Proof. Let u_k be such that $u_k(x,0) = A_k \sin kx$, $u_k(x,1) = B_k \sin kx$, $u_k(0,y) = 0$, $u_k(1,y) = 0$, and $\triangle u = 0$.

We want to solve for u_k. Using separation of variables, we write $u_k(x,y) = F(x)G(y)$. The Laplacian becomes

$$\begin{aligned}\triangle u_k &= \frac{\partial^2 F(x)G(y)}{\partial x^2} + \frac{\partial^2 F(x)G(y)}{\partial y^2} \\ &= F''(x)G(y) + F(x)G''(y) = 0.\end{aligned}$$

Thus, we look for solutions of the form

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.$$

Since those sides depend on different variables, they must be equal to some constant, which we will call λ. Then

$$F''(x) - \lambda F(x) = 0 \text{ and } G''(y) + \lambda G(y) = 0.$$

By our definition, $u_k(x,0) = F(x)G(0) = A_k \sin kx$ and $u(x,1) = F(x)G(1) = B_k \sin kx$. Then $F(x) = a \sin kx$ for some a and $\lambda = -k^2$. By the lemma, $G(y) = \alpha \cosh ky - \beta \sinh ky$ for some $\alpha, \beta \in \mathbb{R}$.

When $y = 0$, we have $\alpha F(x) = A_k \sin kx$, so

$$\alpha F(x) \cosh ky = A_k \sin kx \cosh ky.$$

Similarly, when $y = 1$, $\alpha F(x) \cosh k - \beta F(x) \sinh k = B_k \sin kx$, implying that

$$\beta F(x) \sinh ky = \frac{A_k \cosh k - B_k}{\sinh k} \sin kx \sinh ky.$$

Simplifying,

$$\begin{aligned}F(x)G(y) &= \left(A_k \cosh ky - \frac{A_k \cosh k - B_k}{\sinh k} \sinh ky \right) \sin kx \\ &= \left(A_k \frac{\sinh k \cosh ky - \sinh ky \cosh k}{\sinh k} + B_k \frac{\sinh ky}{\sinh k} \right) \sin kx \\ &= \left(A_k \frac{\sinh k(1-y)}{\sinh k} + B_k \frac{\sinh ky}{\sinh k} \right) \sin kx,\end{aligned}$$

where in the last line we use

$$\begin{aligned}&4 \sinh k \cosh ky - 4 \sinh ky \cosh k \\ &= \left(e^k - e^{-k}\right)\left(e^{ky} + e^{-ky}\right) - \left(e^{ky} - e^{-ky}\right)\left(e^k + e^{-k}\right) \\ &= 2e^{k-ky} - 2e^{ky-y} = 4 \sinh k(1-y).\end{aligned}$$

Define

$$u = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \left(A_k \frac{\sinh k(1-y)}{\sinh k} + B_k \frac{\sinh ky}{\sinh k} \right) \sin kx.$$

Define also

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \text{ and } f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx.$$

Then $u(x,0) = \sum_{k=1}^{\infty} u_k(x,0) = f_0(x)$, and similarly $u(x,1) = f_1(x)$, $u(0,y) = 0$, and $u(1,y) = 0$. Finally, by the linearity of the Laplacian, $\triangle u = 0$.

$\square$

Lemma. *Let f be a twice continuously differentiable function on $\mathbb{R}$ such that $f''(t) - c^2 f(t) = 0$. Then all solutions for f have the form*

$$f(t) = a \cosh ct - b \sinh ct.$$

Proof. Let $g(t) = f(t)\cosh ct - c^{-1} f'(t) \sinh ct$ and $h(t) = f(t)\sinh ct - c^{-1} f'(t)\cosh ct$. Observe that these once differentiable. Differentiating,

$$\begin{aligned}
g'(t) &= f'(t)\cosh ct + cf(t)\sinh ct - c^{-1} f''(t)\sinh ct - f'(t)\cosh ct \\
&= cf(t)\sinh ct - c^{-1} f''(t)\sinh ct \\
&= cf(t)\sin ct - cf(t)\sin ct = 0, \\
h'(t) &= f'(t)\sinh ct + cf(t)\cosh ct - c^{-1} f''(t)\cosh ct - f'(t)\sinh ct \\
&= cf(t)\cosh ct - c^{-1} f''(t)\cosh ct \\
&= cf(t)\cosh ct - cf(t)\cosh ct = 0.
\end{aligned}$$

Thus g and h are constant. Let a and b be constants such that $g(t) = a$ and $h(t) = b$. Then

$$\begin{aligned}
f(t)\cosh^2 ct - c^{-1} f'(t)\sinh ct \cosh ct &= a \cosh ct, \\
f(t)\sinh^2 ct - c^{-1} f'(t)\sinh ct \cosh ct &= b \sinh ct.
\end{aligned}$$

Subtracting,

$$f(t)\cosh^2 ct - f(t)\sinh^2 ct = a\cosh ct - b\sinh ct,$$

which simplifies to $f(t) = a\cosh ct - b\sinh ct$. $\square$