*Proof.*

($\implies$) Let $G$ be a finite cyclic group. Then there exists $g \in G$ such that any element in $G$ can be written $g^n$ for some $n \in \mathbb{Z}$.

Let $N = |G|$ and define $\phi : G \to \mathbb{Z}(N)$ such that $\phi(g^n) = \overline{n}$, where $\overline{m}$ is the class of integers congruent to $m$ modulo $N$.

We first show that $\phi$ is well-defined. Consider the set $\{1, g, \ldots, g^{N-1}\}$. If $g^n = g^m$ for any $0 \leq n < m < N$, then $g^{m-n} = 1$ while $m - n < N$. The cycle would have length less than $N$ and $g$ would not generate $G$, which is a contradiction. Thus the elements are unique and $G = \{1, g, \ldots, g^{N-1}\}$. We also observe that $g^N = 1$, because for all $0 \leq n < N-1$, $0 < 1+n < N$, implying that $gg^n \neq 1$, leaving only $g^{N-1}$ as the inverse of $g$. Let $n, m \in \mathbb{Z}$ be such that $g^n = g^m$. Then $n - m$ is an integer multiple of $N$, so $\phi(g^n) = \phi(g^m)$, as desired.

It is clear that $\phi$ is a homomorphism because for all $n, m \in \mathbb{Z}$, $\phi(g^n g^m) = \phi(g^{n+m}) = \overline{n+m} = \overline{n} + \overline{m} = \phi(g^n) + \phi(g^m)$. Finally, $\phi$ is a bijection because $\left|\{g^0, g^1, \ldots, g^{N-1}\}\right| = \left|\{\overline{0}, \overline{1}, \ldots, \overline{N-1}\}\right| = N$.

($\impliedby$) Let $G$ be a finite abelian group that is isomorphic to $\mathbb{Z}(N)$ for some $N$ with $\phi : \mathbb{Z}(N) \to G$ as the isomorphism.

Observe that $|G| = |\mathbb{Z}(N)| = N$ since $\phi$ is a bijection. Since $\phi$ is a homomorphism, $\left\{\phi^0(1), \phi^1(1), \ldots, \phi^{N-1}(1)\right\} = \{\phi(0), \phi(1), \ldots, \phi(N-1)\}$. The right-hand side set has cardinality $N$ since $\phi$ is a bijection. Thus, the left-hand side also has cardinality $N$ and it is equal to $G$.

$\square$