Cubic Spline

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Contents

1 Generic

For a collection of $n+1$ points, there is n intervals. A cubic spline is a way to calculate the coefficients of a cubic polynomial on every one of these intervals i:

$$g\left(x_i\right) = a_i + b_i\left(r - r_i\right) + c_i\left(r - r_i\right)^2 + d_i\left(r - r_i\right)^3 \tag{1}$$

For $n+1$ points, we define[1] the vector $\mathbf{h}$ of size n as the intervals size:

$$h_i = r_{i+1} - r_i \tag{2}$$

A spline will solve:

$$\mathbf{A}\mathbf{m} = \mathbf{b} \tag{3}$$

A "natural" spline will have zero curvature at its boundaries. Such a spline is given by this definition:

$$\mathbf{A} = \begin{bmatrix}
1 & 0 & & & & & & \\
h_0 & 2\left(h_0 + h_1\right) & h_1 & 0 & & & & \\
0 & h_1 & 2\left(h_1 + h_2\right) & h_2 & 0 & & \cdots & \\
0 & 0 & h_2 & 2\left(h_2 + h_3\right) & h_3 & 0 & & \\
 & & & & \ddots & & & \\
 & & & 0 & h_{n-3} & 2\left(h_{n-3} + h_{n-2}\right) & h_{n-2} & 0 \\
 & & \vdots & & 0 & h_{n-2} & 2\left(h_{n-2} + h_{n-1}\right) & h_{n-1} \\
 & & & & & & 0 & 1
\end{bmatrix} \tag{4}$$

[1] http://people.math.sfu.ca/~stockie/teaching/macm316/notes/splines.pdf

and:

$$\mathbf{b} = 6 \begin{bmatrix} 0 \\ \frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0} \\ \frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1} \\ \vdots \\ \frac{y_{n-1}-y_{n-2}}{h_{n-2}} - \frac{y_{n-2}-y_{n-3}}{h_{n-3}} \\ \frac{y_n-y_{n-1}}{h_{n-1}} - \frac{y_{n-1}-y_{n-2}}{h_{n-2}} \\ 0 \end{bmatrix} \tag{5}$$

After solving for $\mathbf{m}$, the cubic polynomials' coefficients for each sections i is obtained:

$$a_i = y_i \tag{6a}$$

$$b_i = \frac{y_{i+1} - y_i}{h_i} - \frac{h_i m_i}{2} - \frac{h_i}{6}\left(m_{i+1} - m_i\right) \tag{6b}$$

$$c_i = \frac{m_i}{2} \tag{6c}$$

$$d_i = \frac{m_{i+1} - m_i}{6h_i} \tag{6d}$$

2 3 points

Specializing to $n + 1 = 3$ points (or two intervals i), the system to solve:

$$\mathbf{Am} = \mathbf{b} \tag{7}$$

becomes

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ h_0 & 2\left(h_0 + h_1\right) & h_1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = 6 \begin{bmatrix} 0 \\ \frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0} \\ 0 \end{bmatrix} \tag{8}$$

We see that $m_0 = m_2 = 0$ and that:

$$h_0 m_0 + 2\left(h_0 + h_1\right) m_1 + h_1 m_2 = 6\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) \tag{9}$$

$$2\left(h_0 + h_1\right) m_1 = 6\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) \tag{10}$$

$$m_1 = \left(\frac{3}{h_0 + h_1}\right)\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) \tag{11}$$

Inserting this into the coefficients, we get the coefficients for the first interval:

$$a_0 = y_0 \tag{12a}$$

$$b_0 = \frac{y_1 - y_0}{h_0} - \frac{h_0 m_1}{6} \tag{12b}$$

$$c_0 = 0 \tag{12c}$$

$$d_0 = \frac{m_1}{6h_0} \tag{12d}$$

and for the second interval

$$a_1 = y_1 \tag{13a}$$

$$b_1 = \frac{y_2 - y_1}{h_1} - \frac{h_1 m_1}{2} + \frac{h_1 m_1}{6} \tag{13b}$$

$$c_1 = \frac{m_1}{2} \tag{13c}$$

$$d_1 = -\frac{m_1}{6h_1} \tag{13d}$$