Euclidean Planes

Recall that an incidence geometry is called *euclidean* if, given any line ℓ and any point p not on ℓ, there is exactly one line passing through p which is parallel to ℓ. So far we have avoided using any assumptions about the uniqueness of parallel lines, and have been able to prove a good number of interesting results. We will now specialize to the Euclidean case for a while.

Proposition 1 (Converse of the Alternate Interior Angles Theorem)**.** *In a Euclidean plane geometry, if two parallel lines are cut by a transversal, then alternate interior angles formed by the cut are congruent.*

Proof. (copy angle, use AIA, use uniqueness.) □

Proposition 2. *If ℓ and m are parallel and m and t are parallel, then ℓ and t are parallel.*

Proof. We can assume that ℓ and t are distinct (if equal, they are parallel). Suppose BWOC that ℓ and t meet at the unique point x. Since ℓ and m are parallel, x is not on m. By the Euclidean property, there is a unique line s containing x which is parallel to m. But both ℓ and t satisfy this condition, and they are distinct - a contradiction. □

Corollary 3. *If ℓ_1 and ℓ_2 are parallel and m is incident to ℓ_1, then m is incident to ℓ_2.*

Proposition 4. *If ℓ_1 and m are perpendicular, and if ℓ_1 and ℓ_2 are parallel, then ℓ_2 and m are perpendicular.*

Proof. If $\ell_1 = \ell_2$ there's nothing to do. Otherwise m is a transversal and the result follows from the converse of the AIA theorem. □

Proposition 5. *If ℓ_1 and ℓ_2 are parallel, m_1 and ℓ_1 are perpendicular, and m_2 and ℓ_2 are perpendicular, then m_1 and m_2 are parallel.*

Proof. ℓ_2 and m_1 are perpendicular by the converse of AIA, and then m_1 and m_2 are parallel by AIA. □

Construction 6. *Given 3 distinct noncollinear points A, B, and C, there is a unique circle which contains all of them. This circle is called the* circumcircle *of $\triangle ABC$, and its center is the* circumcenter.

Proof. Let ℓ be the perpendicular bisector of $\overline{AB}$ and let m be the perpendicular bisector of $\overline{BC}$. Now ℓ and m must meet, since otherwise $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are parallel (which they aren't, as they meet at the unique point B (since A, B, and C are not collinear)). Moreover they must meet at a *unique* point, say O, since otherwise we can show that $A = C$. Recall that points X on the perpendicular bisector of $\overline{AB}$ have the property that $\overline{AX} \equiv \overline{BX}$. So we have $\overline{AO} \equiv \overline{BO} \equiv \overline{CO}$, and thus $\mathcal{C}_O(A)$ contains A, B, and C. □

Proposition 7.

1. *Opposite angles of a parallelogram are congruent.*
2. *Opposite sides of a parallelogram are congruent.*
3. *The diagonals of a parallelogram bisect each other.*

Proof. For the angles, use AIA and converse of AIA. For the sides, construct a diagonal and use AAS. For the diagonals, use converse of AIA and ASA. □

Proposition 8 (Thales' Theorem)**.** *Suppose A and B are the opposite endpoints of a diameter of a circle centered at O, and that C is a point on this circle distinct from A and B. Then $\angle ACB$ is right. Moreover, $\angle CAB$ is congruent to the bisector of $\angle COB$.*

Proof. Construct the point D on the intersection of $\mathcal{C}_O(A)$ and $\overleftrightarrow{OC}$ by the circle separation axiom. Now $\overline{AC} \equiv \overline{BD}$ using SAS, and similarly $\overline{CB} \equiv \overline{AD}$. Now $\triangle ABC \equiv \triangle BAD$ by SSS, so that $\angle CBA \equiv \angle DAB$. Thus $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are parallel by AIA. Now $\triangle BAC \equiv \triangle DCA$ by SSS, so that $\angle BCA \equiv \angle DAC$. Now $\angle DAC$ and $\angle BCA$ are supplementary by the converse of AIA. So $\angle BCA$ is right.

Now let M be the point on $\overline{BC}$ such that $\overrightarrow{OM}$ bisects $\angle COB$. (Use crossbar.) We have $\angle OCB \equiv \angle OBC$ by Pons Asinorum, so that $\angle OMC \equiv \angle OMB$ by ASA. Thus $\angle CMO$ is right. By AIA, $\overleftrightarrow{OM}$ is parallel to $\overleftrightarrow{AC}$. By the converse of AIA, $\angle OCA \equiv \angle COM$, and $\angle CAO \equiv \angle COM$ by Pons Asinorum. □

Proposition 9 (Converse of Thales' Theorem)**.** *Let A, B, and C be distinct points. If $\angle ACB$ is right, then C is on the circle centered at the midpoint of A and B and passing through A.*

Proof. Let M be the midpoint of A and B, and copy $\overline{MC}$ to the other side of M at D by circle separation. Now $\overline{BC} \equiv \overline{AD}$ by SAS, and similarly $\overline{AC} \equiv \overline{BD}$. So $\triangle ABC \equiv \triangle BAD$ by SSS. Now $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AD}$ by AIA, and so $\angle CAD \equiv \angle ACB$ using the converse of AIA. Now $\triangle CAD \equiv \triangle ACB$ by SAS, so that $\overline{AB} \equiv \overline{CD}$. Thus $\overline{AM} \equiv \overline{CM}$. □

(Here we used a lemma that if two segments are congruent, then their midsegments are congruent.)

Construction 10. *Given a circle $\mathcal{C}_O(A)$ and a point B exterior to this circle, there exist two lines which are tangent to $\mathcal{C}_O(A)$ and which pass through B.*

Proof. Construct the midpoint M of $\overline{BO}$, and construct circle centered at M and passing through O. By the circle cut axiom, $\mathcal{C}_M(O) \cap \mathcal{C}_O(A)$ contains exactly two points, X and Y. Note that $\angle OXB$ and $\angle OYB$ are inscribed on the diameter of a circle, and thus are right; so $\overleftrightarrow{BX}$ and $\overleftrightarrow{BY}$ are tangent to $\mathcal{C}_O(A)$.

(still have to prove these are the only two.) □

Proposition 11 (Inscribed Angle Theorem)**.** *Let A and B be distinct points on a circle centered at O. If C is a point on $\mathcal{C}$ such that C and O are on the same side of $\overleftrightarrow{AB}$, then $\angle ACB$ is congruent to a bisector of $\angle AOB$. In particular, any two such points form congruent angles.*

Altitudes and the Orthocenter

Definition 1. *Let A, B, and C be distinct noncollinear points, and let F be the foot of A on $\overleftrightarrow{BC}$. Then $\overline{AF}$ is called an* altitude *of $\triangle ABC$.*

Proposition 12 (Orthocenter Theorem)**.** *Let A, B, and C be distinct noncollinear points. Then the lines containing the three altitudes of $\triangle ABC$ are concurrent at a point O, called the* orthocenter *of $\triangle ABC$.*

Proof. $\square$