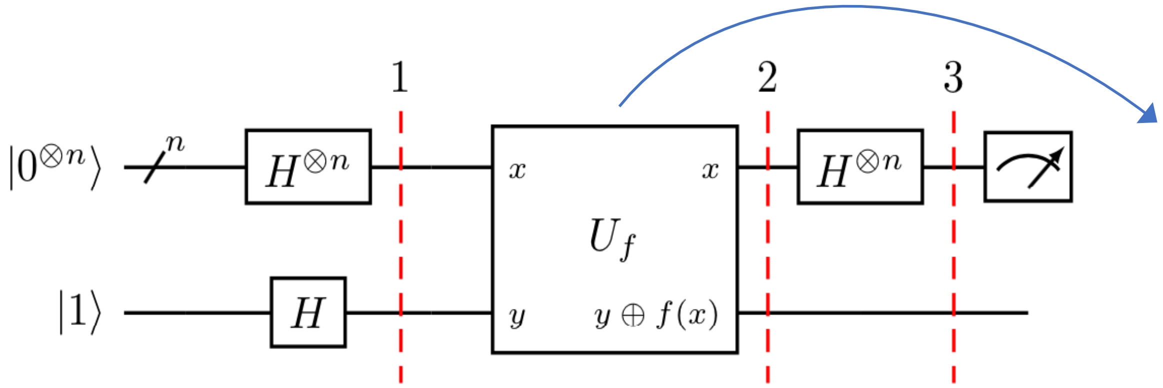
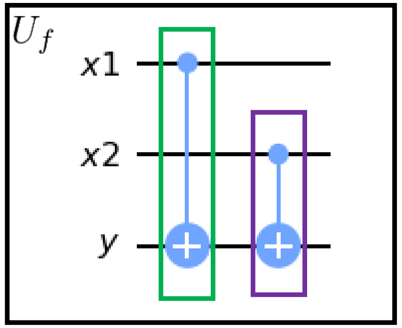
Lecture 20: Permutation Matrices

CS 401: Quantum Computing
Dr. Kell, Spring 2023

Review: Uf gate n = 2 example

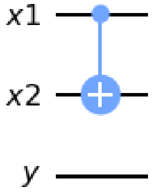




What is the corresponding matrix for this gate?

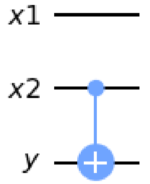
...since it is neither

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$$



nor

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow$$



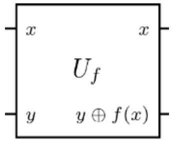
n = 2 Balanced Example

$$f(00) = f(11) = 0$$

$$f(01) = f(10) = 1$$

(XOR function)

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle + \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

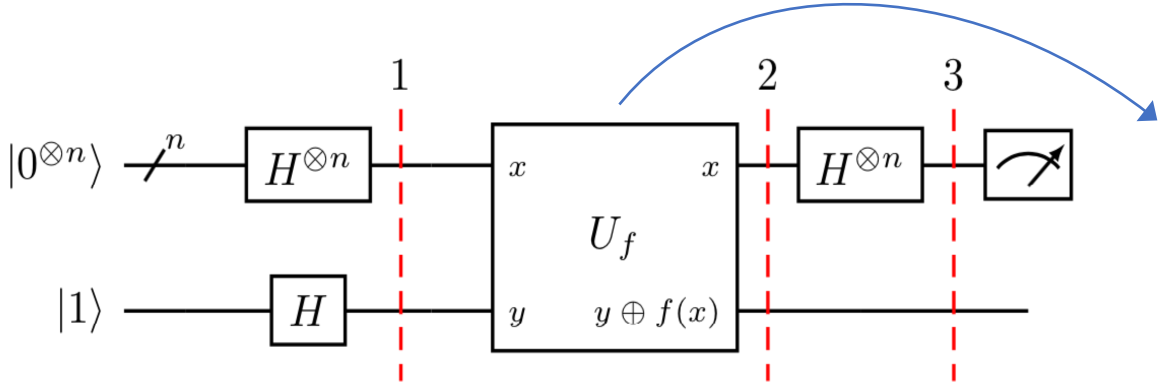


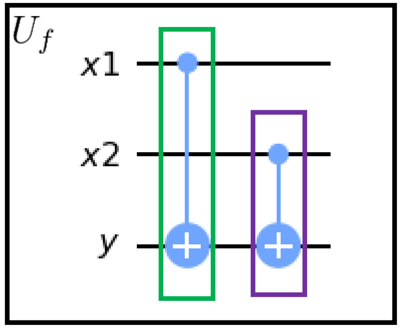
(swap back)

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

$$= \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{011}|010\rangle + \alpha_{010}|011\rangle + \alpha_{101}|100\rangle + \alpha_{100}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

Review: Uf gate n = 2 example

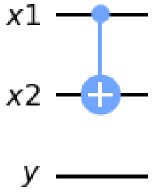




What is the corresponding matrix for this gate?

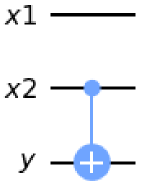
...since it is neither

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow$$



nor

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \otimes \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \longrightarrow$$



This Lecture: systematic way of calculating matrices for 2+ qubit gates.

n = 2 Balanced Example

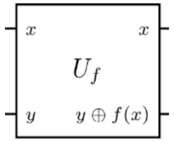
$$f(00) = f(11) = 0$$

$$f(01) = f(10) = 1$$

(XOR function)

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle + \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

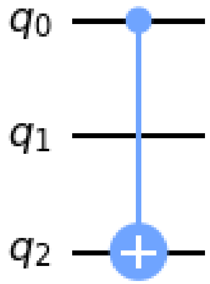
(swap back)



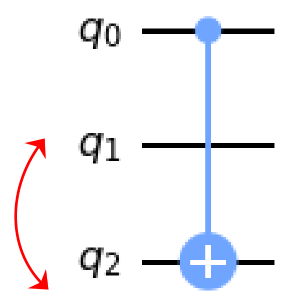
$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

$$= \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{011}|010\rangle + \alpha_{010}|011\rangle + \alpha_{101}|100\rangle + \alpha_{100}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

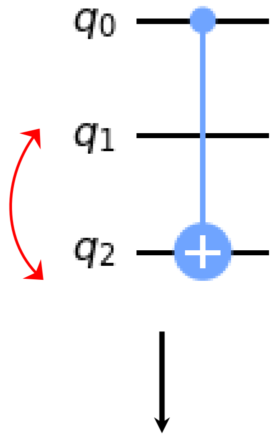
General Approach: Swapping Basis Labels

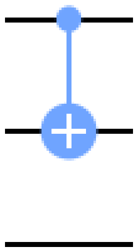


General Approach: Swapping Basis Labels



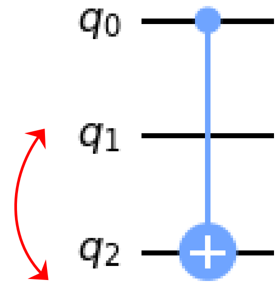
General Approach: Swapping Basis Labels

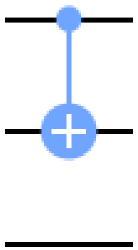




$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

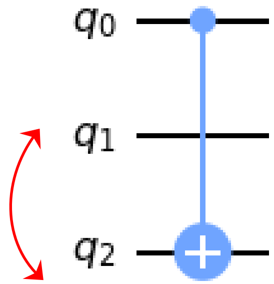
General Approach: Swapping Basis Labels

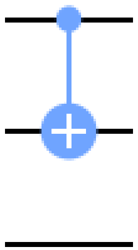




$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

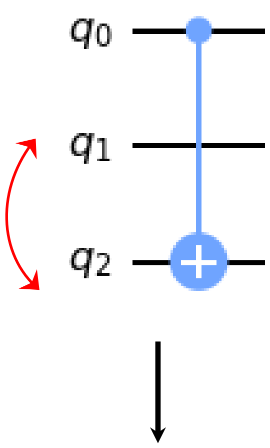
General Approach: Swapping Basis Labels

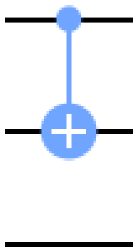




$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} \\ 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} \\ 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&1\end{bmatrix} \\ 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}$$

General Approach: Swapping Basis Labels





$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} =$$

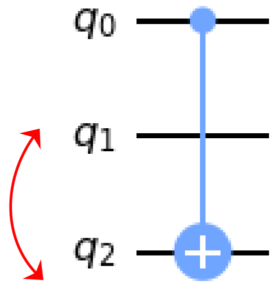
$$\begin{array}{c|c} & |000\rangle \; |001\rangle \; |010\rangle \; |011\rangle \; |100\rangle \; |101\rangle \; |110\rangle \; |111\rangle \\ \hline \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

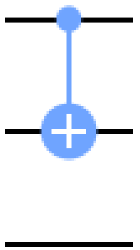
Standard Enumeration

General Approach: Swapping Basis Labels

$$
\begin{array}{c}
|000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle
\end{array}
\rightarrow
\begin{array}{c|cccccccc}
 & |000\rangle & |010\rangle & |001\rangle & |011\rangle & |100\rangle & |110\rangle & |101\rangle & |111\rangle \\
\hline
|000\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
|010\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
|001\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
|011\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
|100\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
|110\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
|101\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
|111\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

"Swapped" Enumeration





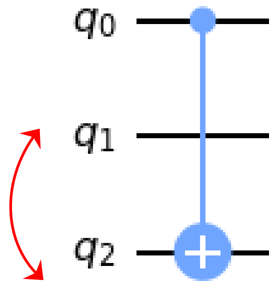
$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =
\begin{bmatrix}
1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{bmatrix} =
$$

$$
\begin{array}{c|cccccccc}
 & |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \\
\hline
|000\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
|001\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
|010\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
|011\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
|100\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
|101\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
|110\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
|111\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

Standard Enumeration

General Approach: Swapping Basis Labels

Idea: applying CX(q0, q1) on swapped enumeration would be the same as CX(q0, q2)



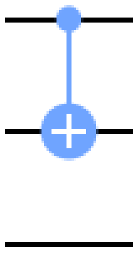
$$\begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} \rightarrow \begin{array}{c} |000\rangle \\ |010\rangle \\ |001\rangle \\ |011\rangle \\ |100\rangle \\ |110\rangle \\ |101\rangle \\ |111\rangle \end{array}$$

$$\begin{array}{c|cccccccc} & |000\rangle & |010\rangle & |001\rangle & |011\rangle & |100\rangle & |110\rangle & |101\rangle & |111\rangle \\ \hline |000\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ |010\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ |001\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ |011\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ |100\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ |110\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ |101\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ |111\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

"Swapped" Enumeration

$$\begin{array}{c|cccccccc} & |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \\ \hline |000\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ |001\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ |010\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ |011\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ |100\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ |101\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ |110\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ |111\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

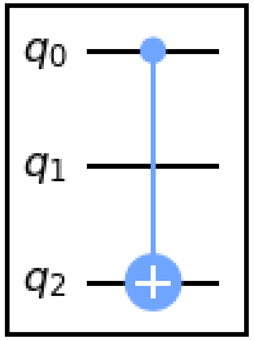
Standard Enumeration



$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} =$$

General Approach: Swapping Basis Labels

More specifically…



$$\longleftrightarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{101} \\ \alpha_{100} \\ \alpha_{111} \\ \alpha_{110} \end{bmatrix} \longleftarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{010} \\ \alpha_{001} \\ \alpha_{011} \\ \alpha_{110} \\ \alpha_{111} \\ \alpha_{100} \\ \alpha_{101} \end{bmatrix} \longleftarrow \begin{array}{c} \\ |000\rangle \\ |010\rangle \\ |001\rangle \\ |011\rangle \\ |100\rangle \\ |110\rangle \\ |101\rangle \\ |111\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} |000\rangle & |010\rangle & |001\rangle & |011\rangle & |100\rangle & |110\rangle & |101\rangle & |111\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \longleftarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{010} \\ \alpha_{001} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{110} \\ \alpha_{101} \\ \alpha_{111} \end{bmatrix} \longleftarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{101} \\ \alpha_{110} \\ \alpha_{111} \end{bmatrix}$$

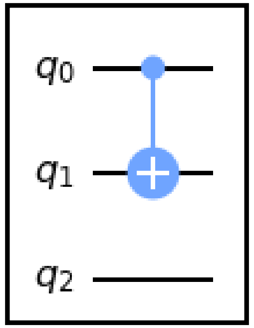
State after CX in Standard Enumeration

State after CX in Swapped Enumeration

State in Swapped Enumeration

State in Standard Enumeration

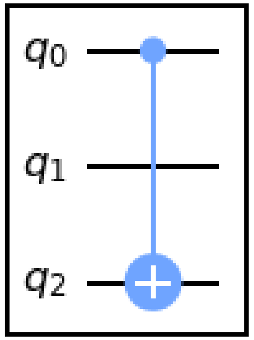
$\updownarrow$



General Approach: Swapping Basis Labels

Question: how do we perform these label swap operations?

More specifically…



$$\longleftrightarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{101} \\ \alpha_{100} \\ \alpha_{111} \\ \alpha_{110} \end{bmatrix} \longleftarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{010} \\ \alpha_{001} \\ \alpha_{011} \\ \alpha_{110} \\ \alpha_{111} \\ \alpha_{100} \\ \alpha_{101} \end{bmatrix} \longleftarrow \begin{array}{c|cccccccc} & |000\rangle & |010\rangle & |001\rangle & |011\rangle & |100\rangle & |110\rangle & |101\rangle & |111\rangle \\ \hline |000\rangle & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ |010\rangle & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ |001\rangle & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ |011\rangle & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ |100\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ |110\rangle & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ |101\rangle & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ |111\rangle & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \longleftarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{010} \\ \alpha_{001} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{110} \\ \alpha_{101} \\ \alpha_{111} \end{bmatrix} \longleftarrow \begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{101} \\ \alpha_{110} \\ \alpha_{111} \end{bmatrix}$$

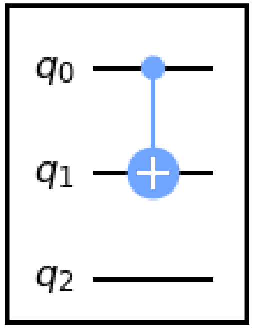
State after CX in Standard Enumeration

State after CX in Swapped Enumeration

State in Swapped Enumeration

State in Standard Enumeration

$\updownarrow$



Answer: Permutation Matrices

Equivalent Ways to Think of Permutations

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 3 & 2 & 5 & 4 & 6 & 7 \end{matrix}$$

shuffling of values

$\longleftrightarrow$

$$\begin{aligned} \pi(0) &\to 0 \\ \pi(1) &\to 1 \\ \pi(2) &\to 3 \\ \pi(3) &\to 2 \\ \pi(4) &\to 5 \\ \pi(5) &\to 4 \\ \pi(6) &\to 6 \\ \pi(7) &\to 7 \end{aligned}$$

1-1 function of a set itself onto itself

Answer: Permutation Matrices

Equivalent Ways to Think of Permutations

0 1 2 3 4 5 6 7

0 1 3 2 5 4 6 7

shuffling of values

$\pi(0) \rightarrow 0$

$\pi(1) \rightarrow 1$

$\pi(2) \rightarrow 3$

$\pi(3) \rightarrow 2$

$\pi(4) \rightarrow 5$

$\pi(5) \rightarrow 4$

$\pi(6) \rightarrow 6$

$\pi(7) \rightarrow 7$

1-1 function of a set itself onto itself

$$\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix}$$

matrix P where $p_{i,j} = 1$ iff ?

Answer: Permutation Matrices

Equivalent Ways to Think of Permutations

0 1 2 3 4 5 6 7

0 1 3 2 5 4 6 7

shuffling of values

$$
\begin{aligned}
\pi(0) &\to 0\\
\pi(1) &\to 1\\
\pi(2) &\to 3\\
\pi(3) &\to 2\\
\pi(4) &\to 5\\
\pi(5) &\to 4\\
\pi(6) &\to 6\\
\pi(7) &\to 7
\end{aligned}
$$

1-1 function of a set itself onto itself

$$
\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix}
$$

matrix P where $p_{i,j} = 1$ iff $\pi(i) = j$

Answer: Permutation Matrices

Equivalent Ways to Think of Permutations

0 1 2 3 4 5 6 7

0 1 3 2 5 4 6 7

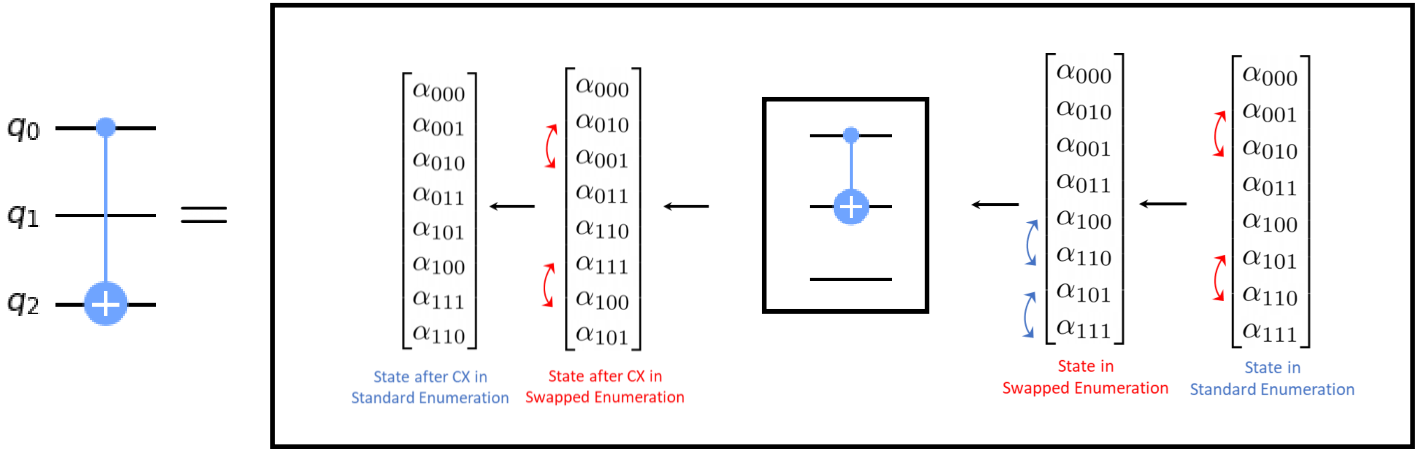
shuffling of values

$\pi(0) \to 0$
$\pi(1) \to 1$
$\pi(2) \to 3$
$\pi(3) \to 2$
$\pi(4) \to 5$
$\pi(5) \to 4$
$\pi(6) \to 6$
$\pi(7) \to 7$

1-1 function of a set itself onto itself

$$\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix}$$

matrix P where $p_{i,j} = 1$ iff $\pi(i) = j$



Answer: Permutation Matrices

matrix multiplication performs label swap

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Equivalent Ways to Think of Permutations

0 1 2 3 4 5 6 7

0 1 3 2 5 4 6 7

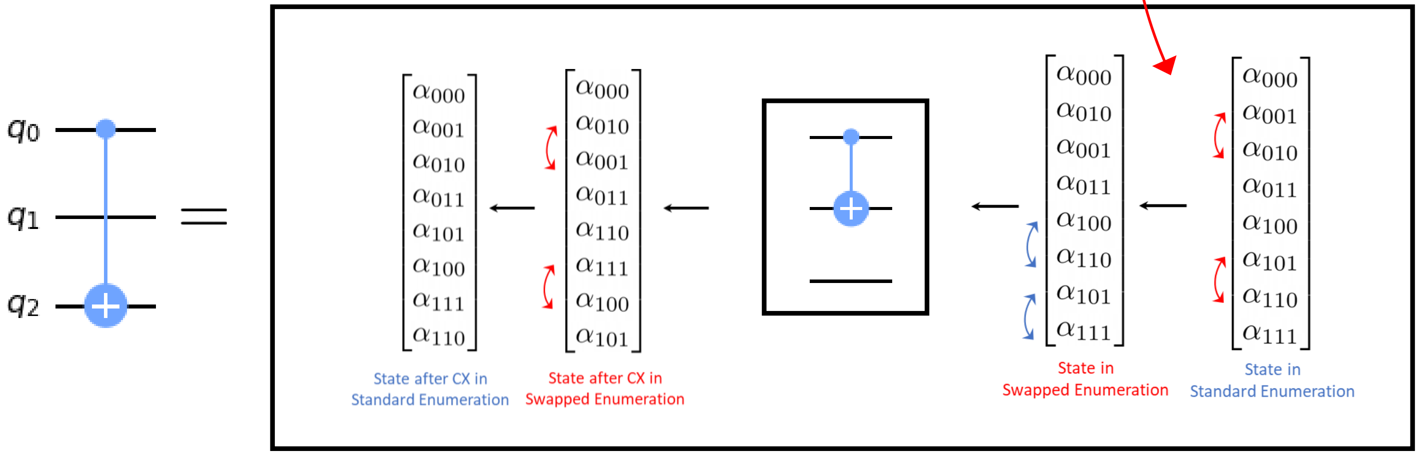
shuffling of values

$$\begin{aligned} \pi(0) &\to 0 \\ \pi(1) &\to 1 \\ \pi(2) &\to 3 \\ \pi(3) &\to 2 \\ \pi(4) &\to 5 \\ \pi(5) &\to 4 \\ \pi(6) &\to 6 \\ \pi(7) &\to 7 \end{aligned}$$

1-1 function of a set itself onto itself

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

matrix P where $p_{i,j} = 1$ iff $\pi(i) = j$



Answer: Permutation Matrices

matrix multiplication performs label swap

$$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}$$

Equivalent Ways to Think of Permutations

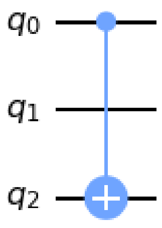
$$\begin{matrix}0&1&2&3&4&5&6&7\\0&1&3&2&5&4&6&7\end{matrix} \longleftrightarrow \begin{aligned}\pi(0)&\to 0\\\pi(1)&\to 1\\\pi(2)&\to 3\\\pi(3)&\to 2\\\pi(4)&\to 5\\\pi(5)&\to 4\\\pi(6)&\to 6\\\pi(7)&\to 7\end{aligned}$$

shuffling of values

1-1 function of a set itself onto itself

$$\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&\boxed{0}&\boxed{1}&0&0&0&0\\0&0&\boxed{1}&\boxed{0}&0&0&0&0\\0&0&0&0&\boxed{0}&\boxed{1}&0&0\\0&0&0&0&\boxed{1}&\boxed{0}&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\2\\3\\4\\5\\6\\7\end{bmatrix}$$

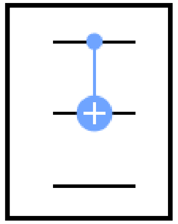
matrix P where $p_{i,j} = 1$ iff $\pi(i) = j$



$=$

$$\begin{bmatrix}\alpha_{000}\\\alpha_{001}\\\alpha_{010}\\\alpha_{011}\\\alpha_{101}\\\alpha_{100}\\\alpha_{111}\\\alpha_{110}\end{bmatrix} \longleftarrow \begin{bmatrix}\alpha_{000}\\\alpha_{010}\\\alpha_{001}\\\alpha_{011}\\\alpha_{110}\\\alpha_{111}\\\alpha_{100}\\\alpha_{101}\end{bmatrix} \longleftarrow$$

State after CX in Standard Enumeration — State after CX in Swapped Enumeration

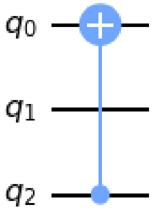


$$\longleftarrow \begin{bmatrix}\alpha_{000}\\\alpha_{010}\\\alpha_{001}\\\alpha_{011}\\\alpha_{100}\\\alpha_{110}\\\alpha_{101}\\\alpha_{111}\end{bmatrix} \longleftarrow \begin{bmatrix}\alpha_{000}\\\alpha_{001}\\\alpha_{010}\\\alpha_{011}\\\alpha_{100}\\\alpha_{101}\\\alpha_{110}\\\alpha_{111}\end{bmatrix}$$

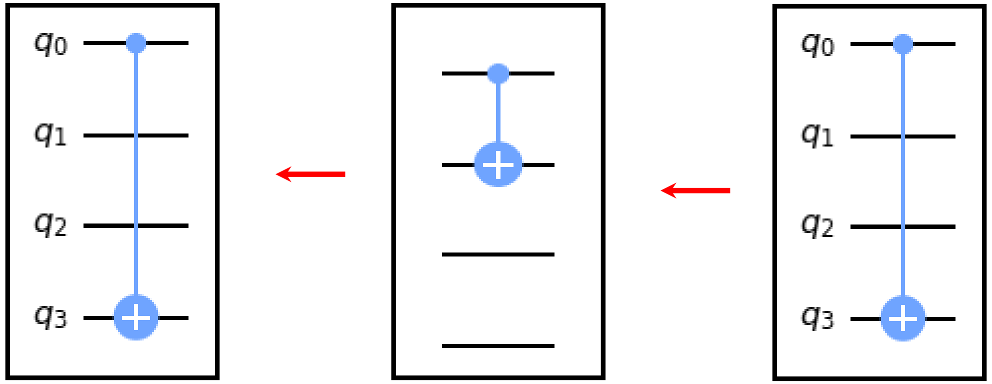
State in Swapped Enumeration — State in Standard Enumeration

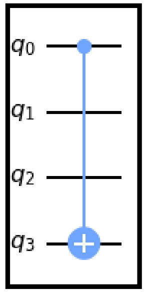
Practice Exercises

1. What's the matrix to performs the "swap back" operation?
2. Prove that any permutation matrix is unitary.
3. Use this method to derive matrix for the gate:



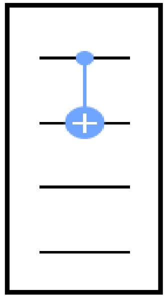
4-qubit Example





$$\leftarrow \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} \underbrace{\left[\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix}\right]}$$

$$P^T = P$$



$$\begin{bmatrix}\alpha_{0000}\\\alpha_{0100}\\\alpha_{0010}\\\alpha_{0110}\\\alpha_{0001}\\\alpha_{0101}\\\alpha_{0011}\\\alpha_{0111}\\\alpha_{1000}\\\alpha_{1100}\\\alpha_{1010}\\\alpha_{1110}\\\alpha_{1001}\\\alpha_{1101}\\\alpha_{1011}\\\alpha_{1111}\end{bmatrix} \leftarrow \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} \begin{bmatrix}\alpha_{0000}\\\alpha_{0001}\\\alpha_{0010}\\\alpha_{0011}\\\alpha_{0100}\\\alpha_{0101}\\\alpha_{0110}\\\alpha_{0111}\\\alpha_{1000}\\\alpha_{1001}\\\alpha_{1010}\\\alpha_{1011}\\\alpha_{1100}\\\alpha_{1101}\\\alpha_{1110}\\\alpha_{1111}\end{bmatrix}$$

$$P$$