Lectures 17-19: Deutsch-Jozsa Algorithm

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Review: Deutsch's Problem

Alice in Wonderland

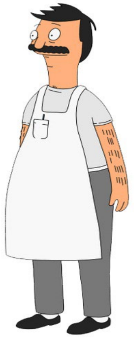
What is f(0)?

What is f(3)?

f(0) = 0

f(3) = 1

Bob in New Jersey



$$f : \{0, 1, \ldots, 2^n - 1\} \to \{0, 1\}$$

1. Alice and Bob are in separate locations. Can only communicate by mail.
2. Bob initially picks a binary-output function f that is either constant or balanced.
3. Alice can send a single value each time in mail for Bob to evaluate. Bob sends back answer.
4. **Alice's Goal:** Determine which kind of function Bob picked in as few exchanges as possible.

Best Possible Strategies for Alice

Classical Deterministic

$2^{n-1} + 1 = O(2^n)$ exchanges

(must try at least half + 1 to be sure)

Quantum Deterministic

n+1 qubits in single exchange!

(… if Bob agrees to evaluate f using quantum circuit)

Classical Randomized

$3 = O(1)$ exchanges $\rightarrow$ correct with prob $= \frac{3}{4}$

$\log_2(n) + 1 = O(\log n)$ $\rightarrow$ correct with prob $= \frac{1}{n}$?

"with high probability"

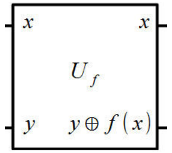
Suppose Bob picks balanced function

Example $n = 3$

Constant	Balanced ✓
$f(0) = 0$	$f(0) = 0$
$f(1) = 0$	$f(1) = 1$
$f(2) = 0$	$f(2) = 1$
$f(3) = 0$	$f(3) = 1$
$f(4) = 0$	$f(4) = 0$
$f(5) = 0$	$f(5) = 0$
$f(6) = 0$	$f(6) = 0$
$f(7) = 0$	$f(7) = 1$
function always evaluates to 0 or 1	half evaluate to 0, half evaluate to 1

Review: Deutsch’s Algorithm

Action of U-f Gate



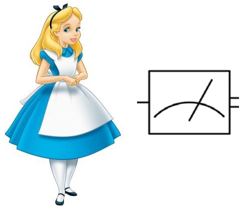
$$\text{on } |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = (-1)^{f(x)}|x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x \in \{0, 1\}$

Before Final Hadamard

$$\pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if constant}$$

$$\pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if balanced}$$



Final State

$$\pm|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if constant}$$

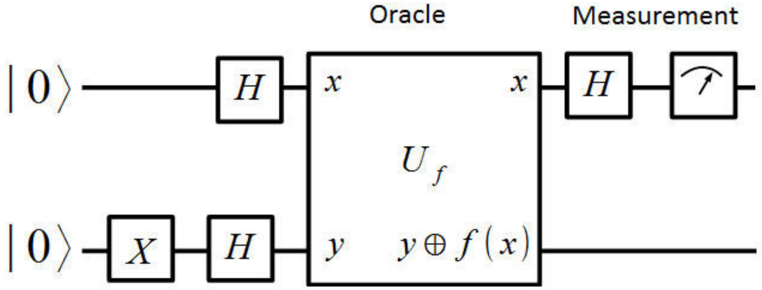
$$\pm|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{if balanced}$$

$\rightarrow$

$x = |0\rangle$
-> constant

$x = |1\rangle$
-> balanced

Deutsch’s Algorithm (n=1)



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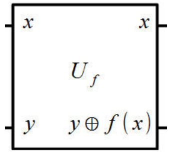

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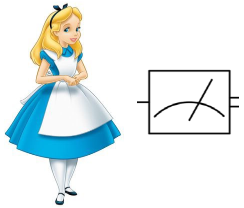


Final State

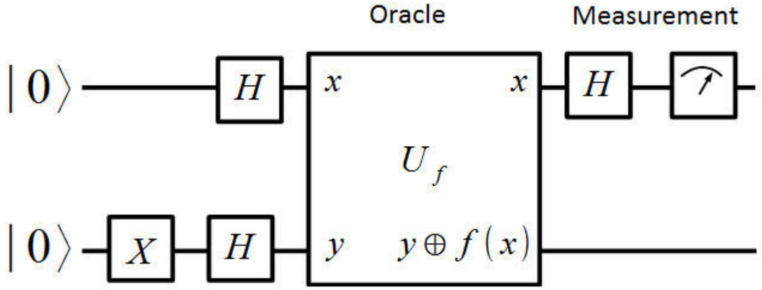
$$\pm|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$ if constant

$$\pm|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$ if balanced

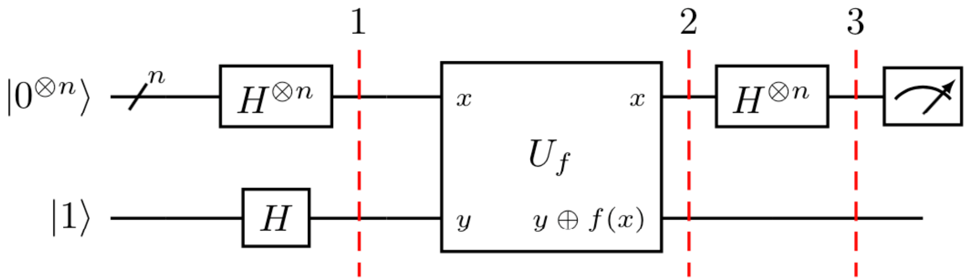
$x = |0\rangle$
-> constant

$x = |1\rangle$
-> balanced

Deutsch's Algorithm (n=1)

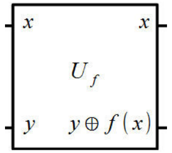


This Lecture: Deutsch-Jozsa Algorithm (general n)



Review: Deutsch’s Algorithm

Action of U-f Gate



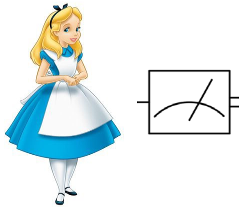
$$\text{on } |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = (-1)^{f(x)}|x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$x \in \{0, 1\}$

Before Final Hadamard

$$\pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$ if constant

$$\pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$ if balanced



Final State

$$\pm|0\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$ if constant

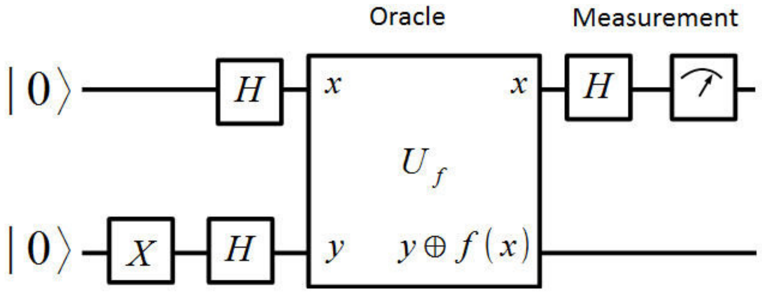
$$\pm|1\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$ if balanced

$\longrightarrow$

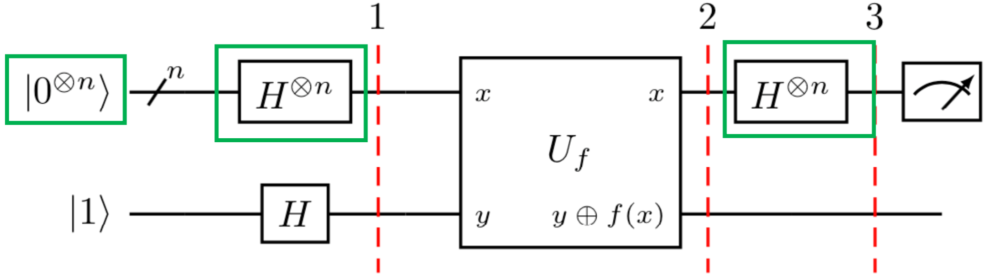
$x = |0\rangle$
-> constant

$x = |1\rangle$
-> balanced

Deutsch’s Algorithm (n=1)



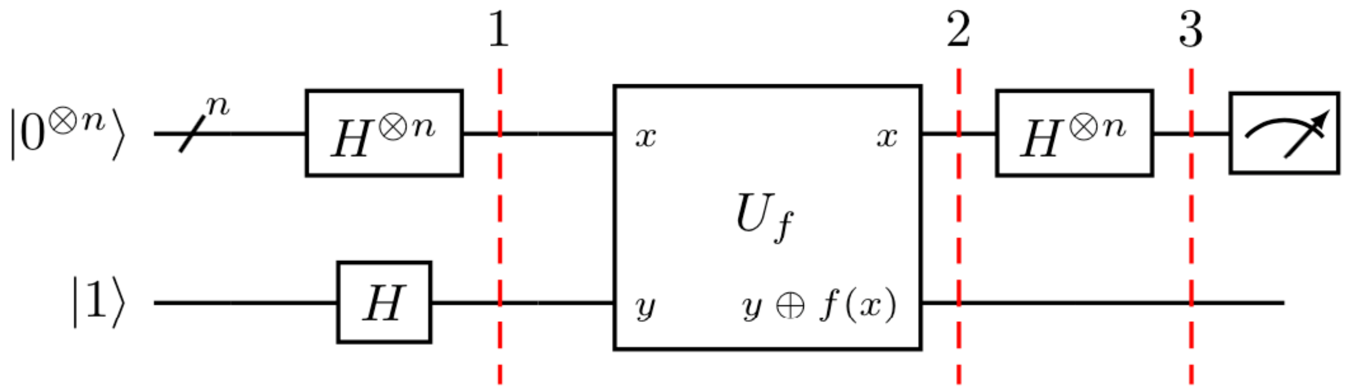
This Lecture: Deutsch-Jozsa Algorithm (general n)



Only Differences:

- x is now an n qubit register.
- Hadamard gates on x are now n-qubit Hadamard gates.

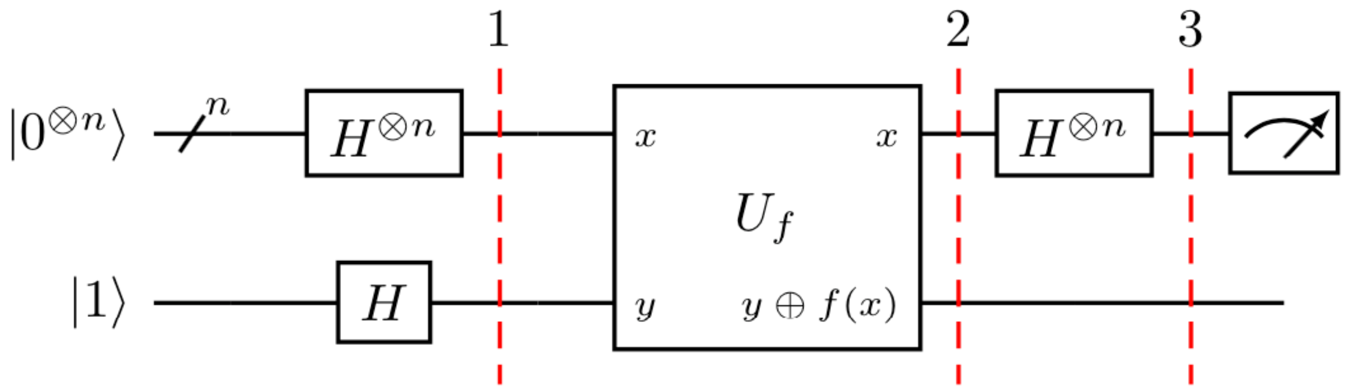
Deutsch-Jozsa Algorithm



	State 0	State 1	State 2
general	$\vert 0\rangle^{\otimes n}\vert 1\rangle$		
n =3 $f(000) = 0$ $f(001) = 1$ $f(010) = 1$ $f(011) = 1$ $f(100) = 0$ $f(101) = 0$ $f(110) = 0$ $f(111) = 1$	$\vert 0001\rangle$		

(balanced)

Deutsch-Jozsa Algorithm



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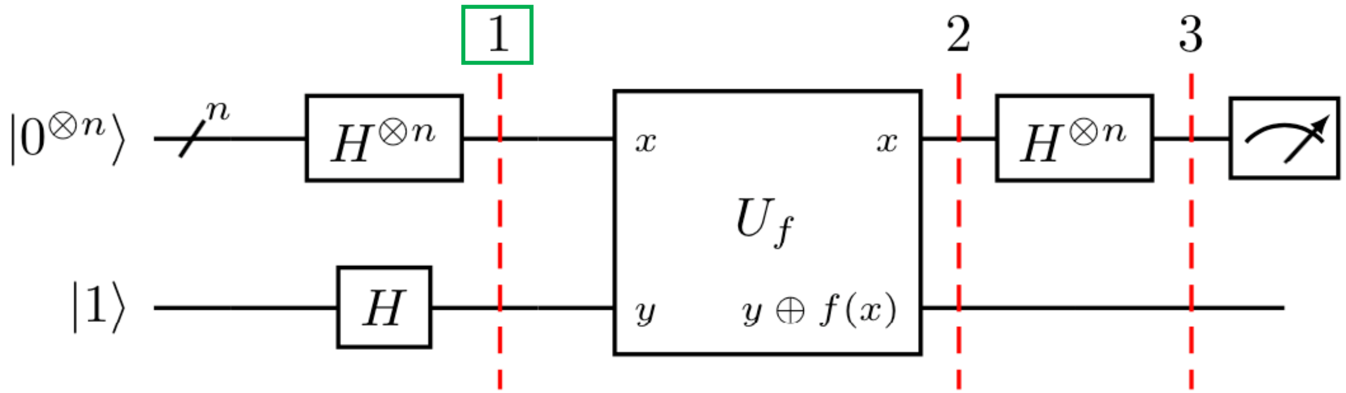
(balanced)

Deutsch-Jozsa Algorithm



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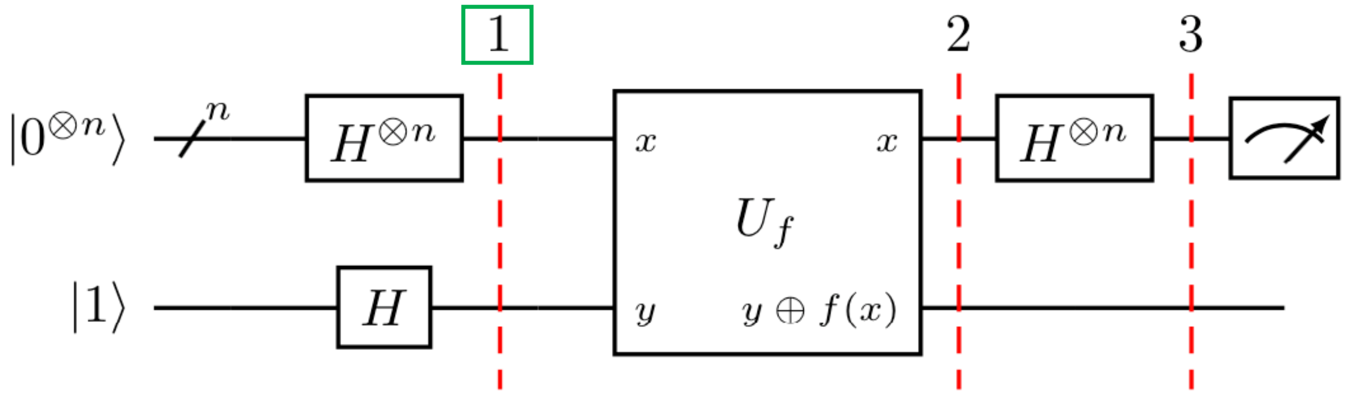
(balanced)

Deutsch-Jozsa Algorithm



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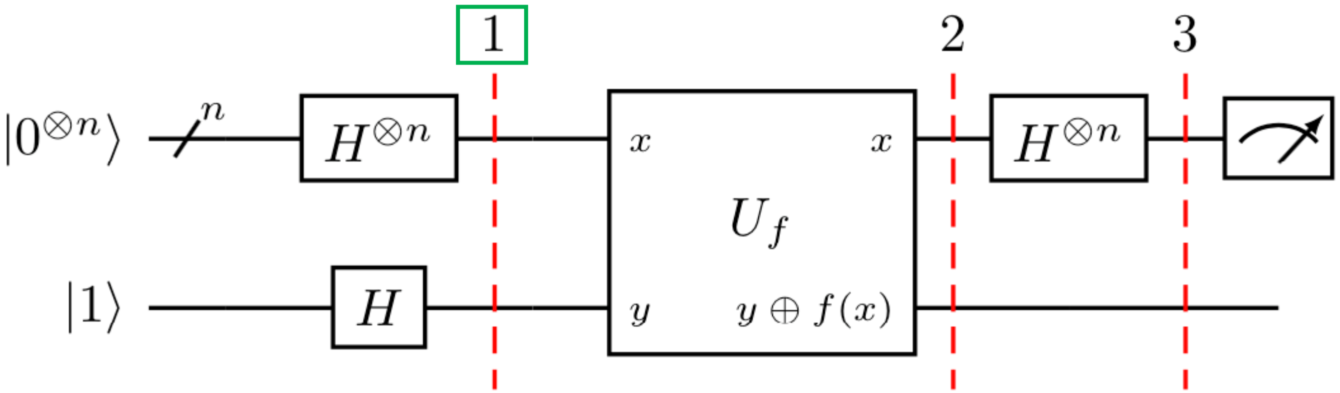
(balanced)

Deutsch-Jozsa Algorithm



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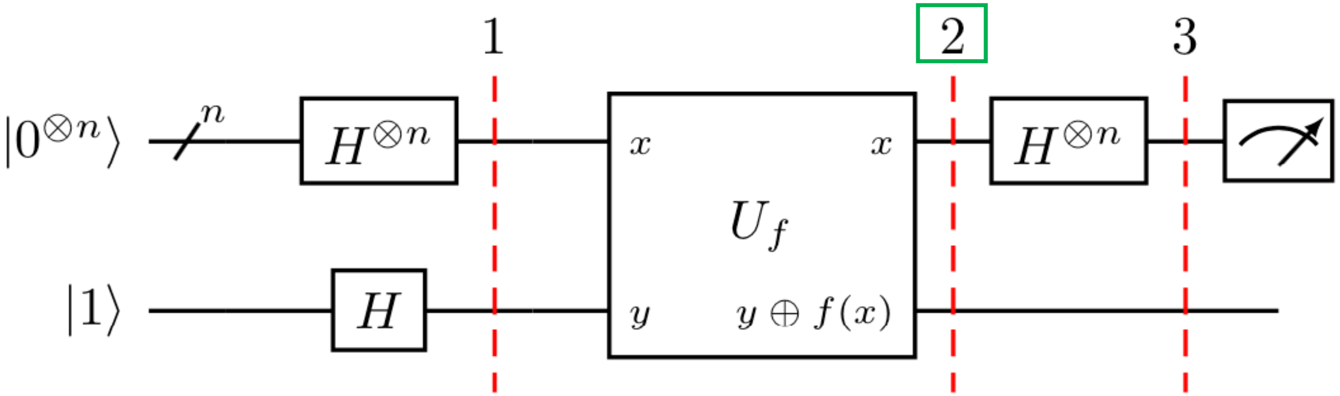
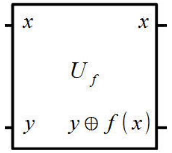


From Last Lecture



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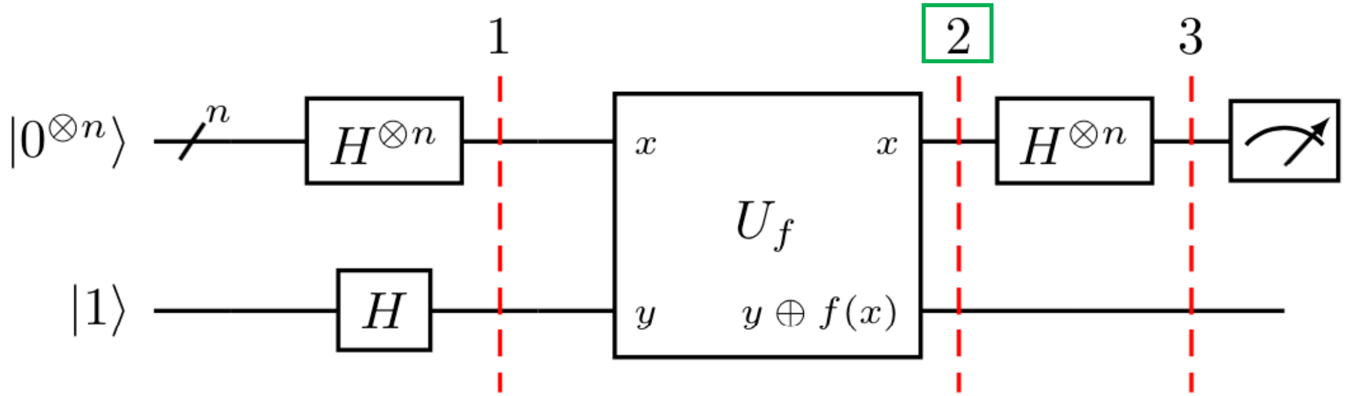
(balanced)

Deutsch-Jozsa Algorithm



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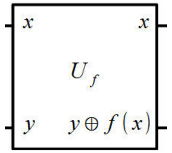

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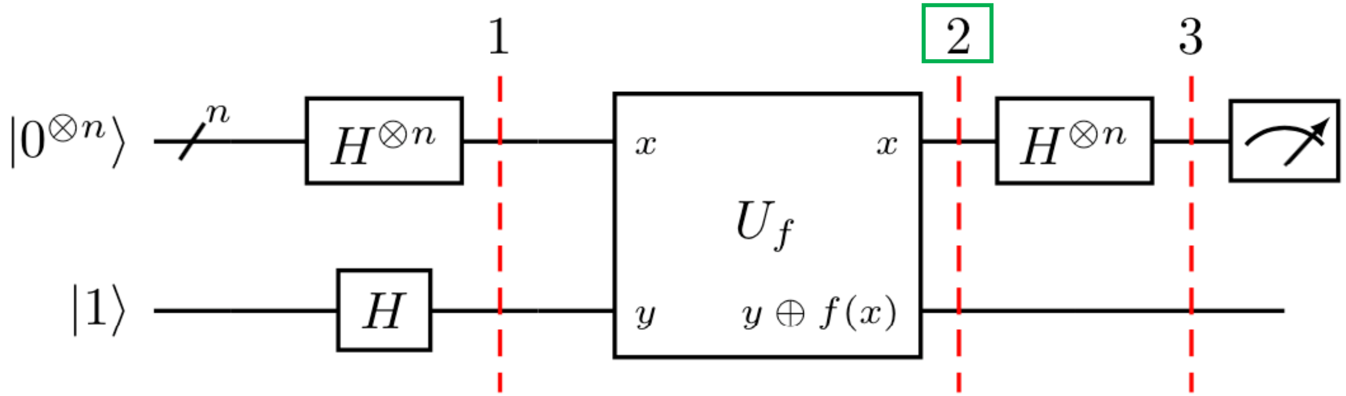
(balanced)

Deutsch-Jozsa Algorithm



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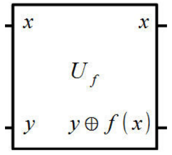

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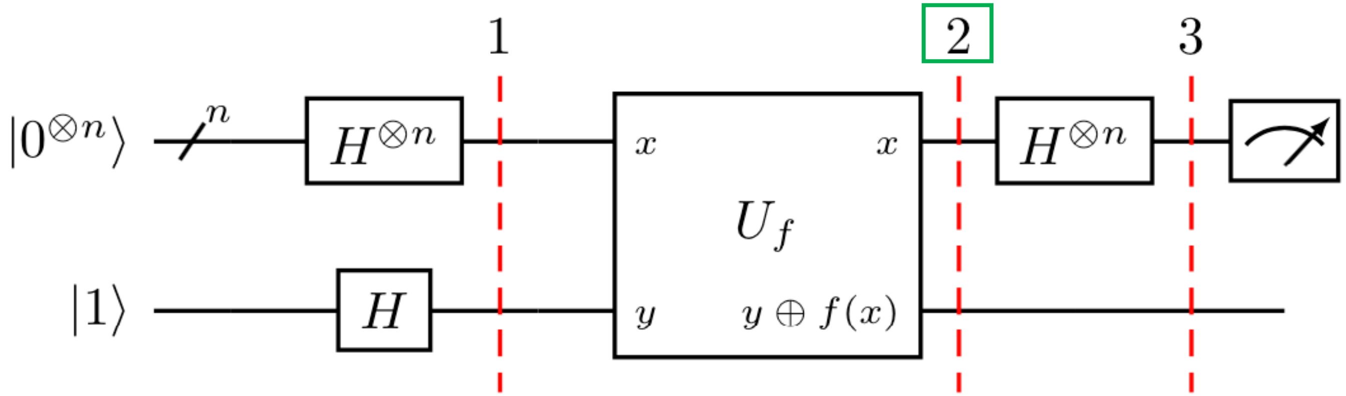
(balanced)

Deutsch-Jozsa Algorithm



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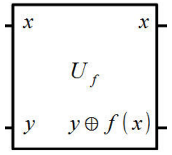

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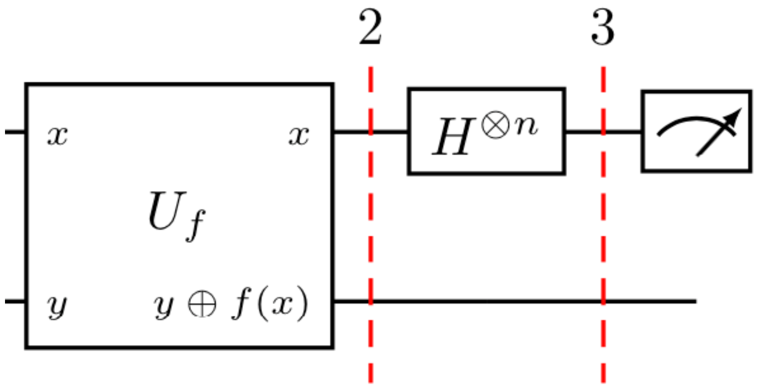
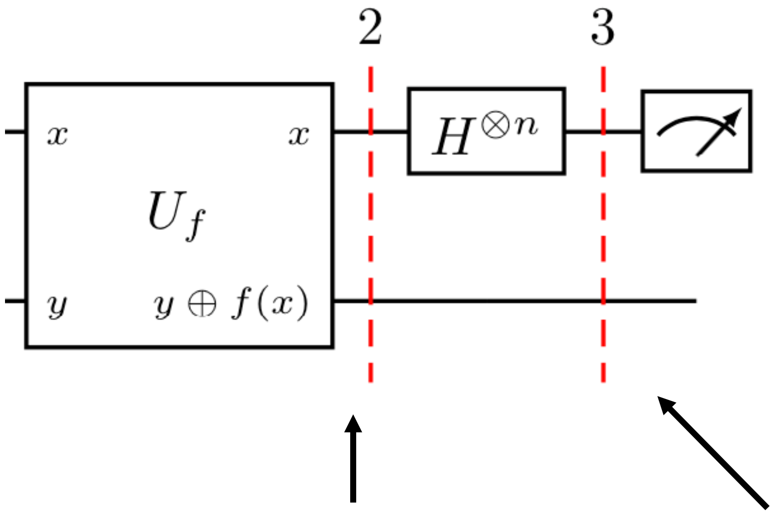
Expression for Final State



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \boxed{H^{\otimes n}} \qquad ?$$

State 2 **State 3**

Expression for Final State



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State 2 **State 3**

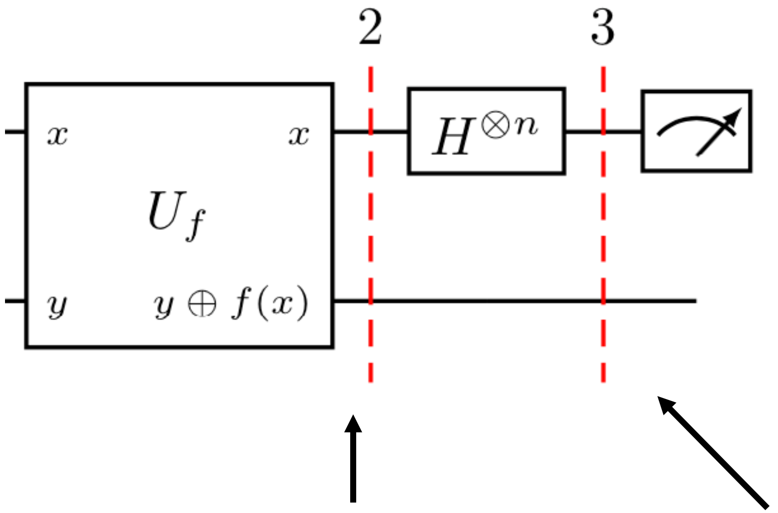
First Step (practice exercise): Find expression for

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1, x_2, \ldots x_n\rangle$$

Hint: Try applying Hadamard to different states where n = 3.

(e.g., calculate $H^{\otimes 3}|000\rangle, H^{\otimes 3}|001\rangle$, etc.)

Expression for Final State



$$\sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \boxed{H^{\otimes n}} \quad \textbf{?}$$

State 2 **State 3**

First Step (practice exercise): Find expression for

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1, x_2, \ldots x_n\rangle$$

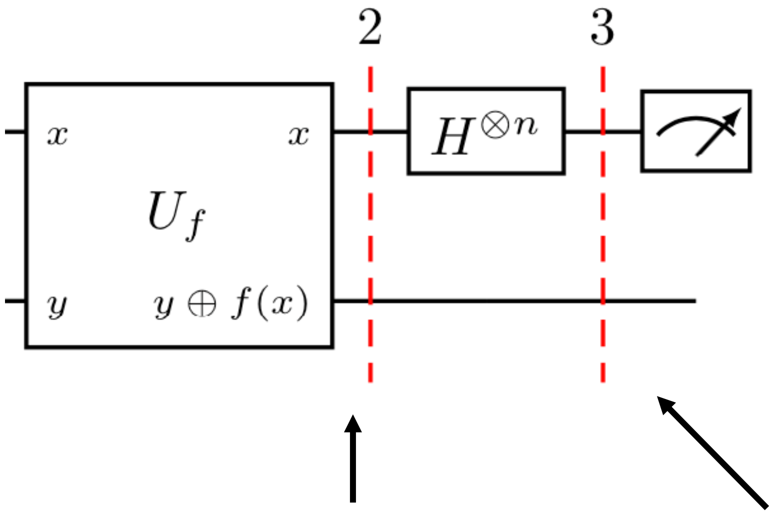
Hint: Try applying Hadamard to different states where n = 3.

(e.g., calculate $H^{\otimes 3}|000\rangle, H^{\otimes 3}|001\rangle$, etc.)

Solution:

$$H^{\otimes n}|011\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Expression for Final State



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \boxed{H^{\otimes n}} \quad \textbf{?}$$

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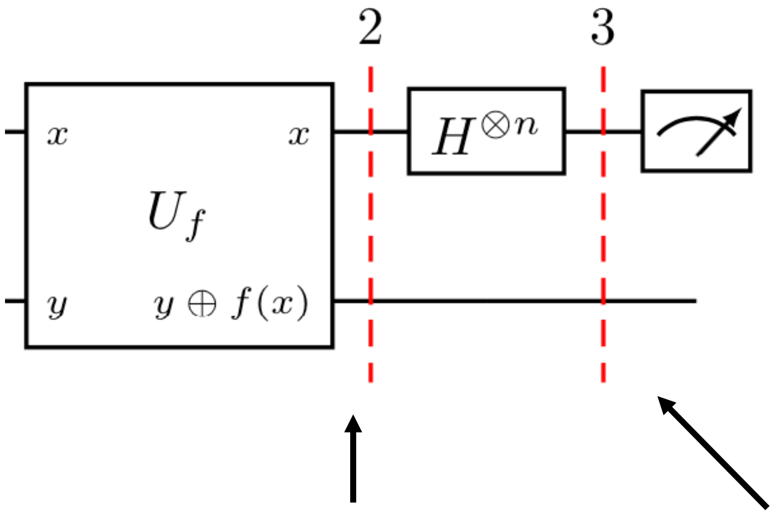
$$= \frac{|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle}{\sqrt{2^3}}$$

Expression for Final State



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}}\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right] \quad -\boxed{H^{\otimes n}}- \quad ?$$

State 2 **State 3**

First Step (practice exercise): Find expression for

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1, x_2, \ldots x_n\rangle$$

Hint: Try applying Hadamard to different states where n = 3.

(e.g., calculate $H^{\otimes 3}|000\rangle, H^{\otimes 3}|001\rangle$, etc.)

Solution:

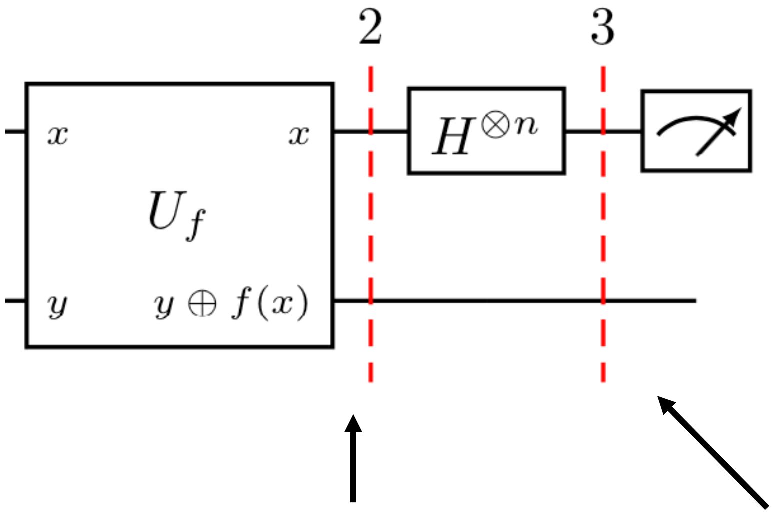
$$H^{\otimes n}|011\rangle = \left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$= \frac{\overset{\text{0 overlap}}{\boxed{\begin{matrix}|011\rangle\\|000\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|001\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|010\rangle\end{matrix}}} + \overset{\text{2 overlap}}{\boxed{\begin{matrix}|011\rangle\\|011\rangle\end{matrix}}} + \overset{\text{0 overlap}}{\boxed{\begin{matrix}|011\rangle\\|100\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|101\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|110\rangle\end{matrix}}} + \overset{\text{2 overlap}}{\boxed{\begin{matrix}|011\rangle\\|111\rangle\end{matrix}}}}{\sqrt{2^3}}$$

Even number of "overlapping 1s" - > positive term

Odd number of "overlapping 1s" - > negative term

Expression for Final State



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \boxed{H^{\otimes n}} \quad \textbf{?}$$

State 2 **State 3**

First Step (practice exercise): Find expression for

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1, x_2, \ldots x_n\rangle$$

Hint: Try applying Hadamard to different states where n = 3.

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$$= \frac{\overset{\text{0 overlap}}{\boxed{\begin{matrix}|011\rangle\\|000\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|001\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|010\rangle\end{matrix}}} + \overset{\text{2 overlap}}{\boxed{\begin{matrix}|011\rangle\\|011\rangle\end{matrix}}} + \overset{\text{0 overlap}}{\boxed{\begin{matrix}|011\rangle\\|100\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|101\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|110\rangle\end{matrix}}} + \overset{\text{2 overlap}}{\boxed{\begin{matrix}|011\rangle\\|111\rangle\end{matrix}}}}{\sqrt{2^3}}$$

Even number of “overlapping 1s” - > positive term

Odd number of “overlapping 1s” - > negative term

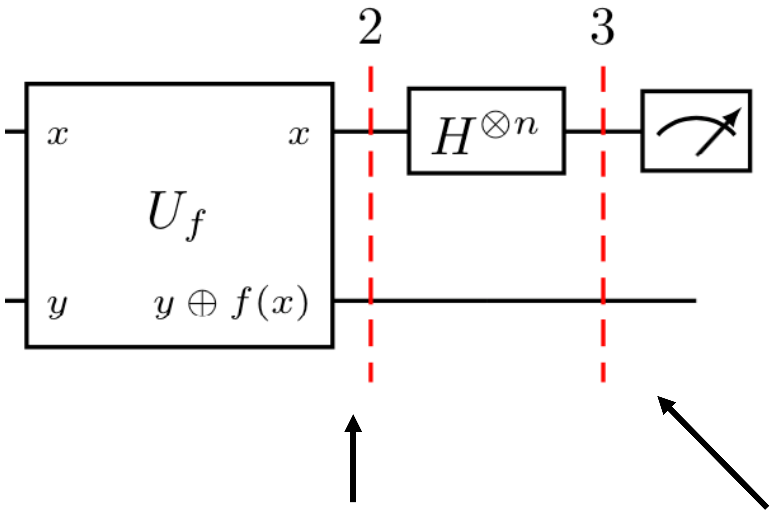
Verbose General Form

“dot product of ket labels”

$$H^{\otimes n}|x_1, \ldots, x_n\rangle = \frac{\sum_{z_1,\ldots,z_n\in\{0,1\}^n} (-1)^{\boxed{x_1 z_1 + \ldots + x_n z_n}} |z_1, \ldots, z_n\rangle}{\sqrt{2^n}}$$

(in above example $|x_1, x_2, x_3\rangle = |011\rangle$)

Expression for Final State



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \boxed{H^{\otimes n}} \quad \textbf{?}$$

State 2 **State 3**

First Step (practice exercise): Find expression for

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1, x_2, \ldots x_n\rangle$$

Hint: Try applying Hadamard to different states where n = 3.

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Solution:

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$$= \frac{\overset{\text{0 overlap}}{\boxed{\begin{matrix}|011\rangle\\|000\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|001\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|010\rangle\end{matrix}}} + \overset{\text{2 overlap}}{\boxed{\begin{matrix}|011\rangle\\|011\rangle\end{matrix}}} + \overset{\text{0 overlap}}{\boxed{\begin{matrix}|011\rangle\\|100\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|101\rangle\end{matrix}}} - \overset{\text{1 overlap}}{\boxed{\begin{matrix}|011\rangle\\|110\rangle\end{matrix}}} + \overset{\text{2 overlap}}{\boxed{\begin{matrix}|011\rangle\\|111\rangle\end{matrix}}}}{\sqrt{2^3}}$$

Even number of “overlapping 1s” - > positive term

Odd number of “overlapping 1s” - > negative term

Verbose General Form

“dot product of ket labels”

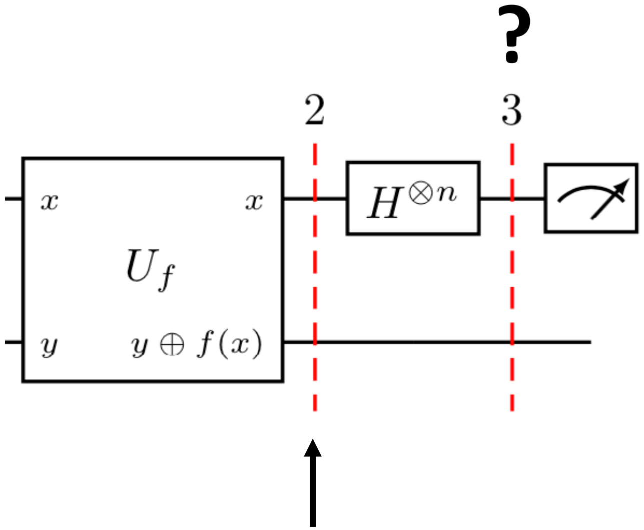
$$H^{\otimes n}|x_1, \ldots, x_n\rangle = \frac{\sum_{z_1,\ldots,z_n\in\{0,1\}^n} (-1)^{\boxed{x_1 z_1 + \ldots + x_n z_n}}|z_1, \ldots, z_n\rangle}{\sqrt{2^n}}$$

(in above example $|x_1, x_2, x_3\rangle = |011\rangle$)

Compact Form

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}} \quad \text{where } |x\rangle = |x_1, \ldots, x_n\rangle$$
$$|z\rangle = |z_1, \ldots, z_n\rangle$$

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{|000\rangle}{\sqrt{2^3}} - \frac{|001\rangle}{\sqrt{2^3}} - \frac{|010\rangle}{\sqrt{2^3}} - \frac{|011\rangle}{\sqrt{2^3}} + \frac{|100\rangle}{\sqrt{2^3}} + \frac{|101\rangle}{\sqrt{2^3}} + \frac{|110\rangle}{\sqrt{2^3}} - \frac{|111\rangle}{\sqrt{2^3}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

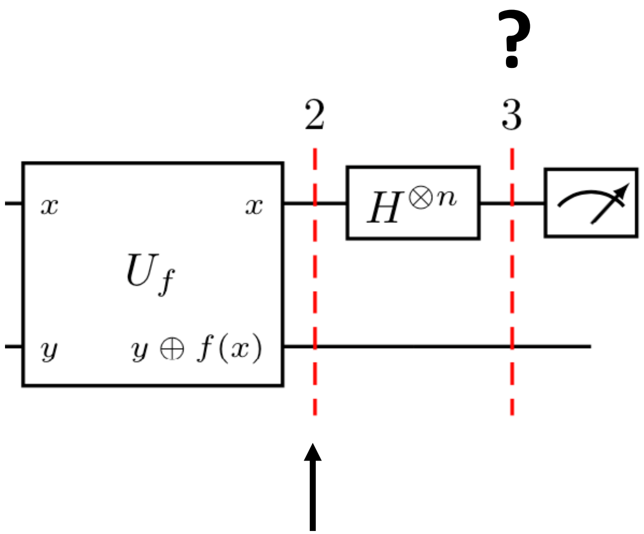
General Hadamard Formula

$$\boxed{H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{|000\rangle}{\sqrt{2^3}} - \frac{|001\rangle}{\sqrt{2^3}} - \frac{|010\rangle}{\sqrt{2^3}} - \frac{|011\rangle}{\sqrt{2^3}} + \frac{|100\rangle}{\sqrt{2^3}} + \frac{|101\rangle}{\sqrt{2^3}} + \frac{|110\rangle}{\sqrt{2^3}} - \frac{|111\rangle}{\sqrt{2^3}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

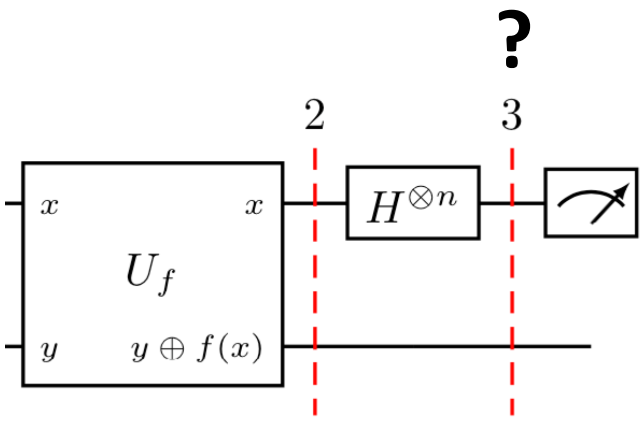
$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left.\begin{matrix}
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]
\end{matrix}\right\} \sum_x$$

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{\boxed{|000\rangle}}{\sqrt{2^3}} - \frac{\boxed{|001\rangle}}{\sqrt{2^3}} - \frac{\boxed{|010\rangle}}{\sqrt{2^3}} - \frac{\boxed{|011\rangle}}{\sqrt{2^3}} + \frac{\boxed{|100\rangle}}{\sqrt{2^3}} + \frac{\boxed{|101\rangle}}{\sqrt{2^3}} + \frac{\boxed{|110\rangle}}{\sqrt{2^3}} - \frac{\boxed{|111\rangle}}{\sqrt{2^3}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

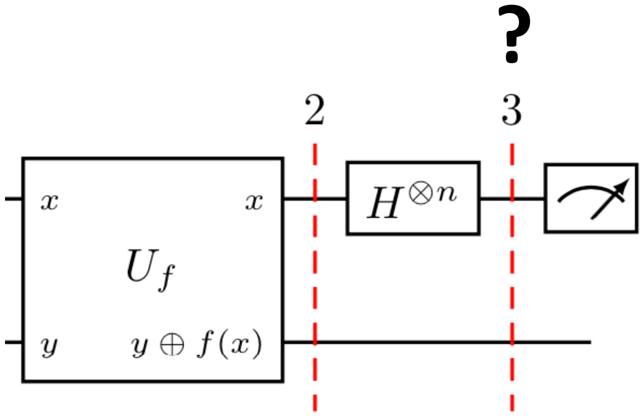
State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Rows calculated from Hadamard formula

$$\sum_x \left\{\begin{aligned}
&\boxed{\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right]} \\
-&\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-&\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-&\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
&\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
&\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
&\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-&\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]
\end{aligned}\right.$$

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\boxed{\frac{|000\rangle}{\sqrt{2^3}}} \boxed{-} \boxed{\frac{|001\rangle}{\sqrt{2^3}}} \boxed{-} \boxed{\frac{|010\rangle}{\sqrt{2^3}}} \boxed{-} \boxed{\frac{|011\rangle}{\sqrt{2^3}}} \boxed{+} \boxed{\frac{|100\rangle}{\sqrt{2^3}}} \boxed{+} \boxed{\frac{|101\rangle}{\sqrt{2^3}}} \boxed{+} \boxed{\frac{|110\rangle}{\sqrt{2^3}}} \boxed{-} \boxed{\frac{|111\rangle}{\sqrt{2^3}}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\boxed{\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

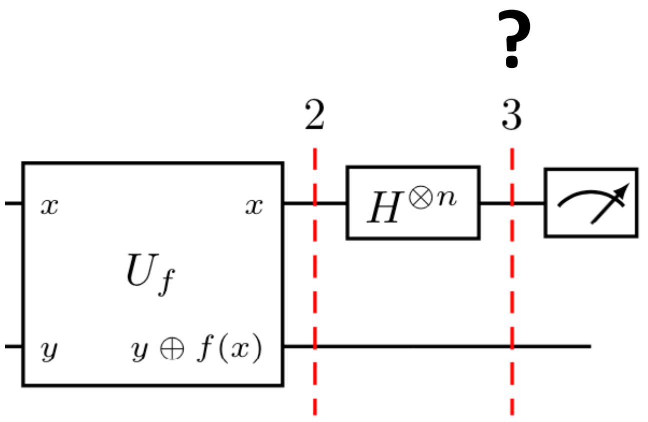
Rows calculated from Hadamard formula

signs for each group determined by:

$(-1)^{f(x)}$

$$\sum_x \left\{ \begin{array}{ll}
\boxed{+}|000\rangle & \boxed{\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right]} \\
\boxed{-}|001\rangle & -\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
\boxed{-}|010\rangle & -\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
\boxed{-}|011\rangle & -\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\boxed{+}|100\rangle & \left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
\boxed{+}|101\rangle & \left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\boxed{+}|110\rangle & \left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\boxed{-}|111\rangle & -\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]
\end{array} \right.$$

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{|000\rangle}{\sqrt{2^3}} - \frac{|001\rangle}{\sqrt{2^3}} - \frac{|010\rangle}{\sqrt{2^3}} - \frac{|011\rangle}{\sqrt{2^3}} + \frac{|100\rangle}{\sqrt{2^3}} + \frac{|101\rangle}{\sqrt{2^3}} + \frac{|110\rangle}{\sqrt{2^3}} - \frac{|111\rangle}{\sqrt{2^3}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Rows calculated from Hadamard formula

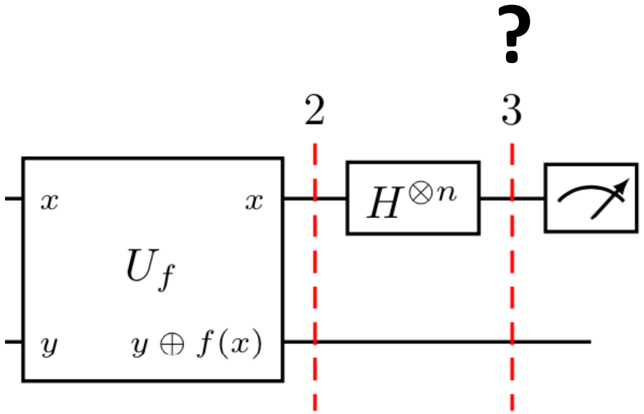
$+|000\rangle$
$-|001\rangle$
$-|010\rangle$
$-|011\rangle$
$+|100\rangle$
$+|101\rangle$
$+|110\rangle$
$-|111\rangle$

signs for each group determined by: $(-1)^{f(x)}$

$$\left.\begin{array}{l}
\left[\frac{|000\rangle}{2^3}+\frac{|001\rangle}{2^3}+\frac{|010\rangle}{2^3}+\frac{|011\rangle}{2^3}+\frac{|100\rangle}{2^3}+\frac{|101\rangle}{2^3}+\frac{|110\rangle}{2^3}+\frac{|111\rangle}{2^3}\right]\\
-\left[\frac{|000\rangle}{2^3}-\frac{|001\rangle}{2^3}+\frac{|010\rangle}{2^3}-\frac{|011\rangle}{2^3}+\frac{|100\rangle}{2^3}-\frac{|101\rangle}{2^3}+\frac{|110\rangle}{2^3}-\frac{|111\rangle}{2^3}\right]\\
-\left[\frac{|000\rangle}{2^3}+\frac{|001\rangle}{2^3}-\frac{|010\rangle}{2^3}-\frac{|011\rangle}{2^3}+\frac{|100\rangle}{2^3}+\frac{|101\rangle}{2^3}-\frac{|110\rangle}{2^3}-\frac{|111\rangle}{2^3}\right]\\
-\left[\frac{|000\rangle}{2^3}-\frac{|001\rangle}{2^3}-\frac{|010\rangle}{2^3}+\frac{|011\rangle}{2^3}+\frac{|100\rangle}{2^3}-\frac{|101\rangle}{2^3}-\frac{|110\rangle}{2^3}+\frac{|111\rangle}{2^3}\right]\\
\left[\frac{|000\rangle}{2^3}+\frac{|001\rangle}{2^3}+\frac{|010\rangle}{2^3}+\frac{|011\rangle}{2^3}-\frac{|100\rangle}{2^3}-\frac{|101\rangle}{2^3}-\frac{|110\rangle}{2^3}-\frac{|111\rangle}{2^3}\right]\\
\left[\frac{|000\rangle}{2^3}-\frac{|001\rangle}{2^3}+\frac{|010\rangle}{2^3}-\frac{|011\rangle}{2^3}-\frac{|100\rangle}{2^3}+\frac{|101\rangle}{2^3}-\frac{|110\rangle}{2^3}+\frac{|111\rangle}{2^3}\right]\\
\left[\frac{|000\rangle}{2^3}+\frac{|001\rangle}{2^3}-\frac{|010\rangle}{2^3}-\frac{|011\rangle}{2^3}-\frac{|100\rangle}{2^3}-\frac{|101\rangle}{2^3}+\frac{|110\rangle}{2^3}+\frac{|111\rangle}{2^3}\right]\\
-\left[\frac{|000\rangle}{2^3}-\frac{|001\rangle}{2^3}-\frac{|010\rangle}{2^3}+\frac{|011\rangle}{2^3}-\frac{|100\rangle}{2^3}+\frac{|101\rangle}{2^3}+\frac{|110\rangle}{2^3}-\frac{|111\rangle}{2^3}\right]
\end{array}\right\}\sum_x$$

Distribute outer sign, collect terms for each basis state and rearrange.

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{|000\rangle}{\sqrt{2^3}} \boxed{-} \frac{|001\rangle}{\sqrt{2^3}} \boxed{-} \frac{|010\rangle}{\sqrt{2^3}} \boxed{-} \frac{|011\rangle}{\sqrt{2^3}} \boxed{+} \frac{|100\rangle}{\sqrt{2^3}} \boxed{+} \frac{|101\rangle}{\sqrt{2^3}} \boxed{+} \frac{|110\rangle}{\sqrt{2^3}} \boxed{-} \frac{|111\rangle}{\sqrt{2^3}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Rows calculated from Hadamard formula

signs for each group determined by:

$(-1)^{f(x)}$

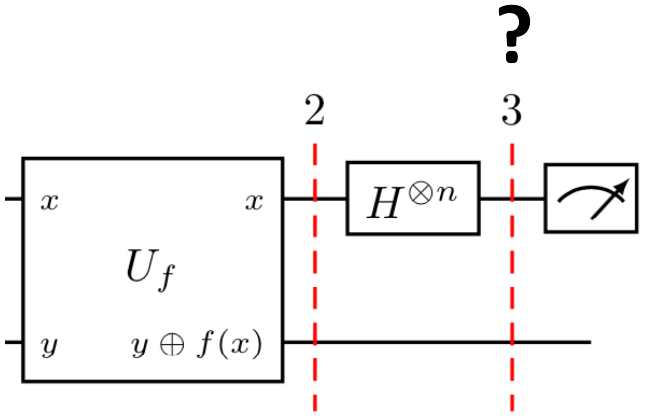
$$\begin{array}{l} +|000\rangle \\ -|001\rangle \\ -|010\rangle \\ -|011\rangle \\ +|100\rangle \\ +|101\rangle \\ +|110\rangle \\ -|111\rangle \end{array}$$

$$\sum_x \left\{\begin{array}{l}
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]
\end{array}\right.$$

$$\frac{0}{2^3}|000\rangle$$

Distribute outer sign, collect terms for each basis state and rearrange.

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{|000\rangle}{\sqrt{2^3}} \boxed{-} \frac{|001\rangle}{\sqrt{2^3}} \boxed{-} \frac{|010\rangle}{\sqrt{2^3}} \boxed{-} \frac{|011\rangle}{\sqrt{2^3}} \boxed{+} \frac{|100\rangle}{\sqrt{2^3}} \boxed{+} \frac{|101\rangle}{\sqrt{2^3}} \boxed{+} \frac{|110\rangle}{\sqrt{2^3}} \boxed{-} \frac{|111\rangle}{\sqrt{2^3}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Rows calculated from Hadamard formula

$$\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right]$$

$+|000\rangle$, $-|001\rangle$, $-|010\rangle$, $-|011\rangle$, $+|100\rangle$, $+|101\rangle$, $+|110\rangle$, $-|111\rangle$

$$-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]$$

$$-\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]$$

$$-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right]$$

$$\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]$$

$$\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right]$$

$$\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right]$$

$$-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]$$

$$\sum_x$$

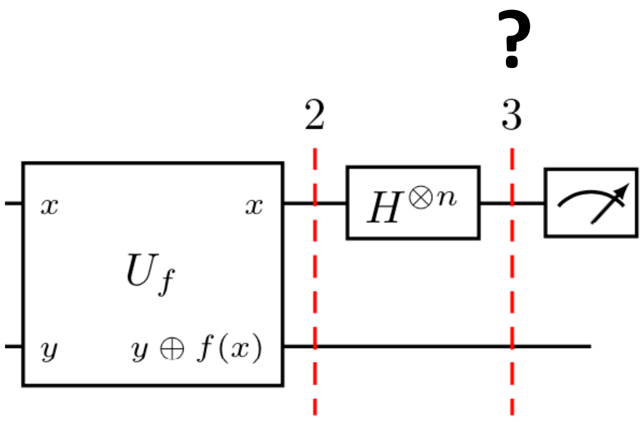
signs for each group determined by:

$(-1)^{f(x)}$

$$\frac{0}{2^3}|000\rangle \qquad \frac{4}{2^3}|001\rangle$$

Distribute outer sign, collect terms for each basis state and rearrange.

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{|000\rangle}{\sqrt{2^3}} - \frac{|001\rangle}{\sqrt{2^3}} - \frac{|010\rangle}{\sqrt{2^3}} - \frac{|011\rangle}{\sqrt{2^3}} + \frac{|100\rangle}{\sqrt{2^3}} + \frac{|101\rangle}{\sqrt{2^3}} + \frac{|110\rangle}{\sqrt{2^3}} - \frac{|111\rangle}{\sqrt{2^3}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Rows calculated from Hadamard formula

signs for each group determined by: $(-1)^{f(x)}$

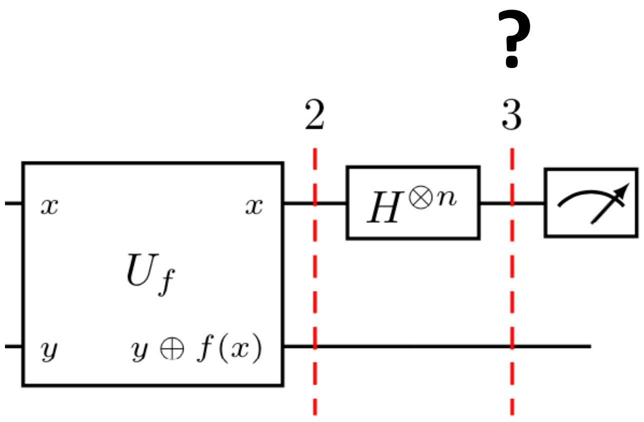
$$+|000\rangle \quad -|001\rangle \quad -|010\rangle \quad -|011\rangle \quad +|100\rangle \quad +|101\rangle \quad +|110\rangle \quad -|111\rangle$$

$$\sum_x \left\{\begin{array}{l}
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]
\end{array}\right\}$$

$$= \frac{0}{2^3}|000\rangle + \frac{4}{2^3}|001\rangle + \frac{4}{2^3}|010\rangle + \frac{0}{2^3}|011\rangle - \frac{4}{2^3}|100\rangle + \frac{0}{2^3}|101\rangle + \frac{0}{2^3}|110\rangle + \frac{4}{2^3}|111\rangle$$

Distribute outer sign, collect terms for each basis state and rearrange.

Final State: Leveraging Interference



$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\left[\frac{\boxed{|000\rangle}}{\sqrt{2^3}} \boxed{-} \frac{\boxed{|001\rangle}}{\sqrt{2^3}} \boxed{-} \frac{\boxed{|010\rangle}}{\sqrt{2^3}} \boxed{-} \frac{\boxed{|011\rangle}}{\sqrt{2^3}} \boxed{+} \frac{\boxed{|100\rangle}}{\sqrt{2^3}} \boxed{+} \frac{\boxed{|101\rangle}}{\sqrt{2^3}} \boxed{+} \frac{\boxed{|110\rangle}}{\sqrt{2^3}} \boxed{-} \frac{\boxed{|111\rangle}}{\sqrt{2^3}}\right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

General Hadamard Formula

$$H^{\otimes n}|x\rangle = \frac{\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

State 3 General Form

$$\sum_x (-1)^{f(x)} \sum_z \frac{(-1)^{x\cdot z}|z\rangle}{2^n}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = \boxed{\sum_z \sum_x \frac{(-1)^{x\cdot z + f(x)}|z\rangle}{2^n}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]}$$

Rows calculated from Hadamard formula

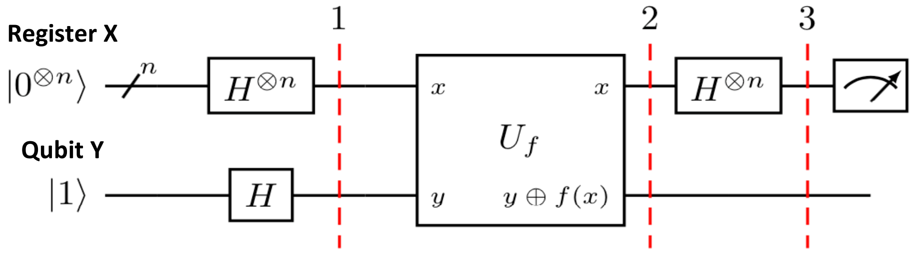
signs for each group determined by: $(-1)^{f(x)}$

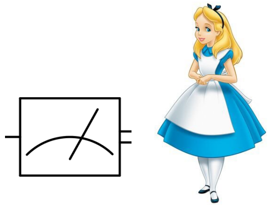
$$\sum_x \left\{ \begin{array}{lr}
+|000\rangle & \left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-|001\rangle & -\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-|010\rangle & -\left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
-|011\rangle & -\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} + \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
+|100\rangle & \left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right] \\
+|101\rangle & \left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} + \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} - \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
+|110\rangle & \left[\frac{|000\rangle}{2^3} + \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} - \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} - \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} + \frac{|111\rangle}{2^3}\right] \\
-|111\rangle & -\left[\frac{|000\rangle}{2^3} - \frac{|001\rangle}{2^3} - \frac{|010\rangle}{2^3} + \frac{|011\rangle}{2^3} - \frac{|100\rangle}{2^3} + \frac{|101\rangle}{2^3} + \frac{|110\rangle}{2^3} - \frac{|111\rangle}{2^3}\right]
\end{array} \right.$$

Final State $\rightarrow$

$$\boxed{\left[\frac{1}{2}|001\rangle + \frac{1}{2}|010\rangle - \frac{1}{2}|100\rangle + \frac{1}{2}|111\rangle\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]}$$

Deutsch-Jozsa Analysis





Final State

$$\sum_z \sum_x \frac{(-1)^{x\cdot z + f(x)} |z\rangle}{2^n} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right]$$

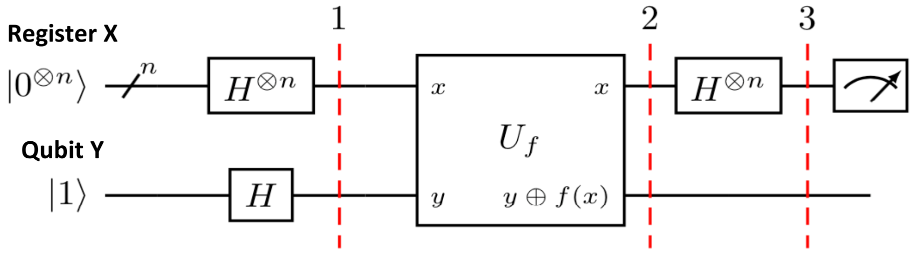
Exercise: Complete the analysis by answering the following questions:

1. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is constant?

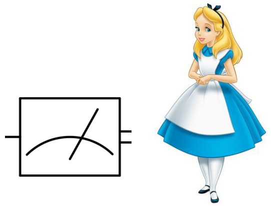
2. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is balanced?

3. Given your answers above, how should Alice translate her measurement outcome to answer to the problem?

Deutsch-Jozsa Analysis



Final State



$$\sum_z \sum_x \frac{(-1)^{x\cdot z+f(x)}|z\rangle}{2^n}\cancel{\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]}$$

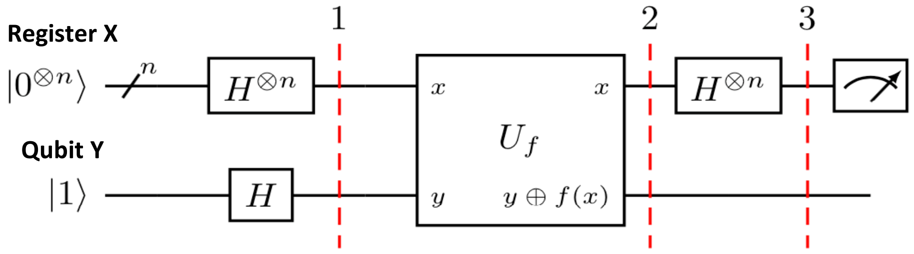
Only measuring register X, so can ignore qubit Y part of state.

$$\sum_z |z\rangle \left[\sum_x \frac{(-1)^{x\cdot z+f(x)}}{2^n}\right]$$

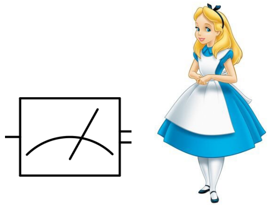
Exercise: Complete the analysis by answering the following questions:

1. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is constant?
2. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is balanced?
3. Given your answers above, how should Alice translate her measurement outcome to answer to the problem?

Deutsch-Jozsa Analysis



Final State



$$\boxed{\sum_z \sum_x \frac{(-1)^{x\cdot z+f(x)}|z\rangle}{2^n} \cancel{\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]}}$$

Only measuring register X, so can ignore qubit Y part of state.

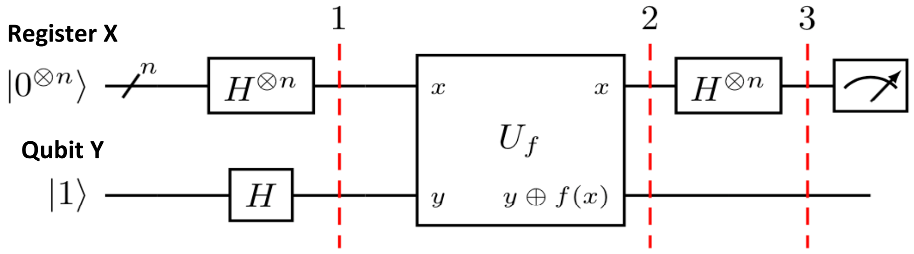
$$\sum_z |z\rangle \left[\sum_x \frac{(-1)^{x\cdot z+f(x)}}{2^n}\right]$$

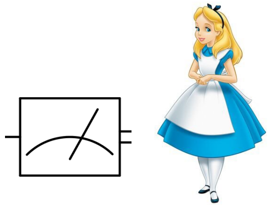
$$\boxed{\text{When } |z\rangle = |0^{\otimes n}\rangle \text{ what does } \sum_x \frac{(-1)^{x\cdot z+f(x)}}{2^n} \text{ equal?}}$$

Exercise: Complete the analysis by answering the following questions:

1. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is constant?
2. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is balanced?
3. Given your answers above, how should Alice translate her measurement outcome to answer to the problem?

Deutsch-Jozsa Analysis





Final State

$$\sum_z \sum_x \frac{(-1)^{x\cdot z + f(x)}|z\rangle}{2^n} \cancel{\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]}$$

Only measuring register X, so can ignore qubit Y part of state.

$$\sum_z |z\rangle \left[\sum_x \frac{(-1)^{x\cdot z + f(x)}}{2^n}\right]$$

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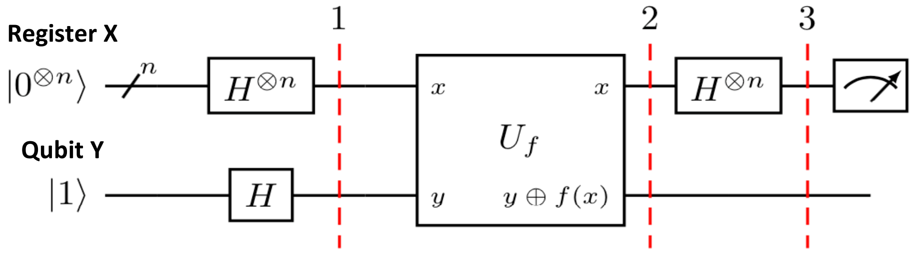
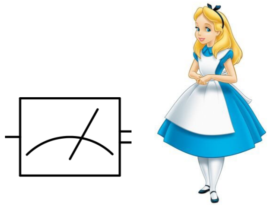
Deutsch-Jozsa Analysis





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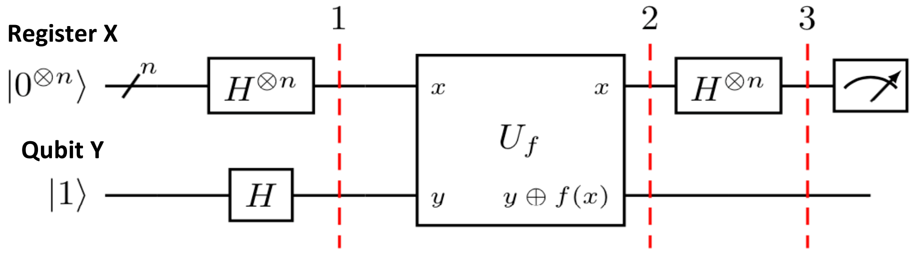

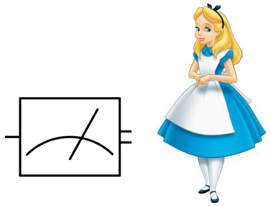

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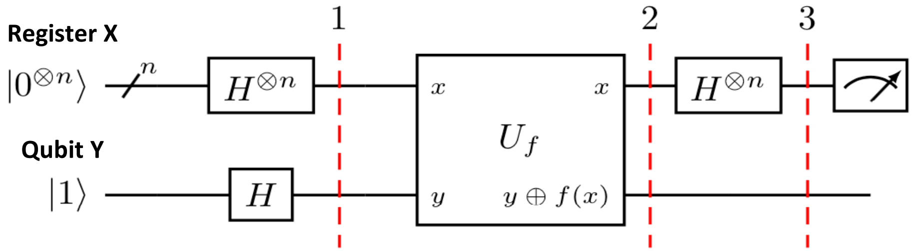

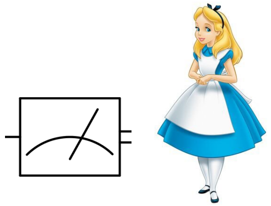

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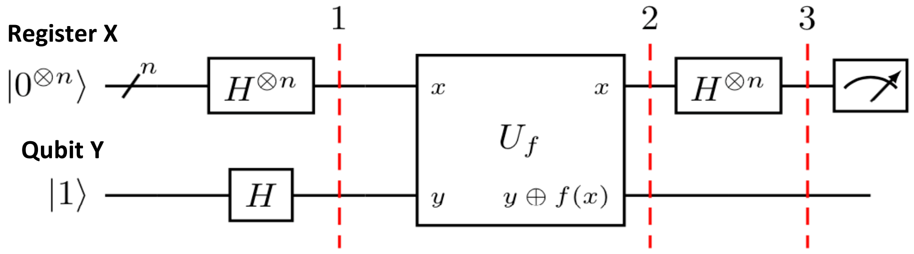

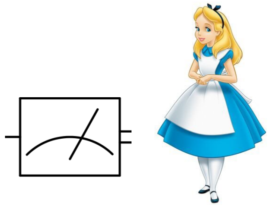

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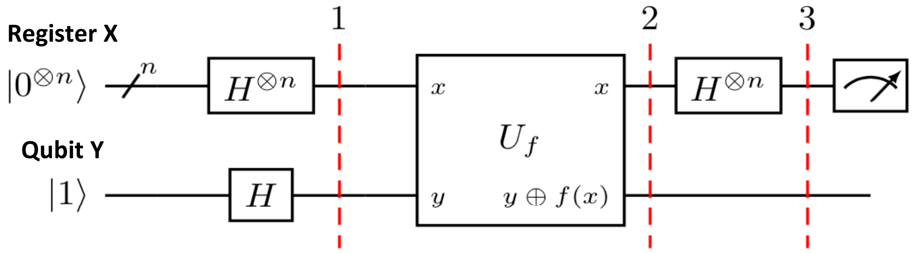

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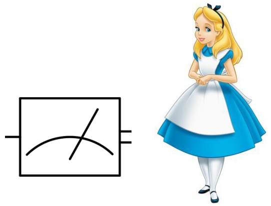

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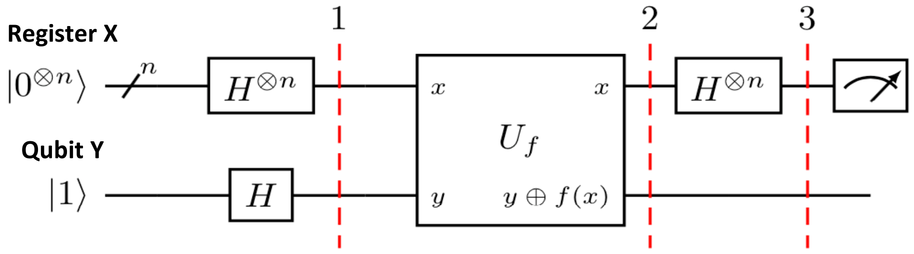

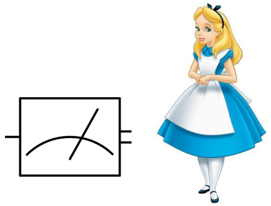

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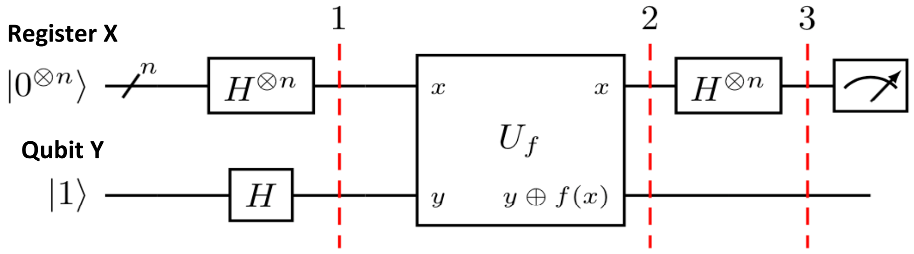

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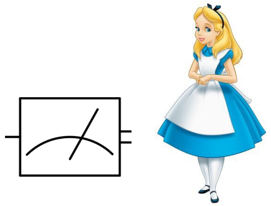

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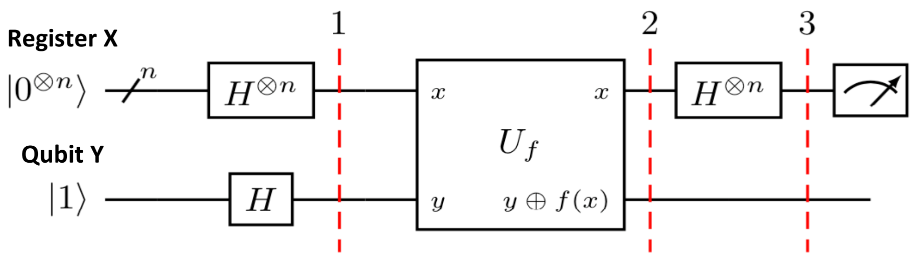

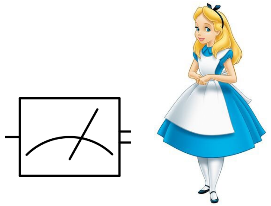

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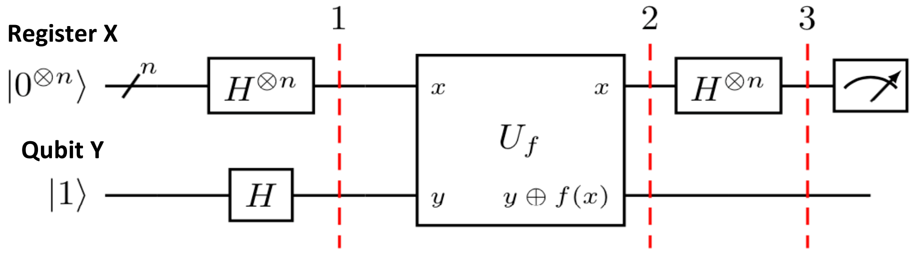

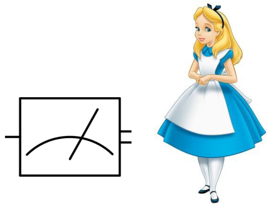

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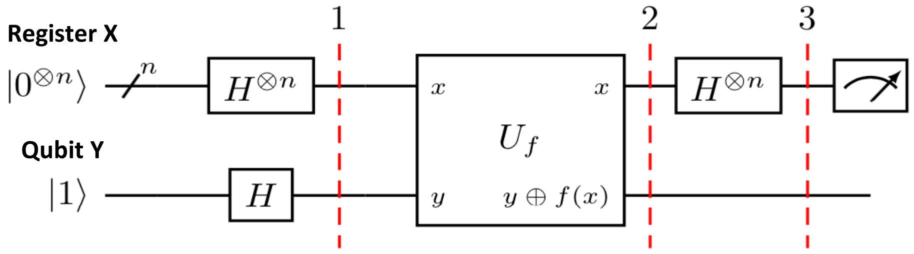

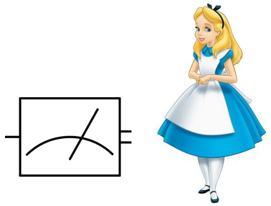

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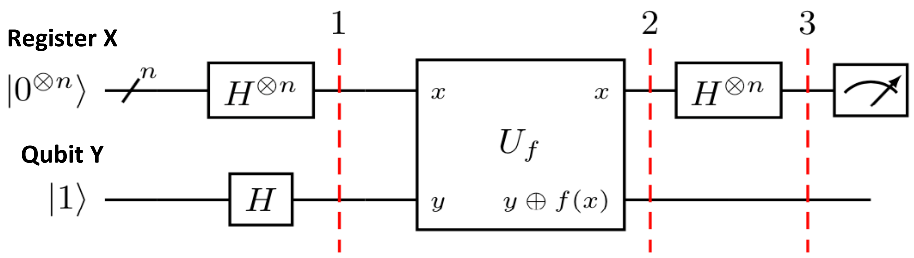

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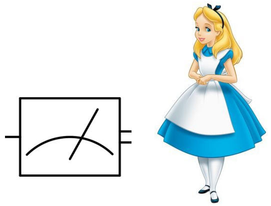

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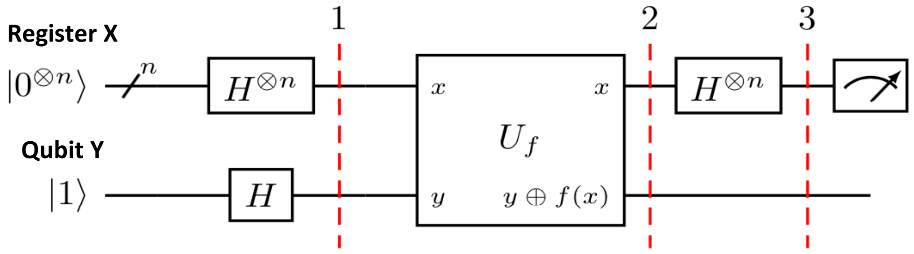
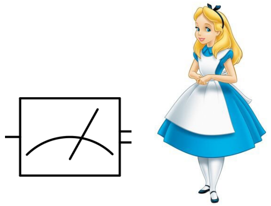
probability 0 for $|0^{\otimes n}\rangle$

Deutsch-Jozsa Analysis





Final State

$$\sum_z \sum_x \frac{(-1)^{x\cdot z + f(x)} |z\rangle}{2^n} \cancel{\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]}$$

Only measuring register X, so can ignore qubit Y part of state.

$$\sum_z |z\rangle \left[\sum_x \frac{(-1)^{x\cdot z + f(x)}}{2^n}\right]$$

When $|z\rangle = |0^{\otimes n}\rangle$ what does $\sum_x \frac{(-1)^{x\cdot z + f(x)}}{2^n}$ equal?

Exercise: Complete the analysis by answering the following questions:

1. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is constant?
2. What will be probability of measuring $|0^{\otimes n}\rangle$ for register X if f is balanced?
3. Given your answers above, how should Alice translate her measurement outcome to answer to the problem?

First Observe: for all x : $\boxed{x \cdot z = 0}$ when $|z\rangle = |0^{\otimes n}\rangle$

$$\boxed{\begin{aligned} |z\rangle &= |0, 0, \dots, 0\rangle \\ |x\rangle &= |x_1, \dots, x_n\rangle \end{aligned}}$$

Always 0 overlapping 1s

Constant f Case: Probability = 1 for measuring $|0^{\otimes n}\rangle$ since:

$$\sum_x \frac{(-1)^{\overset{=0}{\boxed{x\cdot z}} + f(x)}}{2^n} = \begin{cases} \xrightarrow{f(x) = 0 \text{ for all } x} \sum_x \frac{(-1)^{0+0}}{2^n} = \sum_x \frac{1}{2^n} = 2^n\left(\frac{1}{2^n}\right) = 1 \\ \xrightarrow{f(x) = 1 \text{ for all } x} \sum_x \frac{(-1)^{0+1}}{2^n} = \sum_x \frac{-1}{2^n} = 2^n\left(\frac{-1}{2^n}\right) = -1 \end{cases}$$

probability 1 for $|0^{\otimes n}\rangle$

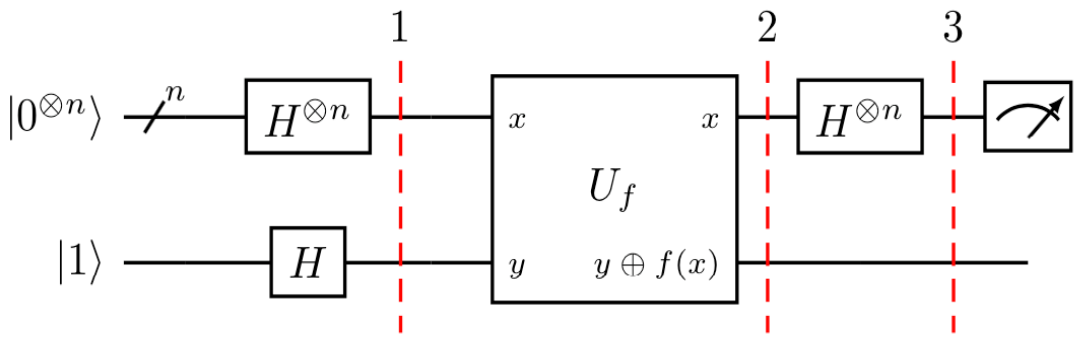
Balanced f Case: Probability = 0 for measuring $|0^{\otimes n}\rangle$ since:

$$\sum_x \frac{(-1)^{\overset{=0}{\boxed{x\cdot z}} + f(x)}}{2^n} = \sum_{x:f(x)=0} \frac{(-1)^{0+f(x)}}{2^n} + \sum_{x:f(x)=1} \frac{(-1)^{0+f(x)}}{2^n} = \frac{2^n}{2}\left(\frac{1}{2^n}\right) + \frac{2^n}{2}\left(\frac{-1}{2^n}\right) = 0$$

probability 0 for $|0^{\otimes n}\rangle$

Alice Output: Constant if measurement is $|0^{\otimes n}\rangle$, otherwise balanced.

Uf Gate for n ≥ 2?



n = 2 Balanced Example

$$f(00) = f(11) = 0$$

$$f(01) = f(10) = 1$$

(XOR function)

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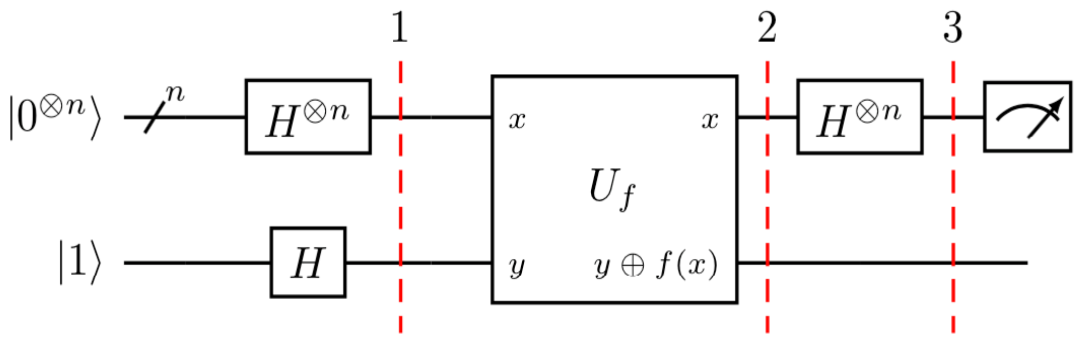

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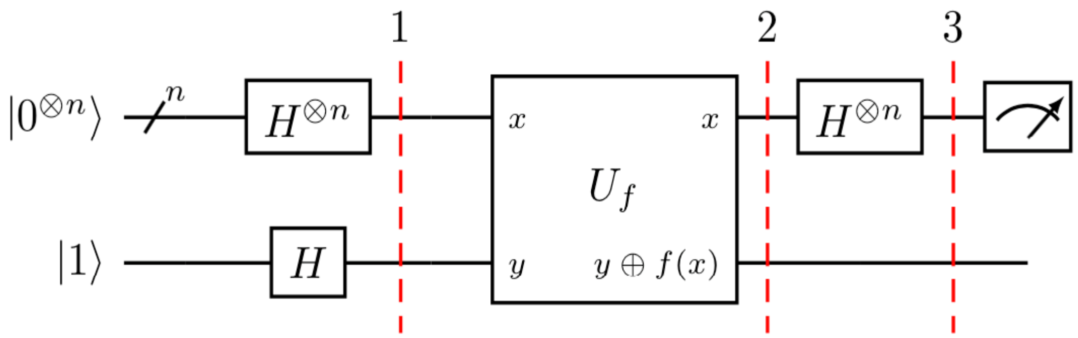

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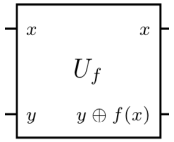

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

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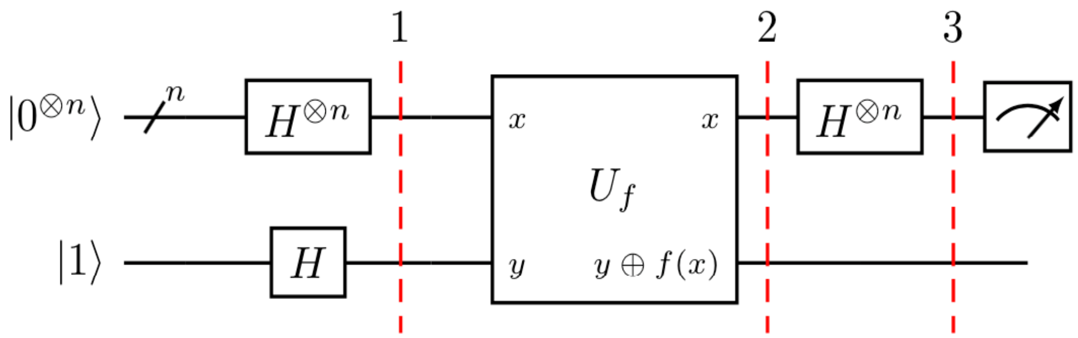

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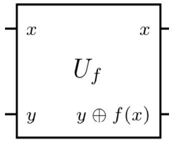

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

$$= \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{011}|010\rangle + \alpha_{010}|011\rangle + \alpha_{101}|100\rangle + \alpha_{100}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

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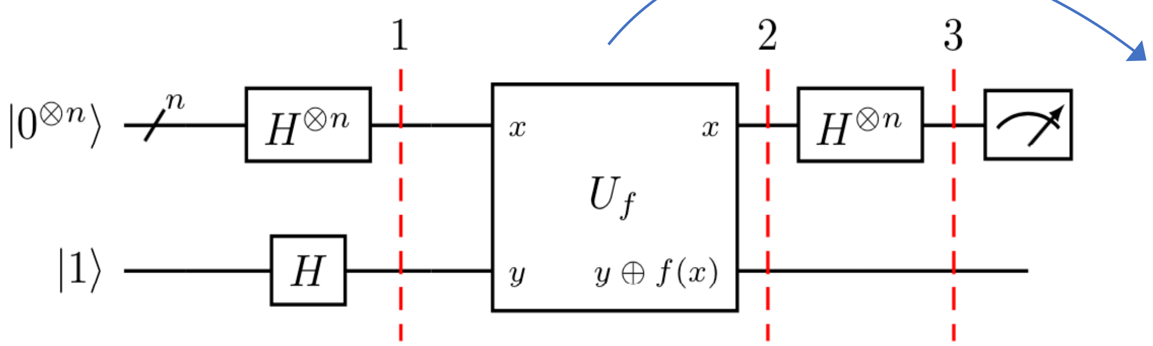

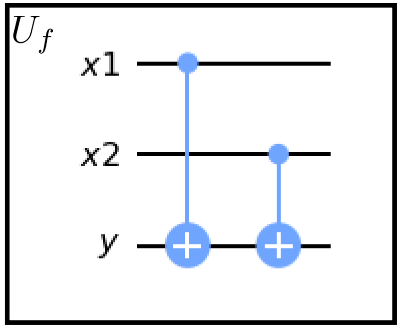

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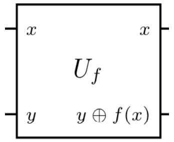

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

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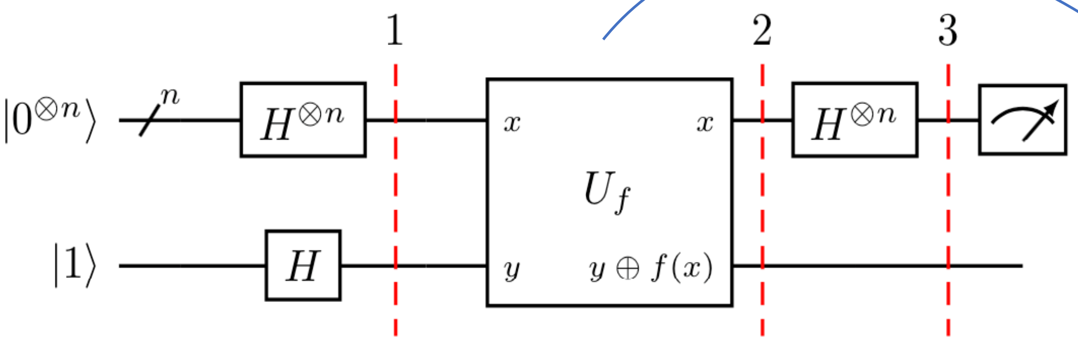

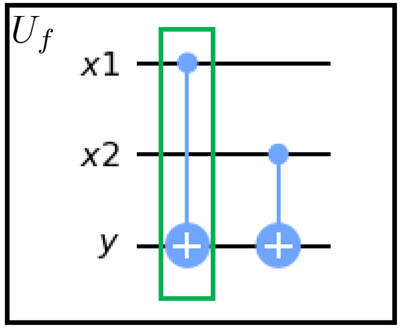

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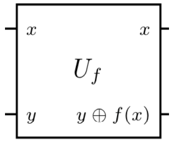

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

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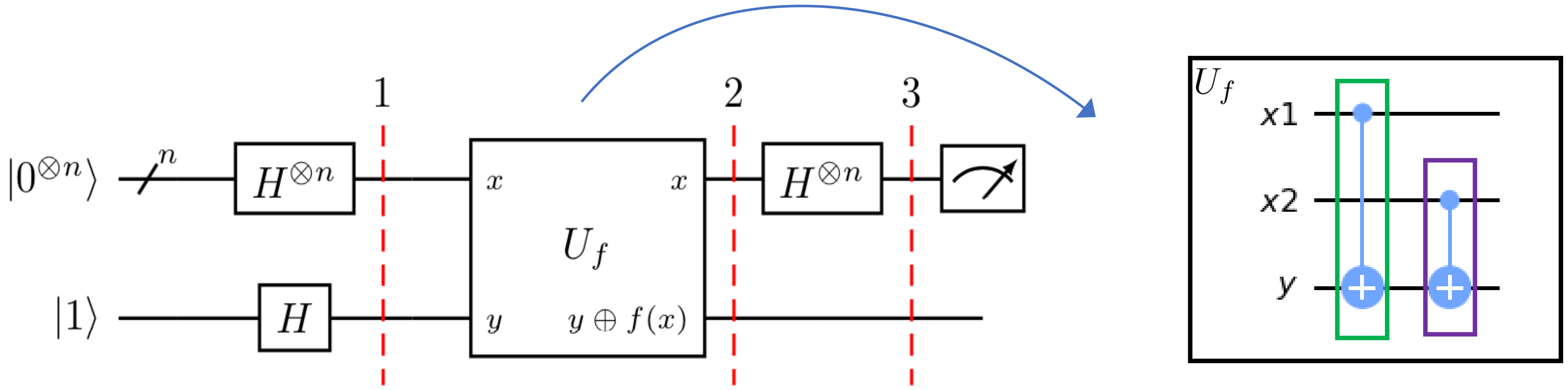
Uf Gate for n ≥ 2?



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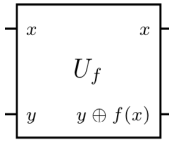

(swap back)

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

$$= \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{011}|010\rangle + \alpha_{010}|011\rangle + \alpha_{101}|100\rangle + \alpha_{100}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

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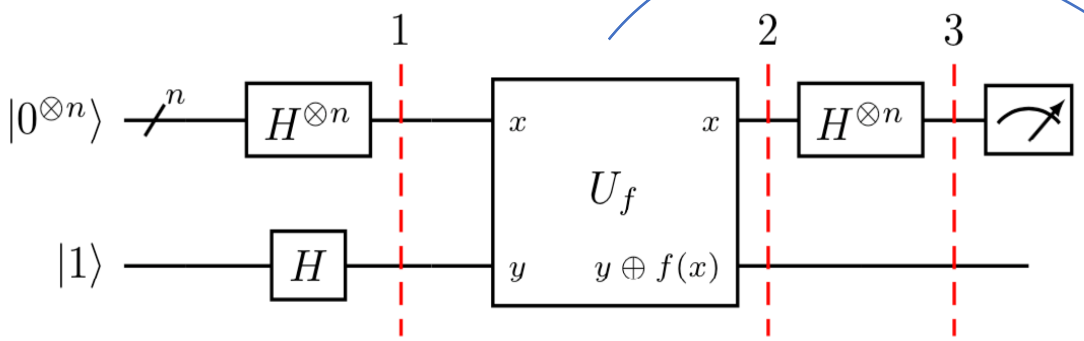

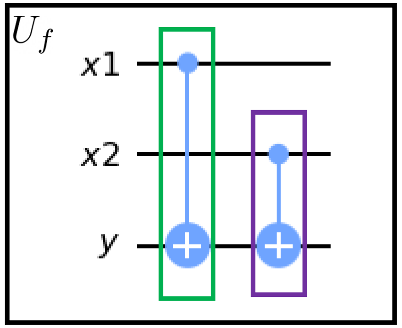

What is the corresponding matrix for this gate?

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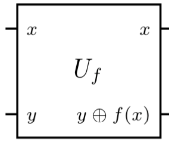

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$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

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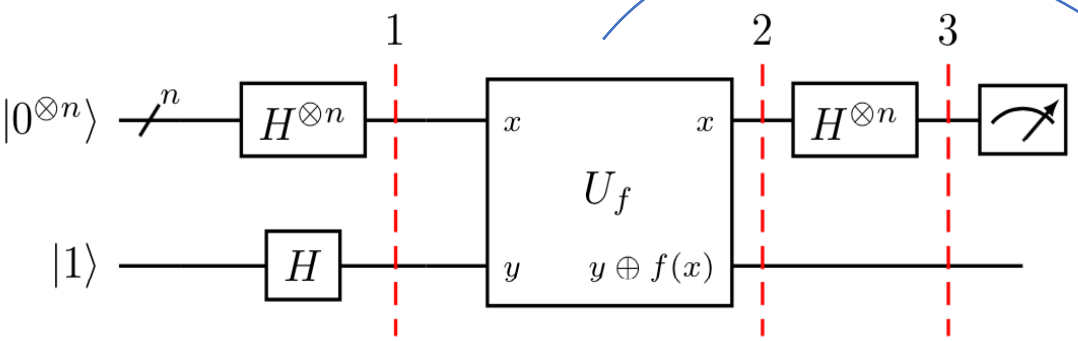

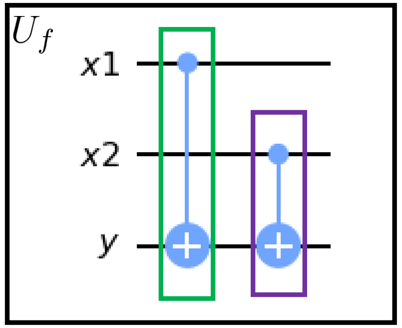

...since it is neither

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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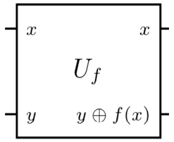

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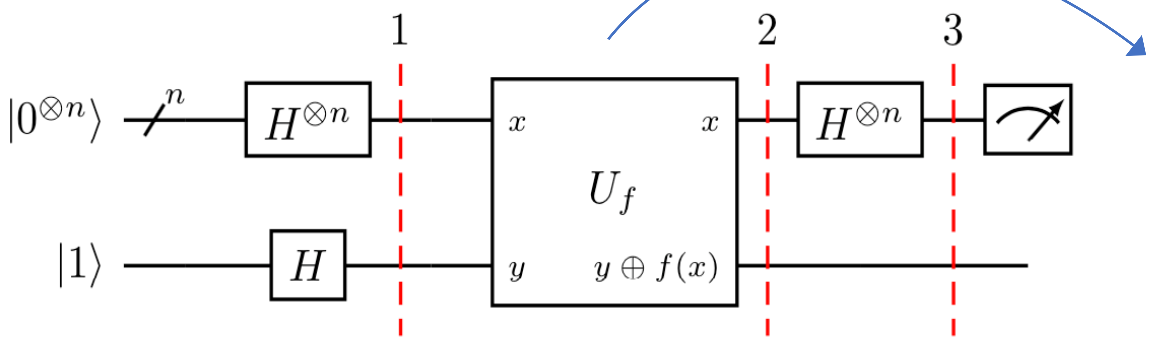

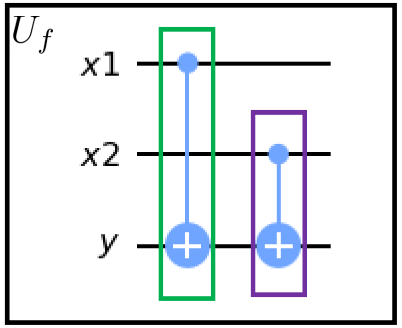

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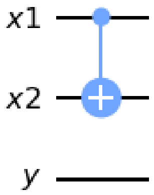
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix} \rightarrow$$



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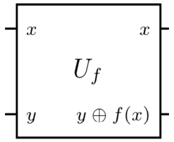

(swap back)

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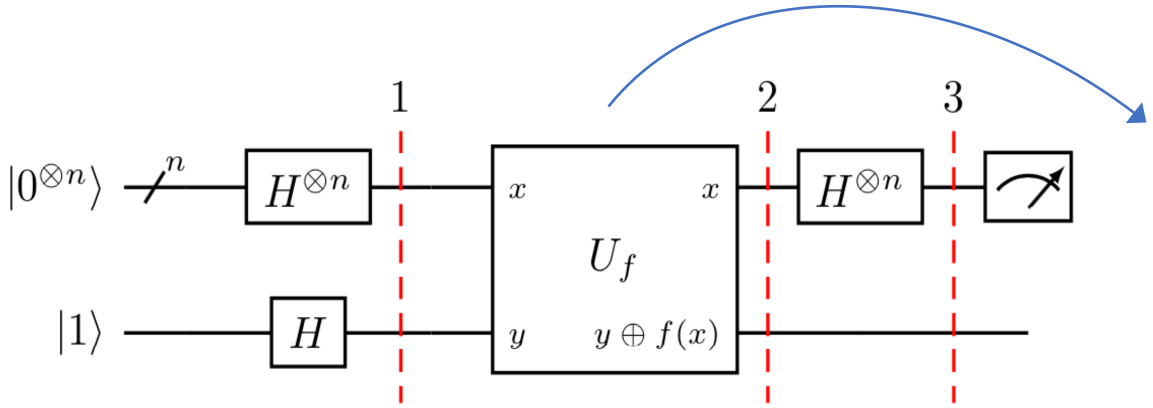

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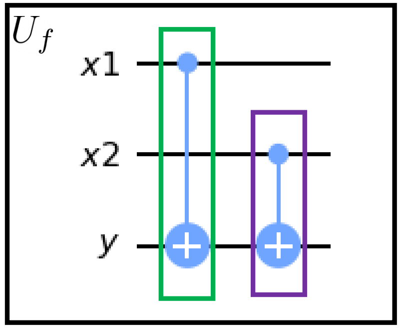

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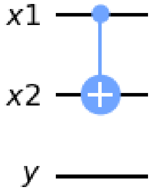

nor

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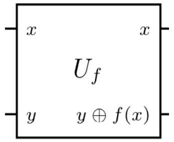

(swap back)

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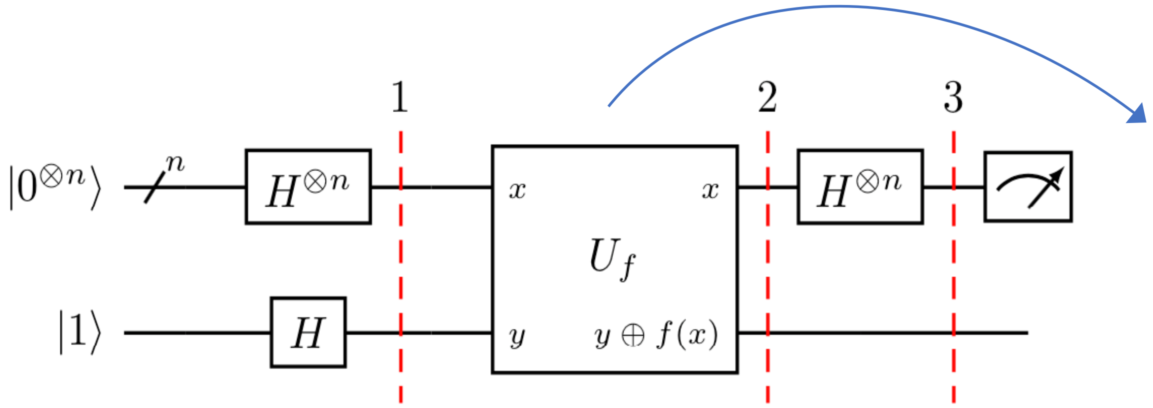

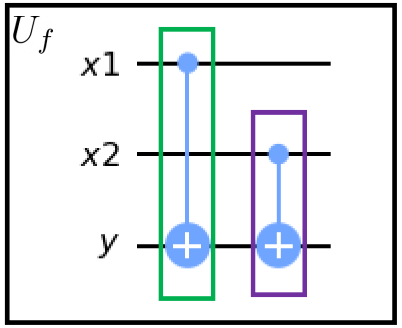

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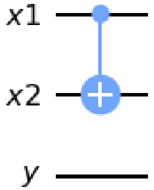
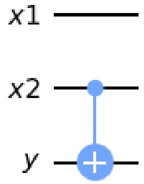


nor

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow$$



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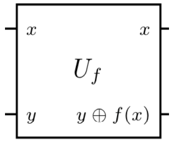
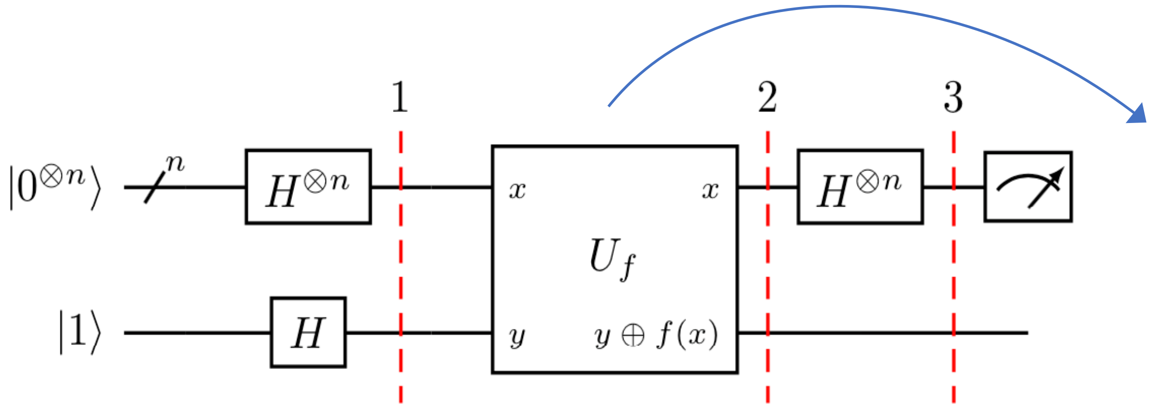


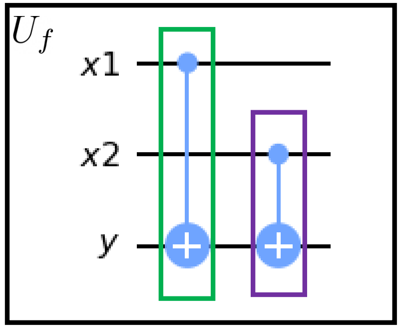
(swap back)

$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

$$= \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{011}|010\rangle + \alpha_{010}|011\rangle + \alpha_{101}|100\rangle + \alpha_{100}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

Uf Gate for n ≥ 2?

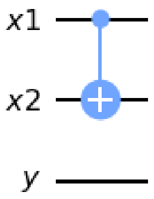




What is the corresponding matrix for this gate?

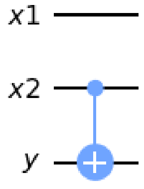
...since it is neither

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$$



nor

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow$$



Next Lecture: want a systematic way of calculating matrices for 2+ qubit gates.

n = 2 Balanced Example

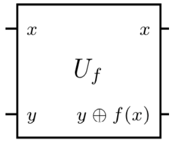
$$f(00) = f(11) = 0$$

$$f(01) = f(10) = 1$$

(XOR function)

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle + \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

(swap back)



$$\alpha_{000}|00, 0 \oplus f(00)\rangle + \alpha_{001}|00, 1 \oplus f(00)\rangle + \alpha_{010}|01, 0 \oplus f(01)\rangle + \alpha_{011}|01, 1 \oplus f(01)\rangle + \alpha_{100}|10, 0 \oplus f(10)\rangle + \alpha_{101}|10, 1 \oplus f(10)\rangle + \alpha_{110}|11, 0 \oplus f(11)\rangle + \alpha_{111}|11, 1 \oplus f(11)\rangle$$

$$= \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{011}|010\rangle + \alpha_{010}|011\rangle + \alpha_{101}|100\rangle + \alpha_{100}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$