Two-Sided Lossless Expanders in the Unbalanced Setting

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Abstract

We present the first explicit construction of *two-sided* lossless expanders in the unbalanced setting (bipartite graphs that have polynomially many more nodes on the left than on the right). Prior to our work, all known explicit constructions in the unbalanced setting achieved only one-sided lossless expansion.

Specifically, we show that the one-sided lossless expanders constructed by Kalev and Ta-Shma (RANDOM'22)—that are based on multiplicity codes introduced by Kopparty, Saraf, and Yekhanin (STOC'11)—are, in fact, two-sided lossless expanders. Moreover, we show that our result is tight, thus completely characterizing the graph of Kalev and Ta-Shma.

Using our unbalanced bipartite expander, we easily obtain lossless (non-bipartite) expander graphs on N vertices with polynomial degree $\ll N$ and expanding sets of size $N^{0.49}$.

1 Introduction

Lossless expanders are graphs in which small sets of vertices have almost as many neighbors as possible. Formally, we say that a d-regular graph $G = (V, E)$ is a (K, A)-*expander* if for all sets $S \subseteq V$ of size at most K we have that $|\Gamma(S)| \geq A\,|S|$ where $\Gamma(S)$ is the neighborhood of S. Generally, we desire that K is as large as possible with $K = \Omega(|V|/d)$. When $A = (1-\varepsilon)d$ for some small ε, we say that G is a (K, ε)-*lossless expander* since only a small fraction of the total number of possible neighbors is lost. It is well-known that a random d-regular graph is a $(K = \gamma n, \varepsilon = 0.01)$-lossless expander with high probability, for some constant $\gamma > 0$.

A reasonable question after seeing this definition is whether other notions of expansion, such as spectral or edge expansion, can be used to derive such graphs. Unfortunately, while Ramanujan graphs (optimal spectral expanders) do have expansion factor arbitrarily close to $A = d/2$, there also exist examples of Ramanujan graphs with expansion factor exactly $A = d/2$, showing that spectral expansion does not necessarily give rise to lossless expansion [Kah95].

The study of lossless expanders has paid special attention to bipartite graphs due to their connection with randomness condensers. A *one-sided lossless* expander is a bipartite graph $G = (L \sqcup R, E)$ where every "small enough" set on the left expands losslessly to the right. It is standard to view lossless condensers and one-sided bipartite lossless expanders as related objects. For this purpose, it is natural to talk about highly unbalanced bipartite graphs in which $|L| \gg |R|$, that is, the neighbor function that takes in a left vertex and the index of a neighbor and outputs the right vertex has a much shorter output length than input length. Current explicit constructions for unbalanced one-sided lossless expanders [TU06; TUZ07; GUV09; KT22]—with the best parameters achieved by [GUV09; KT22] [1] —have found a wide array of applications in

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[1] The lossless expanders of [KT22] have slightly better dependence on constants compared to [GUV09].

coding theory [SS96], extractor constructions [TU06; TUZ07; GUV09; DKSS13], derandomization [DT23],[2] and probabilistic data structures [UW87; BMRV02], with the unbalanced nature of the graph being essential.

We say a bipartite graph is a *two-sided lossless* expander if lossless expansion happens in both directions. Intuitively, in the unbalanced case where $|L| \gg |R|$, lossless expansion is hard to achieve from left to right because there is little room on the smaller side to allow for expansion from the much larger side; on the other hand, the right-to-left expansion seems to be much easier at first sight for the same reason. However, despite aforementioned constructions for one-sided lossless expanders as well as their broad applications, there has been no known explicit construction of unbalanced two-sided lossless expanders before this work. Therefore, it is an extremely intriguing direction to further the theory of lossless expanders. We are not aware of any existing applications of two-sided lossless expanders with a polynomial imbalance between the left and right—we leave this as an interesting open problem—but we believe that a deeper understanding of the structure of these objects is the first step towards making connections with other topics and finding new applications.

In this work, we try to fill the gap, namely, the lack of explicit unbalanced two-sided lossless expanders by closely studying the bipartite graph of [KT22] based on the multiplicity codes of [KSY14]. [3] For simplicity, we refer to this graph as the "KT graph" (see Definition 3.5). As our first main result, we show that the KT graph is a two-sided lossless expander. Moreover, we prove that the expansion is tight up to a constant multiplicative factor. As our second main result, we obtain non-bipartite lossless expanders with high degree by taking the bipartite half of the KT graph.

We note that lossless expanders have been extensively studied in the *balanced* setting as well. We discuss this in Section 1.2.

1.1 Our results

We first define a two-sided lossless expander formally:

Definition 1.1 (Two-sided lossless expander)**.** *We say that a (D_L, D_R)-regular bipartite graph $G = (L \sqcup R, E)$ is a* two-sided (K_L, A_L, K_R, A_R)-expander *if for any subset $S_L \subseteq L$ such that $|S_L| \leq K_L$ we have that $|\Gamma_{\rightarrow}(S_L)| \geq A_L |S_L|$ and similarly that for any subset $S_R \subseteq R$ such that $|S_R| \leq K_R$ we have that $|\Gamma_{\leftarrow}(S_R)| \geq A_R |S_R|$. When $A_L = (1-\varepsilon_L)D_L$ and $A_R = (1-\varepsilon_R)D_R$ for small $\varepsilon_L, \varepsilon_R > 0$, we say that G is a* two-sided $(K_L, \varepsilon_L, K_R, \varepsilon_R)$-lossless expander*.*

With the above definition, we are ready to state the main theorems:

Theorem 1 (Informal version of Theorem 4.8, bipartite two-sided lossless expander)**.** *For infinitely many N and all constant $0 < \delta \leq 0.99$, there exists an explicit, biregular, two-sided $(K_L, \varepsilon_L = 0.01, K_R, \varepsilon_R = 0.01)$ lossless expander $\Gamma_{\rightarrow} : [N] \times [D_L] \to [M]$ where $D_L = \mathrm{poly}(\log N)$, $N^{1.01\delta - o(1)} \leq M \leq D_L \cdot N^{1.01\delta}$, $K_L = N^{\delta}$, and $K_R = \min(O(M/D_L), O(N/(MD_L)))$.*

On the other hand, for every such graph, there exists a set $S \subseteq R$, $|S| = O(K_R)$, on the right side which has less than $1/2 \cdot D_R \cdot |S|$ neighbors on the left.

Remark 1.2. *Because [KT22] has optimal left degree of their bipartite graph (up to polynomial factors), we achieve optimal left-degree as well and, with respect to this, achieve optimal right degree, optimal expansion constant, optimal size of sets of vertices on left side that losslessly expand, and optimal size of sets of vertices on right side that losslessly expand when $M \leq \sqrt{N}$.* [4]

We obtain our second main result by taking the bipartite half (see Section 2.2 for more details) of the [KT22] graph, using the fact that it is a two-sided lossless expander:

[2] [DT23] instantiated Goldreich's PRG [Gol11] with the lossless expander of [KT22].

[3] It is natural to consider whether the unbalanced expander from [GUV09] is a two-sided lossless expander. It will be interesting to determine this since the GUV graph is not even right-regular—see Appendix B for details.

[4] To see that this setting of K_R is indeed optimal, note that in a (D_L, D_R)-biregular graph it must be that $N \cdot D_L = M \cdot D_R$ and so $\frac{M}{D_L} = \frac{N}{D_R}$. Hence, $K_R = O(M/D_L) = O(N/D_R)$, the largest possible size.

Theorem 2 (Informal version of Theorem 6.1, non-bipartite lossless expander)**.** *For infinitely many N and all constant $0 < \delta < 0.99$, there exists an explicit regular $(K, \varepsilon = 0.01)$ lossless expander $G = (V, E)$ where $|V| = N$, the degree is D where $N^{1-1.01\delta} \leq D \leq N^{1-1.01\delta+o(1)}$ and $K = \min\left(N^{\delta}, N^{1-1.01\delta-o(1)}\right)$. Furthermore, G with one vertex is removed, is endowed with a free group action from the multiplicative group $\mathbb{F}_q$, where $q = \mathrm{poly}(\log N)$.*

We realize that though the free group action adds to the structure of this graph, the group action is too small for applications.

Remark 1.3. *When $\delta \leq 0.49$, the value of K is almost optimal (a trivial upper bound is $K \leq N/D$) since in that regime, $K = N^{\delta} \geq (N/D)^{0.99}$.*

One can show that there exist non-bipartite lossless expanders with even constant degree. So, the degree of our lossless graph obtained is far from optimal. Nevertheless, as far as we know, this is the first explicit construction of a regular lossless (non-bipartite) expander with expanding sets of size $N^{0.49}$.

1.2 Lossless expanders in the balanced setting

A parallel line of work considers *balanced* bipartite lossless expanders, with motivations from coding theory. There have been explicit constructions of balanced one-sided lossless expanders [CRVW02; CRT23; Gol24], with which one can construct good error correcting codes [SS96]. There has also been progress in understanding two-sided lossless expanders in the balanced setting. Lossless expansion was shown to be feasible in high-girth regular graphs [MM21; HMMP24], taking the bipartite double cover of which implies a balanced two-sided lossless expander. It was shown in [LH22] that balanced two-sided lossless expanders with constant degree, constant imbalance, and certain algebraic properties have applications to good quantum low-density parity check (qLDPC) codes. Towards this direction, [HMMP24] constructed explicit two-sided lossless expansion for extremely small sets of size $K = \Omega(\exp(\sqrt{\log |V|}))$, which is still not sufficient for the application in [LH22]. Our work in the polynomially unbalanced setting is incomparable to the balanced setting, as it is possible to achieve constant degree in the balanced setting, while having a polynomial imbalance in left and right nodes forces the graph to have non-constant degrees.

Concurrent works Since our paper was made public online, there have been two new papers on constructions of balanced lossless expanders. Chen [Che25] achieved balanced two-sided lossless expansion for polynomial-sized sets as one of their several results; however, their size of expanding sets is not optimal in the balanced setting. Our analysis proves optimal expanding set size in the unbalanced setting (see Remark 1.2). Hsieh, Lin, Mohanty, O'Donnell and Zhang [HLMOZ24] achieved $\frac{3}{5}$-two-sided unique-neighbor expansion in the balanced setting, which is the first explicit construction of balanced two-sided vertex expanders beyond the spectral barrier.

2 Proof Overview

In this section, we first outline the proof of Theorem 1—our two-sided lossless expander. Using it, we construct high degree non-bipartite lossless expanders, proving Theorem 2.

2.1 Two-sided lossless expander

We show that the bipartite graph defined in [KT22] based on multiplicity codes is a two-sided lossless expander. The left-to-right lossless expansion was shown in [KT22]. Our main contribution is showing that the KT graph also expands losslessly from right to left. To do this, we first show that the KT graph is right-regular. Second, for any pair of right vertices, we compute the exact number of common left neighbors they have. Finally, for any not-too-large subset on the right, we lower bound the number of its left neighbors

by using the inclusion-exclusion principle to subtract all possible double counted common left neighbors from the total number of outgoing edges.

We state an informal version of our result and present details on the strategy sketched above.

Theorem 2.1 (Informal version of Theorem 4.1)**.** *For every field $\mathbb{F}_q$ and $n, s \in \mathbb{N}$ with $15 \leq (s+1) < n < char(\mathbb{F}_q)$, and any $\delta > 0$, there exists an explicit bipartite graph $G = (L \sqcup R, E)$ with $L = \mathbb{F}_q^n, R = \mathbb{F}_q^{s+2}$ with left degree $d_L = q$ and right degree $d_R = q^{n-(s+1)}$ such that G is a two-sided (K_L, A_L, K_R, A_R) expander where $K_L = \Omega(q^{s+1})$, $A_L = q - n(s+2)$, $K_R = \delta q^{\min(s+1, n-(s+1))}$, $A_R = \left(1 - O\left(\delta \cdot \frac{q-1}{q}\right)\right) q^{n-(s+1)}$.*

Theorem 1 is obtained from Theorem 2.1 by instantiating the parameters appropriately (see Section 4.3 for more details). We now define the KT graph and then claim that it is a lossless right expander.

Definition 2.2 (The KT graph [KT22])**.** *Let $q, n, s \in \mathbb{N}$ be such that q is a prime power, characteristic of the finite field $\mathbb{F}_q \geq n$ and $s \leq n/2$. We define $G = (L \sqcup R, E)$ where $L = \mathbb{F}_q^n, R = \mathbb{F}_q^{s+2}$. The left degree is q and for any $f \in \mathbb{F}_q^n$ and $y \in \mathbb{F}_q$, the y'th neighbor of f is defined as follows: Identify f as member of $\mathbb{F}_q[X]$ with degree of f at most $n-1$; then, the neighbor $\Gamma_{\rightarrow}(f, y)$ will be $(y, f^{(0)}(y), \ldots, f^{(s)}(y))$ where $f^{(i)}$ is the i'th iterative derivative of f.*

Theorem 2.3 (The KT graph losslessly expands from the right)**.** *The KT graph G is a right (K_R, A_R)-lossless expander where $K_R = \delta \min(|R|, |L| / |R|)$, $\varepsilon_R = O(\delta \cdot \frac{q-1}{q})$ for arbitrary $0 < \delta < 1$. In other words, for any subset $T \subseteq R$, $|T| \leq K_R$, T has at least $(1 - \varepsilon_R) d_R |T|$ neighbors on the left.*

Theorem 2.1 immediately follows from left expansion shown by [KT22] and Theorem 2.3. For the rest of this section, we focus on proving Theorem 2.3 that relies on the following two key lemmas.

Lemma 2.4 (Right regularity)**.** *The KT graph G is right-regular and has right-degree $d_R = q^{n-(s+1)}$.*

Lemma 2.5 (Number of common left neighbors)**.** *For any pair of right-vertices $w_1, w_2 \in \mathbb{F}_q^{s+2}$ such that $w_1 = (y_1, z_1), w_2 = (y_2, z_2)$ where $y_1 \neq y_2 \in \mathbb{F}_q$ and $z_1, z_2 \in \mathbb{F}_q^{s+1}$, we have $|\Gamma_{\leftarrow}(y_1, z_1) \cap \Gamma_{\leftarrow}(y_2, z_2)| = q^{n-(2s+2)}$ if $n \geq 2s+2$ and $|\Gamma_{\leftarrow}(y_1, z_1) \cap \Gamma_{\leftarrow}(y_2, z_2)| \leq 1$ if $n \leq 2s+2$.*

Theorem 2.3 then follows by an application of the inclusion-exclusion principle—subtracting the maximum number of common neighbors between any pair of vertices in T from the total number of edges leaving T—we get the required lower bound on the size of T's left neighborhood.

We now discuss the proof techniques for showing Lemma 2.4 and Lemma 2.5. We start by making a simple but useful observation on the structure of the KT graph G.

Observation 2.6. *Fix $w = (y, z_0, \cdots, z_s) \in R$ and let $f \in L$ be any left-neighbor of w. Then it must be the case that w is the y'th neighbor of f. Now for any $w' \in R$ such that $w' = (y, z'_0, \cdots, z'_s)$, it holds that $f \notin \Gamma_{\leftarrow}(w')$. This is saying that any pair of right vertices (w, w') that come from the same seed [5] must have disjoint left neighborhoods.*

Central to our analysis of the right degree and the number of common left neighbors are the following linear maps.

Definition 2.7. *For $y \in \mathbb{F}_q$, define the map $\psi_y(f) : \mathbb{F}_q^n \to \mathbb{F}_q^{s+1}$ as follows: Interpret $f \in \mathbb{F}_q[X]$ as a degree $\leq n-1$ polynomial and map it to $(f^{(0)}(y), \ldots, f^{(s)}(y))$ where $f^{(i)}$ is the i'th iterative derivative of f.*

We note that ψ_y is a $\mathbb{F}_q$-linear map, for any $y \in \mathbb{F}_q$.

Definition 2.8. *For $y_1, y_2 \in \mathbb{F}_q$, $y_1 \neq y_2$, define the map $\psi_{y_1, y_2}(f) : \mathbb{F}_q^n \to \mathbb{F}_q^{2(s+1)}$ as the concatenation of the respective linear maps, that is, $\psi_{y_1, y_2}(f) = (\psi_{y_1}(f), \psi_{y_2}(f))$.*

Proving the above lemmas (about the KT graph) now boils down to analyzing both ψ_y and ψ_{y_1, y_2} for all $y, y_1, y_2 \in \mathbb{F}_q$.

[5] We sometimes refer to $y \in \mathbb{F}_q$ as the "seed", like in the condensers literature.

1. We show that ψ_y for $y \in \mathbb{F}_q$ is surjective, which along with Observation 2.6 implies Lemma 2.4: For any $w = (y, z_0, \cdots, z_s) \in R$, the set of its left neighbors is $\{f \in L \mid (y, \psi_y(f)) = w\} = \psi_y^{-1}(z_0, \cdots, z_s)$. Therefore, the right degree $D_R = \left|\psi_y^{-1}(z_0, \cdots, z_s)\right| = q^n/q^{s+1} = q^{n-(s+1)}$.

2. We show ψ_{y_1,y_2} is surjective when $n \geq 2s+2$ and injective $n \leq 2s+2$, which implies Lemma 2.5: Similar to above, let $w_1 = (y_1, z_1) \in R$ and $w_2 = (y_2, z_2) \in R$, $y_1 \neq y_2$, be any pair of right vertices from *different* seeds. We extend Observation 2.6 to see that the number of $f \in L$ such that $(y_1, \psi_{y_1}(f)) = w_1$ and $(y_2, \psi_{y_2}(f)) = w_2$ is exactly $\left|\psi_{y_1,y_2}^{-1}(z_1, z_2)\right|$. When $n \geq 2s+2$, this map is surjective, and the number of left neighbors shared by w_1 and w_2 is $q^{n-(2s+2)}$. When $n \leq 2s+2$, this map is injective and the number of shared neighbors is at most 1.

To conclude the proof, we carry out Hermite interpolation (see Lemma 4.5) by applying the Chinese remainder theorem to show the surjectivity and injectivity of the maps ψ_y, ψ_{y_1} and ψ_{y_2}.

2.1.1 Tightness of right expansion

We will show that our parameters from Theorem 2.3 are tight. Throughout this section, we let $P_d \subseteq \mathbb{F}_q[x]$ be the polynomials of degree exactly d and $P_{<d} \subseteq \mathbb{F}_q[x]$ be the vector space over $\mathbb{F}_q$ of polynomials of degree strictly less than d. First, notice that when $|R| > |L|/|R|$, the right expansion is optimal by Remark 1.2. Hence, we only focus on the case when $s+1 < n < 2s+2$ and show the following:

Theorem 2.9 (Informal version of Theorem 5.1)**.** *For $s+1 < n < 2s+2$ and any $0 < \delta \leq 2$, there exists $T \subseteq R$ such that $|T| = \delta q^{n-s-1}$ and $|\Gamma_{\leftarrow}(T)| = (1-\varepsilon)D_R\,|T|$ with $\varepsilon = \delta/4$.*

For fixed $y_1 \neq y_2 \in \mathbb{F}_q$, we will construct $T = T_1 \sqcup T_2$ satisfying the following:

- $|T_1| + |T_2| = (\delta/2)q^{n-s-1}$.
- The first coordinate of every element of T_1 is always y_1 and of T_2 is always y_2.
- For all $t_1 \in T_1$ and $t_2 \in T_2$, there exists a degree less than n polynomial f such that $\Gamma_{\rightarrow}(f, y_1) = t_1$ and $\Gamma_{\rightarrow}(f, y_2) = t_2$.

We then observe that $|\Gamma_{\leftarrow}(T)| = |\Gamma_{\leftarrow}(T_1)| + |\Gamma_{\leftarrow}(T_2)| - |T_1| \cdot |T_2|$. Since the right degree of G is q^{n-s-1}, we have that $|\Gamma_{\leftarrow}(T_1)| = |\Gamma_{\leftarrow}(T_2)| = (\delta/2)q^{2n-2s-2}$. Hence, we compute that $|\Gamma_{\leftarrow}(T)| = (\delta - \delta^2/4)q^{2n-2s-2} = (1-\delta/4)D_R\,|T|$, proving Theorem 2.9.

To construct such T_1, T_2, it suffices to construct $S_1, S_2 \subset \mathbb{F}_q^{s+1}$ with $|S_1| = |S_2| = K/2$ such that $S_1 \times S_2 \subset \psi_{y_1,y_2}(P_{<n})$ (see Lemma 5.4 for a formal claim). Indeed, we can let $T_1 = (y_1, S_1)$ and $T_2 = (y_2, S_2)$ and check that T_1 and T_2 have the desired properties.

Before we show how to construct such S_1 and S_2, we will need to introduce a few algebraic objects. Let $g_1(x) = (x-y_1)^{s+1}, g_2(x) = (x-y_2)^{s+1}$, and let $\varphi : \mathbb{F}_q[x] \to \mathbb{F}_q[x]/g_1 \times \mathbb{F}_q[x]/g_2$ as $\varphi(f) = (f \mod g_1, f \mod g_2)$.

We then prove a structural result regarding φ:

Lemma 2.10 (Informal version of Lemma 5.10)**.** *For $s+1 < d < 2s+2$, we have*

$$\varphi(P_{<d}) = \bigcup_{h \in P_{<d-(s+1)}} \{(f, f + \sigma(h)) \mid f \in P_{\leq s}\}.$$

Here, $\sigma : P_{\leq s} \to P_{\leq s}$ is a specially chosen injective homomorphism that we define later.

To construct such S_1 and S_2, we construct $R_1 \subset \mathbb{F}_q[x]/g_1, R_2 \subset F_q[x]/g_2$ such that $R_1 \times R_2 \subset \varphi(P_{<n})$ (see Lemma 5.8 for formal claim). Once we have such R_1, R_2, we let $S_1 = \psi_{y_1}(R_1)$ and $S_2 = \psi_{y_2}(R_2)$. We then check that if $f \in P_{<n}$ is such that $\varphi(f) = (r_1, r_2)$, then $f(y_1) = r_1(y_1) = s_1$ and $f(y_2) = r_2(y_2) = s_2$, showing that $S_1 \times S_2 \subset \psi_{y_1,y_2}(P_{<d})$, as desired.

We now construct such R_1 and R_2. Let R_1 and R_2 be arbitrary size $K/2$ subsets of $\sigma(P_{<n-(s+1)})$. Note that since σ is injective, $\left|\sigma(P_{<n-(s+1)})\right| = q^{n-s-1} \geq K/2$ and so we can indeed pick such R_1 and R_2. We claim that for any $r_1 \in R_1, r_2 \in R_2$, it holds that $(r_1, r_2) \in \varphi(P_{<n})$. Using Lemma 2.10, it suffices to show that $(r_1, r_2) = (f, f + \sigma(h))$ for some $f \in P_{<n}$ and $h \in P_{n-(s+1)}$. Since $R_1, R_2 \subseteq \sigma(P_{<n-(s+1)})$, we know that $r_1 = \sigma(h_1)$ and $r_2 = \sigma(h_2)$ for some $h_1, h_2 \in P_{<n-(s+1)}$. Consequently,

$$(r_1, r_2) = (\sigma(h_1), \sigma(h_2)) = (\sigma(h_1), \sigma(h_1) + \sigma(h_2 - h_1))$$

as desired. In the last line, we used the fact that σ is a homomorphism and that $h_1, h_2 \in P_{<n-(s+1)}$.

We finally define σ:

Definition 2.11. *Recall that $\sigma : P_{<n-(s+1)} \to P_{\leq s}$. To compute $\sigma(h_1)$, we let $f \in P_{<n}$ be any polynomial such that $f = h_1 g_1 + r_1$ where $r_1 \in P_{\leq s}$ is the remainder of f modulo g_1. We then write $f = h_2 g_2 + r_2$ where $r_2 \in P_{\leq s}$ is the remainder of f modulo g_2. The output of $\sigma(h_1)$ is $r_2 - r_1$.*

At first glance, it seems unclear whether σ is even a well defined function. To help show this, we define $\rho : P_{<n-(s+1)} \to P_{<n-(s+1)}$ such that $\rho(h_1) = h_2$ where h_1, h_2 are defined as above. We first show that ρ is an isomorphism. To do this, we observe that $h_1 g_1 - h_2 g_2 = r_2 - r_1$ has degree at most s, and so it must be that $h_1 g_1$ and $h_2 g_2$ agree on all coefficients corresponding to degree $\geq s + 1$. We compare these coefficients and, using linear algebra, show that we can indeed obtain a unique h_2 from h_1, showing it is indeed a function. This argument in fact directly shows that ρ is an isomorphism.

Once we have that ρ is an isomorphism, we can then define $\sigma(h_1) = h_1 g_1 - \rho(h_2) g_2$. Using this, it easily follows that σ is a homomorphism. From this we obtain Lemma 2.10. By a further counting argument, we can show that σ is injective. For details, see Section 5.2.

2.2 Non-bipartite lossless expander

We show that the bipartite half of the KT graph (from the previous section) yields a non-bipartite regular lossless expander. The bipartite half is an operation of bipartite graphs that transforms them into a non-bipartite graph, and is defined as follows: given a bipartite graph $G = (L \sqcup R, E)$, its bipartite half $G^2[L]$ is a graph with vertex set L where there is an edge $(u, v) \in G^2[L]$ iff u and v share a common neighbor in G.

One nuance of the bipartite half is that applying it to a biregular bipartite graph does not necessarily mean that the bipartite half will be regular itself (although the graph we obtain from [KT22] will indeed be regular). Thus, we must define what it means for a graph to be lossless in this non-regular setting. A natural definition just involves summing the total number of neighbors of a set.

Definition 2.12. *An irregular graph $G = (V, E)$ is a (K, ε)-*lossless expander[6] *if for any set $S \subseteq V$ of size at most K we have that $|\Gamma(S)| \geq (1 - \varepsilon) \sum_{v \in S} d(v)$ where $d(v)$ represents the degree of vertex v.*

A stronger notion of lossless expansion is with respect to the highest degree of a node present in a graph.

Definition 2.13. *An irregular graph $G = (V, E)$ is a* max-degree (K, ε)-lossless expander *if for any set $S \subseteq V$ of size at most K we have that $|\Gamma(S)| \geq (1 - \varepsilon) D |S|$ where $D = \max_{v \in V} d(v)$, the maximum degree of any vertex in G.*

Using this definition, our main observation is that the bipartite half of any two-sided lossless bipartite expander yields a non-bipartite, max-degree lossless expander.

Lemma 2.14 (Lemma 6.3 restated)**.** *Let $G = (L \sqcup R, E)$ be a (D_L, D_R)-regular (K_L, A_L, K_R, A_R)-two-sided lossless expander. Then $G^2[L]$ is a max-degree (K, A)-expander where each node has a degree in $[D_L A_R, D_L D_R]$, and with $K = \min(K_L, K_R / D_L)$ and $A = A_L A_R$.*

[6] We abuse notation between the regular and irregular cases of graphs since this definition of lossless expansion for an irregular graph captures our previous definition of lossless expansion for regular graphs.

The proof of this lemma essentially follows from expanding twice in the underlying two-sided expander G. Since we force our initial set to be at most K_L and K_R/D_L, we are guaranteed that we can use the left-to-right expansion of G and then additionally the right-to-left expansion of G, where at each step we expand by A_L and A_R, respectively.

Finally, we use the bipartite two-sided lossless expander from Theorem 1 as the base graph in Lemma 2.14 to obtain Theorem 2. Luckily, if we use the KT graph as our bipartite two-sided lossless expander, then the resultant graph obtained from taking the bipartite half is indeed regular (see Lemma 6.6 for a proof).

Theorem 2.15 (Informal version of Theorem 6.1)**.** *For infinitely many N and all constant $0 < \delta < 0.99$, there exists an explicit regular $(K, \varepsilon = 0.01)$ lossless expander $G = (V, E)$ where $|V| = N$, the degree is D where $N^{1-1.01\delta} \leq D \leq N^{1-1.01\delta+o(1)}$ and $K = \min\left(N^{\delta}, N^{1-1.01\delta-o(1)}\right)$. Moreover, G is endowed with a free group action from $\mathbb{F}_q$ where $q = \mathrm{poly}(\log N)$ if one vertex is removed.*

In this setting, $A = A_L A_R \approx 0.99 D_L D_R$, implying $G^2[L]$ is indeed a max-degree lossless expander. Additionally, because the vertices in the bipartite half of the KT graph are elements of $\mathbb{F}_q^n$, we get a free group action from $\mathbb{F}_q$ on them by scalar multiplication. One needs to be careful here since $G^2[L]$ contains the zero polynomial vertex; we remove this vertex and observe that removing one vertex still preserves the expansion properties.

Organization We use Section 3 to introduce necessary preliminaries. Then in Section 4.1 we show how our main theorem is proved assuming right regularity and knowing the overlap between two neighborhoods of right vertices. These facts are then proved in Section 4.2. In Section 4.3, we plug in parameters to get our two-sided lossless expander. In Section 5 we prove tightness of our right-to-left expansion analysis of the KT graph. Finally, in Section 6 we show how the bipartite half of the KT graph is a non-bipartite lossless expander with a free group action.

We prove that our constructions are explicit in Appendix A, and discuss why our techniques do not work for the [GUV09] graph in Appendix B.

3 Preliminaries

3.1 Notation

For a function $f \in \mathbb{F}_q[X]$, we we use $f^{(j)}$ to denote the j'th iterated derivative of f. We will often use the notation b_i for $i \in \mathbb{N}$ to refer to the polynomial $x^i \in \mathbb{F}_q[x]$ and we will often use the fact that $(b_0, \ldots, b_n)$ form a basis for the polynomials of degree at most n. For a (d_L, d_R)-biregular bipartite graph $G = (L \sqcup R)$, we use $\Gamma_{\rightarrow} : L \times [D_L] \to R$ to be the function that maps vertices in L to their neighbors in R as given by G; we use $\Gamma_{\leftarrow} : R \times [D_R] \to L$ to be the function that maps vertices in R to their neighbors in L as given by G. Often, we will define graph G by only defining the associated $\Gamma_{\rightarrow}$. When clear from context, we sometimes abuse notation and use $\Gamma_{\rightarrow}(w)$ to denote the right neighborhood of $w \in L$, and similarly $\Gamma_{\leftarrow}(w)$ for the left neighborhood of $w \in R$.

3.2 Lossless expansion

Throughout this paper, we will be focusing on the notion of vertex expansion as opposed to other definitions (e.g., edge, spectral) of expansion. Defining vertex expansion of a regular graph is straightforward.

Definition 3.1. *A D-regular graph $G = (V, E)$ is a* (K, A)-expander *if for all $S \subseteq V$ such that $|S| \leq K$ we have that $|\Gamma(S)| \geq A\,|S|$. If $A = 1 - \varepsilon$, then we say that G is a* (K, ε)-lossless expander*.*

For biregular bipartite graphs, we must consider the degree of each side to define expansion.

Definition 3.2. *A (D_L, D_R)-biregular graph $G = (L \sqcup R, E)$ is a* (K_L, A_L, K_R, A_R)-two-sided expander *if for all $S \subseteq L$ of size at most K_L we have $|\Gamma_{\rightarrow}(S)| \geq A_L\,|S|$ and for all $S \subseteq R$ of size at most K_R we have*

$|\Gamma_{\leftarrow}(S)| \geq A_R |S|$. *If* $A_L = 1 - \varepsilon_L$ *and* $A_R = 1 - \varepsilon_R$, *then we call* G *a* $(K_L, \varepsilon_L, K_R, \varepsilon_R)$-lossless two-sided expander.

For irregular graphs, we can generalize Definition 3.1 in two ways. The first way is considering expansion with respect to the maximum number of neighbors of a set.

Definition 3.3. *An irregular graph* $G = (V, E)$ *is a* (K, ε)-lossless expander *(where we abuse the word "expander" for both regular and irregular graphs) if for any set* $S \subseteq V$ *of size at most* K *we have that* $|\Gamma(S)| \geq (1 - \varepsilon) \sum_{v \in S} d(v)$ *where* $d(v)$ *represents the degree of vertex* v.

The second, stronger notion of lossless expansion is with respect to the highest degree of a node present in a graph.

Definition 3.4. *An irregular graph* $G = (V, E)$ *is a* max-degree (K, ε)-lossless expander *if for any set* $S \subseteq V$ *of size at most* K *we have that* $|\Gamma(S)| \geq (1 - \varepsilon) D |S|$ *where* $D = \max_{v \in V} d(v)$, *the maximum degree of any vertex in* G.

3.3 The KT graph

Throughout the paper, we will use construction of bipartite (left) lossless expanders from [KT22] based on multiplicity codes from [KSY14]. We will often refer to this graph 'the KT graph':

Definition 3.5 (The KT graph)**.** *Let* $q, n, s \in \mathbb{N}$ *be such that* q *is a prime power, characteristic of the finite field* $\mathbb{F}_q \geq n$ *and* $s \leq n/2$. *Define* $G = (L \sqcup R, E)$ *where* $L = \mathbb{F}_q^n, R = \mathbb{F}_q^{s+2}$. *The left degree is* q *and for any* $f \in \mathbb{F}_q^n$ *and* $y \in \mathbb{F}_q$, *the* y*'th neighbor of* f *is defined as follows: Identify* f *as member of* $\mathbb{F}_q[X]$ *with degree of* f *at most* $n-1$ *; then, the neighbor* $\Gamma_{\rightarrow}(f, y)$ *will be* $(y, f^{(0)}(y), \ldots, f^{(s)}(y))$ *where* $f^{(j)}$ *is the* j*'th iterative derivative of* f.

Remark 3.6. *In the paper [KT22], the final lossless expander graph construction slightly differs from ours. While they do construct the KT-graph* G *defined as above and show it has great (left) expanding properties, the final (left) lossless expander graph actually is defined as* $H = (L \sqcup R, E)$ *where* $L = 2^n, R = \mathbb{F}_q^{s+2}$ *and the left degree is* q. H *is constructed by considering the subgraph of* G *induced by vertices on the left side corresponding to* $\{0,1\}^n$. *For us, the final two-sided lossless expander graph will be* G *itself. This is why, our two-sided lossless expander graph has slightly worse parameters (worse constants) compared to the left lossless expander graph from [KT22].*

3.4 A useful inequality

We will use the following inequality based on an application of the Cauchy-Schwarz inequality:

Claim 3.7. *Fix* $n \in \mathbb{N}, S \in \mathbb{R}$. *Let* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *be such that* $\sum_{1 \leq i \leq n} x_i = S$. *Then,*

$$\sum_{1 \leq i < j \leq n} x_i x_j \leq \frac{(n-1)S^2}{2n}$$

Proof. Recall the Cauchy-Schwarz inequality: $\left(\sum_{1 \leq i \leq n} a_i b_i\right)^2 \leq \left(\sum_{1 \leq i \leq n} a_i^2\right) \left(\sum_{1 \leq i \leq n} b_i^2\right)$. We apply this with $a_1 = x_1, \ldots, a_n = x_n$ and $b_1 = b_2 = \cdots = b_n = 1$ to infer that

$$S^2 \leq \left(\sum_{1 \leq i \leq n} x_i^2\right) \cdot n = \left(S^2 - 2 \sum_{1 \leq i < j \leq n} x_i x_j\right) \cdot n$$

Rearranging, we infer that

$$\sum_{1 \leq i < j \leq n} x_i x_j \leq \frac{(n-1)S^2}{2n}$$

as desired. □

3.5 Free group actions on graphs

Here we recall basic notions about group actions on graphs. First, we define an abstract group notion.

Definition 3.8. *Let G be a group and X a set. A* group action $\cdot : G \times X \to X$ *(where we write the $\cdot$ in infix notation) is a function that has the following two properties:*

1. *Identity: The identity element 1_G of G always acts trivially as $1_G \cdot x = x$ for any $x \in X$.*
2. *Compatibility: The group action and multiplication of G are compatible. That is, for any $g, h \in G$ and $x \in X$ we have $(gh) \cdot x = g \cdot (h \cdot x)$ where gh is the product of g and h in G.*

Next, we recall another abstract notion about group actions.

Definition 3.9. *We say that a group action of G on X is* free *if $g \cdot x = x$ for some $x \in X$ implies that $g = 1_G$.*

Finally, we consider what it means for a graph to be invariant with respect to a group action.

Definition 3.10. *Let G be a group and $H = (V, E)$ a graph with a group action from G. We say that H is* G-invariant *if for all $(v, w) \in E$ and $g \in G$ we have that $(g \cdot v, g \cdot w) \in E$.*

4 An Explicit Two-sided Lossless Expander

In this section, we first describe how to prove our main theorem using right regularity and the size of the overlap in neighborhoods between any two right vertices. Then we prove these two facts in Section 4.2.

4.1 Main theorem

Putting together all of our results with the left-to-right expansion of [KT22] yields our main theorem.

Theorem 4.1. *For all finite fields $\mathbb{F}_q$ and $n, s \in \mathbb{N}$ with $15 \leq (s+1) < n < char(\mathbb{F}_q)$, there exists an explicit bipartite graph $G = (L \sqcup R, E)$ with $L = \mathbb{F}_q^n, R = \mathbb{F}_q^{s+2}$, left degree equal to q and right degree $q^{n-(s+1)}$ such that G is a two-sided (K_L, A_L, K_R, A_R) expander with $A_L = q - \frac{n(s+2)}{2} \cdot (qK_L)^{1/(s+2)}$ and $A_R = \left(1 - \frac{K_R}{q^{\min(s+2, n-s)}} \cdot \frac{q-1}{2}\right) q^{n-(s+1)}$.*

Proof. The left-to-right expansion follows from Theorem 3 from [KT22]. The right-to-left expansion follows from Theorem 4.2 below. The explicitness of G follows from Claim A.1. □

Our main achievement is showing the right-to-left expansion of the KT graph in Theorem 4.2 below.

Theorem 4.2. *If $n \geq s + 1$, then the KT graph G in Definition 2.2 is a right (K_{max}, ε)-lossless expander for $K_{max} = \delta q^{s+1}$ and $\varepsilon = \frac{\delta(q-1)}{2q} \cdot q^{\max(2s+2-n, 0)}$ where $0 < \delta < 1$ is arbitrary.*

We prove Theorem 4.2 via the following properties of G:

Lemma 4.3. *When $n \geq s + 1$, G is right-regular and the right degree is $q^{n-(s+1)}$.*

Lemma 4.4. *For any pair of right-vertices w_1, w_2 such that $w_1 = (y_1, z_1), w_2 = (y_2, z_2) \in \mathbb{F}_q^{s+2}$ where $y_1 \neq y_2 \in \mathbb{F}_q$ and $z_1, z_2 \in \mathbb{F}_q^{s+1}$, we have*

$$|\Gamma_{\leftarrow}(y_1, z_1) \cap \Gamma_{\leftarrow}(y_2, z_2)| \leq \begin{cases} q^{n-(2s+2)} & n \geq 2s+2 \\ 1 & n \leq 2s+2 \end{cases}$$

We will prove both these lemmas in Section 4.2.

With the exact right-regularity of G and the number of common left-neighbors shared by any pair of right-vertices generated by different seeds, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Our goal is to show that any right subset $T \subseteq \mathbb{F}_q^{s+2}$ of size at most δq^{s+1} has a neighborhood of size at least $(1-\varepsilon)q^{n-(s+1)}\,|T|$ on the left.

To do this, we consider T as the disjoint union $T = \bigsqcup_{y\in\mathbb{F}_q} T_y$ of buckets $T_y = \{(y,\alpha) : \alpha \in \mathbb{F}_q^{s+1}\}$ where $|T_y| = t_y = \delta_y q^{s+1}$. Let $\delta = \sum_{y\in\mathbb{F}_q} \delta_y$. So, $|T| = \delta q^{s+1}$. By Lemma 4.3, the number of edges leaving T is $|T| \cdot q^{n-(s+1)} = \delta q^n$.

We now consider cases on whether $n \geq 2s+2$ or not:

Case 1. $n \geq 2s+2$.

In this case, $\varepsilon = \frac{\delta(q-1)}{2q}$. By Lemma 4.4, the maximum number of double-counted left vertices is

$$\sum_{\substack{i,j\in[q]\\ i<j}} t_i t_j q^{n-2(s+1)} = \sum_{\substack{i,j\in[q]\\ i<j}} \delta_i q^{s+1}\cdot\delta_j q^{s+1}\cdot q^{n-2(s+1)} = q^n \sum_{\substack{i,j\in[q]\\ i<j}} \delta_i\delta_j \leq q^n\cdot\frac{q-1}{2q}\cdot\delta^2$$

where for the last inequality, we used Claim 3.7. Applying one level of inclusion-exclusion reveals that

$$|\Gamma_{\leftarrow}(T)| \geq \delta q^n - q^n\cdot\frac{q-1}{2q}\cdot\delta^2 = \left(1-\frac{\delta(q-1)}{2q}\right)\delta q^n = (1-\varepsilon)\,q^{n-(s+1)}|T|$$

where the last equality follows because $\varepsilon = \frac{\delta(q-1)}{2q}$.

Case 2. $2s+2 \geq n \geq s+1$.

In this case, $\varepsilon = \frac{\delta(q-1)}{2q}\cdot q^{2s+2-n}$. By Lemma 4.4, the maximum number of double-counted left vertices is

$$\sum_{\substack{i,j\in[q]\\ i<j}} t_i t_j = \sum_{\substack{i,j\in[q]\\ i<j}} \delta_i q^{s+1}\cdot\delta_j q^{s+1} = q^{2s+2}\sum_{\substack{i,j\in[q]\\ i<j}} \delta_i\delta_j \leq q^{2s+2}\cdot\frac{q-1}{2q}\cdot\delta^2$$

where for the last inequality, we used Claim 3.7. We again apply one level of inclusion-exclusion to conclude that

$$|\Gamma_{\leftarrow}(T)| \geq \delta q^n - q^{2s+2}\cdot\frac{q-1}{2q}\cdot\delta^2 = \left(1-\frac{\delta(q-1)}{2q}\cdot q^{2s+2-n}\right)\delta q^n = (1-\varepsilon)\,q^{n-(s+1)}|T|$$

where the last equality follows because $\varepsilon = \frac{\delta(q-1)}{2q}\cdot q^{2s+2-n}$.

□

4.2 Right regularity and bounding common neighbors: Hermite interpolation

In this section, we show the $(q^{n-(s+1)})$-right-regularity of the KT graph G, and bound the number of common left neighbors shared by any pair of right vertices with different seeds. Both tasks are essentially a question of Hermite interpolation—we wish to find polynomials $f \in \mathbb{F}_q[Y]$ of degree at most $n-1$ such that when evaluating at some point $y \in \mathbb{F}_q$, the function value $f(y)$ and its first s derivatives $(f^{(0)}(y), f^{(1)}(y), \cdots, f^{(s)}(y))$ match the values given by the right vertices.

Lemma 4.5 (Hermite interpolation)**.** *Let $y_1, \cdots, y_k \in \mathbb{F}_q$ be distinct, and for $i \in [k]$, let $z_{i,0}, \cdots, z_{i,s} \in \mathbb{F}_q$. Then there exists a unique polynomial $f \in \mathbb{F}_q[Y]$ with degree at most $k(s+1)$ such that $f^{(j)}(y_i) = z_{i,j}$ for $i \in [k]$ and $j \in \{0\}\cup[s]$.*

Proof. Consider the following k congruences,

$$f(Y) \equiv f_1(Y) \mod (Y-y_1)^{s+1},\ \cdots,\ f(Y) \equiv f_k(Y) \mod (Y-y_k)^{s+1}$$

Any polynomial f that satisfies the above k congruences must also satisfy $f^{(j)}(y_i) = z_{i,j}$ for $i \in [k]$ and $j \in \{0\}\cup[s]$— thus solving the interpolation—because $f_i(Y)$ is the order s Taylor polynomial of f at y_i.

We conclude the proof by applying the Chinese remainder theorem for univariate polynomials which asserts that there exists a *unique* polynomial $f \in \mathbb{F}_q[Y]$ of degree at most $k(s+1)$ that satisfies the above congruences.

□

Recall the maps $\psi_{y_1} : \mathbb{F}_q^n \to \mathbb{F}_q^{s+1}$, and $\psi_{y_1,y_2} : \mathbb{F}_q^n \to \mathbb{F}_q^{2s+2}$ where $\psi_y(f) = (f^{(0)}(y), \ldots, f^{(s)}(y))$ and $\psi_{y_1,y_2} = (\psi_{y_1}(f), \psi_{y_2}(f))$. We will use the following fact regarding linearity of derivatives:

Fact 4.6 ([Rit50])**.** *For all $\alpha, \beta \in \mathbb{F}_q$, $f, g \in \mathbb{F}_q[X]$ and $j \geq 0$, it holds that $(\alpha f + \beta g)^{(j)} = \alpha f^{(j)} + \beta g^{(j)}$.*

From this fact, we directly obtain that ψ_y is a linear map:

Corollary 4.7. *For all $y \in \mathbb{F}_q$, ψ_y is an $\mathbb{F}_q$-linear map.*

Now we prove the right-regularity of G (Lemma 4.3) and bound the number of overlapping left neighbors (Lemma 4.4) as special cases of Lemma 4.5.

Proof of Lemma 4.3. Let $w_1 \in \mathbb{F}_q^{s+2}$ where $w_1 = (y_1, z_{1,0}, \cdots, z_{1,s})$. Take $k = 1$ in Lemma 4.5, we get that there exists a unique polynomial f of degree at most $s+1$ such that $f^{(j)}(y_1) = z_{1,j}$ for $j \in \{0\} \cup [s]$. This means the linear map $\psi_{y_1} : \mathbb{F}_q^n \to \mathbb{F}_q^{s+1}$ is surjective. Therefore, the number of left neighbors of any right vertex w_1 is exactly $|\psi_{y_1}^{-1}(z_{1,0}, \cdots, z_{1,s})| = q^{n-(s+1)}$.

□

Proof of Lemma 4.4. Let $w_1, w_2 \in \mathbb{F}_q^{s+2}$ where $w_1 = (y_1, z_{1,0}, \cdots, z_{1,s})$, $w_2 = (y_2, z_{2,0}, \cdots, z_{2,s})$, $y_1 \neq y_2$. Take $k = 2$ in Lemma 4.5, we get that there exists a unique polynomial f of degree at most $2(s+1)$ such that $f^{(j)}(y_i) = z_{i,j}$ for $i \in \{1,2\}, j \in \{0\} \cup [s]$. This means,

- When $n > 2s+2$, $\psi_{y_1,y_2} : \mathbb{F}_q^n \to \mathbb{F}_q^{2s+2}$ is surjective, the number of common neighbors shared by w_1, w_2 is exactly $|\psi_{y_1,y_2}^{-1}(z_{1,0}, \cdots, z_{1,s}, z_{2,0}, \cdots, z_{2,s})| = q^{n-2(s+1)}$.
- When $n \leq 2s+2$, $\psi_{y_1,y_2} : \mathbb{F}_q^n \to \mathbb{F}_q^{2s+2}$ is injective, w_1, w_2 share *at most* 1 common left neighbor.

□

4.3 Plugging in the Parameters

We record our main results regarding two sided lossless expanders:

Theorem 4.8 (Formal version of Theorem 1)**.** *For infinitely many N and all $0 < \delta < 0.99$, there exists an explicit biregular two-sided $(K_L, \varepsilon_L = 0.01, K_R, \varepsilon_R = 0.01)$ lossless expander $\Gamma_{\rightarrow} : [N] \times [D_L] \to [M]$ where $D_L \leq O(\log^{204}(N))$, $N^{1.01\cdot\delta - o(1)} \leq M \leq D_L \cdot N^{1.01\cdot\delta}$, $K_L = N^{\delta}$, $K_R = \frac{1}{50} \cdot (1/D_L) \cdot \min(M, N/M)$.* [7]

These will follow from the following technical lemma:

Lemma 4.9. *Let $\alpha, \varepsilon_L, \varepsilon_R \in (0,1)$ and $K_R, n, k_L, q \in \mathbb{N}$ be such that q is a prime number, $\frac{h^{1+\alpha}}{2} \leq q \leq h^{1+\alpha}$ where $h = (4nk_L/\varepsilon_L)^{1/\alpha}$ and such that both $\frac{4}{k_L}\log(2n/\varepsilon_L) \leq \alpha$ and $k_L(1+\alpha) \leq n$. Then, there exists an explicit biregular $(K_L, \varepsilon_L, K_R, \varepsilon_R)$ two-sided lossless expander $\Gamma_{\rightarrow} : [N] \times [D_L] \to [M]$ where $N = q^n, K_L = q^{k_L}, K_L^{1+\alpha-1/\log(h)} \leq M \leq D_L \cdot K_L^{1+\alpha}$, $D_L \leq O(\log(N)\log(K_L)/\varepsilon_L)^{1+1/\alpha+o(1)}$, $\frac{K_R}{q^{\min(s+2,n-s)}} \cdot \frac{q-1}{2} \leq \varepsilon_R$ where $s+2 = \lceil k_L/\log_q(h) \rceil$.*

We will instantiate this lemma using simple parameters to obtain our main theorems:

Proof of Theorem 4.8. We plug in $\alpha = 0.01, \varepsilon_L = 0.01, \varepsilon_R = 0.01, k_L = \delta n$ in Lemma 4.9 to obtain the desired lossless expander. □

[7] Our theorem statement doesn't have any additional constraint on D_R since it can be uniquely inferred from N, M, D_L.

We finally prove our main technical lemma using two-sided expander from Theorem 4.1:

Proof of Lemma 4.9. As $s+2 = \lceil k_L/\log_q(h) \rceil$, we have that $h^{s+1} \le K_L \le h^{s+2}$. Observe that

$$s+1 < \frac{k_L \log(q)}{\log(h)} \le k_L(1+\alpha) \le n$$

So, we can apply Theorem 4.1 and infer that there exists a graph $\Gamma_{\rightarrow} : \mathbb{F}_q^n \times \mathbb{F}_q \to \mathbb{F}_q^{s+2}$ that is a $(\le h^{s+2}, A_L)$ left expander and $(\le K_R, A_R)$ right expander where $A_L = q - \frac{n(s+2)}{2} \cdot (qh^{s+2})^{1/(s+2)}$ and $A_R = \left(1 - \frac{K_R}{q^{\min(s+2,n-s)}} \cdot \frac{q-1}{2}\right) q^{n-(s+1)}$. Notice that as $K_L \le h^{s+2}$, $\Gamma_{\rightarrow}$ is indeed a (K_L, A_L) expander.

- We first bound the left degree D_L:

$$\begin{aligned} D_L = q \le h^{1+\alpha} &= (4nk_L/\varepsilon_L)^{1+1/\alpha} = (4\log(N)\log(K_L)/\log^2(q)\varepsilon_L)^{1+1/\alpha} \\ &= (4\log(N)\log(K_L)/\varepsilon_L)^{1+1/\alpha} \cdot \log^{2+2/\alpha}(q) \end{aligned}$$

  This implies that

$$D_L = q \le (4\log(N)\log(K_L)/\varepsilon_L)^{1+1/\alpha} \left(\log\left(8\log(N)\log(K_L)/\varepsilon_L\right)\right)^{2+2/\alpha}$$

  Then indeed, $D_L \le O(\log(N)\log(K_L)/\varepsilon_L)^{1+1/\alpha+o(1)}$.

- We now bound the number of right vertices M:

$$M = q^{s+2} \le q \cdot h^{(1+\alpha)(s+1)} \le q \cdot K_L^{1+\alpha}$$

  Additionally,

$$M = q^{s+2} \ge q^{K_L \log(q)/\log(h)} \ge q^{K_L((1+\alpha)(\log h)-1)/\log(h)} = q^{K_L(1+\alpha)-K_L/\log(h)}$$

- We now show lossless expansion from the right side:

$$A_R = \left(1 - \frac{K_R}{q^{s+2}} \cdot \frac{q-1}{2}\right) q^{n-(s+1)} \ge (1-\varepsilon_R) D_R$$

  where the last inequality follows because $\frac{K_R}{q^{\min(s+2,n-s)}} \cdot \frac{q-1}{2} \le \varepsilon_R$.

- We finally show lossless expansion from the left side: First, we note that $s+2 \le 2k_L$. Indeed, $s+2 \le k_L \log_q(h) + 1 = k_L \frac{\log(h)}{\log(q)} + 1 \le k_L(1+\alpha)+1 \le 2k_L$. Then,

$$\begin{aligned} A_L &= q - \frac{n(s+2)}{2} \cdot (qh^{s+2})^{1/(s+2)} \\ &= q - \frac{n(s+2)h}{2} \cdot (q)^{1/(s+2)} \\ &\ge q - nk_L h \cdot (q)^{1/(s+2)} && \text{(since } s+2 \le 2k_L\text{)} \\ &= q - \frac{\varepsilon_L \cdot h^{\alpha}}{4} \cdot h \cdot (q)^{1/(s+2)} && \text{(since } nk_L = (\varepsilon_L \cdot h^{\alpha})/4\text{)} \\ &= q - \varepsilon_L \cdot \frac{h^{1+\alpha}}{4} \cdot (q)^{1/(s+2)} \\ &\ge q - \frac{\varepsilon_L}{2} \cdot q \cdot (q)^{1/(s+2)} && \text{(since } h^{1+\alpha}/2 \le q\text{)} \\ &= q\left(1 - \varepsilon_L \cdot \frac{(q)^{1/(s+2)}}{2}\right) \\ &\ge q(1-\varepsilon_L) \end{aligned}$$

The last inequality $(q)^{1/(s+2)} \leq 2$ follows because we claim that $s+2 \geq \log(q)$. This suffices to prove the last inequality since then $(q)^{1/(s+2)} \leq q^{1/\log(q)} \leq 2$. We indeed compute that

$$s+2 \geq \frac{k_L}{\log_q(h)} \geq \frac{k_L((1+\alpha)\log(h)-1))}{\log(h)} \geq k_L$$

Moreover, as $\alpha \geq \frac{4}{k_L}\log(2n/\varepsilon_L)$, we infer that $k_L \geq \frac{4}{\alpha}\cdot\log(2n/\varepsilon_L)$. Hence indeed,

$$s+2 \geq \frac{4}{\alpha}\cdot\log(2n/\varepsilon_L) \geq \frac{2}{\alpha}\cdot\log(2nk_L/\varepsilon_L) \geq 2\log(h) \geq (1+\alpha)\log(h) \geq \log(q)$$

□

5 Tightness of Our Construction

In this section we show that the right-to-left expansion of Theorem 4.2 is tight. In particular, when $n < 2s+2$ Theorem 4.2 gives a trade-off between the expansion parameter ε and the max size of expanding sets K_{max}. We show that this trade-off is tight up to constants, and thus fully characterize the behavior of the KT graph. Importantly, we show that in the balanced setting where $n = s + O(1)$, the KT graph is *not* a two-sided lossless expander.

Recall that when $n \geq 2s+2$ we know that our result is tight as stated in Remark 1.2. Consequently, our main theorem in this section deals with the regime where $s+1 < n < 2s+2$. In this setting, Theorem 4.2 gives us that sets $S \subseteq R$ on the right of size at most $K_{max} = \delta q^{s+1}$ expand with parameter $\varepsilon = \frac{\delta(q-1)}{2q}\cdot q^{2s+2-n}$. Equivalently, it gives us that sets of size at most $K_{max} = \delta q^{n-s-1}$ expand with parameter $\varepsilon = \frac{\delta(q-1)}{2q}$. Our main theorem in this section upper bounds this expansion.

Theorem 5.1. *When $s+1 < n < 2s+2$, there exists a subset $S \subseteq R$ of the right vertices such that $|S| = K_{max} = \delta q^{n-s-1}$ and $|\Gamma_{\leftarrow}(S)| = (1-\varepsilon)D_R|S|$ with $\varepsilon = \frac{\delta}{4}$ where $\delta > 0$.*

This means that our right-to-left expansion of *Theorem* 4.2 is tight up to a constant factor of 1/2. The proof of our main theorem comes from the construction of two disjoint subsets of right vertices each in different buckets that have the maximum number of overlapping neighbors on the left. Recall that the y-th bucket T_y is defined as $T_y = \{(y,\alpha) : \alpha \in \mathbb{F}_q^{s+1}\}$

Lemma 5.2. *Let $y_1, y_2 \in \mathbb{F}_q$ be arbitrary such that $y_1 \neq y_2$. Then there exist sets $S_1 \subseteq T_{y_1}$ and $S_2 \subseteq T_{y_2}$ such that $|S_1| = |S_2| = \frac{K_{max}}{2}$ and $|\Gamma_{\leftarrow}(S_1) \cap \Gamma_{\leftarrow}(S_2)| = |S_1|\cdot|S_2|$.*

Using this lemma, our main theorem is a result of straightforward computations.

Proof of Theorem 5.1. Take any $y_1, y_2 \in \mathbb{F}_q$ such that $y_1 \neq y_2$ and the S_1 and S_2 from Lemma 5.2. Let $S = S_1 \cup S_2$, so indeed $|S| = K_{max}$. To count $|\Gamma_{\leftarrow}(S)|$, all we need to do is to subtract the number of 2-wise overlaps of left neighbors S_1 and S_2 from the total number of possible neighbors of S_1 and S_2. The latter value is simply $|S|\cdot D_R = K_{max}\cdot D_R$, and the former is given to us by Lemma 5.2 as $|S_1|\cdot|S_2| = \frac{K_{max}^2}{4}$. Therefore, we can compute

$$|\Gamma_{\leftarrow}(S)| = K_{max}\cdot D_R - \frac{K_{max}^2}{4} = K_{max}\cdot D_R\left(1 - \frac{K_{max}}{4D_R}\right),$$

showing that

$$\varepsilon = \frac{K_{max}}{4D_R} = \frac{\delta q^{n-(s+1)}}{4q^{n-(s+1)}} = \frac{\delta}{4}.$$

□

The proof of Lemma 5.2 relies on choosing S_1 and S_2 to contain only the derivatives of polynomials of degree at most $n-(s+1)$. In the following proofs, we will consider vector spaces of polynomials over $\mathbb{F}_q$.

Definition 5.3. *Define the set $P_d = \{f \in \mathbb{F}_q[x] \mid \deg(f) = d\}$ and let $P_{<d} = \{f \in \mathbb{F}_q[x] \mid \deg(f) < d\}$ be a vector space over $\mathbb{F}_q$ with addition and multiplication coming from $\mathbb{F}_q[x]$.*

5.1 Constructing sets with the smallest expansion possible

We prove Lemma 5.2 by restricting the KT graph to the subgraph containing the two buckets T_{y_1} and T_{y_2} and analyzing the behavior of this subgraph. Consequently, we rephrase Lemma 5.2 as follows.

Lemma 5.4 (Technical version of Lemma 5.2)**.** *For any $K \in [q^{n-(s+1)}]$ and distinct $y_1, y_2 \in \mathbb{F}_q$, there exist sets $S_1, S_2 \subseteq \mathbb{F}_q^{s+1}$ such that $|S_1| + |S_2| = K$ and $S_1 \times S_2 \in \psi_{y_1,y_2}(P_{<n})$.*

To create these S_1 and S_2, we actually first create analogous sets in the image of the Chinese Remainder Theorem map, defined below.

Definition 5.5. *For distinct $y_1, y_2 \in \mathbb{F}_q$ let $g_1(x) = (x-y_1)^{s+1}$ and $g_2(x) = (x-y_2)^{s+1}$, let $\pi_1 : \mathbb{F}_q[x] \to \mathbb{F}_q[x]/(g_1)$ and $\pi_2 : \mathbb{F}_q[x] \to \mathbb{F}_q[x]/(g_2)$ be the associated quotient maps. Then let $\varphi : \mathbb{F}_q[x]/(g_1 g_2) \to \mathbb{F}_q[x]/(g_1) \times \mathbb{F}_q[x]/(g_2)$ be defined as $\varphi = \pi_1 \otimes \pi_2$.*

This is exactly the map that the Chinese Remainder Theorem acts on.

Claim 5.6. *The Chinese Remainder Theorem says that φ is an isomorphism of rings.*[8]

Corollary 5.7. *As a consequence, we may also think of φ as an isomorphism of $\mathbb{F}_q$-vector spaces $\varphi : P_{<2s+2} \to P_{<s+1} \times P_{<s+1}$.*

Whereas $\psi_{y_1,y_2}(f)$ tells us about the first s derivatives of f at y_1 and y_2, $\varphi(f)$ tells us about f quotiented by g_1 and g_2. In fact, as we will see, φ and ψ_{y_1,y_2} provide us with the same information about a particular polynomial. With this in mind, we prove a lemma similar to Lemma 5.4 but for φ.

Lemma 5.8. *For any $K \in [q^{n-(s+1)}]$ and distinct $y_1, y_2 \in \mathbb{F}_q$, there exist sets $S_1 \subseteq \mathbb{F}_q[x]/(g_1)$ and $S_2 \subseteq \mathbb{F}_q[x]/(g_2)$ such that $|S_1| + |S_2| = K$ and $S_1 \times S_2 \in \varphi(P_{<n})$.*

Using this relation between φ and ψ_{y_1,y_2}, we prove Lemma 5.4.

Proof of Lemma 5.4. Use Lemma 5.8 to create $R_1 \subseteq \mathbb{F}_q[x]/(g_1)$ and $R_2 \subseteq \mathbb{F}_q[x]/(g_2)$ such that $|R_1|+|R_2| = K$ and $R_1 \times R_2 \in \varphi(P_{<n})$. Let $S_1 = \psi_{y_1} \circ \pi_1^{-1}(R_1)$ and $S_2 = \psi_{y_2} \circ \pi_2^{-1}(R_2)$ where π_1^{-1} and π_2^{-1} are the natural inclusions of $\mathbb{F}_q[x]/(g_1)$ and $\mathbb{F}_q[x]/(g_2)$ into $\mathbb{F}_q[x]$, respectively. We claim that $S_1 \times S_2 \in \psi_{y_1,y_2}(P_{<n})$.

Since the polynomials in the images of π_1^{-1} and π_2^{-1} are of degree strictly less than $s+1$, we have that $\psi_{y_1} \circ \pi_1^{-1}$ and $\psi_{y_2} \circ \pi_2^{-1}$ are injective, so $|S_1|+|S_2| = |R_1|+|R_2| = K$. Moreover, for a pair $(s_1, s_2) \in S_1 \times S_2$, we can take the unique $(r_1, r_2) \in R_1 \times R_2$ such that $s_1 = \psi_{y_1} \circ \pi_1^{-1}(r_1)$ and $s_2 = \psi_{y_2} \circ \pi_2^{-1}(r_2)$. Thus, $(s_1, s_2) = \psi_{y_1,y_2}(\pi_1^{-1}(r_1), \pi_2^{-1}(r_2)) \in \psi_{y_1,y_2}(P_{<n})$, as claimed. □

Our proof of Lemma 5.8 relies on a result about the structure of the image of φ which we prove in Section 5.2 but state here.

Definition 5.9. *For any $b \in \mathbb{F}_q^{s+1}$ define the line $\ell_b = \{(f, f+b) \mid f \in \mathbb{F}_q^{s+1}\}$.*

We now show that the image of φ is actually composed of many of these lines with shifts given by the injective homomorphism $\sigma : P_{<n-(s+1)} \to P_{<s+1}$ that we introduce later in Definition 5.14.

Lemma 5.10. *We show that:*

1. $\varphi(P_{<s+1}) = \ell_0$

[8] For an introduction to the Chinese Remainder Theorem, see Chapter 7.6 of [DF03].

2. *For* $2s+2 > d \geq s+1$ *we have* $\varphi(P_d) = \bigcup_{h \in P_{d-(s+1)}} \ell_{\sigma(h)}$.

Thus, we can conclude that $\varphi(P_{<n}) = \bigcup_{h \in P_{<n-(s+1)}} \ell_{\sigma(h)}$ *where* σ *is an injective homomorphism.*

This structural result on the image of φ allows us to prove Lemma 5.8.

Proof of Lemma 5.8. Let $S_1, S_2 \subseteq \sigma(P_{<n-(s+1)})$ such that $|S_1| = |S_2| = \frac{K_{max}}{2}$. Recall that σ is injective so $\left|\sigma(P_{<n-(s+1)})\right| = \left|P_{<n-(s+1)}\right| = q^{n-(s+1)}$ and we have enough elements to choose from. We claim that any such choice of S_1 and S_2 satisfies the lemma statement.

To prove this claim, we have to show that any pair $(s_1, s_2) \in S_1 \times S_2$ lies in $\varphi(P_{<n})$. By Lemma 5.10, this is equivalent to saying that (s_1, s_2) is of the form $(f, f + \sigma(h))$ for some $f \in P_{<s+1}$ and $h \in P_{<n-(s+1)}$. From our construction, we have that $s_1 = \sigma(h_1)$ and $s_2 = \sigma(h_2)$ for $h_1, h_2 \in P_{<n-(s+1)}$. So we are considering the point $(\sigma(h_1), \sigma(h_2))$. Using the fact that σ is a homomorphism from Claim 5.15, we can rewrite this point in our desired form:

$$\begin{aligned}(\sigma(h_1), \sigma(h_2)) &= (\sigma(h_1), \sigma(h_1) + \sigma(h_2) - \sigma(h_1)) \\ &= (\sigma(h_1), \sigma(h_1) + \sigma(h_2 - h_1)).\end{aligned}$$

Thus, $(s_1, s_2) \in \ell_{\sigma(h_2 - h_1)} \subseteq \varphi(P_{<n})$, as claimed. □

5.2 The structure of the image of φ

We end Section 5 by proving Lemma 5.10, that $\varphi(P_{<n}) = \bigcup_{h \in P_{<n-(s+1)}} \ell_{\sigma(h)}$. To do so we first define a new homomorphism ρ which we will use to define σ later on.

Definition 5.11. *Let* $\rho : P_{<n-(s+1)} \to P_{<n-(s+1)}$ *be defined as follows. Given* $h_1 \in P_{<n-(s+1)}$*, let* $f \in P_{<n}$ *be such that* $f = h_1 g_1 + r_1$ *for some* $r_1 \in P_{<s+1}$*. Then let* $h_2 \in P_{\deg(f)-(s+1)}$ *and* $r_2 \in P_{<s+1}$ *be the unique polynomials such that* $f = h_2 g_2 + r_2$*. We define* $\rho(h_1) := h_2$.

Given this definition, the natural first question is whether ρ is even well-defined since there are q^{s+1} many choices of f that could be used. Lemma 5.12 shows that ρ is well-defined and that it is, in fact, just an invertible linear operator, meaning that ρ is an automorphism of $P_{<n-(s+1)}$.

Lemma 5.12. *Let* $f \in P_{<n}$ *and take* $h_1, h_2, r_1, r_2 \in \mathbb{F}_q[x]$ *to be the unique polynomials such that*

$$\begin{aligned}f &= h_1 g_1 + r_1 \\ f &= h_2 g_2 + r_2\end{aligned}$$

where $\deg(h_1) = \deg(h_2) = \deg(f) - (s+1)$ *and* $\deg(r_1), \deg(r_2) < s+1$.

Then h_2 *can be determined uniquely from* h_1*,* g_1*, and* g_2*. Similarly,* h_1 *can be determined uniquely from* h_2*,* g_1*, and* g_2.

Proof of Lemma 5.12. Rearranging the equations in the lemma statement gives us that

$$h_1 g_1 - h_2 g_2 = r_2 - r_1.$$

Since the degrees of r_1 and r_2 are at most s, we have that $\deg(r_2 - r_1) < s+1$ as well, meaning that $\deg(h_1 g_1 - h_2 g_2) < s+1$. Consequently, the coefficients of x^k for $k \in \{s+1, \ldots, n-1\}$ of $h_1 g_1$ and $h_2 g_2$ must match, which uniquely determines h_2 from h_1.

Formally, if for $t \in \{1, 2\}$ we write $h_t(x) = \sum_{k=0}^{n-1-(s+1)} \eta_k^{(t)} x^k$ for $\eta_k^{(t)} \in \mathbb{F}_q$ and expand out $g_t(x) = \sum_{k=0}^{s+1} \gamma_k^{(t)} x^k$ where $\gamma_k^{(t)} \in \mathbb{F}_q$ and $\gamma_{s+1}^{(t)} = 1$, then for each $k \in \{s+1, \ldots, n-1\}$ we have the linear equation

$$\sum_{j=k-(s+1)}^{n-1-(s+1)} \gamma_{k-j}^{(1)} \eta_{j+s+1}^{(1)} = \sum_{j=k-(s+1)}^{n-1-(s+1)} \gamma_{k-j}^{(2)} \eta_{j+s+1}^{2)}.$$

This yields $n-s-1$ linear equations, which we can write as $M^{(1)}\eta^{(1)} = M^{(2)}\eta^{(2)}$ where for $t \in \{1,2\}$ we let $\eta^{(t)} = (\eta_0^{(t)}, \ldots, \eta_{n-s-2}^{(t)})^\intercal$ and $M^{(t)} \in \mathbb{F}_q^{(n-s-1)\times(n-s-1)}$ is the upper triangular matrix defined as

$$M_{i,j}^{(t)} = \begin{cases} \gamma_{s+1+i-j}^{(t)} & i \le j \\ 0 & i > j \end{cases}$$

Since g_t is monic, meaning that $\gamma_{s+1}^{(t)} = 1$, we clearly see that $\det(M^{(t)}) = 1$, so it is invertible and $\eta^{(2)} = \left(M^{(2)}\right)^{-1} M^{(1)}\eta^{(1)}$. Similarly, $\eta^{(1)} = \left(M^{(1)}\right)^{-1} M^{(2)}\eta^{(2)}$. □

Corollary 5.13. *ρ is an automorphism of $P_{<n-(s+1)}$.*

Proof. Lemma 5.12 shows that we can equivalently define ρ as $\rho = \left(M^{(2)}\right)^{-1} M^{(1)}$, which is an invertible linear transformation. □

Now that we have defined ρ and shown that it is an isomorphism, we use it to build σ.

Definition 5.14. *Let $\sigma : P_{<n-(s+1)} \to P_{<s+1}$ be defined as $\sigma(h) = hg_1 - \rho(h)g_2$. Note that while we consider multiplication here to occur in $\mathbb{F}_q[x]$, Lemma 5.12 tells us that this difference yields a polynomial in $P_{<s+1}$.*

Unsurprisingly, σ is a homomorphism of vector spaces since ρ is one.

Claim 5.15. *Since ρ is an isomorphism, σ is a homomorphism of vector spaces over $\mathbb{F}_q$.*

Proof. The fact that σ is a homomorphism of vector spaces over $\mathbb{F}_q$ follows from ρ being a homomorphism and addition and multiplication commuting in $\mathbb{F}_q[x]$. □

Finally, we are ready to prove Lemma 5.10, that $\varphi(P_{<n}) = \bigcup_{h \in P_{<n-(s+1)}} \ell_{\sigma(h)}$.

Proof of Lemma 5.10. We can directly show the first item by the fact that that the remainder of dividing a polynomial by another of larger degree is the polynomial itself. That is, for any polynomial $f \in P_{<s+1}$, we have that $\pi_1(f) = \pi_2(f) = f$, so $\varphi(f) = (f, f)$. Consequently, going over all f of degree less than $s+1$ gives us $\varphi(P_{<s+1}) = \{(f,f) : f \in P_{<s+1}\} = \ell_0$.

Now, when considering P_d for $2s+2 > d \ge s+1$, quotienting by a degree $s+1$ polynomial does have a non-trivial effect. That is, for $f \in P_d$, there must exist unique $h_1, h_2, r_1, r_2 \in \mathbb{F}_q[x]$ such that

$$f = h_1 g_1 + r_1$$
$$f = h_2 g_2 + r_2$$

where $\deg(h_1) = \deg(h_2) = \deg(f) - (s+1)$ and $\deg(r_1), \deg(r_2) < s+1$. Recall that $g_1(x) = (x - y_1)^{s+1}$ and $g_2(x) = (x - y_2)^{s+1}$. Moreover, $\pi_1(f) = r_1$ and $\pi_2(f) = r_2$. By definition, we have that $h_2 = \rho(h_1)$, meaning that $r_2 = r_1 + h_1 g_1 - \rho(h_1)g_2 = r_1 + \sigma(h_1)$. Therefore, we have $\varphi(f) = (r_1, r_1 + \sigma(h_1))$.

With this in mind, we recall that because h_1 and r_1 uniquely identify f, we can iterate over all elements of P_d by iterating over all $h_1 \in P_{d-(s+1)}$ and $r_1 \in P_{<s+1}$. That is, $P_d = \{h_1 g_1 + r_1 : h_1 \in P_{d-(s+1)}, r_1 \in P_{<s+1}\}$. From this we can conclude that

$$\begin{aligned} \varphi(P_d) &= \{\varphi(f) : f \in P_d\} \\ &= \{\varphi(h_1 g_1 + r_1) : h_1 \in P_{d-(s+1)}, r_1 \in P_{<s+1}\} \\ &= \{(r_1, r_1 + \sigma(h_1)) : h_1 \in P_{d-(s+1)}, r_1 \in P_{<s+1}\} \\ &= \bigcup_{h_1 \in P_{d-(s+1)}} \ell_{\sigma(h_1)}, \end{aligned}$$

as claimed.

Lastly, to see that σ is injective we recall that φ is an isomorphism and count cardinalities. For the left hand side of $\varphi(P_{<n}) = \bigcup_{h \in P_{<n-(s+1)}} \ell_{\sigma(h)}$, we have that $|\varphi(P_{<n})| = |P_{<n}| = q^n$. For the right hand side, we recall that $|\ell_b| = q^{s+1}$ for any b, so $\left|\bigcup_{h \in P_{<n-(s+1)}} \ell_{\sigma(h)}\right| = \left|\sigma(P_{<n-(s+1)})\right| \cdot q^{s+1}$. Therefore, $\left|\sigma(P_{<n-(s+1)})\right| = \frac{q^n}{q^{s+1}} = q^{n-(s+1)}$, showing that σ is injective since $\left|P_{<n-(s+1)}\right| = q^{n-(s+1)}$. □

6 Non-Bipartite Lossless Expander

Here, we show how to transform a two-sided lossless expander into an undirected graph (that is not necessarily bipartite) while retaining lossless expansion. We then apply this transformation to the KT graph to obtain our main theorem.

Theorem 6.1 (Formal version of Theorem 2)**.** *For infinitely many N and all $0 < \delta < 0.99$, there exists an explicit regular $(K, \varepsilon = 0.01)$ lossless expander $G = (V, E)$ where $|V| = N$, the degree is D where $N^{1-1.01\delta} \leq D \leq N^{1-1.01\delta+o(1)}$ and $K = \min\left(N^{\delta}, N^{1-1.01\delta-o(1)}\right)$. Moreover, G with one vertex removed, is endowed with a free group action from $\mathbb{F}_q$, where $q = \text{poly}(\log N)$.*

6.1 Expansion from the bipartite half

Given a two-sided lossless expander, we show how to obtain a (not necessarily bipartite) graph that is also a losslesss expander while inheriting the expansion of this graph. We use the bipartite half transformation defined as follows.

Definition 6.2 (Bipartite half)**.** *Let $G = (L \sqcup R, E)$ be a (D_L, D_R)-regular bipartite graph. Then the* bipartite half $G^2[L] = (L, E^2[L])$ *is defined as $E^2[L] = \{(v, w) \in L \times L \mid w \in \Gamma_{\leftarrow}(\Gamma_{\rightarrow}(v))\}$.*

Next, we show how this transformation retains lossless expansion. For the sake of clarity, we will use $\Gamma_{\rightarrow}$ and $\Gamma_{\leftarrow}$ for the left-to-right and right-to-left neighborhood functions of G and Γ as the neighborhood function of $G^2[L]$.

Lemma 6.3. *Let $G = (L \sqcup R, E)$ be a (D_L, D_R)-regular (K_L, A_L, K_R, A_R)-two-sided lossless expander with $D_L \leq K_R$. Then $G^2[L]$ is a max-degree (K, A)-expander where each node has a degree in $[D_L A_R, D_L D_R]$ and with $K = \min(K_L, K_R/D_L)$ and $A = A_L A_R$.*

Remark 6.4. *While $G^2[L]$ may not be exactly regular, since $A_L = (1 - \varepsilon_L)D_L$ and $A_R = (1 - \varepsilon_R)D_R$, we see that $A = A_L A_R = (1 - \varepsilon_L)(1 - \varepsilon_R)D_L D_R$, meaning that our expansion is with respect to the highest possible degree $D_L D_R$ of any individual vertex.*

Proof of Lemma 6.3. We begin by showing that each node $v \in L$ of $G^2[L]$ has degree in $[D_L A_R, D_L D_R]$. By assumption, we have that $|\Gamma_{\rightarrow}(v)| = D_L \leq K_R$. Thus, by the right-to-left expansion of G, we have that $|\Gamma_{\leftarrow}(\Gamma_{\rightarrow}(v))| \geq D_L A_R$. The upper bound is immediate given that the right degree is D_R so $|\Gamma_{\leftarrow}(\Gamma_{\rightarrow}(v))| \leq D_R |\Gamma_{\rightarrow}(v)| = D_R D_L$.

Next, we prove expansion. Let $S \subseteq L$ be a set of size at most K. Then, because $K \leq K_L$, the left-to-right expansion of G gives us that $|\Gamma_{\rightarrow}(S)| \geq A_L |S|$. To expand a second time, we recall that $K \leq K_R/D_L$, so $|\Gamma_{\rightarrow}(S)| \leq D_L |S| \leq D_L K \leq K_R$, meaning that we can apply the right-to-left expansion of G. This yields $|\Gamma_{\leftarrow}(\Gamma_{\rightarrow}(S))| \geq A_R |\Gamma_{\rightarrow}(S)| \geq A_R A_L |S|$, as claimed. □

In the special case of the KT graph, the bipartite half is regular. To show this, we make the following observation.

Remark 6.5. *The bipartite half of the KT graph G from Definition 3.5 has a succinct representation as $G^2[L] = (L, E^2[L])$ where $E^2[L] = \{(f, g) \mid \exists y \in \mathbb{F}_q,\ \psi_y(f) = \psi_y(g)\}$.*

This allows us to prove the following regularity lemma.

Lemma 6.6. *Let $G^2[L]$ be the bipartite half of the KT graph. Then $G^2[L]$ is regular.*

Proof. Let $T_a^n[f](x) = \sum_{i=0}^{n} \frac{f^{(n)}(a)}{i!}(x - a)^i$ be the n-th Taylor polynomial of f at a. Then we claim that $\psi_y(f) = \psi_y(g)$ for any $y \in \mathbb{F}_q$ if and only if $T_y^s[f](x) = T_y^s[g](x)$ as polynomials. For the forward direction, we note that $\psi_y(f) = \psi_y(g)$ exactly gives us that $f^{(i)}(y) = g^{(i)}(y)$ for $i \in \{0, \ldots, s\}$, immediately implying that $T_y^s[f](x) = T_y^s[g](x)$. Conversely, if $T_y^s[f](x) = T_y^s[g](x)$ as polynomials, then their coefficients must be equivalent. Thus, $f^{(i)}(y) = g^{(i)}(y)$ for $i \in \{0, \ldots, s\}$, meaning that $\psi_y(f) = \psi_y(g)$.

With this claim in hand, we can use Remark 6.5 to see that (f, g) is an edge in $G^2[L]$ if and only if there exists some $y \in \mathbb{F}_q$ such that $T^s_y[f](x) = T^s_y[g](x)$. In other words, the neighbors of f must have the same s-th Taylor polynomial at some $y \in \mathbb{F}_q$. More formally, the neighbor set of f is $\Gamma(f) = \left\{T^s_y[f](x) + \sum_{i=s+1}^{n-1} a_i(x-y)^i \mid a_{s+1}, \ldots, a_n, y \in \mathbb{F}_q\right\}$. Thus, the number of neighbors is $|\Gamma(f)| = \left|\left\{\sum_{i=s+1}^{n-1} a_i(x-y)^i \mid a_{s+1}, \ldots, a_n, y \in \mathbb{F}_q\right\}\right|$, which does not depend on f. Therefore, the degree of any vertex is the same and $G^2[L]$ is regular. □

6.2 Free group action on the bipartite half

Now that we have shown that the bipartite half generally preserves lossless expansion, we will consider it instantiated with the KT graph and show that multiplication by elements of $\mathbb{F}_q$ constitutes a free group action on this resulting graph (with one node removed).

Our action of $\mathbb{F}_q$ on the bipartite half of the KT graph is directly by multiplication in $\mathbb{F}_q$.

Definition 6.7. *Let $G = (L \sqcup R, E)$ be the KT graph and $G^2[L]$ be its bipartite half. We define the action of $\mathbb{F}_q$ on $G^2[L]$ as follows: for any $\alpha, y \in \mathbb{F}_q$ we have $(\alpha \cdot f)(y) = \alpha \cdot f(y)$ where the latter multiplication is in $\mathbb{F}_q$.*

We now show that $G^2[L]$ without the zero polynomial is $\mathbb{F}_q$-invariant and that this is a free group action.

Lemma 6.8. *Let $G = (L \sqcup R, E)$ be the KT graph and $H = G^2[L] \setminus \{0\}$ be its bipartite half without the zero polynomial. Consider the action of $\mathbb{F}_q$ on $G^2[L]$ as defined in Definition 6.7. Then H is $\mathbb{F}_q$-invariant and this action is free.*

Proof. To show that H is $\mathbb{F}_q$-invariant, we must prove that for any $(f, g) \in E^2[L]$ and $\alpha \in \mathbb{F}_q$ we have $(\alpha \cdot f, \alpha \cdot g) \in E^2[L]$. From Remark 6.5 we know that $E^2[L] = \{(f, g) \mid \exists y \in \mathbb{F}_q,\ \psi_y(f) = \psi_y(g)\}$. Thus, we must equivalently show that if $\psi_y(f) = \psi_y(g)$ for some $y \in \mathbb{F}_q$, then $\psi_y(\alpha \cdot f) = \psi_y(\alpha \cdot g)$. This is immediate by Corollary 4.7 because ψ_y being $\mathbb{F}_q$-linear allows us to compute

$$\psi_y(\alpha \cdot f) = \alpha\psi_y(f) = \alpha\psi_y(g) = \psi_y(\alpha \cdot g),$$

showing that $(\alpha \cdot f, \alpha \cdot g) \in E^2[L]$.

Next, we will show that this is a free group action. Consider any f in the vertices of H and $\alpha \in \mathbb{F}_q$. Since H does not contain the zero polynomial, we know that f is not identically zero. Thus, if $\alpha \cdot f = f$, it must be that $\alpha = 1$, showing that the action is indeed free. □

6.3 Plugging in parameters

Finally, we plug in our two-sided lossless expander result from the KT graph to get Theorem 6.1.

Proof of Theorem 6.1. We invoke Lemma 4.9 with $\alpha = 0.01, \varepsilon_L = 0.001, \varepsilon_R = 0.001$ to obtain a (D_L, D_R)-biregular two-sided $(K_L = N^\delta, \varepsilon_L = 0.001, K_R = 0.002 \cdot (1/D_L) \cdot \min(M, N/M), \varepsilon_R = 0.001)$ lossless expander $\Gamma_\to : [N] \times [D_L] \to [M]$ where $D_L \leq O(\log^{204}(N))$ and $N^{1.01\delta - o(1)} \leq M \leq D_L \cdot N^{1.01\delta}$. We then apply Lemma 6.3 to conclude the claim about expansion (since $\varepsilon \geq (1 - \varepsilon_L)(1 - \varepsilon_R)$) and the degree bound. We then apply Lemma 6.6 to show the bipartite half is regular. The claim about the free $\mathbb{F}_q$ action comes from Lemma 6.8 by removing the vertex corresponding to the zero polynomial from the bipartite half of the KT

graph. Lastly, we compute that:

$$\begin{aligned}
K &= \min\left(K_L, K_R/D_L\right) \\
&= \min\left(K_L, (1/500)\cdot(1/D_L^2)\cdot M, (1/500)\cdot(1/D_L^2)\cdot(N/M)\right) \\
&= \min\left(N^{\delta}, (1/500)\cdot N^{1.01\delta-o(1)}, (1/500)\cdot N^{1-1.01\delta-o(1)}\right) \\
&= \min\left(N^{\delta}, (1/500)\cdot N^{1-1.01\delta-o(1)}\right) \\
&= \min\left(N^{\delta}, N^{1-1.01\delta-o(1)}\right)
\end{aligned}$$

and the bound on maximum size of sets that expand follows. Explicitness of this graph follows from Claim A.2. □

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A Explicitness

Claim A.1. *The KT graph G as defined in Definition 3.5 is explicit, i.e., the left neighborhood function $\Gamma_{\rightarrow} : \mathbb{F}_q^n \times \mathbb{F}_q \to \mathbb{F}_q^{s+2}$ and right neighborhood function $\Gamma_{\leftarrow} : \mathbb{F}_q^{s+2} \times \mathbb{F}_q^{n-(s+1)} \to \mathbb{F}_q^n$ can be computed in* $\mathrm{poly}(n, \log(q))$ *time.*

Proof. To compute $\Gamma_{\rightarrow}(f, y)$, we treat f as an element of $\mathbb{F}_q[X]$ of degree at most $n-1$, and map it to $(y, f^{(0)}(y), \ldots, f^{(s)}(y))$. We can compute derivatives of f and evaluate it at y in time $\mathrm{poly}(n, \log(q))$ and hence explicitly compute $\Gamma_{\rightarrow}(f, y)$.

To compute $\Gamma_{\leftarrow}(z, t)$, we proceed as follows. Let $z = (y, w)$ where $y \in \mathbb{F}_q$ and $w \in \mathbb{F}_q^{s+1}$. Then, we need to find f such that $\Gamma_{\rightarrow}(f, y) = z$. Define $\psi_y : \mathbb{F}_q^n \to \mathbb{F}_q^{s+1}$ as $\psi_y(f) = (f^{(0)}(y), \ldots, f^{(s)}(y))$. By Lemma 4.3, ψ_y is a full rank $\mathbb{F}_q$-linear map. As $n > s+1$, kernel of ψ_y has dimension $n-(s+1) > 0$. By considering the matrix associated with ψ_y and using standard linear algebra algorithms, we construct an injective linear map $K_y : \mathbb{F}_q^{n-(s+1)} \to \mathbb{F}_q^n$ in $\mathrm{poly}(n, \log(q))$ time such that the image of K_y is exactly the kernel of ψ_y. Furthermore, using Gaussian elimination on the matrix associated with ψ_y, we can, in $\mathrm{poly}(n, \log(q))$ time, find some $g \in \mathbb{F}_q^n$ such that $\psi_y(g) = w$. Finally, we let $\Gamma_{\leftarrow}(z, t) = K_y(t) + g$. By linearity of ψ_y, we have that $\psi_y(K_y(t) + g) = \psi_y(g) = w$. As K_y is injective, for a fixed z, our computed function maps different t to different outputs as desired. □

Note that given any $n, k, \varepsilon, \alpha$, Lemma 4.9 sets $q = \mathrm{poly}(n, k, 1/\varepsilon)$, so we can deterministically find such a prime q satisfying the requirements in $\mathrm{poly}(n, 1/\varepsilon)$ time as well.

Explicitness of the KT graph also implies explicitness of our non-bipartite lossless expander obtained by taking the bipartite half of the KT graph (and removing the zero vertex):

Claim A.2. *The non-bipartite graph $H = G^2[L \setminus \{0\}]$ as constructed in Theorem 6.1 is explicit. I.e., the neighborhood function $\Gamma : (\mathbb{F}_q^n \setminus \{0\}) \times (\mathbb{F}_q \times \mathbb{F}_q^{n-(s+1)}) \to \mathbb{F}_q^n$ can be computed in* $\mathrm{poly}(n, q)$ *time.*

Proof. Note that since G is not necessarily regular, Γ may sometimes output $\perp$. The guarantee that we will have is that for all $f, g \in \mathbb{F}_q^n$ that are neighbors in H, there will exist unique $y, t \in \mathbb{F}_q \times \mathbb{F}_q^{n-(s+1)}$ such that $\Gamma(f, y, t) = g$.

Let $\Gamma_{\rightarrow}, \Gamma_{\leftarrow}$ be the explicit left and right neighborhood functions of the KT graph as defined in Claim A.1. To compute $\Gamma(f, y, t)$, we first compute $g = \Gamma_{\leftarrow}(\Gamma_{\rightarrow}(f, y), t)$. If $g = 0$, then we output $\perp$. Then, for all $y' < y$, we check whether $\Gamma_{\rightarrow}(f, y') = \Gamma_{\rightarrow}(g, y')$. If they are equal for any such y', we output $\perp$. Otherwise, we output g. This last check is done so that we only output g once as a neighbor of f and otherwise output $\perp$. Explicitness of Γ follows because of explicitness of $\Gamma_{\rightarrow}, \Gamma_{\leftarrow}$ and because the last check has to be done $O(q)$ times. □

B The [GUV09] Graph is Not Right Regular

One may naturally try to show that the predecessor to the KT graph, the [GUV09] graph, is also a two-sided lossless expander. However, it turns out that the [GUV09] graph is not even right regular. To see why, we give the definition of the [GUV09] graph which is similar to the KT graph.

Definition B.1 (The GUV graph, [GUV09])**.** *Let $q, n, m, h \in \mathbb{N}$ be such that q is a prime power greater than h, characteristic of the finite field $\mathbb{F}_q \geq n$ and $m < n$. We define $G = (L \sqcup R, E)$ where $L =$*

$\mathbb{F}_q^n \cong \mathbb{F}_q[x]/(z(x))$ *for some irreducible polynomial* $z(x) \in \mathbb{F}_q(x)$ *of degree* n *and* $R = \mathbb{F}_q^{m+1}$. *The left degree is* q *and for any* $f \in \mathbb{F}_q[x]/(z(x))$ *and* $y \in \mathbb{F}_q$, *the* y*'th neighbor of* f *is defined as* $\Gamma_{\rightarrow}(f, y) = \Big(y, f(y), (f^h \bmod z(x))(y), (f^{h^2} \bmod z(x))(y), \ldots, (f^{h^{m-1}} \bmod z(x))(y)\Big)$.

Our proof of right-regularity - Lemma 4.3 - relied on the fact that the map $\psi_y(f) \mapsto (f^{(0)}(y), \ldots, f^{(s)}(y))$ is full rank over $\mathbb{F}_q$. The analogous GUV map $\varphi_y(f) = (f(y), f^h(y), \ldots, f^{h^{m-1}}(y))$ does not have this property because of two issues. First, it is not necessarily linear over $\mathbb{F}_q$, although it can be made linear over $\mathbb{F}_2$ when q is a power of 2. Second, even over $\mathbb{F}_2$, it does not necessarily have full rank, meaning we cannot guarantee right regularity.

Simulations bear this out. The GUV graph with $q = 2^4$, $n = 4$, $m = 2$, and $h = 2$ has 3072 right vertices with degree 256, 64 with degree 4096, and 960 isolated vertices. For more examples of parameter settings where the GUV graph is not right-regular, we invite the reader to run simulations with our code at `https://github.com/mjguru/GUV-Expander`.