(a) We will prove this in three parts. For parts 1 and 2 we will assume that $s = 0$. In part 1, we will show that

$$\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{n=1}^{\infty} s_n r^n.$$

In part 2, we prove that

$$\lim_{r \to 1} (1-r) \sum_{n=1}^{\infty} s_n r^n = 0.$$

Finally, in part 3, we generalise this result to cases where $s \neq 0$.

1. Let s_n be the partial sum $s_n = \sum_{k=1}^{n} c_k$.
We have

$$\begin{aligned}
(1-r) \sum_{n=1}^{N} s_n r^n &= \sum_{n=1}^{N} s_n r^n - \sum_{n=1}^{N} s_n r^{n+1} \\
&= \sum_{n=1}^{N} s_n r^n - \sum_{n=2}^{N+1} s_{n-1} r^n \\
&= s_1 r + \sum_{n=2}^{N} (s_n - s_{n-1}) r^n - s_N r^{N+1} \\
&= c_1 r + \sum_{n=2}^{N} c_n r^n - s_N r^{N+1} \\
&= \sum_{n=1}^{N} c_n r^n - s_N r^{N+1}.
\end{aligned}$$

Hence

$$\sum_{n=1}^{N} c_n r^n = (1-r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}.$$

By assumption $s_N \to 0$ as $N \to \infty$. Also, $r^{N+1} \to 0$ as $N \to \infty$ because $r \in (0,1)$. Hence, $s_N r^{N+1}$ vanishes as $N \to \infty$.
For all $0 < r < 1$, L.H.S. converges by Dirichlet's test since s_n is bounded and r^n converges monotonically to 0. Then *R.H.S.* also converges and we get

$$\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{n=1}^{\infty} s_n r^n.$$

2. We show that $\lim_{r \to 0} \sum_{n=1}^{\infty} s_n r^n = 0$. We argue by the Moore-Osgood theorem that the limit can be interchanged with the sum.

Firstly we observe that for all N finite, $\lim_{r\to 1}(1-r)\sum_{n=1}^{N} s_n r^n$ exists.

Secondly, we need to show that $\sum_{n=1}^{\infty} s_n r^n$ converges uniformly for all $r \in (0,1)$. We have shown convergence in part 1. To prove that it is uniform, choose some $\epsilon > 0$. Let N be such that $|s_n| < \epsilon$ for all $n > N$. Then for all $r \in (0,1)$,

$$\begin{aligned}
\left|(1-r)\sum_{n=1}^{N} s_n r^n - (1-r)\sum_{n=1}^{\infty} s_n r^n\right| &= \left|(1-r)\sum_{n=N+1}^{\infty} s_n r^n\right| \\
&\leq (1-r)\epsilon \sum_{n=N+1}^{\infty} r^n \\
&= (1-r)\epsilon \frac{r^{N+1}}{1-r} \\
&< \epsilon,
\end{aligned}$$

so the sum converges uniformly.

Hence, we can interchange the limits by the Moore-Osgood theorem and

$$\begin{aligned}
\lim_{r\to 1}\sum_{n=1}^{\infty} c_n r^n &= \lim_{r\to 1}(1-r)\sum_{n=1}^{\infty} s_n r^n \\
&= \sum_{n=1}^{\infty} s_n \lim_{r\to 1}\left(r^n - r^{n+1}\right) \\
&= 0.
\end{aligned}$$

Therefore, $\sum c_n$ is Abel summable to 0.

3. We have shown that if a series converges to 0, then it is also Abel summable to 0. We now generalise this to series that converge to $s \neq 0$.

Let $\sum_{n=1}^{\infty} c_n = s$, then construct $\{c'_n\}_{n=1}^{\infty}$ by $c'_1 = -s$, $c'_2 = c_1$, $c'_3 = c_2$, and so on. In other words we are constructing a new series by prepending $-s$. Then

$$\sum_{n=1}^{\infty} c'_n = -s + \sum_{n=2}^{\infty} c'_n = -s + \sum_{n=1}^{\infty} c_n = 0.$$

Since $\sum c'_n$ converges to 0 it is also Abel summable to 0 and

$$0 = \lim_{r\to 1}\sum_{n=1}^{\infty} c'_n r^n = \lim_{r\to 1} c'_1 r + \lim_{r\to 1}\sum_{n=2}^{\infty} c'_n r^n = -s + \lim_{r\to 1}\sum_{n=1}^{\infty} c_n r^n,$$

so $\sum c_n$ is Abel summable to s. $\square$

(b) Consider the series $\sum_{n=0}^{\infty}(-1)^n$. This series clearly diverges because $\lim_{n\to\infty}(-1)^n$ does not converge to 0. However, this series is Abel summable. For all $r \in [0,1)$ define,

$$A(r) = \sum_{n=0}^{\infty}(-1)^n r^n = \sum_{n=0}^{\infty}(-r)^n = \frac{1}{1+r}$$

where for the last inequality we recognise this as a geometric series. Then $\lim_{r\to 1} A(r) = 1/2$.

(c) The proof is similar to (a). We first assume that $\sigma = 0$. In part 1 we show that

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n.$$

Part 2 we prove that

$$\lim_{r\to 1}(1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n = 0.$$

Finally, part 3 generalises this result to cases where $\sigma \neq 0$.

1. Let s_n be the partial sum $s_n = \sum_{k=1}^{n} c_k$. Let $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k$ be the mean of fit n partial sums.

Following the same argument as in (a), part 1, for $r \in (0,1)$ we have

$$\sum_{n=1}^{N} c^n r^n = (1-r)\sum_{n=1}^{N} s_n r^n + s_N r^{N+1},$$
$$\sum_{n=1}^{N} s^n r^n = (1-r)\sum_{n=1}^{N} n\sigma_n r^n + N\sigma_N r^{N+1}.$$

Then

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n + (1-r)\lim_{n\to\infty} s_n r^{n+1} + \lim_{n\to\infty} n\sigma_n r^{n+1}.$$

By assumption, $\sigma_n \to 0$ as $n \to \infty$. Also, $nr^{n+1} \to 0$ since $r \in (0,1)$. Hence,

$$\lim_{n\to\infty} n\sigma_n r^{n+1} = 0, \tag{1}$$

and the last term vanishes.

To show that the middle term vanishes as well, observe that $s_n = n\sigma_n - (n-1)\sigma_{n-1}$. Then

$$\lim_{n\to\infty} s_n r^{n+1} = \lim_{n\to\infty} n\sigma_n r^{n+1} - \lim_{n\to\infty}(n-1)\sigma_{n-1} r^{n+1} = 0,$$

where the last equality is by (1).

Hence,

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n.$$

Finally, we show that these series converge for all $r \in (0,1)$. For n big enough $|\sigma_n| < 1$, so $|n\sigma_n r^n| < |nr^n|$. Then R.H.S. converges because $\sum |nr^n|$ converges.

2. We show that $\lim_{r\to 1}(1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n = 0$.

From part 1, we know that $(1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$ converges for all $r \in (0,1)$. To show that this convergence is uniform in r, pick an $\epsilon > 0$. Let N be such that $|\sigma_n| < \epsilon$ for all $n > N$. Then for all $r \in (0,1)$,

$$\begin{aligned}
\left|(1-r)^2 \sum_{n=1}^{\infty} N\sigma_n r^n - (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n\right| &= \left|(1-r)^2 \sum_{n=N+1}^{\infty} n\sigma_n r^n\right| \\
&\leq (1-r)^2 \sum_{n=N+1}^{\infty} nr^n\, |\sigma_n| \\
&\leq (1-r)^2 \epsilon \sum_{n=N+1}^{\infty} nr^n \\
&\leq (1-r)^2 \epsilon \sum_{n=1}^{\infty} nr^n \\
&= (1-r)^2 \epsilon \frac{r}{(1-r)^2} \\
&\leq \epsilon.
\end{aligned}$$

Above, we use the fact that for $|r| < 1$,

$$\sum_{n=1}^{M} nr^{n-1} = \frac{d}{dr} \sum_{n=1}^{M} r^n = \frac{d}{dr} \frac{r - r^{M+1}}{1-r} = \frac{Mr^{M+1} - (M+1)r^M + 1}{(1-r)^2}.$$

Limiting $M \to \infty$, we obtain

$$\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}. \tag{2}$$

Then the sum converges uniformly and we can interchange the limits by the Moore-Osgood theorem. We have

$$\lim_{r\to 1} \sum_{n=1}^{\infty} c_n r^n = \lim_{r\to 1} (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \lim_{r\to 1} n\sigma_n r^n (1-r)^2 \\
&= 0
\end{aligned}
$$

as desired.

3. We have shown that if $\sum c_n$ is Cesàro summable to 0, then it is also Abel summable to 0. We now generalise this to series that are Cesàro summable to $\sigma \neq 0$.

If $\sum c_n$ is Cesàro summable to σ, then construct $\{c'_n\}_{n=1}^{\infty}$ by $c'_1 = c_1 - s$, $c'_2 = c_2$, $c'_3 = c_3$, and so on. In other words we are constructing a new series by subtracting s from the first term and leaving the others intact. Then the partial sums σ'_n satisfy $\sigma'_n = \sigma_n - \sigma$, so $\sigma'_n \to 0$ as $n \to \infty$. Since $\sum c'_n$ is Cesàro summable to 0, it is also Abel summable to 0 and

$$
0 = \lim_{r\to 1} \sum_{n=1}^{\infty} c'_n r^n = \lim_{r\to 1} c'_1 r + \lim_{r\to 1} \sum_{n=2}^{\infty} c'_n r^n = c_1 - s + \lim_{r\to 1} \sum_{n=2}^{\infty} c_n r^n,
$$

so $\sum c_n$ is Abel summable to s. $\square$

(d) Consider the sum

$$
\sum_{n=1}^{\infty} (-1)^{n-1} n.
$$

We show that it is Abel summable but not Cesàro summable.

To show Abel summability, we use (2) to find that

$$
\sum_{n=1}^{\infty} (-1)^{n-1} n r^n = \frac{r}{(1+r)^n}.
$$

Then

$$
\lim_{r\to 1} \sum_{n=1}^{\infty} (-1)^{n-1} n r^n = \lim_{r\to 1} \frac{r}{(1+r)^n} = \frac{1}{4},
$$

so $\sum_{n=1}^{\infty} (-1)^{n-1} n$ is Abel summable to $1/4$.

To show that it is not Cesàro summable, observe that the partial sums are

$$
\sigma_n = \begin{cases} (n+1)/2 & \text{for } n \text{ odd}, \\ -n/2 & \text{for } n \text{ even}. \end{cases}
$$

This is easily shown by induction. For n odd, we have $\sigma_n = \sigma_{n-1} + n = -(n-1)/2 + n = (n+1)/2$. For n even, we get $\sigma_n = \sigma_{n-1} - n = n/2 - n = -n/2$. $\sigma_1 = 1$ forms the base case. Then σ_n diverges and $\sum_{n=1}^{\infty} (-1)^{n-1} n$ is not Cesàro summable.