Lectures 7 and 8: Entanglement and Renormalization

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Review: Kronecker Product (Matrix Tensor Product)

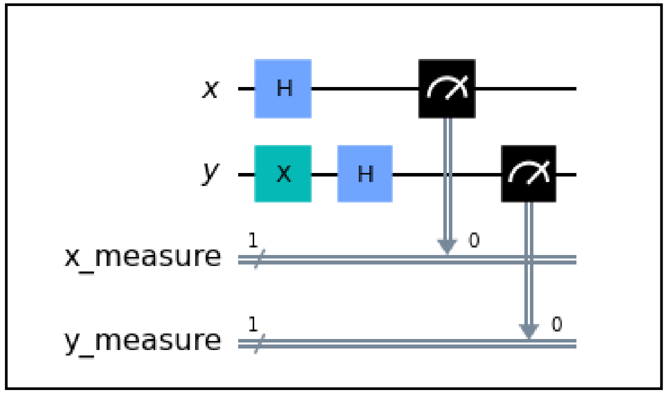
$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

"Insert and multiply" B into the entries of A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} & 0\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \\ 0\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} & 1\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \end{bmatrix}$$

keep x the same... ... apply H-gate to y

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

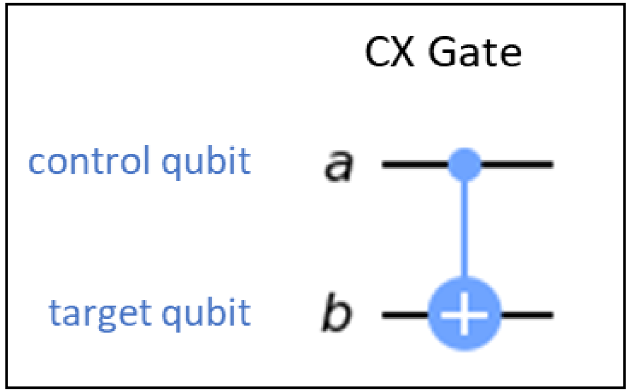


$$\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \boxed{\overset{y\ \text{—}H\text{—}}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}} \overset{y\ \text{—}X\text{—}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \overset{x\ \text{—}H\text{—}}{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

right to left

start state

Review: Controlled-NOT Gate (CX Gate)



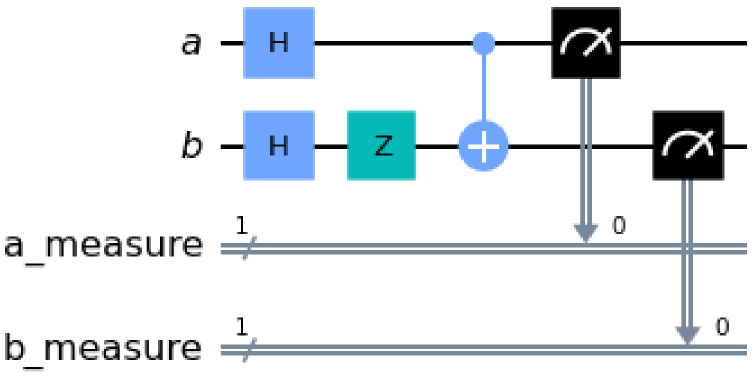
$$\underset{ab}{\alpha|00\rangle} + \underset{ab}{\beta|01\rangle} + \underset{ab}{\gamma|\underline{1}0\rangle} + \underset{ab}{\delta|\underline{1}1\rangle}$$

$$\downarrow$$

$$\underset{ab}{\alpha|00\rangle} + \underset{ab}{\beta|01\rangle} + \underset{ab}{\delta|\underline{1}0\rangle} + \underset{ab}{\gamma|\underline{1}1\rangle}$$

flips amplitudes for states where a =1

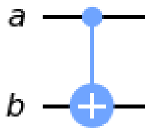
$$\longleftrightarrow \quad \begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & & & & \\ |01\rangle & & \left[\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}\right] & & \\ |10\rangle & & & & \\ |11\rangle & & & & \end{array}$$



start state

$$\underset{ab}{|00\rangle} \xrightarrow{a:\ H} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{b:\ H} \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\xrightarrow{b:\ Z} \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$



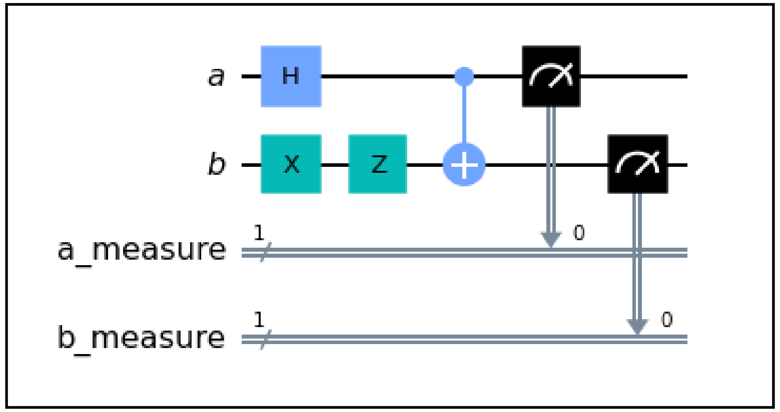
$$\rightarrow \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

Review: Practice Problems

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

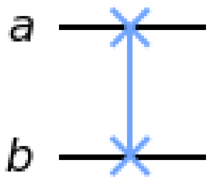
3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$



Circuit 1

4. Design a 2-qubit quantum circuit that performs the following transformation:

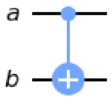
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \longrightarrow \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



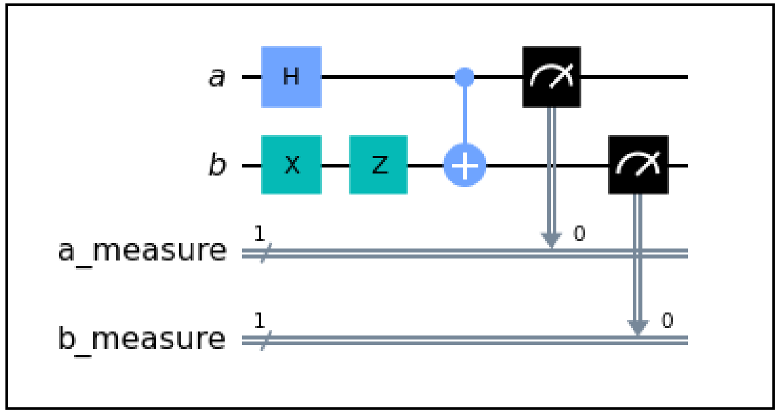
Solutions

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

$$|00\rangle \xrightarrow{H\ (a)} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{X\ (b)} \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \xrightarrow{Z\ (b)} -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle \rightarrow$$



$$\rightarrow -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$



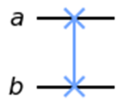
Circuit 1

2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

4. Design a 2-qubit quantum circuit that performs the following transformation:

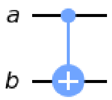
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \qquad \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



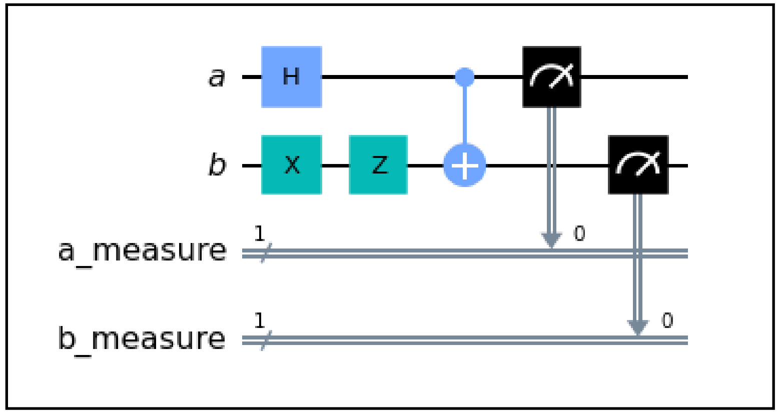
Solutions

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

$$|00\rangle \xrightarrow{a\ \boxed{H}} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{b\ \boxed{X}} \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \xrightarrow{b\ \boxed{Z}} -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle \rightarrow$$



$$\rightarrow -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$



Circuit 1

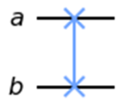
2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

4. Design a 2-qubit quantum circuit that performs the following transformation:

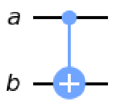
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \qquad \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



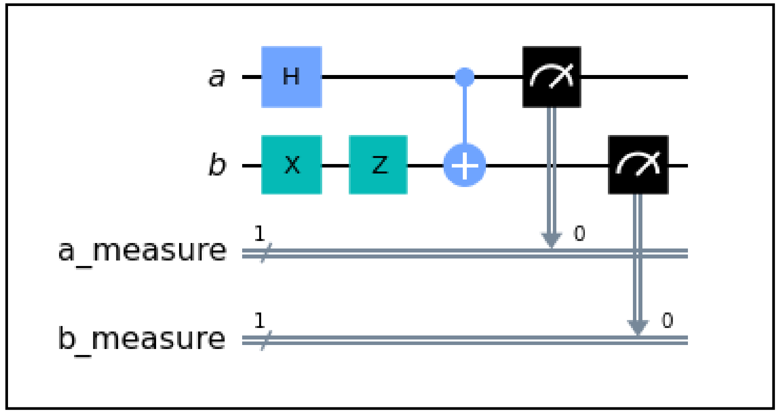
Solutions

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

$$|00\rangle \xrightarrow{a\ \boxed{H}} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{b\ \boxed{X}} \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \xrightarrow{b\ \boxed{Z}} -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle \rightarrow$$



$$\rightarrow -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

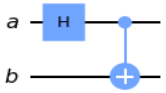


Circuit 1

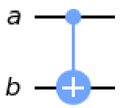
2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$



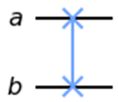
$$|00\rangle \xrightarrow{a\ \boxed{H}} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \rightarrow$$



$$\rightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

4. Design a 2-qubit quantum circuit that performs the following transformation:

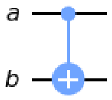
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \qquad \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



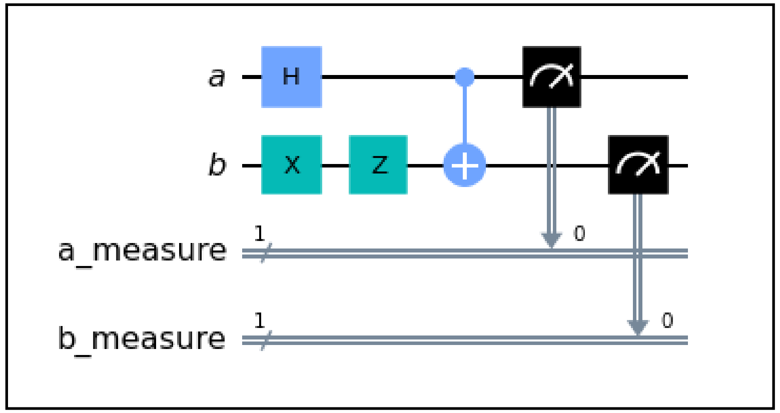
Solutions

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

$$|00\rangle \xrightarrow{H_a} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{X_b} \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \xrightarrow{Z_b} -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle \rightarrow$$



$$\rightarrow -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

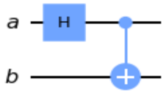


Circuit 1

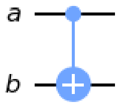
2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$



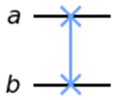
$$|00\rangle \xrightarrow{H_a} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \rightarrow$$



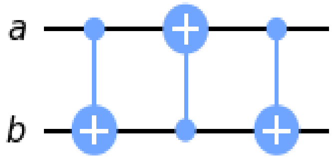
$$\rightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

4. Design a 2-qubit quantum circuit that performs the following transformation:

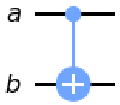
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \qquad \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



Solution

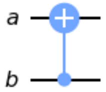


$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow$$



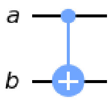
b is now the control bit

$$\alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle \rightarrow$$



$$\alpha|00\rangle + \gamma|01\rangle + \delta|10\rangle + \beta|11\rangle$$

$$\rightarrow$$

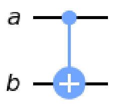


$$\alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$

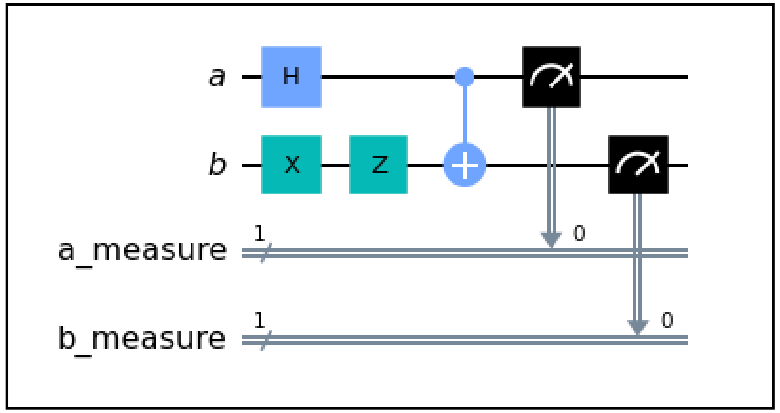
Solutions

1. Show the progression of quantum states using bra-ket notation for Circuit 1.

$$|00\rangle \xrightarrow{\overset{a}{\boxed{H}}} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{\overset{b}{\boxed{X}}} \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle \xrightarrow{\overset{b}{\boxed{Z}}} -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle \rightarrow$$



$$\rightarrow -\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

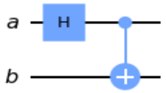


Circuit 1

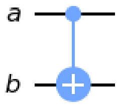
2. Show the sequence matrix multiplications that correspond to the gates in Circuit 1.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

3. Design a 2-qubit quantum circuit whose output state is: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$



$$|00\rangle \xrightarrow{\overset{a}{\boxed{H}}} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \rightarrow$$

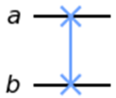


$$\rightarrow \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \rightarrow$$

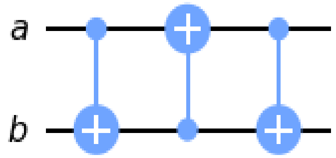
Example of an "entangled state"

4. Design a 2-qubit quantum circuit that performs the following transformation:

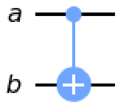
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \qquad \alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$



Solution

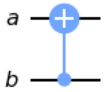


$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow$$



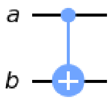
$$\alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle \rightarrow$$

b is now the control bit



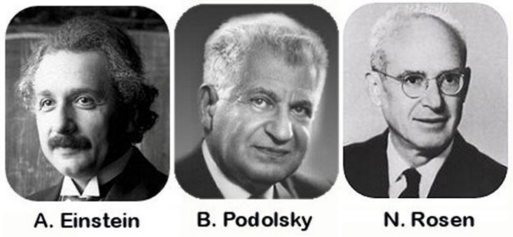
$$\alpha|00\rangle + \gamma|01\rangle + \delta|10\rangle + \beta|11\rangle$$

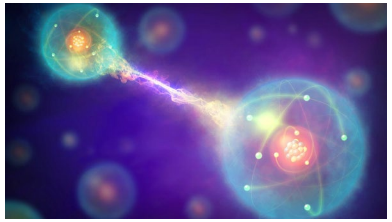
$$\rightarrow$$



$$\alpha|00\rangle + \gamma|01\rangle + \beta|10\rangle + \delta|11\rangle$$

Entanglement





$$|a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \longleftrightarrow \alpha_0\begin{bmatrix}1\\0\end{bmatrix} + \alpha_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix}$$

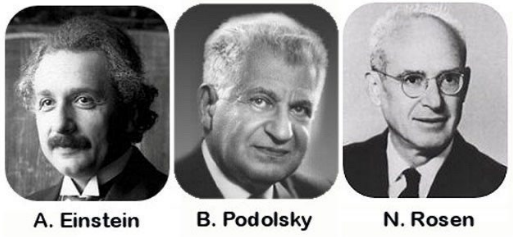
"qubit a"

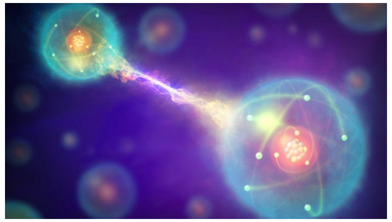
$$|b\rangle = \beta_0|0\rangle + \beta_1|1\rangle \longleftrightarrow \beta_0\begin{bmatrix}1\\0\end{bmatrix} + \beta_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}$$

"qubit b"

What is state of combined system of qubits a and b? (Denoted $|a, b\rangle$)

Entanglement





$$|a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \longleftrightarrow \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

"qubit a"

$$|b\rangle = \beta_0|0\rangle + \beta_1|1\rangle \longleftrightarrow \beta_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

"qubit b"

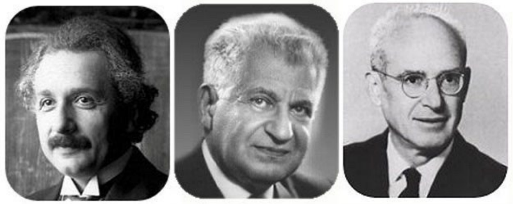
What is state of combined system of qubits a and b? (Denoted $|a, b\rangle$)

$$|a, b\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

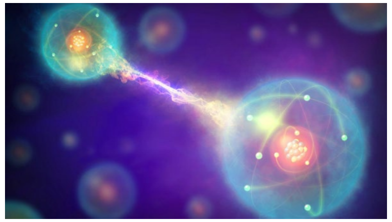
In other words, you calculate Kronecker product (tensor) of the two vectors:

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \\ \alpha_1 \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$

Entanglement



A. Einstein B. Podolsky N. Rosen



$$|a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \longleftrightarrow \alpha_0\begin{bmatrix}1\\0\end{bmatrix} + \alpha_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix}$$

"qubit a"

$$|b\rangle = \beta_0|0\rangle + \beta_1|1\rangle \longleftrightarrow \beta_0\begin{bmatrix}1\\0\end{bmatrix} + \beta_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}$$

"qubit b"

What is state of combined system of qubits a and b? (Denoted $|a, b\rangle$)

$$|a, b\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

In other words, you calculate Kronecker product (tensor) of the two vectors:

$$\begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\\\alpha_1\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\\\alpha_0\beta_1\\\alpha_1\beta_0\\\alpha_1\beta_1\end{bmatrix}$$

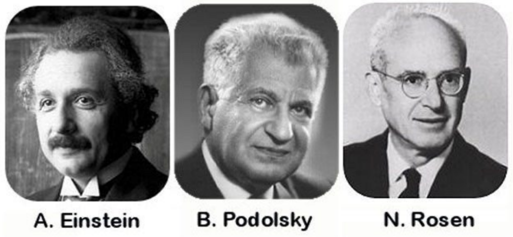
Unentangled Qubits: combined state can be expressed as tensor of individual qubits.

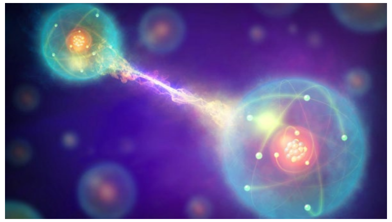
Examples

$$|a, b\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$|a, b\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

Entanglement





$$|a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \longleftrightarrow \alpha_0\begin{bmatrix}1\\0\end{bmatrix} + \alpha_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix}$$

"qubit a"

$$|b\rangle = \beta_0|0\rangle + \beta_1|1\rangle \longleftrightarrow \beta_0\begin{bmatrix}1\\0\end{bmatrix} + \beta_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}$$

"qubit b"

What is state of combined system of qubits a and b? (Denoted $|a,b\rangle$)

$$|a,b\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

In other words, you calculate Kronecker product (tensor) of the two vectors:

$$\begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\\\alpha_1\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\\\alpha_0\beta_1\\\alpha_1\beta_0\\\alpha_1\beta_1\end{bmatrix}$$

Unentangled Qubits: combined state can be expressed as tensor of individual qubits.

Examples

$$|a,b\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$|a,b\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

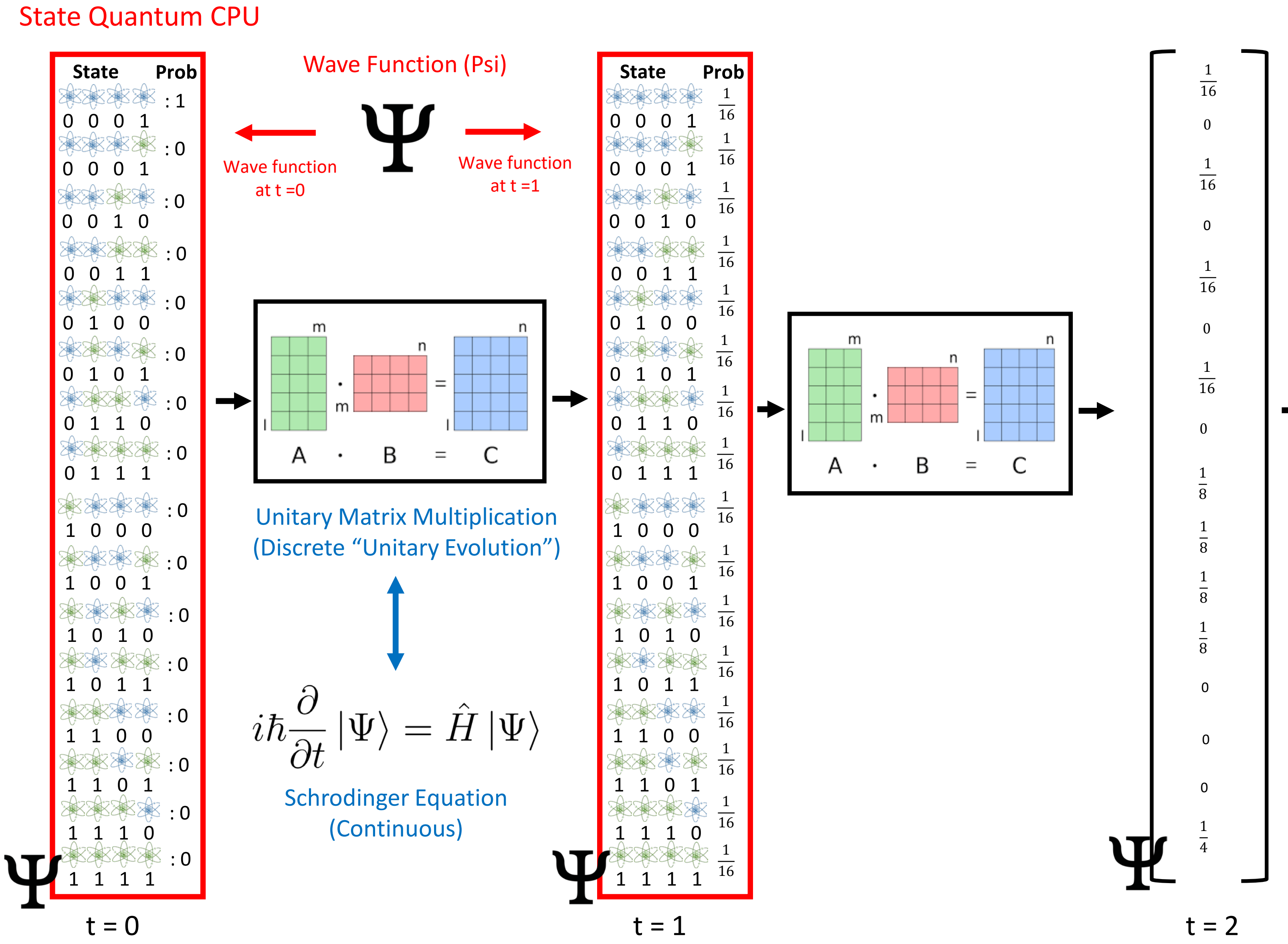
Entangled Qubits: "not unentangled", i.e., no such decomposition exists.

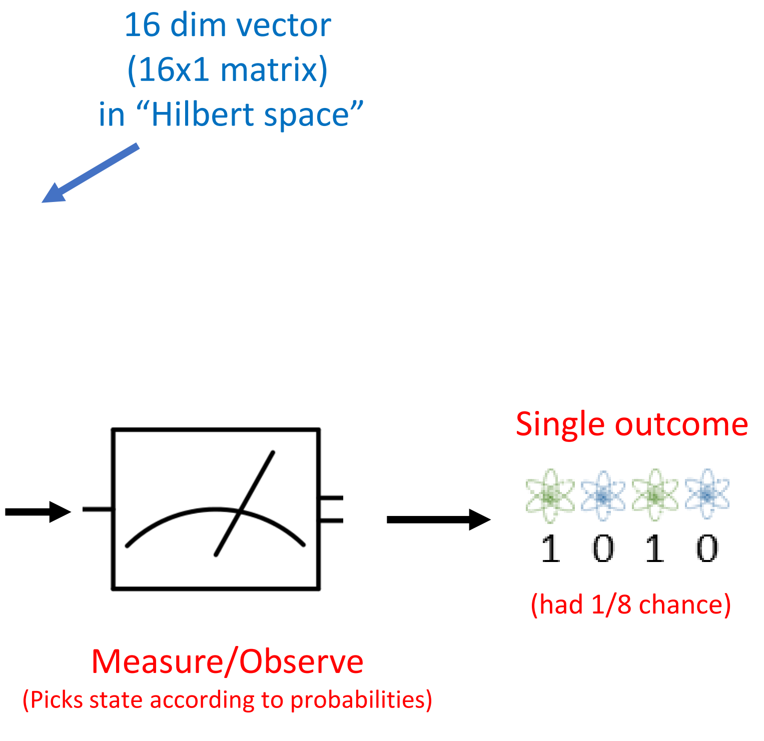
$$|a,b\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

"EPR pair" (Einstein-Pedolsky-Rosen)

Theorem: EPR pair is an entangled state.

Recall: Evolution of Quantum Computers and Terminology





… Well, not exactly

- Probabilities can actually be *negative* or *complex* numbers.
- What happens if we measure some atoms/qubits but not others?

Recall: Evolution of Quantum Computers and Terminology

State Quantum CPU

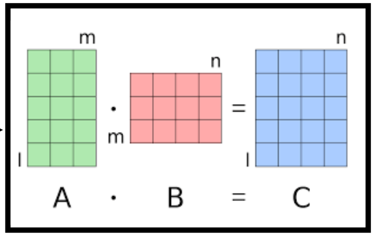
State	Prob
0 0 0 1	: 1
0 0 0 1	: 0
0 0 1 0	: 0
0 0 1 1	: 0
0 1 0 0	: 0
0 1 0 1	: 0
0 1 1 0	: 0
0 1 1 1	: 0
1 0 0 0	: 0
1 0 0 1	: 0
1 0 1 0	: 0
1 0 1 1	: 0
1 1 0 0	: 0
1 1 0 1	: 0
1 1 1 0	: 0
1 1 1 1	: 0

Ψ

t = 0

Wave Function (Psi)

Wave function at t =0 ← Ψ → Wave function at t =1



Unitary Matrix Multiplication
(Discrete "Unitary Evolution")

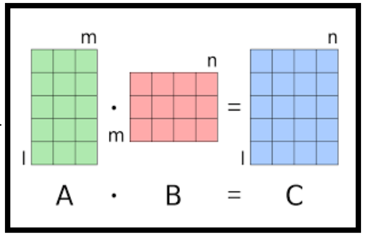
$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$$

Schrodinger Equation
(Continuous)

State	Prob
0 0 0 1	$\frac{1}{16}$
0 0 0 1	$\frac{1}{16}$
0 0 1 0	$\frac{1}{16}$
0 0 1 1	$\frac{1}{16}$
0 1 0 0	$\frac{1}{16}$
0 1 0 1	$\frac{1}{16}$
0 1 1 0	$\frac{1}{16}$
0 1 1 1	$\frac{1}{16}$
1 0 0 0	$\frac{1}{16}$
1 0 0 1	$\frac{1}{16}$
1 0 1 0	$\frac{1}{16}$
1 0 1 1	$\frac{1}{16}$
1 1 0 0	$\frac{1}{16}$
1 1 0 1	$\frac{1}{16}$
1 1 1 0	$\frac{1}{16}$
1 1 1 1	$\frac{1}{16}$

Ψ

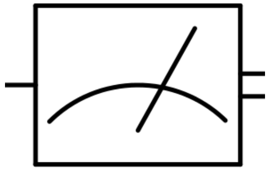
t = 1



$$\Psi \begin{bmatrix} \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

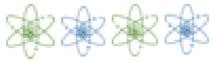
t = 2

16 dim vector
(16x1 matrix)
in "Hilbert space"



Measure/Observe
(Picks state according to probabilities)

Single outcome



1 0 1 0

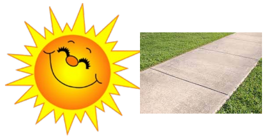
(had 1/8 chance)

… Well, not exactly

- Probabilities can actually be *negative* or *complex* numbers.
- What happens if we measure some atoms/qubits but not others?

Renormalization

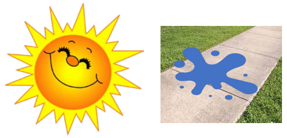
Classic Probability: probabilities for sunny/rainy and wet/dry.



: 90%

Sunny and Dry

Rainy and Dry: 0%



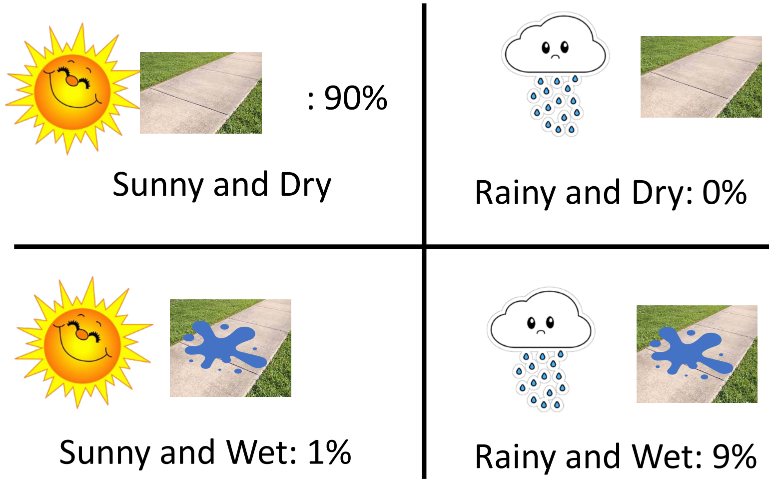
Sunny and Wet: 1%

Rainy and Wet: 9%

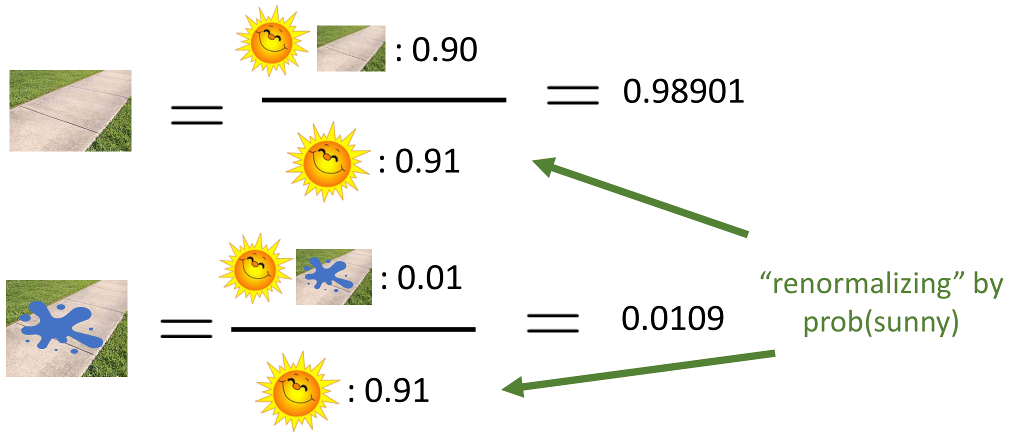
Given it’s sunny, what’s the probability the sidewalk is dry/wet?

Renormalization

Classic Probability: probabilities for sunny/rainy and wet/dry.

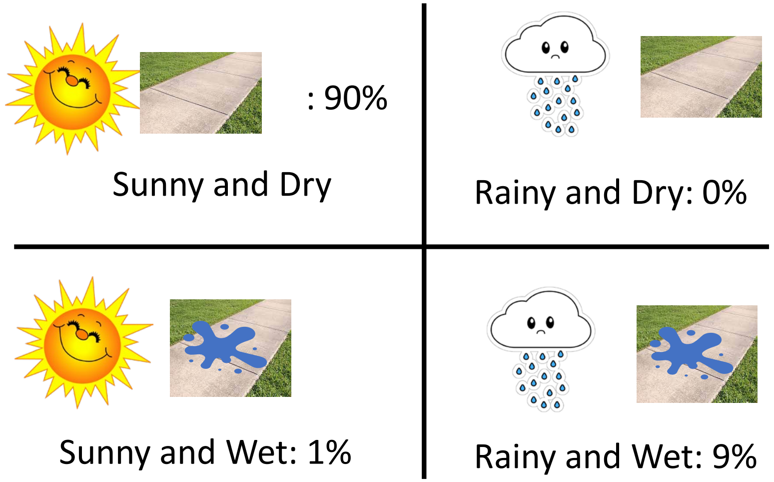


Given it's sunny, what's the probability the sidewalk is dry/wet?

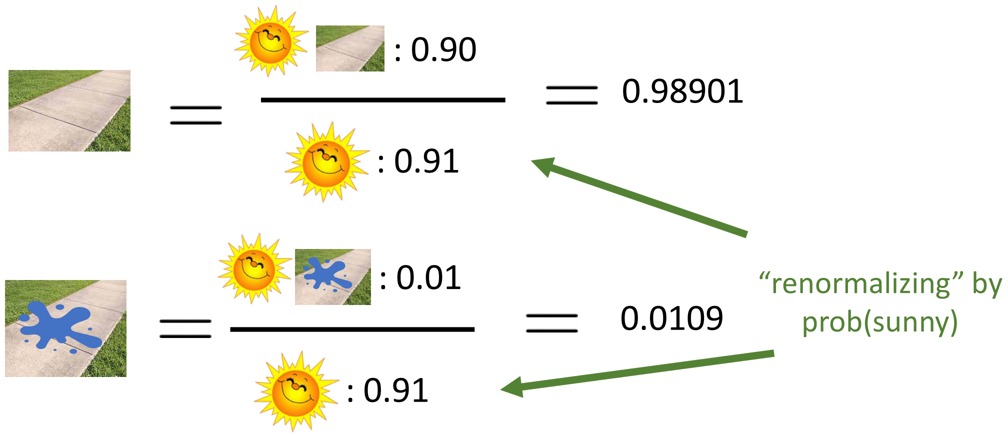


Renormalization

Classic Probability: probabilities for sunny/rainy and wet/dry.



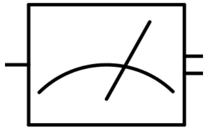
Given it's sunny, what's the probability the sidewalk is dry/wet?



Measurement in Quantum Mechanics: Same idea, but now we renormalize by squared amplitudes instead:

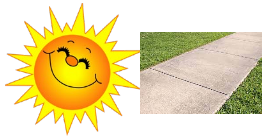
$$|a,b\rangle = \frac{1}{\sqrt{2}}|00\rangle_{ab} + \frac{1}{\sqrt{4}}|01\rangle_{ab} + \frac{1}{4}|10\rangle_{ab} + \frac{\sqrt{3}}{4}|11\rangle_{ab}$$

measure qubit a



Renormalization

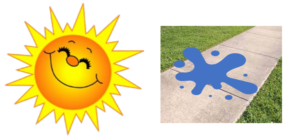
Classic Probability: probabilities for sunny/rainy and wet/dry.



: 90%

Sunny and Dry

Rainy and Dry: 0%



Sunny and Wet: 1%

Rainy and Wet: 9%

Given it's sunny, what's the probability the sidewalk is dry/wet?

=

: 0.90

: 0.91

= 0.98901

=

: 0.01

: 0.91

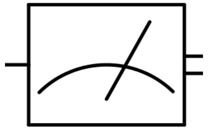
= 0.0109

"renormalizing" by prob(sunny)

Measurement in Quantum Mechanics: Same idea, but now we renormalize by squared amplitudes instead:

$$|a,b\rangle = \frac{1}{\sqrt{2}}|00\rangle_{ab} + \frac{1}{\sqrt{4}}|01\rangle_{ab} + \frac{1}{4}|10\rangle_{ab} + \frac{\sqrt{3}}{4}|11\rangle_{ab}$$

measure qubit a



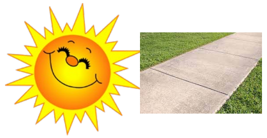
Probabilities for measured state of Qubit a
(now a "classical outcome" once measured)

$|a\rangle = |0\rangle$ with prob: $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{4}}\right)^2 = \frac{3}{4}$

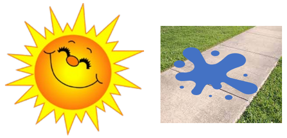
$|a\rangle = |1\rangle$ with prob: $\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{1}{4}$

Renormalization

Classic Probability: probabilities for sunny/rainy and wet/dry.


: 90%
Sunny and Dry


Rainy and Dry: 0%


Sunny and Wet: 1%


Rainy and Wet: 9%

Given it's sunny, what's the probability the sidewalk is dry/wet?


=

: 0.90

: 0.91
= 0.98901


=

: 0.01

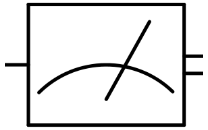
: 0.91
= 0.0109

"renormalizing" by prob(sunny)

Measurement in Quantum Mechanics: Same idea, but now we renormalize by squared amplitudes instead:

$$|a,b\rangle = \boxed{\frac{1}{\sqrt{2}}|00\rangle_{ab} + \frac{1}{\sqrt{4}}|01\rangle_{ab}} + \frac{1}{4}|10\rangle_{ab} + \frac{\sqrt{3}}{4}|11\rangle_{ab}$$

measure qubit a



suppose a = 0

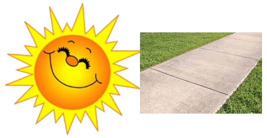
Probabilities for measured state of Qubit a
(now a "classical outcome" once measured)

$$\boxed{|a\rangle = |0\rangle \text{ with prob: } \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{4}}\right)^2 = \frac{3}{4}}$$

$$|a\rangle = |1\rangle \text{ with prob: } \left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{1}{4}$$

Renormalization

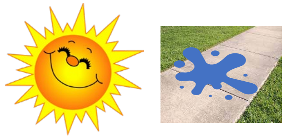
Classic Probability: probabilities for sunny/rainy and wet/dry.



: 90%

Sunny and Dry

Rainy and Dry: 0%



Sunny and Wet: 1%

Rainy and Wet: 9%

Given it's sunny, what's the probability the sidewalk is dry/wet?

 =  : 0.90 /  : 0.91 = 0.98901

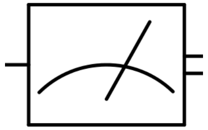
 =  : 0.01 /  : 0.91 = 0.0109

"renormalizing" by prob(sunny)

Measurement in Quantum Mechanics: Same idea, but now we renormalize by squared amplitudes instead:

$$|a, b\rangle = \boxed{\frac{1}{\sqrt{2}}|00\rangle_{ab} + \frac{1}{\sqrt{4}}|01\rangle_{ab}} + \frac{1}{4}|10\rangle_{ab} + \frac{\sqrt{3}}{4}|11\rangle_{ab}$$

measure qubit a



suppose a = 0

Probabilities for measured state of Qubit a
(now a "classical outcome" once measured)

$$\boxed{|a\rangle = |0\rangle \text{ with prob: } \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{4}}\right)^2 = \frac{3}{4}}$$

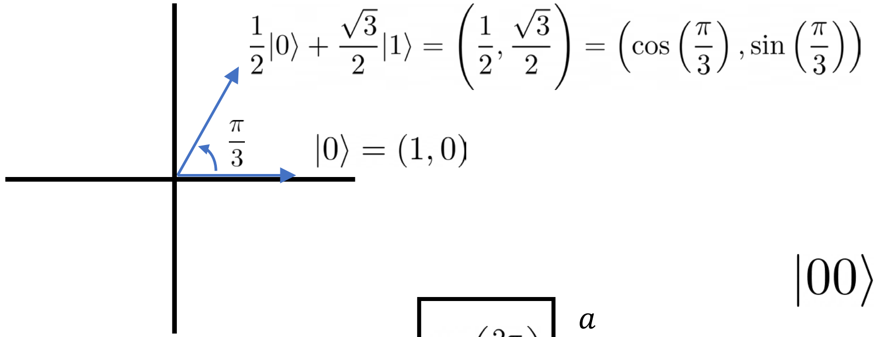
$$|a\rangle = |1\rangle \text{ with prob: } \left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{1}{4}$$

State of qubit b after observing a = 0

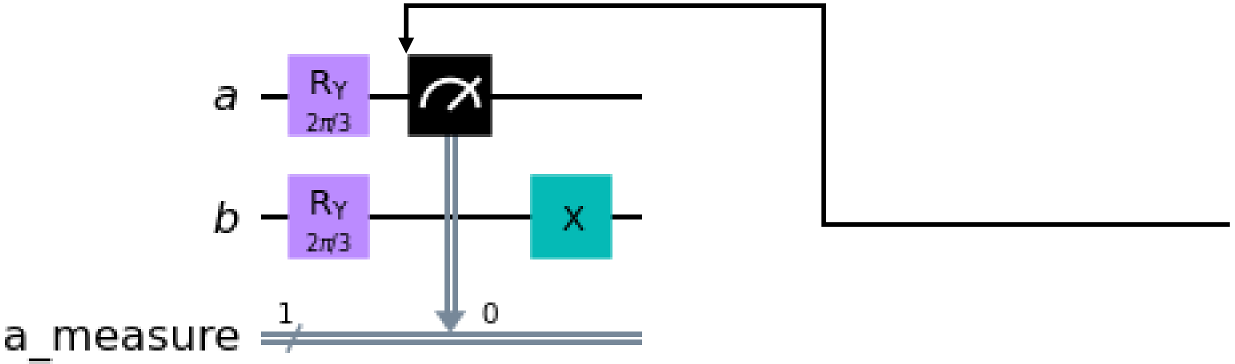
$$|b\rangle = \frac{\frac{1}{\sqrt{2}}}{\sqrt{3/4}}|0\rangle + \frac{\frac{1}{\sqrt{4}}}{\sqrt{3/4}}|1\rangle$$

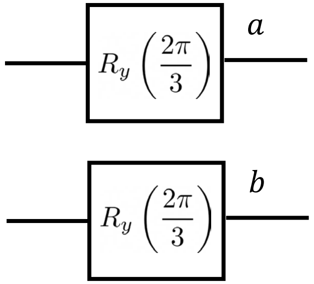
$$= \frac{2}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

Renormalization Practice



What the quantum state at the end of circuit if we measure a = 0?





$$|00\rangle$$

$$\downarrow$$

$$\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|10\rangle$$

$$\downarrow$$

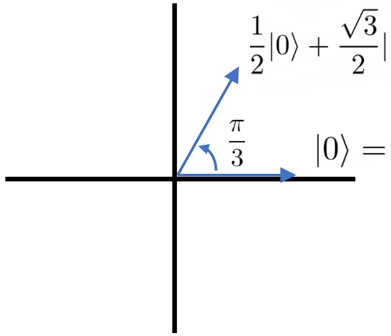
$$\frac{1}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{3}{4}|11\rangle$$

state after rotation gates

$$R_y(\theta) \longleftrightarrow \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Ry-gate (rotate state by theta/2 radians)

Renormalization Practice

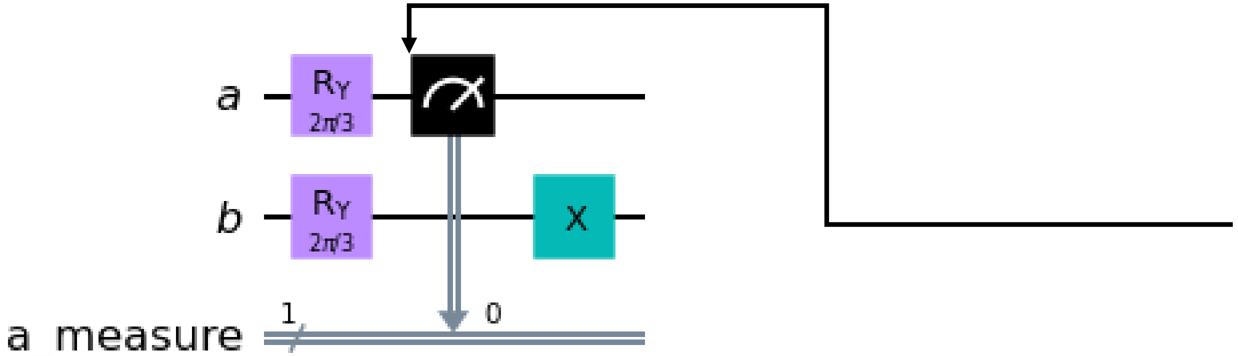


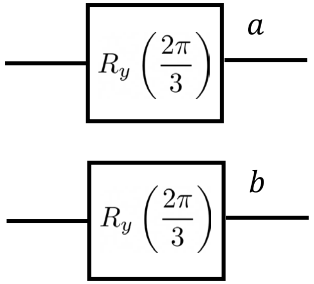
$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)$$

$\frac{\pi}{3}$

$|0\rangle = (1, 0)$

What the quantum state at the end of circuit if we measure a = 0?





$$|00\rangle$$

$$\downarrow$$

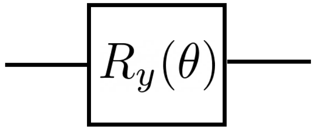
$$\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|10\rangle$$

$$\downarrow$$

$$\frac{1}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{3}{4}|11\rangle$$

state after rotation gates

... now determine final state after renormalization and X gate
(note: derivation of state is not needed to solve problem)



$$\longleftrightarrow \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Ry-gate (rotate state by theta/2 radians)