The Poisson kernel can be expressed as the Fourier series

$$P_r(\theta) = \sum_{-\infty}^{n=\infty} r^{|n|} e^{in\theta}.$$

We define

$$u(r,\theta) = \frac{\partial P_r(\theta)}{\partial \theta}.$$

We can rearrange

$$P_r(\theta) = 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} = 1 + \frac{re^{i\theta}}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}}.$$

Differentiating,

$$u(r,\theta) = \frac{\partial P_r(\theta)}{\partial \theta} = \frac{ire^{i\theta}}{\left(1-re^{i\theta}\right)^2} - \frac{ire^{-i\theta}}{\left(1-re^{-i\theta}\right)^2}.$$

(i) We want to show that $\Delta u = 0$ in the unit disc.

We know that

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We first find $\partial u/\partial r$. Differentiating w.r.t. r,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= ie^{i\theta}\frac{(1-re^{i\theta})^2 + 2re^{i\theta}(1-re^{i\theta})}{(1-re^{i\theta})^4} \\
&\quad - ie^{-i\theta}\frac{(1-re^{-i\theta})^2 + 2re^{-i\theta}(1-re^{-i\theta})}{(1-re^{-i\theta})^4} \\
&= ie^{i\theta}\frac{1+re^{i\theta}}{(1-re^{i\theta})^3} - ie^{-i\theta}\frac{1+re^{-i\theta}}{(1-re^{-i\theta})^3}.
\end{aligned}$$

Differentiating again to get $\partial^2 u/\partial r^2$,

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= ie^{i\theta}\frac{(1-re^{i\theta})^3 e^{i\theta} + 3e^{i\theta}(1-re^{i\theta})^2(1+re^{i\theta})}{(1-re^{i\theta})^6} \\
&\quad - ie^{-i\theta}\frac{(1-re^{-i\theta})^3 e^{-i\theta} + 3e^{-i\theta}(1-re^{-i\theta})^2(1+re^{-i\theta})}{(1-re^{-i\theta})^6} \\
&= 2ie^{2i\theta}\frac{2+re^{i\theta}}{(1-re^{i\theta})^4} - 2ie^{-2i\theta}\frac{2+re^{-i\theta}}{(1-re^{-i\theta})^4}.
\end{aligned}$$

We now find $\partial^2 u/\partial \theta^2$. Differentiating twice,

$$\frac{\partial u}{\partial \theta} = ir\frac{ie^{i\theta}(1-re^{i\theta})^2 + 2ire^{2i\theta}(1-re^{i\theta})}{(1-re^{i\theta})^4}$$

$$
\begin{aligned}
&- ir\frac{-ie^{-i\theta}(1-re^{-i\theta})^2 - 2e^{-i\theta}ire^{-i\theta}(1-re^{-i\theta})}{(1-re^{-i\theta})^4} \\
&= -r\frac{e^{i\theta}+re^{2i\theta}}{(1-re^{i\theta})^3} - r\frac{e^{-i\theta}+re^{-2i\theta}}{(1-re^{-i\theta})^3}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial \theta^2} \\
&= -ir\frac{(e^{i\theta}+2re^{2i\theta})(1-re^{i\theta}) + 3re^{i\theta}(e^{i\theta}+re^{2i\theta})}{(1-re^{i\theta})^4} \\
&\quad + ir\frac{(e^{-i\theta}+2re^{-2i\theta})(1-re^{-i\theta}) + 3re^{-i\theta}(e^{-i\theta}+re^{-2i\theta})}{(1-re^{-i\theta})^4} \\
&= -ir\frac{e^{i\theta}+4re^{2i\theta}+r^2e^{3i\theta}}{(1-re^{i\theta})^4} + ir\frac{e^{-i\theta}+4re^{-2i\theta}+r^2e^{-3i\theta}}{(1-re^{-i\theta})^4}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\Delta u &= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= ie^{i\theta}\frac{1+re^{i\theta}}{r(1-re^{i\theta})^3} - ie^{-i\theta}\frac{1+re^{-i\theta}}{r(1-re^{-i\theta})^3} \\
&\quad + 2ie^{2i\theta}\frac{2+re^{i\theta}}{(1-re^{i\theta})^4} - 2ie^{-2i\theta}\frac{2+re^{-i\theta}}{(1-re^{-i\theta})^4} \\
&\quad - i\frac{e^{i\theta}+4re^{2i\theta}+r^2e^{3i\theta}}{r(1-re^{i\theta})^4} + i\frac{e^{-i\theta}+4re^{-2i\theta}+r^2e^{-3i\theta}}{r(1-re^{-i\theta})^4} \\
&= \frac{ie^{i\theta}}{r(1-re^{i\theta})^4}\Big[(1+re^{i\theta})(1-re^{i\theta}) \\
&\qquad + 2re^{i\theta}(2+re^{i\theta}) - (1+4re^{i\theta}+r^2e^{2i\theta})\Big] \\
&\quad - \frac{ie^{-i\theta}}{r(1-re^{-i\theta})^4}\Big[(1+re^{-i\theta})(1-re^{-i\theta}) \\
&\qquad + 2re^{-i\theta}(2+re^{-i\theta}) - (1+4re^{-i\theta}+r^2e^{-2i\theta})\Big] \\
&= \frac{ie^{i\theta}}{r(1-re^{i\theta})^4}\Big[1-r^2e^{2i\theta}+4re^{i\theta}+2r^2e^{2i\theta}-1-4re^{i\theta}-r^2e^{2i\theta})\Big] \\
&\quad - \frac{ie^{-i\theta}}{r(1-re^{-i\theta})^4}\Big[1-r^2e^{-2i\theta}+4re^{-i\theta}+2r^2e^{-2i\theta} \\
&\qquad -1-4re^{-i\theta}-r^2e^{-2i\theta}\Big] \\
&= 0.
\end{aligned}
$$

$\square$

(ii) Recall that

$$u(r,\theta) = \frac{ire^{i\theta}}{(1-re^{i\theta})^2} - \frac{ire^{-i\theta}}{(1-re^{-i\theta})^2}.$$

We argue by cases on θ.

($e^{i\theta} \neq 1$) Taking the limit,

$$\lim_{r\to 1} u(r,\theta) = \frac{ie^{i\theta}}{(1-e^{i\theta})^2} - \frac{ie^{-i\theta}}{(1-e^{-i\theta})^2}.$$

Since $e^{i\theta} \neq 1$ by assumption, the denominators are non-zero and this limit is well-defined. Simplifying,

$$\lim_{r\to 1} u(r,\theta) = \frac{0}{(1-e^{i\theta})^2(1-e^{-i\theta})^2} = 0.$$

($e^{i\theta} = 1$) Since $e^{i\theta} = 1$ by assumption, u simplifies to

$$u(r,\theta) = \frac{ir}{(1-r)^2} - \frac{ir}{(1-r)^2} = 0.$$

for all $0 \leq r < 1$. Hence $\lim_{r\to 1} u(r,\theta) = 0$. $\square$