(a) We show that $s_n - \sigma_n \to 0$ as $n \to \infty$.

We require the fact that

$$\sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sum_{j=1}^{k} c_j = \sum_{k=1}^{n} (n-k+1) c_k,$$

so

$$s_n - \sigma_n = \sum_{k=1}^{n} c_k - \frac{1}{n} \sum_{k=1}^{n} (n-k+1) c_k = \frac{1}{n} \sum_{k=1}^{n} (k-1) c_k. \tag{1}$$

Pick an arbitrary $\epsilon > 0$. We will show that there exists $N$ such that for all $n > N$, $|s_n - \sigma_n| < \epsilon$.

Since $nc_n \to 0$ by assumption, there exists $M$ such that $k\,|c_k| < \epsilon/2$ for all $k > M$. Let $\rho = \sum_{k=1}^{M} k\,|c_k|$.

Choose $N \geq \max(2\rho/\epsilon, M)$. Then for all $n > N$ we have $n > 2\rho/\epsilon$, so $n\epsilon > \rho + n\epsilon/2$. Hence,

$$\begin{aligned}
n\,|s_n - \sigma_n| &= \left| \sum_{k=1}^{n} (k-1) c_k \right| && \text{(by (1))} \\
&\leq \sum_{k=1}^{n} (k-1)\,|c_k| \\
&\leq \sum_{k=1}^{n} k\,|c_k| \\
&= \rho + \sum_{k=M+1}^{n} k\,|c_k| \\
&\leq \rho + \sum_{k=M+1}^{n} \frac{\epsilon}{2} \\
&= \rho + (n-M)\frac{\epsilon}{2} \\
&\leq \rho + n\frac{\epsilon}{2} \\
&< n\epsilon.
\end{aligned}$$

Dividing both sides by $n$ concludes the proof. □

(b) We are given that $\lim_{k\to\infty} k\,|c_k| = 0$ and that $\{c_k\}_{k=1}^{\infty}$ is Abel summable.

Define

$$A_n(r) = \sum_{k=1}^{n} c_n r^k,$$

$$
\begin{aligned}
A(r) &= \lim_{n\to\infty} A_n(r) = \sum_{k=1}^{n} c_n r^k, \\
\overline{A} &= \lim_{r\to 1} A(r).
\end{aligned}
$$

We want to show that $A(1) = \sum_{k=1}^{\infty} c_k$ converges.

Choose an arbitrary $\epsilon > 0$. We will show that there exists $N$ such that for all $n > N$, $\left|A_n(1) - \overline{A}\right| < \epsilon$.

Let $\rho > 0$ such that $k\,|c_k| < \rho$ for all $k = 1, 2, \ldots$; this bound exists because $\lim_{k\to\infty} k\,|c_k| = 0$. Since $\lim_{t\to 1} A(t) = \overline{A}$, there exists $0 < T < 1$ such that for all $T < t < 1$, $\left|A(t) - \overline{A}\right| < \epsilon/3$. Let $\alpha > \max\{3\rho/\epsilon, 1/(1-T)\} > 0$. Because $\lim_{k\to\infty} k\,|c_k| = 0$, we can find $N$ such that for all $k > N$, $k\,|c_k| < \epsilon/3\alpha$.

Choose an arbitrary $n > N$. We will show that $\left|A_n(1) - \overline{A}\right| < \epsilon$.

Let $r = 1 - 1/\alpha n$. By the triangle inequality

$$
\left|A_n(1) - \overline{A}\right| \le |A_n(1) - A_n(r)| + |A_n(r) - A(r)| + \left|A(r) - \overline{A}\right|.
$$

We tackle these terms one by one.

Firstly,

$$
\begin{aligned}
|A_n(1) - A_n(r)| &= \left|\sum_{k=1}^{n} c_k r^k - \sum_{k=1}^{n} c_k\right| \\
&= \left|\sum_{k=1}^{n} c_k (1 - r^k)\right| \\
&= (1-r)\left|\sum_{k=1}^{n} c_k \sum_{j=0}^{k-1} r^j\right| && (2) \\
&\le (1-r)\sum_{k=1}^{n} k\,|c_k| && (3) \\
&< (1-r) n\rho && (4) \\
&= \frac{1}{\alpha n} n\rho = \frac{\rho}{\alpha} < \frac{\epsilon}{3}. && (5)
\end{aligned}
$$

In eq. (2) we use the fact that $1 - r^k = (1-r)(1 + r + \cdots + r^{k-1})$. In eq. (3) $0 < r^j < 1$ implies $\sum_{j=0}^{k-1} r^j \le k$. Equation (4) is by the bound that defines $\rho$. Finally in eq. (5) we substitute for $r$ and $\alpha$.

Bounding the second term,

$$
|A(r) - A_n(r)| = \left|\sum_{k=n+1}^{\infty} c_k r^k\right|
$$

$$
\begin{aligned}
&< \frac{\epsilon}{3\alpha} \sum_{k=n+1}^{\infty} \frac{1}{k} r^k && (6) \\
&< \frac{\epsilon}{3\alpha n} \sum_{k=n+1}^{\infty} r^k && (7) \\
&= \frac{\epsilon r^{n+1}}{3\alpha n(1-r)} \\
&< \frac{\epsilon}{3\alpha n(1-r)} \\
&= \frac{\epsilon}{3}. && (8)
\end{aligned}
$$

In eq. (6) we use the fact that $k > n > N$, so by our choice of $N$, $k\,|c_k| < \epsilon/3\alpha$. Equation (7) is because $1/k < 1/n$. In eq. (8) we substitute for $r$.

For the final term, we have $\alpha > 1/(1-T)$, so $r = 1 - 1/\alpha n > T$. Then $\left|A(r) - \overline{A}\right| < \epsilon/3$ by our choice of $T$.

Putting it all together,

$$
\begin{aligned}
\left|A_n(1) - \overline{A}\right| &\leq |A_n(1) - A_n(r)| + |A_n(r) - A(r)| + \left|A(r) - \overline{A}\right| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon
\end{aligned}
$$

and $\lim_{n\to\infty} A_n(1) = \sum_{k=1}^{\infty} c_n = \overline{A}$.