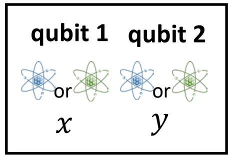
Lecture 6: Kronecker Products and Controlled-NOT Gates

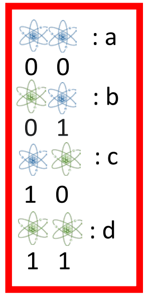
CS 401: Quantum Computing

Dr. Kell, Spring 2023

Review: Two Qubit Circuit

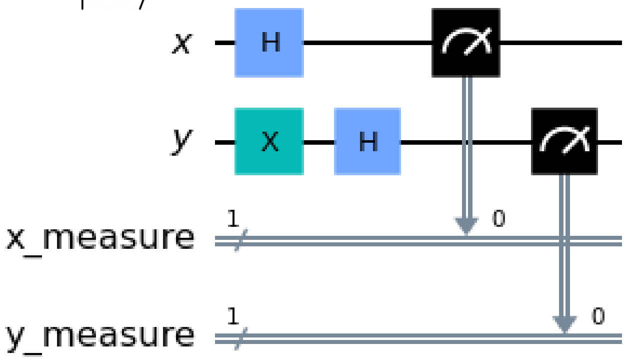


2 qubit -> 4 amplitudes
$a^2 + b^2 + c^2 + d^2 = 1$



$$\longleftrightarrow \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longleftrightarrow a\underset{xy}{|00\rangle} + b\underset{xy}{|01\rangle} + c\underset{xy}{|10\rangle} + d\underset{xy}{|11\rangle}$$

Standard Assumption
starting state = $|00\rangle$



Matrix Multiplication

y —[H]— y —[X]— x —[H]—

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

right to left start state

Bra-ket Notation

$$\underset{xy}{|00\rangle} \quad x\text{—}\boxed{H}\text{—} \quad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \quad y\text{—}\boxed{X}\text{—} \quad \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \quad y\text{—}\boxed{H}\text{—}$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Hadamard just on qubit x.
Amplitude |0> for y qubit stays the same

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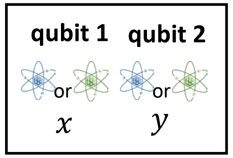

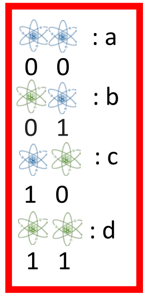

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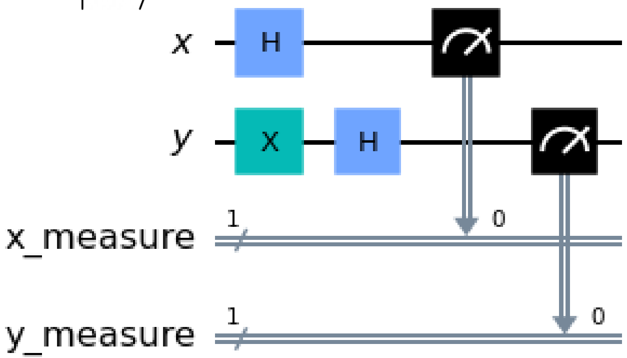

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y —H— y —X— x —H—

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right to left   start state

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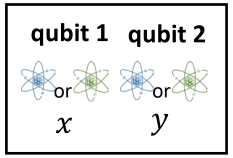

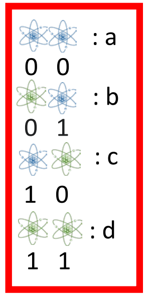

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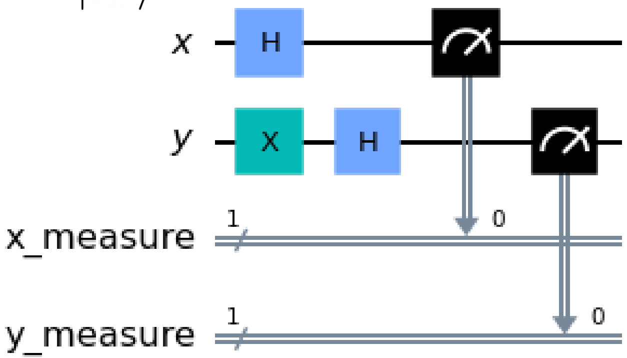

Matrix Multiplication

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How do we calculate these matrices?

right to left

start state

Bra-ket Notation

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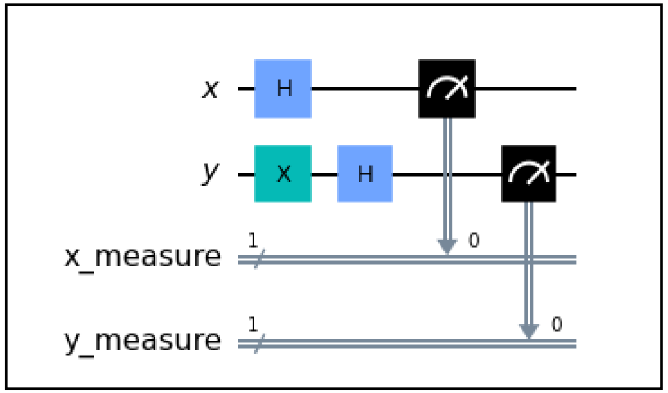
Kronecker Product (Matrix Tensor Product)

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

"Insert and multiply" B into the entries of A



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right to left

start state

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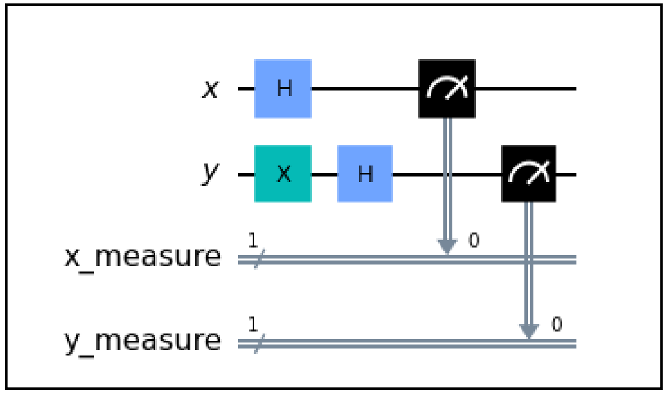
apply H-gate to x... ... keep y the same



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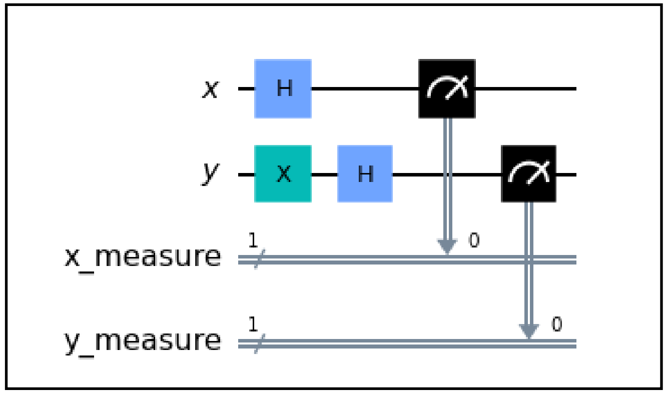

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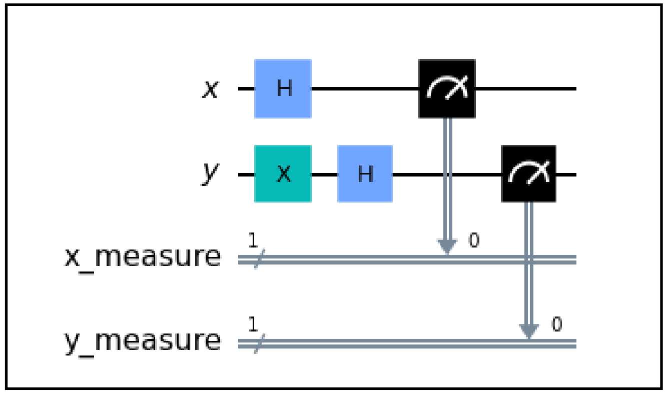

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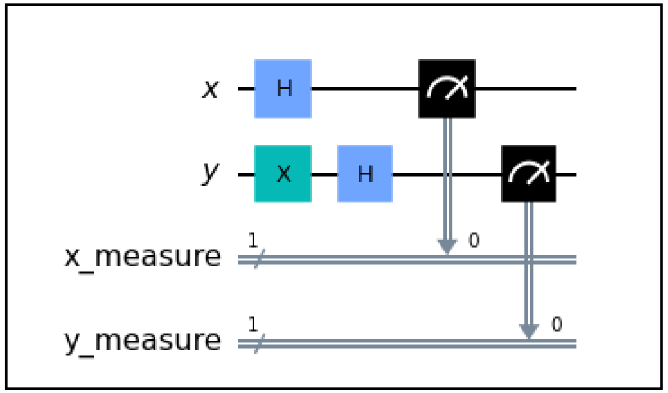
keep x the same… … apply X-gate to y

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



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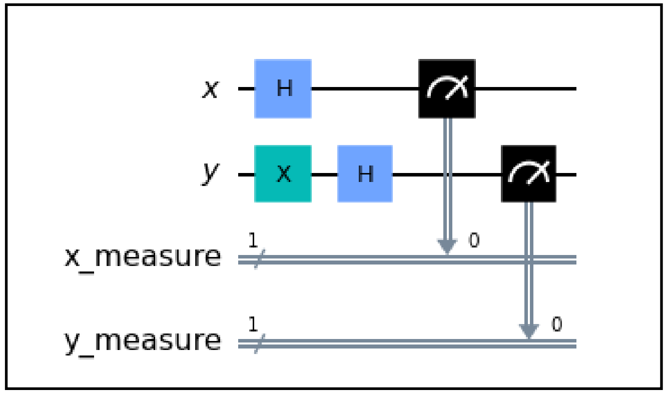

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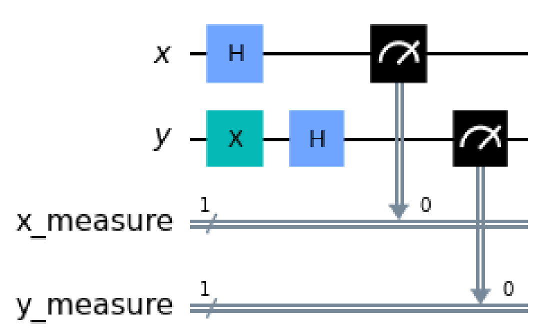

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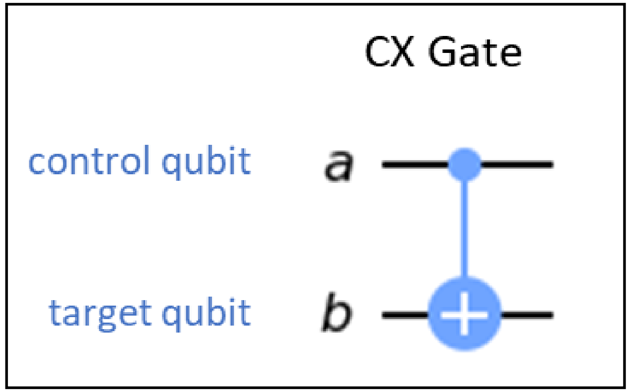
right to left

start state

Controlled-NOT Gate (CX Gate)



$\leftrightarrow$

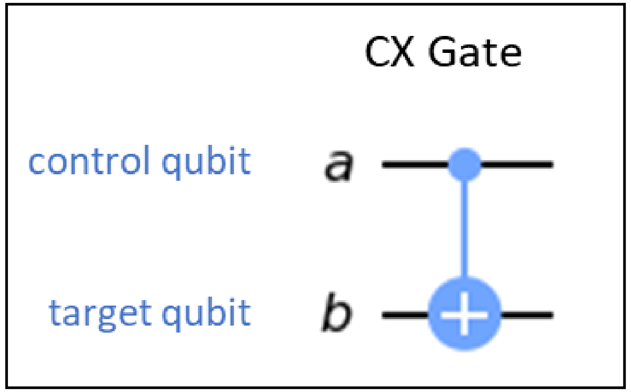
$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \gamma\underset{ab}{|\underline{1}0\rangle} + \delta\underset{ab}{|\underline{1}1\rangle}$$

$$\downarrow$$

$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \delta\underset{ab}{|\underline{1}0\rangle} + \gamma\underset{ab}{|\underline{1}1\rangle}$$

flips amplitudes for states where a =1

Controlled-NOT Gate (CX Gate)



$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \gamma\underset{ab}{|\underline{1}0\rangle} + \delta\underset{ab}{|\underline{1}1\rangle}$$

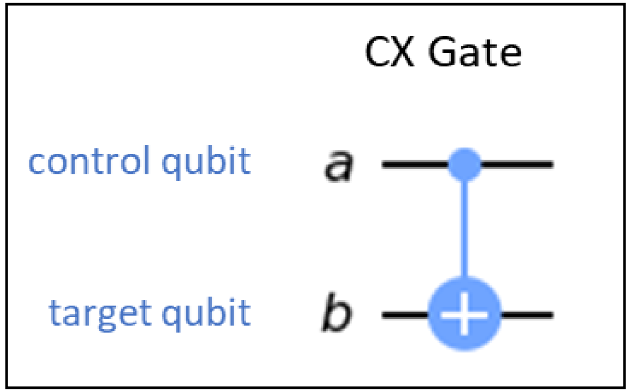
$$\downarrow$$

$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \delta\underset{ab}{|\underline{1}0\rangle} + \gamma\underset{ab}{|\underline{1}1\rangle}$$

flips amplitudes for states where a =1

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & & & & \\ |01\rangle & & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & & \\ |10\rangle & & & & \\ |11\rangle & & & & \end{array}$$

Controlled-NOT Gate (CX Gate)



$\leftrightarrow$

$$\underset{ab}{\alpha|00\rangle} + \underset{ab}{\beta|01\rangle} + \underset{ab}{\gamma|\underline{1}0\rangle} + \underset{ab}{\delta|\underline{1}1\rangle}$$

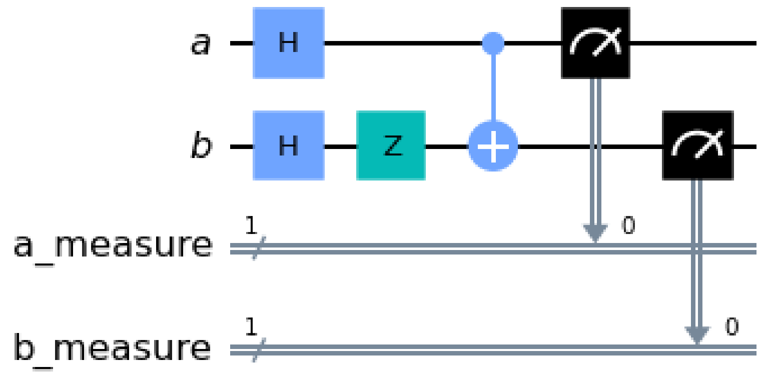
$$\downarrow$$

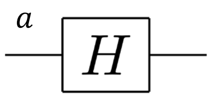
$$\underset{ab}{\alpha|00\rangle} + \underset{ab}{\beta|01\rangle} + \underset{ab}{\delta|\underline{1}0\rangle} + \underset{ab}{\gamma|\underline{1}1\rangle}$$

flips amplitudes for states where a =1

$\leftrightarrow$

$$\begin{array}{cc} & \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

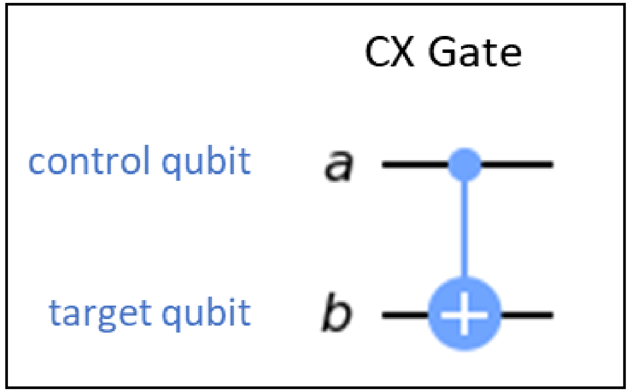




start state

$$\underset{ab}{|00\rangle} \rightarrow$$

Controlled-NOT Gate (CX Gate)



$\leftrightarrow$

$$\alpha|\underset{ab}{00}\rangle + \beta|\underset{ab}{01}\rangle + \gamma|\underset{ab}{\underline{1}0}\rangle + \delta|\underset{ab}{\underline{1}1}\rangle$$

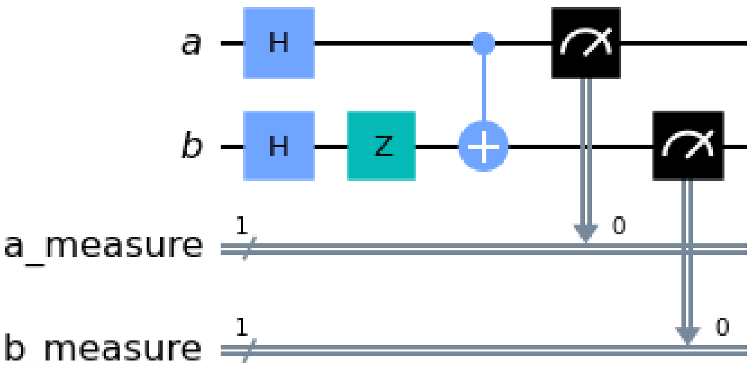
$$\downarrow$$

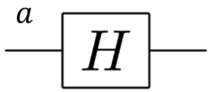
$$\alpha|\underset{ab}{00}\rangle + \beta|\underset{ab}{01}\rangle + \delta|\underset{ab}{\underline{1}0}\rangle + \gamma|\underset{ab}{\underline{1}1}\rangle$$

flips amplitudes for states where a =1

$\leftrightarrow$

$$\begin{array}{c c} & \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

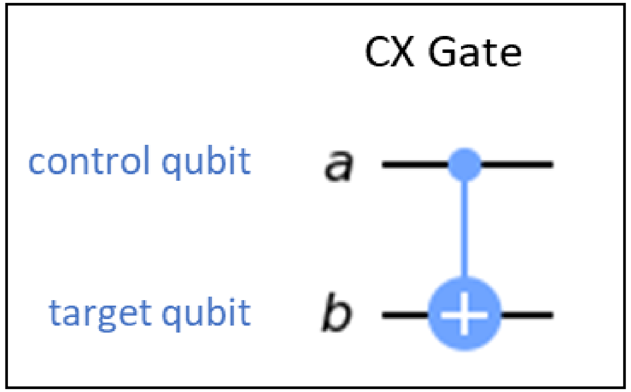




start state

$$|\underset{ab}{00}\rangle \rightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Controlled-NOT Gate (CX Gate)



$\leftrightarrow$

$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \gamma\underset{ab}{|\underline{1}0\rangle} + \delta\underset{ab}{|\underline{1}1\rangle}$$

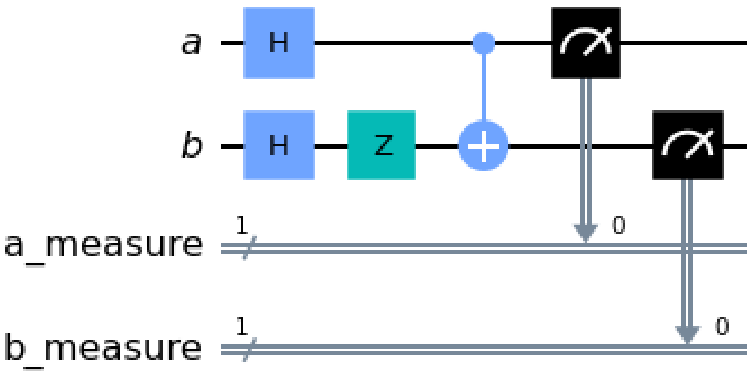
$$\downarrow$$

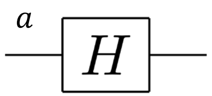
$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \delta\underset{ab}{|\underline{1}0\rangle} + \gamma\underset{ab}{|\underline{1}1\rangle}$$

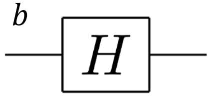
flips amplitudes for states where a =1

$\leftrightarrow$

$$\begin{array}{c c} & \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



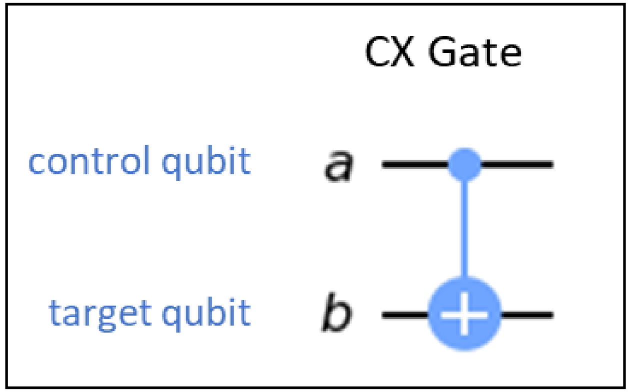




start state

$$\underset{ab}{|00\rangle} \rightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \rightarrow \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

Controlled-NOT Gate (CX Gate)



$\longleftrightarrow$

$$\alpha|\underset{ab}{00}\rangle + \beta|\underset{ab}{01}\rangle + \gamma|\underset{ab}{\underline{1}0}\rangle + \delta|\underset{ab}{\underline{1}1}\rangle$$

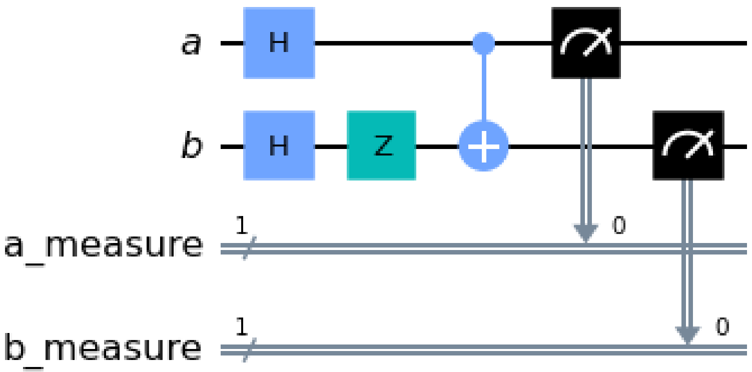
$$\downarrow$$

$$\alpha|\underset{ab}{00}\rangle + \beta|\underset{ab}{01}\rangle + \delta|\underset{ab}{\underline{1}0}\rangle + \gamma|\underset{ab}{\underline{1}1}\rangle$$

flips amplitudes for states where a =1

$\longleftrightarrow$

$$\begin{array}{c c} & \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

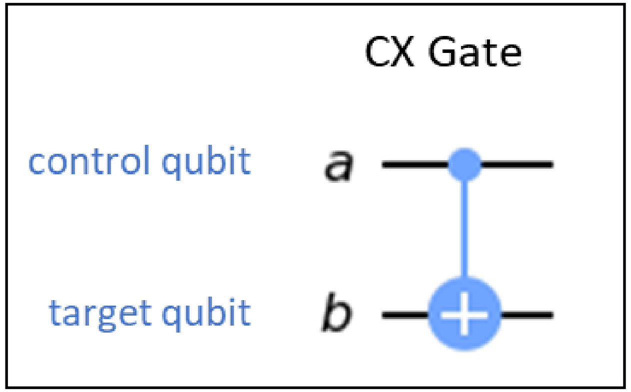


start state

$$|\underset{ab}{00}\rangle \xrightarrow{a\,-\boxed{H}-} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{b\,-\boxed{H}-} \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\xrightarrow{b\,-\boxed{Z}-} \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Controlled-NOT Gate (CX Gate)



$\leftrightarrow$

$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \gamma\underset{ab}{|\underline{1}0\rangle} + \delta\underset{ab}{|\underline{1}1\rangle}$$

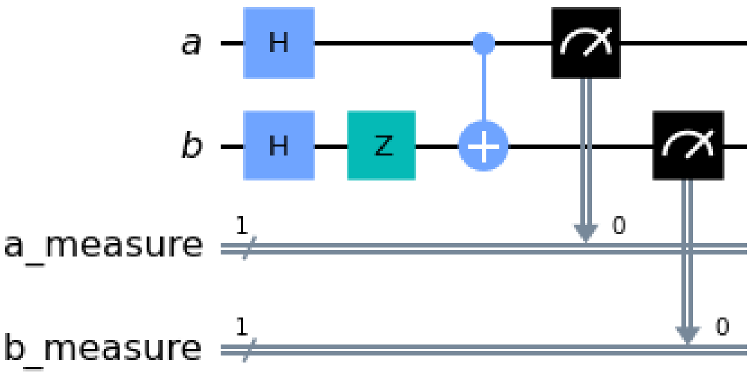
$$\downarrow$$

$$\alpha\underset{ab}{|00\rangle} + \beta\underset{ab}{|01\rangle} + \delta\underset{ab}{|\underline{1}0\rangle} + \gamma\underset{ab}{|\underline{1}1\rangle}$$

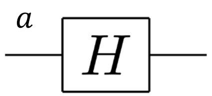
flips amplitudes for states where a =1

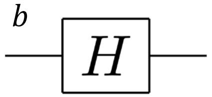
$\leftrightarrow$

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & 1 & 0 & 0 & 0 \\ |01\rangle & 0 & 1 & 0 & 0 \\ |10\rangle & 0 & 0 & 0 & 1 \\ |11\rangle & 0 & 0 & 1 & 0 \end{array}$$

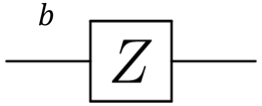


start state

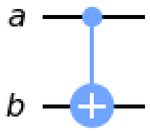




$$\underset{ab}{|00\rangle} \rightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \rightarrow \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$



$$\rightarrow \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$



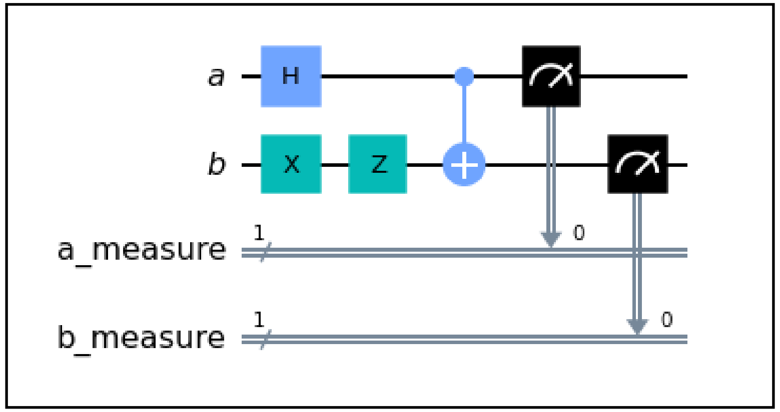
$$\rightarrow \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

Practice Problems

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

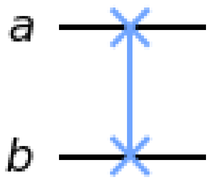
3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$



Circuit 1

4. Design a 2-qubit quantum circuit that performs the following transformation:

$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \longrightarrow \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



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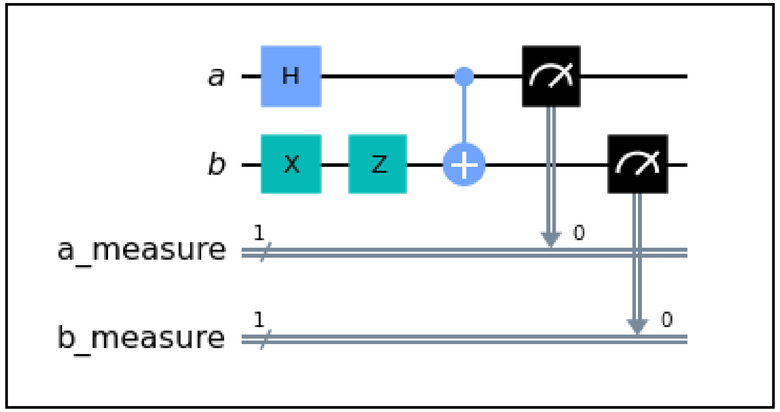

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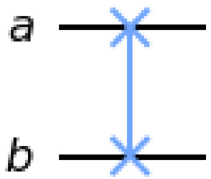

Swap Gate