

- (a) Choose an arbitrary  $\epsilon > 0$ . We will show that there exists  $0 \leq t < 1$  such that for all  $t < r < 1$ ,

$$\left| A_r(f)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| < C\epsilon,$$

where  $C > 0$  is a constant.

We follow the proof of theorem 4.1 in the book.

We can write

$$\begin{aligned} & A_r(f)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \\ &= (f * P_r)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(\theta - t) dt - \frac{f(\theta^+) + f(\theta^-)}{2} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(\theta - t) dt \\ &\quad - \frac{f(\theta^+)}{2\pi} \int_{-\pi}^0 P_r(t) dt - \frac{f(\theta^-)}{2\pi} \int_0^{\pi} P_r(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^0 P_r(t) [f(\theta - t) - f(\theta^+)] dt \\ &\quad + \frac{1}{2\pi} \int_0^{\pi} P_r(t) [f(\theta - t) - f(\theta^-)] dt. \end{aligned} \tag{1}$$

In eq. (1) we use the fact that  $\frac{1}{2\pi} \int_{-\pi}^0 P_r(t) dt = \frac{1}{2\pi} \int_0^{\pi} P_r(t) dt = \frac{1}{2}$ ; this is because  $P_r(t)$  is even in  $t$  and  $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) dt = 1$ .

We tackle the first term. Since  $f$  has a jump discontinuity at  $\theta$ , there exists  $\delta$ , so that  $-\delta < t < 0$  implies  $|f(\theta - t) - f(\theta^+)| < \epsilon$  and  $0 < t < \delta$  implies  $|f(\theta - t) - f(\theta^-)| < \epsilon$ . Then

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_{-\pi}^0 P_r(t) [f(\theta - t) - f(\theta^+)] dt \right| \\ & \leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} |P_r(t)| |f(\theta - t) - f(\theta^+)| dt \\ & \quad + \frac{1}{2\pi} \int_{-\delta}^0 |P_r(t)| |f(\theta - t) - f(\theta^+)| dt \\ & < \frac{B}{2\pi} \int_{-\pi}^{-\delta} |P_r(t)| dt + \frac{\epsilon}{2\pi} \int_{-\delta}^0 |P_r(t)| dt, \end{aligned}$$

where  $B > 0$  is a bound for  $f$ , which exists since  $f$  is integrable. Applying similar steps to the second term, we find

$$\left| \frac{1}{2\pi} \int_0^{\pi} P_r(t) [f(\theta - t) - f(\theta^-)] dt \right|$$

$$< \frac{\epsilon}{2\pi} \int_0^\delta |P_r(t)| dt + \frac{B}{2\pi} \int_\delta^\pi |P_r(t)| dt$$

Summing these,

$$\begin{aligned} & \left| (f * P_r)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| \\ & \leq \left| \frac{1}{2\pi} \int_{-\pi}^0 P_r(t) [f(\theta - t) - f(\theta^+)] dt \right| \\ & \quad + \left| \frac{1}{2\pi} \int_0^\pi P_r(t) [f(\theta - t) - f(\theta^-)] dt \right| \\ & < \frac{\epsilon}{2\pi} \int_{-\delta}^\delta |P_r(t)| dt + \frac{B}{2\pi} \int_{\delta \leq |t| \leq \pi} |P_r(t)| dt \\ & \leq \frac{\epsilon}{2\pi} \int_{-\pi}^\pi |P_r(t)| dt + \frac{B}{2\pi} \int_{\delta \leq |t| \leq \pi} |P_r(t)| dt. \end{aligned}$$

Since  $P_r$  is a good kernel, there exists  $M > 0$  such that  $\int_{-\pi}^\pi |P_r(\theta)| d\theta \leq M$  for all  $0 \leq r < 1$ . Hence, the first term is bounded by  $\epsilon M/2\pi$ . Also by the definition of a good kernel, we have that  $\int_{\delta \leq |\theta| \leq \pi} |P_r(\theta)| d\theta \rightarrow 0$  as  $r \rightarrow 1$ . Then we can find  $0 \leq t < \pi$  such that for all  $t < r < \pi$ ,  $\int_{\delta \leq |\theta| \leq \pi} |P_r(\theta)| d\theta < \epsilon$ .

Then for all  $r \in (t, \pi)$ ,

$$\left| (f * P_r)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| < \frac{\epsilon M}{2\pi} + \frac{\epsilon B}{2\pi} = \frac{M+B}{2\pi} \epsilon,$$

and  $\frac{M+B}{2\pi}$  is a constant.  $\square$

- (b) The  $N^{\text{th}}$  Cesàro mean of the Fourier series at  $\theta$  equals  $(f * F_N)(\theta)$ , where  $F_N$  is the Fejér kernel. It suffices to show that  $(f * F_N)(\theta) \rightarrow (f(\theta^+) + f(\theta^-))/2$  as  $N \rightarrow \infty$ . We follow the proof in (a).

Choose an arbitrary  $\epsilon > 0$ . We can write

$$\begin{aligned} & (f * F_N)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \\ & = \frac{1}{2\pi} \int_{-\pi}^0 F_N(t) [f(\theta - t) - f(\theta^+)] dt \\ & \quad + \frac{1}{2\pi} \int_0^\pi F_N(t) [f(\theta - t) - f(\theta^-)] dt, \end{aligned}$$

because  $\frac{1}{2\pi} \int_{-\pi}^0 F_N(t) dt = \frac{1}{2\pi} \int_0^\pi F_N(t) dt = \frac{1}{2}$  since  $F_N$  is an even good kernel.

Pick a  $\delta$  such that  $-\delta < t < 0$  implies  $|f(\theta - t) - f(\theta^+)| < \epsilon$  and  $0 < t < \delta$  implies  $|f(\theta - t) - f(\theta^-)| < \epsilon$ . Then

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_{-\pi}^0 F_N [f(\theta - t) - f(\theta^+)] dt \right| \\ & < \frac{B}{2\pi} \int_{-\pi}^{-\delta} |F_N(t)| dt + \frac{\epsilon}{2\pi} \int_{-\delta}^0 |F_N(t)| dt \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^{\pi} F_N [f(\theta - t) - f(\theta^+)] dt \right| \\ & < \frac{\epsilon}{2\pi} \int_0^{\delta} |F_N(t)| dt + \frac{B}{2\pi} \int_{\delta}^{\pi} |F_N(t)| dt, \end{aligned}$$

where  $B > 0$  is a bound for  $f$ .

Summing,

$$\begin{aligned} & \left| (f * F_N)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| \\ & \leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |F_N(t)| dt + \frac{B}{2\pi} \int_{\delta \leq |t| \leq \pi} |F_N(t)| dt. \end{aligned}$$

By the properties of good kernels,  $\int_{-\pi}^{\pi} |F_N(t)| dt$  is bounded by  $M > 0$ , and for all  $N$  big enough,  $\int_{\delta \leq |t| \leq \pi} |F_N(t)| dt < \epsilon$ .

Then for all  $N$  big enough,

$$\left| (f * F_N)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| < \frac{\epsilon M}{2\pi} + \frac{\epsilon B}{2\pi} = \frac{M + B}{2\pi} \epsilon,$$

where  $\frac{M+B}{2\pi}$  is a constant. □