Let F be a function on (a, b) with two continuous derivatives.

By Taylor's theorem,

$$F'(y) = F'(x) + (y - x)F''(x) + (y - x)\eta(x)$$

with $\lim_{x \to y} \eta(x) = 0$. Setting

$$\psi(x) = \eta(y - x)$$

we get

$$F'(y) = F'(x) + (y - x)F''(x) + (y - x)\psi(y - x)$$

with $\lim_{h \to 0} \psi(h) = 0$.

Then

$$\begin{aligned}
&F(x + h) - F(x) \\
&= \int_x^{x+h} F'(y)dy \\
&= \int_x^{x+h} F'(x)dy + \int_x^{x+h} (y - x)F''(x)dy + \int_x^{x+h} (y - x)\psi(y - x)dy \\
&= hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h),
\end{aligned}$$

where in the last line we use

$$\begin{aligned}
\int_x^{x+h} (y - x)\psi(y - x)dy &= \int_0^h t\psi(t)dt \\
&= \psi(\eta) \int_0^h tdt \\
&= \frac{h^2}{2}\psi(\eta)
\end{aligned}$$

for some η between 0 and h and set $\varphi(h) = \psi(\eta)/2$. Then $\varphi(h) \to 0$ as $h \to 0$.

Hence,

$$F(x + h) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h)$$

with $\lim_{h \to 0} \varphi(h) = 0$.

Hence

$$\begin{aligned}
&F(x + h) + F(x - h) - 2F(x) \\
&= F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(h) \\
&\qquad + F(x) - hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi(-h) \\
&\qquad - 2F(x) \\
&= h^2F''(x) + h^2\varphi(h) + h^2\varphi(-h).
\end{aligned}$$

Then

$$\begin{aligned}\lim_{h\to 0}\frac{F(x+h)+F(x-h)-2F(x)}{h^2} &= \lim_{h\to 0}\left(F''(x)+\varphi(h)+\varphi(-h)\right)\\ &= F''(x),\end{aligned}$$

where we use the fact that $\varphi(h) \to 0$ as $h \to 0$. $\square$