Technical Report TR-EECS-002 Joint Source and Sensor Localization By Hybrid TDOA-AOA: Localizability Analysis

Yongde Ye and Gang Wang

***Abstract*—This report provides a detailed derivation for the third point of Proposition 1 in Section III of the work [1].**

I. Introduction

Joint source and sensor localization (JSSL) as an innovative idea, effectively avoids the energy and resource consumption associated with traditional sensor networks where sensors must first self-localize. Simultaneously, it significantly enhances the flexibility of sensor deployment and signal source localization.

This report analyzes the localizability of JSSL under minimal source and sensor configurations when using hybrid time-difference-of-arrival (TDOA) and angle-of-arrival (AOA) measurements. It shows that in the 3-D case, JSSL cannot be achieved when one source lies on the line segment connecting two sensors, while the other source lies on the perpendicular bisector plane of that line segment.

II. Measurement Models

Consider a JSSL scenario consists of M sources and N sensors in a k ($k = 2$ or 3)-dimensional space, where the positions of all sources and sensors are unknown. The positions of the m-th source and the n-th sensor are denoted by $\boldsymbol{x}_m^o = [x_{m,(1)^o}, \ldots, x_{m,(k)^o}]^T$ and $\boldsymbol{y}_n^o = [y_{n,(1)}^o, \ldots, y_{n,(k)}^o]^T$ for $m = 1, \ldots, M$ and $n = 1, \ldots, N$, respectively. We consider an outdoor environment where the propagation paths are direct line-of-sight between the sources and sensors. Moreover, we assume that all sensors are time-synchronized and equipped with antenna arrays and processing chips, which has the ability to accurately acquire the TDOA and AOA measurements. The purpose is to use the sensor measurements to locate all sources and the sensors.

Since the positions of all sources and sensors are unknown, we can only determine the relative positions between the sources and sensors with respect to a reference. We choose the first sensor as the reference sensor and set its position $\boldsymbol{y}_1^o = \boldsymbol{0}_{k\times 1}$ without loss of generality. During localization, each source emits a signal and it is received by all sensors. In a multi-source scenario, distinguishing the signals from different sources is critical, which can be realized by letting the sources emit signals at different times or separate frequencies, or with orthogonal codes. With synchronized sensors, the TDOAs between the reference sensor and other sensors for each source can be acquired by cross-correlations. After multiplying by the known signal propagation speed, the TDOAs turn to range-differences (RDs) which can be modeled by

$$\begin{aligned} d_{mn} &= d_{mn}^o + n_{mn}^d \\ &= \|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\| - \|\boldsymbol{x}_m^o\| + n_{mn}^d \\ &= l_{mn}^o - l_{m1}^o + n_{mn}^d, \\ & m = 1, \ldots, M,\ n = 2, \ldots, N, \end{aligned} \tag{1}$$

where $l_{mn}^o = \|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|$ denotes the true range from the source $\boldsymbol{x}_m$ to the sensor $\boldsymbol{y}_n$, and n_{mn}^d is the RD noise.

For AOA measurement in the 2-D case, it is modeled by

$$\begin{aligned} \theta_{mn} &= \operatorname{arctan2}\left(x_{m,(2)}^o - y_{n,(2)}^o, x_{m,(1)}^o - y_{n,(1)}^o\right) + n_{mn}^\theta \\ &= \theta_{mn}^o + n_{mn}^\theta, \\ & m = 1, \ldots, M,\ n = 1, \ldots, N, \end{aligned} \tag{2}$$

where $\theta_{mn}^o = \operatorname{arctan2}\left(x_{m,(2)}^o - y_{n,(2)}^o, x_{m,(1)}^o - y_{n,(1)}^o\right) \in (-\pi, \pi]$ is the true azimuth angle, and n_{mn}^θ is the noise.

For the 3-D case, we have the elevation angle in addition to the azimuth (2),

$$\begin{aligned} \phi_{mn} &= \arctan \frac{x_{m,(3)}^o - y_{n,(3)}^o}{\|\boldsymbol{x}_{m,(1:2)}^o - \boldsymbol{y}_{n,(1:2)}^o\|} + n_{mn}^\phi \\ &= \phi_{mn}^o + n_{mn}^\phi, \\ & m = 1, \ldots, M,\ n = 1, \ldots, N, \end{aligned} \tag{3}$$

where $\phi_{mn}^o = \arctan \frac{x_{m,(3)}^o - y_{n,(3)}^o}{\|\boldsymbol{x}_{m,(1:2)}^o - \boldsymbol{y}_{n,(1:2)}^o\|} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the true elevation angle, and n_{mn}^ϕ is the azimuth noise. The elevation model has an alternative form by expressing $\|\boldsymbol{x}_{m,(1:2)}^o - \boldsymbol{y}_{n,(1:2)}^o\|$ as $(\boldsymbol{x}_{m,(1)}^o - \boldsymbol{y}_{n,(1)}^o)\cos\theta_{mn}^o + (\boldsymbol{x}_{m,(2)}^o - \boldsymbol{y}_{n,(2)}^o)\sin\theta_{mn}^o$ [2].

III. Localizability

In this section, we shall examine the rank of the Jacobian matrix defined by the measurement vector and the unknown variables, which enables us to determine the minimum number of sources and sensors required for JSSL by using hybrid TDOA-AOA measurements, as well as the blind geometries where JSSL is not possible.

Let us group all unknown parameters into the vector $\boldsymbol{\alpha}^o = \left[\boldsymbol{x}_1^{oT}, \ldots, \boldsymbol{x}_M^{oT}, \boldsymbol{y}_2^{oT}, \ldots, \boldsymbol{y}_N^{oT}\right]^T$ containing the source positions and the sensor positions. The Jacobian matrix is

$$\boldsymbol{J} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{\psi}^T(\boldsymbol{\alpha}^o)}{\partial \boldsymbol{\alpha}^o}, \tag{4}$$

where $\boldsymbol{\psi}(\boldsymbol{\alpha}^o)$ is the parametric form of $\boldsymbol{\psi}^o$, given by $\boldsymbol{\psi}^o = [\boldsymbol{d}^{oT}, \boldsymbol{\theta}^{oT}]^T$ for the 2-D case and $\boldsymbol{\psi}^o = [\boldsymbol{d}^{oT}, \boldsymbol{\theta}^{oT}, \boldsymbol{\phi}^{oT}]^T$ for the 3-D case.

Localizability of the JSSL problem requires that $\boldsymbol{J}$ is full rank [3]. In the following, we shall give a detailed proof of the third point of Proposition 1 in [1], i.e., JSSL cannot be achieved when using two sensors and two sources in the 3-D case, if one source lies on the line segment between two sensors and the other source lies on the perpendicular bisector of these two sensors.

The Jacobian matrix of the measurements for $(M=2, N=2)$ in 3-D case is $\boldsymbol{J} = [\boldsymbol{J}_d, \boldsymbol{J}_\theta, \boldsymbol{J}_\phi] \in \mathbb{R}^{9\times 10}$, where

$$\boldsymbol{J}_d = \begin{bmatrix} \nabla^{d_{12}^o}_{\boldsymbol{x}_1^o} & \boldsymbol{0}_{k\times 1} \\ \boldsymbol{0}_{k\times 1} & \nabla^{d_{22}^o}_{\boldsymbol{x}_2^o} \\ \nabla^{d_{12}^o}_{\boldsymbol{y}_2^o} & \nabla^{d_{22}^o}_{\boldsymbol{y}_2^o} \end{bmatrix},\ \boldsymbol{J}_\theta = \begin{bmatrix} \nabla^{\theta_{11}^o}_{\boldsymbol{x}_1^o} & \boldsymbol{0}_{k\times 1} & \nabla^{\theta_{12}^o}_{\boldsymbol{x}_1^o} & \boldsymbol{0}_{k\times 1} \\ \boldsymbol{0}_{k\times 1} & \nabla^{\theta_{21}^o}_{\boldsymbol{x}_2^o} & \boldsymbol{0}_{k\times 1} & \nabla^{\theta_{22}^o}_{\boldsymbol{x}_2^o} \\ \boldsymbol{0}_{k\times 1} & \boldsymbol{0}_{k\times 1} & \nabla^{\theta_{12}^o}_{\boldsymbol{y}_2^o} & \nabla^{\theta_{22}^o}_{\boldsymbol{y}_2^o} \end{bmatrix},$$

$$\boldsymbol{J}_\phi = \begin{bmatrix} \nabla^{\phi_{11}^o}_{\boldsymbol{x}_1^o} & \boldsymbol{0}_{k\times 1} & \nabla^{\phi_{12}^o}_{\boldsymbol{x}_1} & \boldsymbol{0}_{k\times 1} \\ \boldsymbol{0}_{k\times 1} & \nabla^{\phi_{21}^o}_{\boldsymbol{x}_2^o} & \boldsymbol{0}_{k\times 1} & \nabla^{\phi_{22}^o}_{\boldsymbol{x}_2^o} \\ \boldsymbol{0}_{k\times 1} & \boldsymbol{0}_{k\times 1} & \nabla^{\phi_{12}^o}_{\boldsymbol{y}_2^o} & \nabla^{\phi_{22}^o}_{\boldsymbol{y}_2^o} \end{bmatrix}. \tag{5}$$

The expressions of all partial derivatives in (5) are

$$\begin{aligned}
&\nabla^{d_{mn}}_{\boldsymbol{x}_m^T} = \frac{\boldsymbol{x}_m^o - \boldsymbol{y}_n^o}{\|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|} - \frac{\boldsymbol{x}_m^o}{\|\boldsymbol{x}_m^o - \boldsymbol{y}_1^o\|},\ \nabla^{d_{mn}}_{\boldsymbol{y}_n^T} = -\frac{\boldsymbol{x}_m^o - \boldsymbol{y}_n^o}{\|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|},\\
&\nabla^{\theta_{mn}}_{\boldsymbol{x}_m^T} = -\nabla^{\theta_{mn}}_{\boldsymbol{y}_n^T} = \Big[-\frac{\boldsymbol{x}^o_{m,(2)} - \boldsymbol{y}^o_{n,(2)}}{\|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|^2}, \frac{\boldsymbol{x}^o_{m,(1)} - \boldsymbol{y}^o_{n,(1)}}{\|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|^2} \Big],\\
&\nabla^{\phi_{mn}}_{\boldsymbol{x}_m^T} = -\nabla^{\phi_{mn}}_{\boldsymbol{y}_n^T} = \Big[-\frac{(\boldsymbol{x}^o_{m,(1)} - \boldsymbol{y}^o_{n,(1)})(\boldsymbol{x}^o_{m,(3)} - \boldsymbol{y}^o_{n,(3)})}{\|\boldsymbol{x}^o_{m,(1:2)} - \boldsymbol{y}^o_{n,(1:2)}\| \|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|^2},\\
&- \frac{(\boldsymbol{x}^o_{m,(2)} - \boldsymbol{y}^o_{n,(2)})(\boldsymbol{x}^o_{m,(3)} - \boldsymbol{y}^o_{n,(3)})}{\|\boldsymbol{x}^o_{m,(1:2)} - \boldsymbol{y}^o_{n,(1:2)}\| \|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|^2}, \frac{\|\boldsymbol{x}^o_{m,(1:2)} - \boldsymbol{y}^o_{n,(1:2)}\|}{\|\boldsymbol{x}_m^o - \boldsymbol{y}_n^o\|^2} \Big],\\
&\quad \nabla^{d_{m_1 n}}_{\boldsymbol{x}_{m_2}^T} = \nabla^{\boldsymbol{d}_{mn_1}}_{\boldsymbol{y}_{n_2}^T} = \boldsymbol{0}^T_{k\times 1},\ m_1 \neq m_2,\ n_1 \neq n_2,\\
&\quad \nabla^{\theta_{m_1 n}}_{\boldsymbol{x}_{m_2}^T} = \nabla^{\theta_{mn_1}}_{\boldsymbol{y}_{n_2}^T} = \boldsymbol{0}^T_{k\times 1},\ m_1 \neq m_2,\ n_1 \neq n_2,\\
&\quad \nabla^{\phi_{m_1 n}}_{\boldsymbol{x}_{m_2}^T} = \nabla^{\phi_{mn_1}}_{\boldsymbol{y}_{n_2}^T} = \boldsymbol{0}^T_{k\times 1},\ m_1 \neq m_2,\ n_1 \neq n_2.
\end{aligned} \tag{6}$$

For the 3-D Case, we first decompose $\boldsymbol{J}$ into $\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{L}}$, where

$$\begin{aligned}
\bar{\boldsymbol{L}} = \mathrm{diag}(&l_{11}^{-1}l_{12}^{-1}, l_{21}^{-1}l_{22}^{-1}, r_{11}^{-2}, r_{21}^{-2}, r_{12}^{-2}, r_{22}^{-2},\\
&l_{11}^{-2}r_{11}^{-1}, l_{21}^{-2}r_{21}^{-1}, l_{12}^{-2}r_{12}^{-1}, l_{22}^{-2}r_{22}^{-1}).
\end{aligned} \tag{7}$$

$\bar{\boldsymbol{\Omega}}$ can be written as $\bar{\boldsymbol{\Omega}} = [\bar{\boldsymbol{\Omega}}_1, \bar{\boldsymbol{\Omega}}_2]$, where

$$\bar{\boldsymbol{\Omega}}_1 = \begin{bmatrix} \boldsymbol{w}_1 l_{11} - \boldsymbol{x}_1 l_{12} & \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{11}^\theta & \boldsymbol{0}_{k\times 1} \\ \boldsymbol{0}_{k\times 1} & \boldsymbol{w}_2 l_{21} - \boldsymbol{x}_2 l_{22} & \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{21}^\theta \\ -\boldsymbol{w}_1 l_{11} & -\boldsymbol{w}_2 l_{21} & \boldsymbol{0}_{k\times 1} & \boldsymbol{0}_{k\times 1} \end{bmatrix},$$

$$\bar{\boldsymbol{\Omega}}_2 = \begin{bmatrix} \bar{\boldsymbol{v}}_{12}^\theta & \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{11}^\phi & \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{12}^\phi & \boldsymbol{0}_{k\times 1} \\ \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{22}^\theta & \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{21}^\phi & \boldsymbol{0}_{k\times 1} & \bar{\boldsymbol{v}}_{22}^\phi \\ -\bar{\boldsymbol{v}}_{12}^\theta & -\bar{\boldsymbol{v}}_{22}^\theta & \boldsymbol{0}_{k\times 1} & \boldsymbol{0}_{k\times 1} & -\bar{\boldsymbol{v}}_{12}^\phi & -\bar{\boldsymbol{v}}_{22}^\phi \end{bmatrix}, \tag{8}$$

and for $m = 1, 2,\ n = 1, 2,$

$$\begin{aligned}
\boldsymbol{v}_{mn}^\theta &= [-(x_{m,(2)} - y_{n,(2)}), (x_{m,(1)} - y_{n,(1)}), 0]^T,\\
\boldsymbol{v}_{mn}^\phi &= [-(x_{m,(1)} - y_{n,(1)})(x_{m,(3)} - y_{n,(3)}),\\
&\quad -(x_{m,(2)} - y_{n,(2)})(x_{m,(3)} - y_{n,(3)}), r_{mn}^2]^T,
\end{aligned} \tag{9}$$

with $r_{mn} = \|\boldsymbol{x}_{m,(1:2)} - \boldsymbol{y}_{n,(1:2)}\|$. To examine the rank of $\bar{\boldsymbol{\Omega}}$, we perform the following operations. Left multiplying $\bar{\boldsymbol{\Omega}}$ by $\bar{\boldsymbol{E}} = \mathrm{blkdiag}(\bar{\boldsymbol{E}}_1, \bar{\boldsymbol{E}}_2, \boldsymbol{I}_k)$, with

$$\bar{\boldsymbol{E}}_m = \begin{bmatrix} 1 & 0 & 0 \\ x_{m,(1)} & x_{m,(2)} & 0 \\ x_{m,(1)} & x_{m,(2)} & x_{m,(3)} \end{bmatrix}, \tag{10}$$

and then performing several elementary operations to $\bar{\boldsymbol{E}}\bar{\boldsymbol{\Omega}}$ in order, including moving the 3-rd row to the 5-th row, exchanging the 2-nd and 3-rd rows, and moving the 3-rd, 4-th, 7-th, and 8-th columns to the end, we have

$$\bar{\boldsymbol{\Omega}} \sim \begin{bmatrix} \bar{\boldsymbol{\Omega}}_{11} & \bar{\boldsymbol{\Omega}}_{12} \\ \bar{\boldsymbol{\Omega}}_{21} & \boldsymbol{0}_{5\times 4} \end{bmatrix}. \tag{11}$$

The rank of the submatrix $[\bar{\boldsymbol{\Omega}}_{11}, \bar{\boldsymbol{\Omega}}_{12}]$ in (11) is determined by that of $\bar{\boldsymbol{\Omega}}_{12}$, which can be expressed as

$$\bar{\boldsymbol{\Omega}}_{12} = \begin{bmatrix} -x_{1,(2)} & 0 & -x_{1,(1)}x_{1,(3)} & 0 \\ 0 & -x_{2,(2)} & 0 & -x_{2,(1)}x_{2,(3)} \\ 0 & 0 & -r_{11}^2 x_{1,(3)} & 0 \\ 0 & 0 & 0 & -r_{21}^2 x_{2,(3)} \end{bmatrix}, \tag{12}$$

and the value of $\bar{\boldsymbol{\Omega}}_{11}$ is irrelevant. It is straightforward to see that $\bar{\boldsymbol{\Omega}}_{12}$ is full rank when the sources and sensors are not collinear.

In the following, we shall examine the rank of $\bar{\boldsymbol{\Omega}}_{21}$. By letting $e_m = l_{m1}\boldsymbol{x}_m^T(\boldsymbol{x}_m - \boldsymbol{y}_2) - l_{m1}^2 l_{m2}$, $f_m = \boldsymbol{x}_{m,(1:2)} \times \boldsymbol{y}_{2,(1:2)}$, $g_m = -(\boldsymbol{x}_{m,(1:2)}^T \boldsymbol{w}_{m,(1:2)})w_{m,(3)} + r_{m2}^2 x_{m,(3)}$, $m = 1, 2$ and using $\boldsymbol{y}_2 = c\boldsymbol{x}_1$ such that $e_1 = 2(1-c)l_{11}^3 \neq 0$ when $c > 1$ and $f_1 = g_1 = 0$, we can write $\bar{\boldsymbol{\Omega}}_{21}$ into (13) shown at the top of next page. For notation simplicity, let $\boldsymbol{P} = \bar{\boldsymbol{\Omega}}_{21,(2:5,2:5)}$. From (13), we have

$$\begin{aligned}
\det(\boldsymbol{P}) &= p_{11}(-1)^{1+1}\det(\boldsymbol{P}_{11}) + p_{13}(-1)^{1+3}\det(\boldsymbol{P}_{13})\\
&= e_2\det(\boldsymbol{P}_{11}) + f_2\det(\boldsymbol{P}_{13}),
\end{aligned} \tag{14}$$

where $\boldsymbol{P}_{ij}$ donates the cofactor of the matrix expanded along row i and column j, and p_{mn} denotes the element in the m row and n column of matrix $\boldsymbol{P}$. Since

$$\begin{aligned}
\det(\boldsymbol{P}_{11}) &= (-1)^{3+3}p_{44}(p_{22}p_{33} - p_{23}p_{32})\\
&= (-(1-c)^2 r_{11}^2)(-(1-c)x_{1,(2)}(x_{2,(1)} - cx_{1,(1)})\\
&\quad + (1-c)x_{1,(1)}(x_{2,(2)} - cx_{1,(2)}))\\
&= -(1-c)^3(x_{1,(1)}x_{2,(2)} - x_{1,(2)}x_{2,(1)})r_{11}^2,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\det(\boldsymbol{P}_{13}) &= (-1)^{3+1}p_{41}(p_{21}p_{34} - p_{24}p_{32})\\
&\quad + (-1)^{3+3}p_{44}(p_{21}p_{32} - p_{22}p_{31})\\
&= -(x_{2,(3)} - cx_{1,(3)})l_{21}((1-c)^3 x_{1,(3)}x_{1,(2)}^2\\
&\quad + (1-c)^3 x_{1,(3)}x_{1,(1)}^2)\\
&\quad - (1-c)^2 r_{11}^2((x_{2,(1)} - cx_{1,(1)})l_{21}(1-c)x_{1,(1)}\\
&\quad + (1-c)x_{12}(x_{2,(2)} - cx_{12})l_{21})\\
&= -(1-c)^3 l_{21}r_{11}^2 x_{1,(3)}(x_{2,(3)} - cx_{1,(3)})\\
&\quad - (1-c)^3 l_{21}r_{11}^2((\boldsymbol{x}_{1,(1:2)} \times \boldsymbol{x}_{2,(1:2)}) - cr_{11}^2)\\
&= -(1-c)^3 l_{21}r_{11}^2(\boldsymbol{x}_1^T\boldsymbol{x}_2 - cl_{11}^2),
\end{aligned} \tag{16}$$

$$\bar{\boldsymbol{\Omega}}_{21} = \begin{bmatrix} e_1 & 0 & f_1 & 0 & g_1 & 0 \\ 0 & e_2 & 0 & f_2 & 0 & g_2 \\ -w_{1,(1)}l_{11} & -w_{2,(1)}l_{21} & w_{1,(2)} & w_{2,(2)} & w_{1,(1)}w_{1,(3)} & w_{2,(1)}w_{2,(3)} \\ -w_{1,(2)}l_{11} & -w_{2,(2)}l_{21} & -w_{1,(1)} & -w_{2,(1)} & w_{1,(2)}w_{1,(3)} & w_{2,(2)}w_{2,(3)} \\ -w_{1,(3)}l_{11} & -w_{2,(3)}l_{21} & 0 & 0 & -r_{12}^2 & -r_{22}^2 \end{bmatrix}$$
$$= \begin{bmatrix} 2(1-c)l_{11}^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & f_2 & 0 & g_2 \\ -w_{1,(1)}l_{11} & -w_{2,(1)}l_{21} & w_{1,(2)} & w_{2,(2)} & w_{1,(1)}w_{1,(3)} & w_{2,(1)}w_{2,(3)} \\ -w_{1,(2)}l_{11} & -w_{2,(2)}l_{21} & -w_{1,(1)} & -w_{2,(1)} & w_{1,(2)}w_{1,(3)} & w_{2,(2)}w_{2,(3)} \\ -w_{1,(3)}l_{11} & -w_{2,(3)}l_{21} & 0 & 0 & -r_{12}^2 & -r_{22}^2 \end{bmatrix}. \tag{13}$$

we have

$$\begin{aligned}\det(\boldsymbol{P}) &= -e_2(1-c)^3(x_{1,(1)}x_{2,(2)} - x_{1,(2)}x_{2,(1)})r_{11}^2 \\ &\quad - f_2(1-c)^3 l_{21} r_{11}^2(\boldsymbol{x}_1^T\boldsymbol{x}_2 - cl_{11}^2) \\ &= -(1-c)^3 r_{11}^2 l_{21}(x_{1,(1)}x_{2,(2)} - x_{1,(2)}x_{2,(1)})\times \\ &\quad ((l_{21}^2 - c\boldsymbol{x}_2^T\boldsymbol{x}_1) - l_{21}l_{22} - c\boldsymbol{x}_1^T\boldsymbol{x}_2 + c^2 l_{11}^2) \\ &= -(1-c)^3 r_{11}^2(\boldsymbol{x}_{1,(1:2)} \times \boldsymbol{x}_{2,(1:2)}) l_{21}l_{22}(l_{21} - l_{22}).\end{aligned} \tag{17}$$

Obviously, if $l_{21} \neq l_{22}$ and $\boldsymbol{x}_{1,(1:2)} \times \boldsymbol{x}_{2,(1:2)} \neq 0$, then $\det(\bar{\boldsymbol{\Omega}}_{21,(2:5,2:5)}) \neq 0$, and hence, $\mathrm{rank}(\bar{\boldsymbol{\Omega}}_{21}) = 5$ holds.

Next, let us investigate the case when the condition $l_{21} \neq l_{22}$ or $\boldsymbol{x}_{1,(1:2)} \times \boldsymbol{x}_{2,(1:2)} \neq 0$ does not hold.

1) $l_{21} \neq l_{22}$ but $\boldsymbol{x}_{1,(1:2)} \times \boldsymbol{x}_{2,(1:2)} = 0$: If $l_{21} \neq l_{22}$ but $\boldsymbol{x}_{1,(1:2)} \times \boldsymbol{x}_{2,(1:2)} = 0$, by letting $\boldsymbol{x}_{2,(1:2)} = \bar{c}\boldsymbol{x}_{1,(1:2)}$ and $\boldsymbol{Q} = \det([\bar{\boldsymbol{\Omega}}_{21,(2:5,2:3)}, \bar{\boldsymbol{\Omega}}_{21,(2:5,5:6)}])$, we have

$$\begin{aligned}\det(\boldsymbol{Q}) &= q_{11}(-1)^{1+1}\det(\boldsymbol{Q}_{11}) + q_{14}(-1)^{1+4}\det(\boldsymbol{Q}_{14}) \\ &= e_2\det(\boldsymbol{Q}_{11}) - g_2\det(\boldsymbol{Q}_{14}),\end{aligned} \tag{18}$$

where $\boldsymbol{Q}_{ij}$ donates the cofactor of the matrix expanded along row i and column j, and q_{mn} denotes the element in the m row and n column of matrix $\boldsymbol{Q}$. Similarly, we can compute $\det(\boldsymbol{Q}_{11})$ and $\det(\boldsymbol{Q}_{14})$ by

$$\begin{aligned}&\det(\boldsymbol{Q}_{11}) \\ &= (-1)^{3+2}q_{43}(q_{22}q_{34} - q_{24}q_{32}) \\ &\quad + (-1)^{3+3}q_{44}(q_{22}q_{33} - q_{23}q_{32}) \\ &= (1-c)^3(\bar{c}-c)r_{11}^4(x_{2,(3)} - cx_{1,(3)}) \\ &\quad - (1-c)^3(\bar{c}-c)^2 r_{11}^4 x_{1,(3)} \\ &= (1-c)^3(\bar{c}-c)r_{11}^4((x_{2,(3)} - cx_{1,(3)}) - (\bar{c}-c)x_{1,(3)}) \\ &= (1-c)^3(\bar{c}-c)r_{11}^4(x_{2,(3)} - \bar{c}x_{1,(3)}),\end{aligned} \tag{19}$$

$$\begin{aligned}&\det(\boldsymbol{Q}_{14}) \\ &= (-1)^{3+1}q_{41}(q_{22}q_{33} - q_{23}q_{32}) \\ &\quad + (-1)^{3+3}q_{43}(q_{21}q_{32} - q_{22}q_{31}) \\ &= -(1-c)^3 l_{21}r_{11}^2 x_{1,(3)}(x_{2,(3)} - cx_{1,(3)}) \\ &\quad - (1-c)^3(\bar{c}-c)^2 l_{21}r_{11}^4 \\ &= -(1-c)^3 l_{21}r_{11}^2((x_{2,(3)} - cx_{1,(3)})x_{1,(3)} + (\bar{c}-c)r_{11}^2).\end{aligned} \tag{20}$$

Substituting them into $\det(\boldsymbol{Q})$ gives

$$\begin{aligned}&\det(\boldsymbol{Q}) \\ &= e_2((1-c)^3(\bar{c}-c)r_{11}^4(x_{2,(3)} - \bar{c}x_{1,(3)})) \\ &\quad + g_2((1-c)^3 l_{21}r_{11}^2((x_{2,(3)} - cx_{1,(3)})x_{1,(3)} + (\bar{c}-c)r_{11}^2)) \\ &= (1-c)^3(\bar{c}-c)r_{11}^4(x_{2,(3)} - \bar{c}x_{1,(3)})l_{21}(\bar{c}r_{11}^2(\bar{c}-c) \\ &\quad + (x_{2,(3)} - cx_{1,(3)})x_{2,(3)} - l_{21}l_{22}) + (1-c)^3(\bar{c}-c)\times \\ &\quad (\bar{c}x_{1,(3)} - x_{2,(3)})(x_{2,(3)} - cx_{1,(3)})cx_{1,(3)}l_{21}r_{11}^4 \\ &\quad + (1-c)^3(\bar{c}-c)^2(\bar{c}x_{1,(3)} - x_{2,(3)})cl_{21}r_{11}^6 \\ &= (1-c)^3(\bar{c}-c)(x_{2,(3)} - \bar{c}x_{1,(3)})l_{21}r_{11}^4(\bar{c}(\bar{c}-c)r_{11}^2 \\ &\quad + (x_{2,(3)} - cx_{1,(3)})x_{2,(3)} - l_{21}l_{22} \\ &\quad - (x_{2,(3)} - cx_{1,(3)})cx_{1,(3)} - c(\bar{c}-c)r_{11}^2) \\ &= (1-c)^3(\bar{c}-c)(x_{2,(3)} - \bar{c}x_{1,(3)})l_{21}r_{11}^4\times \\ &\quad ((x_{2,(3)} - cx_{1,(3)})^2 + (\bar{c}-c)r_{11}^2 - l_{21}l_{22}) \\ &= (1-c)^3 r_{11}^4 l_{21}l_{22}(\bar{c}-c)(x_{2,(3)} - \bar{c}x_{1,(3)})(l_{21} - l_{22}).\end{aligned} \tag{21}$$

When $\bar{c}x_{1,(3)} - x_{2,(3)} \neq 0$ and $c \neq \bar{c}$, $\mathrm{rank}(\bar{\boldsymbol{\Omega}}_{21}) = 5$ still holds true. Combining the results $\mathrm{rank}(\bar{\boldsymbol{\Omega}}_{12}) = 4$ and $\mathrm{rank}(\bar{\boldsymbol{\Omega}}_{21}) = 5$, we have $\mathrm{rank}(\bar{\boldsymbol{\Omega}}) = 9$, indicates that JSSL can be achieved. Note that when $c = \bar{c}$, the source $\boldsymbol{x}_2$ has the same x- and y-coordinates as the sensor $\boldsymbol{y}_2$. JSSL can still be realized for this special case, although $\bar{\boldsymbol{\Omega}}_{21}$ is rank-deficient.

2) $l_{21} = l_{22}$: If $l_{21} = l_{22}$, we define an unknown vector $\check{\boldsymbol{\beta}} = [\check{\beta}_1, \ldots, \check{\beta}_4]^T$ such that $\bar{\boldsymbol{\Omega}}_{21,(2:5,2:6)}^T\check{\boldsymbol{\beta}} = \boldsymbol{0}$, which can specially be written as

$$e_2\check{\beta}_1 - l_{21}(w_{2,(1)}\check{\beta}_2 + w_{2,(2)}\check{\beta}_3 + w_{2,(3)}\check{\beta}_4) = 0, \tag{22a}$$
$$w_{1,(2)}\check{\beta}_2 - w_{1,(1)}\check{\beta}_3 = 0, \tag{22b}$$
$$f_2\check{\beta}_1 + w_{2,(2)}\check{\beta}_2 - w_{2,(1)}\check{\beta}_3 = 0, \tag{22c}$$
$$w_{1,(1)}w_{1,(3)}\check{\beta}_2 + w_{1,(2)}w_{1,(3)}\check{\beta}_3 - r_{12}^2\check{\beta}_4 = 0, \tag{22d}$$
$$g_2\check{\beta}_1 + w_{2,(1)}w_{2,(3)}\check{\beta}_2 + w_{2,(2)}w_{2,(3)}\check{\beta}_3 - r_{22}^2\check{\beta}_4 = 0. \tag{22e}$$

From (22b), we can obtain $\check{\beta}_3 = (x_{1,(2)}/x_{1,(1)})\check{\beta}_2$. Putting $\check{\beta}_3$ into (22d) yields

$$r_{11}^2\check{\beta}_4 = (x_{1,(1)}x_{1,(3)} + x_{1,(2)}^2\frac{x_{1,(3)}}{x_{1,(1)}})\check{\beta}_2, \tag{23}$$

from which we have

$$\check{\beta}_4 = \frac{x_{1,(3)}}{x_{1,(1)}}\check{\beta}_2. \tag{24}$$

Thus $\check{\beta}_3$ and $\check{\beta}_4$ are written in terms of $\check{\beta}_2$.

Moreover, since $l_{21} = l_{22}$, we have

$$\begin{aligned}
&l_{21}^2 = l_{22}^2\\
&\|\boldsymbol{x}_2 - \boldsymbol{y}_1\|^2 = \|\boldsymbol{x}_2 - \boldsymbol{y}_1\|^2\\
&\|\boldsymbol{x}_2\|^2 = \|\boldsymbol{x}_2\|^2 - 2c\boldsymbol{x}_2^T\boldsymbol{x}_1 + c^2\|\boldsymbol{x}_1\|^2\\
&\boldsymbol{x}_1^T\boldsymbol{x}_2 = \frac{c}{2}l_{11}^2. \qquad (25)
\end{aligned}$$

Then substituting $\check{\beta}_3$, $\check{\beta}_4$ and (25) into (22a) yields

$$\begin{aligned}
\check{\beta}_2 &= \frac{e_2 x_{1,(1)}}{w_{2,(1)}x_{1,(1)} + w_{2,(2)}x_{1,(2)} + w_{2,(3)}x_{1,(3)}}\check{\beta}_1\\
&= \frac{-c\boldsymbol{x}_1^T\boldsymbol{x}_2 x_{1,(1)}}{\boldsymbol{x}_1^T\boldsymbol{x}_2 - cl_{11}^2}\check{\beta}_1\\
&= cx_{1,(1)}\check{\beta}_1. \qquad (26)
\end{aligned}$$

Using (26) in $\check{\beta}_3$ and $\check{\beta}_4$, we can also write them in term of $\check{\beta}_1$:

$$\check{\beta}_3 = cx_{1,(2)}\check{\beta}_1,\ \check{\beta}_4 = cx_{1,(3)}\check{\beta}_1. \qquad (27)$$

Equations (26) and (27) indicate that there exists a nonzero vector $\check{\boldsymbol{\beta}}$ such that $\bar{\boldsymbol{\Omega}}_{21,(2:5,2:6)}^T\check{\boldsymbol{\beta}} = \boldsymbol{0}$. Therefore, $\bar{\boldsymbol{\Omega}}_{21}$ is not full rank when $l_{21} = l_{22}$, and JSSL cannot be achieved.

By summarizing the above results, we conclude that JSSL cannot be achieved when $l_{21} = l_{22}$.

REFERENCES

[1] Y. Ye, G. Wang, and K. C. Ho, "Joint source and sensor localization by hybrid TDOA-AOA: Analysis and solution," *Submitted to IEEE Trans. Wireless Commun.*, Dec. 2025.

[2] Y. Wang and K. C. Ho, "An asymptotically efficient estimator in closed-form for 3-D AOA localization using a sensor network," *IEEE Trans. Wireless Commun.*, vol. 14, no. 12, pp. 6524-6535, Dec. 2015.

[3] K. C. Ho, "Localization through transceivers in unknown constant velocity trajectories," *IEEE Trans. Signal Process.*, vol. 70, pp. 3011-3028, 2022.

[4] G. Strang, *Introduction to Linear Algebra*, 5th ed. Wellesley, MA, USA: Wellesley-Cambridge Press, 2016.