(a) Define
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
We first show that this series converges for every $z \in \mathbb{C}$.

Define
$$a_n = \frac{|z^n|}{n!}.$$
Then
$$\frac{a_{n+1}}{a_n} = \frac{|z^{n+1}|\, n!}{|z^n|\,(n+1)!} = \frac{|z|^{n+1}\, n!}{|z|^n\,(n+1)!} = \frac{|z|}{n+1}.$$
Applying the ratio test,
$$\lim_{n\to\infty} \frac{|z|}{n+1} = 0,$$
so the series $\sum_{n=0}^{\infty} a_n$ converges.

Recalling that $|z^n/n!| = |z^n|/n! = a_n$, we have that
$$e^z = \sum_{n=0}^{z^n}$$
converges by 3(c).

We now show that the convergence is uniform on every bounded subset of $\mathbb{C}$. Pick an arbitrary bounded $S \subset \mathbb{C}$ and an arbitrary $\epsilon > 0$. We will show that there exists an integer M such that for all $N > M$ and $s \in S$,
$$\left| \sum_{n=0}^{N} \frac{z^n}{n!} - e^x \right| < \epsilon. \tag{1}$$
Note that (1) is equivalent to
$$\left| \sum_{n=N+1}^{\infty} \frac{z^n}{n!} \right| < \epsilon \tag{2}$$
after cancelling the first N terms of the series.

Choose c such that $c > |s|$ for all $s \in S$. This is well-defined because S is bounded. We know from above that
$$e^c = \sum_{n=0}^{\infty} \frac{c^n}{n!}$$
converges. Then there exists an integer M such that for all $N > M$,
$$\left| \sum_{n=0}^{N} \frac{c^n}{n!} - e^c \right| < \epsilon,$$

or after cancelling the first N terms of the series,

$$\sum_{n=N+1}^{\infty} \frac{c^n}{n!} < \epsilon.$$

Observe that for all n,

$$\frac{c^n}{n!} > \frac{|z|^n}{n!} = \left|\frac{z^n}{n!}\right|,$$

so

$$\sum_{n=N+1}^{\infty} \left|\frac{z^n}{n!}\right| < \epsilon.$$

For every partial sum from $N+1$ to some N' we have

$$\left|\sum_{n=N+1}^{N'} \frac{z^n}{n!}\right| \leq \sum_{n=N+1}^{N'} \left|\frac{z^n}{n!}\right|$$

by the triangle inequality. Taking the limit,

$$\begin{aligned}\left|\sum_{n=N+1}^{\infty} \frac{z^n}{n!}\right| &\leq \sum_{n=N+1}^{\infty} \left|\frac{z^n}{n!}\right| \\ &< \epsilon,\end{aligned}$$

which matches (2), concluding the proof. $\square$

(b) We first show that the series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely for all z. We have

$$\lim_{n\to\infty} \left|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{z}{n+1}\right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0,$$

so the series converges absolutely by the ratio test.

Observe that for the n^{th} term of the series for $e^{z_1+z_2}$,

$$\begin{aligned}\frac{(z_1+z_2)^n}{n!} &= \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\ &= \sum_{k=0}^{n} \frac{z_1^k}{k!}\frac{z_2^{n-k}}{(n-k)!}.\end{aligned}$$

We thus recognise the series for $e^{z_1+z_2}$ as the Cauchy product of the series for e^{z_1} and e^{z_2}. Since we've shown that these converge absolutely, $e^{z_1+z_2} = e^{z_1}e^{z_2}$. $\square$

(c) We first find the power series of $\cos y$ around 0. We have

$$\begin{aligned} \cos 0 &= \cos 0 = 1, \\ \cos' 0 &= -\sin 0 = 0, \\ \cos'' 0 &= -\cos 0 = -1, \\ \cos''' 0 &= \sin 0 = 0, \\ \cos'''' 0 &= \cos 0 = 1, \end{aligned}$$

and so on. The odd terms are zero, so we can skip them and write our power series as

$$\cos y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}.$$

To prove convergence, we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} y^{2n+2}}{(2n+2)!}}{\frac{(-1)^n y^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{y^2}{(2n+2)(2n+1)} \right| = 0,$$

so the power series converges absolutely by the ratio test.

We repeat the same for $\sin y$:

$$\begin{aligned} \sin 0 &= \sin 0 = 0, \\ \sin' 0 &= \cos 0 = 1, \\ \sin'' 0 &= -\sin 0 = 0, \\ \sin''' 0 &= -\cos 0 = -1, \\ \sin'''' 0 &= \sin 0 = 0, \end{aligned}$$

and so on. Collapsing the even terms, which are zero, we write

$$\sin y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}.$$

Again proving convergence, we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} y^{2n+3}}{(2n+3)!}}{\frac{(-1)^n y^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{y^2}{(2n+3)(2n+2)} \right| = 0,$$

so this power series also converges absolutely by the ratio test.

We combine the power series as

$$\cos y + i \sin y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n i y^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{i^n y^n}{n!},
\end{aligned}
$$

where the n^{th} term of the power series for $\cos y$ becomes the $2n^{\text{th}}$ term of the combined series, and the n^{th} term of the series for $\cos y$ becomes the $2n+1^{\text{th}}$ term of the combined series. We are able to combine the two power series into one because they are absolutely convergent.

Then

$$
\begin{aligned}
\cos y + i \sin y &= \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \\
&= e^{iy},
\end{aligned}
$$

where the last equality is by our definition of complex exponentiation. $\square$

(d) Let $x, y \in \mathbb{R}$. We have

$$
\begin{aligned}
e^{x+iy} &= e^x e^{iy} && \text{(by (b))} \\
&= e^x(\cos y + i \sin y). && \text{(by (c))}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\left|e^{iy}\right| &= |\cos y + i \sin y| \\
&= \sqrt{(\cos y)^2 + (\sin y)^2} \\
&= 1.
\end{aligned} \tag{3}
$$

Then

$$
\begin{aligned}
\left|e^{x+iy}\right| &= \left|e^x e^{iy}\right| && \text{(by (b))} \\
&= |e^x| \left|e^{iy}\right| && \text{(by 1(d))} \\
&= |e^x|. && \text{(by (3))}
\end{aligned}
$$

$\square$

(e) ($\Longrightarrow$) Decompose $z = x + iy$ for $x, y \in \mathbb{R}$. Then $e^z = e^x \cos y + ie^x \sin y$. We set $e^z = 1$. Equating the imaginary components,

$$
e^x \sin y = 0.
$$

Since $e^x > 0$, $\sin y = 0$.
Similarly equating the real components,

$$e^x \cos y = 1.$$

Since $\sin y = 0$, either $\cos y = 1$ or $\cos y = -1$. We know that $\cos y > 0$ since $e^x > 0$ and their product is positive. Hence $\cos y = 1$.
We have $\sin y = 0$ and $\cos y = 1$, so $y = 2\pi k$ for some $k \in \mathbb{Z}$.
Finally $1 = e^x \cos y = e^x$, so $x = 0$.
Thus, $z = x + iy = 2\pi k i$ for some $k \in \mathbb{Z}$.

($\Longleftarrow$) Let k be an arbitrary integer. Let $z = 2\pi k i$. Then

$$\begin{aligned} e^z &= e^{2\pi k i} \\ &= \cos 2\pi k + i \sin 2\pi k && \text{(by (c))} \\ &= 1 + 0i \\ &= 1. \end{aligned}$$

$\square$

(f) Let $z = x + iy$ for some $x, y \in \mathbb{R}$. Let $r = |z|$. Observe that $0 \leq r < \infty$. We show that this is the unique choice of $r \geq 0$ if we wish to represent

$$z = re^{i\theta}$$

for some $\theta \in \mathbb{R}$.
Observe that

$$\begin{aligned} \left|e^{i\theta}\right| &= \sqrt{\cos^2\theta + \sin^2\theta} \\ &= 1, \end{aligned}$$

so

$$\begin{aligned} |z| &= \left|re^{i\theta}\right| \\ &= |r|\left|e^{i\theta}\right| && \text{(by 1(d))} \\ &= |r| \\ &= r. && \text{(we've restricted } r \geq 0) \end{aligned}$$

To pick θ, show that it (along with r) represents z, and show its uniqueness, we argue by cases.

$(x = 0, y = 0)$ $r = 0$. For any choice of $\theta \in \mathbb{R}$,

$$re^{i\theta} = 0e^{i\theta} = 0,$$

so in this degenerate case our choice of $\theta \in \mathbb{R}$ can be completely arbitrary.

$(x = 0, y \neq 0)$ $r = |y|$. We want

$$\begin{aligned} z &= iy \\ &= |y|\, e^{i\theta} \\ &= |y|\,(\cos\theta + i\sin\theta). \end{aligned}$$

Equating the imaginary components, we find $|y|\sin\theta = y$ or $\sin\theta = \operatorname{sgn} y$. Thus

$$\theta = \frac{\pi}{2}\operatorname{sgn} y + 2\pi k$$

for some $k \in \mathbb{Z}$.

This satisfies our equation in the real components as well since $\cos\theta = 0$ for our choice of θ, as required.

$(x \neq 0)$ We want

$$\begin{aligned} z &= x + iy \\ &= re^{i\theta} \\ &= |z|\, e^{i\theta} \\ &= |z|\cos\theta + i\,|z|\sin\theta. \end{aligned}$$

Equating the real and imaginary sides,

$$\begin{aligned} \cos\theta &= \frac{x}{\sqrt{x^2+y^2}}, \\ \sin\theta &= \frac{y}{\sqrt{x^2+y^2}}. \end{aligned}$$

Note that these two equations are sufficient and necessary to obtain θ that represents z.

$\theta = \arctan(y/x) + 2\pi k$ for some arbitrary $k \in \mathbb{Z}$ describes all such θ. $\square$

(g) Multiplying a complex number by i rotates it anticlockwise around the origin by $\pi/2$ radians. More generally, multipying a complex number by $e^{i\theta}$ rotates it anticlockwise around the origin by θ radians.

(h)

$$\begin{aligned} \frac{e^{i\theta} + e^{-i\theta}}{2} &= \frac{\cos\theta + i\sin\theta + \cos\theta + i\sin(-\theta)}{2} && \text{(by (c))} \\ &= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \\ &= \frac{2\cos\theta}{2} \\ &= \cos\theta \end{aligned}$$

$$
\begin{aligned}
\frac{e^{i\theta}-e^{-i\theta}}{2i} &= \frac{\cos\theta + i\sin\theta - \cos\theta - i\sin(-\theta)}{2i} && \text{(by (c))} \\
&= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i} \\
&= \frac{2i\sin\theta}{2i} \\
&= \sin\theta
\end{aligned}
$$

$\square$

(i) Using Euler's identity from (h),

$$
\begin{aligned}
&\cos\theta\cos\vartheta - \sin\theta\sin\vartheta \\
&= \frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\vartheta}+e^{-i\vartheta}\right) - \frac{1}{4i^2}\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\vartheta}-e^{-i\vartheta}\right) \\
&= \frac{1}{4}\left(\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\vartheta}+e^{-i\vartheta}\right) + \left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\vartheta}-e^{-i\vartheta}\right)\right) \\
&= \frac{1}{2}\left(e^{i\theta}e^{i\vartheta}+e^{-i\theta}e^{-i\vartheta}\right) \\
&= \frac{1}{2}\left(e^{i(\theta+\vartheta)}+e^{-i(\theta+\vartheta)}\right) && \text{(by (b))} \\
&= \cos(\theta+\vartheta). && \text{(by (h))}
\end{aligned}
$$

Swapping ϑ for $-\vartheta$ and observing that cos is even and sin is odd shows that

$$\cos(\theta-\vartheta) = \cos\theta\cos\vartheta + \sin\theta\sin\vartheta.$$

Arguing similarly for the sin identities,

$$
\begin{aligned}
&\sin\theta\cos\vartheta + \cos\theta\sin\vartheta \\
&= \frac{1}{4i}\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\vartheta}+e^{-i\vartheta}\right) + \frac{1}{4i}\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\vartheta}-e^{-i\vartheta}\right) \\
&= \frac{1}{4i}\left(\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\vartheta}+e^{-i\vartheta}\right) + \left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\vartheta}-e^{-i\vartheta}\right)\right) \\
&= \frac{1}{2i}\left(e^{i\theta}e^{i\vartheta}-e^{-i\theta}e^{-i\vartheta}\right) \\
&= \frac{1}{2i}\left(e^{i(\theta+\vartheta)}-e^{-i(\theta+\vartheta)}\right) && \text{(by (b))} \\
&= \sin(\theta+\vartheta). && \text{(by (h))}
\end{aligned}
$$

Again swapping ϑ for $-\vartheta$ shows that

$$\sin(\theta-\vartheta) = \sin\theta\cos\vartheta - \cos\theta\sin\vartheta.$$

We list the identities:

$$\cos(\theta+\vartheta) = \cos\theta\cos\vartheta - \sin\theta\sin\vartheta, \tag{4}$$

$$
\begin{aligned}
\cos(\theta - \vartheta) &= \cos\theta\cos\vartheta + \sin\theta\sin\vartheta, && (5)\\
\sin(\theta + \vartheta) &= \sin\theta\cos\vartheta + \cos\theta\sin\vartheta, && (6)\\
\sin(\theta - \vartheta) &= \sin\theta\cos\vartheta - \cos\theta\sin\vartheta. && (7)
\end{aligned}
$$

Subtracting the LHS and RHS of (4) from (5),

$$2\sin\theta\sin\vartheta = \cos(\theta - \vartheta) - \cos(\theta + \vartheta).$$

Similarly adding (6) and (7),

$$2\sin\theta\cos\vartheta = \sin(\theta + \vartheta) + \sin(\theta - \vartheta).$$

□