We first verify that $f(x) = e^{inx}$ is periodic with period 2π. We have

$$\begin{aligned} f(x + 2\pi k) &= e^{in(x+2\pi k)} \\ &= e^{inx+2\pi ikn} \\ &= e^{inx} e^{2\pi ikn} && \text{(by b(b))} \\ &= e^{inx} && (e^{2\pi ikn} = 1 \text{ by 4(e) since } kn \in \mathbb{Z}) \\ &= f(x). \end{aligned}$$

We now show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \tag{1}$$

By cases,

$(n = 0)$

$$\begin{aligned} \int_{-\pi}^{\pi} e^{inx} dx &= \int_{-\pi}^{\pi} e^0 dx \\ &= \int_{-\pi}^{\pi} 1 dx \\ &= 2\pi. \end{aligned}$$

We divide both sides by $1/2\pi$ to obtain our result.

$(n \neq 0)$

$$\begin{aligned} \int_{-\pi}^{\pi} e^{inx} dx &= \int_{-\pi}^{\pi} e^{inx} dx && \text{(here we use } n \neq 0) \\ &= \frac{1}{in} \left[e^{inx} \right]_{-\pi}^{\pi} \\ &= \frac{1}{in} \left(e^{in\pi} - e^{-in\pi} \right) \\ &= \frac{1}{in} \left(f(\pi) - f(-\pi) \right) \\ &= 0. && (f(\pi) = f(-\pi) \text{ from } f\text{'s periodicity}) \end{aligned}$$

Finally, we show that

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos mx \; dx &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases} \\ \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx \; dx &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases} \\ \int_{-\pi}^{\pi} \sin nx \cos mx \; dx &= 0. \end{aligned}$$

To show this, first observe that

$$\begin{aligned}
&e^{i(n-m)x} + e^{i(n+m)x} \\
&= \cos(n-m)x + i\sin(n-m)x + \cos(n+m)x + i\sin(n+m)x \\
&= \cos nx \cos mx + \sin nx \sin mx \qquad \text{(by identities from 4(i))} \\
&\quad + i\sin nx \cos mx - i\cos nx \sin mx \\
&\quad + \cos nx \cos mx - \sin nx \sin mx \\
&\quad + i\sin nx \cos mx + i\cos nx \sin mx \\
&= 2\cos nx \cos mx + 2i \sin nx \cos mx.
\end{aligned}$$

An analogous computation shows that

$$e^{i(n-m)x} - e^{i(n+m)x} = 2\sin nx \sin mx - 2i\cos nx \sin mx.$$

We have

$$\int_{-\pi}^{\pi} e^{i(n+m)x} dx = 0$$

by (1) since $n+m \geq 2$.

Hence

$$\begin{aligned}
&2\int_{-\pi}^{\pi} \cos nx \cos mx \; dx + 2i\int_{-\pi}^{\pi} \sin nx \cos mx \; dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx + \int_{-\pi}^{\pi} e^{i(n+m)x} dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx \\
&= \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m, \end{cases}
\end{aligned}$$

by (1) and

$$\begin{aligned}
&2\int_{-\pi}^{\pi} \sin nx \sin mx \; dx - 2i\int_{-\pi}^{\pi} \cos nx \sin mx \; dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} dx - \int_{-\pi}^{\pi} e^{i(n+m)x} dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x} \\
&= \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m. \end{cases}
\end{aligned}$$

Equating the real and imaginary parts of LHS and RHS, we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

$$
\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}
$$

$$
\int_{-\pi}^{\pi} \sin nx \cos mx \; dx = 0.
$$

$\square$