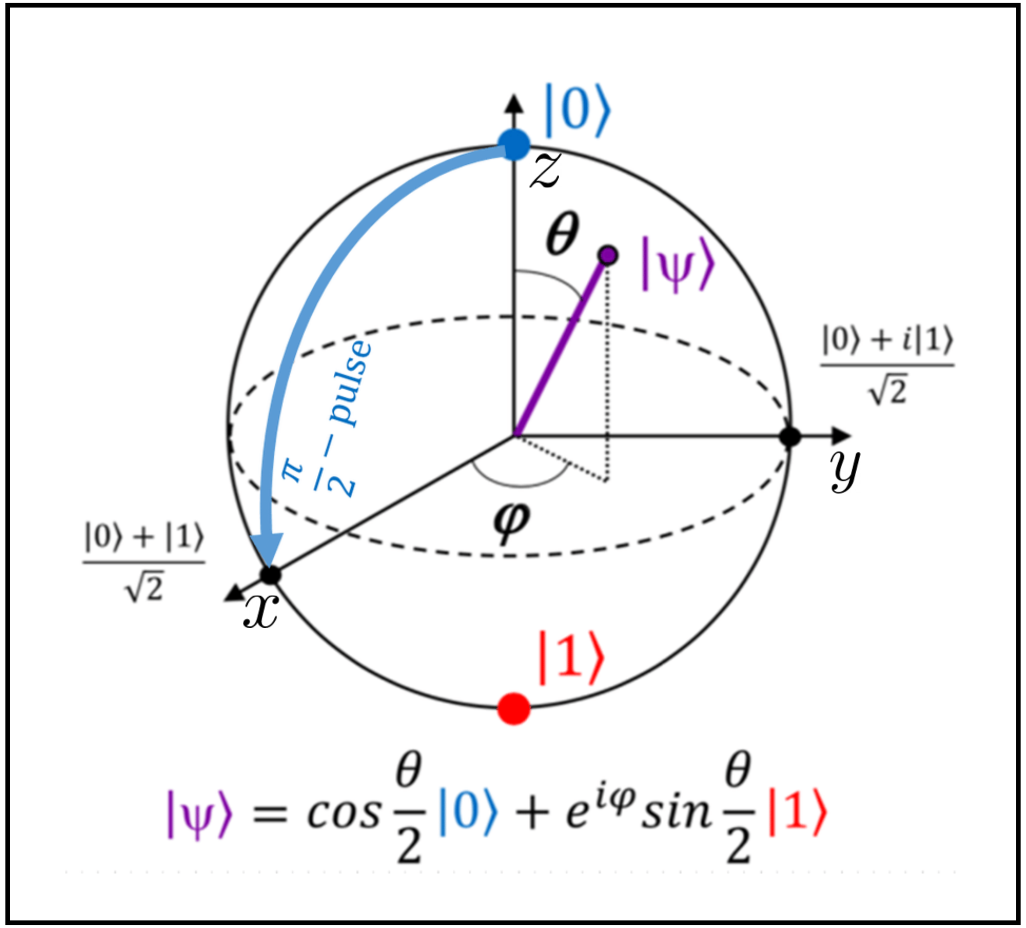
Lectures 13-14: Bloch Spheres

CS 401: Quantum Computing

Dr. Kell, Spring 2023

Bloch Spheres

Bloch Sphere: 3D representation of amplitudes for single qubit.



(Figure courtesy of Jazeari, Beckers, and Tajalli, 2019)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

"equivalent" $\equiv \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$

"Equivalent" in the sense they have the same measurement probabilities, but values of alpha and beta may be different.

Parameter θ **:** determines **latitude** of position on sphere.
Units: radians from the north pole. Ranges $[0, \pi]$.

Parameter ϕ **:** determines **longitude** of position on sphere.
Units: radians from plane defined by x-axis. Ranges $[0, 2\pi]$.

Derivation of Form: $\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$

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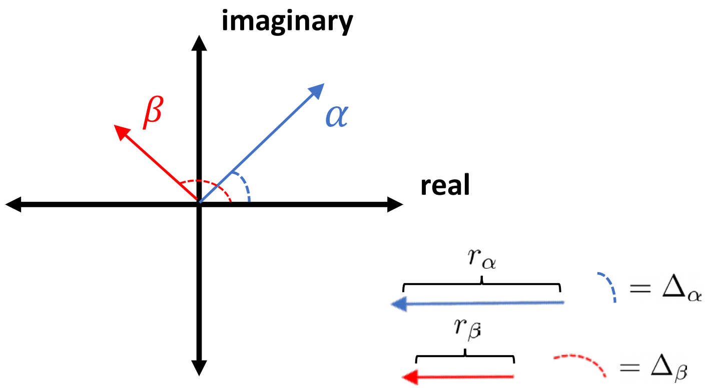

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \boxed{r_\alpha\left[\cos(\Delta_\alpha) + i\sin(\Delta_\alpha)\right]}|0\rangle + \boxed{r_\beta\left[\cos(\Delta_\beta) + i\sin(\Delta_\beta)\right]}|1\rangle$$

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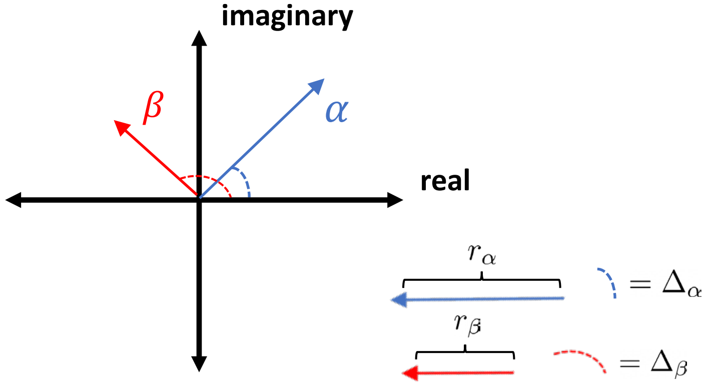

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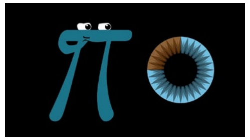

3Blue1Brown Video for intuition: $e^{ix} = \cos(x) + i\sin(x)$

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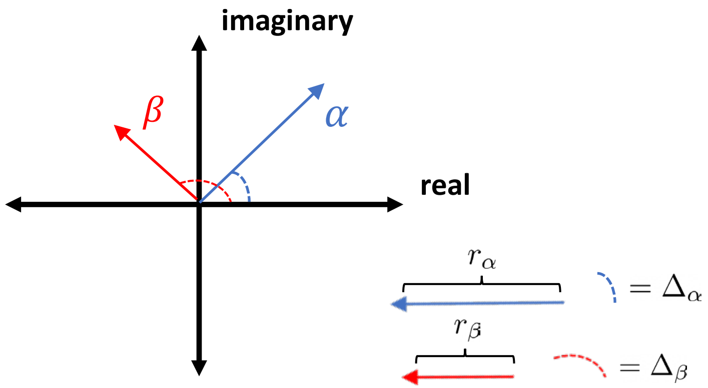

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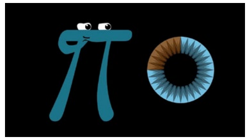

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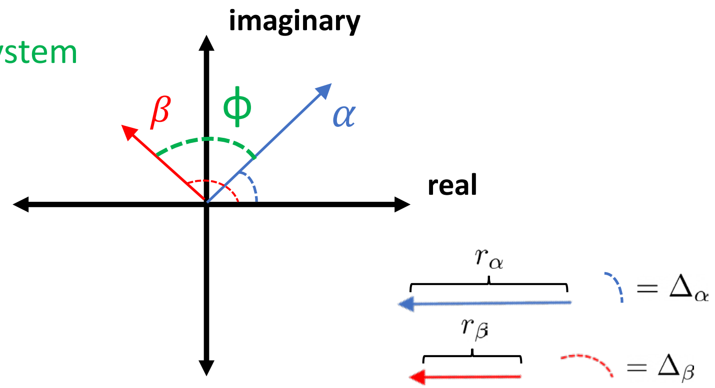
In physics ϕ = phase of qubit's physical system



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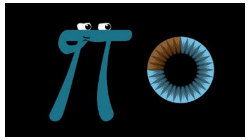

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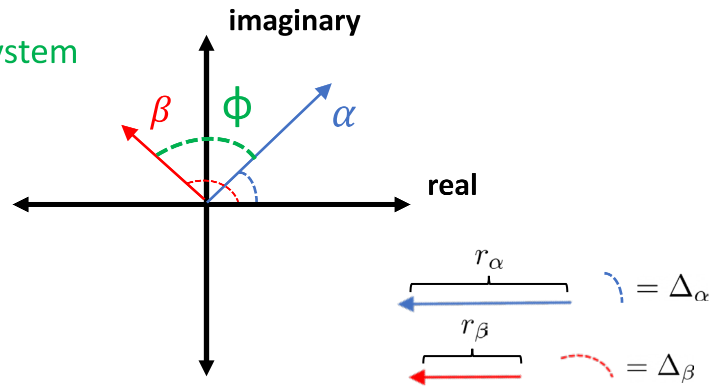

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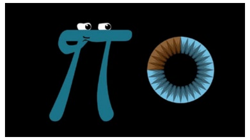

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Does not affect measurement probabilities.

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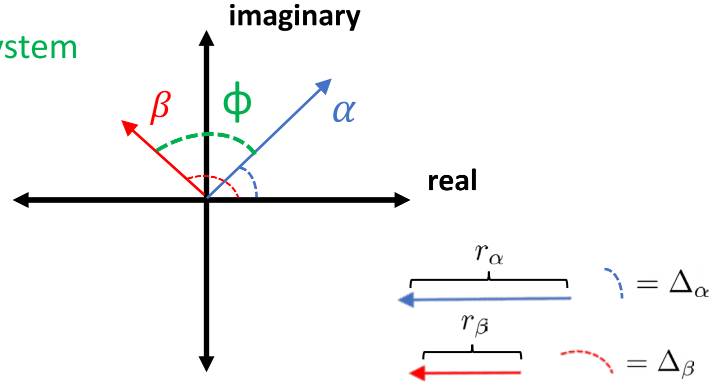

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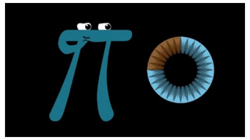

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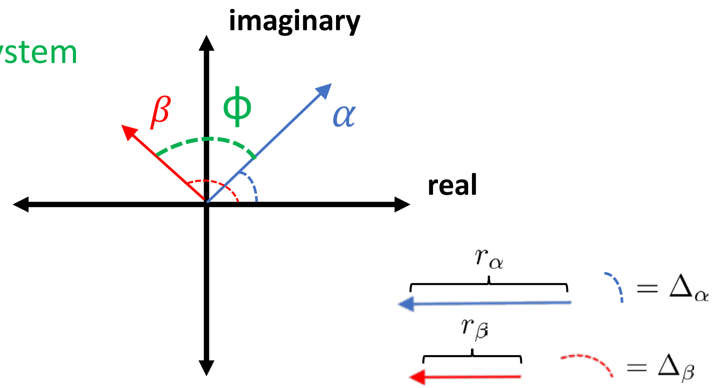

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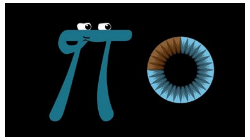

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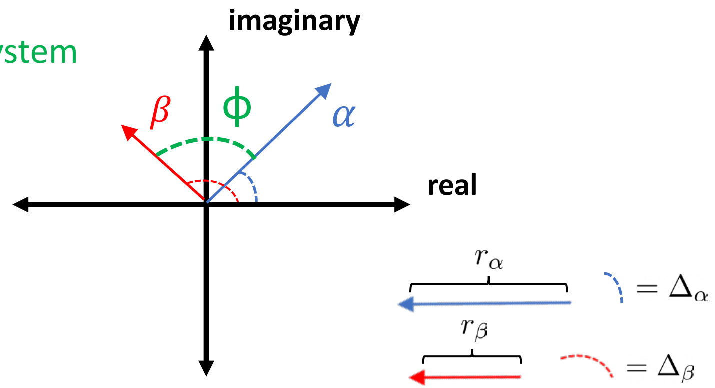

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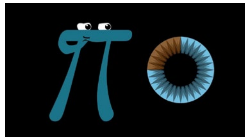

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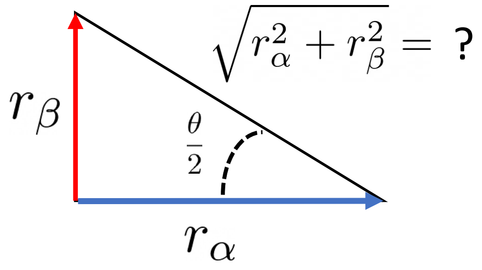

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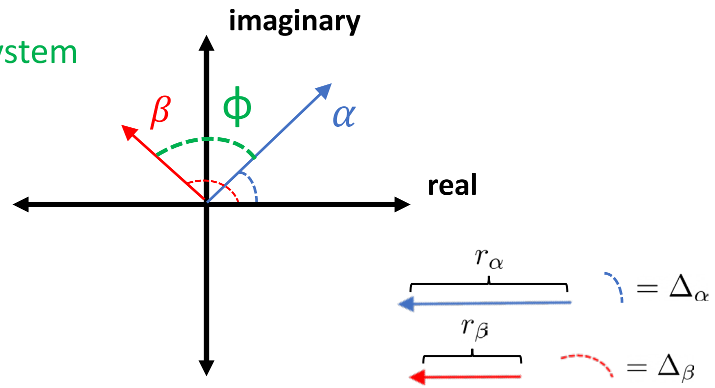

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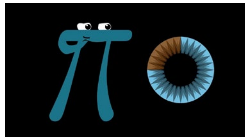

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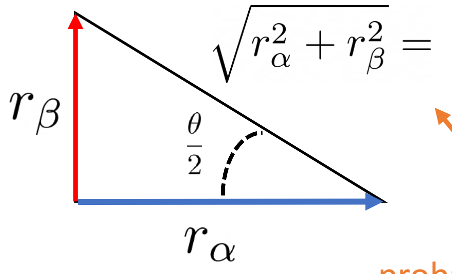


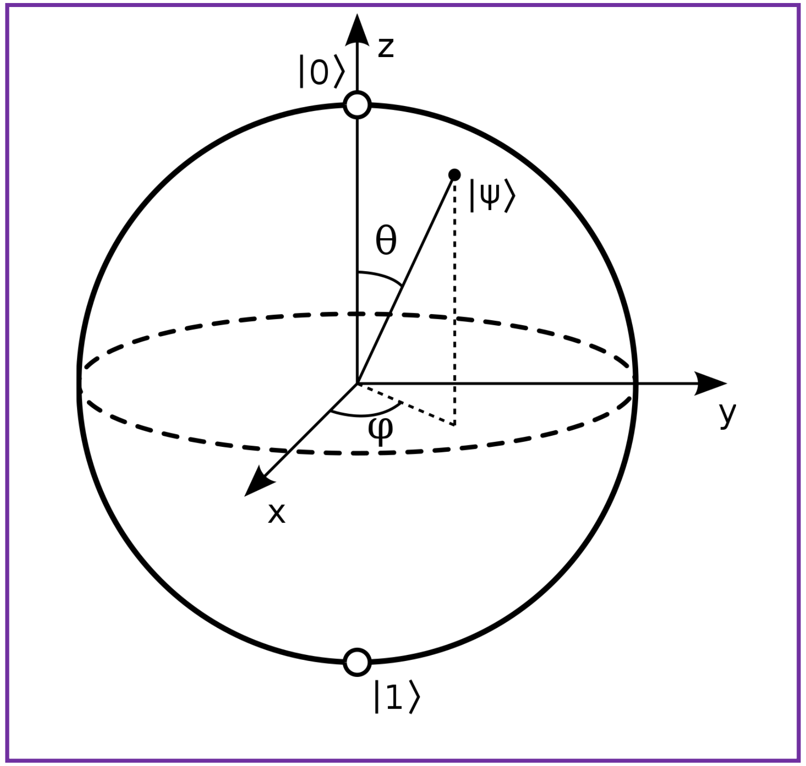
probability rule for qubits

$$\longrightarrow \quad r_\alpha = \cos\left(\frac{\theta}{2}\right) \qquad r_\beta = \sin\left(\frac{\theta}{2}\right)$$

Finding Position on Sphere

How do we determine position of state:

$$|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$



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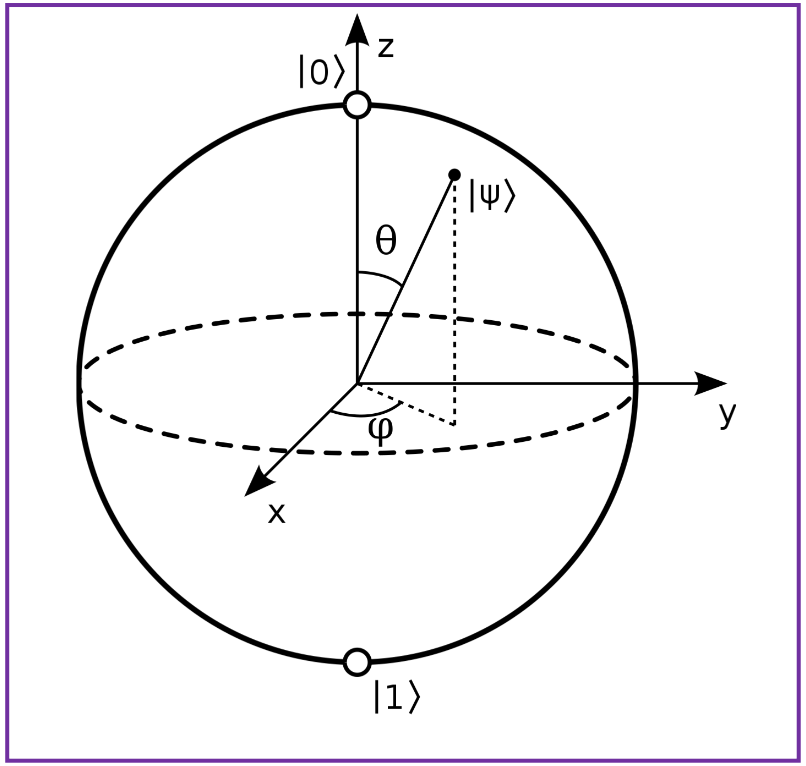
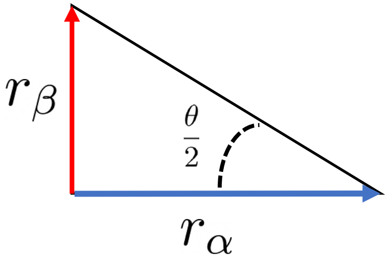


How to find θ :



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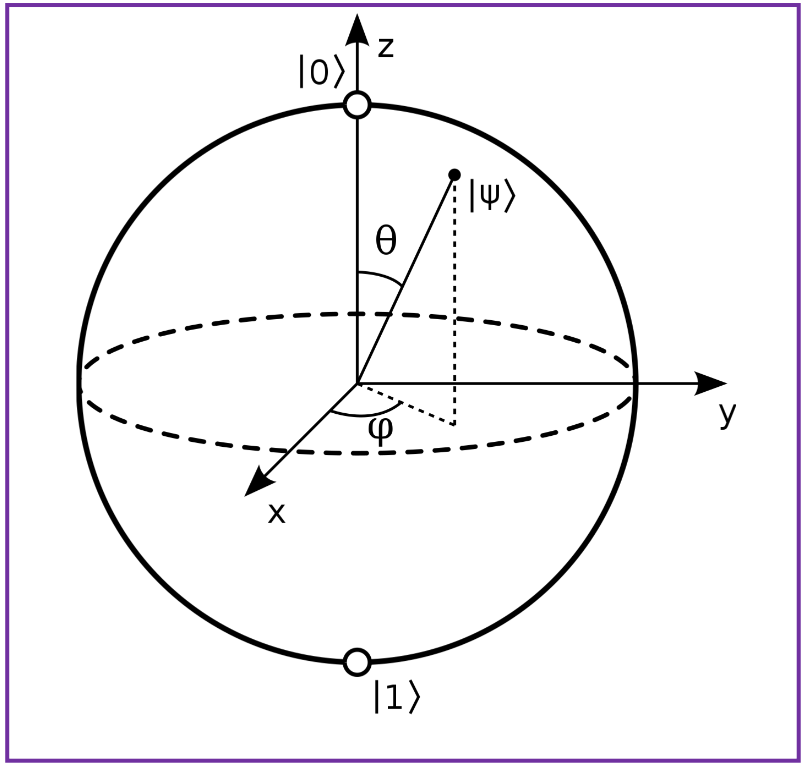

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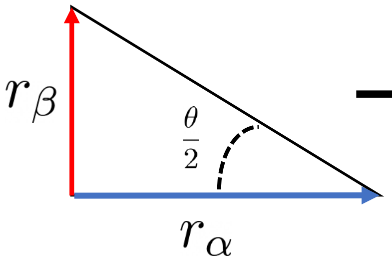

Inverse of cos()

$$\theta = 2\arccos(r_\alpha)$$

or

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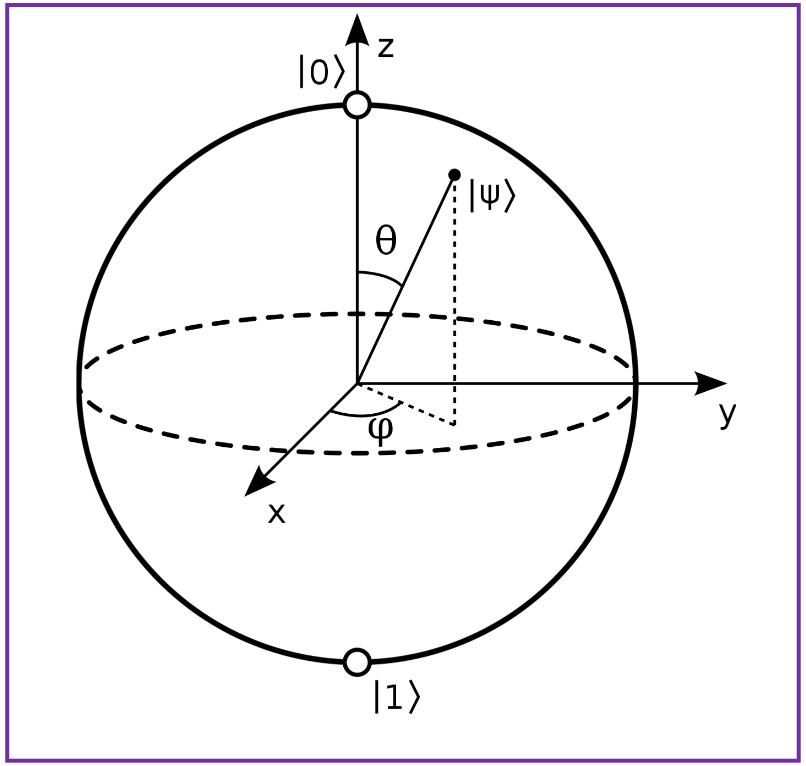

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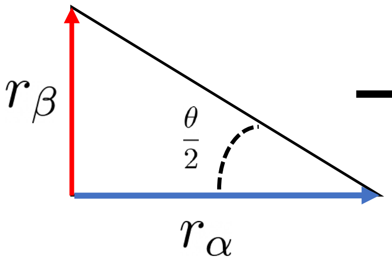

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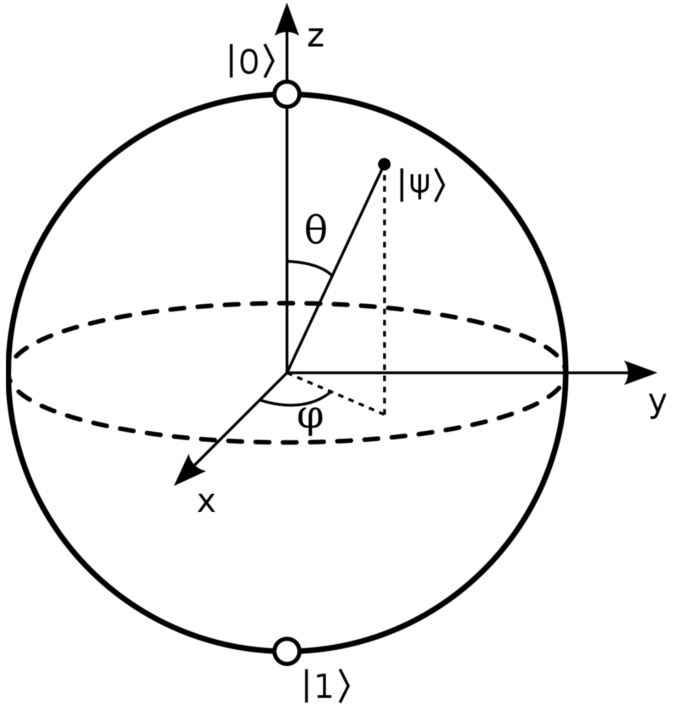

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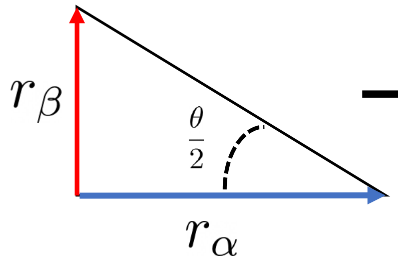

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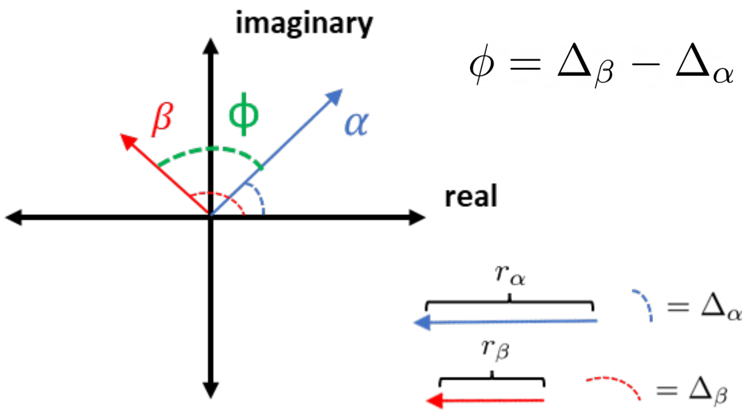

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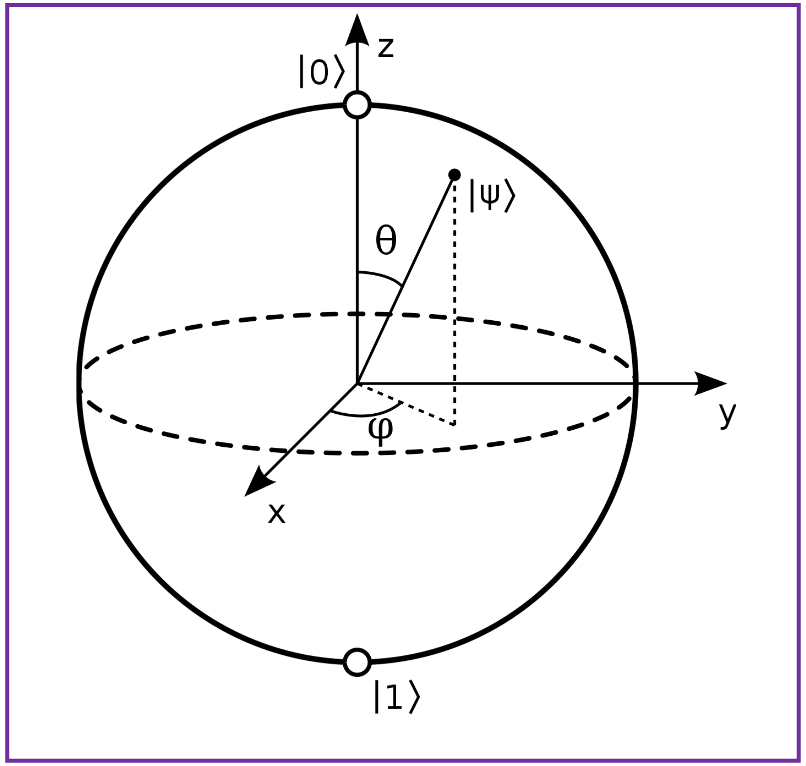

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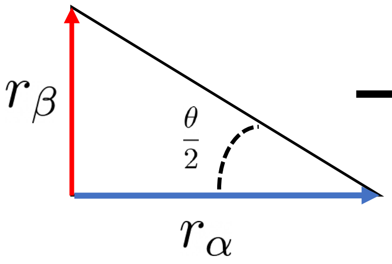

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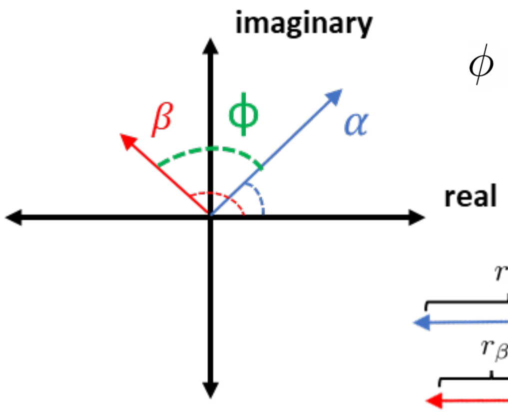

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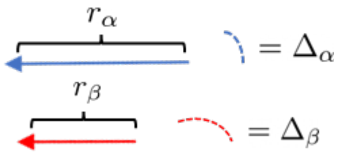

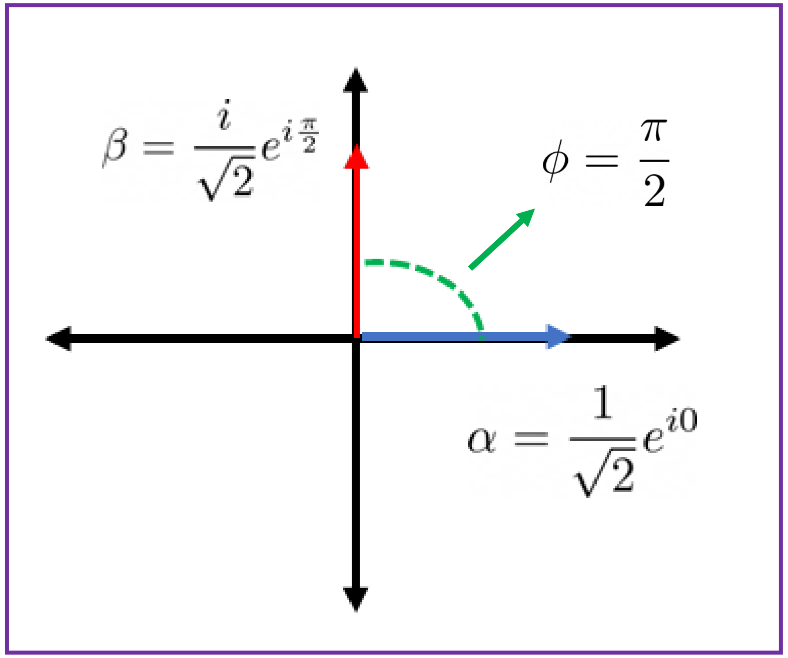

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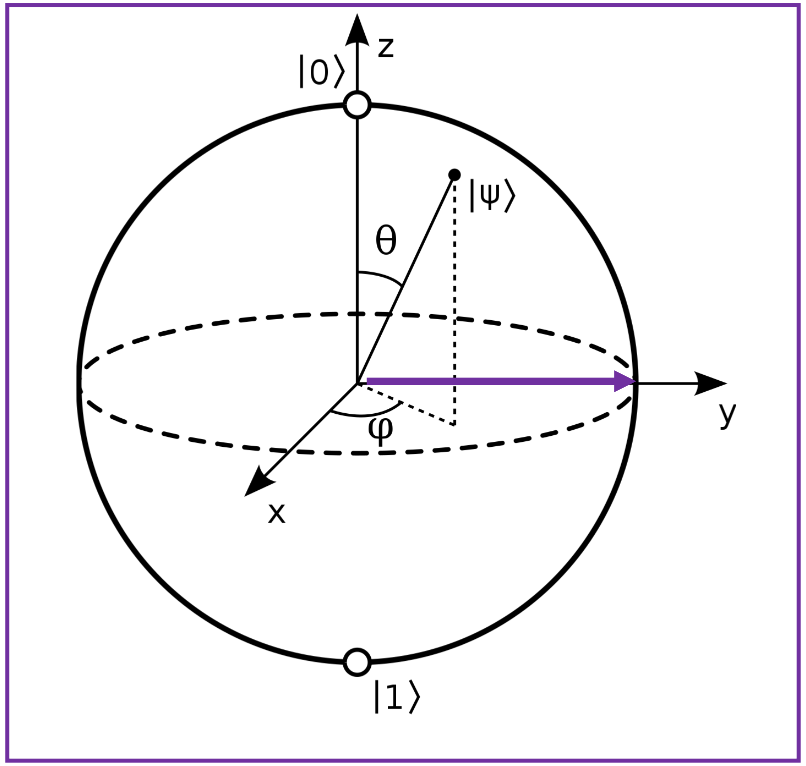

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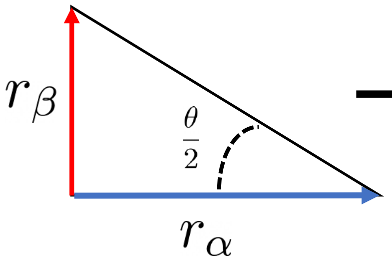

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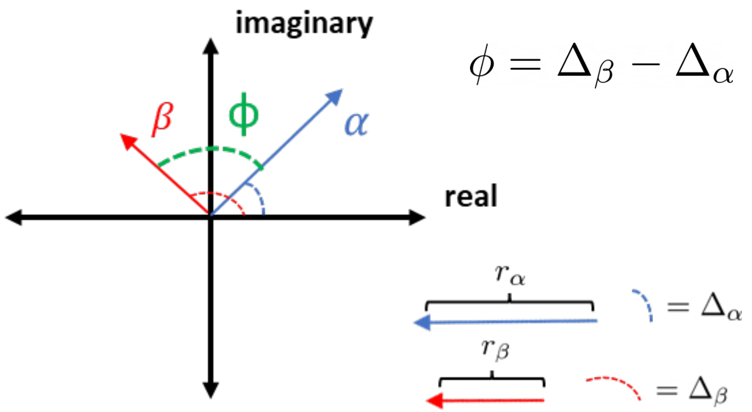

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