Proof. To prove completeness of $\mathbb{C}^d$, consider $\mathbb{R}^{2d}$ and identify every

$$(x_1 + iy_1, x_2 + iy_2, \ldots, x_d + iy_d) \in \mathbb{C}^d$$

with

$$(x_1, y_1, x_2, y_2, \ldots, x_d, y_d) \in \mathbb{R}^{2d}.$$

This transformation is isomorphic under addition and preserves the norm, so it is an isometry.

Hence, it is sufficient to show the completeness of $\mathbb{R}^d$ for all d. Let $\{v_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be a Cauchy sequence. For all n, denote

$$v_n = (v_{n,1}, \ldots, v_{n,d}).$$

For every dimension $k = 1, \ldots, d$ consider the sequence $\{v_{n,k}\}_{n=1}^{\infty}$. It is Cauchy, since for all n, m, $|v_{n,k} - v_{m,k}|$ is bounded by $\|v_n - v_m\|$, so it converges to some $\ell_k \in \mathbb{R}$ by completeness of $\mathbb{R}$.

Let $\ell \in \mathbb{R}^d = (\ell_1, \ldots, \ell_d)$. We decompose the distance as

$$\|v_n - \ell\|^2 = \sum_{k=1}^{d} (v_{n,k} - \ell_k)^2$$

and observe that the RHS goes to 0 as $n \to \infty$ by convergence of $\{v_{n,k}\}_{n=1}^{\infty}$. Hence, v_n converges to ℓ as $n \to \infty$. $\square$