Abstract Algebra
Day 1: The $\mathbb{Z}$ Axioms

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The $\mathbb{Z}$ Axioms

There is a set $\mathbb{Z}$, whose elements are called *integers*, which is equipped with two operations $+$ and $\cdot$ and a binary relation $\leq$ which satisfy the following properties.

The $\mathbb{Z}$ Axioms: Arithmetic

A1. $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.

A2. There is an integer 0 such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{Z}$.

A3. For every $a \in \mathbb{Z}$ there is a unique integer, denoted $-a$, such that $a + (-a) = (-a) + a = 0$.

A4. $a + b = b + a$ for all $a, b \in \mathbb{Z}$.

M. $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{Z}$.

D. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbb{Z}$.

C. $ab = ba$ for all $a, b \in \mathbb{Z}$.

U. There is an integer 1 such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{Z}$.

Z. If $ab = 0$, then either $a = 0$ or $b = 0$ for all $a, b \in \mathbb{Z}$.

The $\mathbb{Z}$ Axioms: Order

P1. $a \leq a$ for all $a \in \mathbb{Z}$.

P2. If $a \leq b$ and $b \leq a$ then $a = b$ for all $a, b \in \mathbb{Z}$.

P3. If $a \leq b$ and $b \leq c$ then $a \leq c$ for all $a, b, c \in \mathbb{Z}$.

P4. Either $a \leq b$ or $b \leq a$ for all $a, b \in \mathbb{Z}$.

O1. If $a \leq b$ then $a + c \leq b + c$ for all $a, b, c \in \mathbb{Z}$.

O2. If $0 \leq a$ and $0 \leq b$ then $0 \leq ab$ for all $a, b \in \mathbb{Z}$.

O3. $0 < 1$.

The $\mathbb{Z}$ Axioms: Well-Ordering Property

We call

$$\mathbb{N} = \{a \in \mathbb{Z} \mid 0 \leq a\}$$

the set of *natural numbers*.

WOP. Every nonempty subset of $\mathbb{N}$ has a $\leq$-least element.

That is, if $S \subseteq \mathbb{N}$ is not empty, there is a natural number $m \in S$ such that $m \leq s$ for all $s \in S$.

Consequences

These 17 axioms uniquely characterize the "integers" we know and love; any other provably true statement about $\mathbb{Z}$ can be derived from them. For example:

- $a \cdot 0 = 0 \cdot a = 0$ for all $a \in \mathbb{Z}$. (Use D)
- $(-1) \cdot a = -a$ for all $a \in \mathbb{Z}$.
- If $a \leq b$ and $0 \leq c$, then $ac \leq bc$. (Use D)
- There is no integer t such that $0 < t < 1$. (Use WOP; $t^2 < t$)
- If $a < b$ then $a + 1 \leq b$
- Exactly one of $a < 0$, $a = 0$, and $a > 0$ is true.
- Every element of $\mathbb{N}$ is either 0 or of the form $n + 1$ where $n \in \mathbb{N}$.
- ... etc.

Principle of Mathematical Induction

Theorem (Induction)

Suppose $B \subseteq \mathbb{N}$ is a subset such that

- $0 \in B$ *(the Base Case) and*
- *If $n \in B$, then $n + 1 \in B$ (the Inductive Step).*

Then $B = \mathbb{N}$.

Theorem (Strong Induction)

Suppose $B \subseteq \mathbb{N}$ is a subset such that

- $0 \in B$ *and*
- *If $k \in B$ for all $0 \leq k \leq n$, then $n + 1 \in B$.*

Then $B = \mathbb{N}$.

Proof: Use WOP. These two statements are equivalent in power, but sometimes Strong Induction is convenient.

Principle of Mathematical Induction: Examples

Proposition

For all natural numbers n, we have

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proposition

For all natural numbers n, we have

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(Hint: Use two base cases, 0 and 1.)