We first show that f is integrable.

Pick an arbitrary $\varepsilon > 0$. Let k be such that

$$\int_0^1 |f_k(x) - f(x)|\, dx < \frac{\varepsilon}{5}.$$

Let $P = \{0 = p_0, \ldots, p_N = 1\}$ be a partition such that

$$\mathcal{U}(P, f_k) - \mathcal{L}(P, f_k) < \frac{\varepsilon}{5} \quad \text{and} \quad \mathcal{U}(P, |f - f_k|) - \mathcal{L}(P, |f - f_k|) < \frac{\varepsilon}{5}.$$

P exists because f_k and $|f - f_k|$ are integrable[1]. Observe that

$$0 \leq \mathcal{L}(P, f) \leq \int_0^1 |f_k(x) - f(x)|\, dx \leq \mathcal{U}(P, f),$$

implying that

$$\mathcal{U}(P, f) \leq \frac{2\varepsilon}{5}.$$

For convenience, define $I_j = [p_{j-1}, p_j]$ for $j = 1, \ldots, N$. We have

$$\begin{aligned}
&\sup_{x \in I_j} f(x) \leq \sup_{x \in I} f_k(x) + \sup_{x \in I} |f_k - f(x)|\,, \text{ and} \\
&\inf_{x \in I_j} f(x) \geq \inf_{x \in I} f_k(x) - \sup_{x \in I} |f_k - f(x)|
\end{aligned}$$

Then

$$\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) \leq \sup_{x \in I} f_k(x) - \inf_{x \in I} f_k(x) + 2 \sup_{x \in I} |f_k - f(x)|\,.$$

Hence,

$$\begin{aligned}
&\mathcal{U}(P, f) - \mathcal{L}(P, f) \\
&= \sum_{j=1}^N |I_j| \sup_{x \in I_j} f(x) - \sum_{j=1}^N |I_j| \inf_{x \in I_j} f(x) \\
&\leq \sum_{j=1}^N |I_j| \sup_{x \in I_j} f_k(x) - \sum_{j=1}^N |I_j| \inf_{x \in I_j} f_k(x) + 2 \sum_{j=1}^N |I_j| \sup_{x \in I_j} |f_k(x) - f(x)| \\
&= \mathcal{U}(P, f_k) - \mathcal{L}(P, f_k) + 2\mathcal{U}(P, |f_k - f|) \\
&< \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, f is integrable and its Fourier coefficients exist.

We have

$$\left| \hat{f}_k(n) - \hat{f}_k(n) \right| = \left| \int_0^1 f_k(x) e^{-2\pi i n x}\, dx - \int_0^1 f(x) e^{-2\pi i n x}\, dx \right|$$

[1] We can find P by finding P_1 and P_2 such that $\mathcal{U}(P_1, f_k) - \mathcal{L}(P_1, f_k) < \varepsilon/5$ and $\mathcal{U}(P_2, g_k) - \mathcal{L}(P_2, g_k) < \varepsilon/5$, and setting $P = P_1 \cup P_2$.

$$
\begin{aligned}
&= \left| \int_0^1 \left(f_k(x) - f(x)\right) e^{-2\pi i n x}\, dx \right| \\
&\leq \int_0^1 \left| \left(f_k(x) - f(x)\right) e^{-2\pi i n x} \right| \, dx \\
&= \int_0^1 \left| f_k(x) - f(x) \right| \, dx.
\end{aligned}
$$

The last integral is independent of n. It tends to 0 as $k \to \infty$ by assumption.

$\square$