(a) We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_{[a,b]}(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_a^b e^{-inx}\,dx.$$

For $n = 0$,

$$\hat{f}(0) = \frac{1}{2\pi}\int_a^b dx = \frac{b-a}{2\pi}.$$

For $n \neq 0$,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_a^b e^{-inx}\,dx \\ &= \left.\frac{ie^{-inx}}{2\pi n}\right|_a^b \\ &= \frac{e^{-ina} - e^{-inb}}{2\pi in}.\end{aligned}$$

Thus,

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}.$$

(b)

Lemma. *Define $S = \bigcup_{n=-\infty}^{\infty}[n + 1/4, n + 3/4]$. In other words, $[n + 1/4, n + 3/4]$ is the closed middle half of the interval between n and $n + 1$, and S is the union of these over all integer n. Let $c \in (0, 1)$. There exists integer $N \geq 1$ such that for every integer M, at least one of $\{cM, \ldots, c(M + N - 1)\}$ lies in S.*

Proof. Fix some c and M. By cases.

($c \leq 1/2$) Let $N = \lceil 1/c\rceil + 1$. Observe that it is independent of M. We show that at least one of $\{cM, \ldots, c(M + N - 1)\}$ lies in S.

If $cM \in S$, then we are done. Assume then that $cM \notin S$.

Let $k = \lceil cM - 1/4\rceil$. Let $r = \lceil (k + 1/4 - cM)/c\rceil$. We claim that $k + 1/4 \leq c(M + r) \leq k + 3/4$ and $0 \leq r < N$.

Trivially, $(k + 1/4)/c \leq \lceil (k + 1/4)/c\rceil$. Observe also that

$$\begin{aligned}\left\lceil \frac{k+1/4}{c}\right\rceil &< \frac{k+1/4}{c} + 1 \\ &= \frac{k+1/4+c}{c} \\ &\leq \frac{k+1/4+1/2}{c} \qquad (0 < c \leq 1/2)\end{aligned}$$

$$= \frac{k+3/4}{c}.$$

Then

$$\begin{aligned}
&\frac{k+1/4}{c} \le \left\lceil \frac{k+1/4}{c} \right\rceil \le \frac{k+3/4}{c} \\
&\implies k + \frac{1}{4} \le c\left\lceil \frac{k+1/4}{c} \right\rceil \le k + \frac{3}{4} \\
&\implies k + \frac{1}{4} \le c\left(M + \left\lceil \frac{k+1/4-cM}{c} \right\rceil\right) \le k + \frac{3}{4} \\
&\implies k + \frac{1}{4} \le c(M+r) \le k + \frac{3}{4}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&cM - \frac{1}{4} \le \left\lceil cM - \frac{1}{4} \right\rceil \\
&\implies 0 \le \left\lceil cM - \frac{1}{4} \right\rceil + \frac{1}{4} - cM \\
&\implies 0 \le \frac{\lceil cM - 1/4 \rceil + 1/4 - cM}{c} && (0 < c) \\
&\implies 0 \le \left\lceil \frac{\lceil cM - 1/4 \rceil + 1/4 - cM}{c} \right\rceil = r
\end{aligned}$$

and

$$\begin{aligned}
&\lceil cM - 1/4 \rceil + 1/4 - cM < 1 \\
&\implies \frac{\lceil cM - 1/4 \rceil + 1/4 - cM}{c} < \frac{1}{c} && (0 < c) \\
&\implies r = \left\lceil \frac{\lceil cM - 1/4 \rceil + 1/4 - cM}{c} \right\rceil < \left\lceil \frac{1}{c} \right\rceil + 1 = N,
\end{aligned}$$

so $0 \le r < N$ as desired.

($c \ge 1/2$) Define $c' = 1 - c$ and observe that $0 < c' < 1/2$.

By the previous case, there exists integer $N \ge 1$ such that for all integer M one of $\{c'M, \ldots, c'(M+N-1)\}$ lies in S. For a fixed M, let $M \le j < M + N$ integer be such that $c'j \in S$.

We have that $c'j \in [n+1/4, n+3/4] \subset S$ for some n, so $j - c'j \in [j-n-1+1/4, j-n-1+3/4] \subset S$.

□

Assume that $a \ne -\pi$, $b \ne \pi$ and $a \ne b$. We show that the Fourier series does not converge absolutely for any x.

Observe that

$$\begin{aligned}
\left|e^{-ina} - e^{-inb}\right| &= |(\cos na - \cos nb) - i(\sin na - \sin nb)| \\
&= \left|-2\sin\frac{na+nb}{2}\sin\frac{na-nb}{2} - 2i\sin\frac{na-nb}{2}\cos\frac{na+nb}{2}\right| \\
&= 2\left|\sin\frac{na-nb}{2}\right|\left|\sin\frac{na+nb}{2} + i\cos\frac{na+nb}{2}\right| \\
&= 2\left|\sin\frac{na-nb}{2}\right| \\
&= 2\left|\sin n\theta_0\right|,
\end{aligned}$$

where $\theta_0 = (b-a)/2$. Then

$$\begin{aligned}
\sum_{n\neq 0}\left|\frac{e^{-ina}-e^{-inb}}{2\pi i n}e^{inx}\right| &= \sum_{n\neq 0}\left|\frac{e^{-ina}-e^{-inb}}{2\pi i n}\right| \\
&= \frac{1}{\pi}\sum_{n\neq 0}\frac{|\sin n\theta_0|}{|n|} \\
&= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n}.
\end{aligned}$$

Since $-\pi < a < b < \pi$, we have that $0 < \theta_0 < \pi$. By the lemma there exists integer $N \geq 1$ such that for all integer M we can find $N \leq k < N+M$ integer with $k\theta_0/\pi \in [j+1/4, j+3/4]$ for some integer j. Then $k\theta_0 \in [j\pi + \pi/4, j\pi + 3\pi/4]$ and $|\sin k\theta_0| > 1/2$.

Since all terms of our series are non-negative, we can rearrange

$$\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n} = \sum_{n=1}^{\infty}\sum_{m=Nn}^{Nn+N-1}\frac{|\sin n\theta_0|}{n}.$$

Since one of $|\sin m\theta_0| > 1/2$ for at least one of $m = Nm, \ldots, Nm+N-1$, we have that $\sum_{m=Nn}^{Nn+N-1}\left|\sin n\theta_0\right|/n \geq 1/2(Nn+N-1) \geq 1/2Nn$. Hence

$$\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n} \geq \frac{1}{2N}\sum_{n=1}^{\infty}\frac{1}{n},$$

which diverges. Hence,

$$\sum_{n\neq 0}\left|\frac{e^{-ina}-e^{-inb}}{2\pi i n}e^{inx}\right| = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n}$$

diverges also. $\square$

(c) From 8. we know that the series

$$\sum_{n\neq 0} \frac{e^{inx}}{n}$$

converges. Since x is an arbitrary real, the series

$$\sum_{n\neq 0} \frac{e^{in(x-a)}}{n} \qquad \text{and} \qquad \sum_{n\neq 0} \frac{e^{in(x-b)}}{n}$$

converge also. Then

$$\sum_{n\neq 0} \frac{e^{in(x-a)}}{n} - \sum_{n\neq 0} \frac{e^{in(x-b)}}{n} = \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{n} e^{inx}$$

converges also and the Fourier series

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}$$

converges. $\square$

When $a = -\pi$ and $b = \pi$, we get

$$\begin{aligned} f(x) &\sim 1 + \sum_{n\neq 0} \frac{e^{in\pi} - e^{-in\pi}}{2\pi in} e^{inx} \\ &= 1 + \sum_{n\neq 0} \frac{(-1)^n - (-1)^n}{2\pi in} e^{inx} \\ &= 1. \end{aligned}$$