Proof. For all n, m with $n < m$, observe that

$$f_m(\theta) - f_n(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq 1/m \\ \log(1/\theta) & \text{for } 1/m < \theta \leq 1/n \\ 0 & \text{for } 1/n < \theta \leq 2\pi. \end{cases}$$

Then

$$\|f_m - f_n\|^2 = \frac{1}{2\pi} \int_{1/m}^{1/n} (\log(1/\theta))^2 \, d\theta.$$

We solve the integral by observing that

$$\frac{d}{d\theta}\Big(\theta(\log(1/\theta))^2 - 2\theta \log(1/\theta) + 2\theta\Big) = (\log(1/\theta))^2,$$

so

$$\begin{aligned} \|f_m - f_n\|^2 &= \frac{(\log n)^2 + 2\log n + 2}{n} - \frac{(\log m)^2 + 2\log m + 2}{m} \\ &< \frac{(\log n)^2 + 2\log n + 2}{n}. \end{aligned}$$

The denominator grows faster than the numerator, so

$$\lim_{n\to\infty} \frac{(\log n)^2 + 2\log n + 2}{n} = 0.$$

Recalling that $m > n$, this implies that $\{f_n\}_{n=1}^{\infty}$ is Cauchy. $\square$