Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable such that

$$f''(t) + c^2 f(t) = 0 \tag{1}$$

with $c \neq 0$.

Let

$$\begin{aligned} g(t) &= f(t)\cos ct - c^{-1} f'(t) \sin ct, \\ h(t) &= f(t)\sin ct + c^{-1} f'(t) \cos ct. \end{aligned}$$

Observe that they are once differentiable. Differentiating,

$$\begin{aligned} g'(t) &= f'(t)\cos ct - cf(t)\sin ct - c^{-1} f''(t)\sin ct - f'(t)\cos ct \\ &= -cf(t)\sin ct - c^{-1} f''(t)\sin ct \\ &= -cf(t)\sin ct + cf(t)\sin ct && \text{(by (1))} \\ &= 0, \\ h'(t) &= f'(t)\sin ct + cf(t)\cos ct + c^{-1} f''(t)\cos ct - f'(t)\sin ct \\ &= cf(t)\cos ct + c^{-1} f''(t)\cos ct \\ &= cf(t)\cos ct - cf(t)\cos ct \\ &= 0. \end{aligned}$$

Thus g and h are constant. Let a and b be constants such that

$$\begin{aligned} g(t) &= f(t)\cos ct - c^{-1} f'(t)\sin ct = a, \\ h(t) &= f(t)\sin ct + c^{-1} f'(t)\cos ct = b. \end{aligned}$$

Then

$$c^{-1} f'(t)\sin ct = f(t)\cos ct - a$$

and

$$f(t)(\sin ct)^2 + c^{-1} f'(t)\sin ct\cos ct = b\sin ct,$$

so

$$\begin{aligned} & f(t)(\sin ct)^2 + (f(t)\cos ct - a)\cos ct = b\sin ct \\ \iff\ & f(t)(\sin ct)^2 + f(t)(\cos ct)^2 = a\cos ct + b\sin ct \\ \iff\ & f(t) = a\cos ct + b\sin ct. \end{aligned}$$

$\square$