(a) $|z|$ is the distance from z to the origin.

(b)

$$\begin{aligned}
&|z| \stackrel{\text{def}}{=} \left(x^2 + y^2\right)^{1/2} = 0 \\
\iff\ & x^2 + y^2 = 0 \\
\iff\ & x^2 = 0 \text{ and } y^2 = 0 \qquad\qquad (x, y \in \mathbb{R}, \text{ so } x^2, y^2 \geq 0) \\
\iff\ & x = 0 \text{ and } y = 0 \\
\iff\ & z \stackrel{\text{def}}{=} x + iy = 0
\end{aligned}$$

$\square$

(c) We have $\lambda z = (\lambda x) + i(\lambda y)$ for some $\lambda \in \mathbb{R}$. Substituting into the definition of the modulus,

$$\begin{aligned}
|\lambda z| &= \left((\lambda x)^2 + (\lambda y)^2\right)^{1/2} \\
&= \left(\lambda^2\right)^{1/2} \left(x^2 + y^2\right)^{1/2} \\
&= |\lambda|\, |z| \, .
\end{aligned}$$

$\square$

(d) Let $z_1 \stackrel{\text{def}}{=} x_1 + iy_1$ and $z_2 \stackrel{\text{def}}{=} x_2 + iy_2$ for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

We first show that $|z_1 z_2| = |z_1|\, |z_2|$. We have

$$\begin{aligned}
z_1 z_2 &= x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2 y_1 y_2 \\
&= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).
\end{aligned}$$

Substituting into the definition of the modulus,

$$\begin{aligned}
|z_1 z_2|^2 &= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 \\
&= \left(x_1^2 x_2^2 - 2 x_1 x_2 y_1 y_2 + y_1^2 y_2^2\right) + \left(x_1^2 y_2^2 + 2 x_1 x_2 y_1 y_2 + x_2^2 y_1^2\right) \\
&= x_1^2 x_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2 \\
&= \left(x_1^2 + y_1^2\right) \left(x_2^2 + y_2^2\right) \\
&= |z_1|^2\, |z_2|^2 \, .
\end{aligned}$$

Taking the square root of both sides concludes the proof.

We now show that $|z_1 + z_2| \leq |z_1| + |z_2|$. We have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

By algebra,

$$\begin{aligned}
& 0 \leq (x_1 y_2 - x_2 y_1)^2 \\
\iff\ & 0 \leq x_1^2 y_2^2 - 2 x_1 x_2 y_1 y_2 + x_2^2 y_1^2
\end{aligned}$$

$$
\begin{aligned}
&\iff 2x_1x_2y_1y_2 \leq x_1^2y_2^2 + x_2^2y_1^2 \\
&\iff x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \leq x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2 \\
&\iff (x_1x_2 + y_1y_2)^2 \leq \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) \\
&\implies x_1x_2 + y_1y_2 \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \qquad \text{(since RHS} \geq 0\text{)} \\
&\iff x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 \\
&\qquad\quad \leq x_1^2 + y_1^2 + 2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} + x_2^2 + y_2^2 \\
&\iff (x_1 + x_2)^2 + (y_1 + y_2)^2 \leq \left(\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}\right)^2 \\
&\iff \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \\
&\iff |z_1 + z_2| \leq |z_1| + |z_2| \,.
\end{aligned}
$$

□

(e) Observe that

$$
\begin{aligned}
\frac{1}{z} &= \frac{1}{x + iy} \\
&= \frac{x - iy}{(x + iy)(x - iy)} \\
&= \frac{x - iy}{x^2 + y^2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|\frac{1}{z}\right| &= \left|\frac{x - iy}{x^2 + y^2}\right| \\
&= \frac{1}{x^2 + y^2}\,|x - iy| \qquad\qquad \text{(by (c))}
\end{aligned}
$$

Observing that the definition of $|z|$ depends only on x^2 and y^2, $|x - iy| = |x + iy| = |z|$. Observe also that $|z|^2 = x^2 + y^2$ by squaring both sides of its definition. Then

$$
\left|\frac{1}{z}\right| = \frac{1}{|z|^2}\,|z| = \frac{1}{|z|}.
$$

□