1 Exercises

1. **(a)** $|z|$ is the distance from z to the origin.

(b)

$$\begin{aligned}
|z| &\overset{\text{def}}{=} \left(x^2+y^2\right)^{1/2} = 0 \\
\iff& x^2+y^2 = 0 \\
\iff& x^2 = 0 \text{ and } y^2 = 0 \qquad (x, y \in \mathbb{R}, \text{ so } x^2, y^2 \geq 0) \\
\iff& x = 0 \text{ and } y = 0 \\
\iff& z \overset{\text{def}}{=} x + iy = 0
\end{aligned}$$

$\square$

(c) We have $\lambda z = (\lambda x) + i(\lambda y)$ for some $\lambda \in \mathbb{R}$. Substituting into the definition of the modulus,

$$\begin{aligned}
|\lambda z| &= \left((\lambda x)^2 + (\lambda y)^2\right)^{1/2} \\
&= \left(\lambda^2\right)^{1/2}\left(x^2+y^2\right)^{1/2} \\
&= |\lambda|\,|z|\,.
\end{aligned}$$

$\square$

(d) Let $z_1 \overset{\text{def}}{=} x_1 + iy_1$ and $z_2 \overset{\text{def}}{=} x_2 + iy_2$ for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
We first show that $|z_1 z_2| = |z_1|\,|z_2|$. We have

$$\begin{aligned}
z_1 z_2 &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\
&= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).
\end{aligned}$$

Substituting into the definition of the modulus,

$$\begin{aligned}
|z_1z_2|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\
&= \left(x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2\right) + \left(x_1^2y_2^2 + 2x_1x_2y_1y_2 + x_2^2y_1^2\right) \\
&= x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2 \\
&= \left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right) \\
&= |z_1|^2\,|z_2|^2\,.
\end{aligned}$$

Taking the square root of both sides concludes the proof.
We now show that $|z_1 + z_2| \leq |z_1| + |z_2|$. We have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

By algebra,

$$0 \leq (x_1y_2 - x_2y_1)^2$$

$$
\begin{aligned}
&\iff 0 \le x_1^2 y_2^2 - 2x_1x_2y_1y_2 + x_2^2 y_1^2 \\
&\iff 2x_1x_2y_1y_2 \le x_1^2 y_2^2 + x_2^2 y_1^2 \\
&\iff x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \le x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2 \\
&\iff (x_1x_2 + y_1y_2)^2 \le \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) \\
&\implies x_1x_2 + y_1y_2 \le \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \qquad \text{(since RHS} \ge 0) \\
&\iff x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 \\
&\qquad \le x_1^2 + y_1^2 + 2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} + x_2^2 + y_2^2 \\
&\iff (x_1 + x_2)^2 + (y_1 + y_2)^2 \le \left(\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}\right)^2 \\
&\iff \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \\
&\iff |z_1 + z_2| \le |z_1| + |z_2| .
\end{aligned}
$$

□

(e) Observe that

$$
\begin{aligned}
\frac{1}{z} &= \frac{1}{x + iy} \\
&= \frac{x - iy}{(x + iy)(x - iy)} \\
&= \frac{x - iy}{x^2 + y^2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|\frac{1}{z}\right| &= \left|\frac{x - iy}{x^2 + y^2}\right| \\
&= \frac{1}{x^2 + y^2}\,|x - iy| \qquad \text{(by (c))}
\end{aligned}
$$

Observing that the definition of $|z|$ depends only on x^2 and y^2, $|x - iy| = |x + iy| = |z|$. Observe also that $|z|^2 = x^2 + y^2$ by squaring both sides of its definition. Then

$$
\left|\frac{1}{z}\right| = \frac{1}{|z|^2}\,|z| = \frac{1}{|z|}.
$$

□

2. **(a)** The complex conjugate $\overline{z}$ is the reflection of z about the x-axis.

(b)

$$
|z|^2 = x^2 + y^2
$$

$$
\begin{aligned}
&= x^2 - i^2y^2 \\
&= (x+iy)(x-iy) \\
&= z\overline{z}.
\end{aligned}
$$

□

(c) Observe that

$$
\begin{aligned}
\frac{1}{z} &= \frac{1}{x+iy} \\
&= \frac{x-iy}{(x+iy)(x-iy)} \\
&= \frac{x-iy}{x^2+y^2}.
\end{aligned}
$$

When z belongs to the unit circle, $x^2+y^2=1$, so $1/z = x-iy = \overline{z}$. □

3. (a) Assume $\{w_n\}_{n=1}^{\infty}$ converges. Let w and w' be its limits. We show that they are equal.

Observe that for all n,

$$
\begin{aligned}
|(w_n - w) - (w_n - w')| &\leq |w_n - w| + |-(w_n - w')| \\
&= |w_n - w| + |w_n - w'|,
\end{aligned}
$$

where the first line is by the triangle inequality.

Since

$$\lim_{n\to\infty} |w_n - w| = 0 \text{ and } \lim_{n\to\infty} |w_n - w'| = 0$$

by assumption,

$$\lim_{n\to\infty} |(w_n - w) - (w_n - w')| = 0$$

by the squeeze theorem (here, the fact that the modulus is non-negative places a lower bound on the limit).

Observe that $(w_n - w) - (w_n - w') = w' - w$, so this sequence is constant. Since its limit is 0, we have that $w' - w = 0$. □

(b) Both directions use the observation that for all complex numbers $z = x + iy$ with $x, y \in \mathbb{R}$,

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&\geq \sqrt{x^2} \qquad (1) \\
&= |x|.
\end{aligned}
$$

A similar argument shows that $|z| \geq |y|$.

($\Longrightarrow$) Let $\{w_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a convergent. We show that it is Cauchy. Decompose $w = t+is$, $w_n = t_n+is_n$ for $t, s, t_n, s_n \in \mathbb{R}$. Dealing with the real and imaginary parts of $w_n - w$ separately we have

$$\lim_{n\to\infty} |t_n - t| \to 0 \text{ and } \lim_{n\to\infty} |s_n - s| \to 0$$

by (1) and the squeeze theorem.
Then $\{t_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ converge, so they are Cauchy.
Pick an arbitrary $\epsilon > 0$. Find N such that $|t_n - t_m| < \epsilon/2$ and $|s_n - s_m| < \epsilon/2$ whenever $n, m > N$. Then

$$\begin{aligned} |w_n - w_n| &= |(t_n - t_m) + i(s_n - s_m)| \\ &\leq |t_n - t_m| + |s_n - s_m| \\ &< \epsilon \end{aligned}$$

by the triangle inequality whenever $n, m > N$.

($\Longleftarrow$) Let $\{w_n\}_{n=1} \subset \mathbb{C}$ be Cauchy. We show that it is convergent. Pick an arbitrary $\epsilon > 0$. Then there exists a positive integer N such that $|w_n - w_m| < \epsilon$ for all $n, m > N$. Decompose $w_n = t_n + is_n$ for $t_n, s_n \in \mathbb{R}$, and decompose w_m similarly. Then $|t_n - t_m| \leq |w_n - w_m| < \epsilon$ $|s_n - s_m| \leq |w_n - w_m| < \epsilon$ by (1). Thus $\{t_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ are Cauchy. It follows that they converge.
Let t and s be the limits of $\{t_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$, respectively. Define $w = t + is$.
We have $|w_n - w| \leq |t_n - t|+|s_n - s|$ by the triangle inequality. We also have $\lim_{n\to\infty} |t_n - t| = 0$ and $\lim_{n\to\infty} |s_n - s| = 0$, which implies $\lim_{n\to\infty}(|t_n - t| + |s_n - s|) = 0$. Then by the squeeze theorem

$$\lim_{n\to\infty} |w_n - w| = 0$$

and $\{w_n\}_{n=1} \subset \mathbb{C}$ converges. $\square$

(c) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative reals such that $\sum_{n=1}^{\infty} a_n$ converges. Let $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence satisfying $|z_n| < a_n$ for all n. We show that $\sum_{n=1}^{\infty} z_n$ converges.

Define $S_N = \sum_{n=1}^{N} z_n$. Our goal is to show that $\{S_N\}_{N=1}^{\infty}$ converges. By (b) it suffices to show that it is Cauchy.

Let $A_N = \sum_{n=1}^{N} a_n$. By assumption, the sequence formed by these partial sums converges, so it is Cauchy.

Pick an arbitrary $\epsilon > 0$.

Then there exists a positive integer M such that for all $N, N' > M$,

$$|A_N - A_{N'}| < \epsilon.$$

W.l.o.g, assume $N > N'$. Observe that

$$\begin{aligned}
&|A_N - A_{N'}| < \epsilon \\
\iff &\left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{N'} a_n\right| < \epsilon \\
\iff &\left|\sum_{n=N'+1}^{N} a_n\right| < \epsilon \\
\iff &\sum_{n=N'+1}^{N} a_n < \epsilon && (a_n \geq 0\ \forall n) \\
\implies &\sum_{n=N'+1}^{N} |z_n| < \epsilon \\
\implies &\left|\sum_{n=N'+1}^{N} z_n\right| < \epsilon && \text{(triangle ineq.)} \\
\iff &\left|\sum_{n=1}^{N} z_n - \sum_{n=1}^{N'} z_n\right| < \epsilon \\
\iff &|S_N - S_{N'}| < \epsilon.
\end{aligned}$$

So $\{S_N\}_{n=1}^{\infty}$ is Cauchy, implying that $\sum_{n=1}^{\infty} z_n$ converges. $\square$

4. (a) Define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We first show that this series converges for every $z \in \mathbb{C}$.
Define

$$a_n = \frac{|z^n|}{n!}.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{\left|z^{n+1}\right| n!}{|z^n|\,(n+1)!} = \frac{|z|^{n+1}\, n!}{|z|^n\,(n+1)!} = \frac{|z|}{n+1}.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|z|}{n+1} = 0,$$

so the series $\sum_{n=0}^{\infty} a_n$ converges.
Recalling that $|z^n/n!| = |z^n|/n! = a_n$, we have that

$$e^z = \sum_{n=0}^{z^n}$$

converges by 3(c).

We now show that the convergence is uniform on every bounded subset of $\mathbb{C}$. Pick an arbitrary bounded $S \subset \mathbb{C}$ and an arbitrary $\epsilon > 0$. We will show that there exists an integer M such that for all $N > M$ and $s \in S$,

$$\left| \sum_{n=0}^{N} \frac{z^n}{n!} - e^x \right| < \epsilon. \tag{2}$$

Note that (2) is equivalent to

$$\left| \sum_{n=N+1}^{\infty} \frac{z^n}{n!} \right| < \epsilon \tag{3}$$

after cancelling the first N terms of the series.

Choose c such that $c > |s|$ for all $s \in S$. This is well-defined because S is bounded. We know from above that

$$e^c = \sum_{n=0}^{\infty} \frac{c^n}{n!}$$

converges. Then there exists an integer M such that for all $N > M$,

$$\left| \sum_{n=0}^{N} \frac{c^n}{n!} - e^c \right| < \epsilon,$$

or after cancelling the first N terms of the series,

$$\sum_{n=N+1}^{\infty} \frac{c^n}{n!} < \epsilon.$$

Observe that for all n,

$$\frac{c^n}{n!} > \frac{|z|^n}{n!} = \left| \frac{z^n}{n!} \right|,$$

so

$$\sum_{n=N+1}^{\infty} \left| \frac{z^n}{n!} \right| < \epsilon.$$

For every partial sum from $N + 1$ to some N' we have

$$\left| \sum_{n=N+1}^{N'} \frac{z^n}{n!} \right| \leq \sum_{n=N+1}^{N'} \left| \frac{z^n}{n!} \right|$$

by the triangle inequality. Taking the limit,

$$\left| \sum_{n=N+1}^{\infty} \frac{z^n}{n!} \right| \leq \sum_{n=N+1}^{\infty} \left| \frac{z^n}{n!} \right|$$

$$< \epsilon,$$

which matches (3), concluding the proof. $\square$

(b) We first show that the series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely for all z. We have

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0,$$

so the series converges absolutely by the ratio test.

Observe that for the n^{th} term of the series for $e^{z_1+z_2}$,

$$\begin{aligned}\frac{(z_1+z_2)^n}{n!} &= \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\ &= \sum_{k=0}^{n} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!}.\end{aligned}$$

We thus recognise the series for $e^{z_1+z_2}$ as the Cauchy product of the series for e^{z_1} and e^{z_2}. Since we've shown that these converge absolutely, $e^{z_1+z_2} = e^{z_1}e^{z_2}$. $\square$

(c) We first find the power series of $\cos y$ around 0. We have

$$\begin{aligned}\cos 0 &= \cos 0 = 1, \\ \cos' 0 &= -\sin 0 = 0, \\ \cos'' 0 &= -\cos 0 = -1, \\ \cos''' 0 &= \sin 0 = 0, \\ \cos'''' 0 &= \cos 0 = 1,\end{aligned}$$

and so on. The odd terms are zero, so we can skip them and write our power series as

$$\cos y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}.$$

To prove convergence, we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} y^{2n+2}}{(2n+2)!}}{\frac{(-1)^n y^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{y^2}{(2n+2)(2n+1)} \right| = 0,$$

so the power series converges absolutely by the ratio test.
We repeat the same for $\sin y$:

$$\begin{aligned}
\sin 0 &= \sin 0 = 0, \\
\sin' 0 &= \cos 0 = 1, \\
\sin'' 0 &= -\sin 0 = 0, \\
\sin''' 0 &= -\cos 0 = -1, \\
\sin'''' 0 &= \sin 0 = 0,
\end{aligned}$$

and so on. Collapsing the even terms, which are zero, we write

$$\sin y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}.$$

Again proving convergence, we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}y^{2n+3}}{(2n+3)!}}{\frac{(-1)^n y^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{y^2}{(2n+3)(2n+2)} \right| = 0,$$

so this power series also converges absolutely by the ratio test.
We combine the power series as

$$\begin{aligned}
\cos y + i \sin y &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n i y^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{i^n y^n}{n!},
\end{aligned}$$

where the n^{th} term of the power series for $\cos y$ becomes the $2n^{\text{th}}$ term of the combined series, and the n^{th} term of the series for $\cos y$ becomes the $2n+1^{\text{th}}$ term of the combined series. We are able to combine the two power series into one because they are absolutely convergent.
Then

$$\begin{aligned}
\cos y + i \sin y &= \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \\
&= e^{iy},
\end{aligned}$$

where the last equality is by our definition of complex exponentiation.
$\square$

(d) Let $x, y \in \mathbb{R}$. We have

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} && \text{(by (b))} \\ &= e^x(\cos y + i \sin y). && \text{(by (c))} \end{aligned}$$

Observe that

$$\begin{aligned} \left|e^{iy}\right| &= |\cos y + i \sin y| \\ &= \sqrt{(\cos y)^2 + (\sin y)^2} \\ &= 1. \end{aligned} \tag{4}$$

Then

$$\begin{aligned} \left|e^{x+iy}\right| &= \left|e^x e^{iy}\right| && \text{(by (b))} \\ &= |e^x| \left|e^{iy}\right| && \text{(by 1(d))} \\ &= |e^x|. && \text{(by (4))} \end{aligned}$$

□

(e) ($\implies$) Decompose $z = x + iy$ for $x, y \in \mathbb{R}$. Then $e^z = e^x \cos y + ie^x \sin y$.
We set $e^z = 1$. Equating the imaginary components,

$$e^x \sin y = 0.$$

Since $e^x > 0$, $\sin y = 0$.
Similarly equating the real components,

$$e^x \cos y = 1.$$

Since $\sin y = 0$, either $\cos y = 1$ or $\cos y = -1$. We know that $\cos y > 0$ since $e^x > 0$ and their product is positive. Hence $\cos y = 1$.
We have $\sin y = 0$ and $\cos y = 1$, so $y = 2\pi k$ for some $k \in \mathbb{Z}$.
Finally $1 = e^x \cos y = e^x$, so $x = 0$.
Thus, $z = x + iy = 2\pi ki$ for some $k \in \mathbb{Z}$.

($\impliedby$) Let k be an arbitrary integer. Let $z = 2\pi ki$. Then

$$\begin{aligned} e^z &= e^{2\pi ki} \\ &= \cos 2\pi k + i \sin 2\pi k && \text{(by (c))} \\ &= 1 + 0i \\ &= 1. \end{aligned}$$

□

(f) Let $z = x + iy$ for some $x, y \in \mathbb{R}$. Let $r = |z|$. Observe that $0 \leq r < \infty$.

We show that this is the unique choice of $r \geq 0$ if we wish to represent

$$z = re^{i\theta}$$

for some $\theta \in \mathbb{R}$.

Observe that

$$\begin{aligned}\left|e^{i\theta}\right| &= \sqrt{\cos^2\theta + \sin^2\theta}\\ &= 1,\end{aligned}$$

so

$$\begin{aligned}|z| &= \left|re^{i\theta}\right| \\ &= |r|\left|e^{i\theta}\right| && \text{(by 1(d))}\\ &= |r| \\ &= r. && \text{(we've restricted } r \geq 0\text{)}\end{aligned}$$

To pick θ, show that it (along with r) represents z, and show its uniqueness, we argue by cases.

$(x = 0, y = 0)$ $r = 0$. For any choice of $\theta \in \mathbb{R}$,

$$re^{i\theta} = 0e^{i\theta} = 0,$$

so in this degenerate case our choice of $\theta \in \mathbb{R}$ can be completely arbitrary.

$(x = 0, y \neq 0)$ $r = |y|$. We want

$$\begin{aligned}z &= iy\\ &= |y|\,e^{i\theta}\\ &= |y|\,(\cos\theta + i\sin\theta).\end{aligned}$$

Equating the imaginary components, we find $|y|\sin\theta = y$ or $\sin\theta = \operatorname{sgn} y$. Thus

$$\theta = \frac{\pi}{2}\operatorname{sgn} y + 2\pi k$$

for some $k \in \mathbb{Z}$.

This satisfies our equation in the real components as well since $\cos\theta = 0$ for our choice of θ, as required.

$(x \neq 0)$ We want

$$\begin{aligned}z &= x + iy\\ &= re^{i\theta}\end{aligned}$$

$$
\begin{aligned}
&= |z|\, e^{i\theta} \\
&= |z| \cos\theta + i\, |z| \sin\theta.
\end{aligned}
$$

Equating the real and imaginary sides,

$$
\begin{aligned}
\cos\theta &= \frac{x}{\sqrt{x^2+y^2}}, \\
\sin\theta &= \frac{y}{\sqrt{x^2+y^2}}.
\end{aligned}
$$

Note that these two equations are sufficient and necessary to obtain θ that represents z.
$\theta = \arctan(y/x) + 2\pi k$ for some arbitrary $k \in \mathbb{Z}$ describes all such θ. $\square$

(g) Multiplying a complex number by i rotates it anticlockwise around the origin by $\pi/2$ radians. More generally, multipying a complex number by $e^{i\theta}$ rotates it anticlockwise around the origin by θ radians.

(h)

$$
\begin{aligned}
\frac{e^{i\theta}+e^{-i\theta}}{2} &= \frac{\cos\theta + i\sin\theta + \cos\theta + i\sin(-\theta)}{2} && \text{(by (c))} \\
&= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \\
&= \frac{2\cos\theta}{2} \\
&= \cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{e^{i\theta}-e^{-i\theta}}{2i} &= \frac{\cos\theta + i\sin\theta - \cos\theta - i\sin(-\theta)}{2i} && \text{(by (c))} \\
&= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i} \\
&= \frac{2i\sin\theta}{2i} \\
&= \sin\theta
\end{aligned}
$$

$\square$

(i) Using Euler's identity from (h),

$$
\begin{aligned}
&\cos\theta\cos\vartheta - \sin\theta\sin\vartheta \\
&= \frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\vartheta}+e^{-i\vartheta}\right) - \frac{1}{4i^2}\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\vartheta}-e^{-i\vartheta}\right) \\
&= \frac{1}{4}\left(\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\vartheta}+e^{-i\vartheta}\right) + \left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\vartheta}-e^{-i\vartheta}\right)\right) \\
&= \frac{1}{2}\left(e^{i\theta}e^{i\vartheta} + e^{-i\theta}e^{-i\vartheta}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left(e^{i(\theta+\vartheta)} + e^{-i(\theta+\vartheta)}\right) && \text{(by (b))}\\
&= \cos(\theta+\vartheta). && \text{(by (h))}
\end{aligned}
$$

Swapping ϑ for $-\vartheta$ and observing that cos is even and sin is odd shows that

$$\cos(\theta - \vartheta) = \cos\theta\cos\vartheta + \sin\theta\sin\vartheta.$$

Arguing similarly for the sin identities,

$$
\begin{aligned}
&\sin\theta\cos\vartheta + \cos\theta\sin\vartheta\\
&= \frac{1}{4i}\left(e^{i\theta} - e^{-i\theta}\right)\left(e^{i\vartheta} + e^{-i\vartheta}\right) + \frac{1}{4i}\left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\vartheta} - e^{-i\vartheta}\right)\\
&= \frac{1}{4i}\left(\left(e^{i\theta} - e^{-i\theta}\right)\left(e^{i\vartheta} + e^{-i\vartheta}\right) + \left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\vartheta} - e^{-i\vartheta}\right)\right)\\
&= \frac{1}{2i}\left(e^{i\theta}e^{i\vartheta} - e^{-i\theta}e^{-i\vartheta}\right)\\
&= \frac{1}{2i}\left(e^{i(\theta+\vartheta)} - e^{-i(\theta+\vartheta)}\right) && \text{(by (b))}\\
&= \sin(\theta+\vartheta). && \text{(by (h))}
\end{aligned}
$$

Again swapping ϑ for $-\vartheta$ shows that

$$\sin(\theta - \vartheta) = \sin\theta\cos\vartheta - \cos\theta\sin\vartheta.$$

We list the identities:

$$
\begin{aligned}
\cos(\theta+\vartheta) &= \cos\theta\cos\vartheta - \sin\theta\sin\vartheta, && (5)\\
\cos(\theta-\vartheta) &= \cos\theta\cos\vartheta + \sin\theta\sin\vartheta, && (6)\\
\sin(\theta+\vartheta) &= \sin\theta\cos\vartheta + \cos\theta\sin\vartheta, && (7)\\
\sin(\theta-\vartheta) &= \sin\theta\cos\vartheta - \cos\theta\sin\vartheta. && (8)
\end{aligned}
$$

Subtracting the LHS and RHS of (5) from (6),

$$2\sin\theta\sin\vartheta = \cos(\theta-\vartheta) - \cos(\theta+\vartheta).$$

Similarly adding (7) and (8),

$$2\sin\theta\cos\vartheta = \sin(\theta+\vartheta) + \sin(\theta-\vartheta).$$

$\square$

5. We first verify that $f(x) = e^{inx}$ is periodic with period 2π. We have

$$
\begin{aligned}
f(x+2\pi k) &= e^{in(x+2\pi k)}\\
&= e^{inx+2\pi ikn}\\
&= e^{inx}e^{2\pi ikn} && \text{(by b(b))}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{inx} && (e^{2\pi ikn} = 1 \text{ by 4(e) since } kn \in \mathbb{Z}) \\
&= f(x).
\end{aligned}
$$

We now show that

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \tag{9}
$$

By cases,

$(n = 0)$

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{inx}dx &= \int_{-\pi}^{\pi} e^{0}dx \\
&= \int_{-\pi}^{\pi} 1dx \\
&= 2\pi.
\end{aligned}
$$

We divide both sides by $1/2\pi$ to obtain our result.

$(n \neq 0)$

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{inx}dx &= \int_{-\pi}^{\pi} e^{inx}dx && \text{(here we use } n \neq 0) \\
&= \frac{1}{in}\left[e^{inx}\right]_{-\pi}^{\pi} \\
&= \frac{1}{in}\left(e^{in\pi} - e^{-in\pi}\right) \\
&= \frac{1}{in}\left(f(\pi) - f(-\pi)\right) \\
&= 0. && (f(\pi) = f(-\pi) \text{ from } f\text{'s periodicity})
\end{aligned}
$$

Finally, we show that

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \; dx &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases} \\
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \; dx &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases} \\
\int_{-\pi}^{\pi} \sin nx \cos mx \; dx &= 0.
\end{aligned}
$$

To show this, first observe that

$$
\begin{aligned}
&e^{i(n-m)x} + e^{i(n+m)x} \\
&= \cos(n-m)x + i\sin(n-m)x + \cos(n+m)x + i\sin(n+m)x
\end{aligned}
$$

$$
\begin{aligned}
&= \cos nx \cos mx + \sin nx \sin mx \qquad \text{(by identities from 4(i))} \\
&\quad + i \sin nx \cos mx - i \cos nx \sin mx \\
&\quad + \cos nx \cos mx - \sin nx \sin mx \\
&\quad + i \sin nx \cos mx + i \cos nx \sin mx \\
&= 2 \cos nx \cos mx + 2i \sin nx \cos mx.
\end{aligned}
$$

An analogous computation shows that

$$e^{i(n-m)x} - e^{i(n+m)x} = 2 \sin nx \sin mx - 2i \cos nx \sin mx.$$

We have

$$\int_{-\pi}^{\pi} e^{i(n+m)x} dx = 0$$

by (9) since $n + m \geq 2$.

Hence

$$
\begin{aligned}
&2 \int_{-\pi}^{\pi} \cos nx \cos mx \; dx + 2i \int_{-\pi}^{\pi} \sin nx \cos mx \; dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx + \int_{-\pi}^{\pi} e^{i(n+m)x} dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx \\
&= \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m, \end{cases}
\end{aligned}
$$

by (9) and

$$
\begin{aligned}
&2 \int_{-\pi}^{\pi} \sin nx \sin mx \; dx - 2i \int_{-\pi}^{\pi} \cos nx \sin mx \; dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx - \int_{-\pi}^{\pi} e^{i(n+m)x} dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} \\
&= \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m. \end{cases}
\end{aligned}
$$

Equating the real and imaginary parts of LHS and RHS, we obtain

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \; dx = 0.$$

$\square$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable such that

$$f''(t) + c^2 f(t) = 0 \tag{10}$$

with $c \neq 0$.

Let

$$\begin{aligned} g(t) &= f(t)\cos ct - c^{-1} f'(t) \sin ct, \\ h(t) &= f(t)\sin ct + c^{-1} f'(t) \cos ct. \end{aligned}$$

Observe that they are once differentiable. Differentiating,

$$\begin{aligned} g'(t) &= f'(t)\cos ct - cf(t)\sin ct - c^{-1}f''(t)\sin ct - f'(t)\cos ct \\ &= -cf(t)\sin ct - c^{-1}f''(t)\sin ct \\ &= -cf(t)\sin ct + cf(t)\sin ct && \text{(by (10))} \\ &= 0, \\ h'(t) &= f'(t)\sin ct + cf(t)\cos ct + c^{-1}f''(t)\cos ct - f'(t)\sin ct \\ &= cf(t)\cos ct + c^{-1}f''(t)\cos ct \\ &= cf(t)\cos ct - cf(t)\cos ct \\ &= 0. \end{aligned}$$

Thus g and h are constant. Let a and b be constants such that

$$\begin{aligned} g(t) &= f(t)\cos ct - c^{-1}f'(t)\sin ct = a, \\ h(t) &= f(t)\sin ct + c^{-1}f'(t)\cos ct = b. \end{aligned}$$

Then

$$c^{-1}f'(t)\sin ct = f(t)\cos ct - a$$

and

$$f(t)(\sin ct)^2 + c^{-1}f'(t)\sin ct\cos ct = b\sin ct,$$

so

$$\begin{aligned} & f(t)(\sin ct)^2 + (f(t)\cos ct - a)\cos ct = b\sin ct \\ \iff\; & f(t)(\sin ct)^2 + f(t)(\cos ct)^2 = a\cos ct + b\sin ct \\ \iff\; & f(t) = a\cos ct + b\sin ct. \end{aligned}$$

$\square$

7.

$$
\begin{aligned}
&A\cos(ct-\varphi) \\
&= A(\cos ct\cos(-\varphi) - \sin ct \sin(-\varphi)) && \text{(by 4(i))} \\
&= A(\cos ct\cos\varphi + \sin ct\sin\varphi) && \text{(cos is even and sin is odd)} \\
&= \sqrt{a^2+b^2}\Big(\frac{a}{\sqrt{a^2+b^2}}\cos ct + \frac{b}{\sqrt{a^2+b^2}}\sin ct\Big) && \text{(substitution)} \\
&= a\cos ct + b\sin ct.
\end{aligned}
$$

$\square$

8. Let F be a function on (a,b) with two continuous derivatives.
By Taylor's theorem,

$$F'(y) = F'(x) + (y-x)F''(x) + (y-x)\eta(x)$$

with $\lim_{x\to y}\eta(x) = 0$. Setting

$$\psi(x) = \eta(y-x)$$

we get

$$F'(y) = F'(x) + (y-x)F''(x) + (y-x)\psi(y-x)$$

with $\lim_{h\to 0}\psi(h) = 0$.
Then

$$
\begin{aligned}
&F(x+h) - F(x) \\
&= \int_x^{x+h} F'(y)dy \\
&= \int_x^{x+h} F'(x)dy + \int_x^{x+h}(y-x)F''(x)dy + \int_x^{x+h}(y-x)\psi(y-x)dy \\
&= hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h),
\end{aligned}
$$

where in the last line we use

$$
\begin{aligned}
\int_x^{x+h}(y-x)\psi(y-x)dy &= \int_0^h t\psi(t)dt \\
&= \psi(\eta)\int_0^h tdt \\
&= \frac{h^2}{2}\psi(\eta)
\end{aligned}
$$

for some η between 0 and h and set $\varphi(h) = \psi(\eta)/2$. Then $\varphi(h) \to 0$ as $h \to 0$.

Hence,

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h)$$

with $\lim_{h\to 0}\varphi(h) = 0$.

Hence

$$\begin{aligned}
&F(x+h) + F(x-h) - 2F(x) \\
&= F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h) \\
&\quad + F(x) - hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(-h) \\
&\quad - 2F(x) \\
&= h^2F''(x) + h^2\varphi(h) + h^2\varphi(-h).
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{h\to 0}\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} &= \lim_{h\to 0}\left(F''(x) + \varphi(h) + \varphi(-h)\right) \\
&= F''(x),
\end{aligned}$$

where we use the fact that $\varphi(h) \to 0$ as $h \to 0$. $\square$

9. We are given that

$$f(x) = \begin{cases} \frac{xh}{p} & \text{for } 0 \le x \le p \\ \frac{h(\pi - x)}{\pi - p} & \text{for } p \le x \le \pi \end{cases}$$

From the formula for the Fourier sine coefficients, we have

$$\begin{aligned}
A_m &= \frac{2}{\pi}\int_0^\pi f(x)\sin mx\ dx \\
&= \frac{2}{\pi}\int_0^p \frac{xh}{p}\sin mx\ dx + \frac{2}{\pi}\int_p^\pi \frac{h(\pi - x)}{\pi - p}\sin mx\ dx,
\end{aligned}$$

where we use the fact that $f(x)$ is piecewise. By algebra,

$$A_m = \frac{2h}{\pi p}\int_0^p x\sin mx\ dx + \frac{2h}{(\pi - p)}\int_p^\pi \sin mx\ dx - \frac{2h}{\pi(\pi - p)}\int_p^\pi x\sin mx\ dx \tag{11}$$

Integrating,

$$\begin{aligned}
\int_p^\pi \sin mx\ dx &= -\frac{1}{m}\left[\cos mx\right]_p^\pi \\
&= \frac{1}{m}\cos mp - \frac{1}{m}\cos \pi m.
\end{aligned}$$

Letting $a, b \in \mathbb{R}$ and integrating by parts,

$$\begin{aligned}
\int_a^b x \sin mx \; dx &= \left[x \int \sin mx \; dx \right]_a^b - \int_a^b \frac{dx}{dx} \int \sin mx \; dx \; dx \\
&= -\frac{1}{m} \left[x \cos mx \right]_a^b + \frac{1}{m} \int_a^b \cos mx \; dx \\
&= -\frac{1}{m} \left[x \cos mx \right]_a^b + \frac{1}{m^2} \left[x \sin mx \right]_a^b \\
&= \frac{1}{m} \left(a \cos ma - b \cos mb \right) + \frac{1}{m^2} \left(\sin mb - \sin ma \right).
\end{aligned}$$

Substituting for a and b,

$$\begin{aligned}
\int_0^p x \sin mx \; dx &= -\frac{p}{m} \cos mp + \frac{1}{m^2} \sin mp \\
\int_p^\pi x \sin mx \; dx &= \frac{p}{m} \cos mp - \frac{\pi}{m} \cos m\pi - \frac{1}{m^2} \sin mp.
\end{aligned}$$

Substituting into (11),

$$\begin{aligned}
\frac{m}{2h} A_m = &-\frac{1}{\pi} \cos mp + \frac{1}{\pi mp} \sin mp \\
&+ \frac{1}{(\pi - p)} \cos mp - \frac{1}{(\pi - p)} \cos \pi m \\
&- \frac{p}{\pi(\pi - p)} \cos mp + \frac{1}{(\pi - p)} \cos m\pi + \frac{1}{\pi m(\pi - p)} \sin mp \\
= &-\frac{1}{\pi} \cos mp + \frac{1}{(\pi - p)} \cos mp - \frac{p}{\pi(\pi - p)} \cos mp \\
&+ \frac{1}{\pi mp} \sin mp + \frac{1}{\pi m(\pi - p)} \sin mp \\
= &\left(\frac{1}{\pi - p} - \frac{1}{\pi} - \frac{p}{\pi(\pi - p)} \right) \cos mp + \left(\frac{1}{\pi mp} + \frac{1}{\pi m(\pi - p)} \right) \sin mp.
\end{aligned}$$

Observe that

$$\frac{1}{\pi - p} - \frac{1}{\pi} - \frac{p}{\pi(\pi - p)} = \frac{\pi - \pi + p - p}{\pi(\pi - p)} = 0$$

and

$$\frac{1}{\pi mp} + \frac{1}{\pi m(\pi - p)} = \frac{\pi - p + p}{\pi mp(\pi - p)} = \frac{1}{mp(\pi - p)},$$

so

$$A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}$$

as desired.

For $0 < h$, $0 < p < \pi$, we have $A_m = 0$ iff $\sin mp = 0$. When $p = \pi/2$, $\sin m\pi/2 = 0$ for $m = 2, 4, \ldots$, so the second, fourth, and so on, harmonics are missing. Similarly, when $p = \pi/3$, $\sin m\pi/3 = 0$ for $m = 3, 6, \ldots$, so the third, sixth, and so on, harmonics are missing.

10. We wish to prove that

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{12}$$

in polar coordinates and also

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2. \tag{13}$$

In both proofs, we'll use

$$\begin{aligned} &\theta = \operatorname{atan2}(y, x), && r = \sqrt{x^2 + y^2}, \\ &x = r\cos\theta, && y = r\sin\theta, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}. \end{aligned}$$

We first prove eq. (12). We are given that

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in Euclidean coordinates.

We want to show that

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates.

By the chain rule,

$$\begin{aligned} \triangle &= \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\right)^2 \\ &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ &\quad + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}. \end{aligned}$$

By the product rule,

$$\begin{aligned}\triangle = & \frac{\partial r}{\partial x}\frac{\partial^2 r}{\partial r \partial x}\frac{\partial}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial r}{\partial x}\frac{\partial^2}{\partial r^2} + \frac{\partial r}{\partial x}\frac{\partial^2 \theta}{\partial r \partial x}\frac{\partial}{\partial \theta} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial \theta}{\partial x}\frac{\partial^2}{\partial r \partial \theta} \\ & + \frac{\partial \theta}{\partial x}\frac{\partial^2 r}{\partial \theta \partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x}\frac{\partial^2}{\partial \theta \partial r} + \frac{\partial \theta}{\partial x}\frac{\partial^2 \theta}{\partial \theta \partial x}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x}\frac{\partial^2}{\partial \theta^2} \\ & + \frac{\partial r}{\partial y}\frac{\partial^2 r}{\partial r \partial y}\frac{\partial}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial r}{\partial y}\frac{\partial^2}{\partial r^2} + \frac{\partial r}{\partial y}\frac{\partial^2 \theta}{\partial r \partial y}\frac{\partial}{\partial \theta} + \frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}\frac{\partial^2}{\partial r \partial \theta} \\ & + \frac{\partial \theta}{\partial y}\frac{\partial^2 r}{\partial \theta \partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial r}{\partial y}\frac{\partial^2}{\partial \theta \partial r} + \frac{\partial \theta}{\partial y}\frac{\partial^2 \theta}{\partial \theta \partial y}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial y}\frac{\partial \theta}{\partial y}\frac{\partial^2}{\partial \theta^2}.\end{aligned}$$

We have

$$\frac{\partial^2 r}{\partial r \partial x} = \frac{\partial}{\partial r}\cos\theta = 0, \qquad \frac{\partial^2 r}{\partial r \partial y} = \frac{\partial}{\partial r}\sin\theta = 0.$$

Using this and collecting terms,

$$\begin{aligned}\triangle = & \left(\frac{\partial \theta}{\partial x}\frac{\partial^2 r}{\partial \theta \partial x} + \frac{\partial \theta}{\partial y}\frac{\partial^2 r}{\partial \theta \partial y}\right)\frac{\partial}{\partial r} \\ + & \left(\frac{\partial r}{\partial x}\frac{\partial^2 \theta}{\partial r \partial x} + \frac{\partial r}{\partial y}\frac{\partial^2 \theta}{\partial r \partial y} + \frac{\partial \theta}{\partial x}\frac{\partial^2 \theta}{\partial \theta \partial x} + \frac{\partial \theta}{\partial y}\frac{\partial^2 \theta}{\partial \theta \partial y}\right)\frac{\partial}{\partial \theta} \\ + & \left(\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right)\frac{\partial^2}{\partial r^2} \\ + & \left(2\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + 2\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}\right)\frac{\partial^2}{\partial r \partial \theta} \\ + & \left(\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right)\frac{\partial^2}{\partial \theta^2}.\end{aligned}$$

We tackle each individually. First observe that

$$\begin{aligned}\frac{\partial^2 \theta}{\partial r \partial x} &= \frac{\sin\theta}{r^2}, & \frac{\partial^2 \theta}{\partial r \partial y} &= -\frac{\cos\theta}{r^2}, \\ \frac{\partial^2 r}{\partial \theta \partial x} &= -\sin\theta, & \frac{\partial^2 r}{\partial \theta \partial y} &= \cos\theta, \\ \frac{\partial^2 \theta}{\partial \theta \partial x} &= -\frac{\cos\theta}{r}, & \frac{\partial^2 \theta}{\partial \theta \partial y} &= -\frac{\sin\theta}{r}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial \theta}{\partial x}\frac{\partial^2 r}{\partial \theta \partial x} + \frac{\partial \theta}{\partial y}\frac{\partial^2 r}{\partial \theta \partial y} &= \frac{\sin\theta}{r}\sin\theta + \frac{\cos\theta}{r}\cos\theta = \frac{1}{r}, \\ \frac{\partial r}{\partial x}\frac{\partial^2 \theta}{\partial r \partial x} + \frac{\partial r}{\partial y}\frac{\partial^2 \theta}{\partial r \partial y}\end{aligned}$$

$$
\begin{aligned}
+\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial\theta\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial\theta\partial y} &= \cos\theta\frac{\sin\theta}{r^2}-\sin\theta\frac{\cos\theta}{r^2} \\
&\qquad +\frac{\sin\theta}{r}\frac{\cos\theta}{r}-\frac{\cos\theta}{r}\frac{\sin\theta}{r}=0, \\
\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2 &= \cos^2\theta+\sin^2\theta=1, \\
2\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}+2\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y} &= -2\cos\theta\frac{\sin\theta}{r}+2\sin\theta\frac{\cos\theta}{r}=0, \\
\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2 &= \left(-\frac{\sin\theta}{r}\right)^2+\left(\frac{\cos\theta}{r}\right)^2=\frac{1}{r^2}.
\end{aligned}
$$

Substituting back,

$$\triangle=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

□

We now show eq. (13). We have

$$
\begin{aligned}
&\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2 \\
&=\left(\frac{\partial r}{\partial x}\frac{\partial u}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial u}{\partial\theta}\right)^2+\left(\frac{\partial r}{\partial y}\frac{\partial u}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial u}{\partial\theta}\right)^2 && \text{(chain rule)} \\
&=\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)^2+\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)^2 && \text{(subst.)} \\
&=\left(\sin^2\theta+\cos^2\theta\right)\left(\frac{\partial u}{\partial r}\right)^2 \\
&\qquad+\left(\frac{2\cos\theta\sin\theta}{r}-\frac{2\cos\theta\sin\theta}{r}\right)\frac{\partial u}{\partial r}\frac{\partial u}{\partial\theta} \\
&\qquad+\left(\frac{\sin^2\theta}{r^2}+\frac{\cos^2\theta}{r^2}\right)\left(\frac{\partial u}{\partial\theta}\right)^2 \\
&=\left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2.
\end{aligned}
$$

□

11. Let $F:(0,\infty)\to\mathbb{R}$ twice differentiable such that

$$r^2F''(x)+rF'(r)-n^2F(r)=0$$

for some $n\in\mathbb{Z}$.

Let $g(r)=F(r)/r^n$. Observe that the denominator is never zero on the domain of F. Then

$$F(r)=r^ng(r),$$

$$\begin{aligned}
F'(r) &= nr^{n-1}g(r) + r^n g'(r), \\
F''(r) &= n(n-1)r^{n-2}g(r) + nr^{n-1}g'(r) + nr^{n-1}g'(r) + r^n g''(r), \\
&= n(n-1)r^{n-2}g(r) + 2nr^{n-1}g'(r) + r^n g''(r).
\end{aligned}$$

Substituting back

$$\begin{aligned}
&r^2 F''(x) + rF'(r) - n^2 F(r) \\
&= r^2 \left(n(n-1)r^{n-2}g(r) + 2nr^{n-1}g'(r) + r^n g''(r)\right) \\
&\qquad + r\left(nr^{n-1}g(r) + r^n g'(r)\right) \\
&\qquad - n^2 r^n g(r) \\
&= n(n-1)r^n g(r) + 2nr^{n+1}g'(r) + r^{n+2}g''(r) \\
&\qquad + nr^n g(r) + r^{n+1}g'(r) \\
&\qquad - n^2 r^n g(r) \\
&= (2n+1)r^{n+1}g'(r) + r^{n+2}g''(r) = 0,
\end{aligned}$$

so

$$(2n+1)g'(r) + rg''(r) = 0.$$

Integrating by parts,

$$\begin{aligned}
\int g'(r)dr &= g(r) + \text{const}, \\
\int rg''(r)dr &= rg'(r) - \int g'(r)dr + \text{const} \\
&= rg'(r) - g(r) + \text{const}.
\end{aligned}$$

Hence

$$(2n+1)g(r) + rg'(r) - g(r) = rg'(r) + 2ng(r) = c$$

for some constant c.

For notational convenience, let $y = g(r)$. Then $g'(r) = dy/dr$ and

$$r\frac{dy}{dr} + 2ny = c.$$

This is separable as

$$\frac{dr}{r} = \frac{dy}{c - 2ny}.$$

We now argue by cases: either $n = 0$ or $n \neq 0$.

$(n = 0)$ We have

$$\frac{dr}{r} = \frac{dy}{c}$$

Integrating,

$$\log r = \frac{1}{c}y + \text{const},$$

so

$$g(r) = y = c \log r + d$$

for some constant d. Then

$$F(r) = r^0 g(r) = c \log r + d,$$

so F is a linear combination of $\log r$ and 1.

$(n \neq 0)$ Integrating,

$$\log r = -\frac{1}{2n} \log |c - 2ny| + \text{const},$$

so

$$\log |c - 2ny| = -2n \log r + \text{const}$$

and

$$2ny - c = dr^{-2n}$$

for some d. Hence,

$$g(r) = y = \frac{d}{2n} r^{-2n} + \frac{c}{2n}.$$

Finally,

$$F(r) = r^n g(r) = \frac{d}{2n} r^{-n} + \frac{c}{2n} r^n,$$

so F is a linear combinatio of r^{-n} and r^n as desired. $\square$

2 Problems

1. *Proof.* Let u_k be such that $u_k(x, 0) = A_k \sin kx$, $u_k(x, 1) = B_k \sin kx$, $u_k(0, y) = 0$, $u_k(1, y) = 0$, and $\triangle u = 0$.

We want to solve for u_k. Using separation of variables, we write $u_k(x, y) = F(x)G(y)$. The Laplacian becomes

$$\begin{aligned} \triangle u_k &= \frac{\partial^2 F(x)G(y)}{\partial x^2} + \frac{\partial^2 F(x)G(y)}{\partial y^2} \\ &= F''(x)G(y) + F(x)G''(y) = 0. \end{aligned}$$

Thus, we look for solutions of the form

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.$$

Since those sides depend on different variables, they must be equal to some constant, which we will call λ. Then

$$F''(x) - \lambda F(x) = 0 \text{ and } G''(y) + \lambda G(y) = 0.$$

By our definition, $u_k(x,0) = F(x)G(0) = A_k \sin kx$ and $u(x,1) = F(x)G(1) = B_k \sin kx$. Then $F(x) = a \sin kx$ for some a and $\lambda = -k^2$. By the lemma, $G(y) = \alpha \cosh ky - \beta \sinh ky$ for some $\alpha, \beta \in \mathbb{R}$.

When $y = 0$, we have $\alpha F(x) = A_k \sin kx$, so

$$\alpha F(x) \cosh ky = A_k \sin kx \cosh ky.$$

Similarly, when $y = 1$, $\alpha F(x) \cosh k - \beta F(x) \sinh k = B_k \sin kx$, implying that

$$\beta F(x) \sinh ky = \frac{A_k \cosh k - B_k}{\sinh k} \sin kx \sinh ky.$$

Simplifying,

$$\begin{aligned}
F(x)G(y) &= \left(A_k \cosh ky - \frac{A_k \cosh k - B_k}{\sinh k} \sinh ky \right) \sin kx \\
&= \left(A_k \frac{\sinh k \cosh ky - \sinh ky \cosh k}{\sinh k} + B_k \frac{\sinh ky}{\sinh k} \right) \sin kx \\
&= \left(A_k \frac{\sinh k(1-y)}{\sinh k} + B_k \frac{\sinh ky}{\sinh k} \right) \sin kx,
\end{aligned}$$

where in the last line we use

$$\begin{aligned}
&4 \sinh k \cosh ky - 4 \sinh ky \cosh k \\
&= \left(e^k - e^{-k}\right)\left(e^{ky} + e^{-ky}\right) - \left(e^{ky} - e^{-ky}\right)\left(e^k + e^{-k}\right) \\
&= 2e^{k-ky} - 2e^{ky-y} = 4 \sinh k(1-y).
\end{aligned}$$

Define

$$u = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \left(A_k \frac{\sinh k(1-y)}{\sinh k} + B_k \frac{\sinh ky}{\sinh k} \right) \sin kx.$$

Define also

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \text{ and } f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx.$$

Then $u(x,0) = \sum_{k=1}^{\infty} u_k(x,0) = f_0(x)$, and similarly $u(x,1) = f_1(x)$, $u(0,y) = 0$, and $u(1,y) = 0$. Finally, by the linearity of the Laplacian, $\triangle u = 0$. $\square$

Lemma. *Let f be a twice continuously differentiable function on $\mathbb{R}$ such that $f''(t) - c^2 f(t) = 0$. Then all solutions for f have the form*

$$f(t) = a \cosh ct - b \sinh ct.$$

Proof. Let $g(t) = f(t)\cosh ct - c^{-1} f'(t) \sinh ct$ and $h(t) = f(t) \sinh ct - c^{-1} f'(t) \cosh ct$. Observe that these once differentiable. Differentiating,

$$\begin{aligned}
g'(t) &= f'(t)\cosh ct + cf(t)\sinh ct - c^{-1} f''(t) \sinh ct - f'(t)\cosh ct \\
&= cf(t)\sinh ct - c^{-1} f''(t) \sinh ct \\
&= cf(t)\sin ct - cf(t)\sin ct = 0, \\
h'(t) &= f'(t)\sinh ct + cf(t)\cosh ct - c^{-1} f''(t) \cosh ct - f'(t)\sinh ct \\
&= cf(t)\cosh ct - c^{-1} f''(t) \cosh ct \\
&= cf(t)\cosh ct - cf(t)\cosh ct = 0.
\end{aligned}$$

Thus g and h are constant. Let a and b be constants such that $g(t) = a$ and $h(t) = b$. Then

$$\begin{aligned}
f(t)\cosh^2 ct - c^{-1} f'(t) \sinh ct \cosh ct &= a \cosh ct, \\
f(t)\sinh^2 ct - c^{-1} f'(t) \sinh ct \cosh ct &= b \sinh ct.
\end{aligned}$$

Subtracting,

$$f(t)\cosh^2 ct - f(t)\sinh^2 ct = a\cosh ct - b\sinh ct,$$

which simplifies to $f(t) = a\cosh ct - b \sinh ct$. $\square$