Let f be a continuous function on the interval $[a, b] \subset \mathbb{R}$.

Pick an arbitrary $\varepsilon > 0$.

Define $g : [-\pi, \pi] \to \mathbb{R}$ by

$$g(x) = f\left(a + \frac{b-a}{\pi}|x|\right).$$

Observe that this maps $g(0) = f(a)$, $g(\pi) = g(-\pi) = b$. g is then a continuous function on the circle.

By corollary 5.4 of asdf, there exists a trigonometric polynomial Q such that

$$|g(x) - Q(x)| < \frac{\epsilon}{2} \quad \text{for all } -\pi \leq x \leq \pi.$$

Let n integer and $c_{-n}, \ldots, c_n \in \mathbb{C}$ be such that

$$Q(x) = c_0 + \sum_{0<|k|\leq n} c_k e^{ikx}.$$

Let $c = \max\{|c_{-n}|, \ldots, |c_{-1}|, |c_1|, \ldots, |c_n|\}$.

Let $R(x)$ be a polynomial such that

$$\left|e^{ix} - R(x)\right| < \frac{\epsilon}{4nc} \quad \text{for all } -n\pi \leq x \leq n\pi.$$

Such R exists because e^{ix} can be approximated by polynomials uniformly on any interval.

Construct

$$Q'(x) = c_0 + \sum_{0<|k|\leq n} c_k R(kx)$$

and observe that it is a polynomial. Then

$$\begin{aligned}
|q(x) - Q'(x)| &\leq |q(x) - Q(x)| + |Q(x) - Q'(x)| \\
&< \frac{\epsilon}{2} + \left|\sum_{0<|k|\leq n} c_k \left(e^{ikx} - R(kx)\right)\right| \\
&\leq \frac{\epsilon}{2} + \sum_{0<|k|\leq n} |c_k| \left|e^{ikx} - R(kx)\right| \\
&< \frac{\epsilon}{2} + \sum_{0<|k|\leq n} |c_k| \frac{\epsilon}{4nc} \\
&\leq \frac{\epsilon}{2} + \sum_{0<|k|\leq n} \frac{\epsilon}{4n} \\
&= \epsilon,
\end{aligned}$$

where we use $-n\pi \leq kx \leq n\pi$ for all $x \in [-\pi, \pi]$ and $-n \leq k \leq n$

Let

$$P(x) = Q'\left(\frac{x-a}{b-a}\pi\right).$$

Then P is a polynomial and

$$|f(x) - P(x)| = |q(x) - Q'(x)| < \epsilon.$$

$\square$