Lemma. *Let f be 2π-periodic and differentiable. Then*

$$\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx$$

for all integer n.

Proof. By parts,

$$\int_{-\pi}^{\pi} f(x)e^{-inx}dx = -\frac{1}{in}\left[f(x)e^{-inx}\right]_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx.$$

We observe that

$$\left[f(x)e^{-inx}\right]_{-\pi}^{\pi} = 0$$

since $f(x)e^{-inx}$ is 2π-periodic. $\square$

Applying the lemma k times, we get

$$\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{(in)^k}\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}dx.$$

Since $f \in C^k$, $f^{(k)}$ is continuous and hence bounded on $[-\pi, \pi]$. Let $0 \leq B$ be such that

$$\left|f^{(k)}\right| \leq B \quad \text{for } x \in [-\pi, \pi].$$

Note that B is independent of n.

Then

$$\begin{aligned}
\left|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right| &= |n|^{-k}\left|\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}dx\right| \\
&\leq |n|^{-k}\int_{-\pi}^{\pi}\left|f^{(k)}(x)e^{-inx}\right|dx \\
&\leq |n|^{-k}\int_{-\pi}^{\pi}\left|f^{(k)}(x)\right|dx \\
&\leq |n|^{-k}\int_{-\pi}^{\pi} B\,dx \\
&= 2\pi\,|n|^{-k}\,B
\end{aligned}$$

Hence,

$$\left|\hat{f}(n)\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right| \leq |n|^{-k}\,B$$

and $\hat{f}(n) = O(1/|n|^k)$. $\square$