(a) Observe that

$$\begin{aligned}
\sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n &= \sum_{n=M}^{N-1} a_{n+1}B_n - \sum_{n=M}^{N-1} a_n B_n \\
&= \sum_{n=M+1}^{N} a_n B_{n-1} - \sum_{n=M}^{N-1} a_n B_n \\
&= a_N B_{N-1} - a_M B_M + \sum_{n=M+1}^{N-1} a_n(B_{n-1}-B_n) \\
&= a_N B_{N-1} - a_M B_M - \sum_{n=M+1}^{N-1} a_n b_n.
\end{aligned}$$

Then

$$\begin{aligned}
&a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n \\
&= a_N B_N - a_M B_{M-1} - a_N B_{N-1} + a_M B_M + \sum_{n=M+1}^{N-1} a_n b_n \\
&= a_N b_N + a_M b_M + \sum_{n=M+1}^{N-1} a_n b_n \\
&= \sum_{n=M}^{N} a_n b_n.
\end{aligned}$$

□

(b) We know that

$$\sum_{n=1}^{N} a_n b_n = a_N B_N - \sum_{n=1}^{N-1}(a_{n+1}-a_n)B_n$$

for all $N = 1, 2, \ldots$.

We are given that B_N is bounded. Let $C > 0$ be such that $|B_n| < C$ for all $n = 0, 1, 2, \ldots$. Pick an arbitrary $\epsilon > 0$. Let M be such that $a_M < \epsilon/2C$. Then for all $N > N' > M$,

$$\begin{aligned}
&\left|\sum_{n=1}^{N} a_n b_n - \sum_{n=1}^{N'} a_n b_n\right| \\
&= \left|a_N B_N - a_{N'} B_{N'} - \sum_{n=1}^{N-1}(a_{n+1}-a_n)B_n + \sum_{n=1}^{N'-1}(a_{n+1}-a_n)B_n\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| a_N B_N - a_{N'} B_{N'} - \sum_{n=N'}^{N-1} (a_{n+1} - a_n) B_n \right| \\
&\leq |a_N B_N| + |a_{N'} B_{N'}| + \sum_{n=N'}^{N-1} |(a_{n+1} - a_n) B_n| \\
&\leq |a_N|\, C + |a_{N'}|\, C + \sum_{n=N'}^{N-1} |a_{n+1} - a_n|\, C \\
&= a_N C + a_{N'} C + \sum_{n=N'}^{N-1} (a_n - a_{n+1}) C \qquad (a_n \geq a_{n+1} \geq 0\ \forall n) \\
&= 2a_{N'} C \\
&\leq 2a_M C \\
&< \epsilon.
\end{aligned}
$$

$\square$