Lecture 5: Multi-qubit Computations

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Review: Different Representations

Amplitude Vector

$$\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

t = 0 | Hadamard Gate H | t = 1 | Z Gate Z | t = 2 | Measure

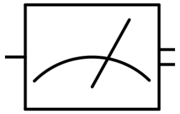
"Bra-ket Notation"

(or Dirac notation )

$$|0\rangle \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix} \qquad |1\rangle \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix}$$

Matrix Multiplications

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$



Matrix Operations (from right to left)

Starting Vector

Quantum Circuit

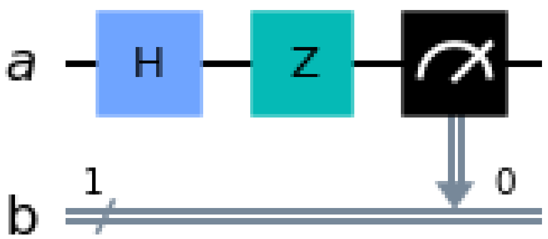
$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix} : \; H$$

Hadamard Gate

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} : \; Z$$

Z Gate

Quantum Register



Classical Register

$$\overbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}^{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}}$$

$$|0\rangle \; — H \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \rightarrow$$

$$— Z \rightarrow \underbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{-1}{\sqrt{2}}|1\rangle}_{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}}$$

Review: Unitary Matrices

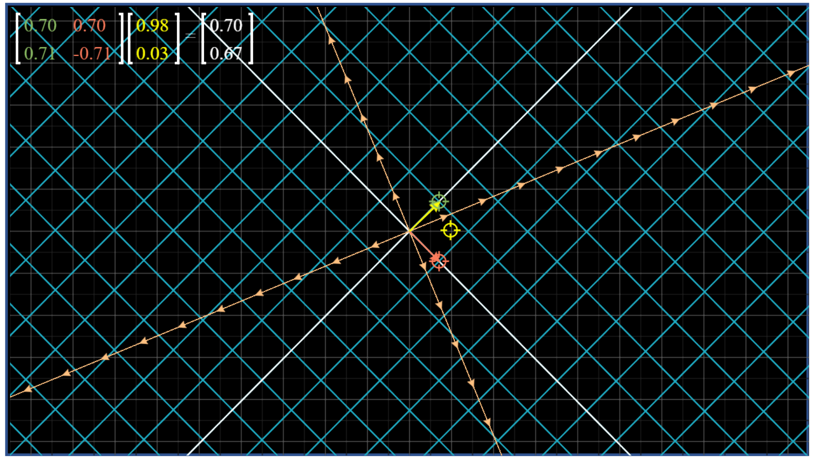
Algebraic Definition: (conjugate) transpose equals inverse.

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{H^T} = \begin{bmatrix} 1/2 + 1/2 & -1/2 + 1/2 \\ -1/2 + 1/2 & 1/2 + 1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

(Identity Matrix)

Geometric Intuition

- Preserves vector lengths.
  (Thus, maintaining square of amplitudes can be used to calculate probabilities)
- Angles between vectors between before and after the transformation are left unchanged.
  (Can only rotate and reflect space)
- Linear transformation visualizer: https://shad.io/MatVis/



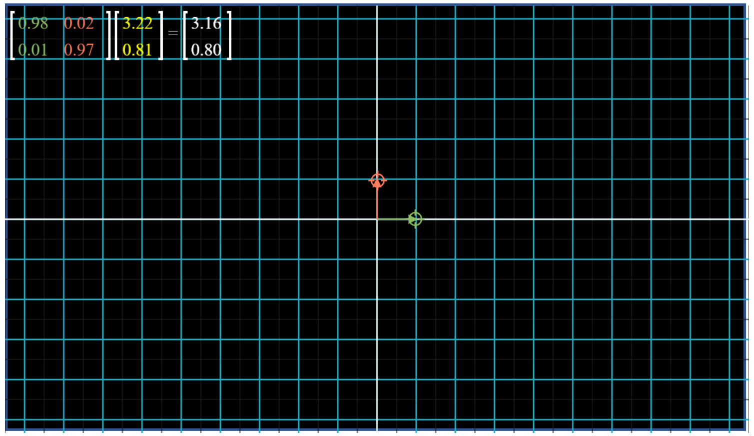
Quick Remark: Matrices as Functions or Linear Transformations

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{H}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \longleftrightarrow H\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right) = \alpha \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \beta \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

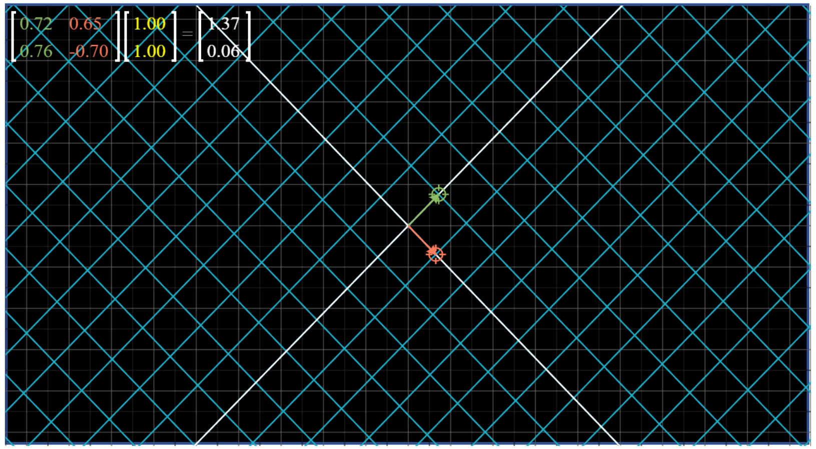
"Multiply matrix H by the vector [α, β]"

"Define function H to takes in a vector as a parameter and returns vector of the same length"

....columns of matrix specify how the function is computed



$\xrightarrow{H}$



" [α, β] corresponds to a point with respect to the standard x-y axes"

" [α, β] when vectors [1/√2, 1/√2] and [1/√2, -1/√2] for our new axes "

Quick Remark: Matrices as Functions or Linear Transformations

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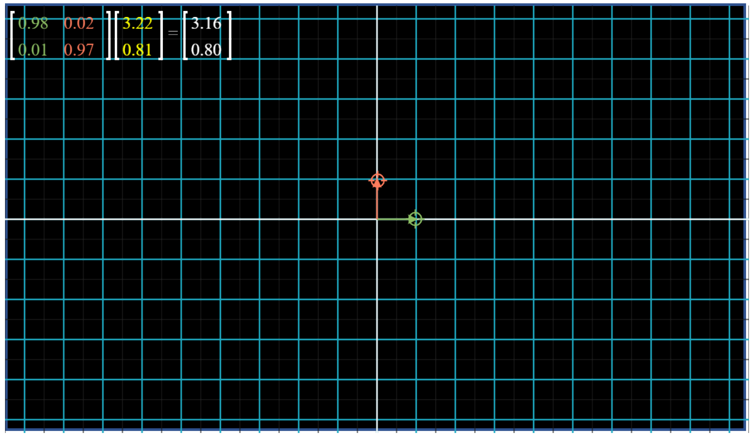

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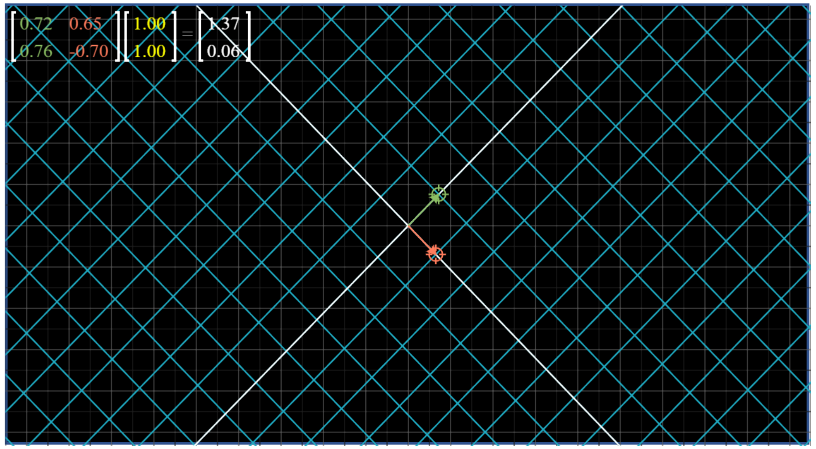

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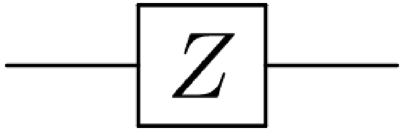
Grant Sanderson

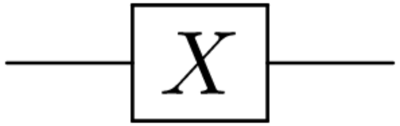
[3Blue1Brown](#)
[*Essence of Linear Algebra* Mini Course](#)

Single Qubit Quantum Gates

Quantum Circuit



Z Gate



X Gate

Hadamard Gate

Matrix

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

Bra-ket Notation

$$\alpha|0\rangle + \beta|1\rangle$$
$$\downarrow$$
$$\alpha|0\rangle - \beta|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle$$
$$\downarrow$$
$$\beta|0\rangle + \alpha|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle$$
$$\downarrow$$
?

Single Qubit Quantum Gates

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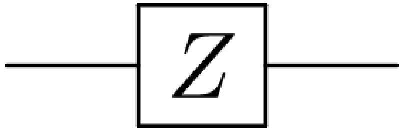

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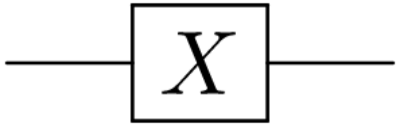

X Gate

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Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

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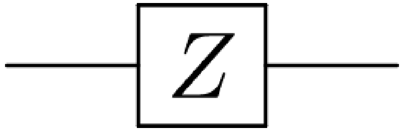
Single Qubit Quantum Gates

Quantum Circuit



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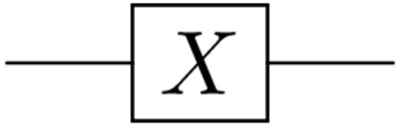

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$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Bra-ket Notation

$$\alpha|0\rangle + \beta|1\rangle$$
$$\downarrow$$
$$\alpha|0\rangle - \beta|1\rangle$$

“negate second amplitude”

$$\alpha|0\rangle + \beta|1\rangle$$
$$\downarrow$$
$$\beta|0\rangle + \alpha|1\rangle$$

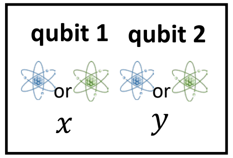
“flip amplitude”

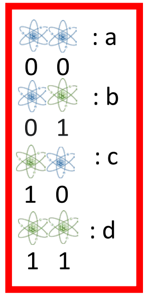
$$\alpha|0\rangle + \beta|1\rangle$$
$$\downarrow$$
$$\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

“create even distribution via unitary operator”

Two Qubit Circuit

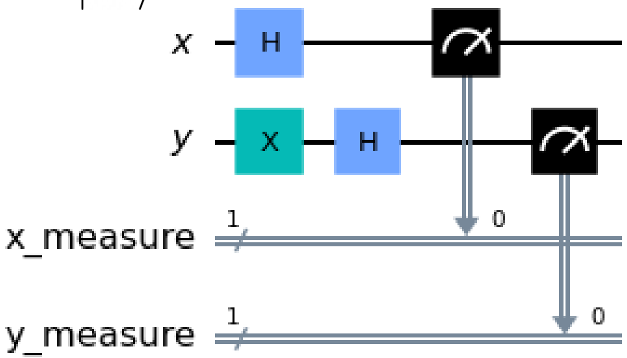
2 qubit -> 4 amplitudes
$a^2 + b^2 + c^2 + d^2 = 1$





$$\longleftrightarrow \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longleftrightarrow a\underset{xy}{|00\rangle} + b\underset{xy}{|01\rangle} + c\underset{xy}{|10\rangle} + d\underset{xy}{|11\rangle}$$

Standard Assumption
starting state = $|00\rangle$

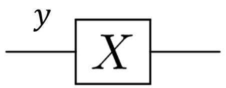


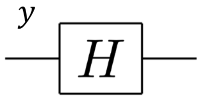
Matrix Multiplication

right to left
start state

Bra-ket Notation

$\underset{xy}{|00\rangle}$





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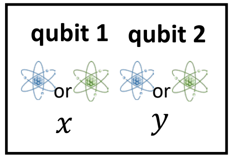

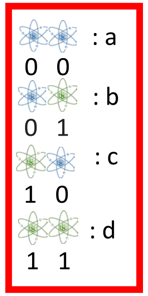

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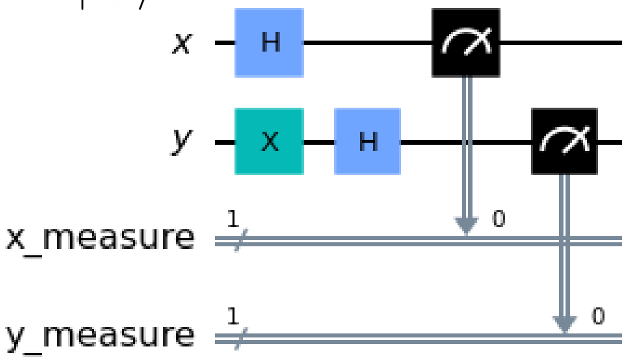


Matrix Multiplication

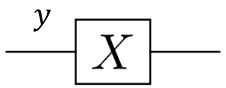
$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

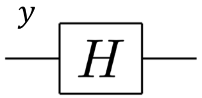
right to left

start state

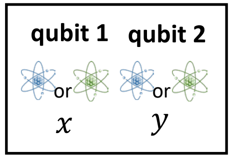
Bra-ket Notation

$\underset{xy}{|00\rangle}$





Two Qubit Circuit



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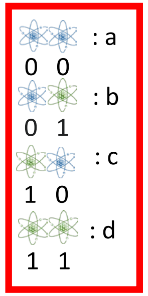

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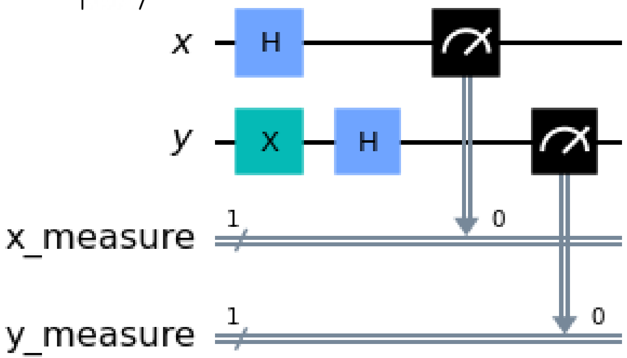


Matrix Multiplication

y — H — y — X — x — H —

$$\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

right to left   start state

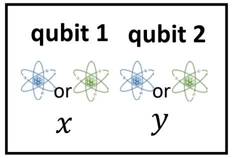
We'll discuss how to derive these matrices in next lecture

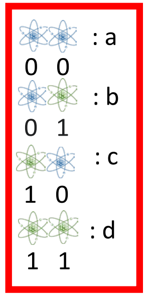
Bra-ket Notation

$\underset{xy}{|00\rangle}$ x — H — y — X — y — H —

Two Qubit Circuit

2 qubit -> 4 amplitudes
$a^2 + b^2 + c^2 + d^2 = 1$





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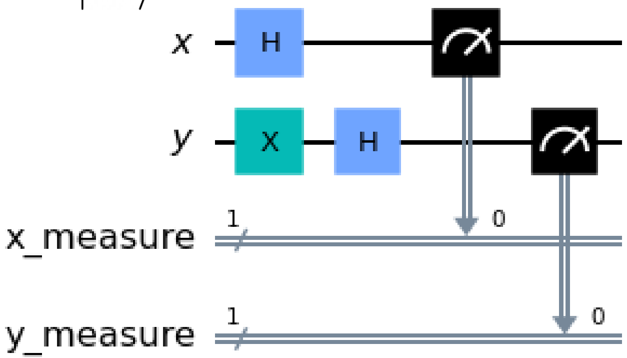

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$\longleftarrow$ right to left

start state

$\longleftarrow$ We'll discuss how to derive these matrices in next lecture

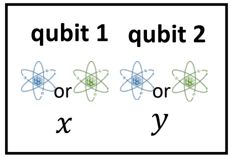
Bra-ket Notation $\longrightarrow$

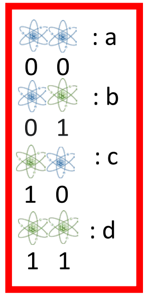
$$\underset{xy}{|00\rangle} \quad \overset{x}{-\boxed{H}-} \quad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \quad \overset{y}{-\boxed{X}-} \qquad\qquad \overset{y}{-\boxed{H}-}$$

Hadamard just on qubit x.
Amplitudes of |0> for y qubit stays the same

Two Qubit Circuit

2 qubit -> 4 amplitudes
$a^2 + b^2 + c^2 + d^2 = 1$





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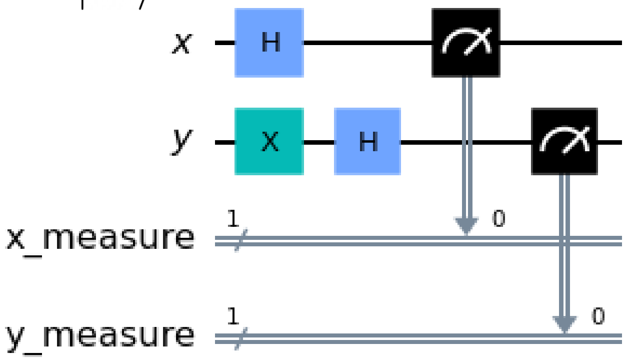
Standard Assumption
starting state = $|00\rangle$



Matrix Multiplication

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right to left

start state

We'll discuss how to derive these matrices in next lecture

Bra-ket Notation

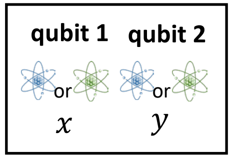
$$\underset{xy}{|00\rangle} \quad x\,-\boxed{H}- \quad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \quad y\,-\boxed{X}- \quad \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \quad y\,-\boxed{H}-$$

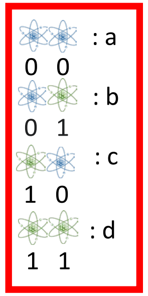
Hadamard just on qubit x.
Amplitudes of |0> for y qubit stays the same

|1> now has
Positive amplitude for y qubit

Two Qubit Circuit

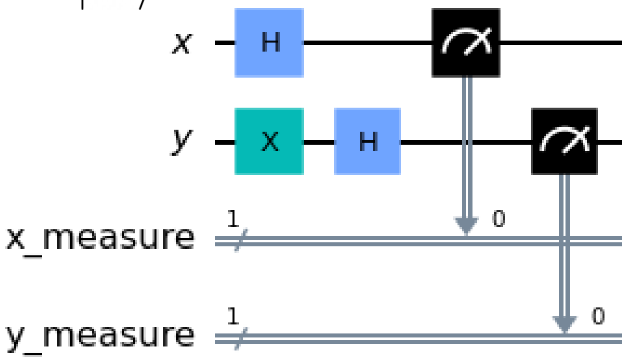
2 qubit -> 4 amplitudes
$a^2 + b^2 + c^2 + d^2 = 1$





$$\longleftrightarrow \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longleftrightarrow a\underset{xy}{|00\rangle} + b\underset{xy}{|01\rangle} + c\underset{xy}{|10\rangle} + d\underset{xy}{|11\rangle}$$

Standard Assumption
starting state = $|00\rangle$



Matrix Multiplication

$$\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{y\text{—}\boxed{H}\text{—}} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{y\text{—}\boxed{X}\text{—}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{x\text{—}\boxed{H}\text{—}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

right to left

start state

We'll discuss how to derive these matrices in next lecture

Bra-ket Notation

$$\underset{xy}{|00\rangle} \;\; x\text{—}\boxed{H}\text{—} \;\; \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \;\; y\text{—}\boxed{X}\text{—} \;\; \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \;\; y\text{—}\boxed{H}\text{—}$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Hadamard just on qubit x.
Amplitudes of |0> for y qubit stays the same

|1> now has
Positive amplitude for y qubit

Hadamard on y part of ket
(only |1> terms have positive amplitude)