Lectures 15-16: Deutsch’s Algorithm

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Deutsch's Problem

Deutsch's Problem

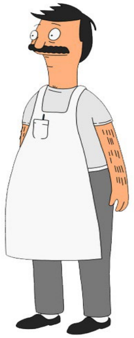
Named after David Deutsch

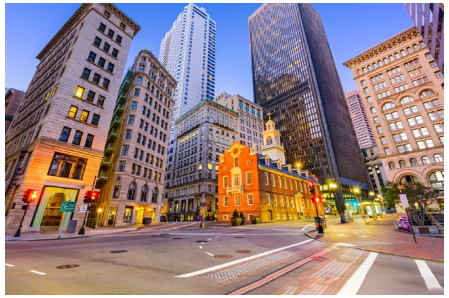
(... many-worlds guy)

Deutsch's Problem

Alice in Amsterdam

Bob in Boston





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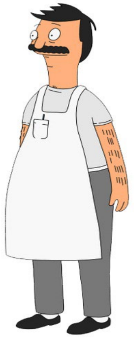
Deutsch's Problem

Alice in Wonderland

Bob in New Jersey



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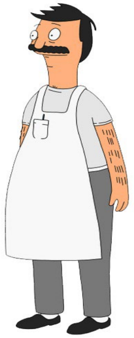

$f : \{0, 1, ..., 2^n - 1\} \rightarrow \{0, 1\}$

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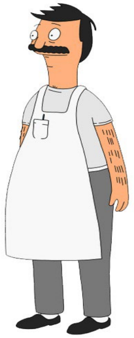

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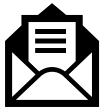

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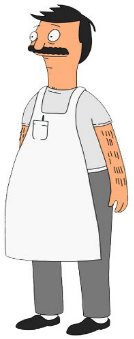

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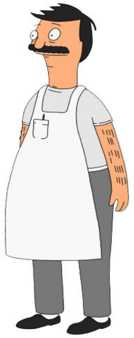

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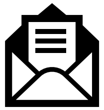
Deutsch's Problem

Alice in Wonderland

What is f(0)?



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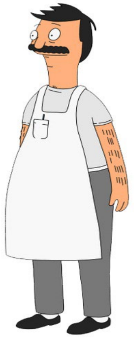

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Alice in Wonderland

What is f(0)?

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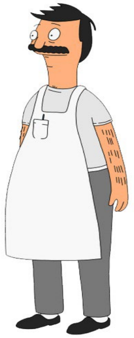
Bob in New Jersey



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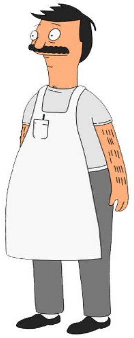

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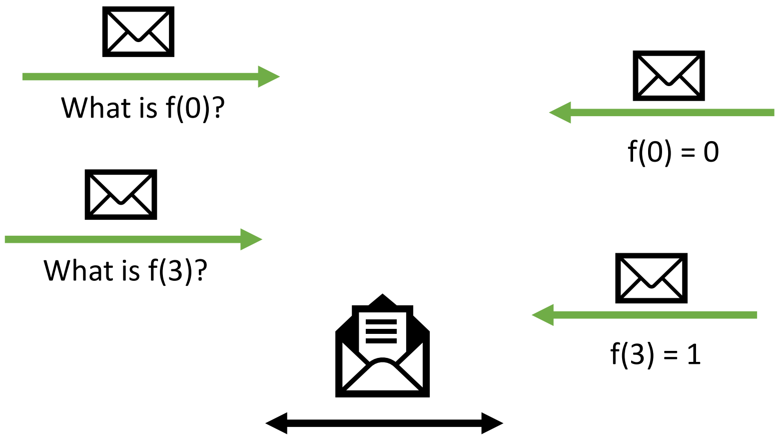

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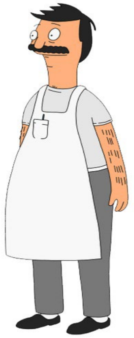

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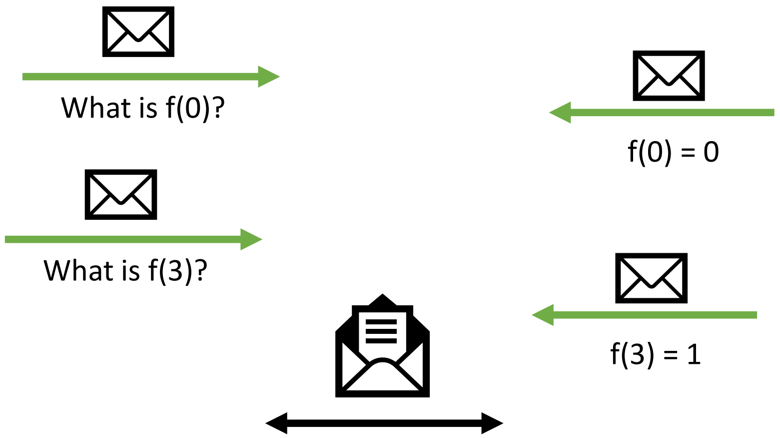

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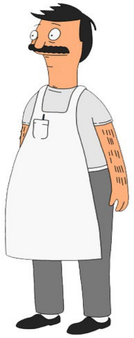

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Best Possible Strategies for Alice

Classical Deterministic	

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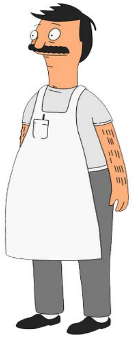

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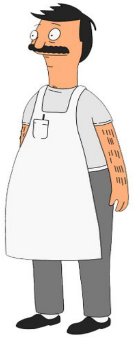

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Classical Deterministic	**Classical Randomized**
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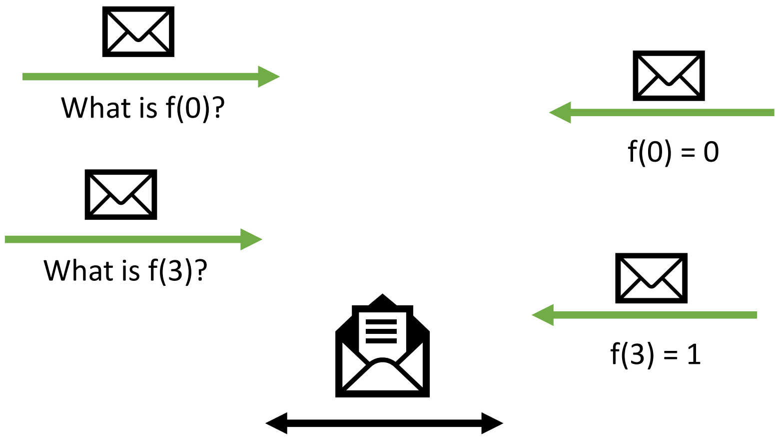

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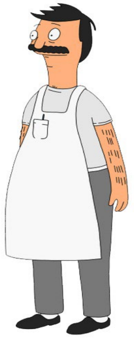

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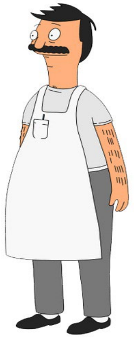

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Example $n = 3$

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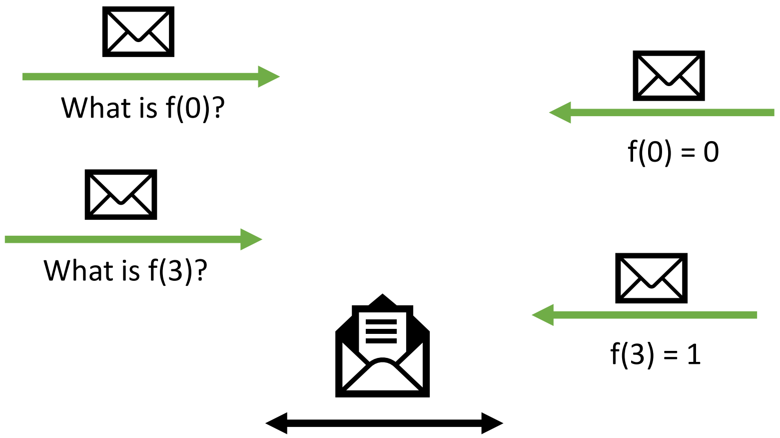

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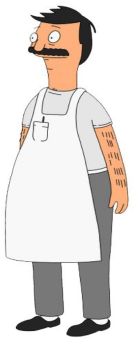

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Best Possible Strategies for Alice

Classical Deterministic

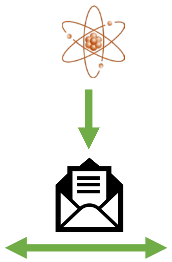
$2^{n-1} + 1 = O(2^n)$ exchanges

(must try at least half + 1 to be sure)

Quantum Deterministic

n+1 qubits in single exchange!

(… if Bob agrees to evaluate f using quantum circuit)



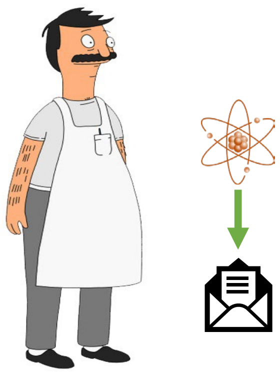
Classical Randomized

$3 = O(1)$ exchanges $\rightarrow$ $\text{correct with prob} = \frac{3}{4}$

$\log_2(n) + 1 = O(\log n)$ $\rightarrow$ $\text{correct with prob} = \frac{1}{n}$?

"with high probability"

Evaluating Function with Quantum Circuit?



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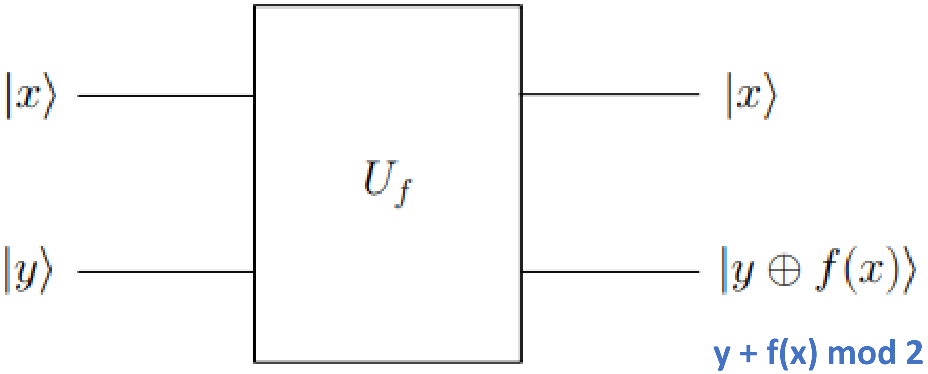

y + f(x) mod 2

"U-f Gate"

Theorem: For any function f, it possible to construct such a matrix that is unitary.
(i.e., quantum computers can perform any computation a classical computer can)

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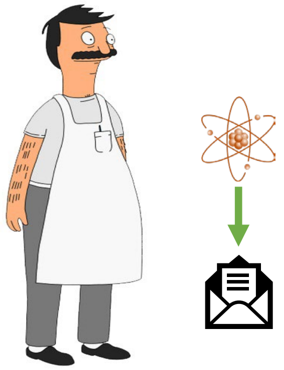

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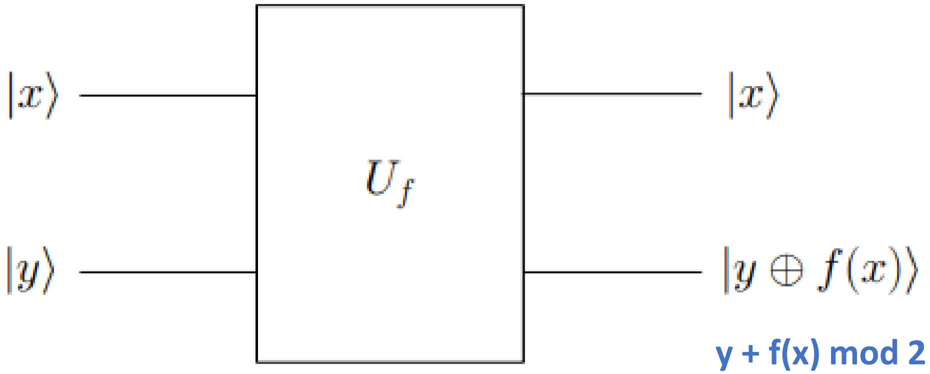

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"U-f Gate"

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Example

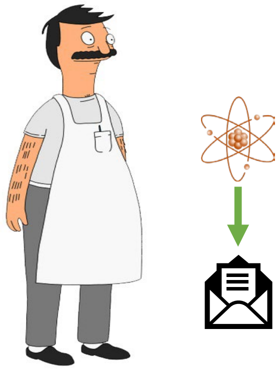
(here $n = 1$)

$$f(0) = 1$$
$$f(1) = 0$$

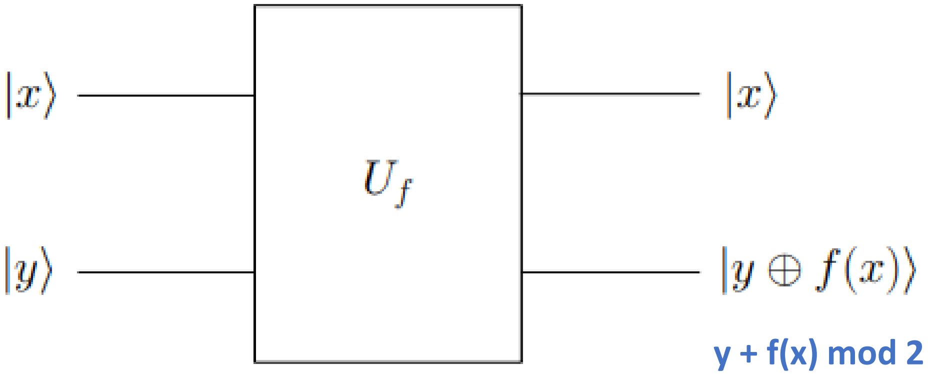
(bit flip)

$$|x, y\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

Evaluating Function with Quantum Circuit?



$$f : \{0, 1, \ldots, 2^n - 1\} \to \{0, 1\}$$



y + f(x) mod 2

"U-f Gate"

Theorem: For any function f, it possible to construct such a matrix that is unitary.
(i.e., quantum computers can perform any computation a classical computer can)

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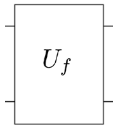

$$\to \alpha|0, 0 \oplus f(0)\rangle + \beta|0, 1 \oplus f(0)\rangle + \gamma|1, 0 \oplus f(1)\rangle + \delta|1, 1 \oplus f(1)\rangle$$

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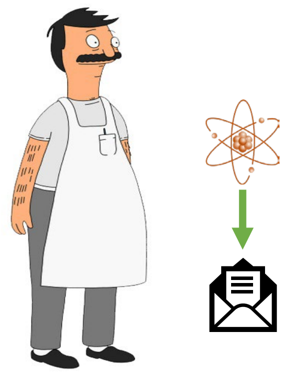

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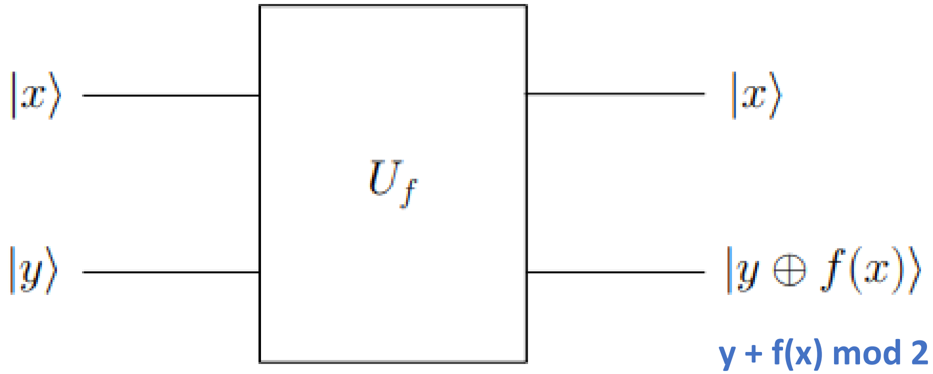

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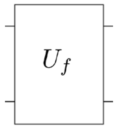

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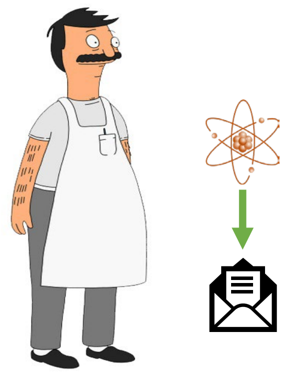

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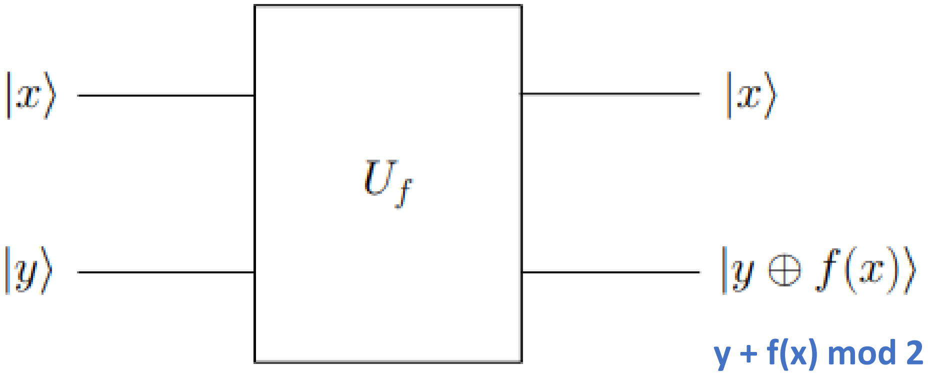

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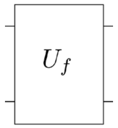

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$$= \alpha|0, 0 \oplus 1\rangle + \beta|0, 1 \oplus 1\rangle + \gamma|1, 0 \oplus 0\rangle + \delta|1, 1 \oplus 0\rangle$$

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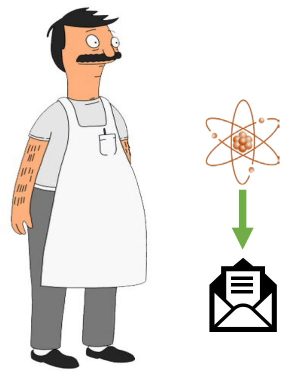

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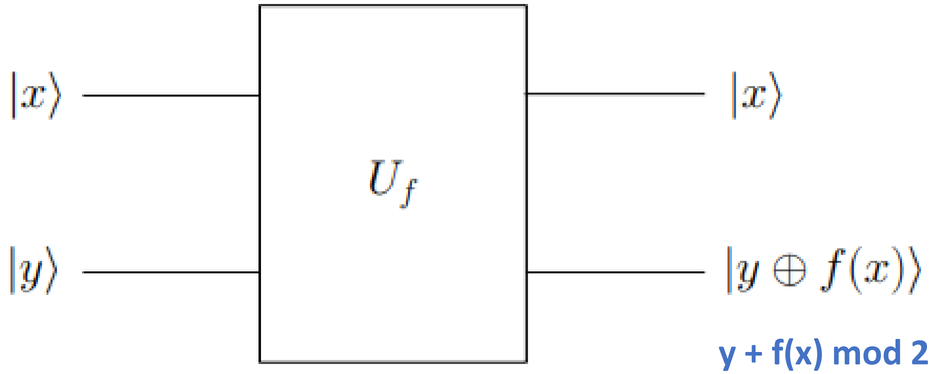

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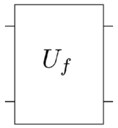

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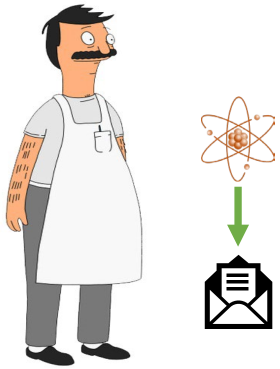

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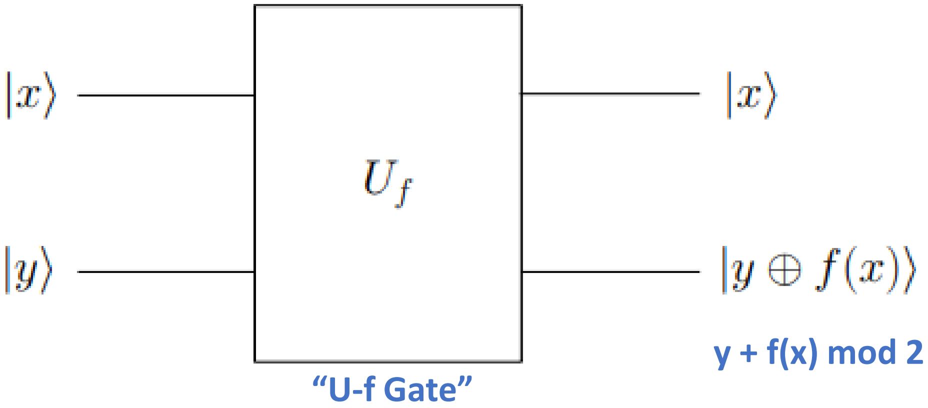

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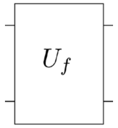

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$$= \alpha|0, \underbrace{0 \oplus 1}_{=1}\rangle + \beta|0, \underbrace{1 \oplus 1}_{=0}\rangle + \gamma|1, \underbrace{0 \oplus 0}_{=0}\rangle + \delta|1, \underbrace{1 \oplus 0}_{=1}\rangle$$

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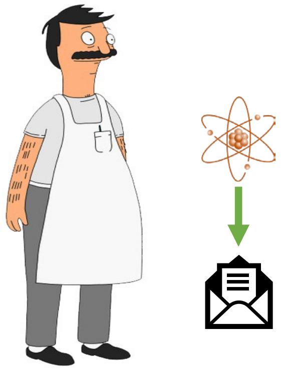

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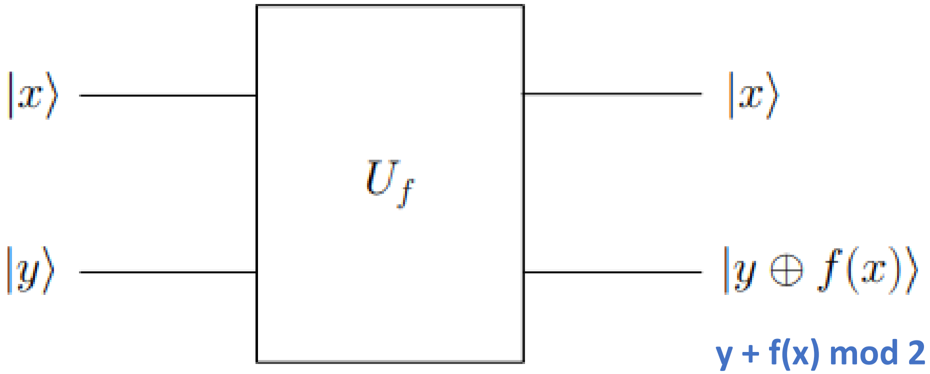

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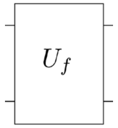

$$\to \alpha|0, 0 \oplus \underbrace{f(0)}_{=1}\rangle + \beta|0, 1 \oplus \underbrace{f(0)}_{=1}\rangle + \gamma|1, 0 \oplus \underbrace{f(1)}_{=0}\rangle + \delta|1, 1 \oplus \underbrace{f(1)}_{=0}\rangle$$

$$= \alpha|\underbrace{0, 0 \oplus 1}_{=1}\rangle + \beta|\underbrace{0, 1 \oplus 1}_{=0}\rangle + \gamma|\underbrace{1, 0 \oplus 0}_{=0}\rangle + \delta|\underbrace{1, 1 \oplus 0}_{=1}\rangle$$

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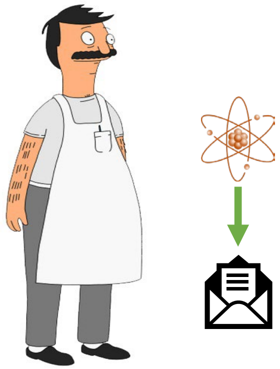

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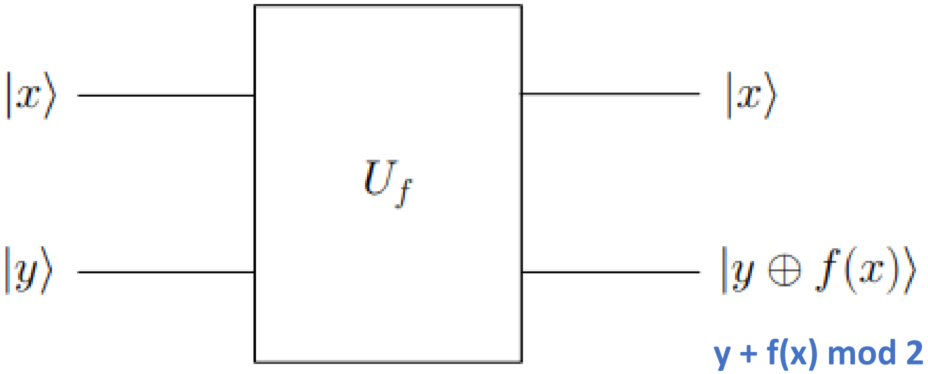

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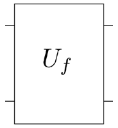

$$|x, y\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

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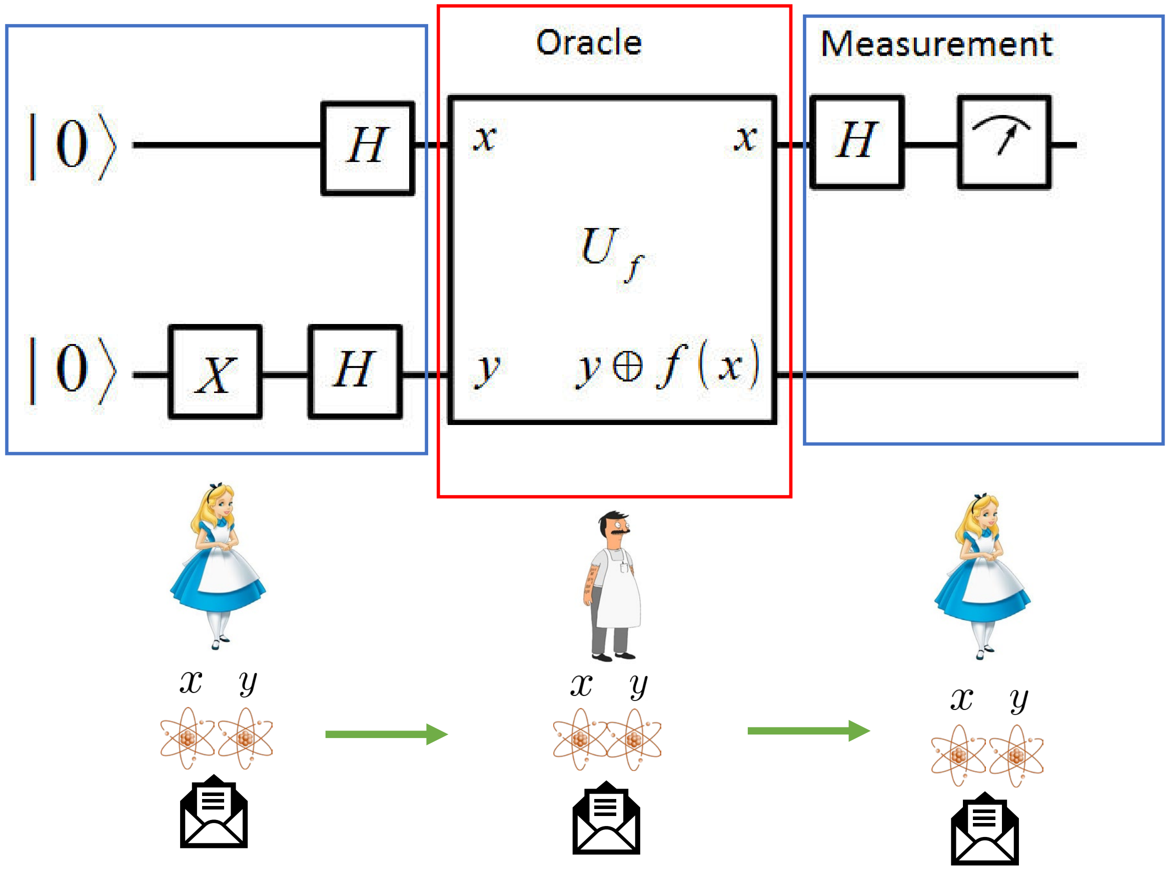
$$= \beta|00\rangle + \alpha|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Unitary Matrix

Deutsch's Algorithm (solves n=1 case)

$n = 1 \implies f : \{0,1\} \to \{0,1\}$



Practice Exercises

(determining what how Alice uses measurement to solve problem)

1. Determine Alice's final quantum state for the following two cases:

$$\boxed{\begin{array}{l} f(0) = 0 \\ f(1) = 1 \end{array}} \quad \text{vs} \quad \boxed{\begin{array}{l} f(0) = 1 \\ f(1) = 1 \end{array}}$$

(possibility for balanced) (possibility for constant)

… based on final state, how does Alice distinguish between two functions after measuring?

2. Follow-up: Design a quantum circuit that implements the U-f gate for each of the functions.

Constant

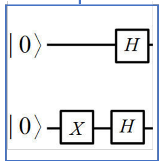
$$f(0) = 0$$
$$f(1) = 0$$

$$f(0) = 1$$
$$f(1) = 1$$

Balanced

$$f(0) = 0$$
$$f(1) = 1$$

Alice Preprocessing



$$f(0) = 1$$
$$f(1) = 0$$

Constant

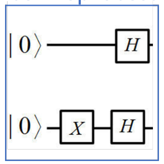
$f(0) = 0$
$f(1) = 0$

$f(0) = 1$
$f(1) = 1$

Balanced

$f(0) = 0$
$f(1) = 1$

Alice Preprocessing



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$f(0) = 1$
$f(1) = 0$

Constant

$$f(0) = 0$$
$$f(1) = 0$$

$$f(0) = 1$$
$$f(1) = 1$$

Balanced

$$f(0) = 0$$
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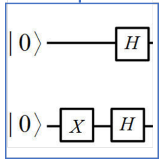
Alice Preprocessing



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

$$f(0) = 1$$
$$f(1) = 0$$

Constant

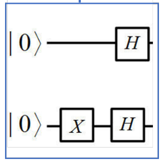
$$f(0) = 0 \qquad f(1) = 0$$

$$f(0) = 1 \qquad f(1) = 1$$

Balanced

$$f(0) = 0 \qquad f(1) = 1$$

Alice Preprocessing



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

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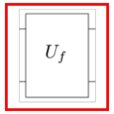


Bob Eval

$$\rightarrow \frac{1}{2}\Big[|0, 0 \oplus f(0)\rangle - |0, 1 \oplus f(0)\rangle + |1, 0 \oplus f(1)\rangle - |1, 1 \oplus f(1)\rangle\Big]$$

$$f(0) = 1 \qquad f(1) = 0$$

Constant

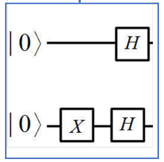
$$\boxed{\begin{aligned} f(0) &= 0 \\ f(1) &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \end{aligned}}$$

Balanced

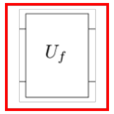
$$\boxed{\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \end{aligned}}$$

Alice Preprocessing



$$\longrightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle - |11\rangle\right]$$



Bob Eval

$$\to \frac{1}{2}\left[|0, \underbrace{0 \oplus f(0)}_{=0}\rangle - |0, \underbrace{1 \oplus f(0)}_{=1}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\right]$$

$$\boxed{\begin{aligned} f(0) &= 1 \\ f(1) &= 0 \end{aligned}}$$

Constant

$$\boxed{\begin{aligned} f(0) &= 0 \\ f(1) &= 0 \end{aligned}}$$

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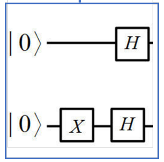

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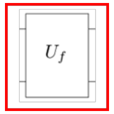
$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$



Bob Eval

$$\rightarrow \frac{1}{2}\Big[\underbrace{|0, 0 \oplus f(0)\rangle}_{=0} - \underbrace{|0, 1 \oplus f(0)\rangle}_{=1} + \underbrace{|1, 0 \oplus f(1)\rangle}_{=1} - \underbrace{|1, 1 \oplus f(1)\rangle}_{=0}\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

$$\boxed{\begin{aligned} f(0) &= 1 \\ f(1) &= 0 \end{aligned}}$$

Constant

$$f(0) = 0$$
$$f(1) = 0$$

$$f(0) = 1$$
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Balanced

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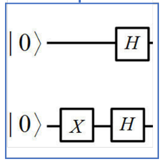

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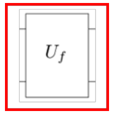

Bob Eval

$$\rightarrow \frac{1}{2}\Big[|0, \underbrace{0 \oplus f(0)}_{=0}\rangle - |0, \underbrace{1 \oplus f(0)}_{=1}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

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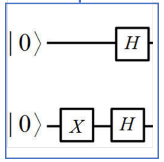

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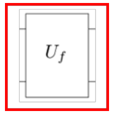

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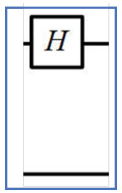
$$\rightarrow \frac{1}{2}\left[|0, \underbrace{0 \oplus f(0)}_{=0}\rangle - |0, \underbrace{1 \oplus f(0)}_{=1}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\right]$$

$$= \frac{1}{2}\left[|00\rangle - |01\rangle - |10\rangle + |11\rangle\right]$$

$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow$$



Alice Postprocessing

$$f(0) = 1$$
$$f(1) = 0$$

Constant

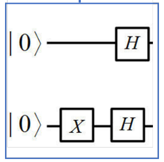
$$f(0) = 0$$
$$f(1) = 0$$

$$f(0) = 1$$
$$f(1) = 1$$

Balanced

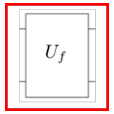
$$f(0) = 0$$
$$f(1) = 1$$

Alice Preprocessing



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

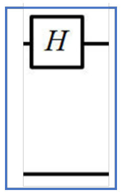


Bob Eval

$$\rightarrow \frac{1}{2}\Big[\underbrace{|0, 0 \oplus f(0)\rangle}_{=0} - \underbrace{|0, 1 \oplus f(0)\rangle}_{=1} + \underbrace{|1, 0 \oplus f(1)\rangle}_{=1} - \underbrace{|1, 1 \oplus f(1)\rangle}_{=0}\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow$$



Alice Postprocessing

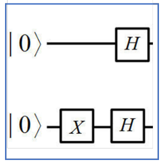
$$\rightarrow |1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(0) = 1$$
$$f(1) = 0$$

Constant

$$f(0) = 0$$
$$f(1) = 0$$

$$f(0) = 1$$
$$f(1) = 1$$



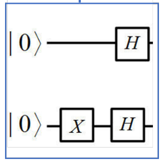
$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

Balanced

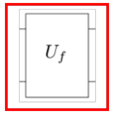
$$f(0) = 0$$
$$f(1) = 1$$

Alice Preprocessing



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

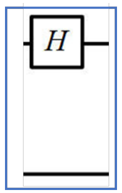


Bob Eval

$$\rightarrow \frac{1}{2}\Big[|0, \underbrace{0 \oplus f(0)}_{=0}\rangle - |0, \underbrace{1 \oplus f(0)}_{=1}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow$$



Alice Postprocessing

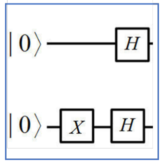
$$\rightarrow |1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(0) = 1$$
$$f(1) = 0$$

Constant

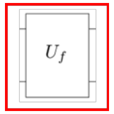
$$f(0) = 0$$
$$f(1) = 0$$

$$f(0) = 1$$
$$f(1) = 1$$



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$



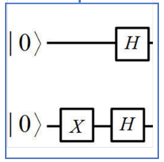
$$\rightarrow \frac{1}{2}\Big[|0, \underbrace{0 \oplus f(0)}_{=1}\rangle - |0, \underbrace{1 \oplus f(0)}_{=0}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\Big]$$

$$= \frac{1}{2}\Big[-|00\rangle + |01\rangle - |10\rangle + |11\rangle\Big]$$

Balanced

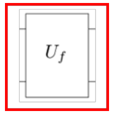
$$f(0) = 0$$
$$f(1) = 1$$

Alice Preprocessing



$$\rightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

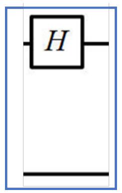


Bob Eval

$$\rightarrow \frac{1}{2}\Big[|0, \underbrace{0 \oplus f(0)}_{=0}\rangle - |0, \underbrace{1 \oplus f(0)}_{=1}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow$$



Alice Postprocessing

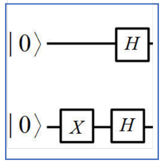
$$\rightarrow |1\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$f(0) = 1$$
$$f(1) = 0$$

Constant

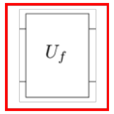
$$f(0) = 0 \qquad f(1) = 0$$

$$f(0) = 1 \qquad f(1) = 1$$



$$\longrightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$



$$\rightarrow \frac{1}{2}\Big[\underbrace{|0, 0 \oplus f(0)\rangle}_{=1} - \underbrace{|0, 1 \oplus f(0)\rangle}_{=0} + \underbrace{|1, 0 \oplus f(1)\rangle}_{=1} - \underbrace{|1, 1 \oplus f(1)\rangle}_{=0}\Big]$$

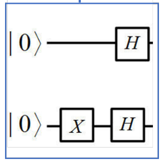
$$= \frac{1}{2}\Big[-|00\rangle + |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Balanced

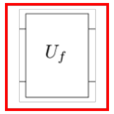
$$f(0) = 0 \qquad f(1) = 1$$

Alice Preprocessing



$$\longrightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

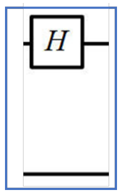


Bob Eval

$$\rightarrow \frac{1}{2}\Big[\underbrace{|0, 0 \oplus f(0)\rangle}_{=0} - \underbrace{|0, 1 \oplus f(0)\rangle}_{=1} + \underbrace{|1, 0 \oplus f(1)\rangle}_{=1} - \underbrace{|1, 1 \oplus f(1)\rangle}_{=0}\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \longrightarrow$$



$$\longrightarrow |1\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

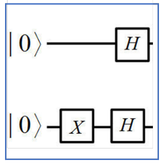
Alice Postprocessing

$$f(0) = 1 \qquad f(1) = 0$$

Constant

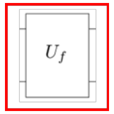
$$f(0) = 0 \quad f(1) = 0$$

$$f(0) = 1 \quad f(1) = 1$$



$$\longrightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

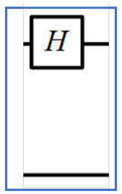
$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$



$$\to \frac{1}{2}\Big[|0, \underbrace{0 \oplus f(0)}_{=1}\rangle - |0, \underbrace{1 \oplus f(0)}_{=0}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\Big]$$

$$= \frac{1}{2}\Big[-|00\rangle + |01\rangle - |10\rangle + |11\rangle\Big]$$

$$= -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \longrightarrow$$

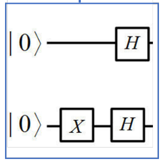


$$\longrightarrow -|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Balanced

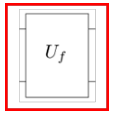
$$f(0) = 0 \quad f(1) = 1$$

Alice Preprocessing



$$\longrightarrow \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big]$$

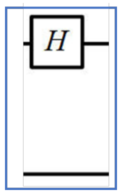


Bob Eval

$$\to \frac{1}{2}\Big[|0, \underbrace{0 \oplus f(0)}_{=0}\rangle - |0, \underbrace{1 \oplus f(0)}_{=1}\rangle + |1, \underbrace{0 \oplus f(1)}_{=1}\rangle - |1, \underbrace{1 \oplus f(1)}_{=0}\rangle\Big]$$

$$= \frac{1}{2}\Big[|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big]$$

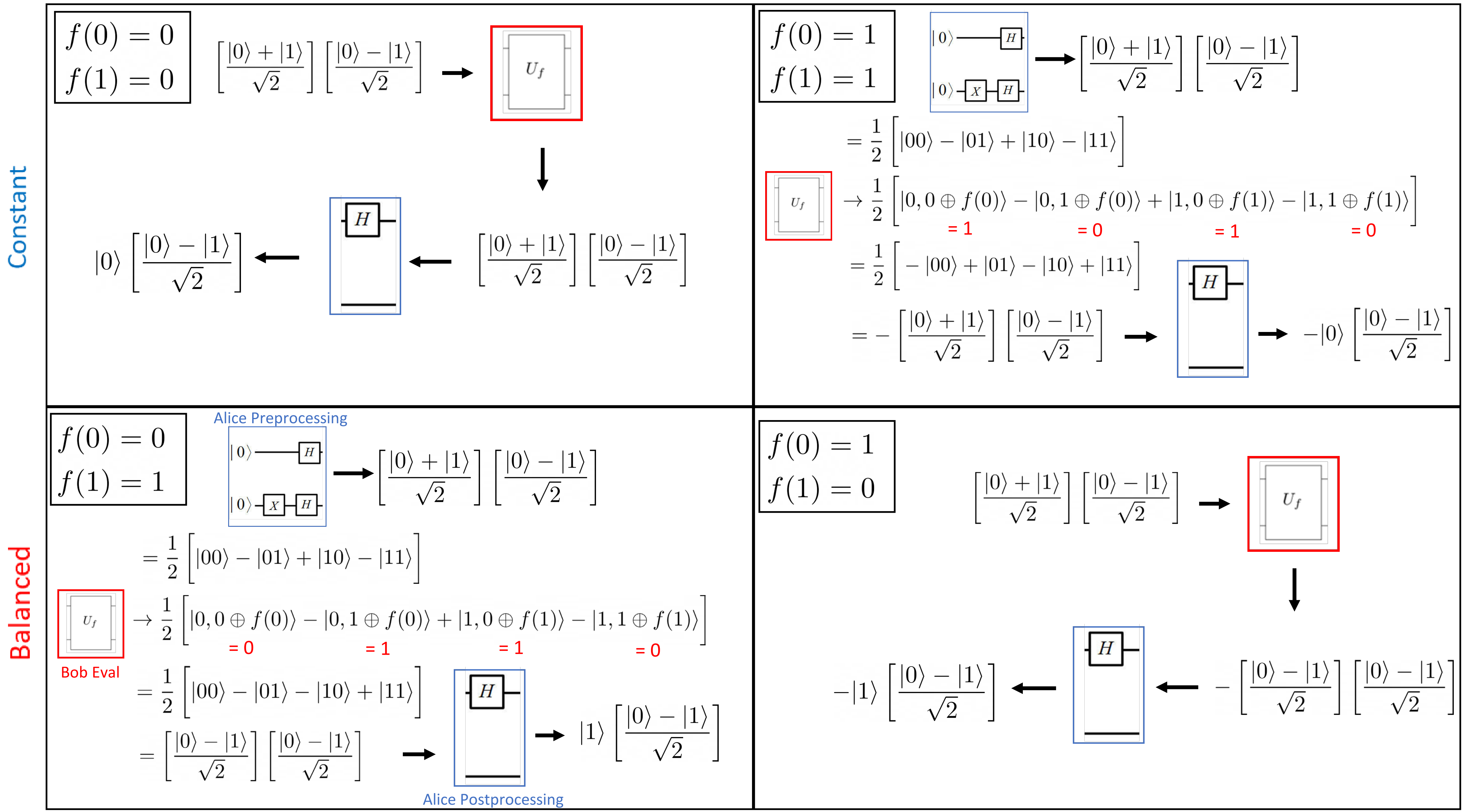
$$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \longrightarrow$$



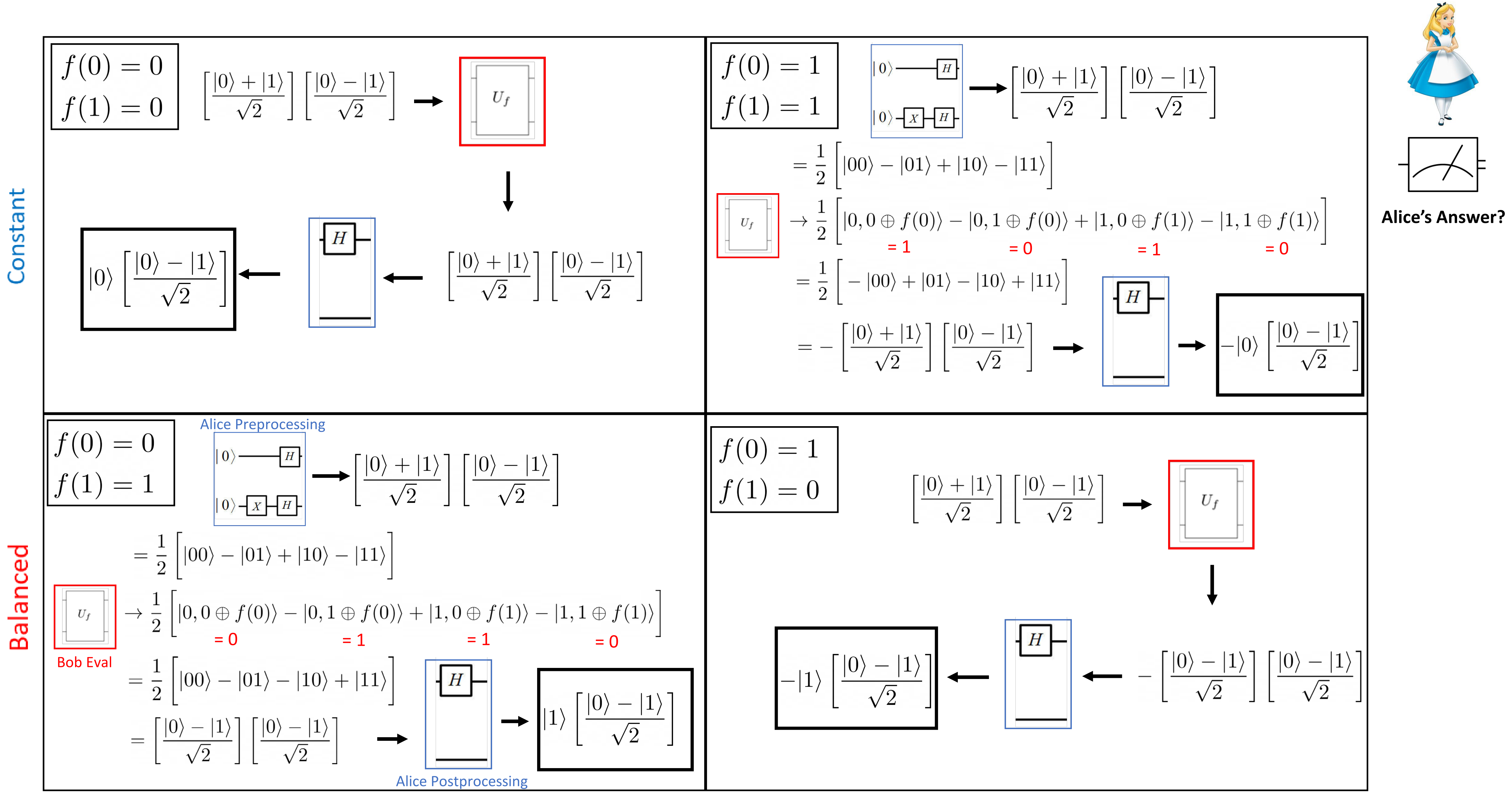
Alice Postprocessing

$$\longrightarrow |1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

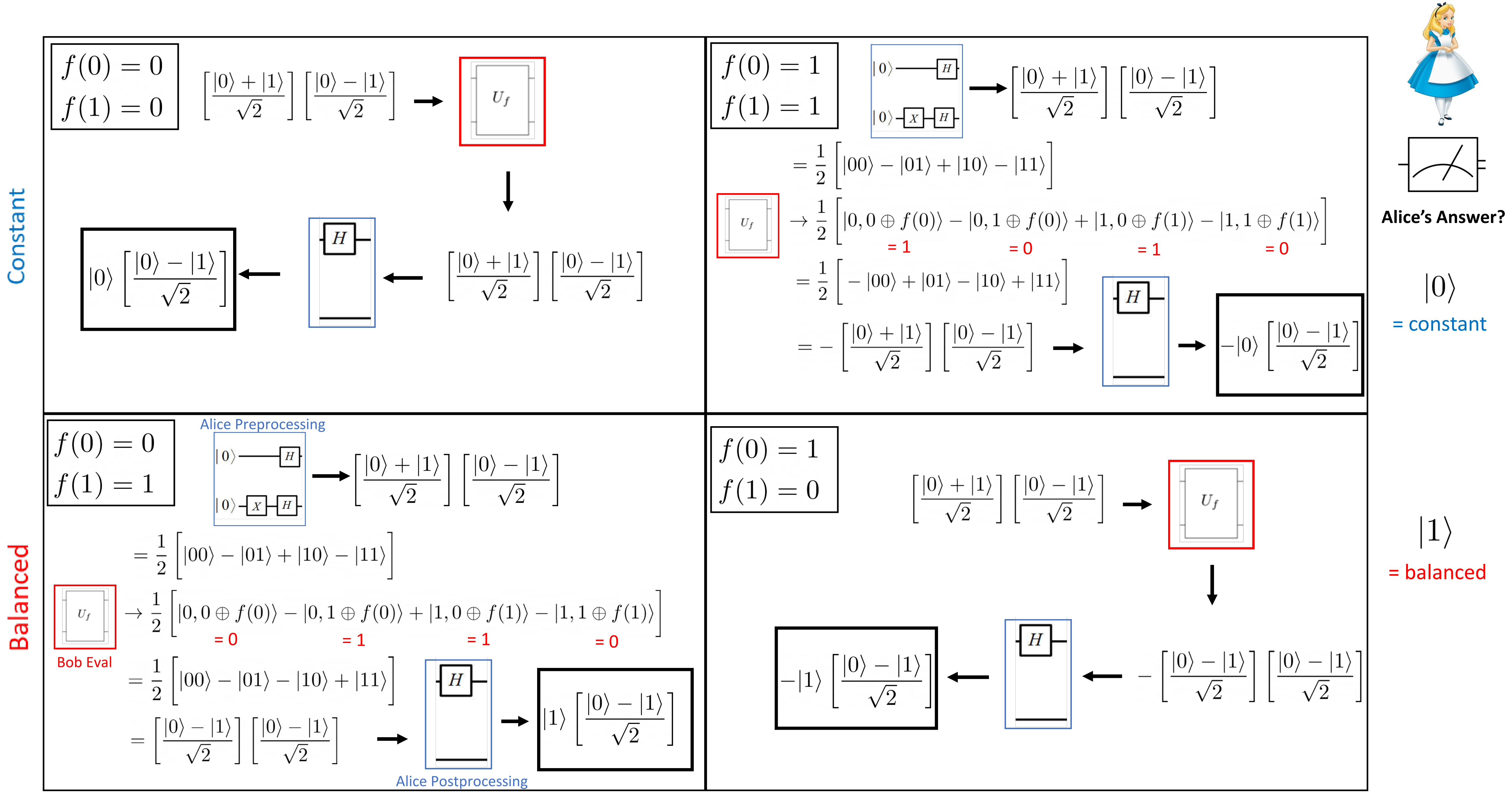
$$f(0) = 1 \quad f(1) = 0$$



Constant
Balanced
f(0) = 0
f(1) = 0
[(|0⟩ + |1⟩)/√2] [(|0⟩ − |1⟩)/√2]
U_f
H
|0⟩ [(|0⟩ − |1⟩)/√2]
f(0) = 1
f(1) = 1
|0⟩
X
= 1/2 [|00⟩ − |01⟩ + |10⟩ − |11⟩]
→ 1/2 [|0, 0 ⊕ f(0)⟩ − |0, 1 ⊕ f(0)⟩ + |1, 0 ⊕ f(1)⟩ − |1, 1 ⊕ f(1)⟩]
= 1
= 0
= 1/2 [− |00⟩ + |01⟩ − |10⟩ + |11⟩]
= − [(|0⟩ + |1⟩)/√2] [(|0⟩ − |1⟩)/√2]
−|0⟩ [(|0⟩ − |1⟩)/√2]
f(0) = 0
f(1) = 1
Alice Preprocessing
Bob Eval
= 1/2 [|00⟩ − |01⟩ − |10⟩ + |11⟩]
= [(|0⟩ − |1⟩)/√2] [(|0⟩ − |1⟩)/√2]
|1⟩ [(|0⟩ − |1⟩)/√2]
Alice Postprocessing
f(0) = 1
f(1) = 0
− [(|0⟩ − |1⟩)/√2] [(|0⟩ − |1⟩)/√2]
−|1⟩ [(|0⟩ − |1⟩)/√2]



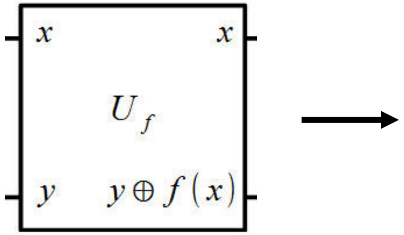
Constant
Balanced
$f(0)=0$ $f(1)=0$
$f(0)=1$ $f(1)=1$
$f(0)=0$ $f(1)=1$
$f(0)=1$ $f(1)=0$
Alice Preprocessing
Bob Eval
Alice Postprocessing
Alice's Answer?
U_f
H
X
$\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
$|0\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
$=\frac{1}{2}\Big[|00\rangle-|01\rangle+|10\rangle-|11\rangle\Big]$
$\rightarrow\frac{1}{2}\Big[|0,0\oplus f(0)\rangle-|0,1\oplus f(0)\rangle+|1,0\oplus f(1)\rangle-|1,1\oplus f(1)\rangle\Big]$
= 1
= 0
= 1
= 0
$=\frac{1}{2}\Big[-|00\rangle+|01\rangle-|10\rangle+|11\rangle\Big]$
$=-\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
$-|0\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
= 0
= 1
= 1
= 0
$=\frac{1}{2}\Big[|00\rangle-|01\rangle-|10\rangle+|11\rangle\Big]$
$=\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
$|1\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
$-\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$
$-|1\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$


Constant
Balanced
f(0) = 0
f(1) = 0
f(0) = 1
f(1) = 1
f(0) = 0
f(1) = 1
f(0) = 1
f(1) = 0
Alice Preprocessing
Bob Eval
Alice Postprocessing
= 1
= 0
= 1
= 0
= 0
= 1
= 1
= 0
Alice's Answer?
|0⟩
= constant
|1⟩
= balanced

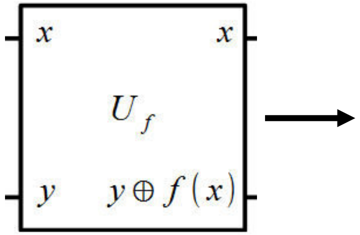
Uf Gate Implementations?

Constant

$f(0) = 0$
$f(1) = 0$

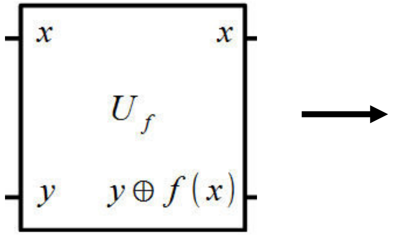


$f(0) = 1$
$f(1) = 1$

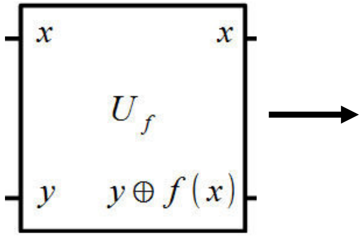


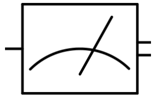
Balanced

$f(0) = 0$
$f(1) = 1$



$f(0) = 1$
$f(1) = 0$





Alice's Answer?

$|0\rangle$

= constant

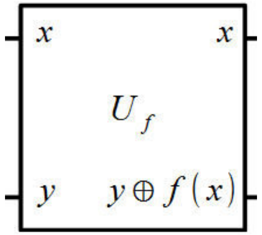
$|1\rangle$

= balanced

Uf Gate Implementations?

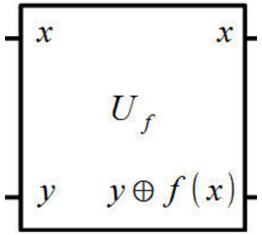
Constant

$$f(0) = 0$$
$$f(1) = 0$$



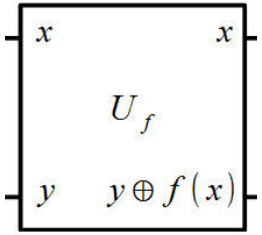
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$f(0) = 1$$
$$f(1) = 1$$

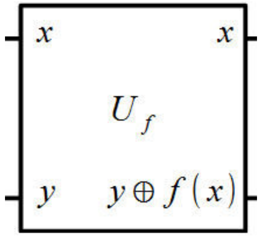


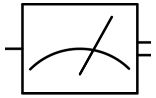
Balanced

$$f(0) = 0$$
$$f(1) = 1$$



$$f(0) = 1$$
$$f(1) = 0$$





Alice's Answer?

$|0\rangle$

= constant

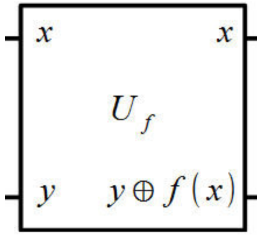
$|1\rangle$

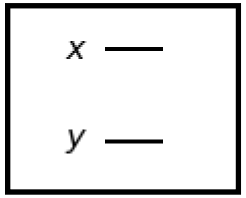
= balanced

Uf Gate Implementations?

Constant

$$f(0) = 0$$
$$f(1) = 0$$

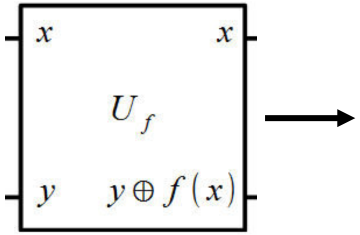




(do nothing)

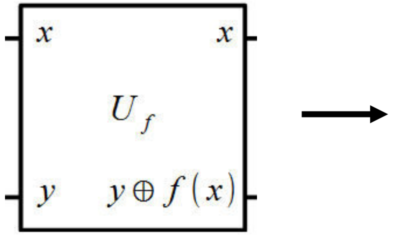
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \to \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$f(0) = 1$$
$$f(1) = 1$$

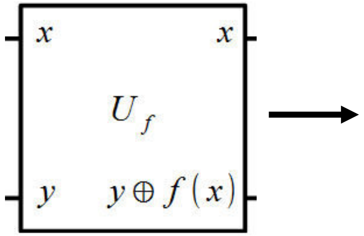


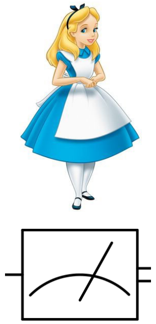
Balanced

$$f(0) = 0$$
$$f(1) = 1$$



$$f(0) = 1$$
$$f(1) = 0$$





Alice's Answer?

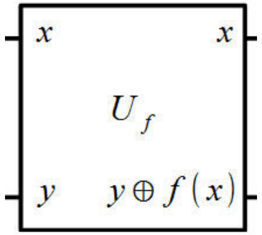
$|0\rangle$ = constant

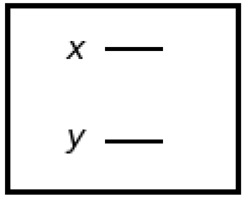
$|1\rangle$ = balanced

Uf Gate Implementations?

Constant

$f(0) = 0$
$f(1) = 0$

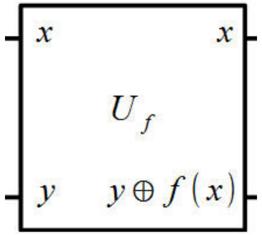




(do nothing)

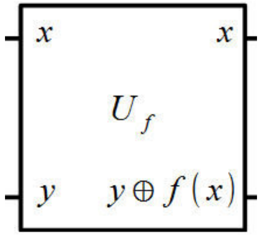
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$f(0) = 1$
$f(1) = 1$



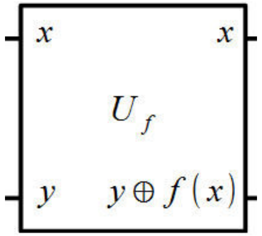
Balanced

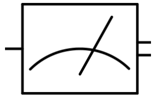
$f(0) = 0$
$f(1) = 1$



$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

$f(0) = 1$
$f(1) = 0$





Alice's Answer?

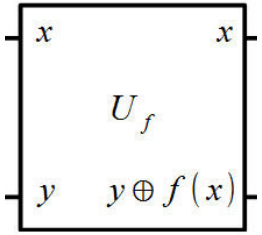
$|0\rangle$ = constant

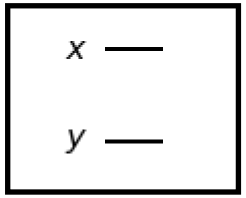
$|1\rangle$ = balanced

Uf Gate Implementations?

Constant

$$f(0) = 0$$
$$f(1) = 0$$

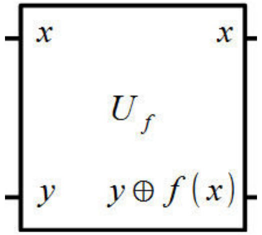




(do nothing)

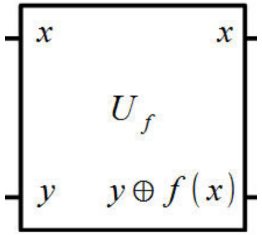
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

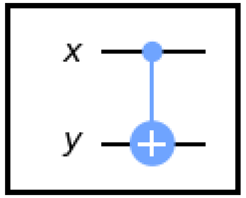
$$f(0) = 1$$
$$f(1) = 1$$



Balanced

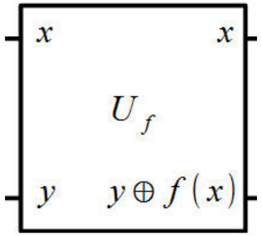
$$f(0) = 0$$
$$f(1) = 1$$

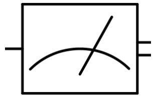




$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

$$f(0) = 1$$
$$f(1) = 0$$





Alice's Answer?

$|0\rangle$

= constant

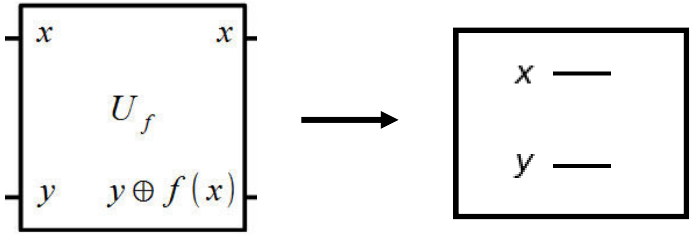
$|1\rangle$

= balanced

Uf Gate Implementations?

Constant

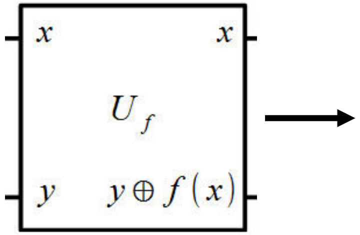
$f(0) = 0$
$f(1) = 0$



(do nothing)

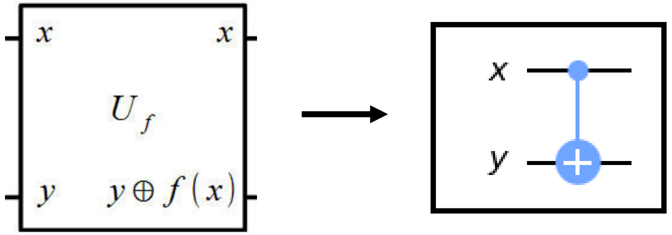
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$f(0) = 1$
$f(1) = 1$



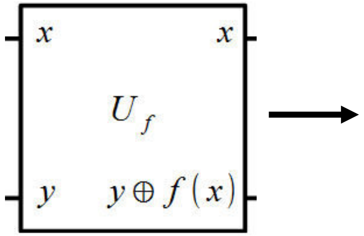
Balanced

$f(0) = 0$
$f(1) = 1$

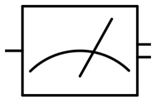


$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

$f(0) = 1$
$f(1) = 0$



$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \beta|00\rangle + \alpha|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

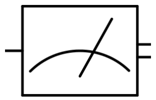


Alice's Answer?

$|0\rangle$
= constant

$|1\rangle$
= balanced

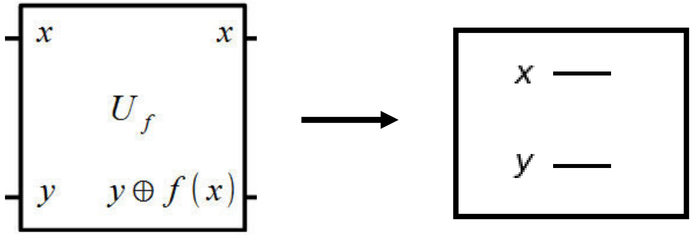
Uf Gate Implementations?



Alice's Answer?

Constant

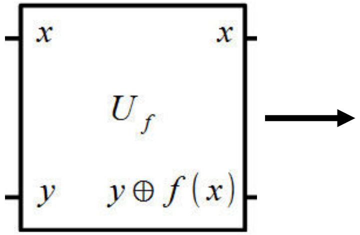
$f(0) = 0$
$f(1) = 0$



(do nothing)

$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$f(0) = 1$
$f(1) = 1$

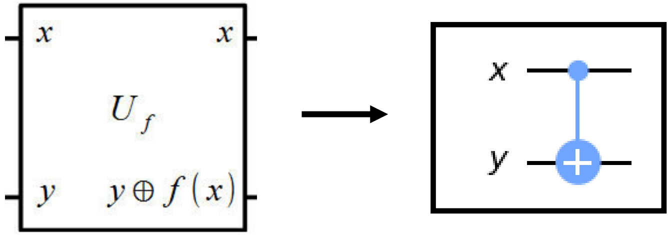


$|0\rangle$

= constant

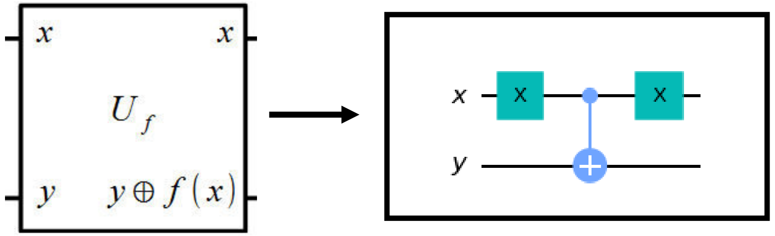
Balanced

$f(0) = 0$
$f(1) = 1$



$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

$f(0) = 1$
$f(1) = 0$



$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \beta|00\rangle + \alpha|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

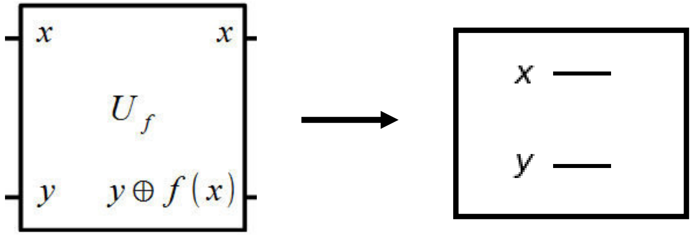
$|1\rangle$

= balanced

Uf Gate Implementations?

Constant

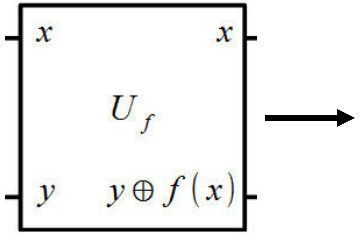
$f(0)=0$
$f(1)=0$



(do nothing)

$$\alpha|00\rangle+\beta|01\rangle+\gamma|10\rangle+\delta|11\rangle \rightarrow \alpha|00\rangle+\beta|01\rangle+\gamma|10\rangle+\delta|11\rangle$$

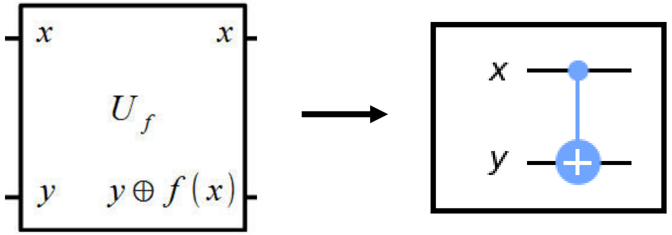
$f(0)=1$
$f(1)=1$



$$\alpha|00\rangle+\beta|01\rangle+\gamma|10\rangle+\delta|11\rangle \rightarrow \beta|00\rangle+\alpha|01\rangle+\delta|10\rangle+\gamma|11\rangle$$

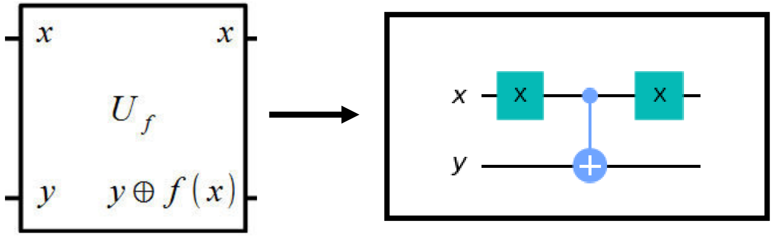
Balanced

$f(0)=0$
$f(1)=1$

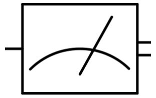


$$\alpha|00\rangle+\beta|01\rangle+\gamma|10\rangle+\delta|11\rangle \rightarrow \alpha|00\rangle+\beta|01\rangle+\delta|10\rangle+\gamma|11\rangle$$

$f(0)=1$
$f(1)=0$



$$\alpha|00\rangle+\beta|01\rangle+\gamma|10\rangle+\delta|11\rangle \rightarrow \beta|00\rangle+\alpha|01\rangle+\gamma|10\rangle+\delta|11\rangle$$



Alice's Answer?

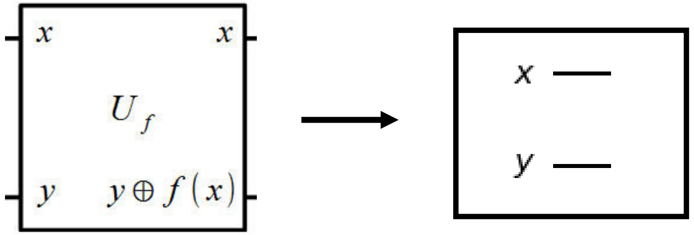
$|0\rangle$ = constant

$|1\rangle$ = balanced

Uf Gate Implementations?

Constant

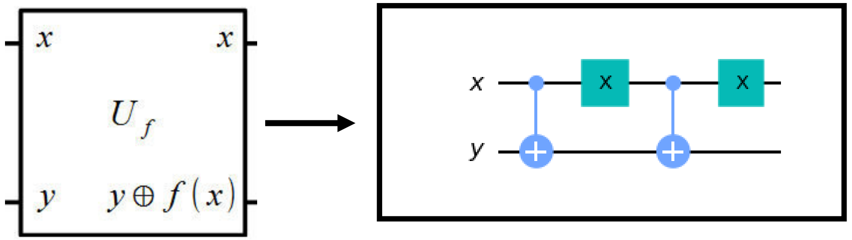
$$f(0) = 0 \quad f(1) = 0$$



(do nothing)

$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

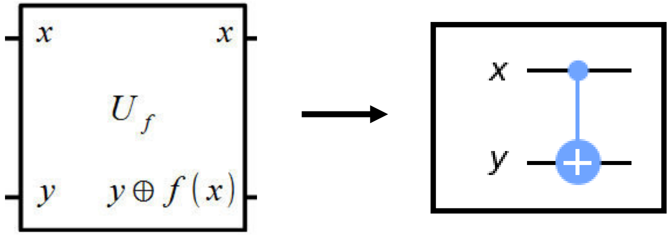
$$f(0) = 1 \quad f(1) = 1$$



$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \beta|00\rangle + \alpha|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

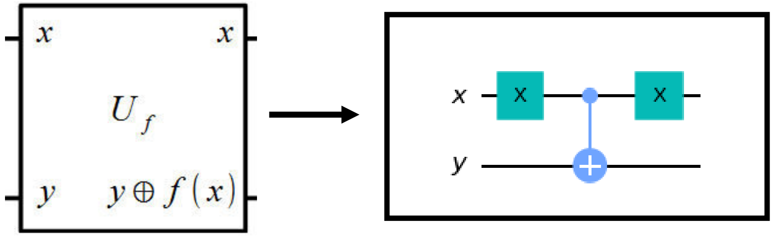
Balanced

$$f(0) = 0 \quad f(1) = 1$$

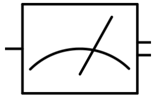


$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

$$f(0) = 1 \quad f(1) = 0$$



$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \beta|00\rangle + \alpha|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

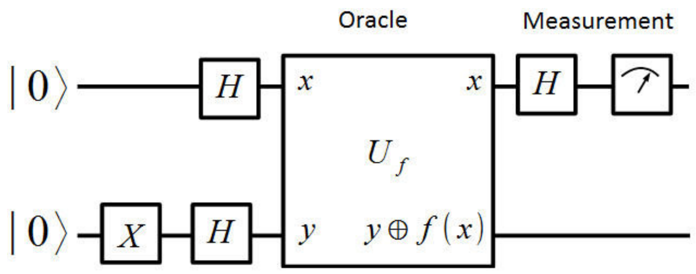


Alice's Answer?

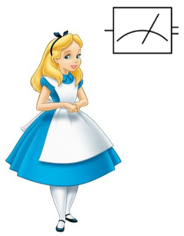
$|0\rangle$ = constant

$|1\rangle$ = balanced

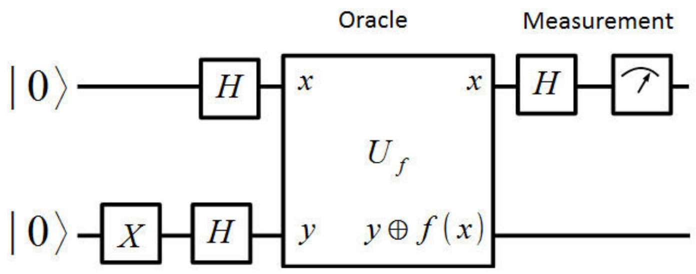
Deutsch's Analysis Summary



Constant	$f(0)=0$, $f(1)=0$ x y $\left[\frac{\lvert 0\rangle+\lvert 1\rangle}{\sqrt{2}}\right]\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right] \rightarrow \lvert 0\rangle\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$ **Before Final Hadamard** **Final State**	$f(0)=1$, $f(1)=1$ $-\left[\frac{\lvert 0\rangle+\lvert 1\rangle}{\sqrt{2}}\right]\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$ $\downarrow$ $-\lvert 0\rangle\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$	$x = \lvert 0\rangle$ -> constant
Balanced	$f(0)=0$, $f(1)=1$ $\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$ $\downarrow$ $\lvert 1\rangle\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$	$f(0)=1$, $f(1)=0$ $-\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$ $\downarrow$ $-\lvert 1\rangle\left[\frac{\lvert 0\rangle-\lvert 1\rangle}{\sqrt{2}}\right]$	$x = \lvert 1\rangle$ -> balanced



Deutsch's Analysis Summary



Constant

$f(0) = 0$
$f(1) = 0$

x ↓ y ↓

$$\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow |0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Before Final Hadamard **Final State**

$f(0) = 1$
$f(1) = 1$

$$-\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |0\rangle$
-> constant

Balanced

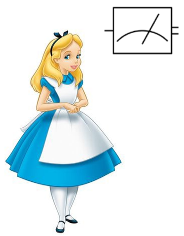
$f(0) = 0$
$f(1) = 1$

$$\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

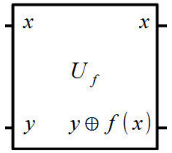
$f(0) = 1$
$f(1) = 0$

$$-\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |1\rangle$
-> balanced



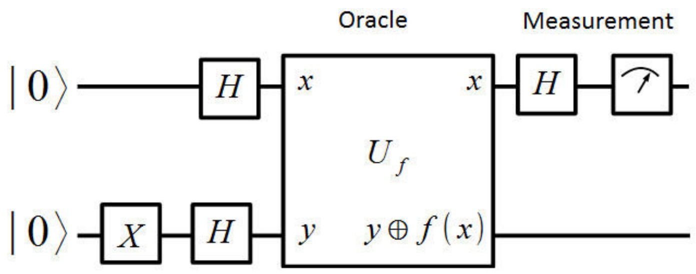
"Compact" Expressions



on $|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] =$

$x \in \{0, 1\}$

Deutsch's Analysis Summary



Constant

$f(0) = 0$
$f(1) = 0$

$$\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow |0\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

x y

Before Final Hadamard

Final State

$f(0) = 1$
$f(1) = 1$

$$-\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|0\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |0\rangle$
-> constant

Balanced

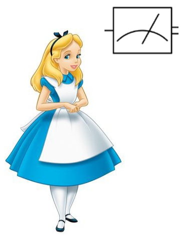
$f(0) = 0$
$f(1) = 1$

$$\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$|1\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

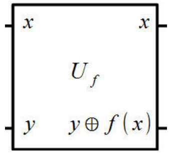
$f(0) = 1$
$f(1) = 0$

$$-\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|1\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |1\rangle$
-> balanced



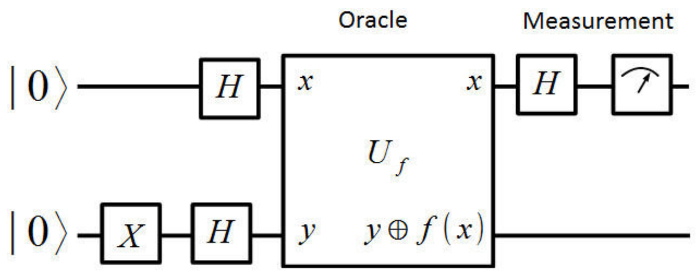
"Compact" Expressions



on $|x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] =$

$x \in \{0, 1\}$

Deutsch's Analysis Summary



Constant

$f(0) = 0$
$f(1) = 0$

$$\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow |0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

x y

Before Final Hadamard

Final State

$f(0) = 1$
$f(1) = 1$

$$-\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |0\rangle$
-> constant

Balanced

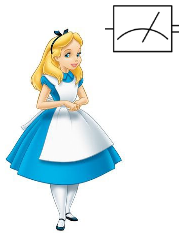
$f(0) = 0$
$f(1) = 1$

$$\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

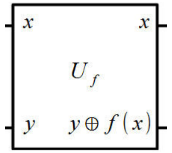
$f(0) = 1$
$f(1) = 0$

$$-\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |1\rangle$
-> balanced



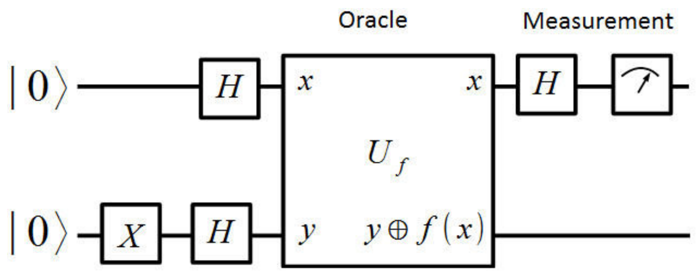
"Compact" Expressions



$$\text{on } |x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = (-1)^{f(x)}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x \in \{0, 1\}$

Deutsch's Analysis Summary



Constant

$f(0) = 0$
$f(1) = 0$

x y

$$\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \rightarrow |0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Before Final Hadamard — **Final State**

$f(0) = 1$
$f(1) = 1$

$$-\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |0\rangle$
-> constant

Balanced

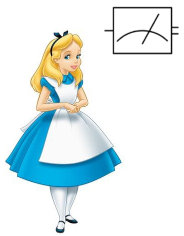
$f(0) = 0$
$f(1) = 1$

$$\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

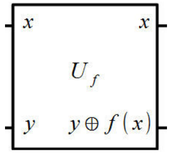
$f(0) = 1$
$f(1) = 0$

$$-\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$
$$\downarrow$$
$$-|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x = |1\rangle$
-> balanced



"Compact" Expressions



$$\text{on } |x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = (-1)^{f(x)}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x \in \{0, 1\}$

Before Final Hadamard

$$\pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if constant}$$

$$\pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if balanced}$$

Final State

$$\pm|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if constant}$$

$$\pm|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if balanced}$$