Define $\tau = 2\pi$.

(a) We know that

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

Pick $N > 0$ and take the partial sum

$$S_N(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}.$$

Then

$$\begin{aligned}
S_N(f)(\theta) &= \hat{f}(0) + \sum_{0<|n|\leq N} \hat{f}(n) e^{in\theta} \\
&= \hat{f}(0) + \sum_{0<n\leq N} \hat{f}(n) e^{in\theta} + \hat{f}(-n) e^{-in\theta} \\
&= \hat{f}(0) + \sum_{0<n\leq N} \hat{f}(n)[\cos n\theta + i \sin n\theta] + \hat{f}(-n)[\cos -n\theta + i \sin -n\theta] \\
&= \hat{f}(0) + \sum_{0<n\leq N} \hat{f}(n)[\cos n\theta + i \sin n\theta] + \hat{f}(-n)[\cos n\theta - i \sin n\theta] \\
&= \hat{f}(0) + \sum_{0<n\leq N} [\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta.
\end{aligned}$$

Limiting $N \to \infty$, we obtain

$$f(\theta) \sim \hat{f}(0) + \sum_{n>0} [\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta.$$

$\square$

(b) Assume that f is even. Then

$$\begin{aligned}
&\tau \hat{f}(n) - \tau \hat{f}(-n) \\
&= \int_{-\tau/2}^{\tau/2} f(\theta) e^{-in\theta} d\theta - \int_{-\tau/2}^{\tau/2} f(\theta) e^{in\theta} d\theta \\
&= \int_{-\tau/2}^{\tau/2} f(\theta) e^{-in\theta} d\theta - \int_{-\tau/2}^{\tau/2} f(-\theta) e^{-in\theta} d\theta \qquad \text{(subst. } \theta := -\theta) \\
&= \int_{-\tau/2}^{\tau/2} [f(\theta) - f(-\theta)]\, e^{-in\theta} d\theta \\
&= 0. \qquad (f(\theta) = f(-\theta))
\end{aligned}$$

Hence $\hat{f}(n) = \hat{f}(-n)$ and by (a),

$$f(\theta) \sim \hat{f}(0) + 2 \sum_{n>0} \hat{f}(n) \cos n\theta.$$

$\square$

(c) Assume that f is odd. Then

$$\begin{aligned}
&\tau\hat{f}(n) + \tau\hat{f}(-n) \\
&= \int_{-\tau/2}^{\tau/2} f(\theta)e^{-in\theta}d\theta + \int_{-\tau/2}^{\tau/2} f(\theta)e^{in\theta}d\theta \\
&= \int_{-\tau/2}^{\tau/2} f(\theta)e^{-in\theta}d\theta + \int_{-\tau/2}^{\tau/2} f(-\theta)e^{-in\theta}d\theta & \text{(subst. } \theta := -\theta) \\
&= \int_{-\tau/2}^{\tau/2} \left[f(\theta) + f(-\theta)\right] e^{-in\theta}d\theta \\
&= 0. & (f(\theta) = -f(-\theta))
\end{aligned}$$

Hence $\hat{f}(n) = -\hat{f}(-n)$ and by (a),

$$f(\theta) \sim \hat{f}(0) + 2i\sum_{n>0} \hat{f}(n)\sin n\theta.$$

$\square$

(d) Suppose that for all $\theta \in \mathbb{R}$, $f(\theta + \tau/2) = f(\theta)$. Let n be odd. Then

$$\begin{aligned}
\hat{f}(n) &= \int_{-\tau/2}^{\tau/2} f(\theta)e^{-in\theta}d\theta \\
&= \int_{-\tau/2}^{0} f(\theta)e^{-in\theta}d\theta + \int_{0}^{\tau/2} f(\theta)e^{-in\theta}d\theta \\
&= \int_{0}^{\tau/2} f(\theta - \tau/2)e^{-in\theta - in\tau/2}d\theta + \int_{0}^{\tau/2} f(\theta)e^{-in\theta}d\theta & \text{(subst. } \theta := \theta + \tau/2) \\
&= \int_{0}^{\tau/2} f(\theta)e^{-in\theta}e^{-in\tau/2}d\theta + \int_{0}^{\tau/2} f(\theta)e^{-in\theta}d\theta \\
&= -\int_{0}^{\tau/2} f(\theta)e^{-in\theta}d\theta + \int_{0}^{\tau/2} f(\theta)e^{-in\theta}d\theta & (e^{in\tau/2} = -1 \text{ for odd } n) \\
&= 0.
\end{aligned}$$

$\square$

(e) ($\implies$) Let f be real-valued. Then for all θ, $\overline{f(\theta)} = f(\theta)$ and

$$\begin{aligned}
\overline{\hat{f}(n)} &= \overline{\int_{-\tau/2}^{\tau/2} f(\theta)e^{-in\theta}d\theta} \\
&= \int_{-\tau/2}^{\tau/2} \overline{f(\theta)e^{-in\theta}}d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\tau/2}^{\tau/2} \overline{f(\theta)} e^{in\theta} d\theta \\
&= \int_{-\tau/2}^{\tau/2} f(\theta) e^{in\theta} d\theta \\
&= \hat{f}(-n).
\end{aligned}
$$

$(\Longleftarrow)$ Let f be such that $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n. Then

$$
\begin{aligned}
\overline{f(\theta)} &\sim \overline{\sum_{-\infty}^{\infty} \hat{f}(n) e^{in\theta}} \\
&= \sum_{-\infty}^{\infty} \overline{\hat{f}(n) e^{in\theta}} \\
&= \sum_{-\infty}^{\infty} \hat{f}(-n) e^{-in\theta} \\
&= \sum_{-\infty}^{\infty} \hat{f}(n) e^{in\theta} \\
&\sim f(\theta).
\end{aligned}
$$

Hence,

$$\overline{f} = f$$

almost everywhere. □