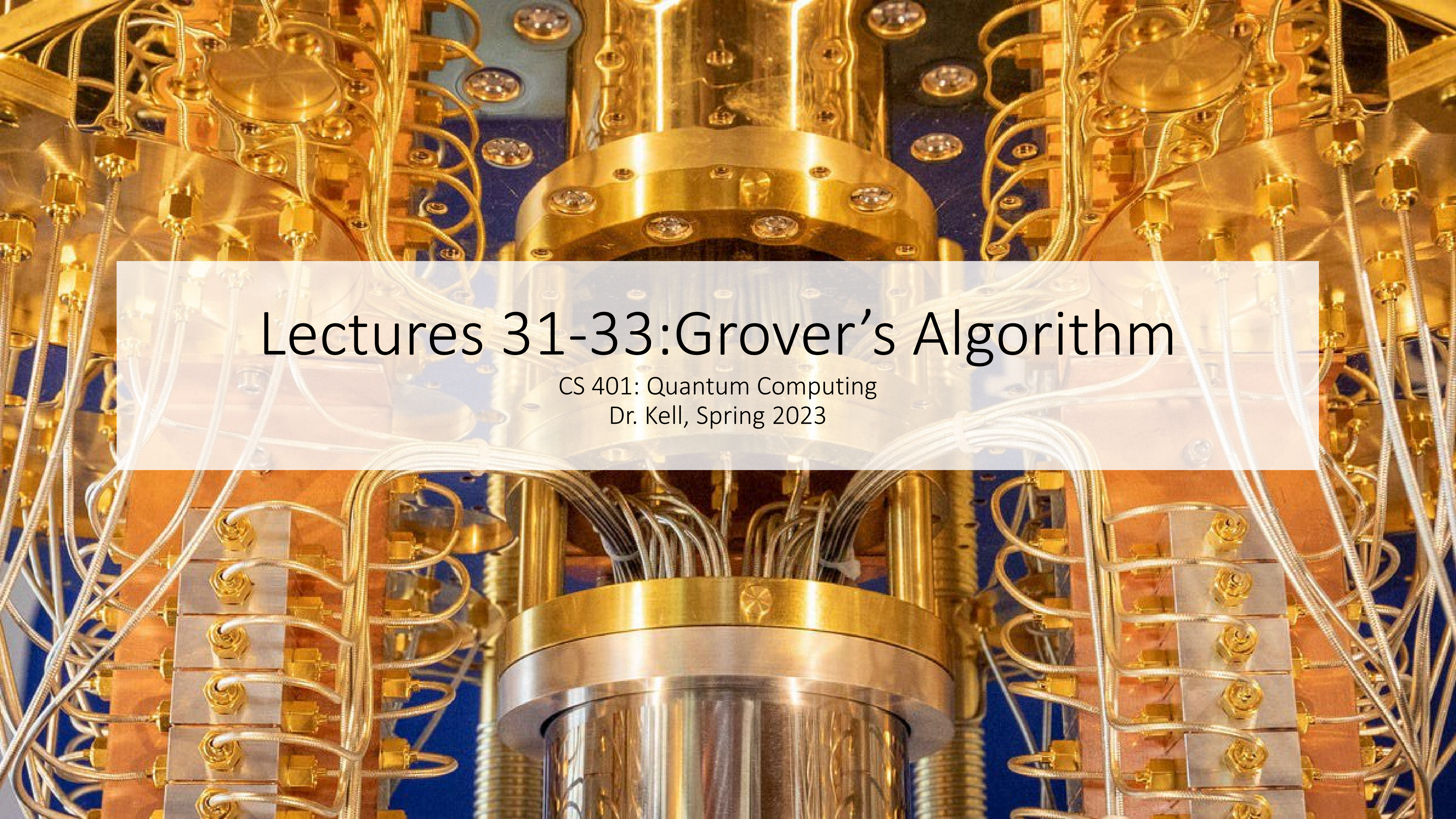


Lectures 31-33:Grover’s Algorithm

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Grover's Algorithm

(Discovered in 1996 by Lov Grover)

Problem: "unstructured" search problem, where we can efficiently verify if we've found a target solution (i.e., a problem in NP).

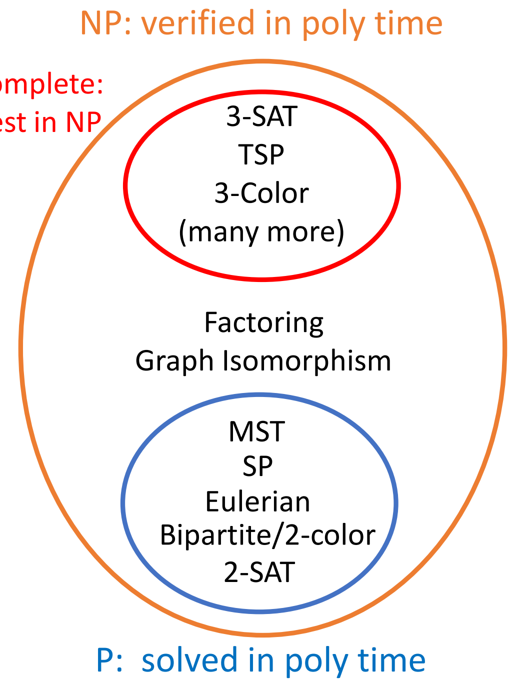
Input: list L, target value
Output: index of target in L

L = [2, 1, 10, 4, 7, 9, 3] → 4
target = 7 (index of 7)

Many applications in cloud quantum computing, databases, etc.

Best Possible Classical Algorithm: O(n)

Grover's Quantum Algorithm: O($n^{1/2}$)



Punchline: N total solutions that can be verified in O(T) time then:

Best Possible Classical Algo: $O(NT)$ time

↓ quadratic speed-up

Best Possible Quantum Algo: $O(\sqrt{NT})$ time

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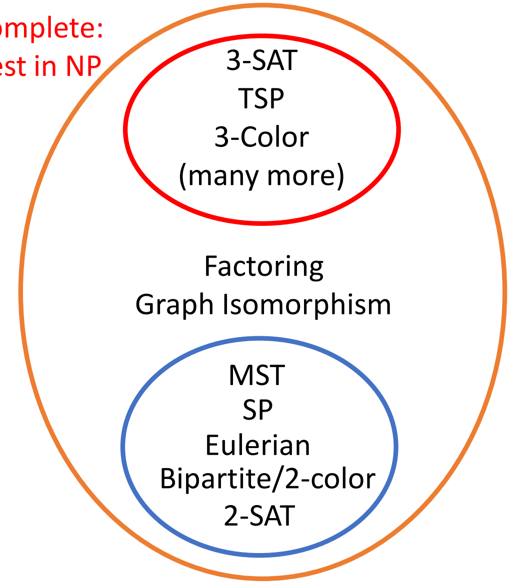

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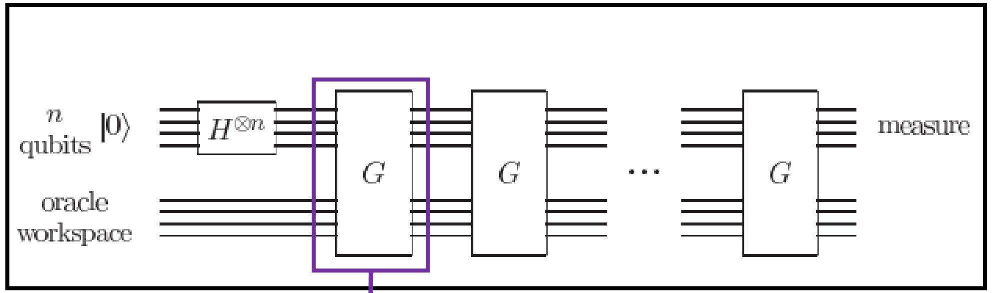
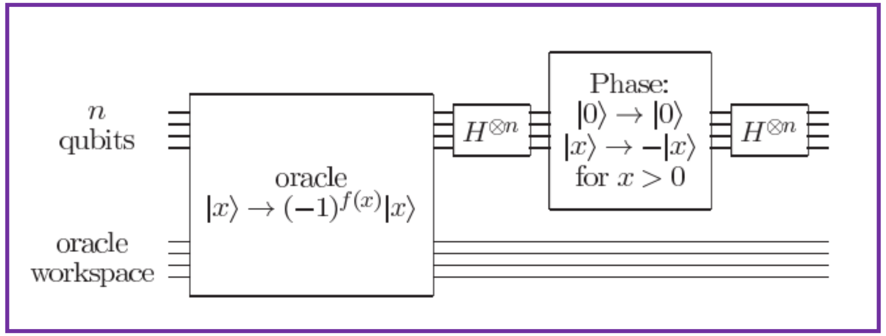
Grover's Algorithm Circuit Outline

Top-level



Grover Iteration



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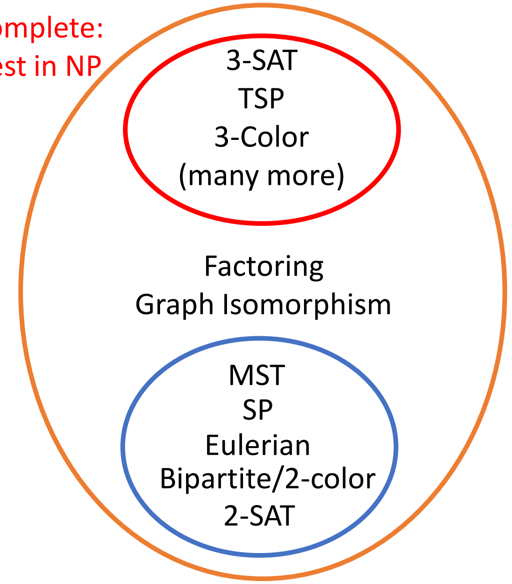
NP: verified in poly time

NP-complete: hardest in NP



P: solved in poly time

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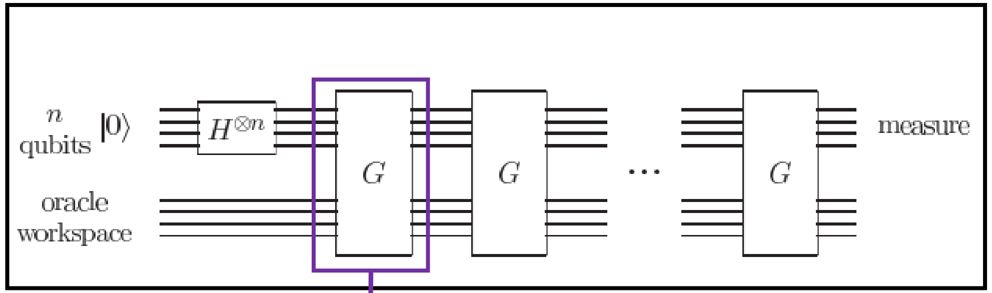
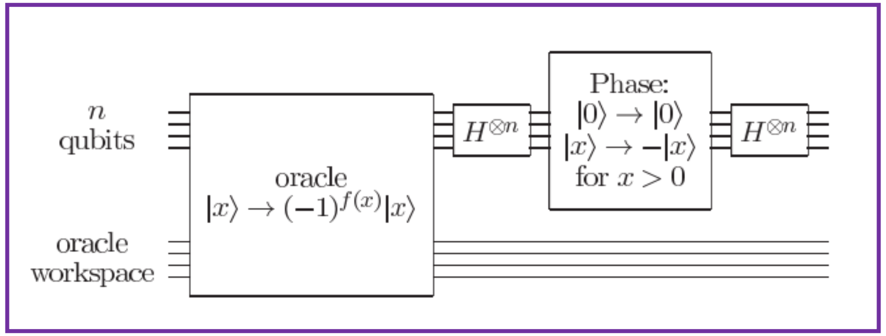
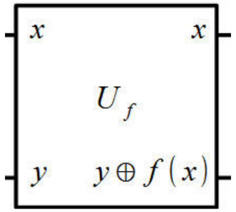


Grover Iteration



Essentially: oracle = Uf gate with $|-\rangle$ qubit

$f(x) = 0 \to \text{not a solution} \quad f(x) = 1 \to \text{is a solution}$



$$\text{on } |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = (-1)^{f(x)}|x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

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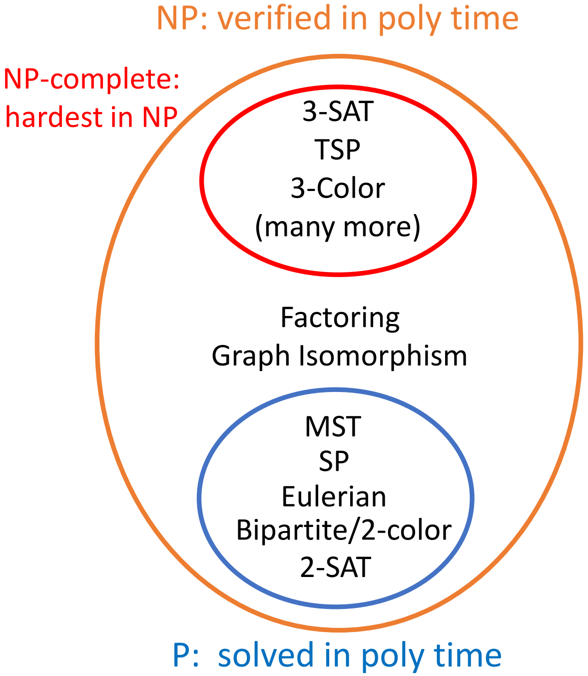

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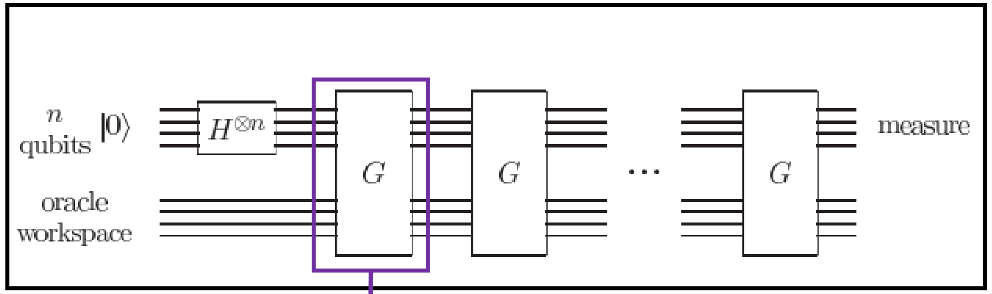
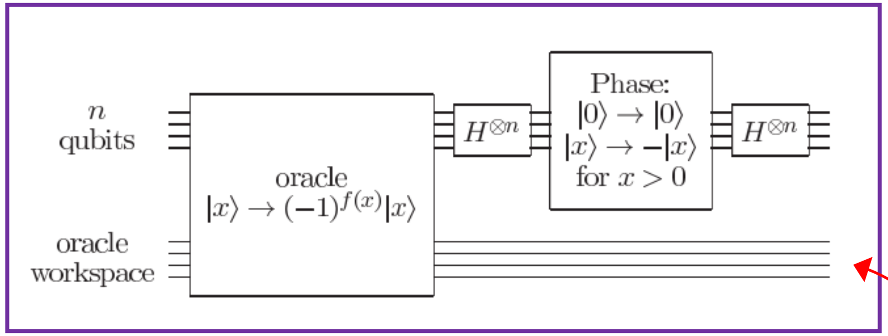


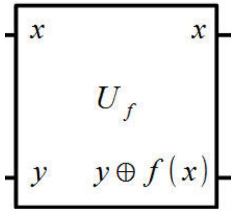
Grover Iteration



... can think of it belonging to workspace

Essentially: oracle = Uf gate with $|-\rangle$ qubit

$f(x) = 0 \to$ not a solution $\quad f(x) = 1 \to$ is a solution

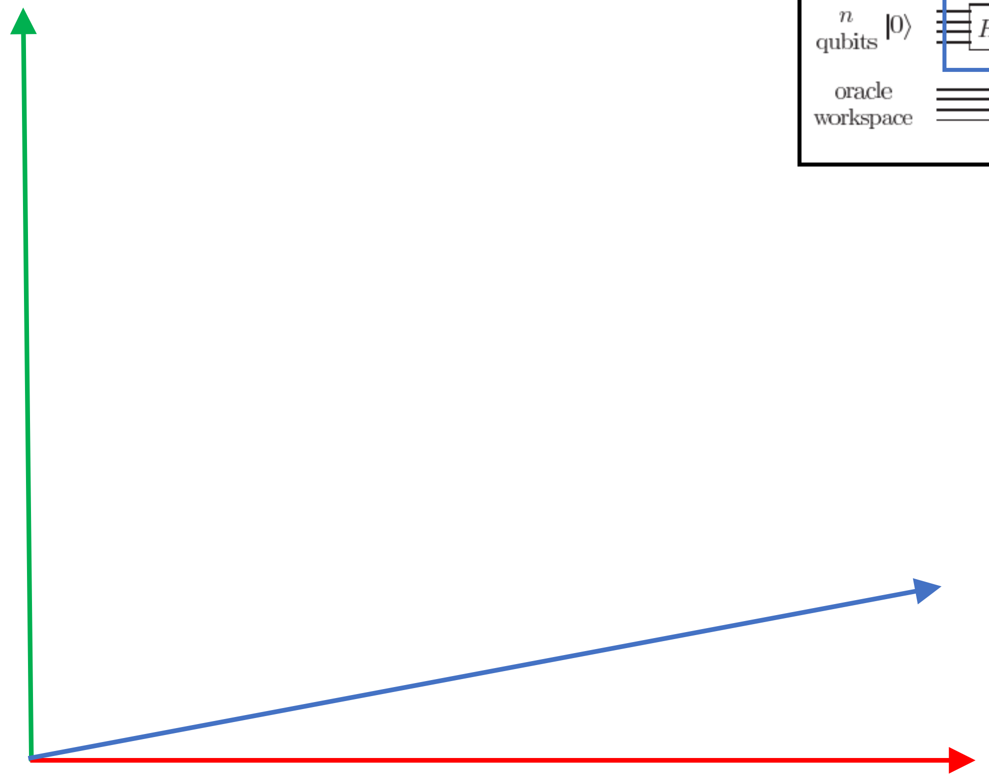


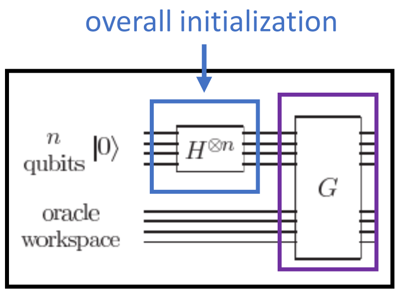
$$\text{on } |x\rangle \cancel{\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]} = (-1)^{f(x)}|x\rangle \cancel{\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]}$$

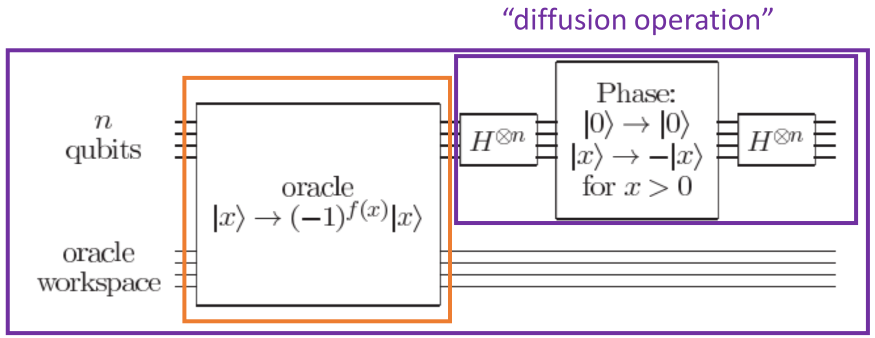
Just like Deustch-Jozsa: y qubit won't be changed throughout

Geometric Interpretation of a Grover Iteration

Example: 3-qubits, target = $|111\rangle$







(in this example: $f(111) = 1, f(x) = 0$ for $|x\rangle \neq |111\rangle$)

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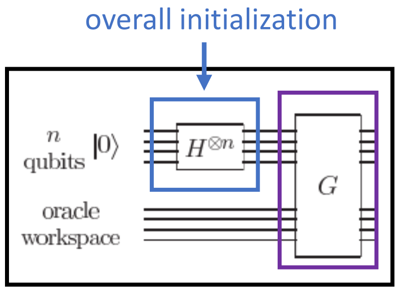

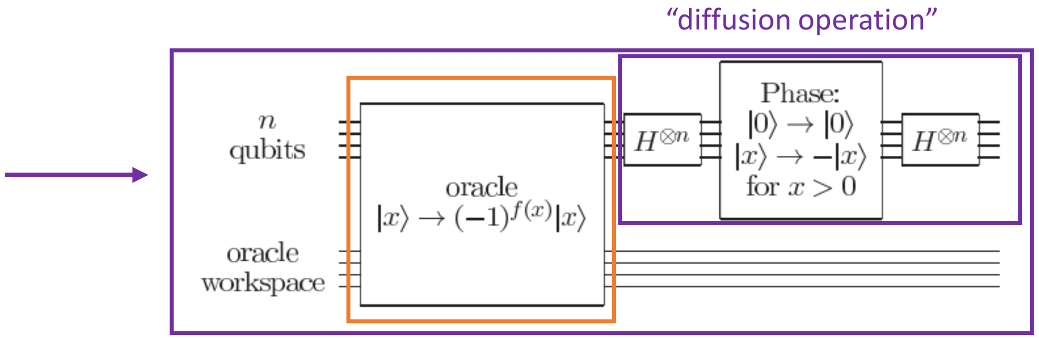

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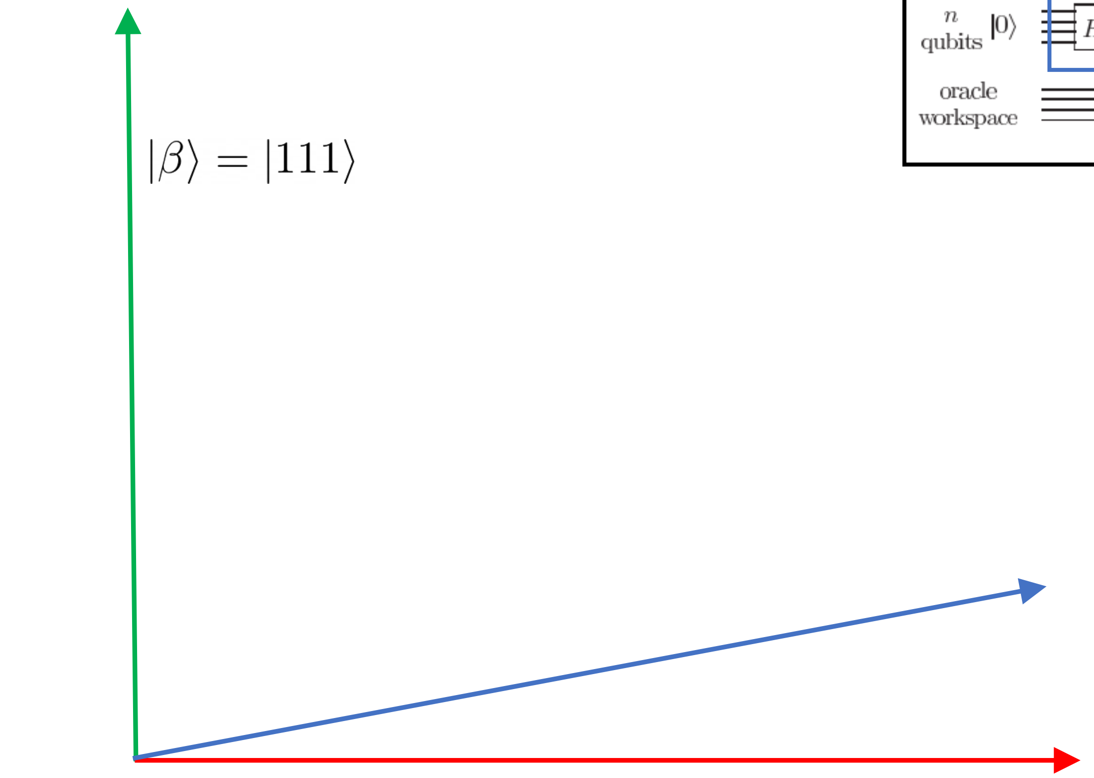

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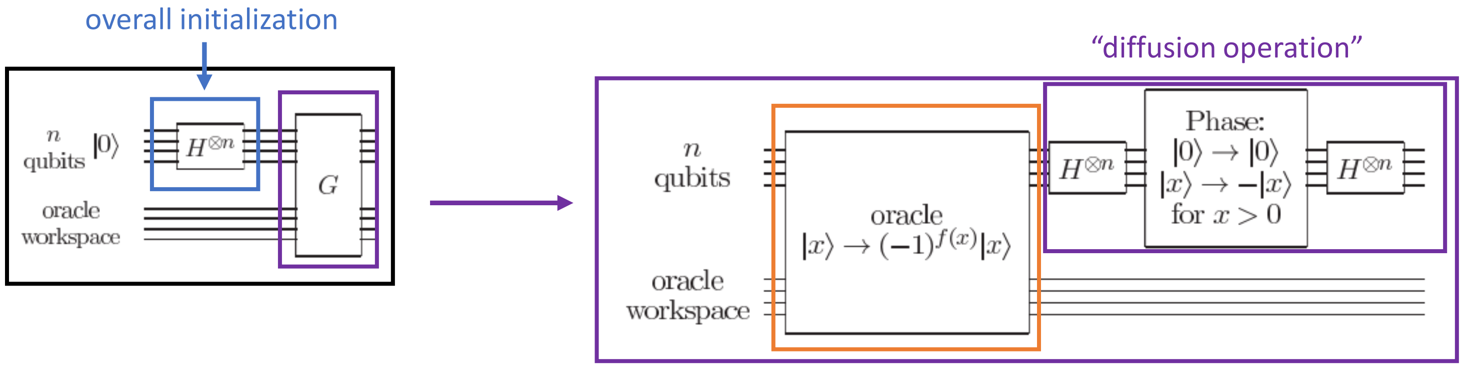

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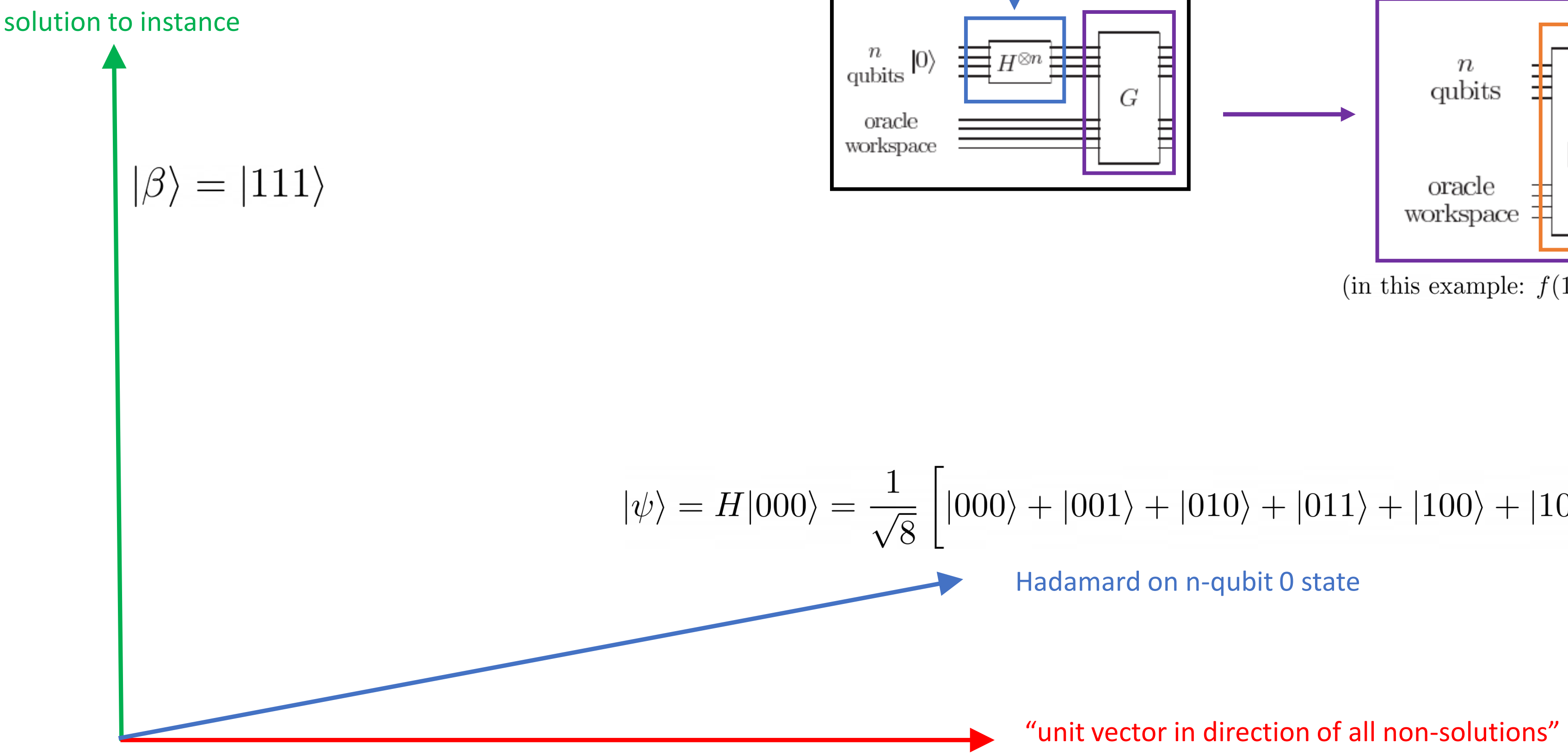

$$|\beta\rangle = |111\rangle$$

$$|\psi\rangle = H|000\rangle = \frac{1}{\sqrt{8}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right]$$

$$|\alpha\rangle = \frac{1}{\sqrt{7}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle\right]$$

Geometric Interpretation of a Grover Iteration

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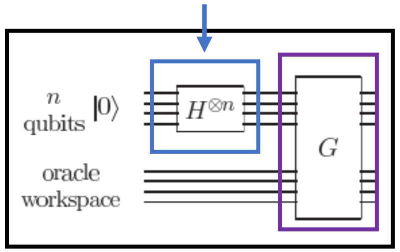
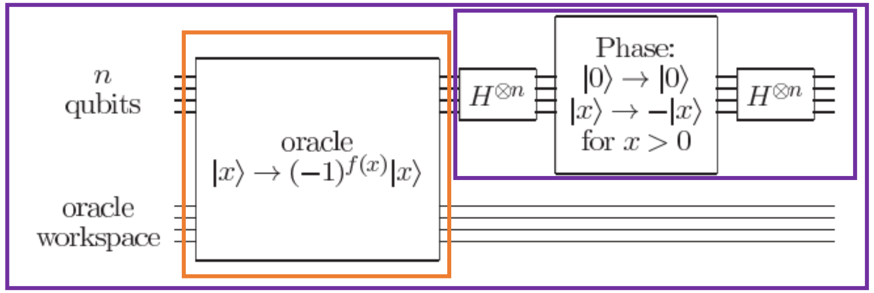
overall initialization



“diffusion operation”



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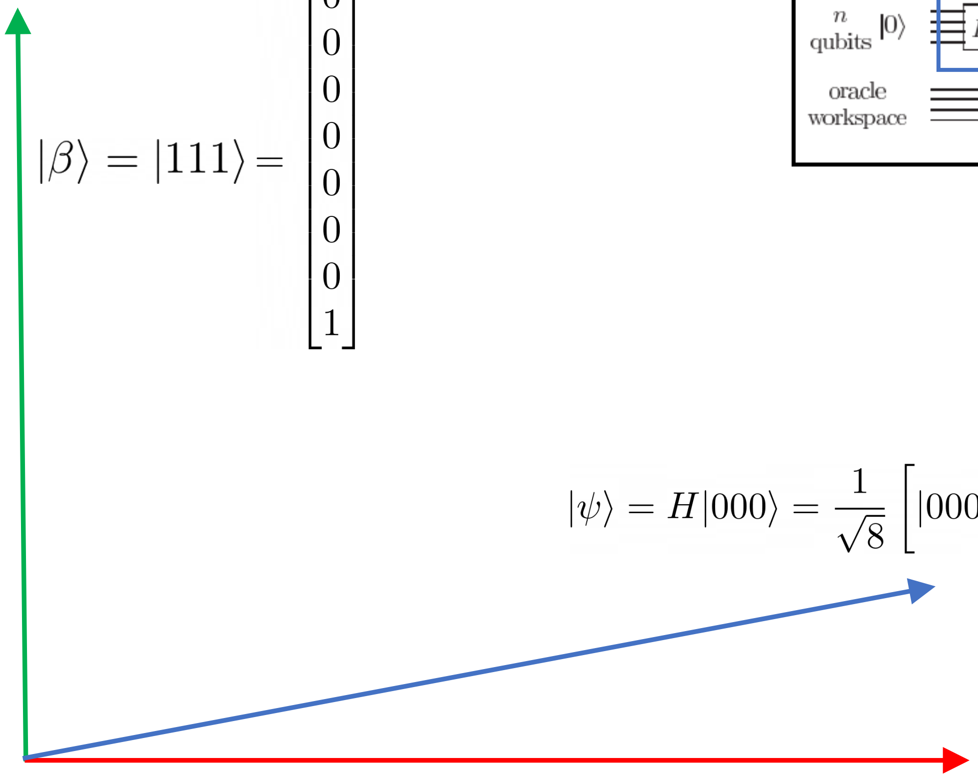

$$|\beta\rangle = |111\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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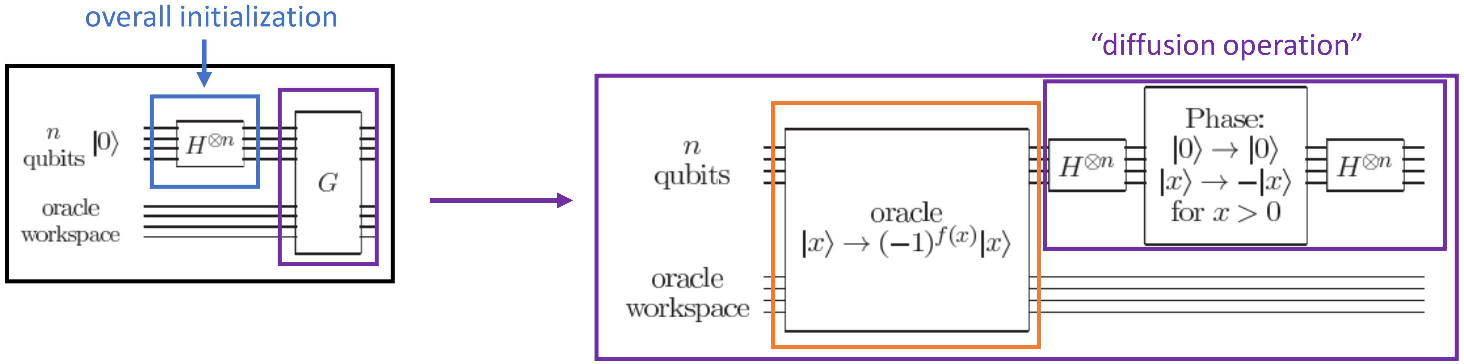

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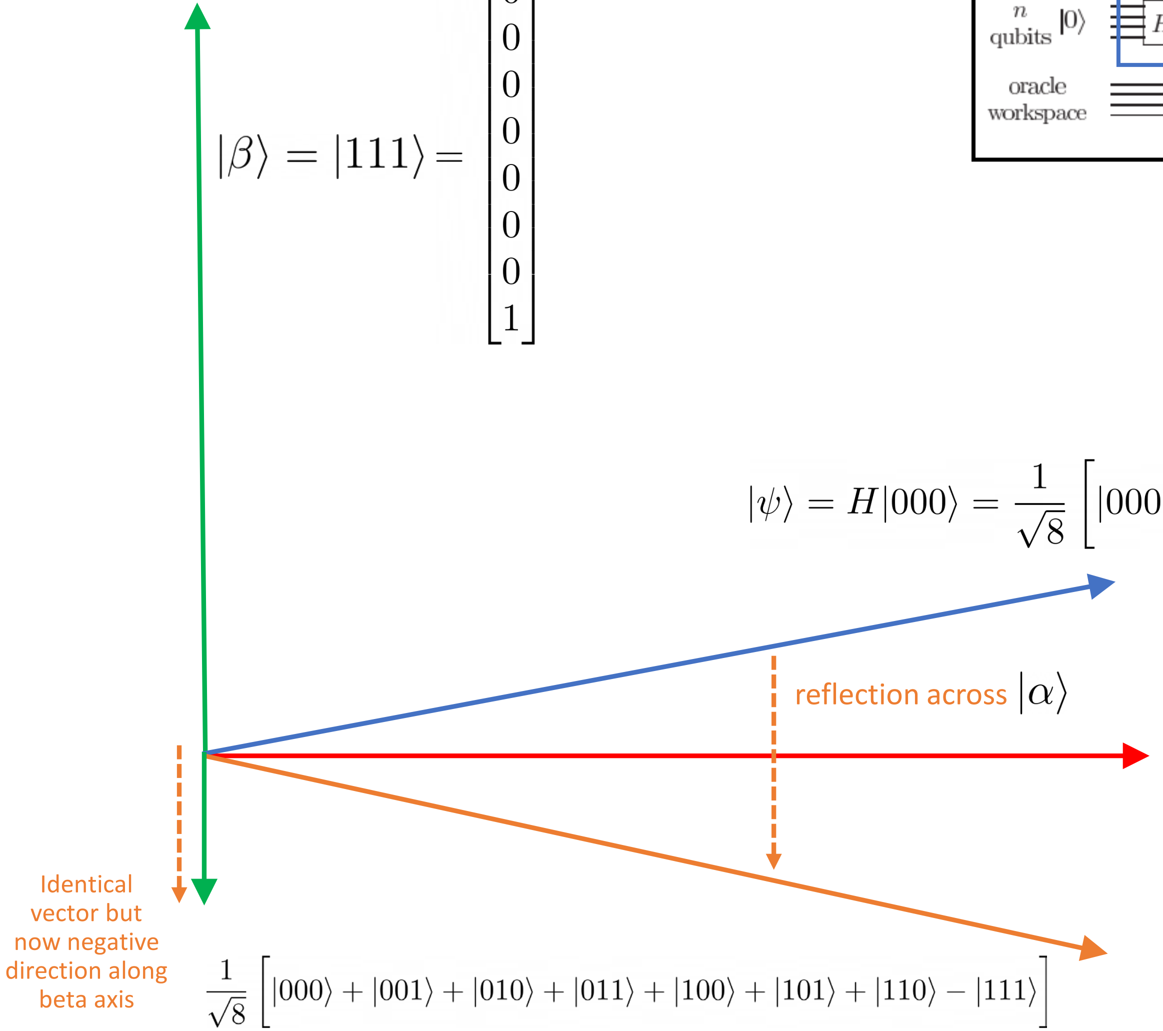

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$$\frac{1}{\sqrt{8}}\Big[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle\Big]$$

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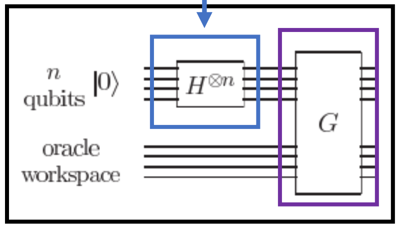

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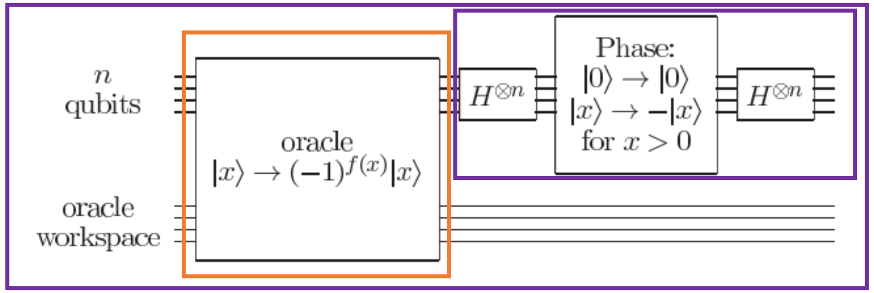

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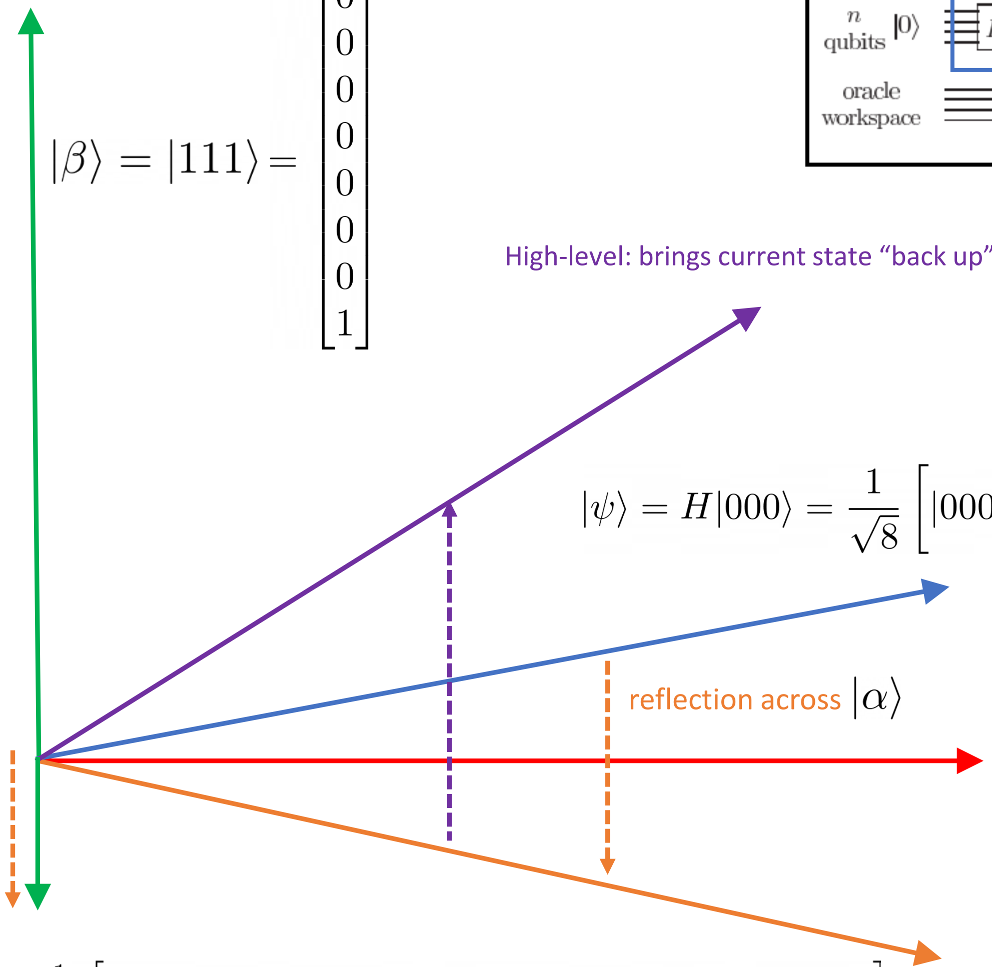
solution to instance

$$|\beta\rangle = |111\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



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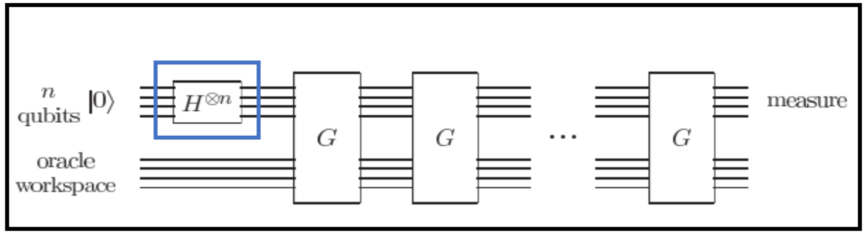
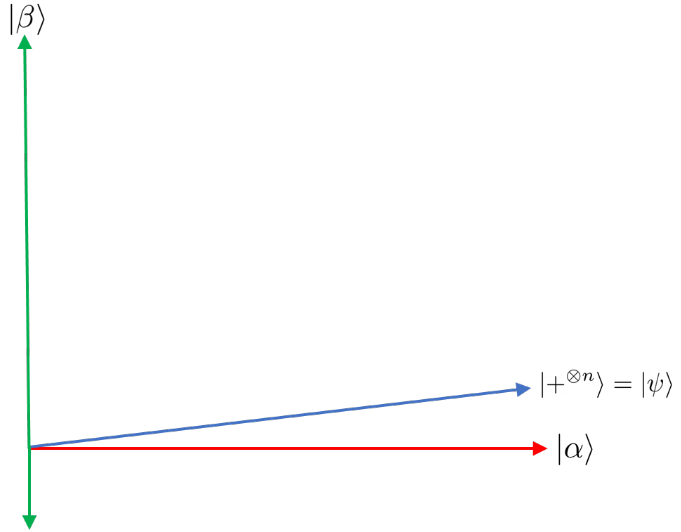
$$\frac{1}{\sqrt{8}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle\right]$$

Geometric Interpretation of a Grover Iteration

Thus: state of circuit makes progress toward beta with each application of G.





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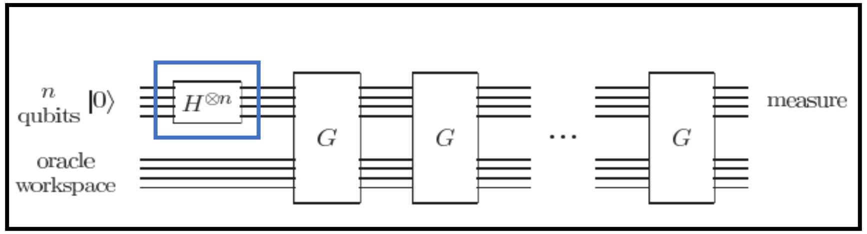

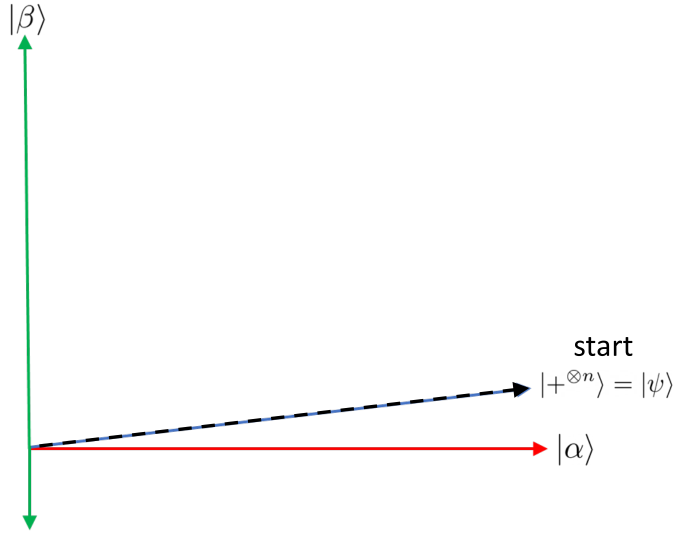

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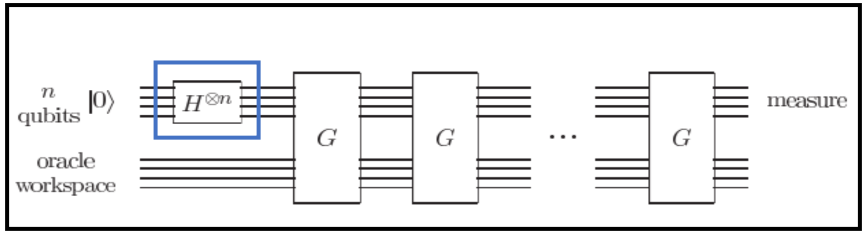

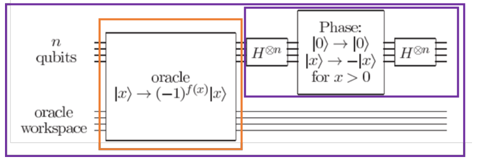

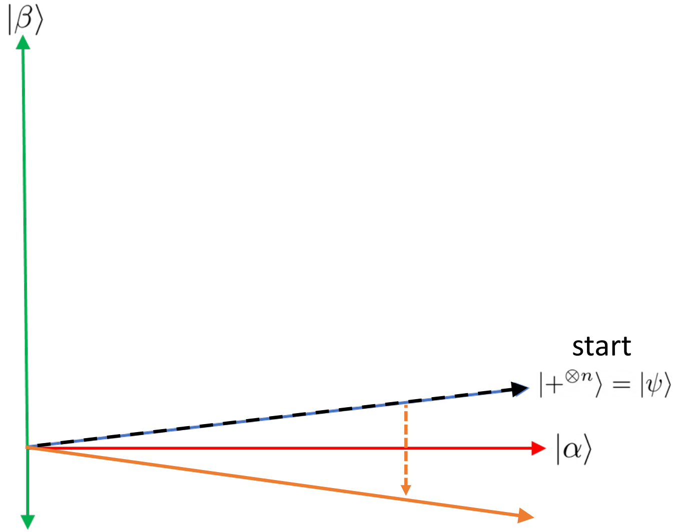

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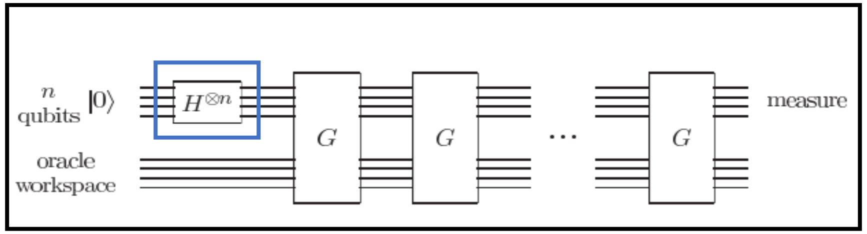

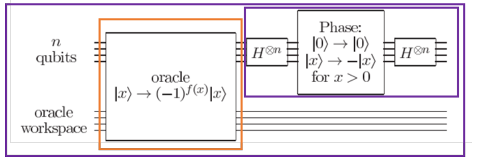

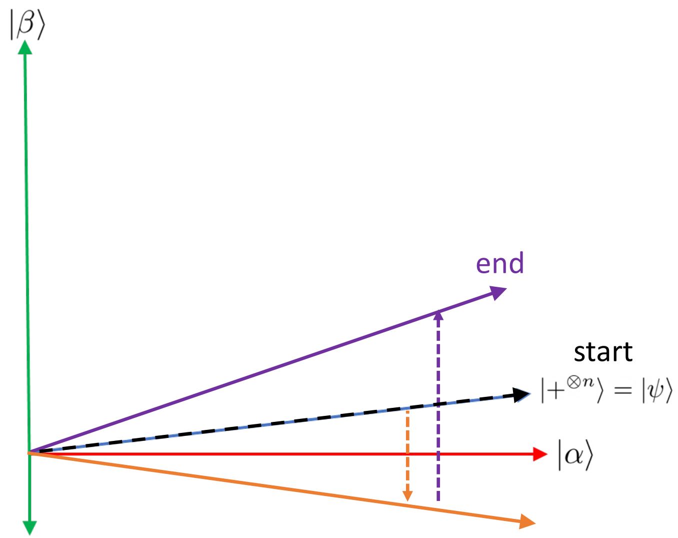

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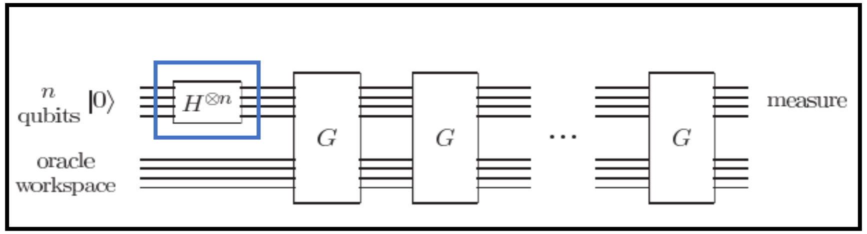

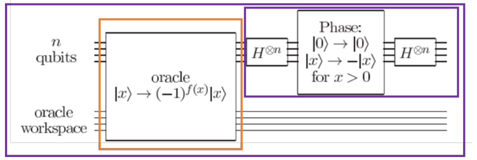

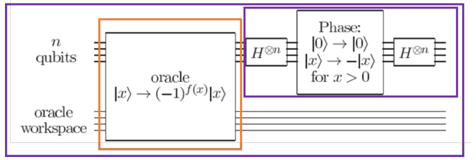

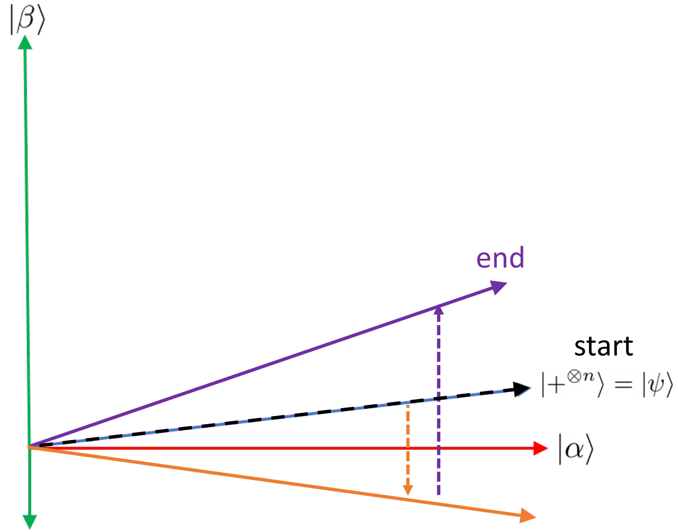

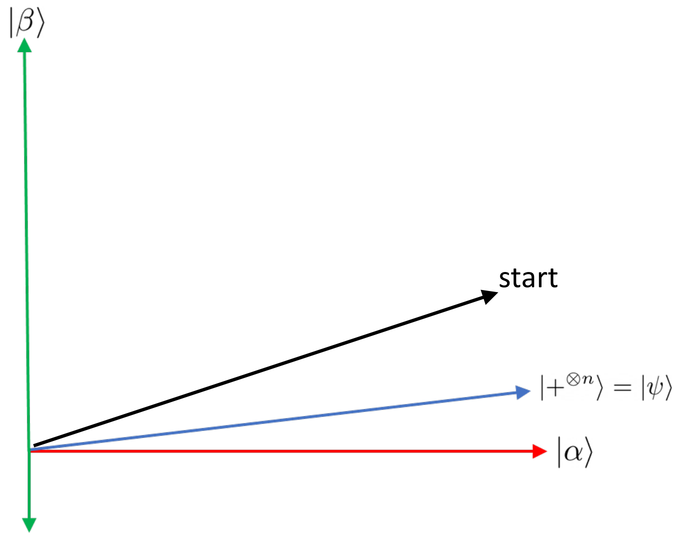

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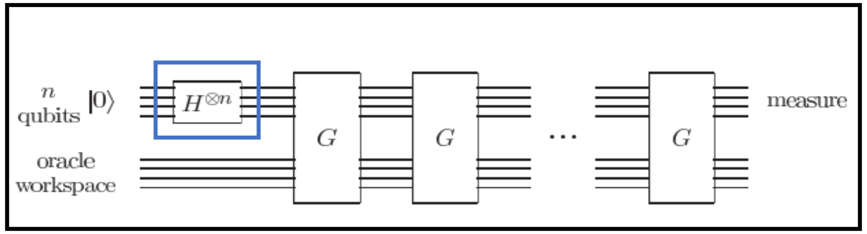

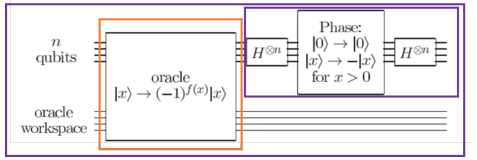

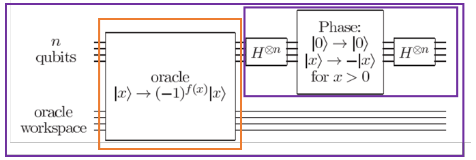

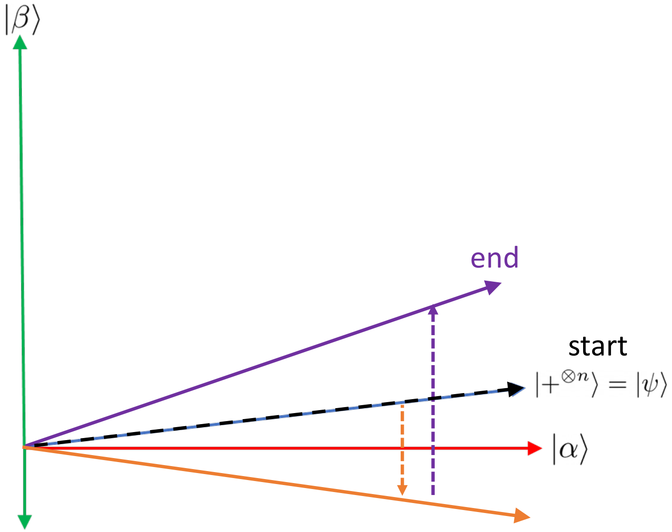

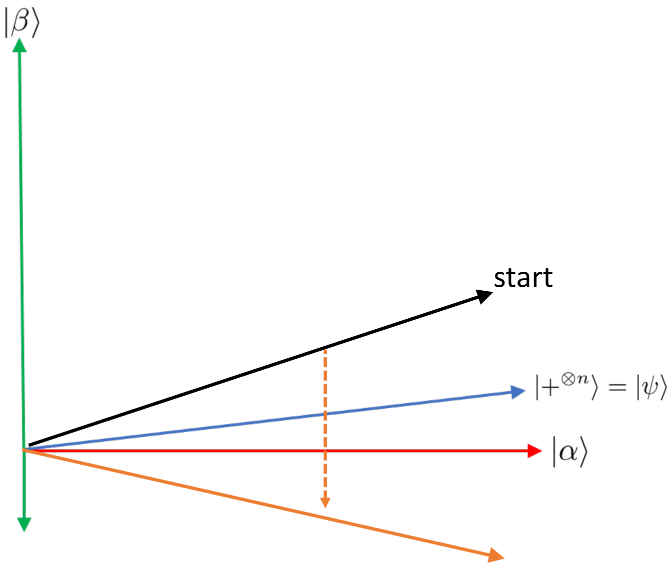

Geometric Interpretation of a Grover Iteration

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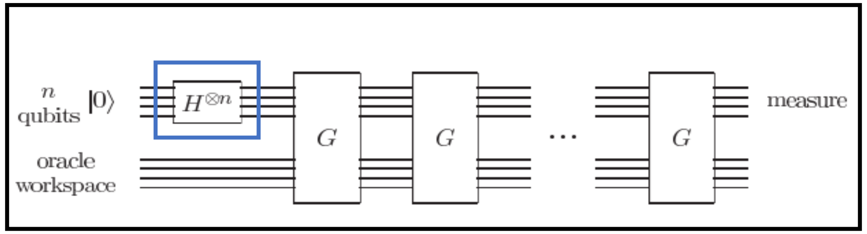

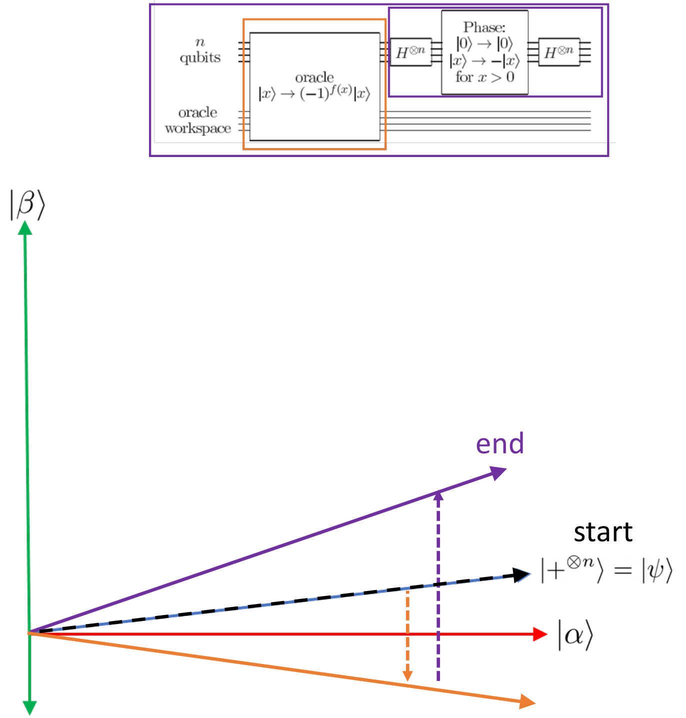

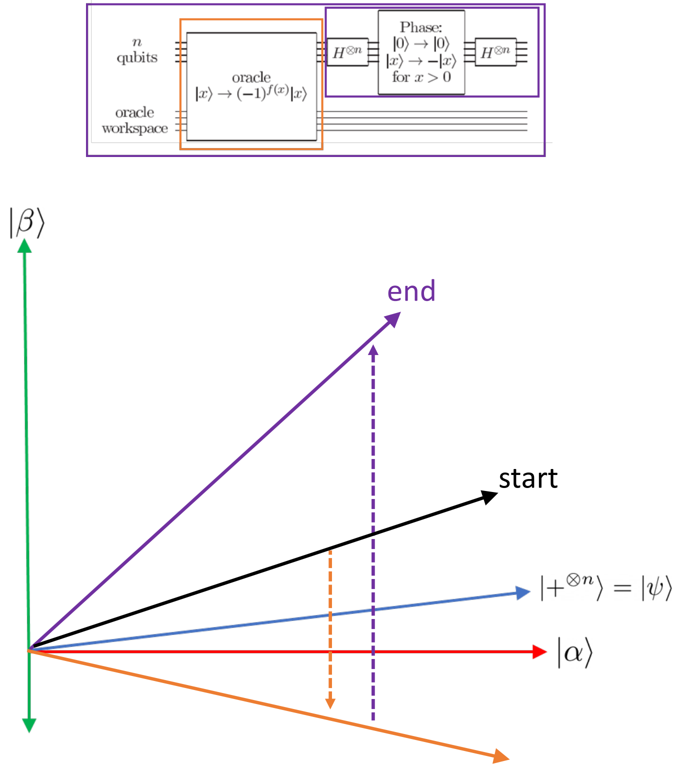

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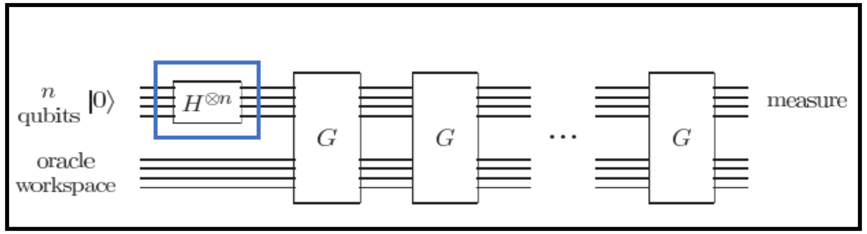

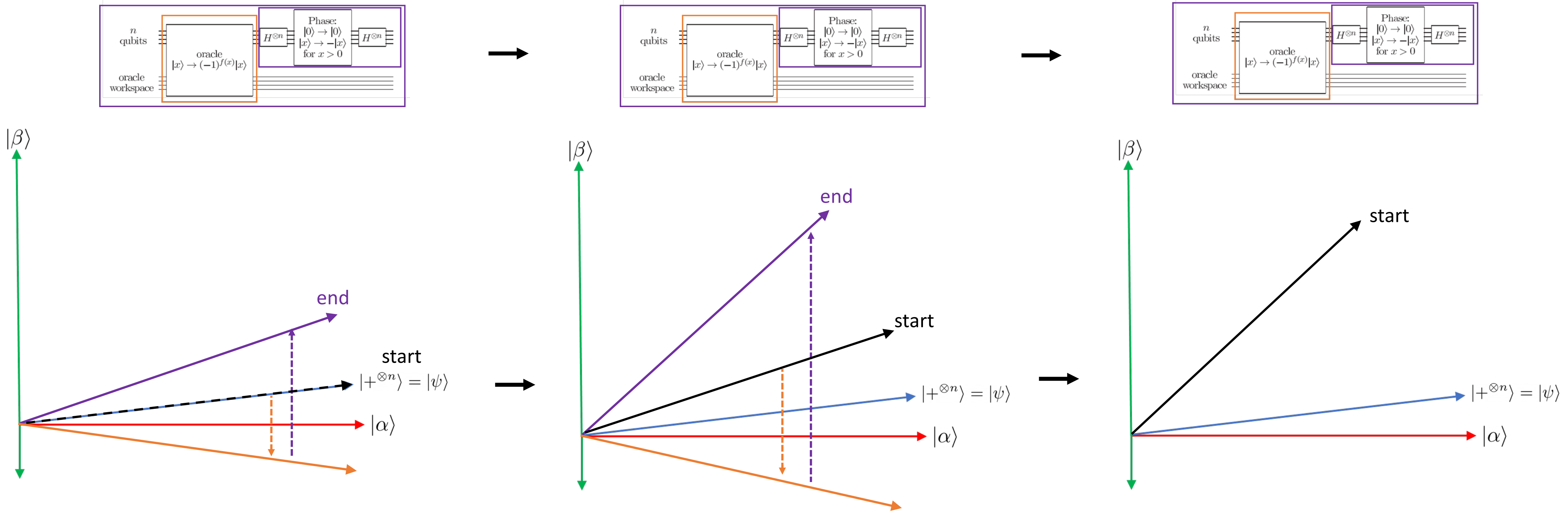

Geometric Interpretation of a Grover Iteration

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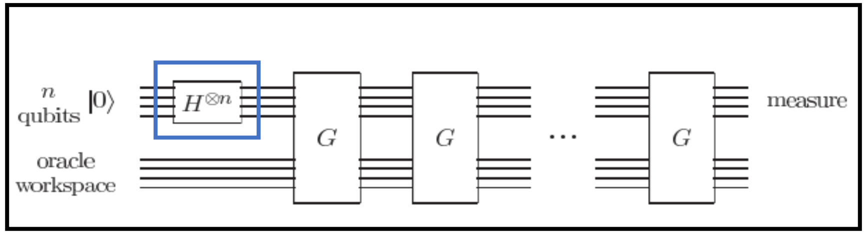

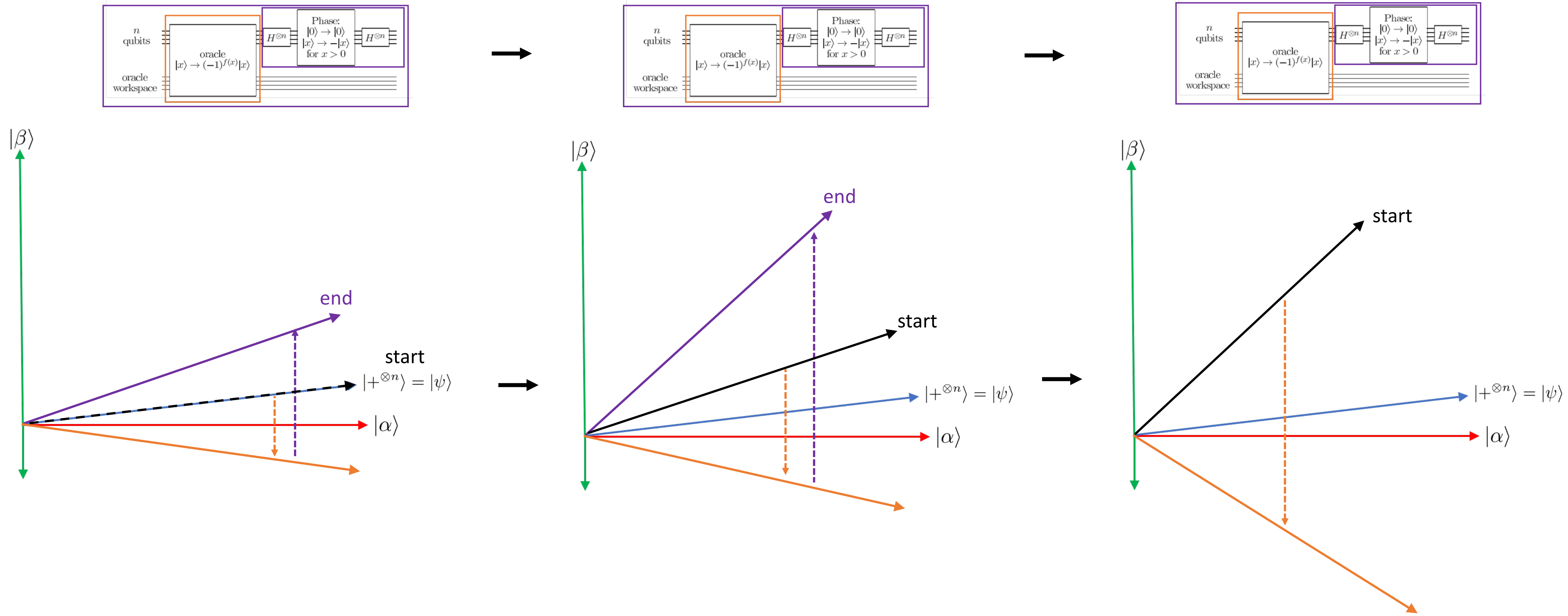

Geometric Interpretation of a Grover Iteration

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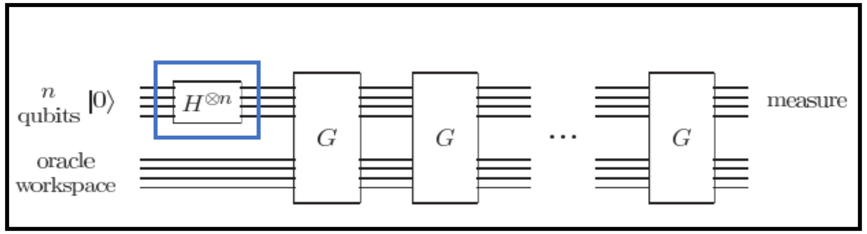

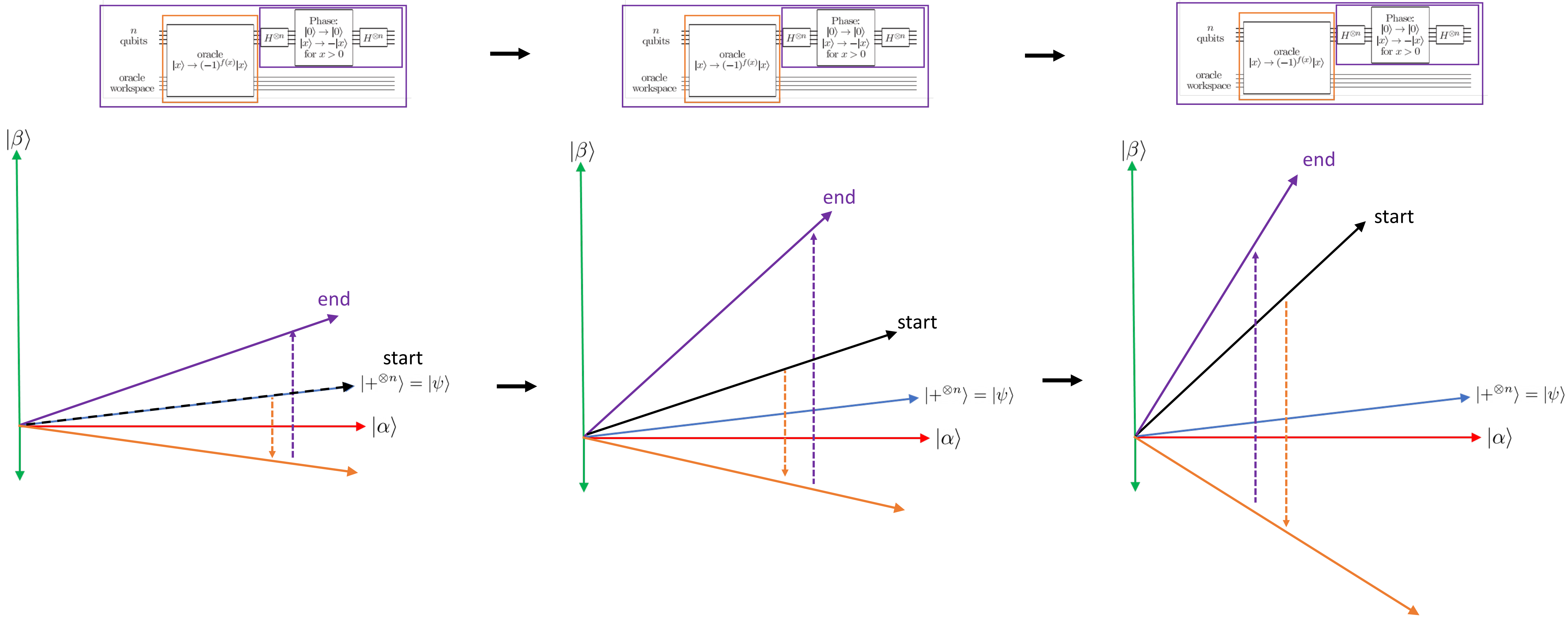

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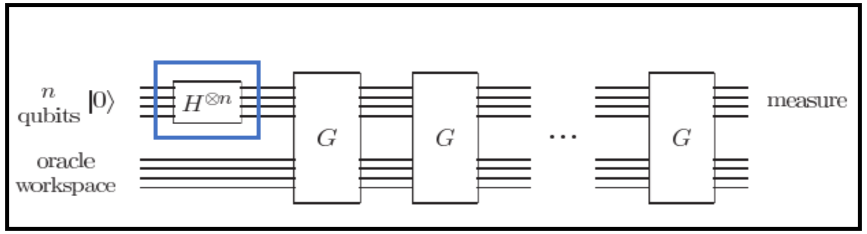
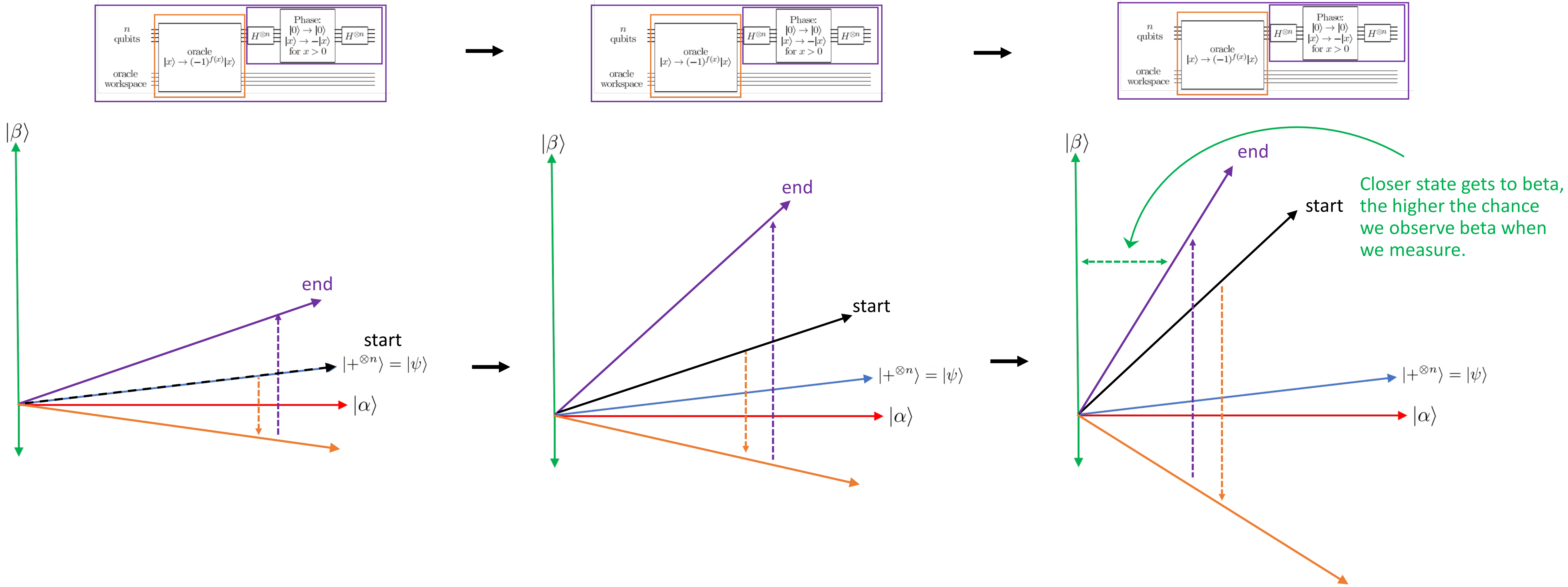
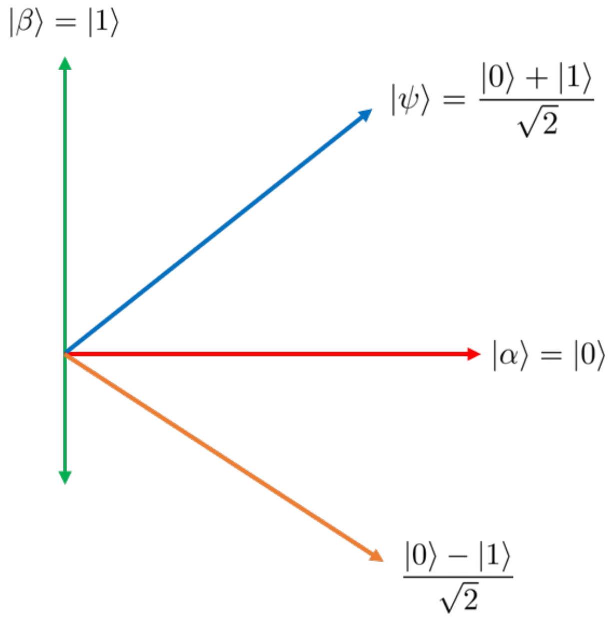




Geometric Interpretation of Diffusion Operator

one qubit, target = $|1\rangle$



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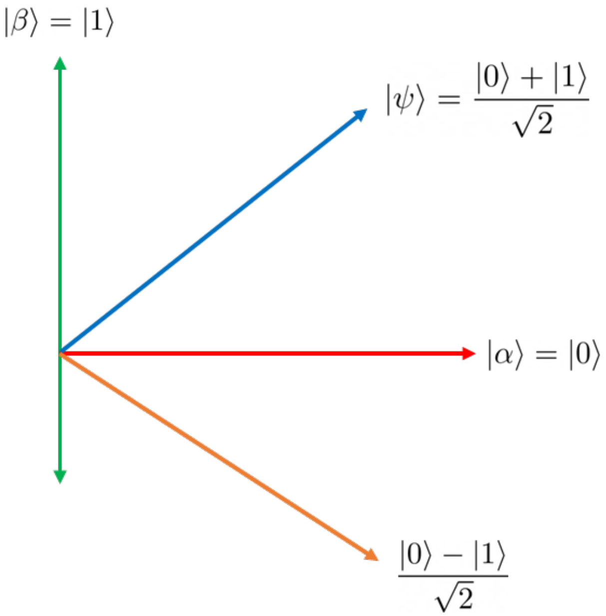

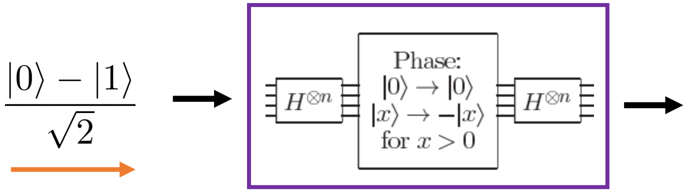

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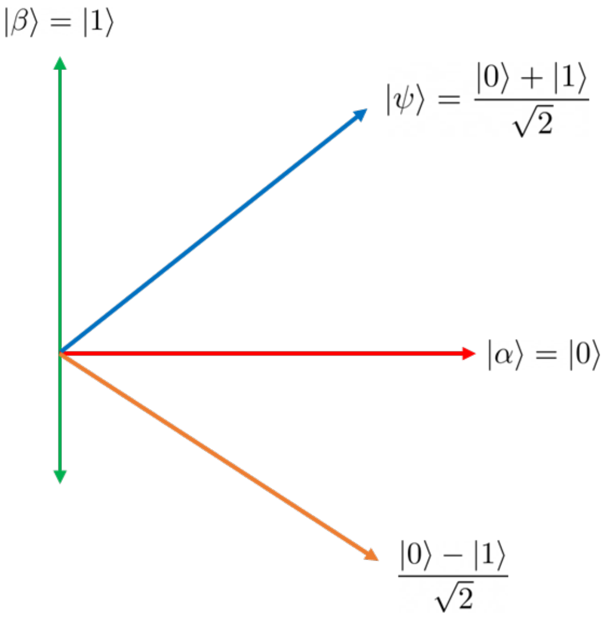
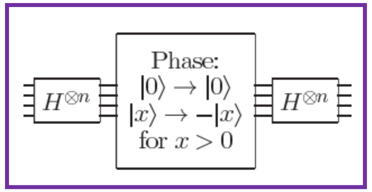


$$\frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow$$



$\rightarrow$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

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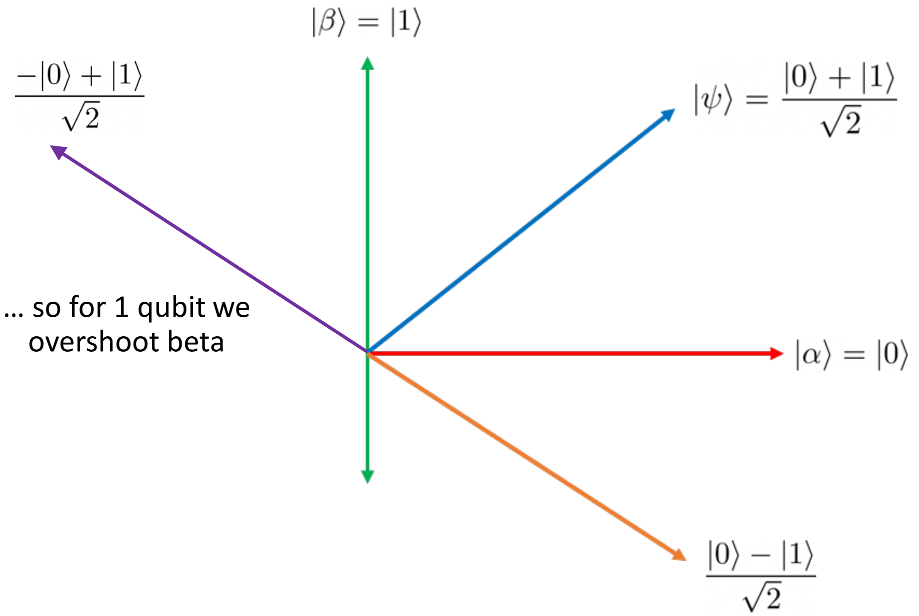
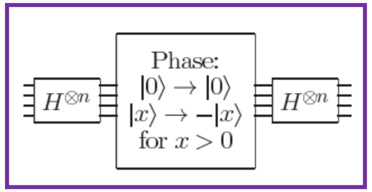


$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ →



→ $\frac{-|0\rangle + |1\rangle}{\sqrt{2}}$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Geometric Interpretation of Diffusion Operator

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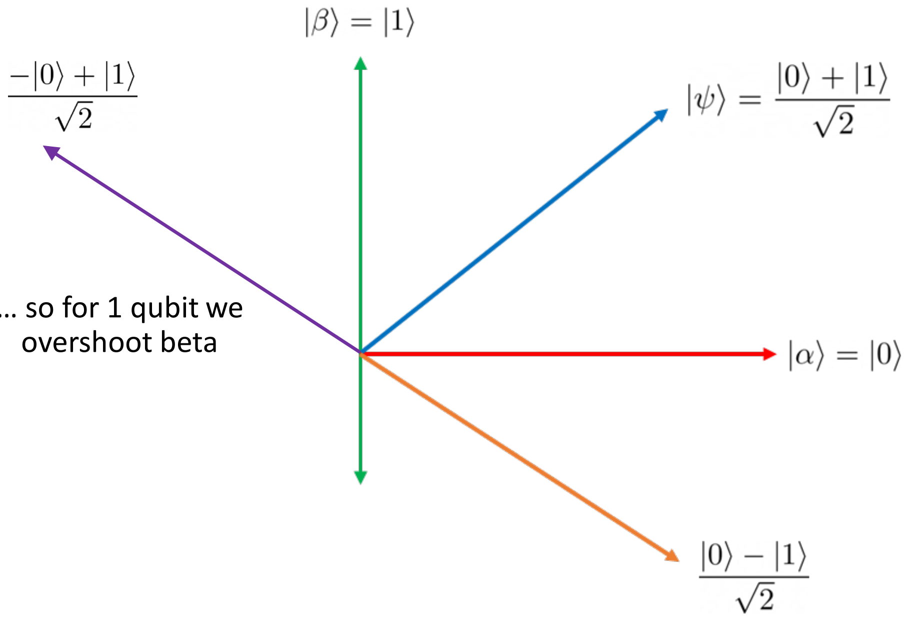
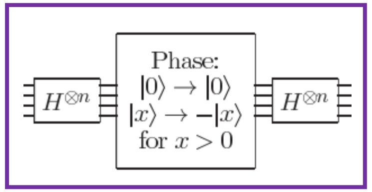


... so for 1 qubit we overshoot beta

$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ →



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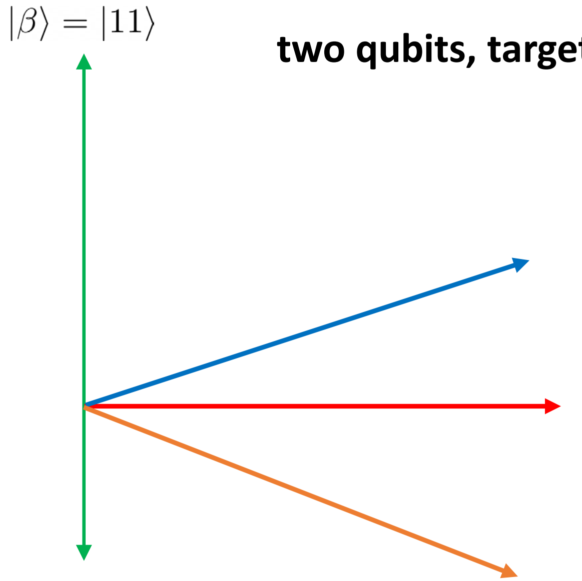
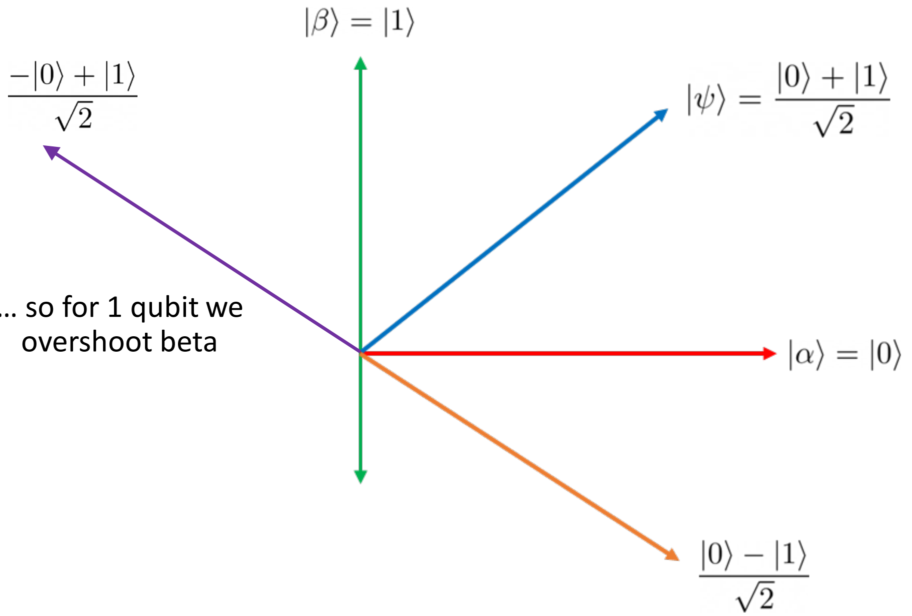
two qubits, target = $|11\rangle$

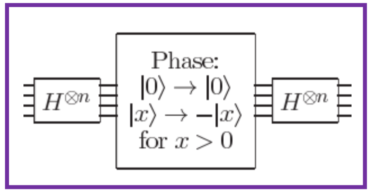


Geometric Interpretation of Diffusion Operator

one qubit, target = $|1\rangle$



$\frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow$



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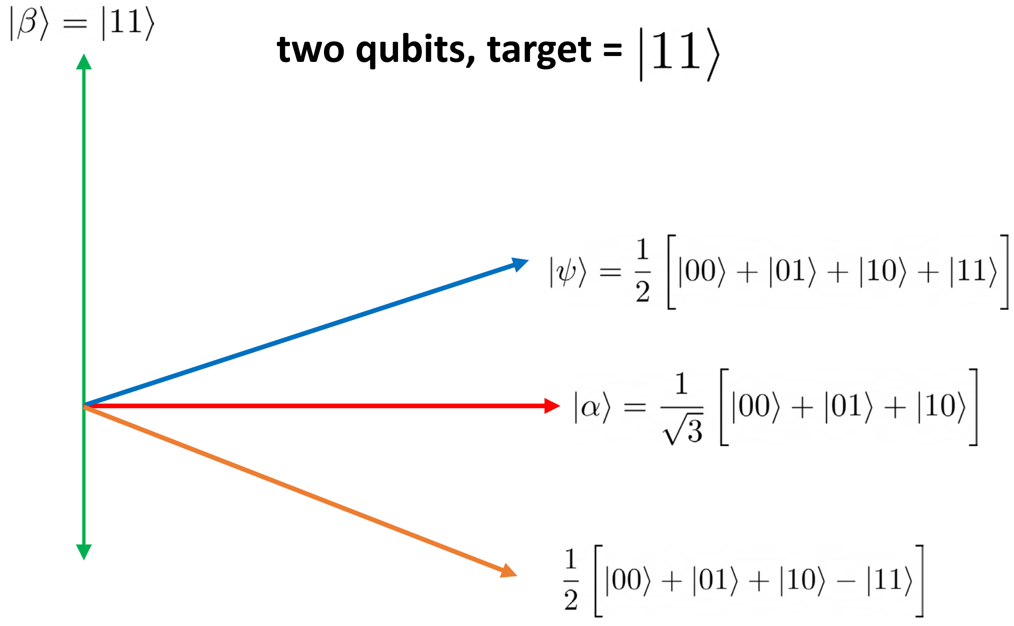
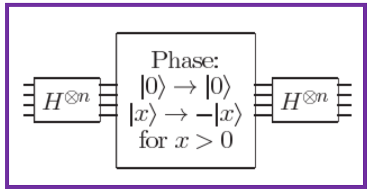


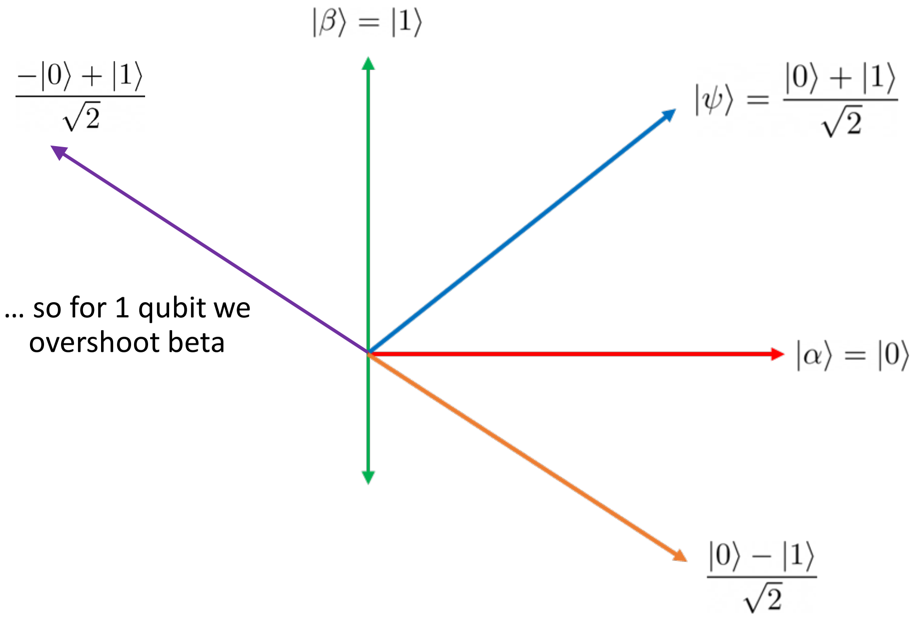
$\frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle - |11\rangle\right] \rightarrow$



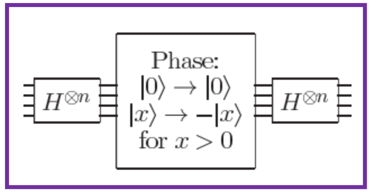
$\rightarrow$

Geometric Interpretation of Diffusion Operator

one qubit, target = $|1\rangle$



$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ →



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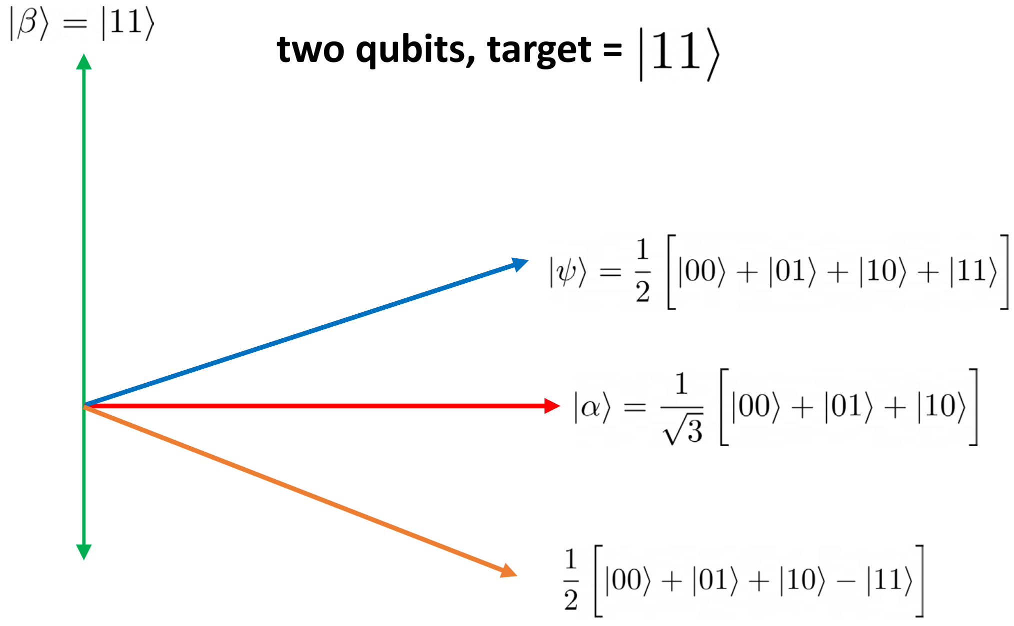
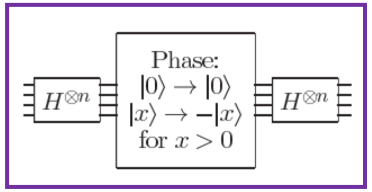


$\frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle - |11\rangle\right]$ →

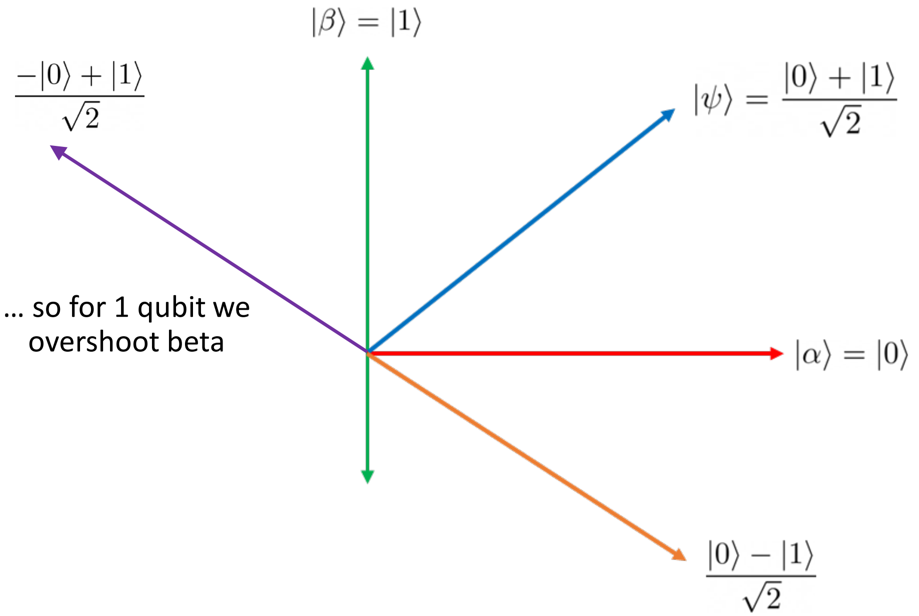


→

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Geometric Interpretation of Diffusion Operator

one qubit, target = $|1\rangle$



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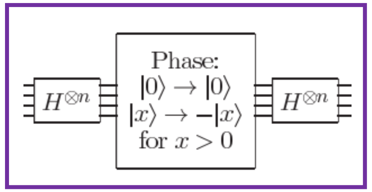

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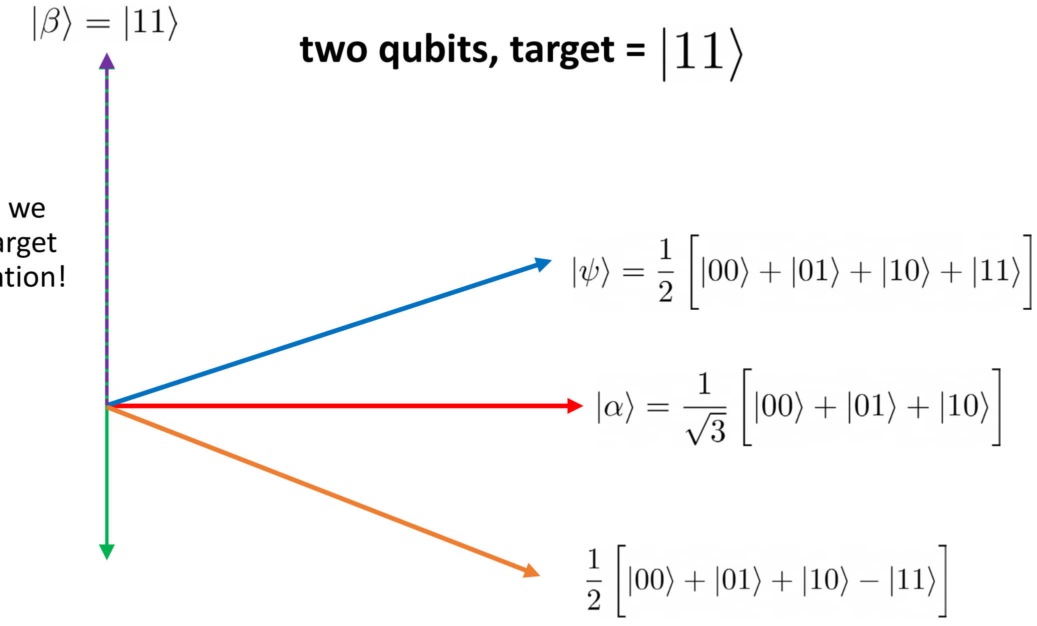
two qubits, target = $|11\rangle$

For 2 qubits, we always find target after one iteration!



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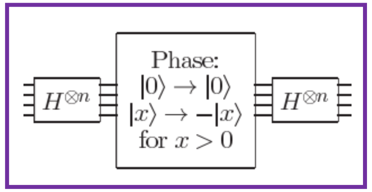
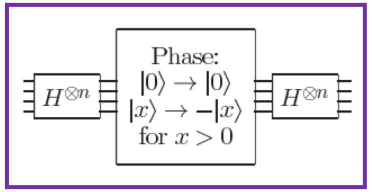


→ $|11\rangle$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Geometric Interpretation of Diffusion Operator

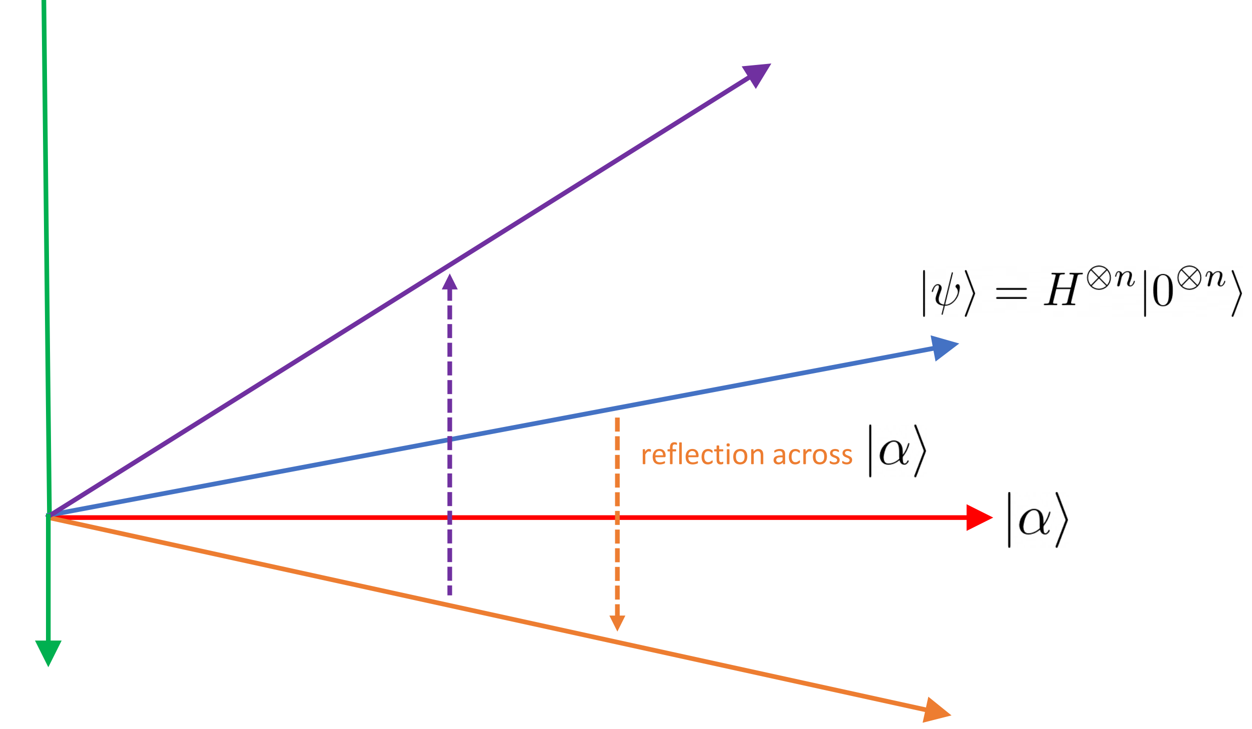
In General



$$= H^{\otimes n}\left(2|0^{\otimes n}\rangle\langle 0^{\otimes n}| - I\right) H^{\otimes n}$$

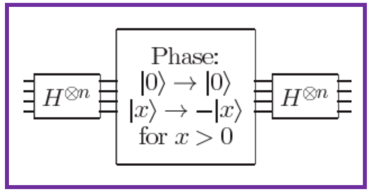
$$= \left(2|\psi\rangle\langle\psi| - I\right)$$

$$=$$



Geometric Interpretation of Diffusion Operator

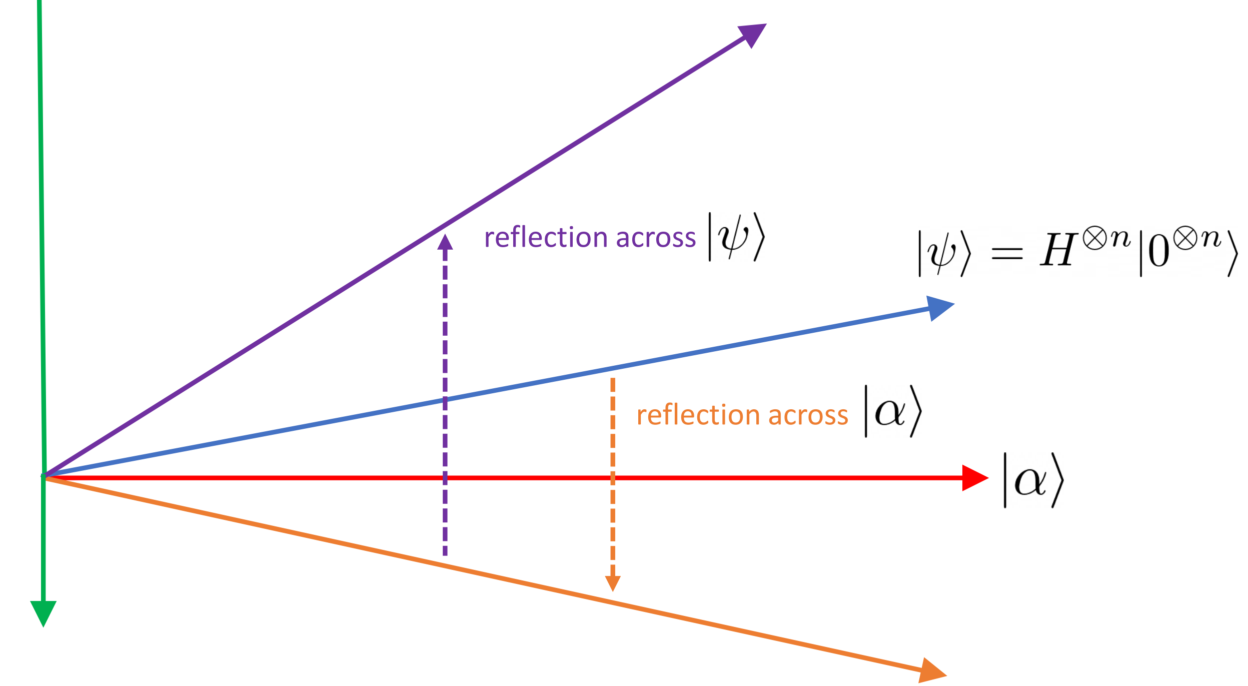
In General



$$= H^{\otimes n}\left(2|0^{\otimes n}\rangle\langle 0^{\otimes n}| - I\right) H^{\otimes n}$$

$$= \left(2|\psi\rangle\langle\psi| - I\right)$$

$=$ reflection across $|\psi\rangle$



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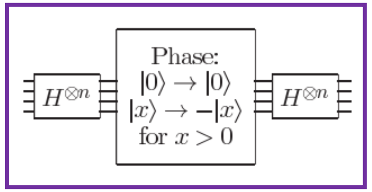
In General



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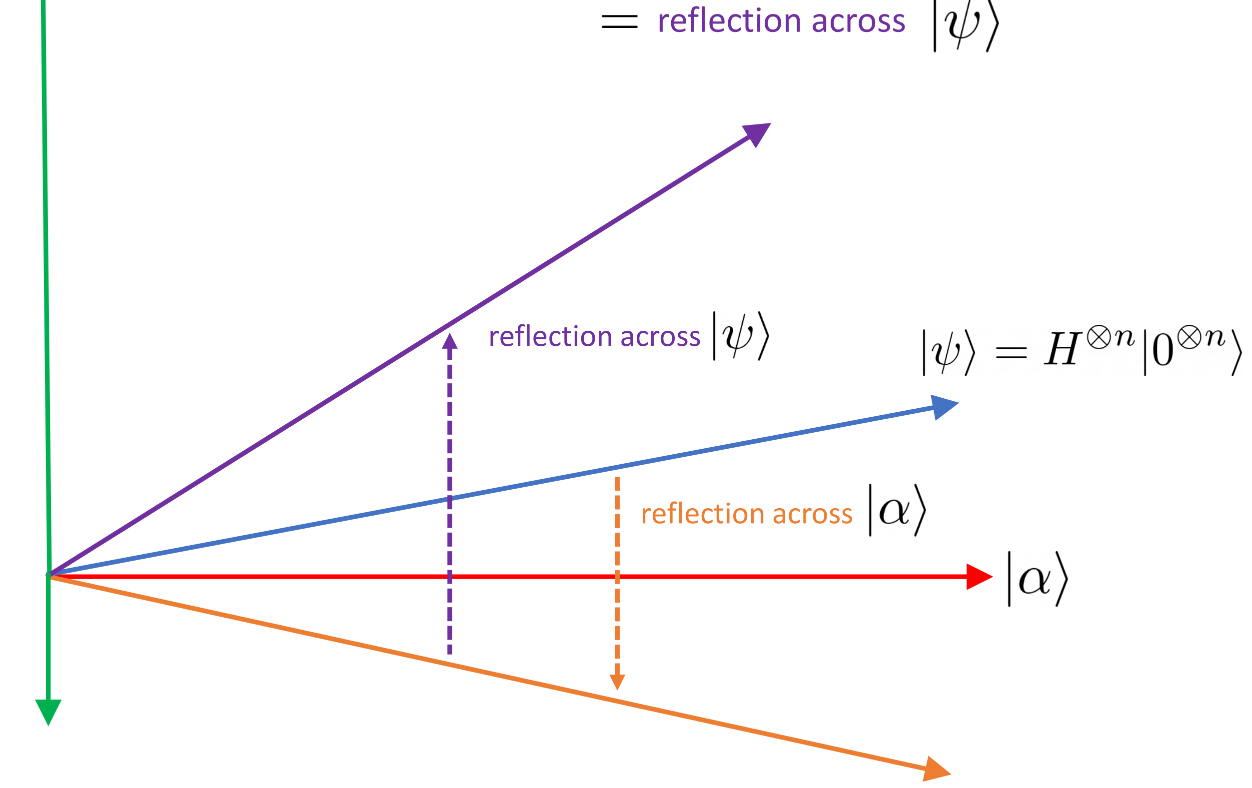
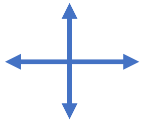
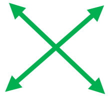
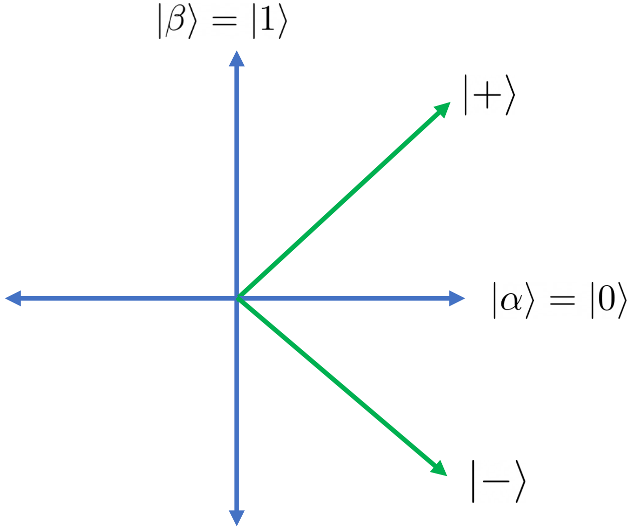


Easiest way to see why: Change of Basis

Recall: one qubit picture (using color scheme from CHSH game)

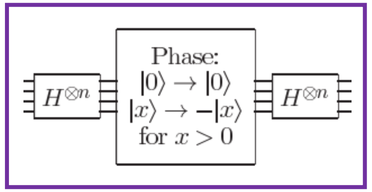
 : $|0\rangle$ vs $|1\rangle$ basis  : $|+\rangle$ vs $|-\rangle$ basis



$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Geometric Interpretation of Diffusion Operator

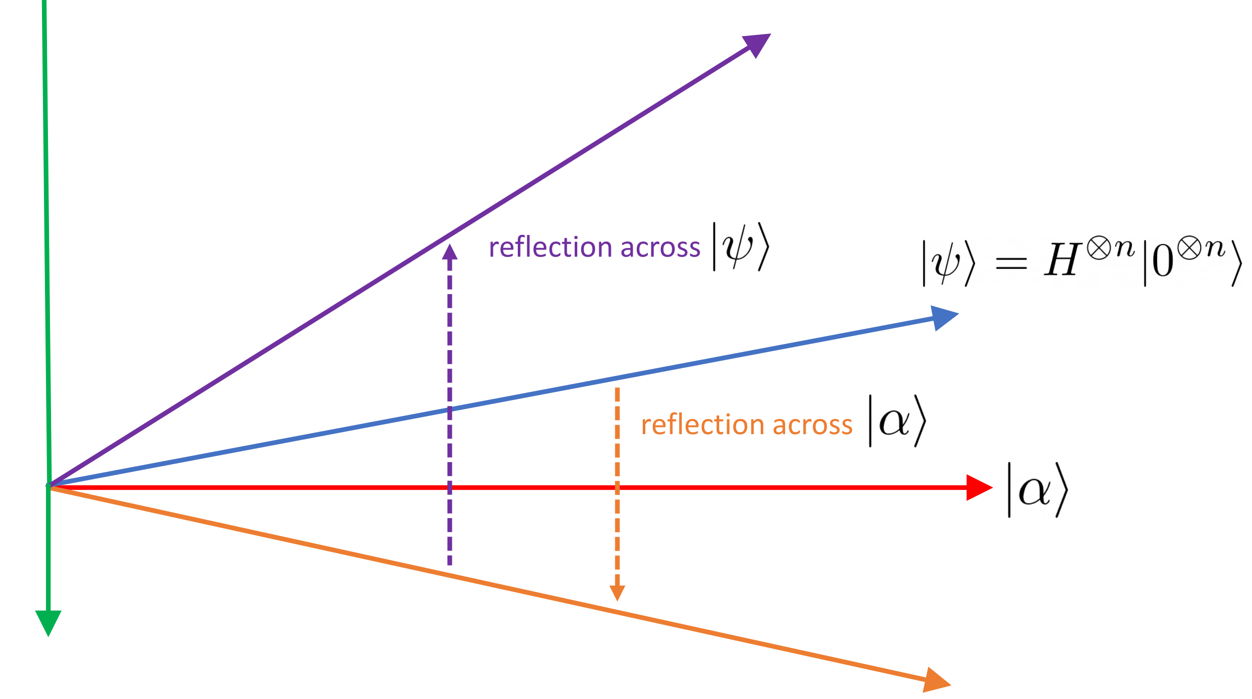
In General



$$= H^{\otimes n}\,(2|0^{\otimes n}\rangle\langle 0^{\otimes n}| - I)\,H^{\otimes n}$$

$$= (2|\psi\rangle\langle\psi| - I)$$

= reflection across $|\psi\rangle$



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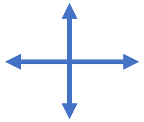
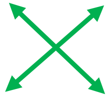
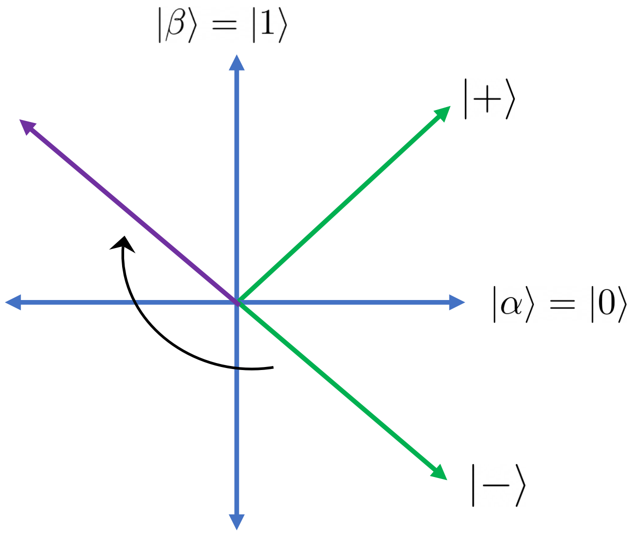
 : $|0\rangle$ vs $|1\rangle$ basis

 : $|+\rangle$ vs $|-\rangle$ basis



change back	"reflect over x-axis"	change to H basis
$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Number of Iterations: Finding Rotation Angle

$|\beta\rangle = |1^{\otimes n}\rangle$

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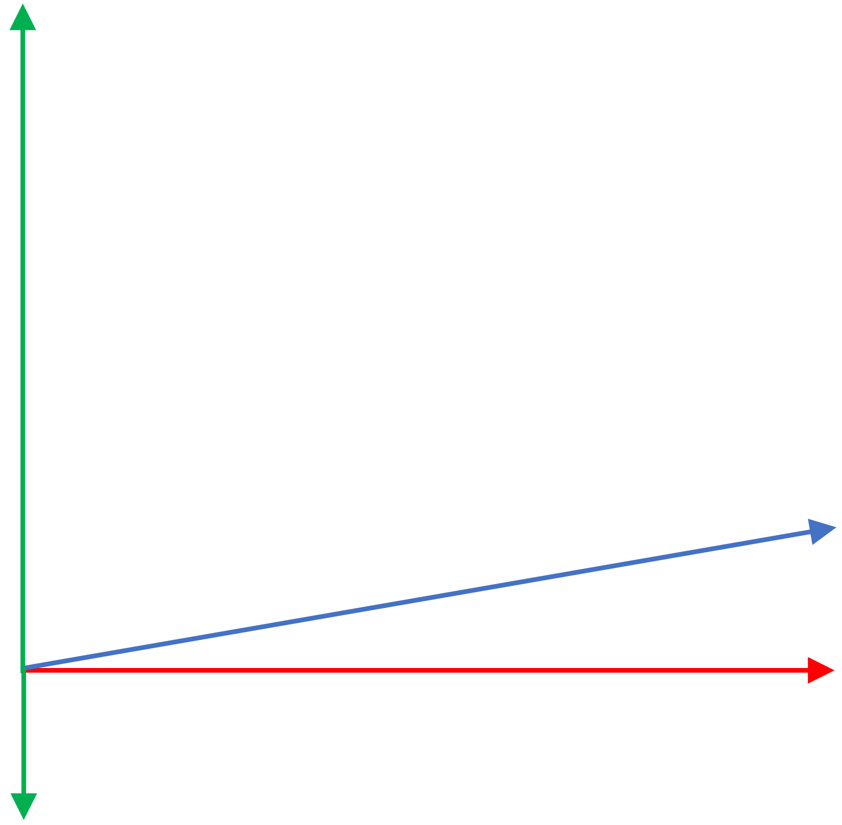

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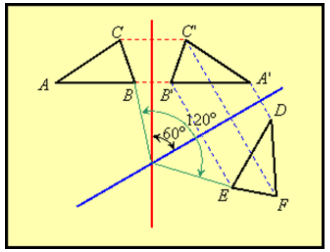
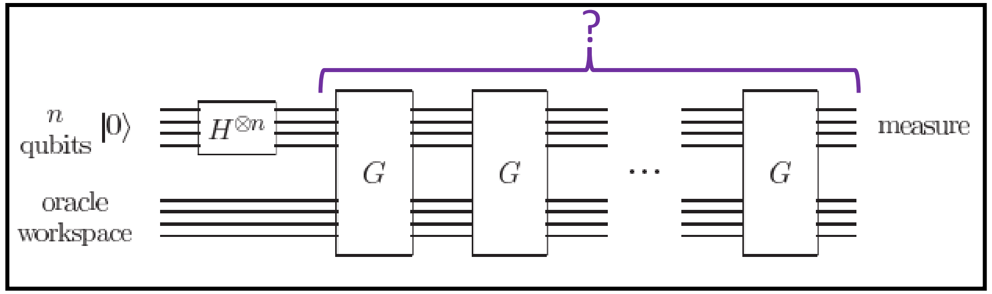


(figure courtesy of ceemrr.com)

Thus: number Grover iterations ~ angle of rotation



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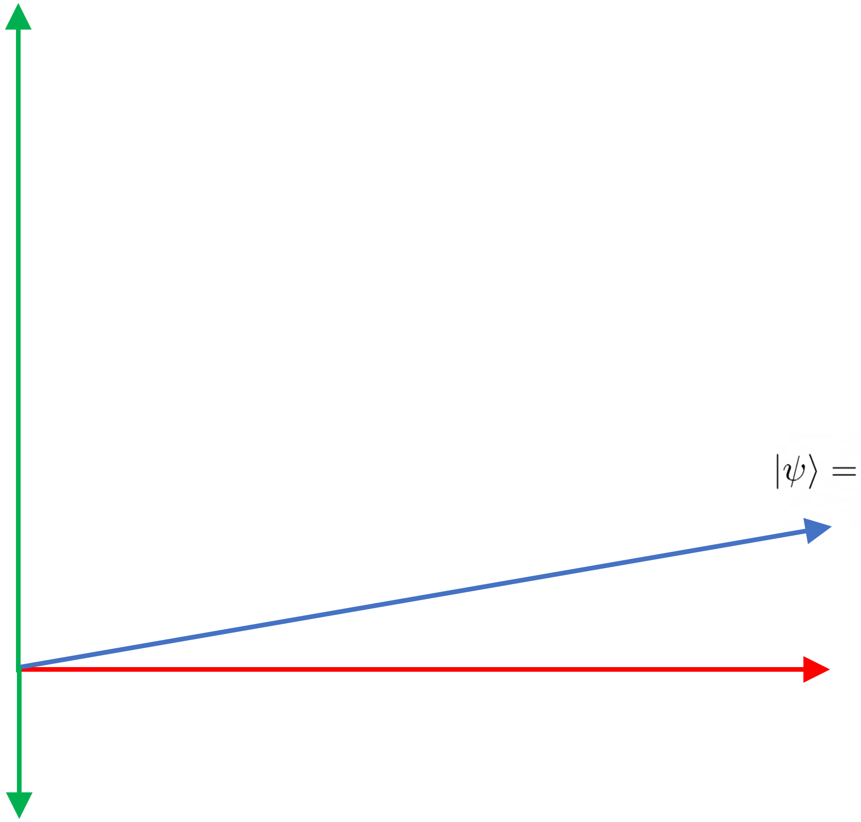

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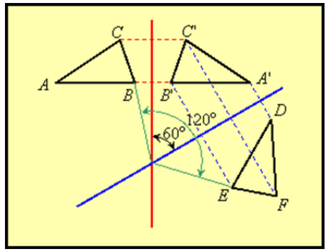

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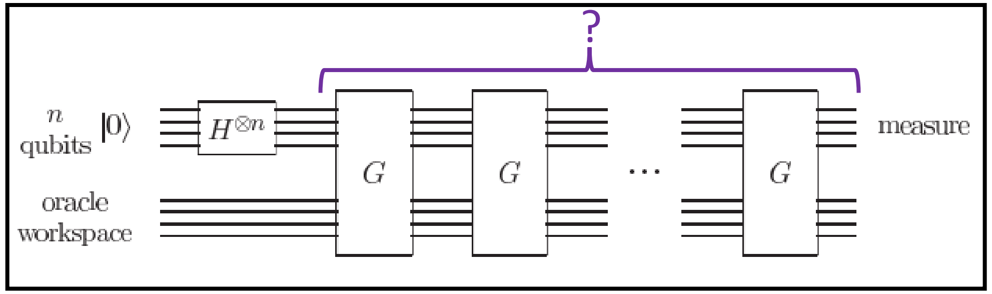

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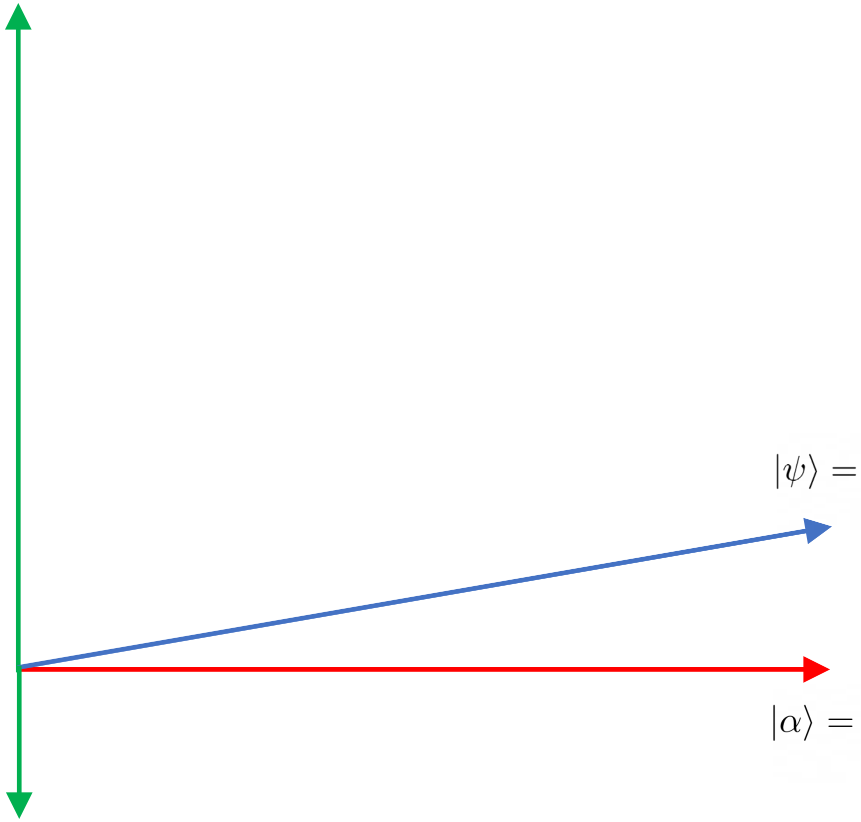

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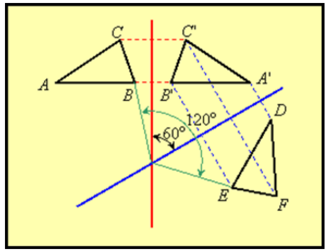

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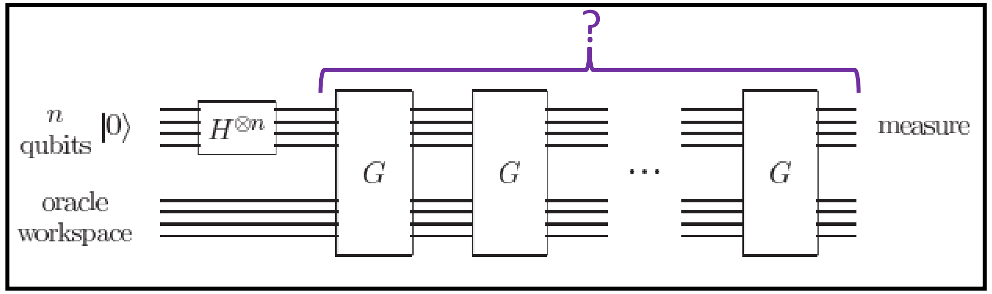

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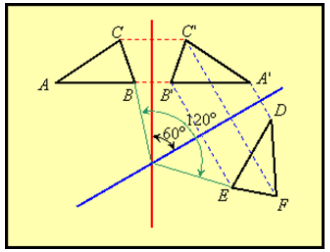

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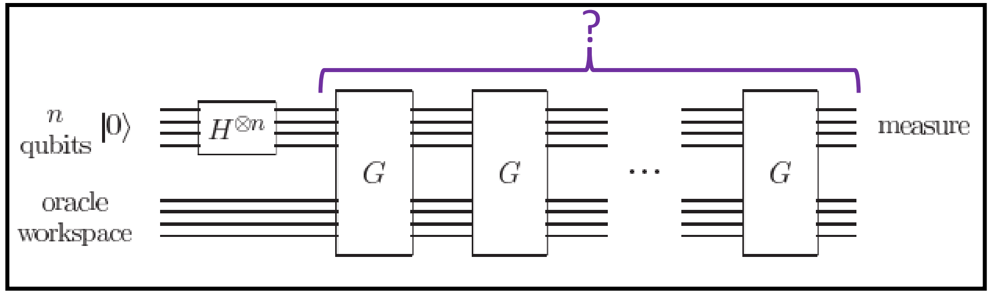

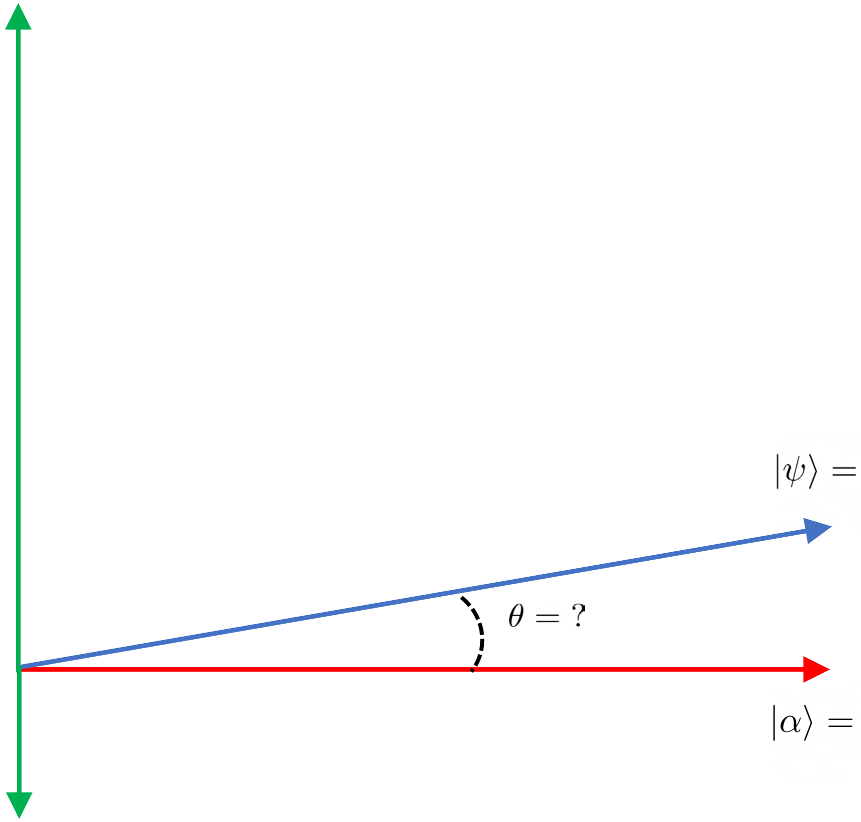

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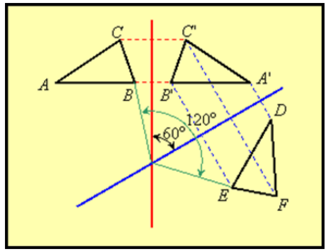

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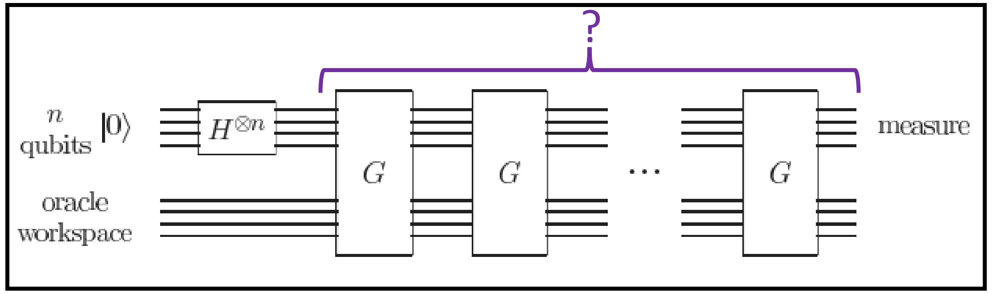

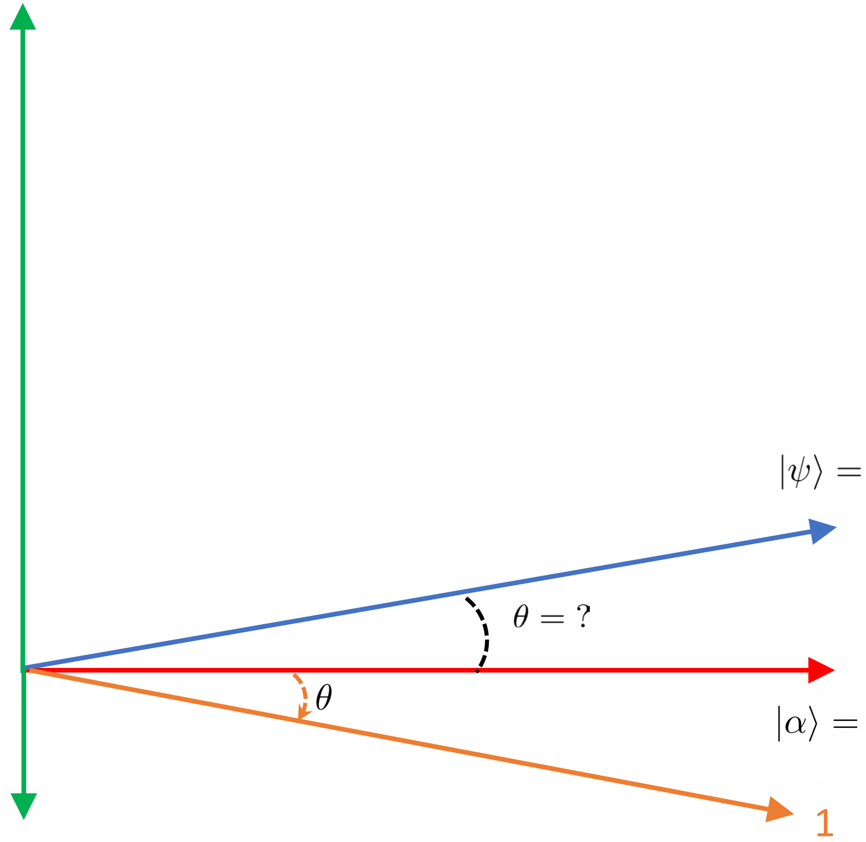

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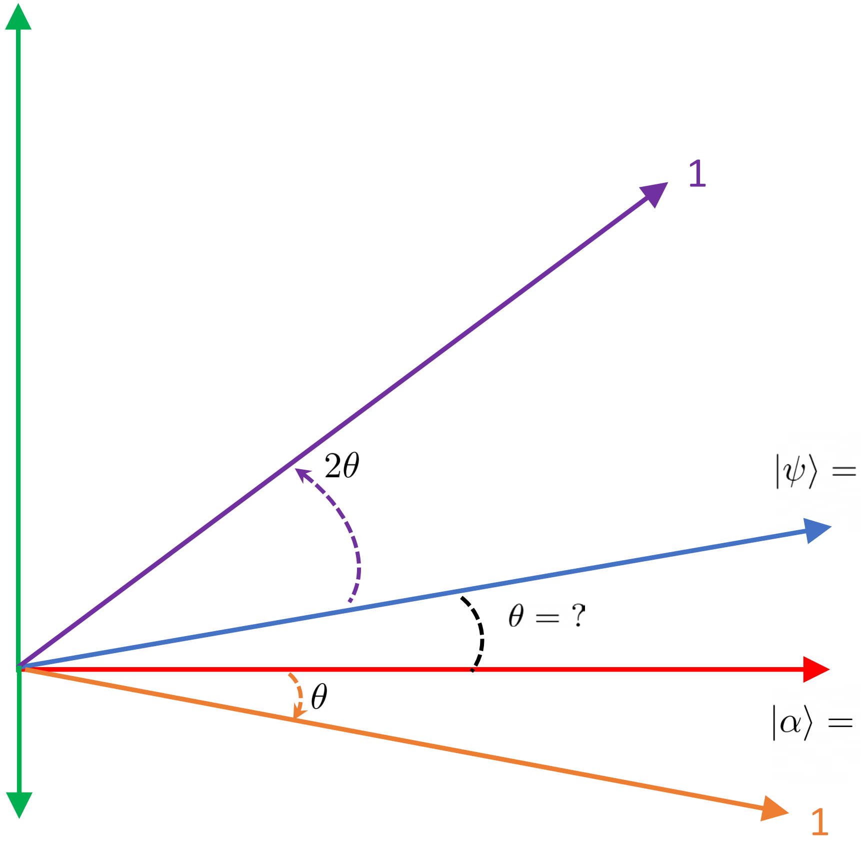

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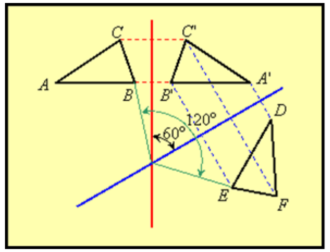

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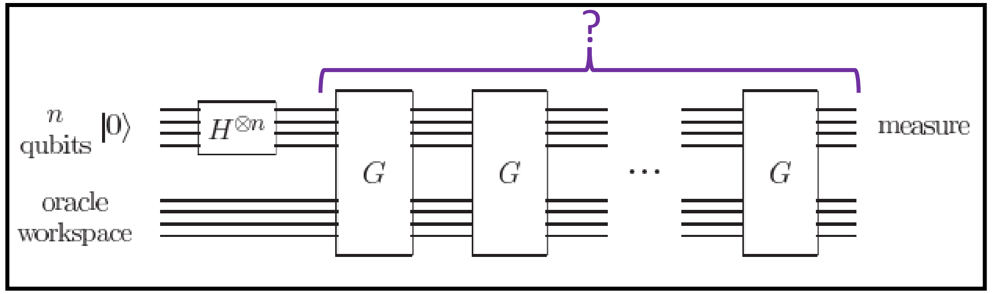

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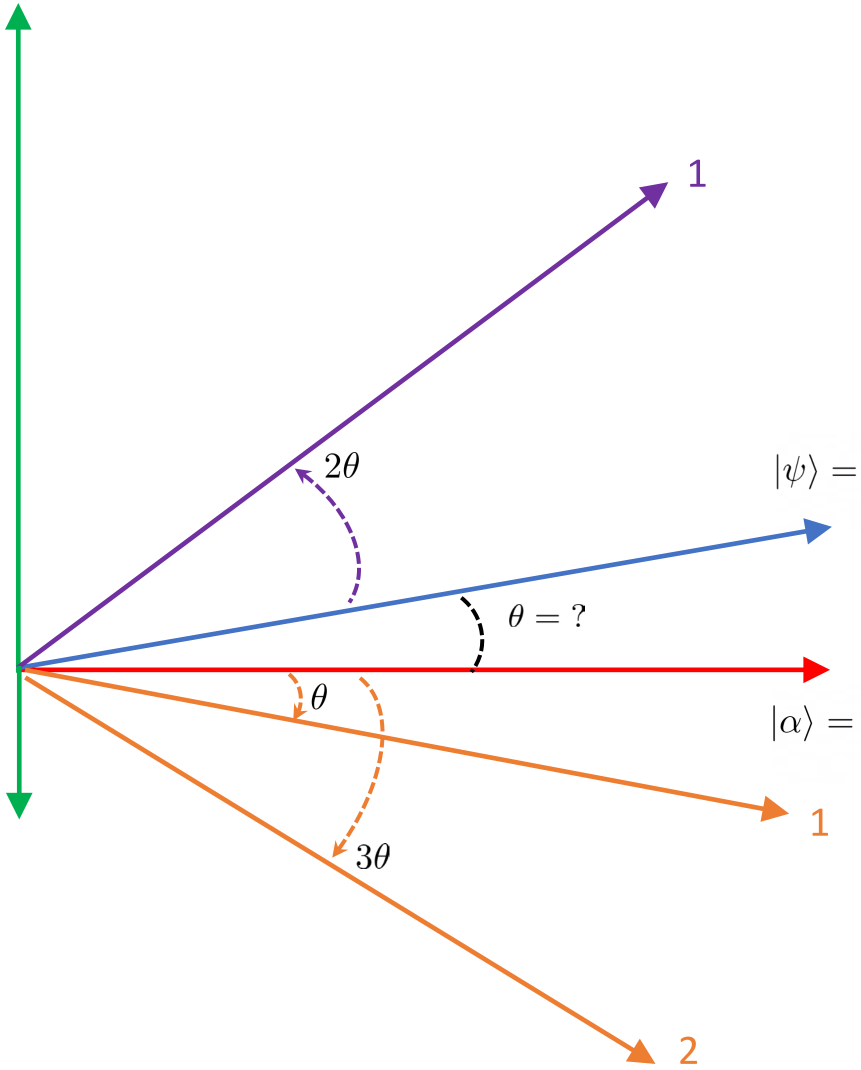

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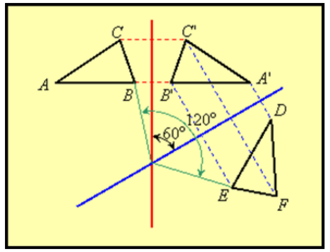

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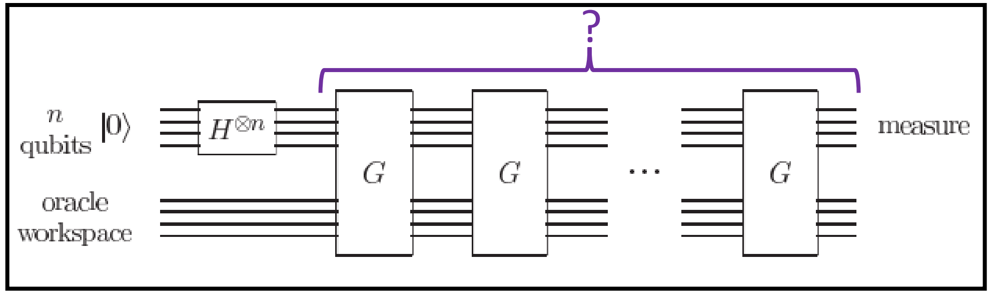

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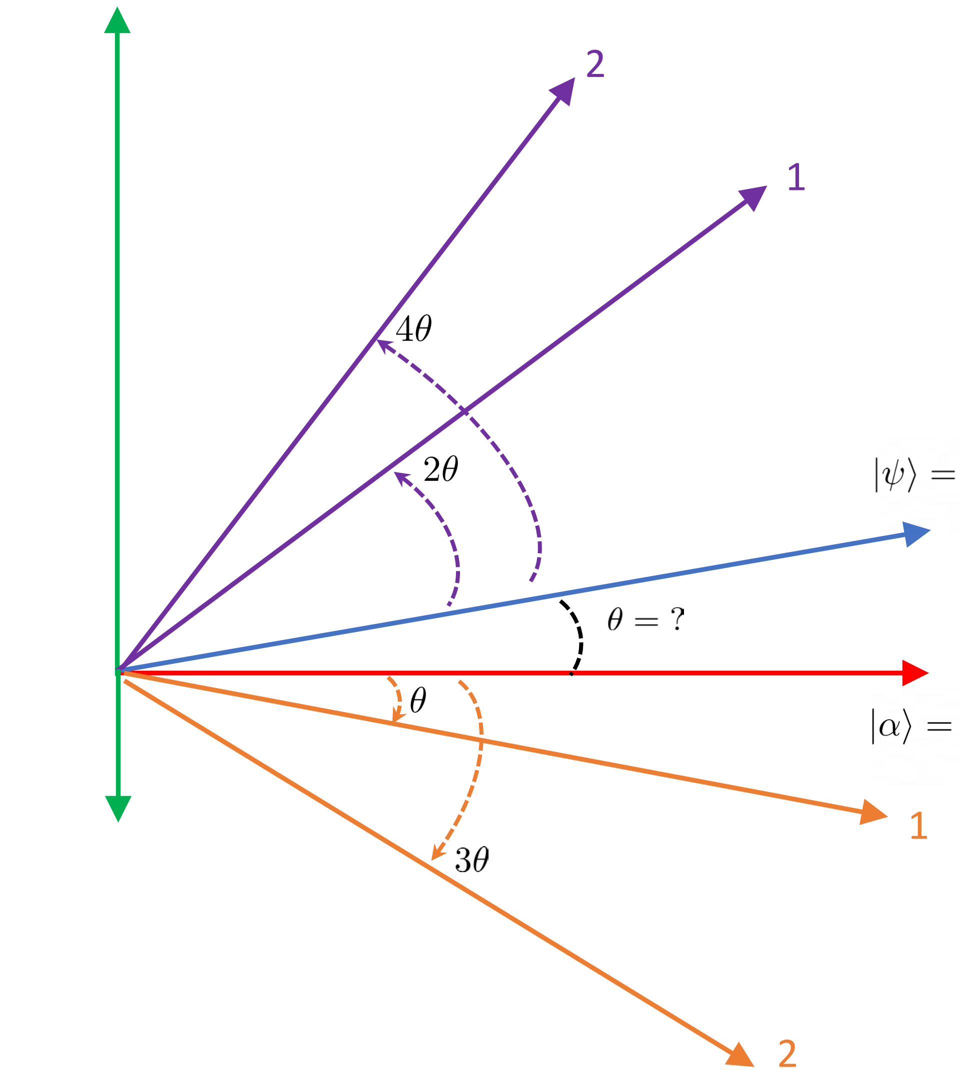

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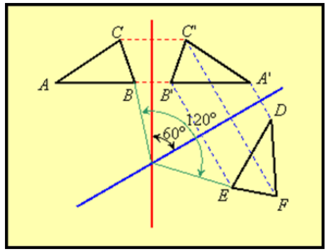

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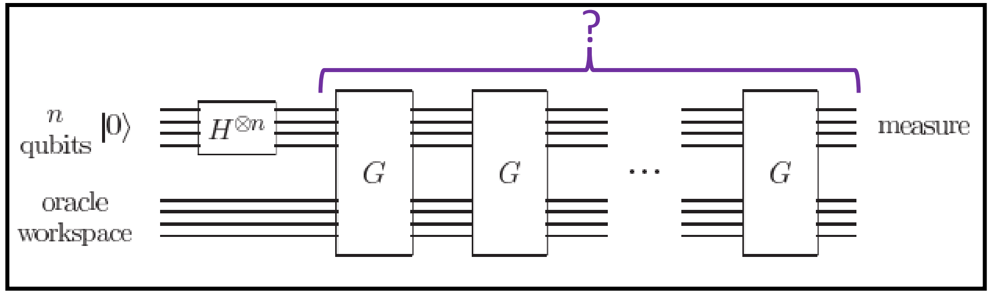

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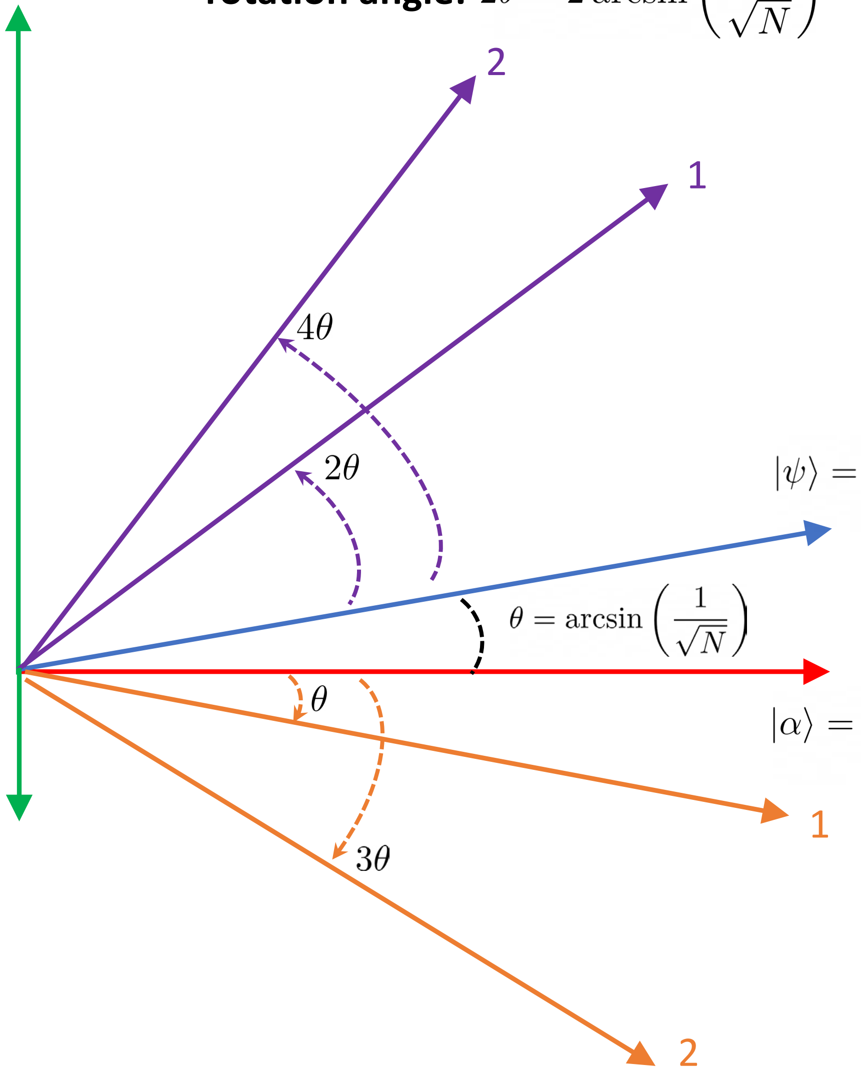

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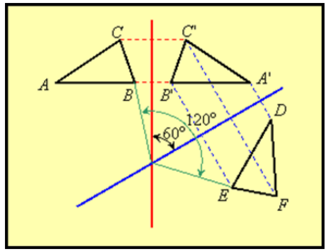

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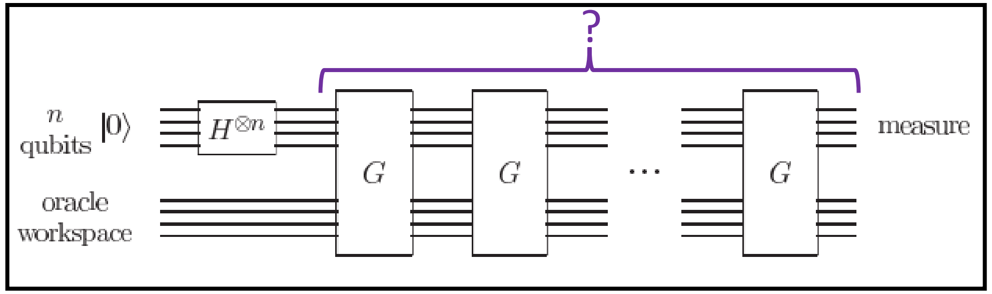

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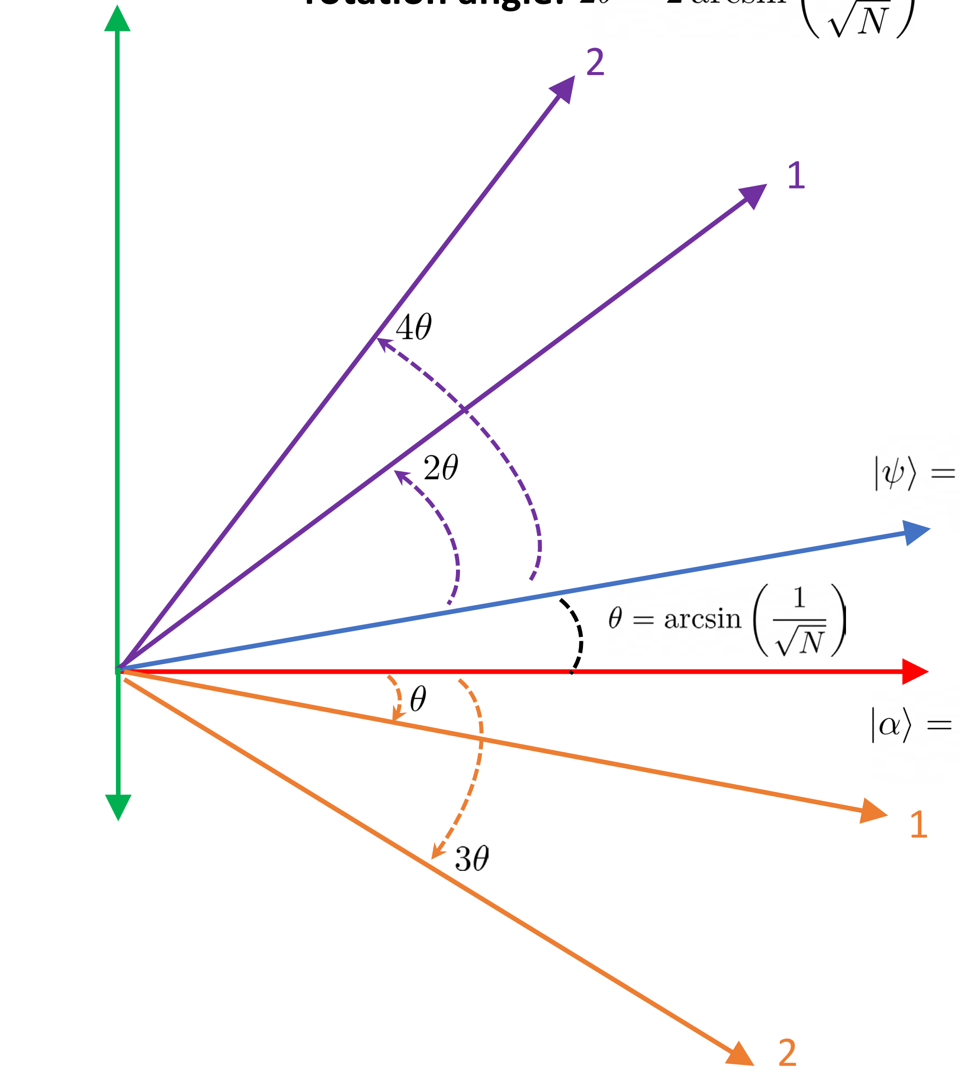

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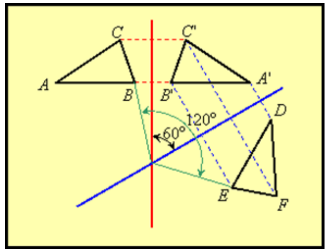

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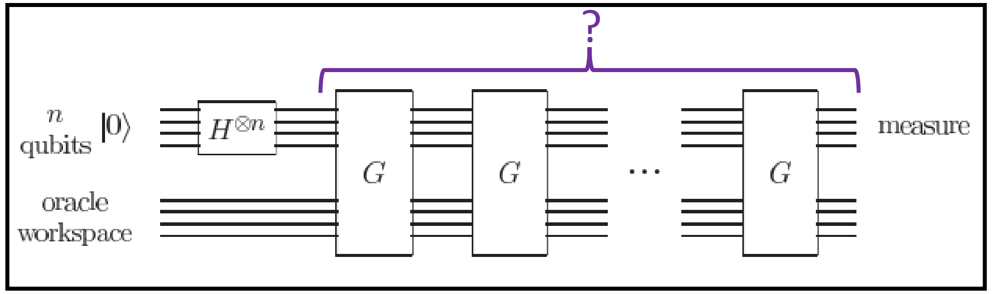

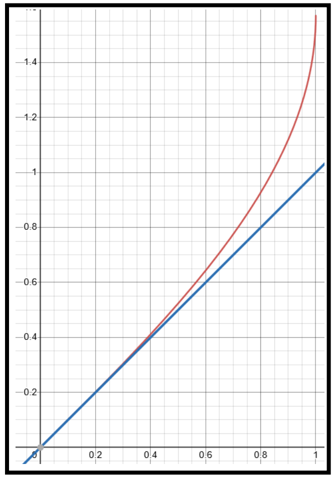

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and

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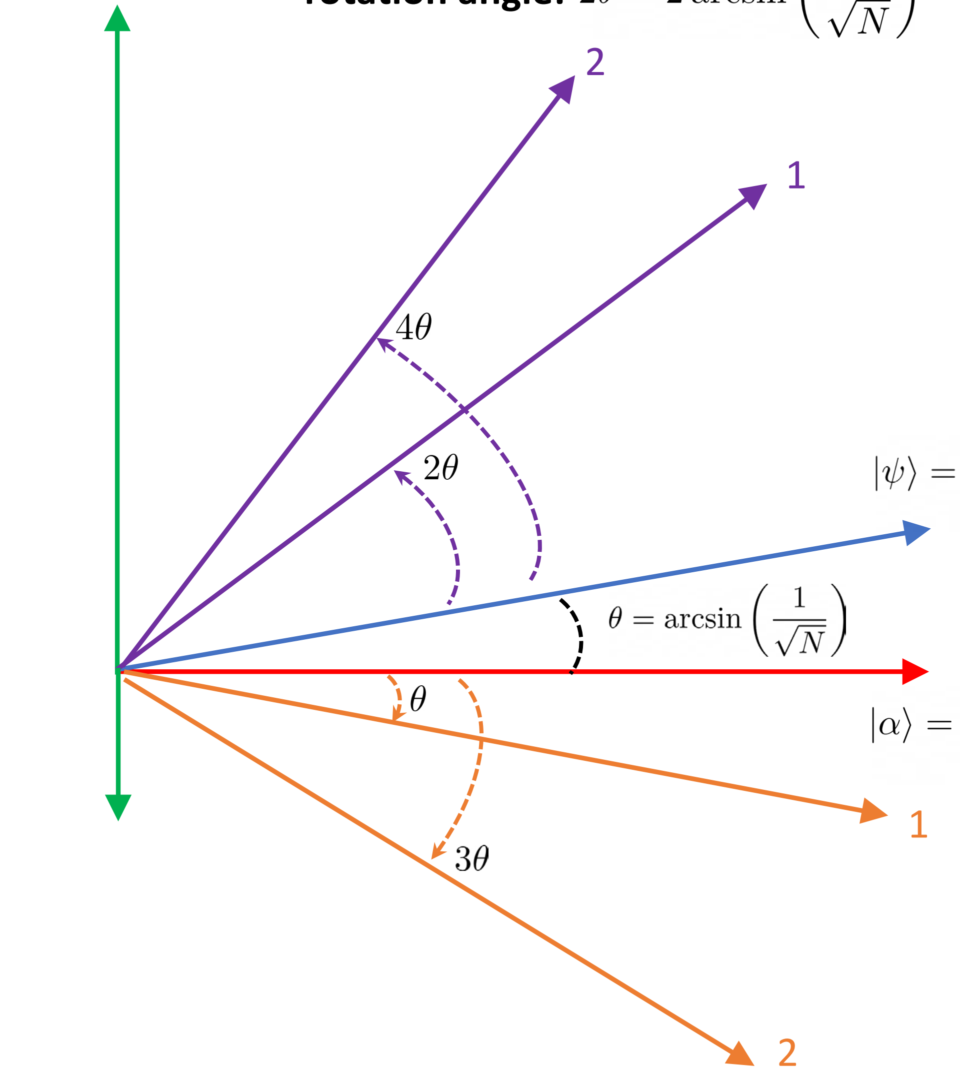

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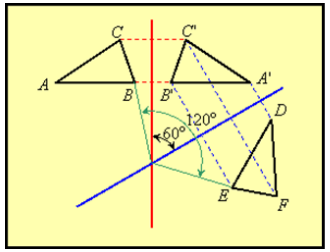

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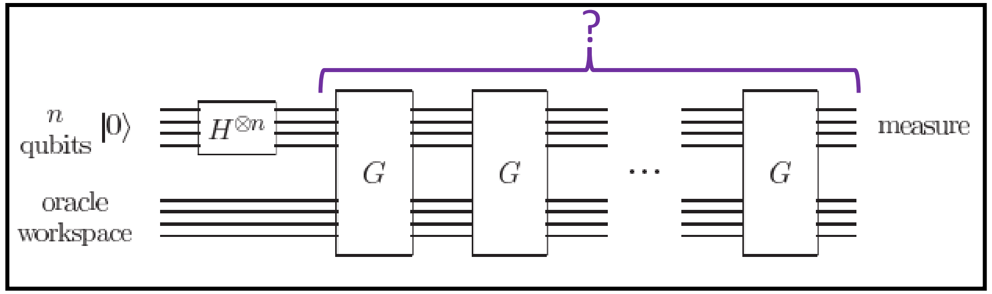

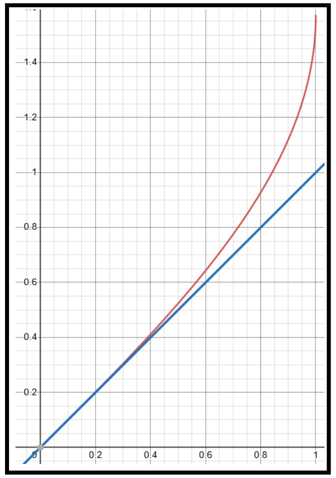

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$$\leq \frac{\frac{\pi}{2}}{2\theta} \leq \frac{\frac{\pi}{2}}{2\arcsin\left(\frac{1}{\sqrt{N}}\right)} \leq \frac{\frac{\pi}{2}}{\frac{2}{\sqrt{N}}} = \frac{\pi}{4}\sqrt{N} = O\left(\sqrt{N}\right)$$