The Fourier coefficients of f are

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Observing that f is odd, this becomes

$$\hat{f}(n) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi i} \int_{0}^{\pi} f(x) \sin nx \, dx.$$

For $n = 0$, we get $\hat{f}(n) = 0$. Computing the integral for $n \neq 0$,

$$\begin{aligned} \hat{f}(n) &= \frac{1}{\pi i} \int_0^{\pi} \left(\frac{\pi}{2} - \frac{x}{2} \right) \sin nx \, dx \\ &= \frac{1}{\pi i} \frac{\pi n + \sin \pi n}{2n^2} \\ &= \frac{1}{2ni}. && (\sin \pi n = 0) \end{aligned}$$

Thus

$$f(x) \sim \frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n} = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{inx} - e^{-inx} \right).$$

We now show that this series converges for all x. For $x = 0$, the Fourier series converges to $0 = f(0)$ because $\hat{f}(n)$ is odd. For $x \neq 0$, we argue by Dirichlet's test. Let $a_n = 1/n$. We can see that a_n converges to 0 monotonically as $n \to \infty$. Let $b_n(x) = e^{inx} - e^{-inx}$ and $B_N(x) = \sum_{n=1}^{N} b_n(x)$. $B_N(x)$ is the N^{th} Dirichlet kernel and

$$B_N(x) = D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(x/2)}.$$

by section 1.1, example 4. The denominator $\sin(x/2)$ is constant and the numerator is bounded by $|\sin((N + 1/2)x)| \leq 1$, so $B_N(x)$ is bounded for a given x. Hence, by Dirichlet's test,

$$\sum_{n=1}^{\infty} a_n b_n = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{inx} - e^{-inx} \right) = \frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$$

converges. $\square$