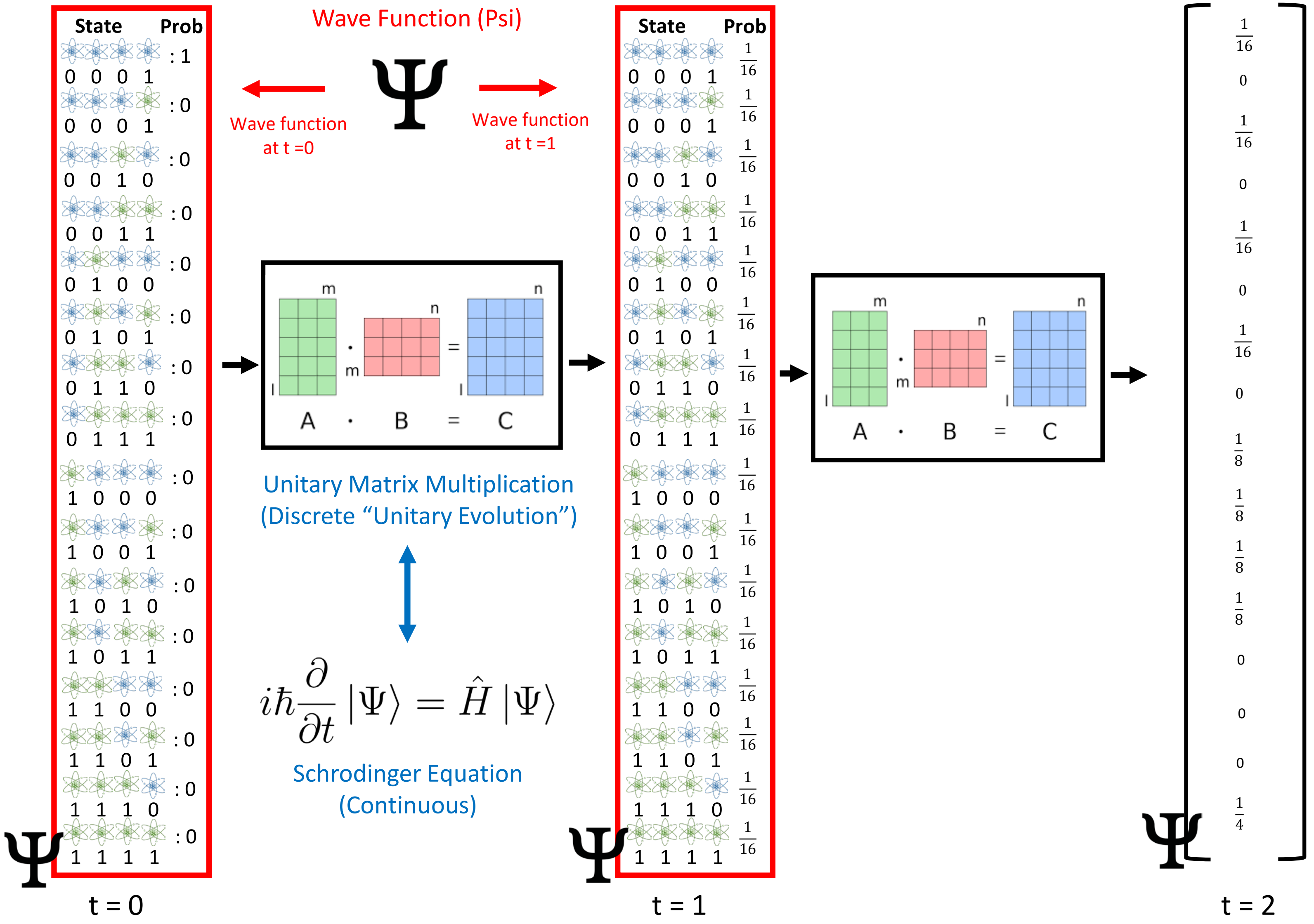
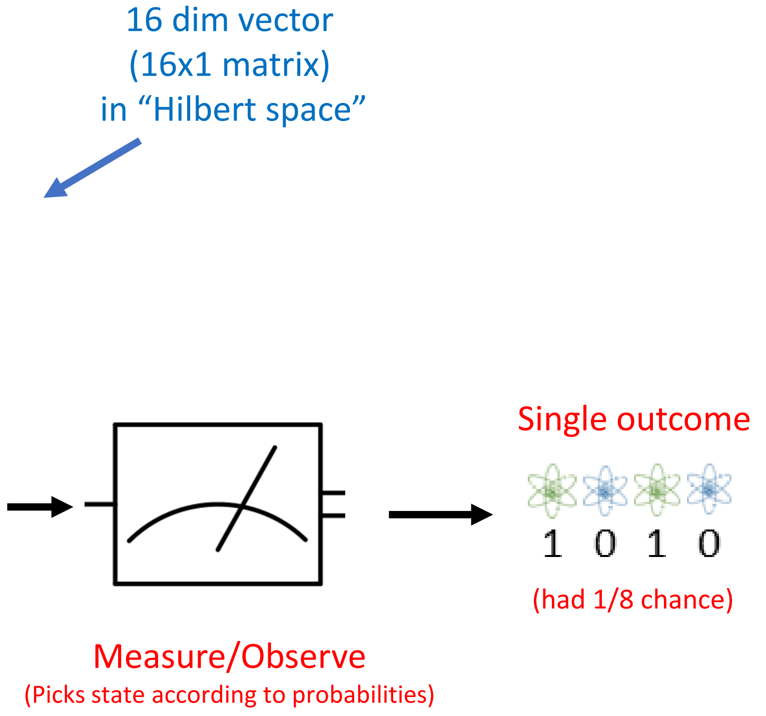
Lecture 4: Superpositions, Amplitudes, and Unitary Operators

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Review: Evolution Quantum Computers and Terminology





… Well, not exactly

- Probabilities can actually be *negative* or *complex* numbers.
- What happens if we measure some atoms/qubits but not others?

Review: Evolution Quantum Computers and Terminology

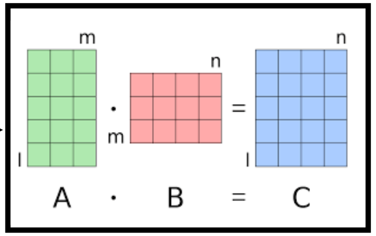
State	Prob
0 0 0 0	: 1
0 0 0 1	: 0
0 0 1 0	: 0
0 0 1 1	: 0
0 1 0 0	: 0
0 1 0 1	: 0
0 1 1 0	: 0
0 1 1 1	: 0
1 0 0 0	: 0
1 0 0 1	: 0
1 0 1 0	: 0
1 0 1 1	: 0
1 1 0 0	: 0
1 1 0 1	: 0
1 1 1 0	: 0
1 1 1 1	: 0

Ψ

t = 0

Heisenberg

Helgoland (1924)



Unitary Matrix Multiplication
(Discrete "Unitary Evolution")

Schrodinger (1926)

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$$

Schrodinger Equation
(Continuous)

State	Prob
0 0 0 1	$\frac{1}{16}$
0 0 0 1	$\frac{1}{16}$
0 0 1 0	$\frac{1}{16}$
0 0 1 1	$\frac{1}{16}$
0 1 0 0	$\frac{1}{16}$
0 1 0 1	$\frac{1}{16}$
0 1 1 0	$\frac{1}{16}$
0 1 1 1	$\frac{1}{16}$
1 0 0 0	$\frac{1}{16}$
1 0 0 1	$\frac{1}{16}$
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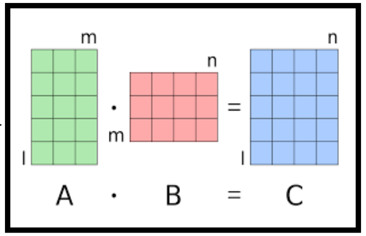
Ψ

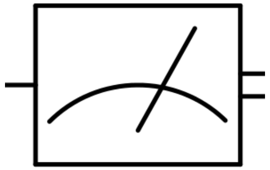
t = 1



$$\Psi = \begin{bmatrix} \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

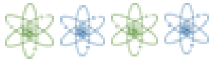
t = 2

16 dim vector
(16x1 matrix)
in "Hilbert space"



Measure/Observe
(Picks state according to probabilities)

Single outcome



1 0 1 0

(had 1/8 chance)

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- Probabilities can actually be *negative* or *complex* numbers.
- What happens if we measure some atoms/qubits but not others?

Canonical Problems with Quantum Advantage

Problem 1: Factoring Integers

Input: integer x.
Output: non-trivial factors of x.

x = 54 ⟶ 2, 3, 6, 9, 18, 27

Best Classical Algorithm: $O(2^n)$ for n bit numbers

Shor's Quantum Algorithm: O(poly(n))

Many cryptography schemes (e.g., RSA) rely on exponential runtime for the problem.

Problem 2: Search Problem

Input: list L, target value
Output: index of target in L

L = [2, 1, 10, 4, 7, 9, 3] ⟶ 4
target = 7 (index of 7)

Many applications in cloud quantum computing, databases, etc.

Canonical Problems with Quantum Advantage

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Best Possible Classical Algorithm: O(n)

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Many applications in cloud quantum computing, databases, etc.

Best Possible Classical Algorithm: O(n)

Grover's Quantum Algorithm: O($n^{1/2}$)

Review: Evolution Quantum Computers and Terminology

State Quantum CPU

Ψ — t = 0

State	Prob
0 0 0 1	: 1
0 0 0 1	: 0
0 0 1 0	: 0
0 0 1 1	: 0
0 1 0 0	: 0
0 1 0 1	: 0
0 1 1 0	: 0
0 1 1 1	: 0
1 0 0 0	: 0
1 0 0 1	: 0
1 0 1 0	: 0
1 0 1 1	: 0
1 1 0 0	: 0
1 1 0 1	: 0
1 1 1 0	: 0
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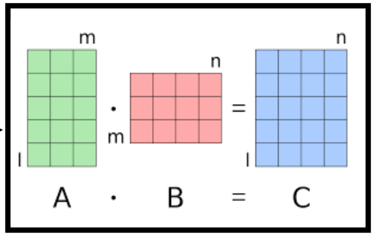
Wave Function (Psi)

Ψ

Wave function at t =0 ← Ψ → Wave function at t =1



Unitary Matrix Multiplication
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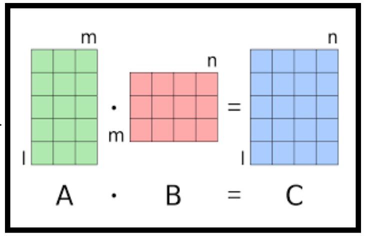

$$\Psi = \begin{bmatrix} \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{16} \\ 0 \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

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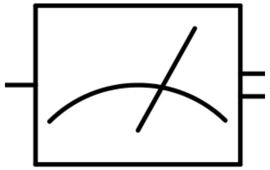

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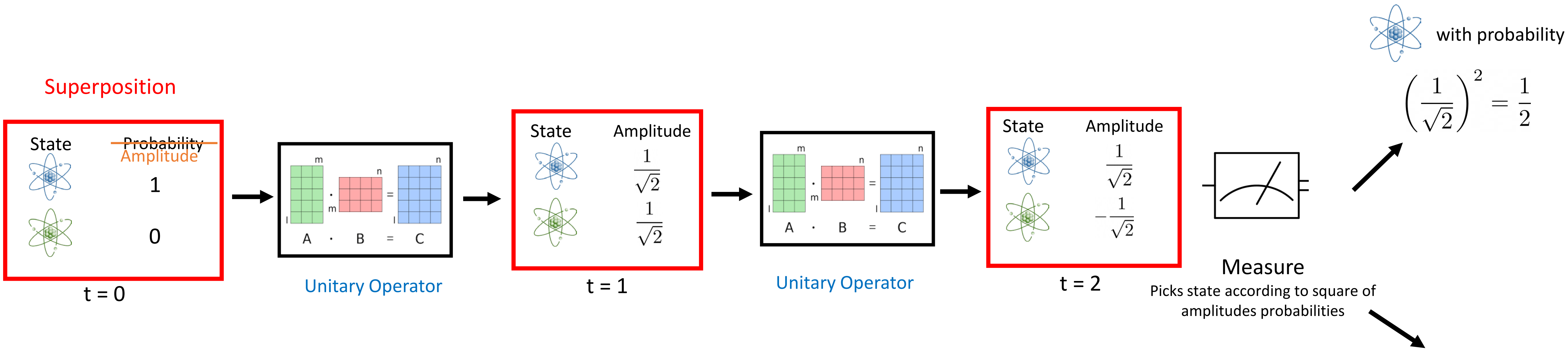
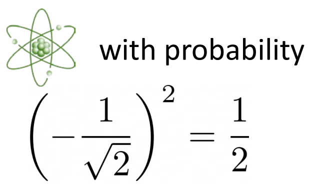
Superpositions and Amplitudes in Single Qubit Computation

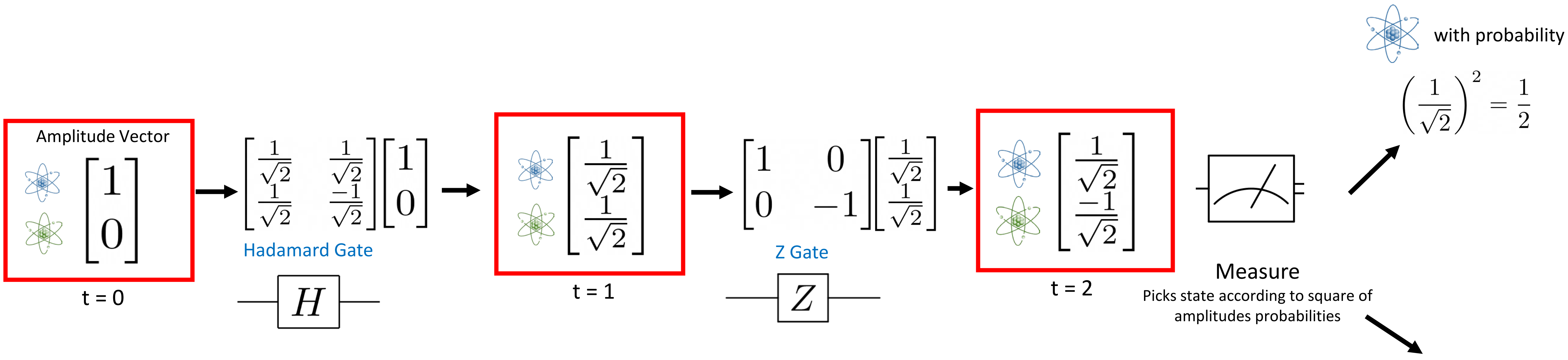


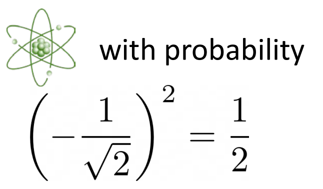


Distributions and Probabilities -> Superpositions and Amplitudes

- Sum of *squared* amplitudes must be 1.
- Probabilities are calculated from squared values of amplitudes.
- Note this is a valid method for maintaining/calculating probabilities even if amplitudes are **negative** (or even complex).

Superpositions and Amplitudes in Single Qubit Computation





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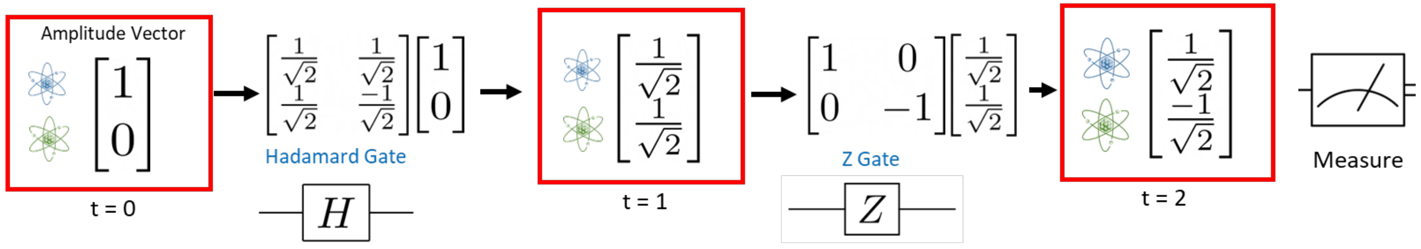
Different Representations



Matrix Multiplications

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

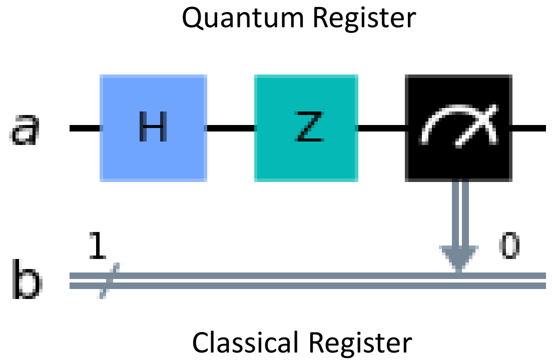
Matrix Operations (from right to left)

Starting Vector

Quantum Circuit

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$: H Hadamard Gate

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$: Z Z Gate



Different Representations

Amplitude Vector

$$\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

t = 0 | Hadamard Gate H | t = 1 | Z Gate Z | t = 2 | Measure

“Bra-ket Notation”

(or Dirac notation )

$$|0\rangle \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix} \qquad |1\rangle \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix}$$

Matrix Multiplications

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

Matrix Operations (from right to left)

Starting Vector

Quantum Circuit

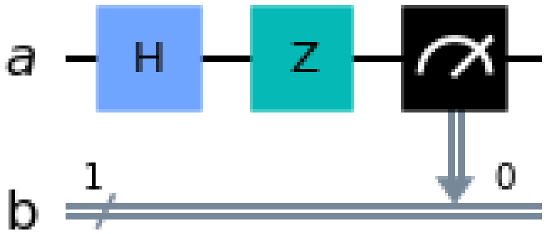
$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix} : \; H$$

Hadamard Gate

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} : \; Z$$

Z Gate

Quantum Register



Classical Register

Different Representations

Amplitude Vector

$$\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

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Matrix Operations (from right to left)

Starting Vector

$$|0\rangle \; \text{—}\boxed{H}\text{—} \rightarrow \overbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}^{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}} \rightarrow$$

$$\text{—}\boxed{Z}\text{—} \rightarrow \underbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{-1}{\sqrt{2}}|1\rangle}_{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}}$$

Quantum Circuit

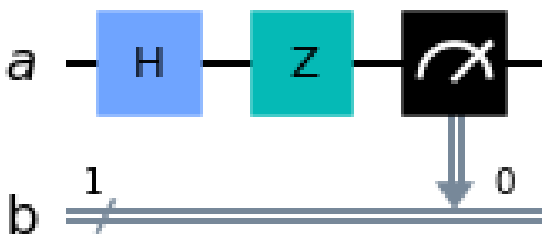
$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix} : \text{—}\boxed{H}\text{—}$$

Hadamard Gate

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} : \text{—}\boxed{Z}\text{—}$$

Z Gate

Quantum Register



Classical Register

Valid Matrix Operations: Unitary Matrices

Algebraic Definition: (conjugate) transpose equals inverse.

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{H^T} =$$

Valid Matrix Operations: Unitary Matrices

Algebraic Definition: (conjugate) transpose equals inverse.

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{H^T} = \begin{bmatrix} 1/2 + 1/2 & -1/2 + 1/2 \\ -1/2 + 1/2 & 1/2 + 1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

(Identity Matrix)

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(Identity Matrix)

Geometric Intuition

- Preserves vector lengths.
  (Thus, maintaining square of amplitudes can be used to calculate probabilities)
- Angles between vectors between before and after the transformation are left unchanged.
  (Can only rotate and reflect space)
- Linear transformation visualizer: https://shad.io/MatVis/