Technical Report: Modified DAC Method for Moving Rigid Body Localization Using TOA Only

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***Abstract*—This report presents the modified divide-and-conquer (DAC) method for moving rigid body localization (MRBL) using time-of-arrival (TOA) measurements only. This method is an extension of the DAC method for stationary rigid body localization presented in [1, Section III].**

I. Introduction

Moving rigid body localization (MRBL) based on time-of-arrival (TOA) measurements has important applications in wireless communications, navigation, and tracking systems. The detailed background of MRBL can be found in [2].

II. TOA Measurement Model

Consider a rigid body moving in three-dimensional space with translational velocity $\boldsymbol{v}^o$, angular velocity $\boldsymbol{\omega}^o$, initial position $\boldsymbol{t}^o$, and initial rotation angle $\boldsymbol{\xi}^o$. The rigid body carries N sensors whose positions in the local coordinate system are known, denoted as $\boldsymbol{c}_i, i=1,\ldots,N$, while their global coordinates are unknown. At the initial time, the global coordinate of the i-th sensor is given by

$$\boldsymbol{s}_i^o = \boldsymbol{Q}^o \boldsymbol{c}_i + \boldsymbol{t}^o, \tag{1}$$

where $\boldsymbol{Q}^o \in SO(3) = \{\boldsymbol{Q} \in \mathbb{R}^{3\times 3} : \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}, \det(\boldsymbol{Q}) = 1\}$ denotes the rotation matrix corresponding to $\boldsymbol{\xi}^o$.

The system includes M anchors with known positions $\boldsymbol{a}_m, m=1,\ldots,M$, which perform TOA measurements on the sensors at fixed time intervals T, for a total of K measurements. At the k-th measurement, the global coordinate of the i-th sensor is expressed as

$$\boldsymbol{s}_{i,k}^o = \Delta\boldsymbol{Q}_k^o \boldsymbol{Q}^o \boldsymbol{c}_i + \boldsymbol{t}^o + kT\boldsymbol{v}^o, \tag{2}$$

where $\Delta\boldsymbol{Q}_k^o$ is the rotation matrix corresponding to the rotation $kT\boldsymbol{\omega}^o$.

Multiplying the TOA measurements by the signal propagation speed yields the distance between the i-th sensor and the m-th anchor at time kT:

$$\begin{aligned} d_{im,k} &= d_{im,k}^o + n_{im,k} = \|\boldsymbol{s}_{i,k}^o - \boldsymbol{a}_m\| + n_{im,k} \\ i &= 1,\ldots,\ N,\ m=1,\ldots,M,\ k=1,\ldots,K, \end{aligned} \tag{3}$$

where $d_{im,k}^o = \|\boldsymbol{s}_{i,k}^o - \boldsymbol{a}_m\|$ is the true distance, and $n_{im,k} \sim \mathcal{N}(0, \sigma_{im,k}^2)$ denotes the measurement noise.

III. Modified DAC Method for MRBL in 3-D

This section presents the modified DAC method briefly. The first step estimates the global sensor positions using the TOA measurements, and the second step estimates the rigid body parameters using the estimated global sensor positions. Since it is very similar to the original DAC method in [1, Section III], we only give the key steps in the following.

A. Step 1: Estimation of the Global Sensor Positions

We square both sides of the distance equation (3) and neglect the second-order noise terms to obtain:

$$d_{im,k}^2 \simeq \|\boldsymbol{s}_{i,k}^o\|^2 - 2\boldsymbol{a}_m^T \boldsymbol{s}_{i,k}^o + \|\boldsymbol{a}_m\|^2 + 2d_{im,k}^o n_{im,k}, \tag{4}$$

The transformed model (4) is related to the global sensor positions and it can be used to estimate them. More specifically, by letting $\boldsymbol{\theta}_1 = [\boldsymbol{s}_{i,k}^{oT}, \|\boldsymbol{s}_{i,k}^o\|^2]^T$ for any $i \in \{1,\ldots,N\}$ and $k \in \{1,\ldots,K\}$, (4) can be written in a matrix form as

$$\boldsymbol{h}_1 - \boldsymbol{G}_1\boldsymbol{\theta}_1 = \boldsymbol{B}_1\boldsymbol{\nu}_1, \tag{5}$$

where

$$\begin{aligned} \boldsymbol{G}_1 &= \begin{bmatrix} -2\boldsymbol{a}_1^T & 1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_M^T & 1 \end{bmatrix},\ \boldsymbol{h}_1 = \begin{bmatrix} d_{i1,k}^2 - \|\boldsymbol{a}_1\|^2 \\ \vdots \\ d_{iM,k}^2 - \|\boldsymbol{a}_M\|^2 \end{bmatrix}, \\ \boldsymbol{B}_1 &= 2\mathrm{diag}\left\{d_{i1,k}^o,\ldots,d_{iM,k}^o\right\}, \\ \boldsymbol{\nu}_1 &= [n_{i1,k},\ldots,n_{iM,k}]^T. \end{aligned} \tag{6}$$

The LWLS solution to (5) is

$$\hat{\boldsymbol{\theta}}_1 = (\boldsymbol{G}_1^T\boldsymbol{W}_1\boldsymbol{G}_1)^{-1}\boldsymbol{G}_1^T\boldsymbol{W}_1\boldsymbol{h}_1, \tag{7}$$

where $\boldsymbol{W}_1 = (\boldsymbol{B}_1\boldsymbol{R}_1\boldsymbol{B}_1^T)^{-1}$ with $\boldsymbol{R}_1$ being the covariance matrix of $\boldsymbol{\nu}_1$. It should be noted that $\boldsymbol{B}_1$ contains the true range values. We first set $\boldsymbol{W}_1$ to the identity matrix to obtain an initial estimate by (7), and then update $\boldsymbol{W}_1$ using this initial estimate.

The relations among the elements of $\hat{\boldsymbol{\theta}}_1$ are not considered in (7). From their relations, we can form the following model:

$$\boldsymbol{h}_2 - \boldsymbol{G}_2\boldsymbol{\theta}_2 = \boldsymbol{\nu}_2, \tag{8}$$

where

$$\boldsymbol{h}_2 = \begin{bmatrix} \hat{\boldsymbol{\theta}}_{1,(1:3)} \odot \hat{\boldsymbol{\theta}}_{1,(1:3)} \\ \hat{\boldsymbol{\theta}}_{1,(4)} \end{bmatrix},\ \boldsymbol{G}_2 = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{1}^T \end{bmatrix},\ \boldsymbol{\theta}_2 = [\boldsymbol{s}_{i,k}^o \odot \boldsymbol{s}_{i,k}^o]. \tag{9}$$

Its LWLS solution is

$$\hat{\boldsymbol{\theta}}_2 = (\boldsymbol{G}_2^T\boldsymbol{W}_2\boldsymbol{G}_2)^{-1}\boldsymbol{G}_2^T\boldsymbol{W}_2\boldsymbol{h}_2, \tag{10}$$

where $\boldsymbol{W}_2 = [\boldsymbol{S}(\boldsymbol{G}_1^T\boldsymbol{W}_1\boldsymbol{G}_1)^{-1}\boldsymbol{S}]^{-1}$ with $\boldsymbol{S} = \mathrm{diag}\{[2\hat{\boldsymbol{\theta}}_{1,(1:3)}^T, 1]^T\}$.

The sensor position estimate at time k is

$$\hat{\boldsymbol{s}}_{i,k} = \mathrm{diag}(\mathrm{sgn}(\hat{\boldsymbol{\theta}}_{1,(1:3)}))[\sqrt{\hat{\theta}_{2,(1)}}, \sqrt{\hat{\theta}_{2,(2)}}, \sqrt{\hat{\theta}_{2,(3)}}]^T, \tag{11}$$

and the estimation covariance matrix is

$$\mathrm{cov}(\hat{\boldsymbol{s}}_{i,k}) = \boldsymbol{B}_2^{-1}(\boldsymbol{G}_2^T\boldsymbol{W}_2\boldsymbol{G}_2)^{-1}\boldsymbol{B}_2^{-T}, \tag{12}$$

where $\boldsymbol{B}_2 = 2\mathrm{diag}\{\hat{\boldsymbol{s}}_{i,k}\}$.

B. Step 2: Estimation of the Rigid Body Parameters

In this step, we estimate the rigid body parameters using the estimated global sensor positions given in Step 1. Different from the original DAC method presented in [1, Section III], we cannot obtain an closed-form solution by performing singular-value decomposition (SVD) in this step. Instead, we propose to apply semidefinite relaxation (SDR) to solve the problem in this step.

To relate $\boldsymbol{s}_{i,k}^o$ with the unknowns, we first manipulate the expression. Since the measurement interval T is small, $kT\boldsymbol{\omega} = [\varpi_1, \varpi_2, \varpi_3]^T$ is also small. Therefore, the rotation matrix $\Delta\boldsymbol{Q}_k^o$ can be approximated using the small-angle approximations $\cos\varphi \simeq 1$ and $\sin\varphi \simeq \varphi$ as

$$\begin{aligned}
&\Delta\boldsymbol{Q}_k^o \\
&= \begin{bmatrix} c_{\varpi_2}c_{\varpi_3} & c_{\varpi_2}s_{\varpi_3} & -s_{\varpi_2} \\ s_{\varpi_1}s_{\varpi_2}c_{\varpi_3} - c_{\varpi_1}s_{\varpi_3} & s_{\varpi_1}s_{\varpi_2}s_{\varpi_3} + c_{\varpi_1}c_{\varpi_3} & c_{\varpi_2}s_{\varpi_1} \\ c_{\varpi_1}s_{\varpi_2}c_{\varpi_3} + s_{\varpi_1}s_{\varpi_3} & c_{\varpi_1}s_{\varpi_2}s_{\varpi_3} - s_{\varpi_1}c_{\varpi_3} & c_{\varpi_2}c_{\varpi_1} \end{bmatrix} \\
&\simeq \begin{bmatrix} 1 & \varpi_3 & -\varpi_2 \\ -\varpi_3 & 1 & \varpi_1 \\ \varpi_2 & -\varpi_1 & 1 \end{bmatrix} = \boldsymbol{I} - kT[\boldsymbol{\omega}^o]^\times, \qquad (13)
\end{aligned}$$

where $s_\varphi = \sin\varphi^o$, $c_\varphi = \cos\varphi^o$ and

$$[\boldsymbol{\omega}^o]^\times = \begin{bmatrix} 0 & -\omega_{(3)}^o & \omega_{(2)}^o \\ \omega_{(3)}^o & 0 & -\omega_{(1)}^o \\ -\omega_{(2)}^o & \omega_{(1)}^o & 0 \end{bmatrix}. \qquad (14)$$

Substituting (13) into (2) gives

$$\begin{aligned}
\boldsymbol{s}_{i,k}^o &\simeq (\boldsymbol{I} - kT[\boldsymbol{\omega}^o]^\times)\boldsymbol{Q}^o\boldsymbol{c}_i + \boldsymbol{t}^o + kT\boldsymbol{v}^o \\
&= \boldsymbol{Q}^o\boldsymbol{c}_i - kT[\boldsymbol{\omega}^o]^\times\boldsymbol{Q}^o\boldsymbol{c}_i + \boldsymbol{t}^o + kT\boldsymbol{v}^o \\
&= \boldsymbol{Q}^o\boldsymbol{c}_i - kT\boldsymbol{P}^o\boldsymbol{c}_i + \boldsymbol{t}^o + kT\boldsymbol{v}^o, \qquad (15)
\end{aligned}$$

where $\boldsymbol{P}^o = [\boldsymbol{\omega}^o]^\times\boldsymbol{Q}^o$. Using the property $\mathrm{vec}(\boldsymbol{Y}_1\boldsymbol{Y}_2\boldsymbol{Y}_3) = (\boldsymbol{Y}_3^T \otimes \boldsymbol{Y}_1)\mathrm{vec}(\boldsymbol{Y}_2)$, (15) can be rewritten as

$$\boldsymbol{s}_{i,k}^o = (\boldsymbol{c}_i^T \otimes \boldsymbol{I})\mathrm{vec}(\boldsymbol{Q}^o) - kT(\boldsymbol{c}_i^T \otimes \boldsymbol{I})\mathrm{vec}(\boldsymbol{P}^o) + \boldsymbol{t}^o + kT\boldsymbol{v}^o, \qquad (16)$$

According to (16), the following constrained weighted least squares (CWLS) problem can be formulated using $\hat{\boldsymbol{s}}_{i,k}$:

$$\begin{aligned}
&\min_{\boldsymbol{y}} \ (\hat{\boldsymbol{A}}\boldsymbol{z} - \hat{\boldsymbol{s}})^T\hat{\boldsymbol{W}}(\hat{\boldsymbol{A}}\boldsymbol{z} - \hat{\boldsymbol{s}}) \qquad (17a) \\
&\text{s.t. } \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}, \qquad (17b) \\
&\quad \det(\boldsymbol{Q}) = 1, \qquad (17c)
\end{aligned}$$

where the weighting matrix

$$\begin{aligned}
\hat{\boldsymbol{W}} = &\mathrm{Diag}\{\mathrm{cov}(\hat{\boldsymbol{s}}_{1,1}), \ldots, \mathrm{cov}(\hat{\boldsymbol{s}}_{N,1}), \ldots, \mathrm{cov}(\hat{\boldsymbol{s}}_{1,K}), \ldots, \\
&\mathrm{cov}(\hat{\boldsymbol{s}}_{N,K})\}^{-1},
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{z} &= [\mathrm{vec}(\boldsymbol{Q})^T, \mathrm{vec}(\boldsymbol{P})^T, \boldsymbol{t}^T, \boldsymbol{v}^T]^T, \\
\hat{\boldsymbol{s}} &= [\hat{\boldsymbol{s}}_{1,1}^T, \ldots, \hat{\boldsymbol{s}}_{N,1}^T, \ldots, \hat{\boldsymbol{s}}_{1,K}^T, \ldots, \hat{\boldsymbol{s}}_{N,K}^T]^T, \\
\hat{\boldsymbol{A}} &= [\hat{\boldsymbol{A}}_{1,1}^T, \ldots, \hat{\boldsymbol{A}}_{N,1}^T, \ldots, \hat{\boldsymbol{A}}_{1,K}^T, \ldots, \hat{\boldsymbol{A}}_{N,K}^T]^T, \qquad (18)
\end{aligned}$$

with $\hat{\boldsymbol{A}}_{i,k} = [(\boldsymbol{c}_i^T \otimes \boldsymbol{I}), -(\boldsymbol{c}_i^T \otimes kT\boldsymbol{I}), \boldsymbol{I}, kT\boldsymbol{I}]$.

Next, we perform SDR to solve the CWLS problem. To do so, we first introduce an auxiliary matrix $\boldsymbol{Z} = \boldsymbol{z}\boldsymbol{z}^T$, which has the following equivalent form:

$$\boldsymbol{Z} = \boldsymbol{z}\boldsymbol{z}^T \Longleftrightarrow \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & 1 \end{bmatrix} \succeq \boldsymbol{0}, \ \mathrm{rank}\left(\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & 1 \end{bmatrix}\right) = 1.$$

Using $\boldsymbol{Z}$, the constraint (17b) can be expressed as

$$\begin{aligned}
&\mathrm{tr}\{\boldsymbol{Z}_{(1:3,1:3)}\} = 1, \ \mathrm{tr}\{\boldsymbol{Z}_{(4:6,4:6)}\} = 1, \\
&\mathrm{tr}\{\boldsymbol{Z}_{(7:9,7:9)}\} = 1, \ \mathrm{tr}\{\boldsymbol{Z}_{(1:3,4:6)}\} = 0, \\
&\mathrm{tr}\{\boldsymbol{Z}_{(1:3,7:9)}\} = 0, \ \mathrm{tr}\{\boldsymbol{Z}_{(4:6,7:9)}\} = 0. \qquad (19)
\end{aligned}$$

Ignoring the rank-1 and determinant-1 constraints, problem (17) can be relaxed as the following semidefinite programming (SDP) problem:

$$\begin{aligned}
&\min_{\boldsymbol{Z},\boldsymbol{z}} \ \mathrm{tr}\{\hat{\boldsymbol{A}}^T\hat{\boldsymbol{W}}\hat{\boldsymbol{A}}\boldsymbol{Z} - 2\hat{\boldsymbol{s}}^T\hat{\boldsymbol{W}}\hat{\boldsymbol{A}}\boldsymbol{z} + \hat{\boldsymbol{s}}^T\hat{\boldsymbol{W}}\hat{\boldsymbol{s}}\} \\
&\text{s.t. } (19), \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & 1 \end{bmatrix} \succeq \boldsymbol{0}. \qquad (20)
\end{aligned}$$

After solving (20), the same orthogonalization and refinement steps described in [2, Section IV] are followed to obtain the refined DAC solution.

References

[1] S. Chen and K. C. Ho, "Accurate localization of a rigid body using multiple sensors and landmarks," *IEEE Trans. Signal Process.*, vol. 63, no. 24, pp. 6459-6472, Dec. 2015.

[2] Y. Hu, G. Wang, and K. C. Ho, "Moving rigid body localization using TOA only," *IEEE Trans. Wireless Commun.*, Submitted.