

It is intuitive that the optimal portfolio trades toward an aim portfolio, which is a weighted average of future expected Markowitz portfolios. This means that persistent signals are given more weight as they affect the Markowitz portfolio for a longer time period. This result is seen most clearly in Equation (14). The aim portfolio is seen to be almost of the same form as the Markowitz portfolio, expect that the signals are scaled down by $(I + a/\gamma\Phi)^{-1}$. Given that Φ contains the signals' mean-reversion rates, this means that quickly mean-reverting signals are scaled down more while more persistent ones receive more weight.

The trading rate given in (13) is remarkably simple. Naturally, the trading rate is decreasing in the transaction cost λ and increasing in risk aversion γ. Indeed, for a patient agent with $\rho \approx 0$, we see that the trading rate is approximately $\sqrt{\gamma/\lambda}$.

Example. To illustrate the optimal portfolio choice with frictions, we consider a specific

example. Figure 1 plots the evolution of the Markowitz portfolio in an economy with a single asset, say an equity-market index. The agent must decide on his equity allocation in light of his time-varying estimate of the equity premium while being mindful of transaction costs. To do this, he constructs an aim portfolio as seen in the figure. The aim portfolio is a more conservative version of the Markowitz portfolio due to transaction costs and because the agent anticipates that the Markowitz portfolio will mean-revert. Finally, the agent's optimal portfolio, also plotted in Figure 1, smoothly moves toward the aim portfolio, thus saving on transaction costs while capturing most of the benefits on the Markowtiz portfolio.

Interesting, we see that there are times when the optimal portfolio is below the Markowitz portfolio and above the aim, such that the optimal strategy is selling (i.e., negative $dx/dt = \tau$) even though the best risk-return trade-off is above. This selling is motivated by the agent's anticipation that the Markowitz portfolio will go down in the future, and, to save on transaction costs, it is cheaper to start selling gradually already now.

While it may not be easy to see in the figure, there are times when the aim portfolio is farther away from the Markowitz portfolio and other times when it is closer. This distance depends on which signal is driving the current high Markowitz portfolio — a persistent signal increases the aim portfolio more than a signal that will quickly mean-revert, which the signal's mean-reversion rates are irrelevant for Markowitz portfolio (and have not been studied by the portfolio choice literature more broadly, with the exception of Gârleanu and Pedersen (2013)).

Figure 2 illustrates the optimal portfolio choice with two assets. There are several differences in this illustration. First, the horizonal axis is now the allocation to asset one and the vertical axis the allocation to asset two. Second, rather than considering how the optimal portfolio evolves over time as shocks hit the economy, we consider its *expected* path.

We see that the Markowitz portfolio is expected to mean-revert along a concave curve. The concavity reflects that the signal that currently predicts a high return of asset 2 is more persistent. The current aim portfolio lies in the convex hull of the path of the expected future Markowitz portfolios. The optimal portfolio trade in the direction of the aim and

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it expected to eventually approach the Markowitz portfolio. Intuitively, the initial trading process focuses on buying shares of asset two, which is expected to have a high return over an extended time period.

1.2 Temporary and Persistent Transaction Costs

We modify the set-up above by adding persistent transaction costs. Specifically, the agent transacts at price $\bar{p}_t = p_t + D_t$, where the distortion D_t evolves according to

$$dD_t = -RD_t\, dt + C\, dx_t = -RD_t\, dt + C\tau_t\, dt. \tag{15}$$

where the scalar R is the price resiliency. The agent's objective now becomes

$$\max_{(\tau_s)_{s \geq t}} E_t \int_t^\infty e^{-\rho(s-t)} \left(x_s^\top \left(Bf_s - (r^f + R)D_s + C\tau_s \right) - \frac{\gamma}{2} x_s^\top \Sigma x_s - \frac{1}{2} \tau_s^\top \Lambda \tau_s \right) ds. \tag{16}$$

This objective function is similar to the one from above, but it has some new terms that multiply the position x_s. Now the expected excess return of prices that include persistent distortions, $\bar{p}_s = p_s + D_s$, is given by the expected excess return of p_s, which is Bf_s as before, plus the expected excess return of D_s, which is given by (15) in excess of the risk-free rate r^f. It might appear odd that the agent seems to benefit from buying and pushing the price higher, but this benefit leads to a loss as the distortion D decays.

It is no longer true in general that the objective (16) is concave in $\{\tau_t\}_t$, since the gain from the immediate increase in the mark-to-market value of the portfolio may exceed the loss from the (discounted) round-trip transaction costs . We therefore have to restrict attention to parameter configurations for which the objective is, indeed, concave. The fact that such configurations — with $C \neq 0$ — exist is ensured by Lemma 1, which provides sufficient conditions for concavity.

Lemma 1 *The objective function* (16) *is concave in* $\{\tau_t\}_t$ *if the persistent-impact matrix C is symmetric positive definite and $\gamma \geq (\rho - r^f)\|\Sigma^{-\frac{1}{2}} C \Sigma^{-\frac{1}{2}}\|$. The latter condition is automatically satisfied if $\rho \leq r^f$.*

We conjecture, as before, a value function that is quadratic in the endogenous state variable (x_t, D_t) and the factor f_t. Specifically, we write

$$V(x, D, f) = -\frac{1}{2}x^\top A_{xx}x + x^\top A_{xy}D + \frac{1}{2}D^\top A_{DD}D + x^\top A_x(f) + D^\top A_D(f) + A_{ff}(f). \quad (17)$$

Under an appropriate transversality condition, the value function exists and must have this form. We are concentrating on the optimal trading strategy.

Proposition 2 *(i) The optimal trading intensity has the form*

$$\tau_t = \bar{M}^{rate}\left(\bar{M}_f^{aim}(f_t) + \bar{M}_D^{aim}D_t - x_t\right) \quad (18)$$

for appropriate matrices $\bar{M}^{rate}$ and $\bar{M}_D^{aim}$ and function $\bar{M}_f^{aim}$ solving an ODE given in the proof.

(ii) An equivalent representation of the portion $\bar{M}_f^{aim}(f_t)$ of the aim due to f is

$$\bar{M}_f^{aim}(f) = N_2 \int_0^\infty e^{-N_1 t} N_3 \, \mathrm{E}\left[Markowitz_t | f_0 = f\right] dt \quad (19)$$

for matrices N_i given explicitly in the appendix.

We see that the optimal portfolio choice continues to have the same intuitive characteristics as in the model with only temporary impact costs. The optimal portfolio trades toward an aim, which now depends both on the current signals and the current persistent price distortions. The current signals affect the aim through a combination of their implied current and future Markowitz portfolios.

1.3 Purely Persistent Costs

The set-up is as above, but now we take $\Lambda = 0$. Under this assumption, it no longer follows that x_t has to be of the form $dx_t = \tau_t \, dt$ for some τ. Indeed, with purely persistent price-impact costs, the optimal portfolio policy can have jumps and infinite quadratic variation (i.e., "wiggle" like a Brownian motion).

As before, D is the price distortion and it evolves as

$$dD_t = -RD_t\, dt + C\, dx_t. \tag{20}$$

We define the objective of the trader to be

$$E_t \int_t^\infty e^{-\rho(s-t)} \left(x_s^\top \left(\alpha_s - (r^f + R)D_s \right) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \tag{21}$$
$$+ E_t \int_t^\infty e^{-\rho(s-t)} x_{s-}^\top C\, dx_s + \frac{1}{2} E_t \int_t^\infty e^{-\rho(s-t)} d\, [x, Cx]_s.$$

The terms in the first row of (21) are as before. Also, the first term in the second row is as before, although here it is written more generally. Indeed, $x_{s-}^\top C\, dx_s = x_s^\top C\tau_s ds$ when the portfolio is continuous and of bounded variation as it was above. This term captures the mark-to-market profit on the old position due to the market impact of the new trade, as before.

The last term is new. It records the instantaneous mark-to-market gain on the just-purchased units dx_s. Specifically, the new trade moves the price distortion by Cdx_s and we assume that the trade is executed at an average of the pre- and post-trade prices, which leads to a mark-to-market profit of $\frac{1}{2}$ times the price move. As the price distortion eventually disappears, this short-term gain is more than reversed later.[8]

A helpful observation in this case is that making a large trade Δx over an infinitesimal time interval has an easily described impact on the value function. In fact, the ability to liquidate one's position instantaneously, and then take a new position, at no cost relative to trading directly to the new position, implies

$$V(x, D, f) = V(0, D - Cx, f) - \frac{1}{2} x^\top C x. \tag{22}$$

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[8]One could alternatively assume that the entire new trade is executed at the new price, thus eliminating this term. However, such an assumption would imply that the objective function has no solution since a trader would prefer arbitrarily fast, but continuous and of bounded variation, trades rather than the solution that we derive. These strategies would be arbitrarily close to the optimal strategy that we derive. Other alternative assumptions suffer from similar issues. Under our assumption, a concavity result similar to Lemma 1 holds.

We prove this intuitive conjecture by providing a verification argument for the optimal control and value function that we propose.

Proposition 3 *(i) A quadratic value function exists of the form* (A.42) *in the appendix. The optimal portfolio is given by*

$$x_t = \bar{M}_{0,f}Bf_t + \bar{M}_{1,f}(f_t) - \bar{M}_D(D_- - Cx_-), \tag{23}$$

where the matrices $\bar{M}_{0,f}$ and $\bar{M}_D$ are given in the Appendix in terms of solutions to algebraic Riccati equations, and $M_{1,f}$ in terms of an appropriate ODE. (ii) It holds that

$$\bar{M}_{0,f}Bf_t + \bar{M}_{1,f}(f) = \hat{N}_0 Markowitz_0 + \hat{N}_2 \int_0^\infty e^{-\hat{N}_1 t}\hat{N}_3 \, \mathrm{E}\left[Markowitz_t|f_0 = f\right]dt \tag{24}$$

for appropriate matrices $\hat{N}_i$ given in the appendix.

We see that the optimal strategy is qualitatively different from the strategies that we derived above. Indeed, with purely persistent costs, the optimal strategy is no longer to trade toward an aim, but, rather, to choose a portfolio directly based on the current signals. Further, while the optimal portfolio continues to depend on the current and future expected Markowitz portfolios, the current one now has a distinct impact as seen in Equation (24).

Example. Figure 3 illustrates this result graphically. We see that the optimal portfolio has quadratic variation. It follows the Markowitz portfolio, but moderates the position to economize on persistent transaction costs.

2 Discrete-Time Foundation

In this section we consider the discrete-time foundation for our continuous-time model. We first review how the model is solved in discrete time (Section 2.1), then show how transaction costs arise as compensation to dealers (Section 2.2), and finally derive this section's main

result, namely that the discrete-time solution approaches the appropriate continuous-time solution as the length of time Δt between trading dates goes to zero (Section 2.3).

One of the central issues in this section is how each parameter depends on Δt. For the statistical-distribution parameters, the dependence on Δt is standard from the literature on how to discretize a continuous-time model, of course. For instance, the variance of a shock is linear in Δt, and so on. The one parameter where the literature does *not* offer guidance as to its dependence on Δt is the transaction cost. Further, how transaction costs depend on Δt is crucially important for understanding how one should optimally trade in modern markets where one can trade almost arbitrarily frequently. Hence, while Section 2.2 is decidedly stylized, it should really be viewed as a way to understand the economics of how transaction costs depend on time frequencies.

2.1 Discrete-Time Model and Solution

We start by presenting a discretely-sampled version of the continuous-time model. Securities are now traded at dates indexed by $t \in \{0, 1, 2, ...\}$, corresponding to calendar times $0, \Delta t, 2\Delta t, \ldots$, where Δt is the length of the time periods. The securities' price changes between times t and $t+1$ in excess of the risk-free return, $p_{t+1} - (1 + r^f \Delta t)p_t$, are collected in an $S \times 1$ vector r_{t+1}. As before, excess returns can be predicted by the factors f_t:

$$r_{t+1} \;\; = \;\; B f_t \Delta t + u_{t+1}, \tag{25}$$

where u_{t+1} is the unpredictable zero-mean noise term with variance $\text{var}_t(u_{t+1}) = \Sigma \Delta t$. Naturally, the returns' mean and variance scale linearly in time, Δt. The return-predicting factor f_t is known to the investor at time t and it evolves according to

$$\Delta f_{t+1} \;\; = \;\; -\Phi f_t \Delta t + \varepsilon_{t+1}, \tag{26}$$

where $\Delta f_{t+1} = f_{t+1} - f_t$ is the change in the factors, Φ is the matrix of mean-reversion coefficients, and ε_{t+1} is the factor shock with variance $\text{var}_t(\varepsilon_{t+1}) = \Omega \Delta t$. (We note that

we have imposed Assumption A1 to simplify the dynamics of f, but this is just for ease of exposition as our results extend more generally.)

An investor in the economy faces transaction costs. The transaction cost (TC) associated with trading $\Delta x_t = x_t - x_{t-1}$ shares is given by

$$TC(\Delta x_t) = \frac{1}{2}\Delta x_t^\top \Lambda(\Delta t)\Delta x_t, \tag{27}$$

where $\Lambda(\Delta t)$ is the matrix of transitory market impact costs. The literature does not offer guidance for how $\Lambda(\Delta t)$ depends on Δt. To address this issue, Section 2.2 provides this dependence of transaction costs on Δt in a model of endogenous dealer behavior.

The investor's objective is to choose the dynamic trading strategy $(x_0, x_1, ...)$ to maximize the present value of all future expected excess returns, penalized for risks and trading costs:

$$\max_{x_0, x_1,...} E_0 \left[\sum_t (1 - \rho\Delta t)^{t+1} \left(x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma \Delta t x_t \right) - \frac{(1 - \rho\Delta t)^t}{2} \Delta x_t^\top \Lambda \Delta x_t \right], \tag{28}$$

where the discount rate is $\rho\Delta t$ with $\rho \in (0,1)$, and γ is the risk-aversion coefficient (which naturally does not depend on Δt).

Gârleanu and Pedersen (2013) solve this discrete-time model using dynamic programming, but we re-derive the solution here for completeness. The value function $V(x_{t-1}, f_t)$ measures the value of entering period t with a portfolio of x_{t-1} securities and observing return-predicting factors f_t. It solves the Bellman equation:

$$V(x_{t-1}, f_t) = \max_{x_t} \left\{ -\frac{1}{2}\Delta x_t^\top \Lambda(\Delta t)\Delta x_t + (1 - \rho\Delta t)\left(x_t^\top E_t[r_{t+1}] - \frac{\gamma}{2} x_t^\top \Sigma x_t \Delta t + E_t[V(x_t, f_{t+1})] \right) \right\}. \tag{29}$$

The model has a unique solution and can be solved explicitly:

Proposition 4 (Discrete-Time Solution with Transitory Costs) *The optimal portfo-*

lio x_t tracks an "aim portfolio," $M^{aim}(\Delta t) f_t$, with trading rate $M^{rate}(\Delta t)$:

$$\Delta x_t = M^{rate}(\Delta t) \left(M^{aim}(\Delta t) f_t - x_{t-1} \right), \tag{30}$$

where the coefficients are given by the value-function coefficients, made explicit in the appendix:

$$M^{rate}(\Delta t) = \Lambda^{-1}(\Delta t) A_{xx}(\Delta t) \tag{31}$$

$$M^{aim}(\Delta t) = A_{xx}(\Delta t)^{-1} A_{xf}(\Delta t). \tag{32}$$

Transitory and persistent transaction costs.

To study the more general case featuring both transitory and persistent transaction costs, we extend the model by letting the price be given by $\bar{p}_t = p_t + D_t$ and the investor incur the cost associated with the persistent price distortion D_t in addition to the temporary trading cost TC from before. Hence, the price $\bar{p}_t$ is the sum of the price p_t without the persistent effect of the investor's own trading (as before) and the new term D_t, which captures the accumulated price distortion due to the investor's (previous) trades. Trading an amount Δx_t pushes prices by $C \Delta x_t$ such that the price distortion becomes $D_t + C \Delta x_t$, where $C(\Delta t)$ is Kyle's lambda for persistent price moves. Further, the price distortion mean reverts at a speed (or "resiliency") $R(\Delta t)$. Section 2.2 shows how C and R depend on Δt. Given the persistent price impact and resilience, the price distortion at the following date $(t + 1)$ is

$$D_{t+1} = (I - R)(D_t + C \Delta x_t). \tag{33}$$

The investor's objective is as before, with a natural modification due to the price distor-

tion:

$$E_0 \left[\sum_t (1 - \rho\Delta t)^{t+1} \left(x_t^\top \left[Bf_t - \left(R + r^f\right) \left(D_t + C\Delta x_t\right)\right] \Delta t - \frac{\gamma}{2} x_t^\top \Sigma x_t \Delta t \right) \right.$$
$$\left. + (1 - \rho\Delta t)^t \left(-\frac{1}{2} \Delta x_t^\top \Lambda \Delta x_t + x_{t-1}^\top C\Delta x_t + \frac{1}{2} \Delta x_t^\top C\Delta x_t \right) \right]. \tag{34}$$

Let us explain the various new terms in this objective function. The first term is the position x_t times the expected excess return of the price $\bar{p}_t = p_t + D_t$ given inside the inner square brackets. As before, the expected excess return of p_t is Bf_t. The expected excess return due to the post-trade price distortion is

$$D_{t+1} - (1 + r^f \Delta t)(D_t + C\Delta x_t) = -(R + r^f)(D_t + C\Delta x_t)\Delta t. \tag{35}$$

The second term is the penalty for taking risk as before. The three terms on the second line of (34) are discounted at $(1 - \rho)^t$ because these cash flows are incurred at time t, not time $t + 1$. The first of these is the temporary transaction cost as before. The second reflects the mark-to-market gain from the old position x_{t-1} from the price impact of the new trade, $C\Delta x_t$. The last term reflects that the traded shares Δx_t are assumed to be executed at the average price distortion, $D_t + \frac{1}{2}C\Delta x_t$. Hence, the traded shares Δx_t earn a mark-to-market gain of $\frac{1}{2}\Delta x_t^\top C\Delta x_t$ as the price moves up an additional $\frac{1}{2}C\Delta x_t$.

The value function is now quadratic in the extended state variable $(x_{t-1}, y_t) \equiv (x_{t-1}, f_t, D_t)$:

$$V(x, y) = -\frac{1}{2} x^\top A_{xx} x + x^\top A_{xy} y + \frac{1}{2} y^\top A_{yy} y + A_0.$$

As before, there exists a unique solution to the Bellman equation and the following proposition characterizes the optimal portfolio strategy.

Proposition 5 (General Discrete-Time Solution) *The optimal portfolio x_t is*

$$\Delta x_t = M^{rate}(\Delta t) \left(M^{aim}(\Delta t) y_t - x_{t-1} \right), \tag{36}$$

which tracks an aim portfolio, $M^{aim}(\Delta t) y_t$, that depends on the return-predicting factors and

the price distortion, $y_t = (f_t, D_t)$. The coefficient matrices $M^{rate}(\Delta t)$ and $M^{aim}(\Delta t)$, which depend on the length Δt of the time periods, are stated in the appendix.

2.2 Foundation for Transaction-Cost Specifications

We next consider the economic foundation for the quadratic transaction cost, the dependence on the trading frequency, and the limit as the trading frequency increases.

Transitory transaction costs.

To obtain a temporary price impact of trades endogenously, we consider an economy populated by three types of investors: (i) the trader whose optimization problem we study in the paper, referred throughout this section as "the trader," (ii) "market makers," who act as intermediaries, and (iii) "end users," on whom market makers eventually unload their positions as described below.

The temporary price impact is due to the market makers' inventories. We assume that there are a mass-one continuum of market makers indexed by the set $[0, h]$ and they arrive for the first time at the market at a time equal to their index. The market operates only at discrete times Δt apart,[9] and the market makers trade at the first trading opportunity. Once they trade — say, at time t — market makers must spend h units of time gaining access to end users. At time $t + h$, therefore, they unload their inventories at a price p_{t+h} described below, and rejoin the market immediately thereafter. It follows that at each market trading date there is always a mass $\frac{\Delta t}{h}$ of competing market makers that clear the market.

The price p, the competitive price of end users, follows an exogenous process and corresponds to the fundamental price in the body of the paper. Market makers take this price as given and trade a quantity q to maximize a quadratic utility:

$$\max_q \left\{ \hat{E}_t \left[q^\top (p_{t+h} - e^{rh} \hat{p}_t) \right] - \frac{\gamma^M}{2} Var_t \left[q^\top (p_{t+h} - e^{rh} \hat{p}_t) \right] \right\}, \tag{37}$$

where $\hat{p}_t$ is the market price at time t and r is the (continuously-compounded) risk-free rate

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[9]We make the simplifying assumption that $\frac{h}{\Delta t}$ is an integer.

over the horizon. $\hat{E}$ denotes expectations under the probability measure obtained from the market makers' beliefs using their (normalized) marginal utilities corresponding to $q = 0$ as Radon-Nikodym derivative. Consequently,

$$\hat{E}_t\left[p_{t+h}\right] = e^{rh}p_t,$$

so that the maximization problem becomes

$$\max_q \left\{ q(p_t - \hat{p}_t) - e^{-rh}\frac{\gamma^M}{2}Var_t\left[qp_{t+h}\right] \right\}. \tag{38}$$

The price $\hat{p}$ is set so as to satisfy the market-clearing condition

$$0 = \Delta x_t + q\frac{\Delta t}{h}. \tag{39}$$

Combining the market clearing condition and the market marker's optimality condition, we get the following expression, where we also use that the variance of h-periods-ahead prices (denoted V_h) can be easily calculated since p is exogenous and Gaussian,[10]

$$\hat{p}_t = p_t + e^{-rh}\gamma^M V_h \frac{\Delta x_t}{\Delta t}h. \tag{40}$$

Consequently, if the trader trades an amount Δx_t, he trades at the unit price of p_t and pays an additional transaction cost of

$$e^{-rh}\gamma^M \Delta x_t^\top V_h \frac{\Delta x_t}{\Delta t}h,$$

which has the quadratic form posited in Section 2.1.

Two cases suggest themselves naturally when considering the choice for the holding period h as a function of Δt. In the first case, a decreasing Δt is thought of as an improvement in the trading technology, attention, etc., of all market participants, and therefore h decreases as Δt does — in its simplest form, $h = \Delta t$, which yields a transaction cost of the order Δt^2.

---

[10]The resulting value is $V_h = \Sigma h + BN_h\Omega N_h^\top B^\top$, where $N_h = \int_0^h \int_u^h e^{-\Phi(t-u)}dt\,du = \Phi^{-1}h - \Phi^{-2}\left(I - e^{-\Phi h}\right)$ if Φ is invertible. (Note that the first term, Σh, is of order h, while the second of order h^2.)

Generally, as long as $h \to 0$ as $\Delta t \to 0$, the transaction costs also vanishes.

The second case is that of a constant h: the dealers need a fixed amount of time to lay off a position regardless of the frequency with which our original traders access the market. It follows, in this case, that the price impact does not vanish as Δt becomes small: in the continuous-time limit ($\Delta t \to 0$), the per-unit-of-time transaction cost is proportional to

$$\lim_{\Delta t \to 0} \frac{\Delta x_t^\top}{\Delta t} V_h h \frac{\Delta x_t}{\Delta t} = \tau^\top V_h h \tau, \tag{41}$$

as assumed in Section 1.1. One can therefore interpret Δt in this case as the frequency with which the researcher observes the world, which does not impact (to the first order) equilibrium quantities — in particular, flow trades and costs. We summarize our results as follows:

Proposition 6 (Time Dependence of Transitory Transaction Costs)

(i) If dealers need a fixed amount of calendar time to lay off their inventory, then the transitory market-impact parameter $\Lambda(\Delta t)$ is of order $1/\Delta t$, $\Lambda(\Delta t) = \Lambda/\Delta t$.

(ii) If dealers can lay off their inventory during each time period, then the transitory market-impact parameter $\Lambda(\Delta t)$ is of order Δt, $\Lambda(\Delta t) = \Lambda \Delta t$.

Persistent transaction costs.

We model persistent price impact costs with a similar model, but with a different specification of the market makers. Consider therefore the same model as in the previous section, but suppose now that market makers do not hold their inventories for a deterministic number h of time units, but rather manage to deplete them, through trade with end users at price p, at a constant rate ψ. Thus, between two trading dates with the trader, a market maker's inventory evolves according to

$$\Delta I_t = -\psi I_{t-1} \Delta t + q_t, \tag{42}$$

where, in equilibrium,

$$q_t = \Delta x_t.$$

The market makers continue to maximize a quadratic objective:

$$\max_{\{q_s\}_{s \geq t}} \left\{ \hat{E}_t \sum_{s \geq t} e^{-r(s-t)} \left(\psi I_{s-1}^\top p_s \Delta t - q_s^\top \hat{p}_s - \frac{\gamma^M}{2} I_{s-1}^\top V_{\Delta t} I_{s-1} \right) \right\}, \tag{43}$$

subject to (42) and expectations about q described below. Note that the market maker's objective depends (positively) on the expected cash flows $\psi I_s^\top p_{s-1} \Delta t - q_s^\top \hat{p}_s$ due to future trades with the end user and the trader and negatively on the risk of his inventory.

We assume that market makers cannot predict the trader's order flow Δx. More specifically, according to their probability distribution,

$$\hat{E}\left[\Delta x_t \,|\, \mathcal{F}_s, s < t \right] = 0 \tag{44}$$

$$\hat{E}\left[(\Delta x_t)^2 \,|\, \mathcal{F}_s, s < t \right] = v. \tag{45}$$

Moments of q_s and I_s follow immediately.

The first-order condition with respect to q_t is

$$0 = \hat{E}_t \sum_{s \geq t} e^{-r(s-t)} \left(\psi p_s^\top - \gamma^M I_{s-1}^\top \frac{V_{\Delta t}}{\Delta t} \right) \frac{\partial I_{s-1}}{\partial q_t} \Delta t - \hat{p}_t^\top. \tag{46}$$

Using the fact that $\frac{\partial I_s}{\partial q_t} = (1 - \psi \Delta t)^{s-t}$ for $s \geq t$, the first-order condition yields

$$\hat{p}_t = \hat{E}_t \sum_{s > t} e^{-r(s-t)} (1 - \psi \Delta t)^{s-t-1} \left(\psi p_s - \gamma^M \frac{V_{\Delta t}}{\Delta t} I_{s-1} \right) \Delta t. \tag{47}$$

Using the facts that $\hat{E}_t[e^{-r(s-t)} p_s] = p_t$ and $\hat{E}_t[I_s] = (1 - \psi \Delta t)^{s-t} I_t$, we obtain

$$\hat{p}_t = p_t - \kappa_I I_t \tag{48}$$

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for a constant matrix

$$\kappa_I = \sum_{n=0}^{\infty} e^{-r(n+1)\Delta t}(1 - \psi\Delta t)^{2n\Delta t}\gamma^M \frac{V_{\Delta t}}{\Delta t}\Delta t. \tag{49}$$

The price $\hat{p}_t$ is only the price at the end of trading date t — the price at which the last unit of the q_t shares is traded. We assume that, during the trading date, orders of infinitesimal size come to market sequentially and the market makers' expectation is that the remainder of date-t order flow aggregates to zero — thus, the order flow is a martingale. It follows that the price paid for the kth percentile of the order flow q_t is $p_t - \kappa_I(I_{t-1} + kq_t)$. This mechanism ensures that round-trip trades over very short intervals do not have transaction-cost implications.

This price specification is the same as in Section 2.1, with $\Lambda = 0$ and $D_t = -\kappa_I I_t$:

$$
\begin{aligned}
D_{t+1} &= -\kappa_I I_{t+1} \\
&= -\kappa_I \left(I_t - \psi I_t \Delta t + \Delta x_t \right) \\
&= D_t - \kappa_I \psi \kappa_I^{-1} \Delta t D_t - \kappa_I \Delta x_t \\
&\equiv (I - R(\Delta t))\left(D_t + C\Delta x_t \right).
\end{aligned}
\tag{50}
$$

We summarize the implications for the specification of the price impact faced by the trader in the following.

Proposition 7 (Time Dependence of Persistent Transaction Costs) *The resiliency parameter R is of order Δt, $R(\Delta t) = R\Delta t$. The persistent market impact C does not depend on Δt.*

Transitory and persistent transaction costs.

The two types of price impact can obtain simultaneously in this model so that we can have both kinds of transaction costs and consider their separate convergence to continuous time using Propositions 6–7. To see this, consider for instance an economy with the trader and *two* kinds of market makers. The trader transacts with the first group of market makers. After a

period of length h, these market makers clear their inventories with a second group of market makers, who specialize in locating end users and trading with them. This second group of market makers deplete their inventories only gradually (at a constant rate as above), giving rise to a persistent impact. The trader must compensate both groups of market makers for the risk taken, resulting in the two price-impact components.

2.3   Convergence as Length of Time Periods Vanishes

We now show that the continuous-time model and its solution are the limit of their discrete-time analogues. Our micro foundation for transaction costs highlights that there are two important cases that lead to different continuous-time limits, as seen in Proposition 6.

Proposition 8 *(i) Suppose that dealers need a fixed amount of calendar time to lay off their inventory, so that $\Lambda(\Delta t) = \Lambda/\Delta t$. Consider any continuous-time strategy x_t and the discretely sampled counterparts $x_t^{(\Delta t)}$ at the model frequency $1/\Delta t$. Then, as $\Delta t \to 0$, the objective (34) in the general discrete-time model with transitory and persistent transaction costs tends to the continuous-time objective (16) for any strategy x_t satisfying $dx_t = \tau_t dt$. For all other strategies the limit objective equals negative infinity.*

Furthermore, the optimal discrete-time trading strategy tends to the continuous-time solution from Proposition 2. In particular, the continuous-time matrix coefficients $\bar{M}^{rate}$ and $\bar{M}^{speed}$ are the limits of the discrete-time coefficients M^{rate} and M^{speed} as follows:

$$\lim_{\Delta t \to 0} \frac{M^{rate}(\Delta t)}{\Delta t} = \bar{M}^{rate} \tag{51}$$

$$\lim_{\Delta t \to 0} M^{aim}(\Delta t) = \bar{M}^{aim}. \tag{52}$$

(ii) Suppose that dealers can lay off their inventory each time period so that $\Lambda(\Delta t) = \Lambda \Delta t$. Then, for any continuous-time strategy x_t, the objective (34) evaluated at the discretely-sampled $x_t^{(\Delta t)}$ tends to the continuous-time objective (21) of the continuous-time model with purely persistent costs. The optimal discrete-time trading strategy converges to the continuous-time strategy described in Proposition 3.

The proposition establishes that, for small Δt, the discrete-time model is fundamentally the same as one of the two continuous-time models introduced in Section 1.2, respectively Section 1.3. A key observation is that the limit model, and consequently the qualitative properties of the optimal strategy, are different depending on the behavior of the function $\Lambda(\Delta t)$ as Δt vanishes, and thus on the nature of the liquidity provision by the intermediaries. Specifically, if intermediaries hold their risky inventories over periods of time of fixed length as Δt goes to zero, then the model with transitory costs and smooth optimal trading obtains in the limit. On the other hand, if holding periods shrink towards zero with Δt, then there are no transitory costs in the limit and the optimal trading has non-zero quadratic variation.

Example. This convergence result in illustrated in Figure 4. The figure plots the optimal position in discrete time when $\Delta t = 1$ and $\Delta t = 0.25$ and in continuous time. The parameters are chosen consistently under the assumption of a fixed calendar time to lay off inventory. Also, the outcome of the random shocks are chosen consistently in the sense that the discrete-time models use the discretized versions of the shocks to the return-predicting signals f_t. Figure 4 illustrates how discrete-time trading corresponds to a step-function for the portfolio. As the trading frequency increases, the step function becomes smoother and, in the limit, converges to the continuous-time solution as shown.

3   Time-Varying Volatility or Risk Aversion

Much of the tractability of the framework is preserved if one lets the risk aversion, transaction costs, or return variance vary over time. Specifically, the results derived above continue to hold, except that the value-function coefficients are functions of the time-varying parameter.[11] We illustrate this statement in the simplest setting in which the price impact is purely transitory and Assumptions A1–A2 hold. The novel assumption is that, rather than having a constant variance-covariance matrix for shocks u to returns, we now consider the case of time-varying volatility. Specifically, let $\text{var}_t(du_t) = v_t \bar{\Sigma}$ where the positive process v_t evolves

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[11]We continue to assume a Markovian structure.

according to

$$dv_t = \mu_v(v_t)dt + \sigma_v(v_t)dw_t. \tag{53}$$

Here, w_t is a (one-dimensional) Wiener process, possibly correlated with ε and u.

We conjecture the value function to be quadratic in (x, f), but with coefficients that depend on v:

$$V(x, f, v) = -\frac{1}{2}x^\top A_{xx}(v)x + x^\top A_{xf}(v)f + \frac{1}{2}f^\top A_{ff}(v)f + A_0(v). \tag{54}$$

The HJB equation provides (second-order) differential equations for the coefficient functions — in particular, A_{xx} and A_{xf}, which determine the trading strategy. In the special case $\mu_v(v) = \bar{\mu}_v v$,[12] these ODEs can be solved explicitly: A_{xx} is linear in v and A_{xf} is constant. A more empirically relevant case, however, is that of a mean-reverting volatility level v. The following proposition records some properties of the ensuing optimal trading strategy.

Proposition 9 (Stochastic volatility) *Suppose that the drift $\mu_v(\cdot)$ of v_t is a decreasing function which crosses zero on the support of v. Then there exists a cut-off value $\hat{v}$ such that (i) the trading rate satisfies $M^{rate}(v) \geq M^{rate}(\hat{v}) \geq M^{rate}(v')$ whenever $v \leq \hat{v} \leq v'$; and (ii) for $v_t \leq \hat{v}$, $M^{rate}(v_t)$ is higher than it would be if v was constant and equal to v_t for $s \geq t$, and conversely for $v_t \geq \hat{v}$.*

The proposition shows how the volatility mean reversion impacts the trading strategy. In particular, if v_t is low enough, so that it is expected to increase, then the trading intensity is higher than if v were to stay constant: the higher utility cost due to increased future volatility, which is persistent, is mitigated by trading more currently, before the trading cost increases along with the volatilty.

We also note that, under Assumption A1, letting $\mathrm{var}_t(du_t)$ be proportional to v_t and letting Λ_t and γ_t be proportional to v_t are isomorphic. The results also hold if $\Lambda_t = \lambda\Sigma$ is constant (this is the same as letting only γ depend on v), as is intuitive.

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[12]One can ensure that v is bounded by letting $\sigma_v(v)$ be zero outside some interval.

4   Equilibrium Implications

In this section we study the restrictions placed on a security's return properties by the market equilibrium. More specifically, we consider a situation in which an investor facing transaction costs absorbs a residual supply specified exogenously and analyze the relationship implied between the characteristics of the supply dynamics and the excess return.

For simplicity, we consider a model set in continuous time, as detailed in Section 1.1, featuring one security in which $L \geq 1$ groups of (exogenously given) noise traders hold positions z_t^l (net of the aggregate supply) given by

$$dz_t^l \;=\; \kappa \left(f_t^l - z_t^l \right) dt \tag{55}$$

$$df_t^l \;=\; -\psi_l f_t^l dt + d\varepsilon_t^l. \tag{56}$$

It follows that the aggregate noise-trader holding, $z_t = \sum_l z_t^l$, satisfies

$$dz_t \;=\; \kappa \left(\sum_{l=1}^{L} f_t^l - z_t \right) dt. \tag{57}$$

We conjecture that the investor's inference problem is as studied in Section 1.1, where f given by $f \equiv (f^1, ..., f^L, z)$ is a linear return predictor and B is to be determined. We verify the conjecture and find B as part of Proposition 10 below.

Given the definition of f, the mean-reversion matrix Φ is given by

$$\Phi \;=\; \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa & -\kappa & \cdots & \kappa \end{pmatrix}. \tag{58}$$

Suppose that the only other investors in the economy are the investors considered in Section 1.1, facing transaction costs given by $\Lambda = \lambda \sigma^2$. In this simple context, an equilibrium is defined as a price process and market-clearing asset holdings that are optimal for all agents given the price process. Since the noise traders' positions are optimal by assumption as

specified by (55)–(56), the restriction imposed by equilibrium is that the dynamics of the price are such that, for all t,

$$x_t = -z_t \tag{59}$$

$$dx_t = -dz_t. \tag{60}$$

Using the results in Proposition 1, these equilibrium conditions lead to

$$\frac{a}{\lambda}\sigma^{-2}B(a\Phi + \gamma I)^{-1} + \frac{a}{\lambda}e_{L+1} = -\kappa(\mathbf{1} - 2e_{L+1}), \tag{61}$$

where $e_{L+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{L+1}$ and $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^{L+1}$. It consequently follows that, if the investor is to hold $-z_t = -f_t^{L+1}$ at time t for all t, then the factor loadings must be given by

$$B = \sigma^2 \left[-\frac{\lambda}{a}\kappa(\mathbf{1} - 2e_{L+1}) - e_{L+1} \right] (a\Phi + \gamma I). \tag{62}$$

For $l \leq L$, we calculate B_l further as

$$\begin{aligned} B_l &= -\sigma^2\kappa(\lambda\psi_l + \lambda\gamma a^{-1} + \lambda\kappa - a) \\ &= -\lambda\sigma^2\kappa(\psi_l + \rho + \kappa), \end{aligned} \tag{63}$$

while

$$B_{L+1} = \sigma^2(\rho\lambda\kappa + \lambda\kappa^2 - \gamma). \tag{64}$$

Using this, it is straightforward to see the following key equilibrium implications:

Proposition 10 *The market is in equilibrium if and only if $x_0 = -z_0$ and the security's expected excess return is given by*

$$\frac{1}{dt}E_t[dp_t - r^f p_t \, dt] = \sum_{l=1}^{L} \lambda\sigma^2\kappa(\psi_l + \rho + \kappa)(-f_t^l) + \sigma^2(\rho\lambda\kappa + \lambda\kappa^2 - \gamma)z_t. \tag{65}$$

The coefficients $\lambda\sigma^2\kappa(\psi_k+\rho+\kappa)$ are positive and increase in the mean-reversion parameters ψ_k and κ and in the trading costs $\lambda\sigma^2$. In other words, noise trader selling ($f_t^k < 0$) increases expected excess returns, and especially so if its mean reversion is faster and if the trading cost is larger.

Naturally, noise-trader selling increases the expected excess return, while noise-trader buying lowers it, since the arbitrageurs need to be compensated to take the other side of the trade. Interestingly, the effect is larger when trading costs are larger and for noise-trader shocks with faster mean reversion because such shocks are associated with larger trading costs for the arbitrageurs.

5   Conclusion and Broader Implications

This paper provides a general framework for optimal portfolio choice with frictions and multiple time-varying signals about expected returns. While the framework is very general, allowing rich dynamics for returns and signals, it is nevertheless highly tractable. Indeed, the optimal portfolio is derived as an intuitive closed-form expression.

The optimal portfolio strategy trades gradually toward an aim portfolio that depends on current on future expected optimal optimal portfolios in the absence of transaction costs. Hence, financial frictions imply that signals' dynamics are important, in particular their persistence over time. Intuitively, a signal is given more weight if is it more persistent, since a longer-lasting effect should be incorporated more in light of frictions.

We show how our continuous-time model is approached by discrete-time models of vanishing time-period length if the model parameters are scaled appropriately. The key innovation in this respect is to determine the correct time-scaling of the transaction-cost parameter. We provide an economic foundation for this time-scaling of transaction costs, and show that the convergence happens naturally in this economic setting. Further, we derive implications for equilibrium expected returns, showing why high-frequency movements in expected returns are larger than low-frequency movements, as documented empirically. Finally, as

we elaborate below, the model's tractability makes it a powerful tool with many potential applications in other areas of economics and, indeed, even more broadly.

General dynamic models. Before outlining a few specific applications, we note that many dynamic models in the social sciences are special cases of the linear-quadratic framework or can be approximated well by this framework. At a high level, our model shows that, with frictions and multiple signals with varying mean-reversion rates, the optimal strategy moves gradually toward an aim that overweights persistent information. Further, our model shows how the answer is robust to the frequency of policy changes when the policy parameters are scaled appropriately depending on the time horizon.

Macroeconomics. Many macroeconomic models rely on the linear-quadratic framework (see, e.g., Ljungqvist and Sargent (2004)). As an illustration, consider an economy with different signals about total factor productivity (TFP) and capital adjustment costs. In this case, our model can be applied to show how to gradually adjust the capital stock towards an aim that overweights persistent signals about TFP shocks.

Monetary policy. The linear-quadratic framework has also been employed extensively in models of monetary economics (Benigno and Woodford (2003)). Our model can be recast as describing a central bank receiving multiple signals about inflation pressures, e.g., across regions, and facing adjustment costs (capturing what is often termed "policy inertia"). In this case, our results mean that monetary policy should move gradually towards an aim that optimally weights the different signals. A highly persistent signal of deflationary pressures in southern Europe should be weighted more heavily than a transitory signal of inflation in (an equal-sized region of) Germany.

Political economy. As another potential application, the model could describe a political party receiving different signals from various constituents. In this case, our model's insight shows how the party should move its politics gradually toward an aim that optimally incorporates all signals, giving more weight to persistent political trends and less to shorter-lived fads. Let us sketch how to capture this in our model, as this framework may be less standard in political economics. A political party must choose its views x_t on each of several

issues, e.g., x_t^1 is the view on economic policy, x_t^2 is the view on social issues, and so on. The party receives signals f_t about the views of different constituents, which can be aggregated to a vector of average views about all the issues, Gf_t. The policy maker faces quadratic costs of deviating from the current average view:

$$(x_t - Gf_t)'\Sigma(x_t - Gf_t) = -x_t'Bf_t + x_t'\Sigma x_t + f_t'G'\Sigma Gf_t,$$

where $B = -2\Sigma G$. The first two terms correspond to our objective function and the last can be ignored as it is independent of the choice x_t. Further, the party faces quadratic costs of changing its views (the cost of "flip-flopping"), making the model a specific case of our framework.

Microeconomic model of product design. Consider a monopolistic firm, which must choose the design x_t of its product, where x_t^1 could be the color, x_t^2 the marketing expense, etc. Customers' preferences for different products change over time such that the firm faces the following demand curve:

$$Demand(price; x_t, f_t) = x_t'Hf_t \cdot price^{-s}.$$

Here, $s > 1$ is the price elasticity, H is a matrix with positive elements, and f_t is a positive process capturing how consumers value each product attribute. Renting a machine that can produce the good with design x_t costs $1/2 x_t'\Sigma x_t$ and the marginal production cost is c. Given a product design, the profit is derived from the optimal price setting:

$$x_t'Bf_t := x_t'Hf_t \cdot \max_{price} price^{-s} \cdot (price - c).$$

With a quadratic cost of changing the product design, we see that this model is special case of our general framework. Hence, our results show that the product design should be adjusted towards a combination of the signals of consumer tastes that gives higher weight to the more persistent trends.

In summary, the model presents a highly tractable framework that gives rise to several

insights concerning optimal trading in financial markets, and it can be applied to other dynamic problems featuring frictions and signals of varying persistence.

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A   Proofs

Proof of Proposition 1.   This proposition is a special case of Proposition 2, but its simplicity allows for the relaxation the constraints placed on other parameters to obtain a well-behaved problem. Specifically, it is immediate that the objective (5) is strictly concave in $\{\tau_s\}_s$. To prove optimality with an infinite horizon, we impose a transversality condition on any admissible strategy, namely that

$$\lim_{T \to \infty} \mathrm{E}_t[e^{-\rho(T-t)} x_T] = 0. \tag{A.1}$$

We also impose appropriate conditions ensuring that f is stationary.

The Hamilton-Jacoby-Bellman (HJB) equation is

$$\rho V = \sup_\tau \left\{ x^\top B f - \frac{\gamma}{2} x^\top \Sigma x - \frac{1}{2} \tau^\top \Lambda \tau + \frac{\partial V}{\partial x} \tau + \frac{\partial V}{\partial f} \mu_f + \frac{1}{2} tr \left(\Omega \frac{\partial^2 V}{\partial f \partial f^\top} \right) \right\}. \tag{A.2}$$

Maximizing this expression with respect to the trading intensity results in

$$\tau = \Lambda^{-1} \frac{\partial V}{\partial x}^\top .$$

Given the conjectured form (6) of the value function, the optimal choice τ equals

$$\tau_t = -\Lambda^{-1} A_{xx} x_t + \Lambda^{-1} A_x(f_t).$$

Once this expression is inserted in the HJB equation, it results in the following equations defining the value-function coefficients (using the symmetry of A_{xx}):

$$-\rho A_{xx} = A_{xx} \Lambda^{-1} A_{xx} - \gamma \Sigma \tag{A.3}$$

$$\rho A_x(f) = -A_{xx} \Lambda^{-1} A_x(f) + \mathcal{D} A_x(f) + B f \tag{A.4}$$

$$\rho A(f) = A_{xf}^\top \Lambda^{-1} A_{xf} + \mathcal{D} A_{ff}. \tag{A.5}$$

Pre- and post-multiplying (A.3) by $\Lambda^{-\frac{1}{2}}$, we obtain

$$-\rho Z \;=\; Z^2 + \frac{\rho^2}{4}I - U, \tag{A.6}$$

that is,

$$\left(Z + \frac{\rho}{2}I\right)^2 = U, \tag{A.7}$$

where

$$Z \;=\; \Lambda^{-\frac{1}{2}}A_{xx}\Lambda^{-\frac{1}{2}} \tag{A.8}$$

$$U \;=\; \gamma\Lambda^{-\frac{1}{2}}\Sigma\Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4}I. \tag{A.9}$$

This leads to

$$Z \;=\; -\frac{\rho}{2}I + U^{\frac{1}{2}} \;\geq\; 0, \tag{A.10}$$

implying that

$$A_{xx} \;=\; -\frac{\rho}{2}\Lambda + \Lambda^{\frac{1}{2}}\left(\gamma\Lambda^{-\frac{1}{2}}\Sigma\Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4}\right)^{\frac{1}{2}}\Lambda^{\frac{1}{2}}. \tag{A.11}$$

The value of A_x follows as a solution to the ODE (A.4). Note that, using the Feynman-Kac formula, A_x can be written as

$$A_x(f) = \mathrm{E}\left[\int_0^\infty e^{-(\rho + A_{xx}\Lambda^{-1})t}B f_t\, dt\,|\,f_0 = f\right] \tag{A.12}$$

$$= \int_0^\infty e^{-(\gamma\Sigma)A_{xx}^{-1}t}(\gamma\Sigma)\,\mathrm{E}\left[M_t|f_0 = f\right] dt \tag{A.13}$$

$$= \int_0^\infty (\gamma\Sigma)e^{-A_{xx}^{-1}(\gamma\Sigma)t}\,\mathrm{E}\left[M_t|f_0 = f\right] dt, \tag{A.14}$$

where the second equality holds because of (A.3).

If $\mu_f(f) = -\phi f$, then $A_x(f) = A_{xf}f$ and $A(f) = f^\top A_{ff}f$, and (A.4)–(A.5) become

$$\rho A_{xf} = -A_{xx}\Lambda^{-1}A_{xf} - A_{xf}\Phi + B \tag{A.15}$$

$$\rho A_{ff} = A_{xf}^\top \Lambda^{-1}A_{xf} - 2A_{ff}\Phi. \tag{A.16}$$

The solution for A_{xf} follows from Equation (A.15), using the general rule that $\mathrm{vec}(XYZ) = (Z^\top \otimes X)\,\mathrm{vec}(Y)$:

$$\mathrm{vec}(A_{xf}) = \left(\rho I + \Phi^\top \otimes I_K + I_S \otimes (A_{xx}\Lambda^{-1})\right)^{-1}\mathrm{vec}(B).$$

If $\Lambda = \lambda\Sigma$, then $A_{xx} = a\Sigma$ with

$$-\rho a = a^2\frac{1}{\lambda} - \gamma, \tag{A.17}$$

with solution

$$a = -\frac{\rho}{2}\lambda + \sqrt{\gamma\lambda + \frac{\rho^2}{4}\lambda^2}. \tag{A.18}$$

In this case, (A.4) yields

$$\begin{aligned}
A_{xf} &= B\left(\rho I + \frac{a}{\lambda}I + \Phi\right)^{-1} \\
&= B\left(\frac{\gamma}{a}I + \Phi\right)^{-1},
\end{aligned}$$

where the last equality uses (A.17).

Then, we have

$$\tau_t = \frac{a}{\lambda}\left[\Sigma^{-1}B\left(a\Phi + \gamma I\right)^{-1}f_t - x_t\right]. \tag{A.19}$$

It is clear from (A.18) that $\frac{a}{\lambda}$ decreases in λ and increases in γ. ∎

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Proof of Lemma 1. The trader's utility dependence on τ is given by

$$\int_0^\infty e^{-\rho s} \left(x_s^\top \left(B f_s - (r + R) D_s + C \tau_s \right) - \frac{\gamma}{2} x_s^\top \Sigma x_s - \frac{1}{2} \tau_s^\top \Lambda \tau_s \right) ds, \tag{A.20}$$

which is clearly concave if

$$\int_0^\infty e^{-\rho s} \left(x_s^\top \left(-(r + R) D_s + C \tau_s \right) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \tag{A.21}$$

is. To see the latter fact, we start by evaluating[13]

$$- \int_0^\infty e^{-\rho s} x_s^\top (r + R) D_s = - \int_0^\infty e^{-\rho s} \left(\int_0^s \tau_t^\top dt \, (r + R) \int_0^s e^{-R(s-u)} C \tau_u \, du \right) ds$$

$$= - \frac{r + R}{\rho + R} \int_0^\infty \int_0^\infty \tau_t^\top e^{Ru - (R+\rho)(t \vee u)} C \tau_u \, du \, dt \tag{A.22}$$

and break down (A.22) in two term depending on whether $u < t$ or vice-versa. On the set $u \geq t$ we obtain the integral

$$\int_0^\infty \int_0^u \tau_t^\top e^{-\rho u} C \tau_u \, dt \, du = \int_0^\infty x_u^\top e^{-\rho u} C \tau_u \, du \tag{A.23}$$

$$= \left[x_u^\top e^{-\rho u} C x_u \right]_0^\infty - \int_0^\infty x_u^\top e^{-\rho u} C \left(\tau_u - \rho x_u \right) du \tag{A.24}$$

$$= \frac{1}{2} \int_0^\infty \rho e^{-\rho u} x_u^\top C x_u \, du, \tag{A.25}$$

where the second equality follows by integration by parts and the third by solving the equation implicit in the second.

The integral on the set $u < t$ is similarly shown to be negative definite. To ensure the concavity of (A.21), it therefore suffices that

$$\frac{1}{2} \left(1 - \frac{r + R}{\rho + R} \right) \int_0^\infty \rho e^{-\rho u} x_u^\top C x_u \, du - \frac{\gamma}{2} \int_0^\infty \rho e^{-\rho u} x_u^\top \Sigma x_u \, du \tag{A.26}$$

---

[13]We take $x_0 = D_0 = 0$ without loss of generality, since these values do not affect the concavity of the objective.

is concave, or

$$\gamma > \frac{\rho - r}{\rho + R} \left\| \Sigma^{-\frac{1}{2}} C \Sigma^{-\frac{1}{2}} \right\|_2.$$ (A.27)

∎

Proof of Proposition 2. In this case, the conjectured value function is

$$V = -\frac{1}{2} x^\top A_{xx} x + x^\top A_{xD} D + \frac{1}{2} D^\top A_{DD} D + x^\top A_x(f) + D^\top A_D(f) + A(f).$$ (A.28)

Given the HJB equation

$$\rho V = \sup_\tau \left\{ -\frac{\gamma}{2} x^\top \Sigma x + x^\top (Bf + C\tau) - \frac{1}{2} \tau^\top \Lambda \tau + \tau^\top V_x^\top + (C\tau - RD)^\top V_D^\top + \right.$$ (A.29)
$$\left. x^\top \mathcal{D} A_x(f) + D^\top \mathcal{D} A_D(f) + \mathcal{D} A(f) \right\},$$

the optimal trade follows as

$$\tau = \Lambda^{-1} \left(A_x(f) + C^\top A_D(f) + \left(A_{xD} + C^\top A_{DD} \right) D - \left(A_{xx} + C^\top + C^\top A_{Dx} \right) x \right)$$
$$\equiv \bar{M}^{rate} \left(\bar{M}_f^{aim}(f) + \bar{M}_D^{aim} D - x \right).$$ (A.30)

Plugging (A.30) in (A.29) and then proceeding as in part (i), we obtain

$$\left[A_x(f)^\top \quad A_D(f)^\top \right]^\top = \mathrm{E} \left[\int_0^\infty e^{-N_1 t} \left[B^\top \quad 0 \right]^\top f_t \, dt \middle| f_0 = f \right]$$
$$= \int_0^\infty e^{-N_1 t} \left[\gamma \Sigma \quad 0 \right]^\top \mathrm{E} \left[M_t | f_0 = f \right] dt,$$ (A.31)

where

$$N_1 = \rho + \begin{bmatrix} A_{xx} - A_{xD} C - C \\ -A_{xD}^\top - A_{DD} C \end{bmatrix} \Lambda^{-1} \begin{bmatrix} I & C^\top \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R^\top \end{bmatrix}.$$ (A.32)

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Equation (12) follows immediately with

$$N_2 = \left(A_{xx} + C^\top + C^\top A_{Dx}\right)^{-1} \begin{bmatrix} I & C^\top \end{bmatrix} \tag{A.33}$$

$$N_3 = \begin{bmatrix} \gamma\Sigma & 0 \end{bmatrix}^\top. \tag{A.34}$$

To write explicitly the Riccati equations for the constant value-function coefficients and the ODEs for the ones that depend (non-linearly) on f, we match the coefficients in the HJB equation evaluated at the optimal τ.

The constant coefficient matrices solve the system

$$
\begin{aligned}
-\rho A_{xx} &= -\gamma\Sigma + Q_x^\top \Lambda^{-1} Q_x \\
\rho A_{xD} &= Q_x^\top \Lambda^{-1} Q_D - A_{xD} R \\
\rho A_{DD} &= Q_D^\top \Lambda^{-1} Q_D - A_{DD} R - R^\top A_{DD},
\end{aligned}
\tag{A.35}
$$

while the ODE system is

$$
\begin{aligned}
\rho A_x(f) &= Bf - Q_x^\top \Lambda^{-1} Q_f + \mathcal{D} A_x(f) \\
\rho A_D(f) &= Q_D^\top \Lambda^{-1} Q_f - R^\top A_{Df} + \mathcal{D} A_D(f) \\
\rho A(f) &= \frac{1}{2} Q_f^\top \Lambda^{-1} Q_f + \mathcal{D} A(f).
\end{aligned}
\tag{A.36}
$$

Here we used the notation

$$
\begin{aligned}
Q_x &= -A_{xx} + C^\top A_{xD}^\top + C^\top \\
Q_D &= A_{xD} + C^\top A_{DD} \\
Q_f &= A_x(f) + C A_D(f).
\end{aligned}
\tag{A.37}
$$

We note that the equations above have to be solved simultaneously for A_{xx}, A_{xD}, and A_{DD}; there is no closed-form solution in general. The complication is due to the fact that current trading affects the persistent price component D (that is, $C \neq 0$). Furthermore, the ODEs for A_x and A_D are coupled, but the solution can be written reasonably simply

as (A.31) above. ∎

Proof of Proposition 3.

(iii) Let's start with the complete problem:

$$
V(x_t, D_t, f_t) = E_t \int_t^\infty e^{-\rho(s-t)} \left(x_s^\top (Bf_s - (r+R)D_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \tag{A.38}
$$
$$
+ E_t \int_t^\infty e^{-\rho(s-t)} x_{s-}^\top C dx_s + \frac{1}{2} E_t \int_t^\infty e^{-\rho(s-t)} d[x_s, Cx_s].
$$

As is customary with such problems, we write and solve the HJB equation, then use the fact that it is satisfied to provide a so-called verification argument for the proposed optimal control and value function. We also use the conjecture (22) and introduce the notation $\hat{V}(D, f) = V(0, D, f)$, so that

$$
V(x, D, f) = \hat{V}(D - Cx, f) - \frac{1}{2} x^\top Cx. \tag{A.39}
$$

Note that

$$
d(D_s - Cx_s) = -RD_s ds, \tag{A.40}
$$

so that $D^0 \equiv D - Cx$ is a continuous and finite-variation process.

The HJB equation is

$$
0 = \sup_{\Delta x, \mu, \sigma} \left\{ x^\top \left(Bf - (r^f + R)D \right) - \frac{\gamma}{2} x^\top \Sigma x + x_-^\top C \frac{1}{dt} E_t [dx_t] + \frac{1}{2} \frac{1}{dt} E_t d[x_t, Cx_t] \right.
$$
$$
- \rho \hat{V} + \frac{1}{2} \rho x^\top Cx + \hat{V}_D(-RD) + \hat{V}_f \mu_f \tag{A.41}
$$
$$
- x_-^\top C \frac{1}{dt} E_t [dx_t] - \frac{1}{2} \frac{1}{dt} E_t d[x_t, Cx_t] + \frac{1}{dt} E_t d[f_t, V_{ff} f_t]
$$
$$
\left. + \frac{1}{\Delta t} E_t \left[x_-^\top C \Delta x + \frac{1}{2} \Delta x^\top C \Delta x + \hat{V}(D^0, f_- + \Delta f) - \hat{V}(D^0, f_-) \right] \right\}.
$$

Here, we suppressed the notational dependence on time and also wrote x_- for x_{t-} and similarly D_- and f_-.

We conjecture a quadratic form for the value function $\hat{V}$:

$$\hat{V} = \frac{1}{2}D^{0\top}A_{DD}D^0 + D^{0\top}A_D(f) + A(f), \tag{A.42}$$

which leads to the simplification

$$0 = \sup_x \left\{ -\rho\hat{V} + \frac{\rho}{2}x^\top Cx + x^\top Bf - x^\top(r^f + R)(D^0 + Cx) - \frac{\gamma}{2}x^\top\Sigma x \right.$$
$$\left. - \hat{V}_D R(D^0 + Cx) + D^{0\top}\mathcal{D}A_D(f) + \mathcal{D}A(f) \right\}. \tag{A.43}$$

We remark on the fact that (A.43) has the standard continuous-time form. The first two terms in (A.43) equal the value function decay rate $-\rho V(\hat{x}, \hat{D}, f)$, while the remaining terms represent the flow benefit from taking position x for the next infinitesimal time period: the expected excess return, the distortion decay summed with the opportunity cost of funds (the risk-free rate), from which the position x will suffer over dt, the risk cost, and the change over time in V induced by the decay of D and of f, as well as the convexity and jump adjustments for f. Note that, in order for the problem to be well defined, it is necessary that $\rho C < 2(r^f + R)C + \gamma\Sigma$ — otherwise, the agent gains too much from pushing the prices up currently relative to the perceived cost of the risk and the decay in the distortion.

In order to write down the solution, let

$$J = \frac{1}{2}(J_0 + J_0^\top) \tag{A.44}$$
$$J_0 = \gamma\Sigma + (2R + 2r^f - \rho)C \tag{A.45}$$
$$j = Bf - C^\top R^\top A_D(f) - \left(C^\top R^\top A_{DD} + r^f + R\right)D^0. \tag{A.46}$$

It follows that

$$x = J^{-1}\left(Bf - (r^f + R)D^0 - C^\top R^\top \hat{V}_D^\top\right) \tag{A.47}$$
$$= J^{-1}j \tag{A.48}$$

and the HJB equation becomes

$$\rho \hat{V} \;=\; \frac{1}{2} j^\top J^{-1} j - \left(D^{0\top} A_{DD} + A_D(f)^\top \right) R D^0 + D^{0\top} \mathcal{D} A_D(f) + \mathcal{D} A(f). \qquad \text{(A.49)}$$

The constant matrix A_{DD} is computed in the usual way:

$$\rho A_{DD} = \left(A_{DD} R C + r^f + R^\top \right) J^{-1} \left(C^\top R^\top A_{DD} + r^f + R \right) - A_{DD} R - R^\top A_{DD}. \qquad \text{(A.50)}$$

The coefficient function $A_D(f)$ satisfies the (integro-)differential equation

$$\rho A_D(f) = \left(A_{DD} R C + r^f + R^\top \right) J^{-1} \left(C^\top R^\top A_D(f) - Bf \right) - R^\top A_D(f) - \mathcal{D} A_D(f), \qquad \text{(A.51)}$$

and has the representation

$$A_D(f) = - \int_0^\infty \hat{N}_2 e^{-\hat{N}_1 t} B \, \mathrm{E}\left[f_t | f_0 = f \right] \qquad \text{(A.52)}$$

$$= - \int_0^\infty \hat{N}_2 e^{-\hat{N}_1 t} \hat{N}_3 \, \mathrm{E}\left[M_t | f_0 = f \right], \qquad \text{(A.53)}$$

with

$$\hat{N}_1 = \rho + R^\top - \left(A_{DD} R C + r^f + R^\top \right) J^{-1} \qquad \text{(A.54)}$$

$$\hat{N}_2 = \left(A_{DD} R C + r^f + R^\top \right) J^{-1} C^\top R^\top. \qquad \text{(A.55)}$$

To prove that the proposed solution does, indeed, solve the trader's problem, we follow a verification argument. Let $\hat{x}$ be an arbitrary trading strategy (satisfying technical transversality conditions) and V quadratic, defined by (22) and the coefficients A. Since it holds

generally that

$$e^{-\rho t}V(\hat{x}_t, \hat{D}_t, f_t) = e^{-\rho t}\hat{V}(\hat{D}_t - C\hat{x}_t, f_t) - \frac{1}{2}e^{-\rho t}\hat{x}_t^\top C\hat{x}_t$$

$$= e^{-\rho T}\hat{V}(\hat{D}_T - C\hat{x}_T, f_T) - \frac{1}{2}e^{-\rho T}\hat{x}_T^\top C\hat{x}_T$$

$$- \int_t^T d\left(e^{-\rho s}\hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2}e^{-\rho s}\hat{x}_s^\top C\hat{x}_s\right),$$

it would be sufficient that $\lim_{T\to\infty} e^{-\rho T}E_t\left[\hat{V}(\hat{D}_T - C\hat{x}_T, f_T) - \frac{1}{2}\hat{x}_T^\top C\hat{x}_T\right] = 0,$

$$E_t\left[-\int_t^T d\left(e^{-\rho s}\hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2}e^{-\rho s}\hat{x}_s^\top C\hat{x}_s\right)\right]$$

$$\geq E_t\int_t^T e^{-\rho s}\left(\hat{x}_s^\top\left(Bf_s - (r + R)\hat{D}_s\right) - \frac{\gamma}{2}\hat{x}_s^\top\Sigma\hat{x}_s\right)ds \tag{A.56}$$

$$+ E_t\int_t^T e^{-\rho s}\hat{x}_{s-}^\top Cd\hat{x}_s + \frac{1}{2}E_t\int_t^T e^{-\rho s}d[\hat{x}_s, C\hat{x}_s],$$

and that the inequality holds with equality at the conjectured optimum control.

Ito's lemma implies

$$d\left(\hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2}\hat{x}_s^\top C\hat{x}_s\right)$$

$$= \hat{V}_D(d\hat{D}_s - Cd\hat{x}_s) + \hat{V}_f(df_s - \Delta f_s) - \hat{x}_s^\top Cd\hat{x}_s - \frac{1}{2}d[\hat{x}_s, C\hat{x}_s] + \tag{A.57}$$

$$\frac{1}{2}d[f_s, \hat{V}_{ff}f_s] - \frac{1}{2}\Delta f_s^\top\hat{V}_{ff}\Delta f_s + \hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \hat{V}(\hat{D}_s - C\hat{x}_s, f_{s-}).$$

Taking conditional expectations of (A.57), one gets

$$-\hat{V}_D R\hat{D}_s - \hat{x}_s^\top C E_s \frac{1}{ds}[d\hat{x}_s] - \frac{1}{2}\frac{1}{ds}d[\hat{x}_s, C\hat{x}_s] + (\hat{D}_s - C\hat{x}_s)^\top \mathcal{D}A_D(f_s) + \mathcal{D}A(f_s),$$

the negative of which we wish to be larger than

$$\frac{\rho}{2}\hat{x}_s^\top C\hat{x}_s - \rho\hat{V} + \hat{x}_s^\top\left(Bf_s - (r + R)\hat{D}_s\right) - \frac{\gamma}{2}\hat{x}_s^\top\Sigma\hat{x}_s + \hat{x}_{s-}^\top C\frac{1}{ds}E_s[d\hat{x}_s] + \frac{1}{2}\frac{1}{ds}E_sd[\hat{x}_s, C\hat{x}_s].$$

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This outcome is ensured by the HJB equation (A.43), which is satisfied by $\hat{V}$ and $\hat{x}$, for any value of $\hat{D}^0$. Furthermore, x satisfies the constraint with equality. ∎

Proof of Proposition 4. The proof is the same as in Gârleanu and Pedersen (2013), and it follows the standard procedure of assuming the function form (in this case, quadratic) of the value function, computing the optimal control conditional on this function, and then calculating the coefficients that yield the value function as a fixed point. For completeness, we record the resulting value-function coefficients. With $\bar{\rho} = 1 - \rho\Delta t$ and $\bar{\Lambda} = \bar{\rho}^{-1}\Lambda(\Delta t)$,

$$A_{xx} = \left(\bar{\rho}\gamma\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{\frac{1}{2}} + \frac{1}{4}\left(\rho^2\bar{\Lambda}^2 + 2\rho\gamma\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{\frac{1}{2}} + \gamma^2\bar{\Lambda}^{\frac{1}{2}}\Sigma\bar{\Lambda}^{-1}\Sigma\bar{\Lambda}^{\frac{1}{2}} \right) \right)^{\frac{1}{2}} \Delta t \tag{A.58}$$
$$- \frac{1}{2}(\rho\bar{\Lambda} + \gamma\Sigma)\Delta t$$
$$\text{vec}(A_{xf}) = \bar{\rho}\left(I - \bar{\rho}(I - \Phi)^{\top} \otimes (I - A_{xx}\Lambda(\Delta t)^{-1}) \right)^{-1} \text{vec}((I - A_{xx}\Lambda(\Delta t)^{-1})B). \tag{A.59}$$

∎

Proof of Proposition 5. The situation is the same as for Proposition 4. Here, however, we would like to record the equations defining the value-function coefficients, to use in the proof of convergence to the continuous-time solution.

With the additional definitions[14]

$$\Pi = \begin{bmatrix} \Phi & 0 \\ 0 & R \end{bmatrix}\Delta t,$$
$$\tilde{C} = (1 - R\Delta t)\begin{bmatrix} 0 \\ C \end{bmatrix}, \tag{A.60}$$
$$\tilde{B} = \begin{bmatrix} B & -(R + r^f) \end{bmatrix}\Delta t,$$

---

[14]We omit notational dependence on Δt for simplicity.

the unknown matrices have to satisfy the system of equations

$$-\bar{\rho}^{-1}A_{xx} \;=\; S_x^\top J^{-1}S_x - \bar{\Lambda} - \bar{\rho}^{-1}C + \tilde{C}^\top A_{yy}\tilde{C} \tag{A.61}$$

$$\bar{\rho}^{-1}A_{xy} \;=\; S_x^\top J^{-1}S_y - \tilde{C}^\top A_{yy}(I - \Pi) \tag{A.62}$$

$$\bar{\rho}^{-1}A_{yy} \;=\; S_y^\top J^{-1}S_y + (I - \Pi)^\top A_{yy}(I - \Pi), \tag{A.63}$$

where the matrices J, S_x, and S_y are explicit functions of the unknown coefficients:

$$J \;=\; \gamma\Sigma\Delta t + \bar{\Lambda} + \left(2(R + r^f)\Delta t - \bar{\rho}^{-1}\right)C + A_{xx} - 2A_{xy}\tilde{C} - \tilde{C}^\top A_{yy}\tilde{C} \tag{A.64}$$

$$S_x \;=\; \bar{\Lambda} + (R + r^f)\Delta t C - A_{xy}\tilde{C} - \tilde{C}^\top A_{yy}\tilde{C} \tag{A.65}$$

$$S_y \;=\; \tilde{B} + A_{xy}(I - \Pi) + \tilde{C}^\top A_{yy}(I - \Pi). \tag{A.66}$$

The optimal position x_t can be written as

$$x_t \;=\; x_{t-1} + \left(I - J^{-1}S_x\right)\left(\left(I - J^{-1}S_x\right)^{-1}\left(J^{-1}S_y\right)y_t\right), \tag{A.67}$$

so that

$$M^{rate} = I - J^{-1}S_x \tag{A.68}$$

$$M^{aim} = \left(I - J^{-1}S_x\right)^{-1}\left(J^{-1}S_y\right). \tag{A.69}$$

■

Proof of Proposition 6. In the text. ■

Proof of Proposition 7. In the text. ■

Proof of Proposition 8. Consider evaluating the objective (34) at $x_t^{(\Delta t)}$ and letting Δt go to zero. The only non-standard terms in the objective are the ones involving transaction

costs:

$$E_0 \left[\sum_t -(1-\rho\Delta t)^{t+1} x_t^\top \left(R + r^f \right) (D_t + C\Delta x_t) \Delta t \right.$$

$$\left. + (1-\rho\Delta t)^t \left(-\frac{1}{2}\Delta x_t^\top \Lambda(\Delta t)\Delta x_t + x_{t-1}^\top C\Delta x_t + \frac{1}{2}\Delta x_t^\top C\Delta x_t \right) \right]. \qquad \text{(A.70)}$$

Under regularity conditions, the sum of the terms not involving Λ tends to

$$E_0 \int_t^\infty -e^{-\rho t} \left(x_t^\top (r^f + R)D_t \right) dt \qquad \text{(A.71)}$$

$$+ E_0 \int_0^\infty e^{-\rho t} x_{t-}^\top C dx_t + \frac{1}{2} E_0 \int_0^\infty e^{-\rho t} d\,[x, Cx]_t \,.$$

If we write $\Lambda(\Delta t) = \Lambda s(\Delta t)$ for some scalar function s and let $\tau_t = \frac{1}{dt} E_t[dx_t]$, then we can express the remaining limit term as

$$-\frac{1}{2} E_0 \int_0^\infty e^{-\rho t} \tau_t^\top \Lambda \tau_t dt \times \lim_{\Delta t \to 0} (s(\Delta t)\Delta t) - \frac{1}{2} E_0 \int_0^\infty e^{-\rho t} d[x, \Lambda x]_t \times \lim_{\Delta t \to 0} s(\Delta t). \qquad \text{(A.72)}$$

We note that, under the assumption of part (i), i.e., $s(\Delta t) = \Delta t^{-1}$, the first term is non-zero while the second is infinite if x has non-zero quadratic variation. Under the assumption of part (ii), i.e., $s(\Delta t) = \Delta t$, both terms are zero, so that the objective coincides with (21).

The only element that may need proving, therefore, is that the optimal trade in the discrete-time model has a well-defined continuous-time limit; for if the limit exists, it has to be optimal, or else its discretely sampled counterpart can be improved upon, at least for Δt small enough. We know that the discrete-time optimal trade is given as a quadratic function of the exogenous process $\{f_t\}_t$, so the only claim to prove is that the sequence of matrix tuples (A_{xx}, A_{xy}, A_{yy}), which generate the coefficients of this function, has a (finite) limit.

We can achieve this goal through direct manipulation of the Riccati equations defining the discrete-time coefficients and taking the limit. The precise details depend on the case of the proposition and the coefficient, but the general idea is the same. We illustrate for the matrix A_{xx} under the assumption $\Lambda(\Delta t) = \Delta t^{-1}\Lambda$.

We work with the characterization of solutions provided in the proof of Proposition 5. We show that, as $\Delta t \to 0$, Equation (A.61) tends to its counterpart in (A.35), which implies that the solutions also do.

We first rewrite this equation as

$$
\begin{aligned}
-A_{xx} &= \bar{\rho} S_x^\top J^{-1} S_x - \Lambda(\Delta t) - C + \bar{\rho}\tilde{C}^\top A_{yy}\tilde{C} && \text{(A.73)} \\
&= \bar{\rho}(S_x - J)^\top J^{-1}(S_x - J) - \Lambda(\Delta t) - \bar{\rho}J + 2\bar{\rho}S_x + \bar{\rho}\tilde{C}^\top A_{yy}\tilde{C}, && \text{(A.74)}
\end{aligned}
$$

and then rearrange it, using (A.64) and (A.65), as

$$
-A_{xx}\left(1 - \bar{\rho}\right) = \bar{\rho}(S_x - J)^\top J^{-1}(S_x - J) - \bar{\rho}\gamma\Sigma\Delta t. \tag{A.75}
$$

Dividing through by Δt and ignoring terms in Δt in $S_x - J$ and $J\Delta t$ — note that $\bar{\Lambda}(\Delta t)\Delta t \to \Lambda$ as $\Delta t \to 0$ — we obtain

$$
-\rho A_{xx} = -\gamma\Sigma + \left(A_{xx} - A_{xy}\tilde{C} - C\right)\Lambda^{-1}\left(A_{xx} - \tilde{C}^\top A_{xy}^\top - C^\top\right), \tag{A.76}
$$

the same as in continuous time.

Once the value-function coefficients in discrete time are established to have as limit their counterparts in continuous time, one proceeds by noting that, when letting Δt go to 0, the rate term M^{rate} is given by

$$
\Lambda^{-1}\left(A_{xx} - C - \tilde{C}A_{xy}^\top\right), \tag{A.77}
$$

while the aim term M^{aim} by

$$
\left(A_{xx} - C^\top A_{xy}^\top - C^\top\right)^{-1}\left(A_{xy} + \tilde{C}^\top A_{yy}\right). \tag{A.78}
$$

These expressions are the same as obtained in continuous time. ∎

Proof of Proposition 9. We start from the HJB equation, which reads

$$0 = \sup_{\tau_s} \left\{ x_s^\top B f_s - \frac{\gamma}{2} x_s^\top \bar{\Sigma} x_s v_s - \frac{\lambda}{2} \tau_s^\top \bar{\Sigma} \tau_s v_s - \rho V + V_x \tau_s - V_f \Phi f + \frac{1}{2} tr \left(V_{ff} \Omega \right) \right.$$
$$\left. + V_v \mu_v + \frac{1}{2} V_{vv} \sigma_v^2 + V_{vf} \frac{d}{ds} [f, v]_s \right\}, \tag{A.79}$$

with

$$V_v = -\frac{1}{2} x^\top A'_{xx} x + x^\top A'_{xf} f + \frac{1}{2} f^\top A'_{ff} f + A'_0 \tag{A.80}$$

$$V_{vv} = -\frac{1}{2} x^\top A''_{xx} x + x^\top A''_{xf} f + \frac{1}{2} f^\top A''_{ff} f + A''_0. \tag{A.81}$$

Similarly to the special case of Proposition 1, we conjecture and verify that $A_{xx}(v) = a(v)\bar{\Sigma}$. Under the assumption, collecting the terms in (A.79) that are quadratic in x gives rise to the ODE

$$0 = \lambda^{-1} \frac{a^2}{v^2} + \rho \frac{a}{v} - \gamma - \frac{a'}{v} \mu_v - \frac{1}{2} \frac{a''}{v} {\sigma_v}^2. \tag{A.82}$$

The first observation we make is that a is increasing, since the value function is unambiguously decreasing in v. Let a_0 be the constant solving $\lambda^{-1} a^2 + \rho a = \gamma$ (i.e., given by (A.18)) and v_z the point where $\mu_v(v_z) = 0$.

Suppose now that, for some value v, $a(v) = a_0 v$. If $\mu_v(v) < 0$, then (A.82) implies that $a''(v) > 0$. If, furthermore, $a'(v) \geq a_0$, then $a(v') > a_0 v'$ for all $v' > v$. Thus, if a crosses $a_0 v$ from below once $\mu_v < 0$, then it remains above $a_0(v)$ for all v. On the other hand, for v sufficiently high, $a(v) < a_0 v$. This statement holds because $a(v)$ is bounded above by the utility generated by any suboptimal policy. In particular, consider a policy with

$$M^{rate}(v) = \lambda^{-1} a_0, \tag{A.83}$$

and let $\hat{a}$ be the resulting value-function coefficient. Then $\hat{a}$ solves

$$0 = -\frac{a_0^2}{\lambda} + \rho\frac{\hat{a}}{v} - \gamma + 2\frac{a_0}{\lambda}\frac{\hat{a}}{v} - \frac{\hat{a}'}{v}\mu_v - \frac{1}{2}\frac{\hat{a}''}{v}{\sigma_v}^2, \tag{A.84}$$

or

$$\hat{a}(v) = \int_0^\infty e^{-(\rho+2\lambda^{-1}a_0)t}\left(\lambda^{-1}a_0^2 + \gamma\right)\mathrm{E}_0[v_t|v_0 = v]\,dt. \tag{A.85}$$

It follows that, for $v > v_z$,

$$\hat{a}(v) > \frac{\lambda^{-1}a_0^2 + \gamma}{\rho + 2\lambda^{-1}a_0}v \tag{A.86}$$

$$= a_0v. \tag{A.87}$$

In conclusion, for $v > v_z$, a cannot cross the line a_0v from below.

Consider now a point $v < v_z$ where a crosses a_0v from above. Here it must be the case that $a''(v) < 0$, so that $a'(v) < a_0$ for all $v' \in (v, v_z)$. Thus a cannot cross a_0v on $(0, v_z)$ again after crossing downwards the first time. Finally, it is obvious that $a(0) > 0$. (If v is bounded below away from zero a.s., then, as long as σ_v is zero at the lower bound $\underline{v}$, (A.82) ensures that $a(\underline{v}) > a_0\underline{v}$.)

In conclusion, $a(v)$ starts above a_0v and ends below it, and can never cross it upwards. The unique crossing point is the desired $\hat{v}$. The conclusion of the proposition follows immediately from the fact that the trading rate with constant v is $\lambda^{-1}a_0$. ∎

Proof of Proposition 10. Suppose that $E_t[dp_t - r^f p_t\,dt] = Bf_t\,dt$ with B given by (62) and apply Proposition 1 to conclude that, if $x_t = -f_t^{K+1}$, then $dx_t = -df_t^{K+1}$. The comparative-static results are immediate. ∎

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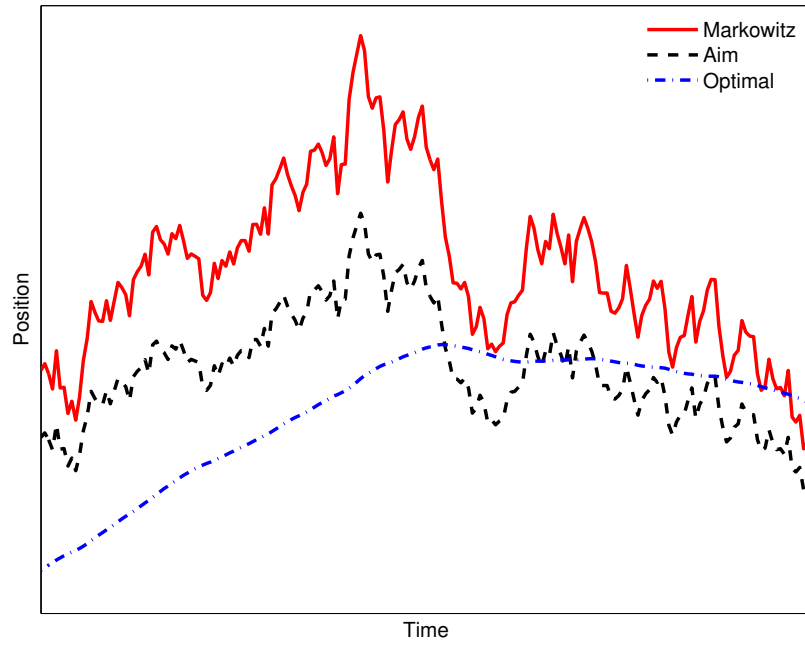
Figure 1: **Optimal portfolio with one asset and temporary impact costs.**
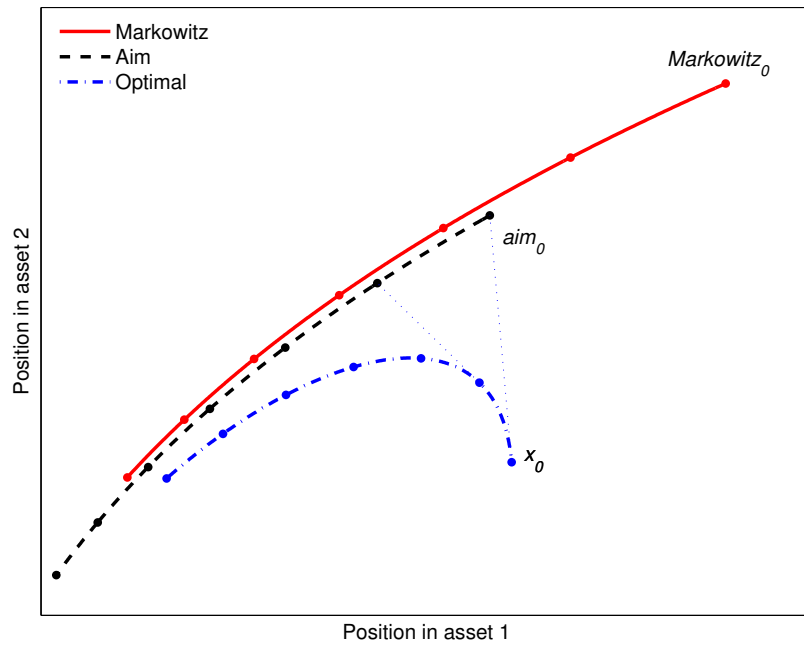
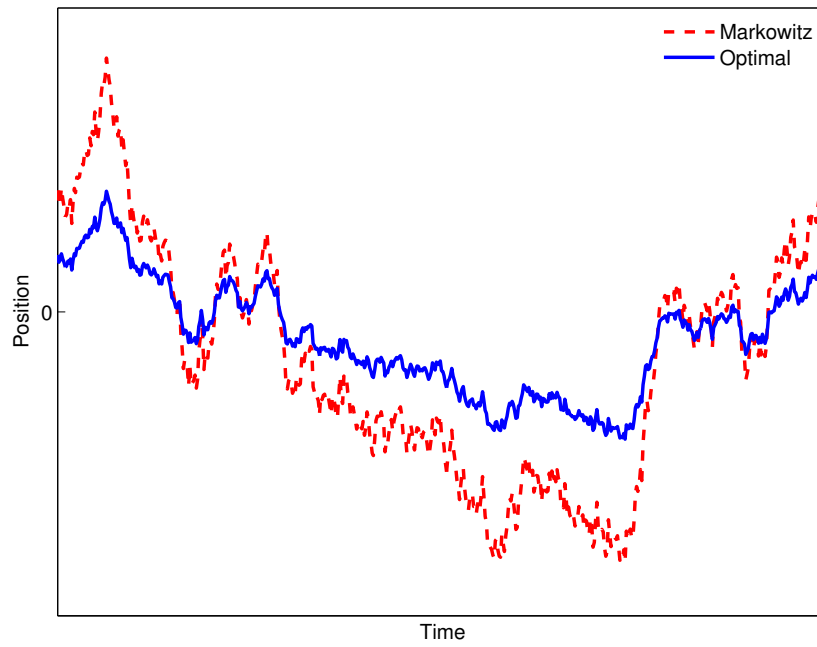Figure 2: **Expected optimal portfolio with two assets.**

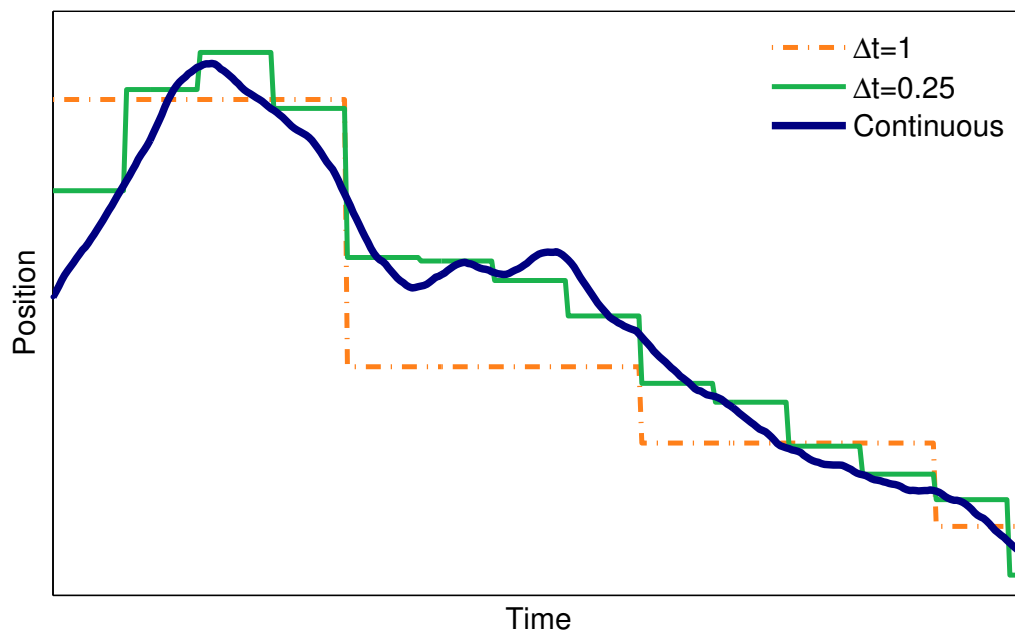Figure 3: **Optimal portfolio with one asset and purely persistent.**

Figure 4: **Convergence of discrete time to continuous time.**