Define $\tau = 2\pi$.

Changing variables $u = x - \tau$, we have

$$\int_{a+\tau}^{b+\tau} f(x)dx = \int_a^b f(u+\tau)du = \int_a^b f(u)du, \tag{1}$$

where in the last equality we use the fact that f is τ-periodic.

Setting $a = a' - \tau, b = b' - \tau$ in (1), we obtain

$$\int_{a'}^{b'} f(x)dx = \int_{a'-\tau}^{b-\tau} f(x)dx.$$

Hence,

$$\int_a^b f(x)dx = \int_{a+\tau}^{b+\tau} f(x)dx = \int_{a-\tau}^{b-\tau} f(x)dx.$$

Similarly, we have

$$\begin{aligned}
\int_a^{\tau+a} f(x)dx &= \int_a^{\tau} f(x)dx + \int_{\tau}^{\tau+a} f(x)dx && \text{(F.T.C.)} \\
&= \int_a^{\tau} f(x)dx + \int_{\tau}^{\tau+a} f(x-\tau)dx && \text{(periodicity)} \\
&= \int_a^{\tau} f(x)dx + \int_0^{a} f(u)du && (u = x - \tau) \\
&= \int_0^{\tau} f(x)dx. && (2)
\end{aligned}$$

Also,

$$\int_0^{\tau} f(x+a)dx = \int_a^{\tau+a} f(u)du = \int_0^{\tau} f(x)dx$$

where the second inequality is by (2).

Offsetting a by $\tau/2$, we also obtain

$$\int_{-\tau/2}^{\tau/2} f(x+a)dx = \int_{-\tau/2}^{\tau/2} f(x)dx = \int_{-\tau/2+a}^{\tau/2+a} f(x)dx$$

as desired.