# Relations and Functions

**Definition 1** (Function)**.** *If $A$ and $B$ are sets, then the subsets of $A \times B$ are called* relations*. Suppose $\varphi \subseteq A \times B$ is a relation.*

- *We say that $\varphi$ is* total *if for every $a \in A$, there is a $b \in B$ such that $(a, b) \in \varphi$.*
- *We say that $\varphi$ is* well-defined *if whenever $a \in A$ and $b_1, b_2 \in B$ such that $(a, b_1), (a, b_2) \in \varphi$, in fact $b_1 = b_2$.*
- *We say that $\varphi$ is a* function *or* mapping *from $A$ to $B$ if it is both total and well-defined. We denote this by $\varphi : A \to B$.*
- *If $\varphi : A \to B$ is a function and $a \in A$, then there is a unique $b \in B$ such that $(a, b) \in \varphi$; we call this $b$ the* image *of $a$ under $\varphi$, and denote it $\varphi(a)$.*

**Definition 2.** *Let $\varphi : A \to B$ be a function.*

- *We say that $\varphi$ is* injective *if whenever $\varphi(a_1) = \varphi(a_2)$, in fact $a_1 = a_2$.*
- *We say that $\varphi$ is* surjective *if for every $b \in B$, there is an element $a \in A$ such that $\varphi(a) = b$.*

**Definition 3** (Kernel)**.** *Let $\varphi : A \to B$ be a function. The relation*

$$\mathsf{ker}\ \varphi = \{(x, y) \mid \varphi(x) = \varphi(y)\}$$

*is called the* kernel *of $\varphi$.*