The Poisson kernel can be expressed as the Fourier series

$$P_r(\theta) = \sum_{-\infty}^{n=\infty} r^{|n|} e^{in\theta}.$$

We define

$$u(r,\theta) = \frac{\partial P_r(\theta)}{\partial \theta}.$$

We can rearrange

$$P_r(\theta) = 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} = 1 + \frac{re^{i\theta}}{1 - re^{i\theta}} + \frac{re^{-i\theta}}{1 - re^{-i\theta}}.$$

Differentiating,

$$u(r,\theta) = \frac{\partial P_r(\theta)}{\partial \theta} = \frac{ire^{i\theta}}{\left(1 - re^{i\theta}\right)^2} - \frac{ire^{-i\theta}}{\left(1 - re^{-i\theta}\right)^2}.$$

(i) We want to show that $\Delta u = 0$ in the unit disc.

We know that

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

We first find $\partial u/\partial r$. Differentiating w.r.t. r,

$$\begin{split} \frac{\partial u}{\partial r} &= ie^{i\theta} \frac{(1 - re^{i\theta})^2 + 2re^{i\theta}(1 - re^{i\theta})}{(1 - re^{i\theta})^4} \\ &- ie^{-i\theta} \frac{(1 - re^{-i\theta})^2 + 2re^{-i\theta}(1 - re^{-i\theta})}{(1 - re^{-i\theta})^4} \\ &= ie^{i\theta} \frac{1 + re^{i\theta}}{(1 - re^{i\theta})^3} - ie^{-i\theta} \frac{1 + re^{-i\theta}}{(1 - re^{-i\theta})^3}. \end{split}$$

Differentiating again to get $\partial^2 u/\partial r^2$,

$$\begin{split} \frac{\partial^2 u}{\partial r^2} &= ie^{i\theta} \frac{(1-re^{i\theta})^3 e^{i\theta} + 3e^{i\theta} (1-re^{i\theta})^2 (1+re^{i\theta})}{(1-re^{i\theta})^6} \\ &- ie^{-i\theta} \frac{(1-re^{-i\theta})^3 e^{-i\theta} + 3e^{-i\theta} (1-re^{-i\theta})^2 (1+re^{-i\theta})}{(1-re^{-i\theta})^6} \\ &= 2ie^{2i\theta} \frac{2+re^{i\theta}}{(1-re^{i\theta})^4} - 2ie^{-2i\theta} \frac{2+re^{-i\theta}}{(1-re^{-i\theta})^4}. \end{split}$$

We now find $\partial^2 u/\partial \theta^2$. Differentiating twice,

$$\frac{\partial u}{\partial \theta} = ir \frac{ie^{i\theta}(1 - re^{i\theta})^2 + 2ire^{2i\theta}(1 - re^{i\theta})}{(1 - re^{i\theta})^4}$$

$$\begin{split} &-ir\frac{-ie^{-i\theta}(1-re^{-i\theta})^2-2e^{-i\theta}ire^{-i\theta}(1-re^{-i\theta})}{(1-re^{-i\theta})^4}\\ &=-r\frac{e^{i\theta}+re^{2i\theta}}{(1-re^{i\theta})^3}-r\frac{e^{-i\theta}+re^{-2i\theta}}{(1-re^{-i\theta})^3} \end{split}$$

and

$$\begin{split} \frac{\partial^2 u}{\partial \theta^2} \\ &= -ir \frac{(e^{i\theta} + 2re^{2i\theta})(1 - re^{i\theta}) + 3re^{i\theta}(e^{i\theta} + re^{2i\theta})}{(1 - re^{i\theta})^4} \\ &\quad + ir \frac{(e^{-i\theta} + 2re^{-2i\theta})(1 - re^{-i\theta}) + 3re^{-i\theta}(e^{-i\theta} + re^{-2i\theta})}{(1 - re^{-i\theta})^4} \\ &\quad = -ir \frac{e^{i\theta} + 4re^{2i\theta} + r^2e^{3i\theta}}{(1 - re^{i\theta})^4} + ir \frac{e^{-i\theta} + 4re^{-2i\theta} + r^2e^{-3i\theta}}{(1 - re^{-i\theta})^4}. \end{split}$$

Finally

$$\begin{split} \Delta u &= \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\ &= ie^{i\theta} \frac{1 + re^{i\theta}}{r(1 - re^{i\theta})^3} - ie^{-i\theta} \frac{1 + re^{-i\theta}}{r(1 - re^{-i\theta})^3} \\ &\quad + 2ie^{2i\theta} \frac{2 + re^{i\theta}}{(1 - re^{i\theta})^4} - 2ie^{-2i\theta} \frac{2 + re^{-i\theta}}{(1 - re^{-i\theta})^4} \\ &\quad - i \frac{e^{i\theta} + 4re^{2i\theta} + r^2e^{3i\theta}}{r(1 - re^{i\theta})^4} + i \frac{e^{-i\theta} + 4re^{-2i\theta} + r^2e^{-3i\theta}}{r(1 - re^{-i\theta})^4} \\ &= \frac{ie^{i\theta}}{r(1 - re^{i\theta})^4} \Big[(1 + re^{i\theta})(1 - re^{i\theta}) \\ &\quad + 2re^{i\theta}(2 + re^{i\theta}) - (1 + 4re^{i\theta} + r^2e^{2i\theta}) \Big] \\ &\quad - \frac{ie^{-i\theta}}{r(1 - re^{-i\theta})^4} \Big[(1 + re^{-i\theta})(1 - re^{-i\theta}) \\ &\quad + 2re^{-i\theta}(2 + re^{-i\theta}) - (1 + 4re^{-i\theta} + r^2e^{-2i\theta}) \Big] \\ &= \frac{ie^{i\theta}}{r(1 - re^{i\theta})^4} \Big[1 - r^2e^{2i\theta} + 4re^{i\theta} + 2r^2e^{2i\theta} - 1 - 4re^{i\theta} - r^2e^{2i\theta}) \Big] \\ &\quad - \frac{ie^{-i\theta}}{r(1 - re^{-i\theta})^4} \Big[1 - r^2e^{-2i\theta} + 4re^{i\theta} + 2r^2e^{-2i\theta} \\ &\quad - 1 - 4re^{-i\theta} - r^2e^{-2i\theta} \Big] \\ &= 0. \end{split}$$

(ii) Recall that

$$u(r,\theta) = \frac{ire^{i\theta}}{(1-re^{i\theta})^2} - \frac{ire^{-i\theta}}{(1-re^{-i\theta})^2}.$$

We argue by cases on θ .

 $(e^{i\theta} \neq 1)$ Taking the limit,

$$\lim_{r\to 1} u(r,\theta) = \frac{ie^{i\theta}}{(1-e^{i\theta})^2} - \frac{ie^{-i\theta}}{(1-e^{-i\theta})^2}.$$

Since $e^{i\theta} \neq 1$ by assumption, the denominators are non-zero and this limit is well-defined. Simplifying,

$$\lim_{r \to 1} u(r, \theta) = \frac{0}{(1 - e^{i\theta})^2 (1 - e^{-i\theta})^2} = 0.$$

 $(e^{i\theta} = 1)$ Since $e^{i\theta} = 1$ by assumption, u simplifies to

$$u(r,\theta) = \frac{ir}{(1-r)^2} - \frac{ir}{(1-r)^2} = 0.$$

for all $0 \le r < 1$. Hence $\lim_{r \to 1} u(r, \theta) = 0$.