(a) We are given that

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Choose a countable dense sequence $\{r_n\}$ in $(0, 1]$, for example, an enumeration of $(0, 1] \cap \mathbb{Q}$. Define $F : [0, 1] \to \mathbb{R}$ with

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n).$$

We want to show that F is integrable and that it is discontinuous at every point of $\{r_n\}$.

Integrability It is sufficient to show that F is monotonic. Then F is integrable since it is bounded[1] and monotonic on a closed interval.

Choose x and x' with $x \leq x'$. For all n, $f(x - r_n) = 1$ implies $r_n \leq x$, so $r_n \leq x'$ and $f(x' - r_n) = 1 = f(x - r_n)$. Since f is either 0 or 1, $f(x - r_n)/n^2 \leq f(x' - r_n)/n^2$. Therefore $F(x) \leq F(x')$ since all terms of the series for $F(x)$ are bounded by those of $F(x')$.

Discontinuity We will show that F is discontinous at all r_n. Choose an arbitrary $\delta > 0$. Set $x = \max\{0, r_n - \delta/2\}$, so $|r_n - x| \leq \delta/2 < \delta$[2]. But $|F(r_n) - F(x)| \geq 1/n^2$. This can be seen by noting that for all m, $f(x - r_m) \leq f(r_n - r_m)$ since $x < r_n$, and in particular, $0 = f(x - r_n) < f(r_n - r_n) = 1$. Then the series for $F(r_n)$ includes all the terms that the series for $F(x)$ includes, but $F(r_n)$ also has a $1/n^2$ term that $F(x)$ is missing. □

(b) Choose a countable dense sequence $\{r_n\}$ in $[0, 1]$. Let

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n)$$

with

$$g(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We want to show that F is integrable, discontinuous at every point of $\{r_n\}$, and non-monotonic in any subinterval of $[0, 1]$.

For convenience, define $g_n : [0, 1] \to \mathbb{R}$ by $g_n(x) = g(x - r_n)$.

[1] F is bounded above by $\sum_{n=1}^{\infty} 1/n^2$ and below by 0 since the codomain of f is $\{0, 1\}$.
[2] We never have $x = r_n$ because $\delta > 0$ and $r_n > 0$.

Integrability It suffices to show that the set of discontinuities of F has measure zero.

Let $S = [0,1] \setminus \{r_n\}_{n=1}^{\infty}$. We will show that F is continuous at every $x \in S$. Choose an abitrary $x \in S$ and $\epsilon > 0$.

Let N be a positive integer such that $3^{-N} < \epsilon/2$. For each $n = 1, \ldots, N$, $g_n(x)$ is continuous everywhere except at $x = r_n$. Since $x \neq r_n$ by our choice of S, there exists δ_n such that $|g_n(x) - g_n(x')| < \epsilon$ whenever $|x - x'| < \delta_n$.

Set $\delta = \min\{\delta_1, \ldots, \delta_N\}$. Then for all $x' \in [0,1]$, $|x - x'| < \delta$ implies

$$\begin{aligned} |F(x) - F(x')| &\leq \sum_{n=1}^{\infty} 3^{-n} \left|g_n(x) - g_n(x')\right| \\ &= \sum_{n=1}^{N} 3^{-n} \left|g_n(x) - g_n(x')\right| \\ &\qquad + \sum_{n=N+1}^{\infty} 3^{-n} \left|g_n(x) - g_n(x')\right|. \end{aligned}$$

The first term is bounded by

$$\sum_{n=1}^{N} 3^{-n} \left|g_n(x) - g_n(x')\right| < \epsilon \sum_{n=1}^{N} 3^{-n} = \epsilon \frac{1 - 3^{-N}}{2} < \frac{\epsilon}{2}$$

by our choice of δ. The second term is bounded by

$$\sum_{n=N+1}^{\infty} 3^{-n} \left|g_n(x) - g_n(x')\right| \leq 2 \sum_{n=N+1}^{\infty} 3^{-n} = 3^{-N} < \frac{\epsilon}{2}$$

because g_n is bounded by 1 and by our choice of N. Hence,

$$|F(x) - F(x')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then the set of discontinuities is a subset of the countable set $[0,1] \setminus S = \{r_n\}_{n=1}^{\infty}$, and therefore has measure zero.

Discontinuity Choose some positive integer n. We will show that F is not continuous at r_n. Choose an arbitary $\delta > 0$.

Let

$$h(x) = \sum_{k=1}^{n-1} 3^{-k} g_k(x) + 3^{-n-1} g_{n+1}(x).$$

Our choice of h is continuous at r_n as a finite sum of functions that are continuous at r_n. Hence, there exists $\delta_h > 0$ such that $|x - r_n| < \delta_h$ implies $|h(x) - h(r_n)| < 3^{-n-1}$. Refine our choice of δ as $\delta' = \min\{\delta, \delta_h\}$.

We can choose an integer m such that $-\delta' < 1/\pi(m+1/2) < \delta'$ and $0 \le r_n + 1/\pi(m+1/2) \le 1$. Let $x = r_n + 1/\pi(m+1/2)$ and observe that $|r_n - x| < \delta' \le \delta'$.

Then by the reverse triangle inequality,

$$\begin{aligned}|F(r_n) - F(x)| \ge 3^{-n} &|g_n(r_n) - g_n(x)| \\ &- |h(r_n) - h(x)| \\ &- \sum_{k=n+2}^{\infty} 3^{-k} |g_k(r_n) - g_k(x)| .\end{aligned}$$

We simplify the first term with

$$\begin{aligned}3^{-n} |g_n(r_n) - g_n(x)| &= 3^{-n} \left| g(0) - g\left(\frac{1}{m(\pi+1/2)}\right)\right| \\ &= 3^{-n} \left| 0 - \sin\pi\left(m + \frac{1}{2}\right)\right| \\ &= 3^{-n}.\end{aligned}$$

By our choice of δ_h and thus δ', $|h(r_n) - h(x)| < 3^{-n-1}$. Finally, $|g_k|$ has a bound at 1, so

$$\sum_{k=n+2}^{\infty} 3^{-k} |g_k(r_n) - g_k(x)| \le 2 \sum_{k=n+2}^{\infty} 3^{-k} = 3^{-n-1}.$$

Combining these bounds,

$$|F(r_n) - F(x)| > 3^{-n} - 3^{-n-1} - 3^{-n-1} = 3^{-n-1},$$

which is a constant not dependent on δ.

Non-monotonicity Choose arbitrary $0 \le a < b \le 1$. We will find $a < y < y' < b$ so that $F(y) < F(y')$ and also $a < z < z' < b$ with $F(z) > F(z')$.

Since $\{r_k\}_{k=1}^{\infty}$ is dense in $[0,1]$, $(a,b) \cap \{r_k\}_{k=1}^{\infty}$ is nonempty. Let n be the index of the first element of $\{r_k\}_{k=1}^{\infty}$ contained in (a,b); that is, $r_n \in (a,b)$ but $r_k \notin (a,b)$ for $k = 1, \ldots, n-1$.

Set $y = z = r_n$. Define $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \sum_{k=1}^{n-1} 3^{-k} g_k(x) + 3^{-n-1} g_{n+1}(x).$$

We have that h is continuous at r_n as a sum of continuous functions. Then there exists $\delta > 0$ such that $|x - r_n| < \delta$ implies $|h(x) - h(r_n)| < 3^{-n-1}$ and δ is small enough such that $(r_n - \delta, r_n + \delta) \subseteq (a,b)$. Let $y' = r_n +$

$1/(2k_y+1/2)\pi$, where $k_y > 0$ is an integer such that $1/(2k_y+1/2)\pi < \delta$. Similarly, let $z' = r_n + 1/(2k_z + 3/2)\pi$, for integer $k_z > 0$ such that $1/(2k_z+3/2)\pi < \delta$.

Then

$$\begin{aligned} F(y) - F(y') = 3^{-n} g_n(r_n) - 3^{-n} g_n\left(r_n + \frac{1}{\left(2k_y + \frac{1}{2}\right)\pi}\right) \\ + h(y) - h(y') \\ + \sum_{k=n+2}^{\infty} 3^{-k} g_k(y) - \sum_{k=n+2}^{\infty} 3^{-k} g_k(y'). \end{aligned}$$

The first term can be simplified as

$$\begin{aligned} &3^{-n} g_n(r_n) - 3^{-n} g_n\left(r_n + \frac{1}{\left(2k_y + \frac{1}{2}\right)\pi}\right) \\ &= 3^{-n}\sin 0 - 3^{-n}\sin\left(2k_y + \frac{1}{2}\right)\pi \\ &= -3^{-n} \end{aligned}$$

The second term can be bounded with $|h(y) - h(y')| < 3^{-n-1}$ by our choice of δ and hence y'. We bound the last term as

$$\begin{aligned} &\left|\sum_{k=n+2}^{\infty} 3^{-k} g_k(y) - \sum_{k=n+2}^{\infty} 3^{-k} g_k(y')\right| \\ &\leq \sum_{k=n+2}^{\infty} 3^{-k}\left|g_k(y)\right| \sum_{k=n+2}^{\infty} 3^{-k}\left|g_k(y')\right| \\ &\leq 2\sum_{k=n+2}^{\infty} 3^{-k} \\ &= 3^{-n-1} \end{aligned}$$

Lastly,

$$\begin{aligned} F(y) - F(y') \leq 3^{-n} g_n(r_n) - 3^{-n} g_n\left(r_n + \frac{1}{\left(2k_y + \frac{1}{2}\right)\pi}\right) \\ + |h(y) - h(y')| \\ + \left|\sum_{k=n+2}^{\infty} 3^{-k} g_k(y) - \sum_{k=n+2}^{\infty} 3^{-k} g_k(y')\right| \\ < -3^{-n} + 3^{-n-1} + 3^{-n-1} = -3^{-n-1}, \end{aligned}$$

so $F(y) < F(y')$.

Similarly, for z, the first term simplifies to

$$3^{-n}g_n(r_n) - 3^{-n}g_n\left(r_n + \frac{1}{\left(2k_z + \frac{3}{2}\right)\pi}\right) = 3^{-n},$$

the second term has a bound of $|h(z) - h(z')| < 3^{-n-1}$, and the last term has can be bounded with

$$\left|\sum_{k=n+2}^{\infty} 3^{-k}g_k(z) - \sum_{k=n+2}^{\infty} 3^{-k}g_k(z')\right| = 3^{-n-1}.$$

These combine to

$$\begin{aligned} F(z) - F(z') \geq 3^{-n}g_n(r_n) - 3^{-n}g_n\left(r_n + \frac{1}{\left(2k_z + \frac{1}{2}\right)\pi}\right) \\ - |h(z) - h(z')| \\ - \left|\sum_{k=n+2}^{\infty} 3^{-k}g_k(z) - \sum_{k=n+2}^{\infty} 3^{-k}g_k(z')\right| \\ > 3^{-n} - 3^{-n-1} - 3^{-n-1} = 3^{-n-1}, \end{aligned}$$

so $F(z) > F(z')$. □

(c) We show that F is integrable on every interval in $\mathbb{R}$ but is discontinuous whenever $x = m/2n$ for odd $m \in \mathbb{Z}$ and nonzero $n \in \mathbb{Z}$.

Integrability Since F is periodic, it suffices to show that it is integrable on $[-1/2, 1/2]$. Define a sequence of partitions $P_1, P_2, \ldots$ by

$$P_n = \left\{-1/2, -1/2 + \frac{1}{n!}, \ldots, 1/2 - \frac{1}{n!}, 1/2\right\} \subset [-1/2, 1/2].$$

Let $F_n(x) = \sum_{k=1}^{n}(kx)/k^2$ denote the partial sums. Then

$$\begin{aligned} \mathcal{U}(P_n, F) - \mathcal{L}(P_n, F) \leq [\mathcal{U}(P_n, F_n) - \mathcal{L}(P_n, F_n)] \\ + [\mathcal{U}(P_n, F - F_n) - \mathcal{L}(P_n, F - F_n)]. \end{aligned}$$

We can easily bound

$$\mathcal{U}(P_n, F - F_n) - \mathcal{L}(P_n, F - F_n) \leq 2\sum_{k=n+1}^{\infty}\left|\frac{(kx)}{k^2}\right| \leq \sum_{k=n+1}^{\infty}\frac{1}{k^2},$$

so $\mathcal{U}(P_n, F - F_n) - \mathcal{L}(P_n, F - F_n) \to 0$ as $n \to \infty$.

To bound $\mathcal{U}(P_n, F_n) - \mathcal{L}(P_n, F_n)$, observe that each $x \mapsto (kx)/k^2$ is piecewise affine with breaks at $\{-1/2, -1/2 + 1/k, \ldots, 1/2 - 1/k, 1/2\} \subseteq P_n$

when $n \geq k$. Thus F_n is affine within the interior of each interval in the partitioning by P_n, with a gradient of $\sum_{k=1}^{n} k/k^2 = H_n$, the n^{th} harmonic number. Each interval has width $1/n!$, yielding a height of $H_n/n!$ and an area of $H_n/(n!)^2$. Finally, since there are $n!$ intervals in our partitioning, $\mathcal{U}(P_n, F_n) - \mathcal{L}(P_n, F_n) = H_n/n!$, which tends to 0 as $n \to \infty$.[3]

Hence, $\mathcal{U}(P_n, F) - \mathcal{L}(P_n, F) \to 0$ as $n \to \infty$ and F is integrable.

Discontinuity Let $x = m/2n$ for $m, n \in \mathbb{Z}$ with m odd and $n \neq 0$. We show that the right limit $\lim_{t \to x^+} F(t)$ exists and does not equal $F(x)$.

We can bound

$$F(t) = \sum_{k=1}^{\infty} \frac{(kt)}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2},,$$

so the series converges uniformly in t. We may thus swap the limits and

$$\lim_{t \to x^+} F(t) = \sum_{k=1}^{\infty} \lim_{t \to x^+} \frac{(kt)}{k^2}.$$

For each k, $\lim_{t \to x^+} (kt)/k^2$ equals either $(xt)/k^2$ or $1/2k^2$. If every term equals the former, then $\lim_{t \to x^+} F(t) = F(x)$, and if $\lim_{t \to x^+} (kt)/k^2 = 1/2k^2$ for at least one k, then $F(x) > \lim_{t \to x^+} F(t)$.

Consider $k = n$. We can write $1/2 - m/2n = (n - m)/2n$. Since n and m are odd $n - m$ is even, and $1/2 - x$ is an integer multiple of $1/n$. Thus $\lim_{t \to x^+} (nt)/n^2 = 1/2n^2$ and $F(x) > \lim_{t \to x^+} F(t)$, so F is discontinuous at x. $\square$

[3]If we insist on following exactly the definition of $\mathcal{L}$ and $\mathcal{U}$ in the appendix, then we need to formally account for the discontinuities at interval endpoints. There is one jump of length 1, two of length 1/4, three of length 1/9, and so on, potentially overlapping. A jump of length ℓ may increase the height of a bounding box by at most ℓ. Since each bounding box has width $1/n!$, this pedantic threatment of jump points increases $\mathcal{U}(P_n, F_n) - \mathcal{L}(P_n, F_n)$ by at most $(\sum_{k=1}^{n} k/k^2)/n! = H_n/n!$, which tends to 0 as $n \to \infty$.