Equivalence Relations

Recall:

Definition 1 (Equivalence Relation)**.** *Let A be a set and let $\sigma \subseteq A \times A$ be a relation on A. We say that σ is an* equivalence relation *if the following hold.*

1. *$a\sigma a$ for all $a \in A$. (Reflexivity)*
2. *If $a\sigma b$ then $b\sigma a$ for all $a, b \in A$. (Symmetry)*
3. *If $a\sigma b$ and $b\sigma c$, then $a\sigma c$ for all $a, b, c \in A$. (Transitivity)*

If σ is an equivalence on A and $a \in A$, then the set

$$[a]_\sigma = \{b \in A \mid a\sigma b\}$$

is called the equivalence class *of a.*

Definition 2 (Partition)**.** *Let A be a set, and let $P \subseteq 2^A$ be a collection of subsets of A. We say that P is a* partition *of A if the following hold.*

1. *$\bigcup P = A$ (P is collectively exhaustive)*
2. *If $C_1, C_2 \in P$ are distinct, then $C_1 \cap C_2 = \varnothing$.*

Equivalence relations and partitions are related in a fundamental way.

Theorem 1. *Let A be a set.*

1. *If σ is an equivalence relation, then the set*

   $$A/\sigma = \{[a] \mid a \in A\}$$

   is a partition of A.

2. *If P is a partition of A, then the relation*

   $$\sigma = \{(x, y) \mid y \in A, x \in [y]\}$$

   is an equivalence on A.

That is, given an equivalence we can build a partition, and given a partition we can build an equivalence relation.

Definition 3.

- *Let A be a set and σ an equivalence on A. The mapping $\pi_\sigma : A \to A/\sigma$ given by $\varphi(a) = [a]$ is called the* natural projection *of A onto A/σ.*
- *Let $\varphi : A \to B$ be a function. The relation*

  $$\mathsf{ker}\ \varphi = \{(x, y) \mid \varphi(x) = \varphi(y)\}$$

  is an equivalence on A, called the kernel *of φ.*

Theorem 2 (First Isomorphism Theorem for Sets)**.** *Let $f : A \to B$ be a function. There is a unique function $\overline{f}$ such that $f = \overline{f} \circ \pi_{\mathsf{ker}\ f}$.*