(a) Choose an arbitrary $\epsilon > 0$. We will show that there exists $0 \le t < 1$ such that for all $t < r < 1$,

$$\left| A_r(f)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| < C\epsilon,$$

where $C > 0$ is a constant.

We follow the proof of theorem 4.1 in the book.

We can write

$$\begin{aligned}
& A_r(f)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \\
&= (f * P_r)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(\theta - t) dt - \frac{f(\theta^+) + f(\theta^-)}{2} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(\theta - t) dt \qquad (1) \\
&\quad - \frac{f(\theta^+)}{2\pi} \int_{-\pi}^{0} P_r(t) dt - \frac{f(\theta^-)}{2\pi} \int_{0}^{\pi} P_r(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{0} P_r(t) \left[f(\theta - t) - f(\theta^+) \right] dt \\
&\quad + \frac{1}{2\pi} \int_{0}^{\pi} P_r(t) \left[f(\theta - t) - f(\theta^-) \right] dt.
\end{aligned}$$

In eq. (1) we use the fact that $\frac{1}{2\pi} \int_{-\pi}^{0} P_r(t) dt = \frac{1}{2\pi} \int_{0}^{\pi} P_r(t) dt = \frac{1}{2}$; this is because $P_r(t)$ is even in t and $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) dt = 1$.

We tackle the first term. Since f has a jump discontinuity at θ, there exists δ, so that $-\delta < t < 0$ implies $|f(\theta - t) - f(\theta^+)| < \epsilon$ and $0 < t < \delta$ implies $|f(\theta - t) - f(\theta^-)| < \epsilon$. Then

$$\begin{aligned}
& \left| \frac{1}{2\pi} \int_{-\pi}^{0} P_r(t) \left[f(\theta - t) - f(\theta^+) \right] dt \right| \\
&\le \frac{1}{2\pi} \int_{-\pi}^{-\delta} |P_r(t)| \left| f(\theta - t) - f(\theta^+) \right| dt \\
&\quad + \frac{1}{2\pi} \int_{-\delta}^{0} |P_r(t)| \left| f(\theta - t) - f(\theta^+) \right| dt \\
&< \frac{B}{2\pi} \int_{-\pi}^{-\delta} |P_r(t)| \, dt + \frac{\epsilon}{2\pi} \int_{-\delta}^{0} |P_r(t)| \, dt,
\end{aligned}$$

where $B > 0$ is a bound for f, which exists since f is integrable. Applying similar steps to the second term, we find

$$\left| \frac{1}{2\pi} \int_{0}^{\pi} P_r(t) \left[f(\theta - t) - f(\theta^-) \right] dt \right|$$

$$< \frac{\epsilon}{2\pi}\int_0^\delta |P_r(t)|\,dt + \frac{B}{2\pi}\int_\delta^\pi |P_r(t)|\,dt$$

Summing these,

$$\begin{aligned}
&\left|(f*P_r)(\theta) - \frac{f(\theta^+)+f(\theta^-)}{2}\right| \\
&\le \left|\frac{1}{2\pi}\int_{-\pi}^0 P_r(t)\left[f(\theta-t)-f(\theta^+)\right]dt\right| \\
&\quad + \left|\frac{1}{2\pi}\int_0^\pi P_r(t)\left[f(\theta-t)-f(\theta^-)\right]dt\right| \\
&< \frac{\epsilon}{2\pi}\int_{-\delta}^{\delta} |P_r(t)|\,dt + \frac{B}{2\pi}\int_{\delta\le|t|\le\pi} |P_r(t)|\,dt \\
&\le \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |P_r(t)|\,dt + \frac{B}{2\pi}\int_{\delta\le|t|\le\pi} |P_r(t)|\,dt.
\end{aligned}$$

Since P_r is a good kernel, there exists $M > 0$ such that $\int_{-\pi}^{\pi}|P_r(\theta)|\,d\theta \le M$ for all $0 \le r < 1$. Hence, the first term is bounded by $\epsilon M/2\pi$. Also by the definition of a good kernel, we have that $\int_{\delta\le|\theta|\le\pi}|P_r(\theta)|\,d\theta \to 0$ as $r \to 1$. Then we can find $0 \le t < \pi$ such that for all $t < r < \pi$, $\int_{\delta\le|\theta|\le\pi}|P_r(\theta)|\,d\theta < \epsilon$.

Then for all $r \in (t, \pi)$,

$$\left|(f*P_r)(\theta) - \frac{f(\theta^+)+f(\theta^-)}{2}\right| < \frac{\epsilon M}{2\pi} + \frac{\epsilon B}{2\pi} = \frac{M+B}{2\pi}\epsilon,$$

and $\frac{M+B}{2\pi}$ is a constant. □

(b) The N^{th} Cesàro mean of the Fourier series at θ equals $(f*F_N)(\theta)$, where F_N is the Fejér kernel. It suffices to show that $(f*F_N)(\theta) \to (f(\theta^+) + f(\theta^-))/2$ as $N \to \infty$. We follow the proof in (a).

Choose an arbitrary $\epsilon > 0$. We can write

$$\begin{aligned}
&(f*F_N)(\theta) - \frac{f(\theta^+)+f(\theta^-)}{2} \\
&= \frac{1}{2\pi}\int_{-\pi}^0 F_N(t)\left[f(\theta-t)-f(\theta^+)\right]dt \\
&\quad + \frac{1}{2\pi}\int_0^\pi F_N(t)\left[f(\theta-t)-f(\theta^-)\right]dt,
\end{aligned}$$

because $\frac{1}{2\pi}\int_{-\pi}^0 F_N(t)dt = \frac{1}{2\pi}\int_0^\pi F_N(t)dt = \frac{1}{2}$ since F_N is an even good kernel.

Pick a δ such that $-\delta < t < 0$ implies $|f(\theta - t) - f(\theta^+)| < \epsilon$ and $0 < t < \delta$ implies $|f(\theta - t) - f(\theta^-)| < \epsilon$. Then

$$\begin{aligned} &\left| \frac{1}{2\pi} \int_{-\pi}^{0} F_N \left[f(\theta - t) - f(\theta^+) \right] dt \right| \\ &< \frac{B}{2\pi} \int_{-\pi}^{-\delta} |F_N(t)| \, dt + \frac{\epsilon}{2\pi} \int_{-\delta}^{0} |F_N(t)| \, dt \end{aligned}$$

and

$$\begin{aligned} &\left| \frac{1}{2\pi} \int_{0}^{\pi} F_N \left[f(\theta - t) - f(\theta^+) \right] dt \right| \\ &< \frac{\epsilon}{2\pi} \int_{0}^{\delta} |F_N(t)| \, dt + \frac{B}{2\pi} \int_{\delta}^{\pi} |F_N(t)| \, dt, \end{aligned}$$

where $B > 0$ is a bound for f.

Summing,

$$\begin{aligned} &\left| (f * F_N)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| \\ &\leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |F_N(t)| \, dt + \frac{B}{2\pi} \int_{\delta \leq |t| \leq \pi} |F_N(t)| \, dt. \end{aligned}$$

By the properties of good kernels, $\int_{-\pi}^{\pi} |F_N(t)| \, dt$ is bounded by $M > 0$, and for all N big enough, $\int_{\delta \leq |t| \leq \pi} |F_N(t)| \, dt < \epsilon$.

Then for all N big enough,

$$\left| (f * F_N)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2} \right| < \frac{\epsilon M}{2\pi} + \frac{\epsilon B}{2\pi} = \frac{M + B}{2\pi} \epsilon,$$

where $\frac{M+B}{2\pi}$ is a constant. $\square$