We wish to prove that

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{1}$$

in polar coordinates and also

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2. \tag{2}$$

In both proofs, we'll use

$$\begin{aligned} \theta &= \operatorname{atan2}(y, x), & r &= \sqrt{x^2+y^2}, \\ x &= r\cos\theta, & y &= r\sin\theta, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}. \end{aligned}$$

(1) We are given that

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in Euclidean coordinates.

We want to show that

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates.

By the chain rule,

$$\begin{aligned} \triangle &= \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\right)^2 \\ &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ &\quad + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}. \end{aligned}$$

By the product rule,

$$\begin{aligned} \triangle = {} & \frac{\partial r}{\partial x}\frac{\partial^2 r}{\partial r \partial x}\frac{\partial}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial r}{\partial x}\frac{\partial^2}{\partial r^2} + \frac{\partial r}{\partial x}\frac{\partial^2 \theta}{\partial r \partial x}\frac{\partial}{\partial \theta} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial \theta}{\partial x}\frac{\partial^2}{\partial r \partial \theta} \\ & + \frac{\partial \theta}{\partial x}\frac{\partial^2 r}{\partial \theta \partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x}\frac{\partial^2}{\partial \theta \partial r} + \frac{\partial \theta}{\partial x}\frac{\partial^2 \theta}{\partial \theta \partial x}\frac{\partial}{\partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x}\frac{\partial^2}{\partial \theta^2} \end{aligned}$$

$$+\frac{\partial r}{\partial y}\frac{\partial^2 r}{\partial r\partial y}\frac{\partial}{\partial r}+\frac{\partial r}{\partial y}\frac{\partial r}{\partial y}\frac{\partial^2}{\partial r^2}+\frac{\partial r}{\partial y}\frac{\partial^2\theta}{\partial r\partial y}\frac{\partial}{\partial\theta}+\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}\frac{\partial^2}{\partial r\partial\theta}$$
$$+\frac{\partial\theta}{\partial y}\frac{\partial^2 r}{\partial\theta\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial r}{\partial y}\frac{\partial^2}{\partial\theta\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial\theta\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\theta}{\partial y}\frac{\partial\theta}{\partial y}\frac{\partial^2}{\partial\theta^2}.$$

We have

$$\frac{\partial^2 r}{\partial r\partial x}=\frac{\partial}{\partial r}\cos\theta=0,\qquad\qquad \frac{\partial^2 r}{\partial r\partial y}=\frac{\partial}{\partial r}\sin\theta=0.$$

Using this and collecting terms,

$$\begin{aligned}
\triangle = {} & \left(\frac{\partial\theta}{\partial x}\frac{\partial^2 r}{\partial\theta\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2 r}{\partial\theta\partial y}\right)\frac{\partial}{\partial r}\\
+ & \left(\frac{\partial r}{\partial x}\frac{\partial^2\theta}{\partial r\partial x}+\frac{\partial r}{\partial y}\frac{\partial^2\theta}{\partial r\partial y}+\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial\theta\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial\theta\partial y}\right)\frac{\partial}{\partial\theta}\\
+ & \left(\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right)\frac{\partial^2}{\partial r^2}\\
+ & \left(2\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x}+2\frac{\partial r}{\partial y}\frac{\partial\theta}{\partial y}\right)\frac{\partial^2}{\partial r\partial\theta}\\
+ & \left(\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right)\frac{\partial^2}{\partial\theta^2}.
\end{aligned}$$

We tackle each individually. First observe that

$$\begin{aligned}
\frac{\partial^2\theta}{\partial r\partial x}&=\frac{\sin\theta}{r^2}, & \frac{\partial^2\theta}{\partial r\partial y}&=-\frac{\cos\theta}{r^2},\\
\frac{\partial^2 r}{\partial\theta\partial x}&=-\sin\theta, & \frac{\partial^2 r}{\partial\theta\partial y}&=\cos\theta,\\
\frac{\partial^2\theta}{\partial\theta\partial x}&=-\frac{\cos\theta}{r}, & \frac{\partial^2\theta}{\partial\theta\partial y}&=-\frac{\sin\theta}{r}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial\theta}{\partial x}\frac{\partial^2 r}{\partial\theta\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2 r}{\partial\theta\partial y}&=\frac{\sin\theta}{r}\sin\theta+\frac{\cos\theta}{r}\cos\theta=\frac{1}{r},\\
\frac{\partial r}{\partial x}\frac{\partial^2\theta}{\partial r\partial x}+\frac{\partial r}{\partial y}\frac{\partial^2\theta}{\partial r\partial y}\\
+\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial\theta\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial\theta\partial y}&=\cos\theta\frac{\sin\theta}{r^2}-\sin\theta\frac{\cos\theta}{r^2}\\
&\quad+\frac{\sin\theta}{r}\frac{\cos\theta}{r}-\frac{\cos\theta}{r}\frac{\sin\theta}{r}=0,\\
\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2&=\cos^2\theta+\sin^2\theta=1,
\end{aligned}$$

$$
\begin{aligned}
2\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + 2\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} &= -2\cos\theta\frac{\sin\theta}{r} + 2\sin\theta\frac{\cos\theta}{r} = 0, \\
\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 &= \left(-\frac{\sin\theta}{r}\right)^2 + \left(\frac{\cos\theta}{r}\right)^2 = \frac{1}{r^2}.
\end{aligned}
$$

Substituting back,

$$
\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.
$$

□

We now show that

$$
\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.
$$

We have

$$
\begin{aligned}
&\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \\
&= \left(\frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial \theta}\right)^2 && \text{(chain rule)} \\
&= \left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)^2 + \left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)^2 && \text{(subst.)} \\
&= \left(\sin^2\theta + \cos^2\theta\right)\left(\frac{\partial u}{\partial r}\right)^2 \\
&\qquad + \left(\frac{2\cos\theta\sin\theta}{r} - \frac{2\cos\theta\sin\theta}{r}\right)\frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta} \\
&\qquad + \left(\frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2}\right)\left(\frac{\partial u}{\partial \theta}\right)^2 \\
&= \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.
\end{aligned}
$$

□