(a) *Proof.* Consider the function $f : [0, 2\pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq \pi \\ 1 & \text{if } x = \pi. \end{cases}$$

Then $|f(x)|^2 = f(x)$. We integrate it by considering the sequence of partitions $\{P_n\}_{n=1}^{\infty}$ with $P_n = \{0, \pi - 1/n, \pi + 1/n, 2\pi\}$. The lower sum is $L_n = 0$ and the upper sum $U_n = 2/n$ converges to 0 as $n \to \infty$. Hence, $f \in \mathcal{R}$ and $\|f\| = 0$. $\square$

(b) *Proof.* We argue by contrapositive. Consider an integrable function $f : [0, 2\pi] \to \mathbb{R}$ that is continuous at some $x_0 \in [0, 2\pi]$ with $f(x_0) \neq 0$.

Since composition with a continuous functions preserves continuity, $g(x) := |f(x)|^2 \geq 0$ is also continuous at x_0 and $g(x_0) > 0$. Let $\epsilon = g(x_0)/2 > 0$. There exists some $\delta > 0$ such that $g(x_0) - \epsilon < g(x)$ for all $x_0 - \delta < x < x_0 + \delta$. Hence, we've established a lower bound of $g(x_0) - \epsilon = g(x_0)/2 > 0$ for g in the interval $(x_0 - \delta, x_0 + \delta)$. Then

$$\|f\| = \int_0^{2\pi} g(x)\ dx \geq \int_{x_0-\delta}^{x_0+\delta} g(x)\ dx \geq \delta g(x_0) > 0.$$

$\square$

(c) *Proof.* We argue by contrapositive. Consider an integrable function $f : [0, 2\pi] \to \mathbb{R}$ such that $\|f\| > 0$.

Let $g(x) = |f(x)|^2$ and observe that $2\pi \|f\|^2 = \int_0^{2\pi} g(x)\ dx > 0$. There exists a partition $P = \{0 = x_0 < \cdots < x_N = 2\pi\}$ such that the lower sum

$$L = \sum_{n=1}^{N} \Big[\inf_{x \in [x_{n-1}, x_n]} g(x) \Big] (x_n - x_{n-1})$$

is positive since the lower sum converges to $\|f\| > 0$ with sufficiently fine partitions. Every element of the sum is non-negative, so there exists n such that $\inf_{x \in [x_{n-1}, x_n]} g(x) > 0$.

Since f is integrable, the set of its discontinuities has measure 0. The interval $[x_{n-1}, x_n]$ has measure $x_n - x_{n-1} > 0$, so f cannot be discontinuous in that entire interval. There exists $x_{n-1} \leq y \leq x_n$ such that f is continuous at y.

By our bound on g,

$$g(y) \geq \inf_{x \in [x_{n-1}, x_n]} g(x) > 0.$$

Then $f(y) \neq 0$, so f does not vanish at all points of continuity. $\square$