Let $F : (0, \infty) \to \mathbb{R}$ twice differentiable such that

$$r^2 F''(x) + rF'(r) - n^2 F(r) = 0$$

for some $n \in \mathbb{Z}$.

Let $g(r) = F(r)/r^n$. Observe that the denominator is never zero on the domain of F. Then

$$\begin{aligned}
F(r) &= r^n g(r), \\
F'(r) &= nr^{n-1} g(r) + r^n g'(r), \\
F''(r) &= n(n-1)r^{n-2} g(r) + nr^{n-1} g'(r) + nr^{n-1} g'(r) + r^n g''(r), \\
&= n(n-1)r^{n-2} g(r) + 2nr^{n-1} g'(r) + r^n g''(r).
\end{aligned}$$

Substituting back

$$\begin{aligned}
&r^2 F''(x) + rF'(r) - n^2 F(r) \\
&= r^2 \left(n(n-1)r^{n-2} g(r) + 2nr^{n-1} g'(r) + r^n g''(r) \right) \\
&\quad + r \left(nr^{n-1} g(r) + r^n g'(r) \right) \\
&\quad - n^2 r^n g(r) \\
&= n(n-1)r^n g(r) + 2nr^{n+1} g'(r) + r^{n+2} g''(r) \\
&\quad + nr^n g(r) + r^{n+1} g'(r) \\
&\quad - n^2 r^n g(r) \\
&= (2n+1)r^{n+1} g'(r) + r^{n+2} g''(r) = 0,
\end{aligned}$$

so

$$(2n+1)g'(r) + rg''(r) = 0.$$

Integrating by parts,

$$\begin{aligned}
\int g'(r) dr &= g(r) + \text{const}, \\
\int rg''(r) dr &= rg'(r) - \int g'(r) dr + \text{const} \\
&= rg'(r) - g(r) + \text{const}.
\end{aligned}$$

Hence

$$(2n+1)g(r) + rg'(r) - g(r) = rg'(r) + 2ng(r) = c$$

for some constant c.

For notational convenience, let $y = g(r)$. Then $g'(r) = dy/dr$ and

$$r\frac{dy}{dr} + 2ny = c.$$

This is separable as

$$\frac{dr}{r} = \frac{dy}{c - 2ny}.$$

We now argue by cases: either $n = 0$ or $n \neq 0$.

($n = 0$) We have

$$\frac{dr}{r} = \frac{dy}{c}$$

Integrating,

$$\log r = \frac{1}{c} y + \text{const},$$

so

$$g(r) = y = c \log r + d$$

for some constant d. Then

$$F(r) = r^0 g(r) = c \log r + d,$$

so F is a linear combination of $\log r$ and 1.

($n \neq 0$) Integrating,

$$\log r = -\frac{1}{2n} \log |c - 2ny| + \text{const},$$

so

$$\log |c - 2ny| = -2n \log r + \text{const}$$

and

$$2ny - c = dr^{-2n}$$

for some d. Hence,

$$g(r) = y = \frac{d}{2n} r^{-2n} + \frac{c}{2n}.$$

Finally,

$$F(r) = r^n g(r) = \frac{d}{2n} r^{-n} + \frac{c}{2n} r^n,$$

so F is a linear combinatio of r^{-n} and r^n as desired. $\square$