(a) Assume $\{w_n\}_{n=1}^{\infty}$ converges. Let w and w' be its limits. We show that they are equal.

Observe that for all n,

$$\begin{aligned}|(w_n - w) - (w_n - w')| &\leq |w_n - w| + |-(w_n - w')| \\ &= |w_n - w| + |w_n - w'|,\end{aligned}$$

where the first line is by the triangle inequality.

Since

$$\lim_{n\to\infty} |w_n - w| = 0 \text{ and } \lim_{n\to\infty} |w_n - w'| = 0$$

by assumption,

$$\lim_{n\to\infty} |(w_n - w) - (w_n - w')| = 0$$

by the squeeze theorem (here, the fact that the modulus is non-negative places a lower bound on the limit).

Observe that $(w_n - w) - (w_n - w') = w' - w$, so this sequence is constant. Since its limit is 0, we have that $w' - w = 0$. □

(b) Both directions use the observation that for all complex numbers $z = x+iy$ with $x, y \in \mathbb{R}$,

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2} \\ &\geq \sqrt{x^2} \\ &= |x|.\end{aligned} \tag{1}$$

A similar argument shows that $|z| \geq |y|$.

($\implies$) Let $\{w_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a convergent. We show that it is Cauchy.

Decompose $w = t + is$, $w_n = t_n + is_n$ for $t, s, t_n, s_n \in \mathbb{R}$. Dealing with the real and imaginary parts of $w_n - w$ separately we have

$$\lim_{n\to\infty} |t_n - t| \to 0 \text{ and } \lim_{n\to\infty} |s_n - s| \to 0$$

by (1) and the squeeze theorem.

Then $\{t_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ converge, so they are Cauchy.

Pick an arbitrary $\epsilon > 0$. Find N such that $|t_n - t_m| < \epsilon/2$ and $|s_n - s_m| < \epsilon/2$ whenever $n, m > N$. Then

$$\begin{aligned}|w_n - w_n| &= |(t_n - t_m) + i(s_n - s_m)| \\ &\leq |t_n - t_m| + |s_n - s_m| \\ &< \epsilon\end{aligned}$$

by the triangle inequality whenever $n, m > N$.

($\Longleftarrow$) Let $\{w_n\}_{n=1} \subset \mathbb{C}$ be Cauchy. We show that it is convergent.

Pick an arbitrary $\epsilon > 0$. Then there exists a positive integer N such that $|w_n - w_m| < \epsilon$ for all $n, m > N$. Decompose $w_n = t_n + is_n$ for $t_n, s_n \in \mathbb{R}$, and decompose w_m similarly. Then $|t_n - t_m| \leq |w_n - w_m| < \epsilon$ $|s_n - s_m| \leq |w_n - w_m| < \epsilon$ by (1). Thus $\{t_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ are Cauchy. It follows that they converge.

Let t and s be the limits of $\{t_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$, respectively. Define $w = t + is$.

We have $|w_n - w| \leq |t_n - t| + |s_n - s|$ by the triangle inequality. We also have $\lim_{n\to\infty} |t_n - t| = 0$ and $\lim_{n\to\infty} |s_n - s| = 0$, which implies $\lim_{n\to\infty}(|t_n - t| + |s_n - s|) = 0$. Then by the squeeze theorem

$$\lim_{n\to\infty} |w_n - w| = 0$$

and $\{w_n\}_{n=1} \subset \mathbb{C}$ converges. $\square$

(c) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative reals such that $\sum_{n=1}^{\infty} a_n$ converges. Let $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence satisfying $|z_n| < a_n$ for all n. We show that $\sum_{n=1}^{\infty} z_n$ converges.

Define $S_N = \sum_{n=1}^{N} z_n$. Our goal is to show that $\{S_N\}_{N=1}^{\infty}$ converges. By (b) it suffices to show that it is Cauchy.

Let $A_N = \sum_{n=1}^{N} a_n$. By assumption, the sequence formed by these partial sums converges, so it is Cauchy.

Pick an arbitrary $\epsilon > 0$.

Then there exists a positive integer M such that for all $N, N' > M$,

$$|A_N - A_{N'}| < \epsilon.$$

W.l.o.g, assume $N > N'$. Observe that

$$\begin{aligned}
&|A_N - A_{N'}| < \epsilon \\
&\Longleftrightarrow \left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{N'} a_n \right| < \epsilon \\
&\Longleftrightarrow \left| \sum_{n=N'+1}^{N} a_n \right| < \epsilon \\
&\Longleftrightarrow \sum_{n=N'+1}^{N} a_n < \epsilon && (a_n \geq 0\ \forall n) \\
&\Longrightarrow \sum_{n=N'+1}^{N} |z_n| < \epsilon
\end{aligned}$$

$$
\begin{aligned}
&\implies \left|\sum_{n=N'+1}^{N} z_n\right| < \epsilon && \text{(triangle ineq.)} \\
&\iff \left|\sum_{n=1}^{N} z_n - \sum_{n=1}^{N'} z_n\right| < \epsilon \\
&\iff |S_N - S_{N'}| < \epsilon.
\end{aligned}
$$

So $\{S_N\}_{n=1}^{\infty}$ is Cauchy, implying that $\sum_{n=1}^{\infty} z_n$ converges. $\square$