Lecture 37-39: Quantum Fourier Transform

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Quantum Fourier Transform: Overview

Classical Fourier Transform (8-dimensions)

For a given vector $\vec{a}$ compute:

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^{2\cdot 2} & \omega^{2\cdot 3} & \omega^{2\cdot 4} & \omega^{2\cdot 5} & \omega^{2\cdot 6} & \omega^{2\cdot 7} \\ 1 & \omega^3 & \omega^{3\cdot 2} & \omega^{3\cdot 3} & \omega^{3\cdot 4} & \omega^{3\cdot 5} & \omega^{3\cdot 6} & \omega^{3\cdot 7} \\ 1 & \omega^4 & \omega^{4\cdot 2} & \omega^{4\cdot 3} & \omega^{4\cdot 4} & \omega^{4\cdot 5} & \omega^{4\cdot 6} & \omega^{4\cdot 7} \\ 1 & \omega^5 & \omega^{5\cdot 2} & \omega^{5\cdot 3} & \omega^{5\cdot 4} & \omega^{5\cdot 5} & \omega^{5\cdot 6} & \omega^{5\cdot 7} \\ 1 & \omega^6 & \omega^{6\cdot 2} & \omega^{6\cdot 3} & \omega^{6\cdot 4} & \omega^{6\cdot 5} & \omega^{6\cdot 6} & \omega^{6\cdot 7} \\ 1 & \omega^7 & \omega^{7\cdot 2} & \omega^{7\cdot 3} & \omega^{7\cdot 4} & \omega^{7\cdot 5} & \omega^{7\cdot 6} & \omega^{7\cdot 7} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

$$\left(\text{where } \omega = e^{\frac{2\pi i}{8}} \text{ is } n\text{th primative root of unity}\right)$$

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whereas run time of classical FFT: $O(N\log N) \rightarrow O(2^n \cdot n)$

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Classical Fourier Transform (8-dimensions)

For a given vector $\vec{a}$ compute:

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^{2\cdot2} & \omega^{2\cdot3} & \omega^{2\cdot4} & \omega^{2\cdot5} & \omega^{2\cdot6} & \omega^{2\cdot7} \\ 1 & \omega^3 & \omega^{3\cdot2} & \omega^{3\cdot3} & \omega^{3\cdot4} & \omega^{3\cdot5} & \omega^{3\cdot6} & \omega^{3\cdot7} \\ 1 & \omega^4 & \omega^{4\cdot2} & \omega^{4\cdot3} & \omega^{4\cdot4} & \omega^{4\cdot5} & \omega^{4\cdot6} & \omega^{4\cdot7} \\ 1 & \omega^5 & \omega^{5\cdot2} & \omega^{5\cdot3} & \omega^{5\cdot4} & \omega^{5\cdot5} & \omega^{5\cdot6} & \omega^{5\cdot7} \\ 1 & \omega^6 & \omega^{6\cdot2} & \omega^{6\cdot3} & \omega^{6\cdot4} & \omega^{6\cdot5} & \omega^{6\cdot6} & \omega^{6\cdot7} \\ 1 & \omega^7 & \omega^{7\cdot2} & \omega^{7\cdot3} & \omega^{7\cdot4} & \omega^{7\cdot5} & \omega^{7\cdot6} & \omega^{7\cdot7} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

$\left(\text{where } \omega = e^{\frac{2\pi i}{8}} \text{ is } n\text{th primative root of unity}\right)$

In bra-ket notation:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle$$

$$= \sum_{x\in\{0,1\}^3} a_x|x\rangle \xrightarrow{\text{Fourier Transform}} \frac{1}{\sqrt{8}}\sum_{x\in\{0,1\}^3} b_x|x\rangle$$

$$\text{where } b_x = \sum_{y\in\{0,1\}^3} a_y\omega^{xy}|y\rangle$$

Quantum Fourier Transform (3 qubits)

$$\text{Given quantum state: } \sum_{x\in\{0,1\}^3} \alpha_x|x\rangle$$

$$= \alpha_0|000\rangle + \alpha_1|001\rangle + \alpha_2|010\rangle + \alpha_3|011\rangle + \alpha_4|100\rangle + \alpha_5|101\rangle + \alpha_6|110\rangle + \alpha_7|111\rangle$$

$$\Big\downarrow \text{QFT}$$

$$\sum_{x\in\{0,1\}^3} \beta_x|x\rangle \text{ where } \beta_x = \sum_{y\in\{0,1\}^3} \alpha_y\omega^{xy}|y\rangle$$

So pretty much the same, except…

Upside

FT matrix is unitary and can be implemented using $O(\log^2 N)$ quantum gates.

n qubits $\rightarrow 2^n = N \rightarrow \log^2 N = O(n^2)$ …poly-time in number of qubits!

whereas run time of classical FFT: $O(N\log N) \rightarrow O(2^n \cdot n)$

Downsides

- Can't actually see the FT vector - it's "hidden" in amplitudes!
- Can only measure one outcome (why QFT is sometimes called *Fourier sampling*)

Step 1: Matrix Formulation for Classical FFT

Goal: compute multiplication

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^{2\cdot2} & \omega^{2\cdot3} & \omega^{2\cdot4} & \omega^{2\cdot5} & \omega^{2\cdot6} & \omega^{2\cdot7} \\ 1 & \omega^3 & \omega^{3\cdot2} & \omega^{3\cdot3} & \omega^{3\cdot4} & \omega^{3\cdot5} & \omega^{3\cdot6} & \omega^{3\cdot7} \\ 1 & \omega^4 & \omega^{4\cdot2} & \omega^{4\cdot3} & \omega^{4\cdot4} & \omega^{4\cdot5} & \omega^{4\cdot6} & \omega^{4\cdot7} \\ 1 & \omega^5 & \omega^{5\cdot2} & \omega^{5\cdot3} & \omega^{5\cdot4} & \omega^{5\cdot5} & \omega^{5\cdot6} & \omega^{5\cdot7} \\ 1 & \omega^6 & \omega^{6\cdot2} & \omega^{6\cdot3} & \omega^{6\cdot4} & \omega^{6\cdot5} & \omega^{6\cdot6} & \omega^{6\cdot7} \\ 1 & \omega^7 & \omega^{7\cdot2} & \omega^{7\cdot3} & \omega^{7\cdot4} & \omega^{7\cdot5} & \omega^{7\cdot6} & \omega^{7\cdot7} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

Last Week: Polynomial Formulation via Divide and Conquer

Figure 2.7 The fast Fourier transform (polynomial formulation)

<u>function FFT</u>(A, ω)
Input: Coefficient representation of a polynomial $A(x)$
of degree $\leq n-1$, where n is a power of 2
ω, an nth root of unity
Output: Value representation $A(\omega^0), \ldots, A(\omega^{n-1})$

if $\omega = 1$: return $A(1)$
express $A(x)$ in the form $A_e(x^2) + xA_o(x^2)$
call FFT(A_e, ω^2) to evaluate A_e at even powers of ω
call FFT(A_o, ω^2) to evaluate A_o at even powers of ω
for $j = 0$ to $n-1$:
  compute $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$

return $A(\omega^0), \ldots, A(\omega^{n-1})$

Step 1: Matrix Formulation for Classical FFT

Goal: compute multiplication

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^{2\cdot 2} & \omega^{2\cdot 3} & \omega^{2\cdot 4} & \omega^{2\cdot 5} & \omega^{2\cdot 6} & \omega^{2\cdot 7} \\ 1 & \omega^3 & \omega^{3\cdot 2} & \omega^{3\cdot 3} & \omega^{3\cdot 4} & \omega^{3\cdot 5} & \omega^{3\cdot 6} & \omega^{3\cdot 7} \\ 1 & \omega^4 & \omega^{4\cdot 2} & \omega^{4\cdot 3} & \omega^{4\cdot 4} & \omega^{4\cdot 5} & \omega^{4\cdot 6} & \omega^{4\cdot 7} \\ 1 & \omega^5 & \omega^{5\cdot 2} & \omega^{5\cdot 3} & \omega^{5\cdot 4} & \omega^{5\cdot 5} & \omega^{5\cdot 6} & \omega^{5\cdot 7} \\ 1 & \omega^6 & \omega^{6\cdot 2} & \omega^{6\cdot 3} & \omega^{6\cdot 4} & \omega^{6\cdot 5} & \omega^{6\cdot 6} & \omega^{6\cdot 7} \\ 1 & \omega^7 & \omega^{7\cdot 2} & \omega^{7\cdot 3} & \omega^{7\cdot 4} & \omega^{7\cdot 5} & \omega^{7\cdot 6} & \omega^{7\cdot 7} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

$\boxed{}$: even indices $\boxed{}$: odd indices

Last Week: Polynomial Formulation via Divide and Conquer

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Input: Coefficient representation of a polynomial $A(x)$
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ω, an nth root of unity
Output: Value representation $A(\omega^0), \ldots, A(\omega^{n-1})$

if $\omega = 1$: return $A(1)$
express $A(x)$ in the form $A_e(x^2) + xA_o(x^2)$
call FFT(A_e, ω^2) to evaluate A_e at even powers of ω
call FFT(A_o, ω^2) to evaluate A_o at even powers of ω
for $j = 0$ to $n-1$:
compute $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$

return $A(\omega^0), \ldots, A(\omega^{n-1})$

Step 1: Matrix Formulation for Classical FFT

Goal: compute multiplication

$$\frac{1}{\sqrt{8}}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7\\
1 & \omega^2 & \omega^{2\cdot 2} & \omega^{2\cdot 3} & \omega^{2\cdot 4} & \omega^{2\cdot 5} & \omega^{2\cdot 6} & \omega^{2\cdot 7}\\
1 & \omega^3 & \omega^{3\cdot 2} & \omega^{3\cdot 3} & \omega^{3\cdot 4} & \omega^{3\cdot 5} & \omega^{3\cdot 6} & \omega^{3\cdot 7}\\
1 & \omega^4 & \omega^{4\cdot 2} & \omega^{4\cdot 3} & \omega^{4\cdot 4} & \omega^{4\cdot 5} & \omega^{4\cdot 6} & \omega^{4\cdot 7}\\
1 & \omega^5 & \omega^{5\cdot 2} & \omega^{5\cdot 3} & \omega^{5\cdot 4} & \omega^{5\cdot 5} & \omega^{5\cdot 6} & \omega^{5\cdot 7}\\
1 & \omega^6 & \omega^{6\cdot 2} & \omega^{6\cdot 3} & \omega^{6\cdot 4} & \omega^{6\cdot 5} & \omega^{6\cdot 6} & \omega^{6\cdot 7}\\
1 & \omega^7 & \omega^{7\cdot 2} & \omega^{7\cdot 3} & \omega^{7\cdot 4} & \omega^{7\cdot 5} & \omega^{7\cdot 6} & \omega^{7\cdot 7}
\end{bmatrix}
\begin{bmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7 \end{bmatrix}$$

$\boxed{}$: even indices $\boxed{}$: odd indices

$$=$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & \omega^2 & \omega^4 & \omega^6 & \omega^1 & \omega^3 & \omega^5 & \omega^7\\
1 & \omega^{2\cdot 2} & \omega^{2\cdot 4} & \omega^{2\cdot 6} & \omega^2 & \omega^{2\cdot 3} & \omega^{2\cdot 5} & \omega^{2\cdot 7}\\
1 & \omega^{3\cdot 2} & \omega^{3\cdot 4} & \omega^{3\cdot 6} & \omega^3 & \omega^{3\cdot 3} & \omega^{3\cdot 5} & \omega^{3\cdot 7}\\
1 & \omega^{4\cdot 2} & \omega^{4\cdot 4} & \omega^{4\cdot 6} & \omega^4 & \omega^{4\cdot 3} & \omega^{4\cdot 5} & \omega^{4\cdot 7}\\
1 & \omega^{5\cdot 2} & \omega^{5\cdot 4} & \omega^{5\cdot 6} & \omega^5 & \omega^{5\cdot 3} & \omega^{5\cdot 5} & \omega^{5\cdot 7}\\
1 & \omega^{6\cdot 2} & \omega^{6\cdot 4} & \omega^{6\cdot 6} & \omega^6 & \omega^{6\cdot 3} & \omega^{6\cdot 5} & \omega^{6\cdot 7}\\
1 & \omega^{7\cdot 2} & \omega^{7\cdot 4} & \omega^{7\cdot 6} & \omega^7 & \omega^{7\cdot 3} & \omega^{7\cdot 5} & \omega^{7\cdot 7}
\end{bmatrix}
\begin{bmatrix} a_0\\ a_2\\ a_4\\ a_6\\ a_1\\ a_3\\ a_5\\ a_7 \end{bmatrix}$$

Last Week: Polynomial Formulation via Divide and Conquer

Figure 2.7 The fast Fourier transform (polynomial formulation)

function FFT(A, ω)
Input: Coefficient representation of a polynomial $A(x)$
of degree $\leq n-1$, where n is a power of 2
ω, an nth root of unity
Output: Value representation $A(\omega^0), \ldots, A(\omega^{n-1})$

if $\omega = 1$: return $A(1)$
express $A(x)$ in the form $A_e(x^2) + xA_o(x^2)$
call FFT(A_e, ω^2) to evaluate A_e at even powers of ω
call FFT(A_o, ω^2) to evaluate A_o at even powers of ω
for $j = 0$ to $n-1$:
compute $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$

return $A(\omega^0), \ldots, A(\omega^{n-1})$

Step 1: Matrix Formulation for Classical FFT

Goal: compute multiplication

$$
\frac{1}{\sqrt{8}}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\
1 & \omega^2 & \omega^{2\cdot 2} & \omega^{2\cdot 3} & \omega^{2\cdot 4} & \omega^{2\cdot 5} & \omega^{2\cdot 6} & \omega^{2\cdot 7} \\
1 & \omega^3 & \omega^{3\cdot 2} & \omega^{3\cdot 3} & \omega^{3\cdot 4} & \omega^{3\cdot 5} & \omega^{3\cdot 6} & \omega^{3\cdot 7} \\
1 & \omega^4 & \omega^{4\cdot 2} & \omega^{4\cdot 3} & \omega^{4\cdot 4} & \omega^{4\cdot 5} & \omega^{4\cdot 6} & \omega^{4\cdot 7} \\
1 & \omega^5 & \omega^{5\cdot 2} & \omega^{5\cdot 3} & \omega^{5\cdot 4} & \omega^{5\cdot 5} & \omega^{5\cdot 6} & \omega^{5\cdot 7} \\
1 & \omega^6 & \omega^{6\cdot 2} & \omega^{6\cdot 3} & \omega^{6\cdot 4} & \omega^{6\cdot 5} & \omega^{6\cdot 6} & \omega^{6\cdot 7} \\
1 & \omega^7 & \omega^{7\cdot 2} & \omega^{7\cdot 3} & \omega^{7\cdot 4} & \omega^{7\cdot 5} & \omega^{7\cdot 6} & \omega^{7\cdot 7}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}
$$

: even indices   : odd indices

$$
=
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega^1 & \omega^3 & \omega^5 & \omega^7 \\
1 & \omega^{2\cdot 2} & \omega^{2\cdot 4} & \omega^{2\cdot 6} & \omega^2 & \omega^{2\cdot 3} & \omega^{2\cdot 5} & \omega^{2\cdot 7} \\
1 & \omega^{3\cdot 2} & \omega^{3\cdot 4} & \omega^{3\cdot 6} & \omega^3 & \omega^{3\cdot 3} & \omega^{3\cdot 5} & \omega^{3\cdot 7} \\
1 & \omega^{4\cdot 2} & \omega^{4\cdot 4} & \omega^{4\cdot 6} & \omega^4 & \omega^{4\cdot 3} & \omega^{4\cdot 5} & \omega^{4\cdot 7} \\
1 & \omega^{5\cdot 2} & \omega^{5\cdot 4} & \omega^{5\cdot 6} & \omega^5 & \omega^{5\cdot 3} & \omega^{5\cdot 5} & \omega^{5\cdot 7} \\
1 & \omega^{6\cdot 2} & \omega^{6\cdot 4} & \omega^{6\cdot 6} & \omega^6 & \omega^{6\cdot 3} & \omega^{6\cdot 5} & \omega^{6\cdot 7} \\
1 & \omega^{7\cdot 2} & \omega^{7\cdot 4} & \omega^{7\cdot 6} & \omega^7 & \omega^{7\cdot 3} & \omega^{7\cdot 5} & \omega^{7\cdot 7}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \\ a_1 \\ a_3 \\ a_5 \\ a_7 \end{bmatrix}
$$

Last Week: Polynomial Formulation via Divide and Conquer

Figure 2.7 The fast Fourier transform (polynomial formulation)

function FFT(A, ω)
Input: Coefficient representation of a polynomial $A(x)$
of degree $\leq n-1$, where n is a power of 2
ω, an nth root of unity
Output: Value representation $A(\omega^0), \ldots, A(\omega^{n-1})$

if $\omega = 1$: return $A(1)$
express $A(x)$ in the form $A_e(x^2) + xA_o(x^2)$
call FFT(A_e, ω^2) to evaluate A_e at even powers of ω
call FFT(A_o, ω^2) to evaluate A_o at even powers of ω
for $j = 0$ to $n-1$:
  compute $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$

return $A(\omega^0), \ldots, A(\omega^{n-1})$

Step 1: Matrix Formulation for Classical FFT

Goal: compute multiplication

$$
\frac{1}{\sqrt{8}}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\
1 & \omega^2 & \omega^{2\cdot 2} & \omega^{2\cdot 3} & \omega^{2\cdot 4} & \omega^{2\cdot 5} & \omega^{2\cdot 6} & \omega^{2\cdot 7} \\
1 & \omega^3 & \omega^{3\cdot 2} & \omega^{3\cdot 3} & \omega^{3\cdot 4} & \omega^{3\cdot 5} & \omega^{3\cdot 6} & \omega^{3\cdot 7} \\
1 & \omega^4 & \omega^{4\cdot 2} & \omega^{4\cdot 3} & \omega^{4\cdot 4} & \omega^{4\cdot 5} & \omega^{4\cdot 6} & \omega^{4\cdot 7} \\
1 & \omega^5 & \omega^{5\cdot 2} & \omega^{5\cdot 3} & \omega^{5\cdot 4} & \omega^{5\cdot 5} & \omega^{5\cdot 6} & \omega^{5\cdot 7} \\
1 & \omega^6 & \omega^{6\cdot 2} & \omega^{6\cdot 3} & \omega^{6\cdot 4} & \omega^{6\cdot 5} & \omega^{6\cdot 6} & \omega^{6\cdot 7} \\
1 & \omega^7 & \omega^{7\cdot 2} & \omega^{7\cdot 3} & \omega^{7\cdot 4} & \omega^{7\cdot 5} & \omega^{7\cdot 6} & \omega^{7\cdot 7}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}
$$

: even indices : odd indices

$$
=
\left[\begin{array}{cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega^1 & \omega^3 & \omega^5 & \omega^7 \\
1 & \omega^{2\cdot 2} & \omega^{2\cdot 4} & \omega^{2\cdot 6} & \omega^2 & \omega^{2\cdot 3} & \omega^{2\cdot 5} & \omega^{2\cdot 7} \\
1 & \omega^{3\cdot 2} & \omega^{3\cdot 4} & \omega^{3\cdot 6} & \omega^3 & \omega^{3\cdot 3} & \omega^{3\cdot 5} & \omega^{3\cdot 7} \\
\hline
1 & \omega^{4\cdot 2} & \omega^{4\cdot 4} & \omega^{4\cdot 6} & \omega^4 & \omega^{4\cdot 3} & \omega^{4\cdot 5} & \omega^{4\cdot 7} \\
1 & \omega^{5\cdot 2} & \omega^{5\cdot 4} & \omega^{5\cdot 6} & \omega^5 & \omega^{5\cdot 3} & \omega^{5\cdot 5} & \omega^{5\cdot 7} \\
1 & \omega^{6\cdot 2} & \omega^{6\cdot 4} & \omega^{6\cdot 6} & \omega^6 & \omega^{6\cdot 3} & \omega^{6\cdot 5} & \omega^{6\cdot 7} \\
1 & \omega^{7\cdot 2} & \omega^{7\cdot 4} & \omega^{7\cdot 6} & \omega^7 & \omega^{7\cdot 3} & \omega^{7\cdot 5} & \omega^{7\cdot 7}
\end{array}\right]
\begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \\ a_1 \\ a_3 \\ a_5 \\ a_7 \end{bmatrix}
$$

Last Week: Polynomial Formulation via Divide and Conquer

Figure 2.7 The fast Fourier transform (polynomial formulation)

```
function FFT(A, ω)
Input:  Coefficient representation of a polynomial A(x)
        of degree ≤ n − 1, where n is a power of 2
        ω, an nth root of unity
Output:  Value representation A(ω^0), ..., A(ω^(n−1))

if ω = 1:  return A(1)
express A(x) in the form A_e(x^2) + x A_o(x^2)
call FFT(A_e, ω^2) to evaluate A_e at even powers of ω
call FFT(A_o, ω^2) to evaluate A_o at even powers of ω
for j = 0 to n − 1:
   compute  A(ω^j) = A_e(ω^(2j)) + ω^j A_o(ω^(2j))

return A(ω^0), ..., A(ω^(n−1))
```

$$
= \underbrace{\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega^2 & \omega^4 & \omega^6 \\
1 & \omega^{2\cdot 2} & \omega^{2\cdot 4} & \omega^{2\cdot 6} \\
1 & \omega^{3\cdot 2} & \omega^{3\cdot 4} & \omega^{3\cdot 6}
\end{bmatrix}}_{\text{top half}}
\begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{bmatrix}
+
\underbrace{\begin{bmatrix}
1 & 1 & 1 & 1 \\
\omega^1 & \omega^3 & \omega^5 & \omega^7 \\
\omega^2 & \omega^{2\cdot 3} & \omega^{2\cdot 5} & \omega^{2\cdot 7} \\
\omega^3 & \omega^{3\cdot 3} & \omega^{3\cdot 5} & \omega^{3\cdot 7}
\end{bmatrix}}_{\text{top half}}
\begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ a_7 \end{bmatrix}
$$

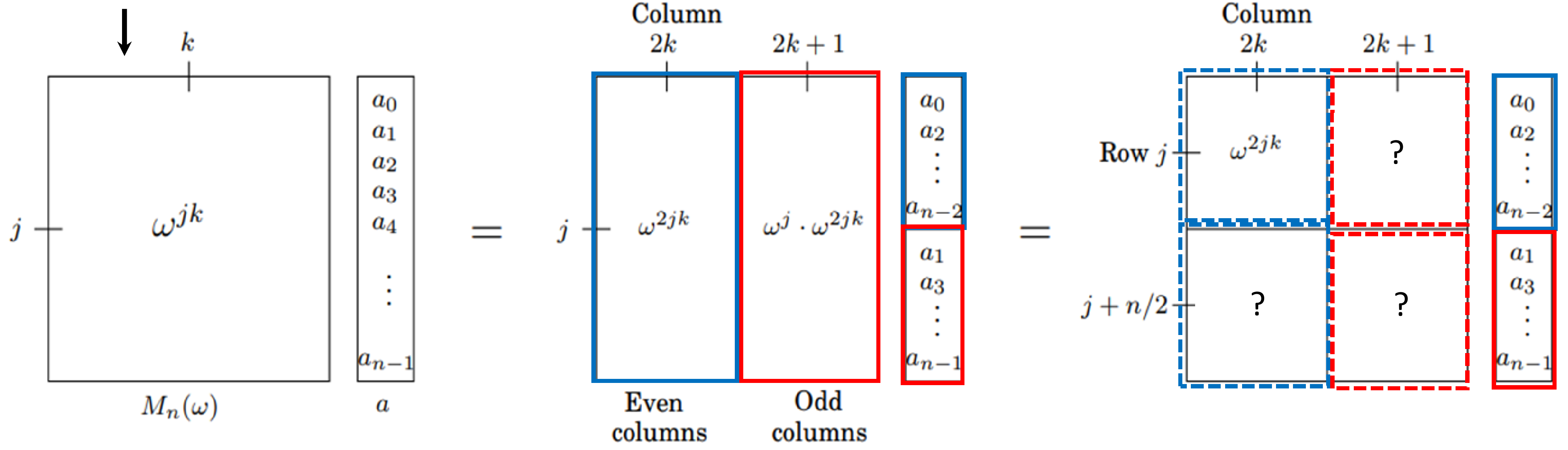
$$
\underbrace{\begin{bmatrix}
1 & \omega^{4\cdot 2} & \omega^{4\cdot 4} & \omega^{4\cdot 6} \\
1 & \omega^{5\cdot 2} & \omega^{5\cdot 4} & \omega^{5\cdot 6} \\
1 & \omega^{6\cdot 2} & \omega^{6\cdot 4} & \omega^{6\cdot 6} \\
1 & \omega^{7\cdot 2} & \omega^{7\cdot 4} & \omega^{7\cdot 6}
\end{bmatrix}}_{\text{bottom half}}
\begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{bmatrix}
+
\underbrace{\begin{bmatrix}
\omega^4 & \omega^{4\cdot 3} & \omega^{4\cdot 5} & \omega^{4\cdot 7} \\
\omega^5 & \omega^{5\cdot 3} & \omega^{5\cdot 5} & \omega^{5\cdot 7} \\
\omega^6 & \omega^{6\cdot 3} & \omega^{6\cdot 5} & \omega^{6\cdot 7} \\
\omega^7 & \omega^{7\cdot 3} & \omega^{7\cdot 5} & \omega^{7\cdot 7}
\end{bmatrix}}_{\text{bottom half}}
\begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ a_7 \end{bmatrix}
$$

four multiplications of dimension 4

Step 1: Matrix Formulation for Classical FFT

(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)

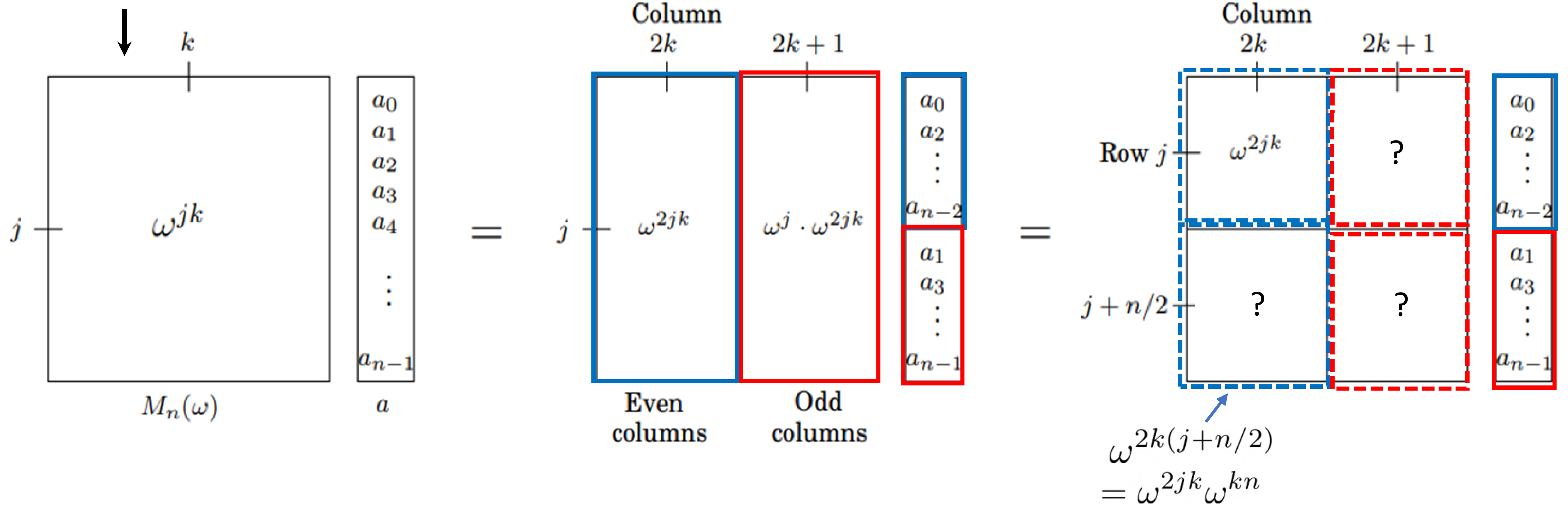
$M_n = n$ dimensional Fourier transform



Step 1: Matrix Formulation for Classical FFT

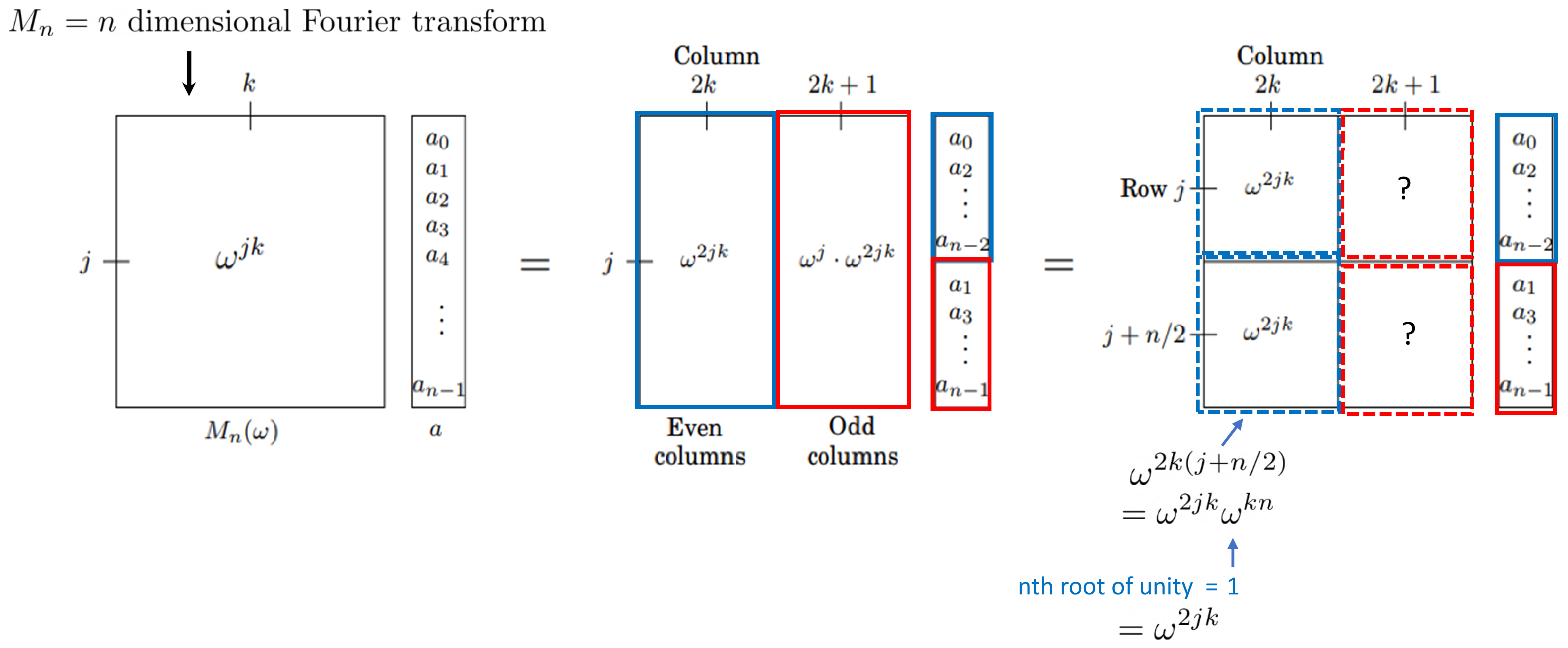
(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)

$M_n = n$ dimensional Fourier transform



Step 1: Matrix Formulation for Classical FFT

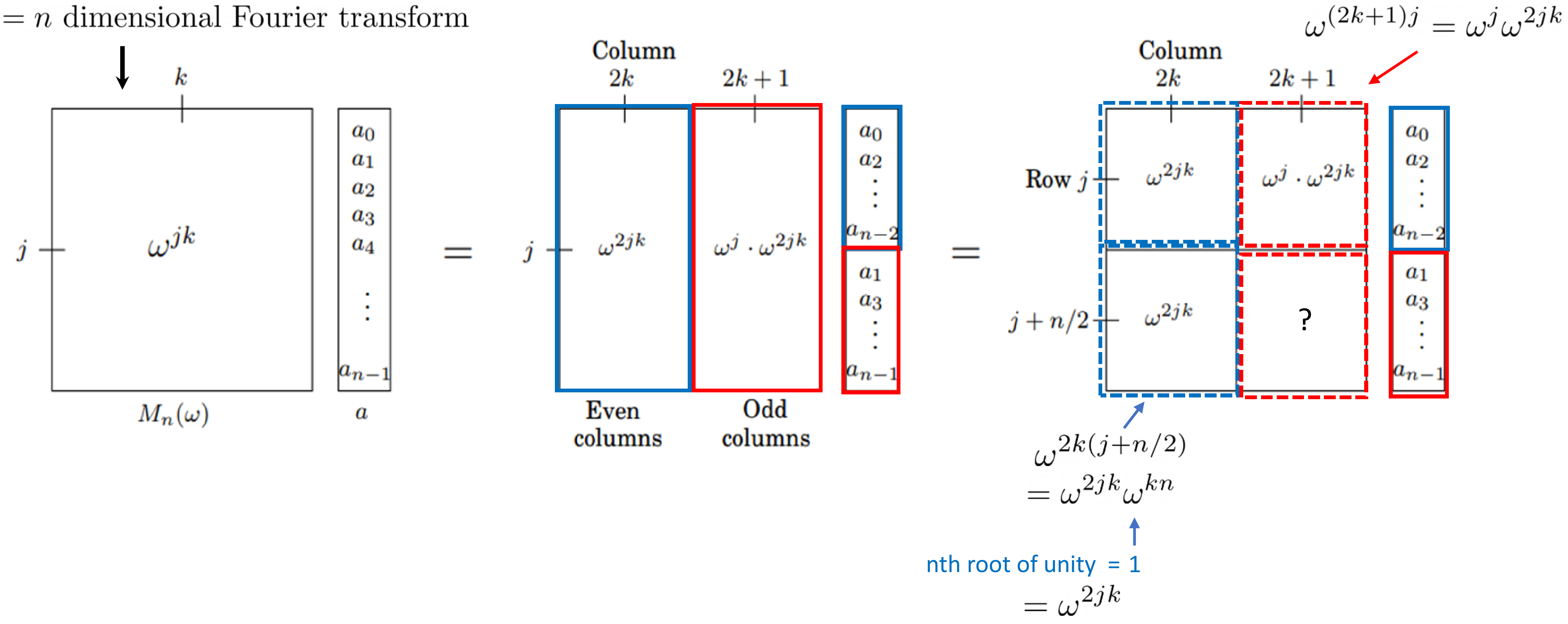
(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)



Step 1: Matrix Formulation for Classical FFT

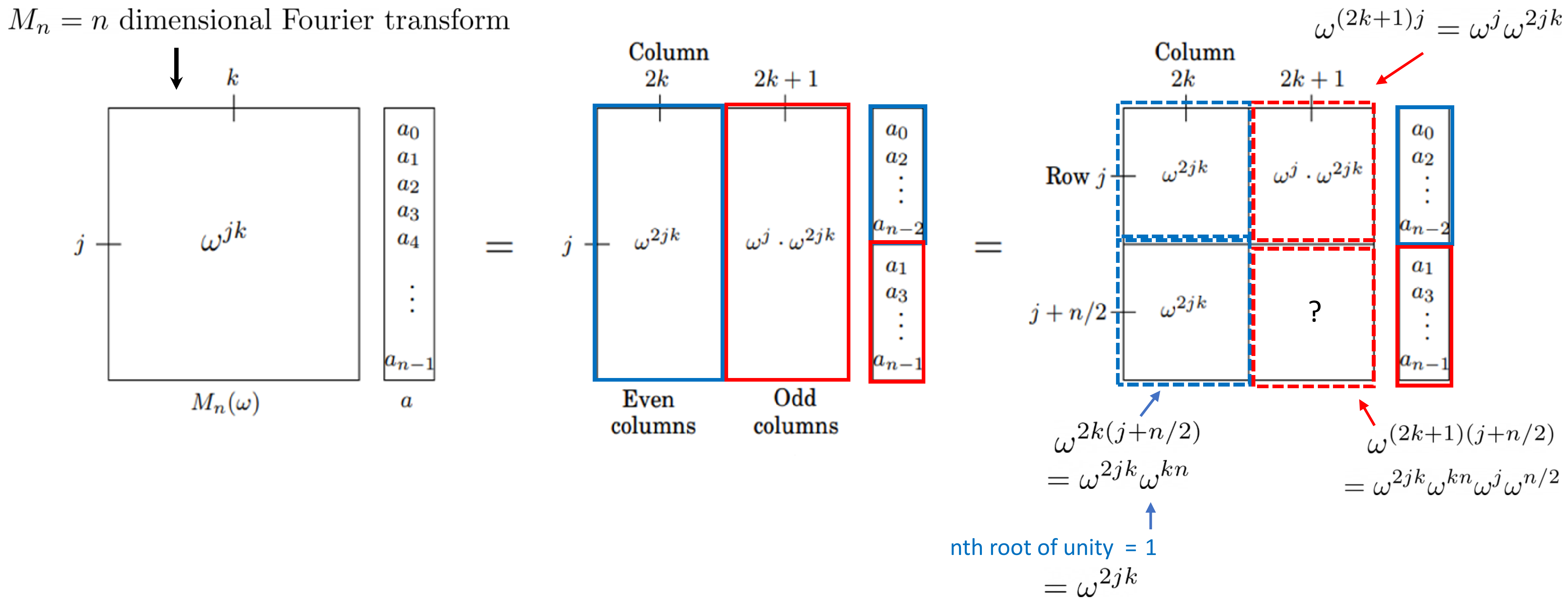
(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)

$M_n = n$ dimensional Fourier transform



Step 1: Matrix Formulation for Classical FFT

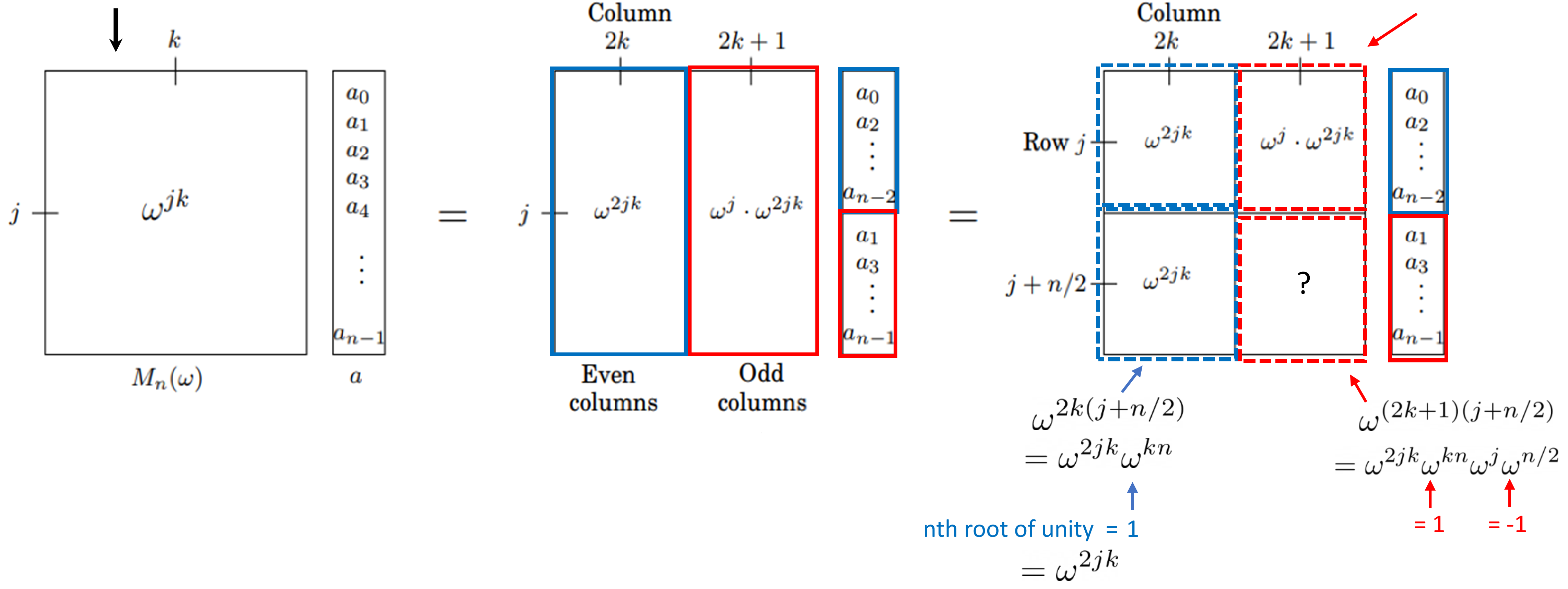
(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)



Step 1: Matrix Formulation for Classical FFT

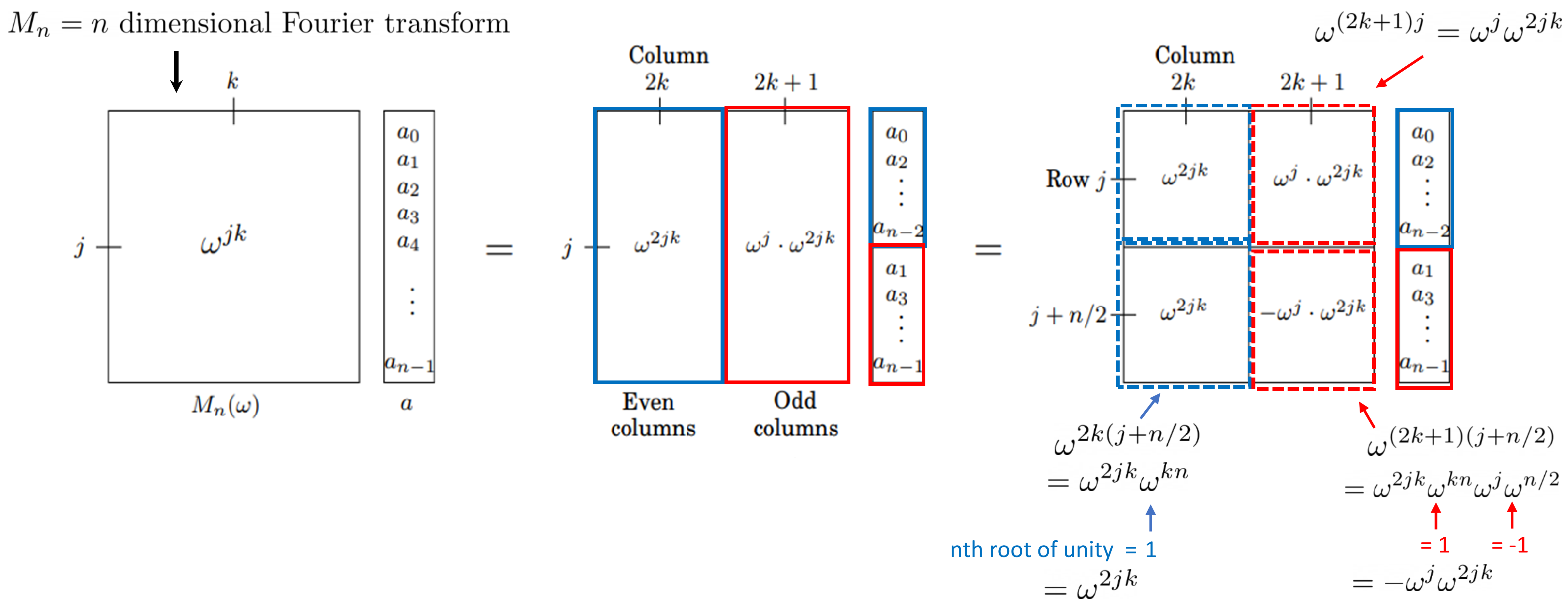
(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)

$M_n = n$ dimensional Fourier transform



Step 1: Matrix Formulation for Classical FFT

(Figures courtesy Dasgupta, Papadimitriou, and Vazirani 2006)



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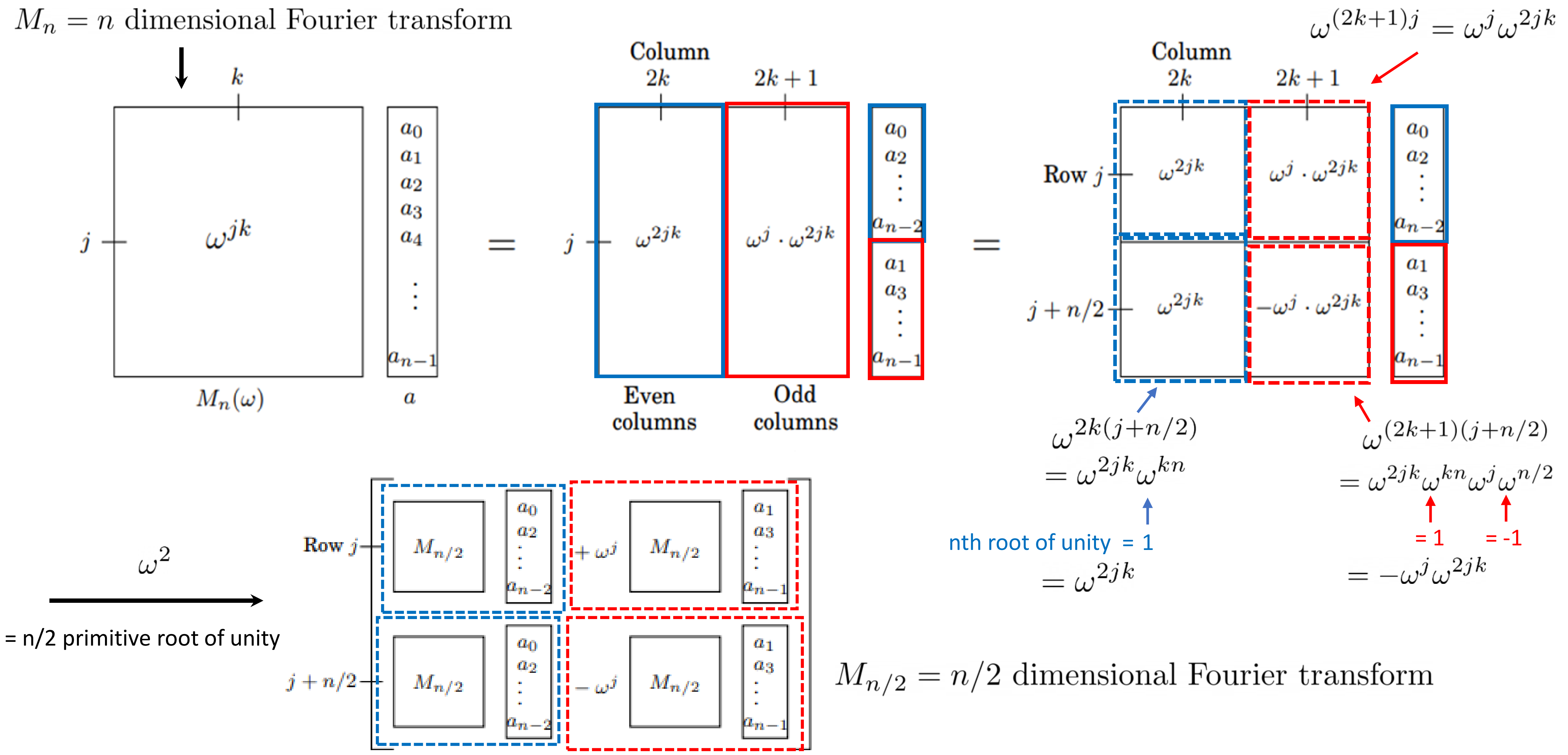

Step 1: Matrix Formulation for Classical FFT

Figure 2.9 The fast Fourier transform

<u>function FFT</u>(a, ω)

Input: An array $a = (a_0, a_1, \ldots, a_{n-1})$, for n a power of 2
A primitive nth root of unity, ω

Output: $M_n(\omega)\, a$

if $\omega = 1$: return a

$\boxed{(s_0, s_1, \ldots, s_{n/2-1}) = \text{FFT}((a_0, a_2, \ldots, a_{n-2}), \omega^2)}$

$\boxed{(s'_0, s'_1, \ldots, s'_{n/2-1}) = \text{FFT}((a_1, a_3, \ldots, a_{n-1}), \omega^2)}$

for $j = 0$ to $n/2 - 1$:

  $\boxed{r_j = s_j + \omega^j s'_j}$

  $\boxed{r_{j+n/2} = s_j - \omega^j s'_j}$

return $(r_0, r_1, \ldots, r_{n-1})$

$$\boxed{\begin{bmatrix} & & \\ & M_{n/2} & \\ & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \\ \vdots \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n/2-1} \end{bmatrix}}$$

even index sub-problem

$$\boxed{\begin{bmatrix} & & \\ & M_{n/2} & \\ & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} s'_0 \\ s'_1 \\ \vdots \\ s'_{n/2-1} \end{bmatrix}}$$

odd index sub-problem

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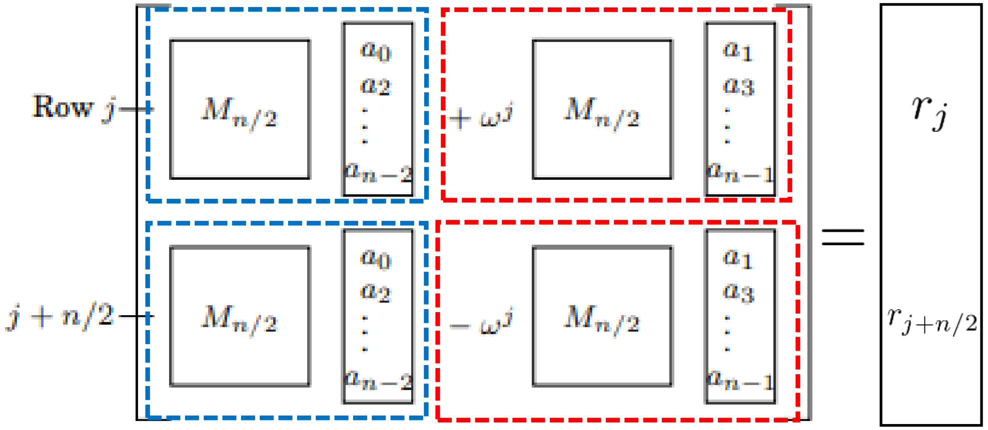

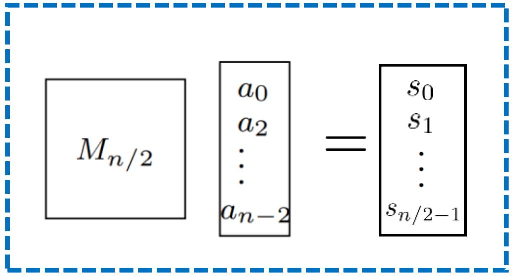

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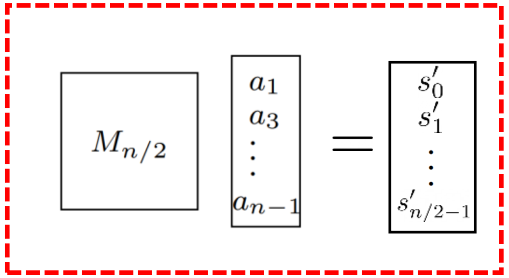

odd index sub-problem

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Figure 2.9 The fast Fourier transform

function FFT(a, ω)
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Output: $M_n(\omega)\, a$

if $\omega = 1$: return a
$(s_0, s_1, \ldots, s_{n/2-1}) = \text{FFT}((a_0, a_2, \ldots, a_{n-2}), \omega^2)$
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for $j = 0$ to $n/2 - 1$:
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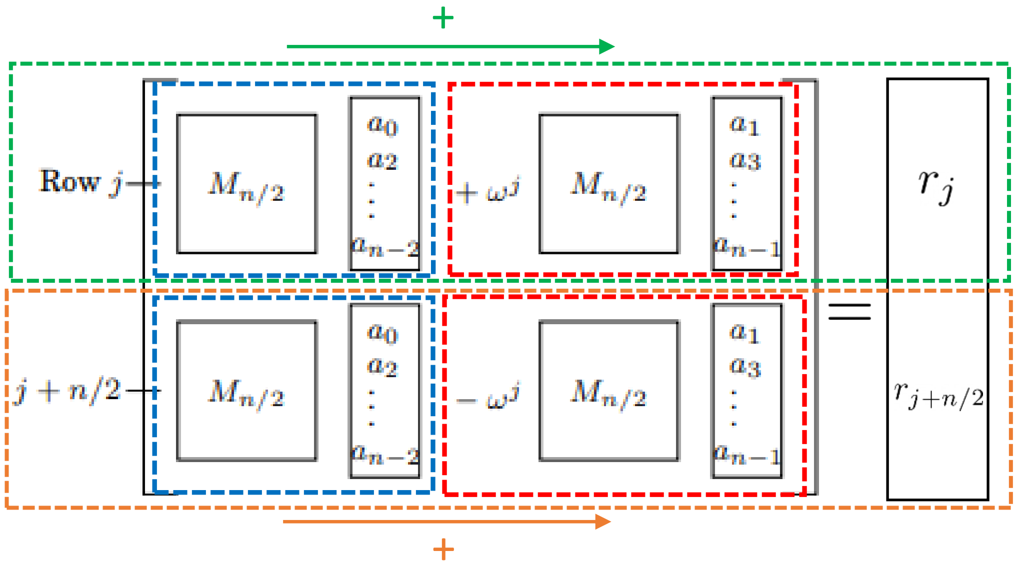
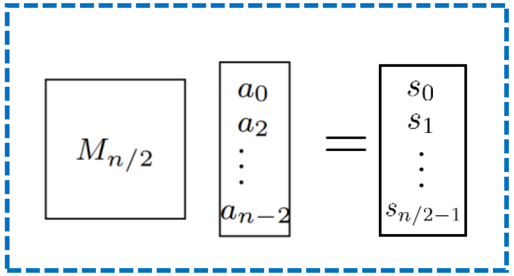
  $r_{j+n/2} = s_j - \omega^j s'_j$
return $(r_0, r_1, \ldots, r_{n-1})$





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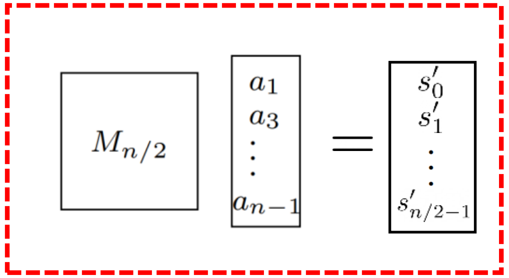

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Step 1: Matrix Formulation for Classical FFT

Figure 2.9 The fast Fourier transform

function FFT(a, ω)
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A primitive nth root of unity, ω
Output: $M_n(\omega)\, a$

if $\omega = 1$: return a
$(s_0, s_1, \ldots, s_{n/2-1}) = \text{FFT}((a_0, a_2, \ldots, a_{n-2}), \omega^2)$
$(s'_0, s'_1, \ldots, s'_{n/2-1}) = \text{FFT}((a_1, a_3, \ldots, a_{n-1}), \omega^2)$
for $j = 0$ to $n/2 - 1$:
  $r_j = s_j + \omega^j s'_j$
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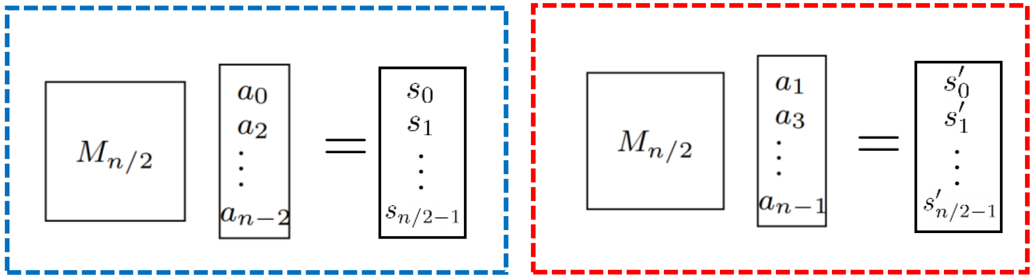
return $(r_0, r_1, \ldots, r_{n-1})$



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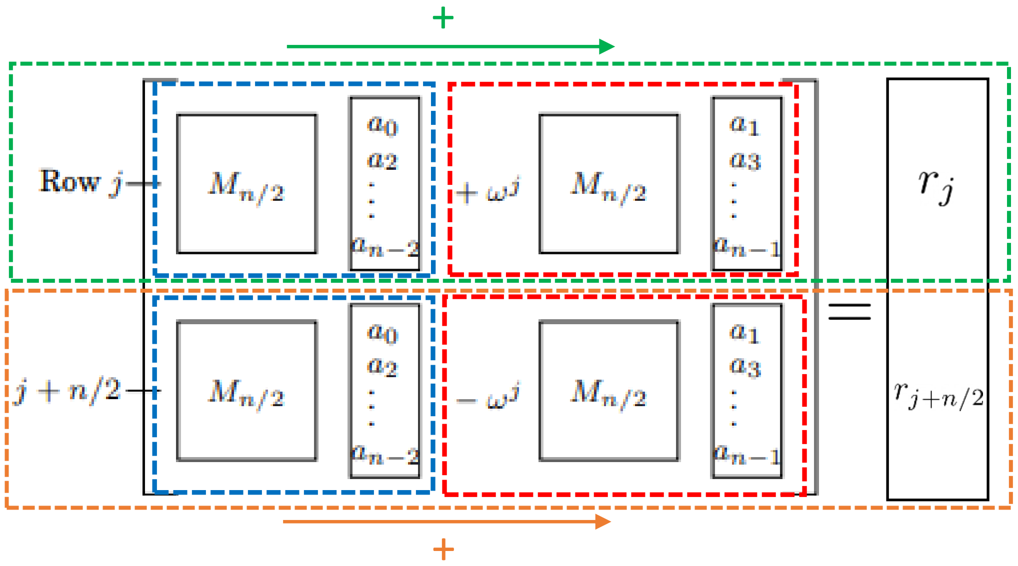

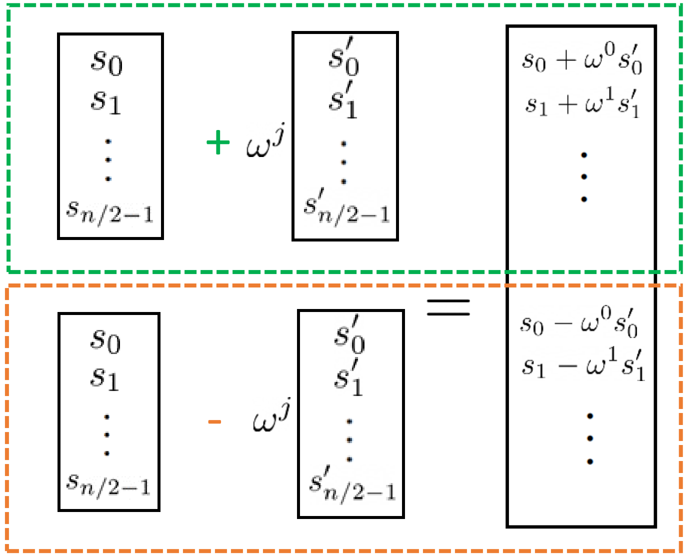

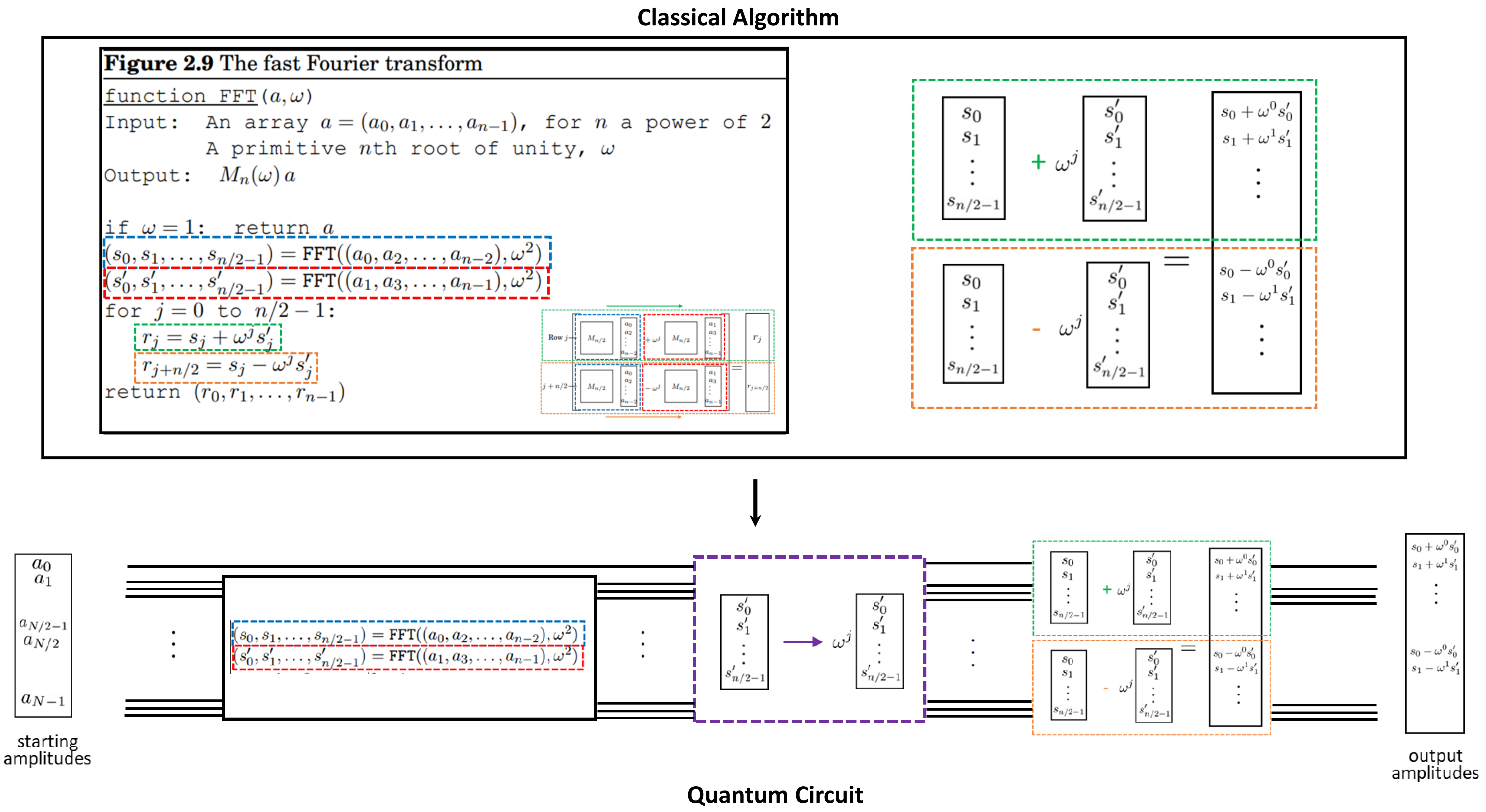

Classical Algorithm
Figure 2.9 The fast Fourier transform
function FFT(a, ω)
Input: An array a = (a0, a1, ..., an−1), for n a power of 2
A primitive nth root of unity, ω
Output: Mn(ω) a
if ω = 1: return a
(s0, s1, ..., sn/2−1) = FFT((a0, a2, ..., an−2), ω²)
(s′0, s′1, ..., s′n/2−1) = FFT((a1, a3, ..., an−1), ω²)
for j = 0 to n/2 − 1:
rj = sj + ω^j s′j
rj+n/2 = sj − ω^j s′j
return (r0, r1, ..., rn−1)
starting amplitudes
output amplitudes
Quantum Circuit