Proof. Consider the sequence $\{A_n\}_{n=1}^{\infty}$ where $A_n = \{a_{n,k}\}_{k\in\mathbb{Z}}$.

For every $k \in \mathbb{Z}$, the sequence $\{a_{n,k}\}_{n=1}^{\infty}$ is Cauchy since $|a_{n,k} - a_{m,k}|$ is bounded by $\|A_n - A_m\|$ for all $n, m \in \{1, 2, \dots\}$. By completeness of $\mathbb{C}$[1] $\{a_{n,k}\}_{n=1}^{\infty}$ converges to some b_k.

Let $B = (\dots, b_{-1}, b_0, b_1, \dots)$. Since $\{A_n\}_{n=1}^{\infty}$ is Cauchy, for all $\epsilon > 0$ there exists N such that for all $n, m > N$,

$$\sum_{k=-K}^{K} |a_{n,k} - a_{m,k}|^2 \leq \|A_n - A_m\|^2 < \epsilon.$$

Limiting $m \to \infty$, we obtain

$$\sum_{k=-K}^{K} |a_{n,k} - b_k|^2 \leq \epsilon.$$

By the monotonic convergence theorem, we can limit $K \to \infty$ and obtain

$$|A_n - B|^2 \leq \epsilon,$$

so $|A_n - B| \to 0$ as $n \to \infty$.

Finally, we must prove that $B \in \ell^2(\mathbb{Z})$. Choose some $\epsilon > 0$ and pick n such that $\|A_n - B\| < \epsilon$. Then for every K

$$\sqrt{\sum_{k=-K}^{K} |b_k|^2} \leq \|A_n\| + \|A_n - B\| < \|A_n\| + \epsilon.$$

Limiting K to infinity, we find that

$$\|B\| \leq \|A_n\| + \epsilon < \infty,$$

so $B \in \ell^2(\mathbb{Z})$. $\square$

[1] See exercise 1.