Name: ______________________

Calculus 1: Review (Test 1)

1. Compute the following limit.

$$\lim_{x \to 5} \left(5x^2 + 3x + 8\right)$$

Solution: Since this expression is a polynomial, we can find the limit as x approaches 5 by evaluating at 5.

$$\begin{aligned} \lim_{x \to 5} \left(5x^2 + 3x + 8\right) &= 5(5)^2 + 3(5) + 8 \\ &= 125 + 15 + 8 \\ &= 148. \end{aligned}$$

2. Compute the following limit.

$$\lim_{x \to 7} \frac{x^2 - 3x - 28}{x - 7}$$

Solution: Note that this rational function is not defined at $x = 7$. However, factoring the numerator we have

$$\begin{aligned} \lim_{x \to 7} \frac{x^2 - 3x - 28}{x - 7} &= \lim_{x \to 7} \frac{(x-7)(x+4)}{x-7} \\ &= \lim_{x \to 7} \frac{\cancel{(x-7)}(x+4)}{\cancel{x-7}} \\ &= \lim_{x \to 7} (x+4) \\ &= 11. \end{aligned}$$

3. Compute the following limit.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

Solution: Note that this expression is not defined if $x = 3$. However, we can factor the numerator as a difference of squares and cancel.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} (x+3) = 6$$

4. Compute the following limit.

$$\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$$

Solution: Note that this expression is not defined if $x = 4$. However, we can factor the numerator as a difference of squares.

$$\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} - 2} = \lim_{x \to 4} (\sqrt{x} + 2) = 4$$

5. Compute the following limit.

$$\lim_{x\to 26} \frac{\sqrt{x-1}-5}{x-26}$$

Solution: Note that this expression is not defined if $x = 26$. But also note that if we multiply the numerator by its radical conjugate, something nice happens:

$$\left(\sqrt{x-1}-5\right)\left(\sqrt{x-1}+5\right) = x-1-25 = x-26.$$

Let's try multiplying by 1, but write 1 as $\sqrt{x-1}+5$ over itself.

$$\begin{aligned}
\lim_{x\to 26} \frac{\sqrt{x-1}-5}{x-26} &= \lim_{x\to 26} \left(\frac{\sqrt{x-1}-5}{x-26} \cdot \frac{\sqrt{x-1}+5}{\sqrt{x-1}+5}\right) \\
&= \lim_{x\to 26} \frac{x-26}{(x-26)(\sqrt{x-1}+5)} \\
&= \lim_{x\to 26} \frac{1}{\sqrt{x-1}+5} \\
&= \frac{1}{10}
\end{aligned}$$

6. Compute the following limit.

$$\lim_{x\to 28} \frac{\sqrt{x-3}-5}{x-28}$$

Solution: Note that this expression is not defined if $x = 28$. But also note that if we multiply the numerator by its radical conjugate, something nice happens:

$$\left(\sqrt{x-3}-5\right)\left(\sqrt{x-3}+5\right) = x-3-25 = x-28.$$

Let's try multiplying by 1, but write 1 as $\sqrt{x-3}+5$ over itself.

$$\begin{aligned}
\lim_{x\to 28} \frac{\sqrt{x-3}-5}{x-28} &= \lim_{x\to 28} \left(\frac{\sqrt{x-3}-5}{x-28} \cdot \frac{\sqrt{x-3}+5}{\sqrt{x-3}+5}\right) \\
&= \lim_{x\to 28} \frac{x-28}{(x-28)(\sqrt{x-3}+5)} \\
&= \lim_{x\to 28} \frac{1}{\sqrt{x-3}+5} \\
&= \frac{1}{10}
\end{aligned}$$

7. Compute the following limit.

$$\lim_{x\to 2} \frac{x^3-2x^2-9x+18}{x-2}$$

Solution: Note that this rational function is not defined if $x = 2$. However, 2 is a root of both the numerator and the denominator, so we can factor (either by grouping or using long or synthetic division) and cancel.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^3-2x^2-9x+18}{x-2} &= \lim_{x\to 2} \frac{x^2(x-2)-9(x-2)}{x-2} \\
&= \lim_{x\to 2} \frac{(x-2)(x^2-9)}{x-2} \\
&= \lim_{x\to 2} (x^2-9) \\
&= -5
\end{aligned}$$

8. Compute the following limit.
$$\lim_{x\to -4} \frac{x^3+6x^2+5x-12}{x+4}$$
Solution: Note that this rational function is not defined if $x=-4$. However, -4 is a root of both the numerator and the denominator, so we can factor (using either long or synthetic division) and cancel.
$$\lim_{x\to -4} \frac{x^3+6x^2+5x-12}{x+4} = \lim_{x\to -4} \frac{(x+4)(x^2+2x-3)}{x+4} = \lim_{x\to -4}(x^2+2x-3) = 5$$

9. Compute the following limit.
$$\lim_{x\to 0} \left|x^2+2x-4\right|$$
Solution: The function which we are computing the limit of here is continuous everywhere; so to find the limit, we can simply evaluate this function at $x=0$. Thus
$$\lim_{x\to 0} \left|x^2+2x-4\right| = |-4| = 4$$
So the limit is $\boxed{4}$.

10. Compute the following limit.
$$\lim_{x\to 5} \left|\frac{x^3-125}{x-5}\right|$$
Solution: Note that this expression is not defined if $x=5$ because of the $x-5$ in the denominator. However, we can factor the numerator as a difference of cubes and cancel.
$$\lim_{x\to 5} \left|\frac{x^3-125}{x-5}\right| = \lim_{x\to 5} \left|\frac{(x-5)(x^2+5x+25)}{x-5}\right| = \lim_{x\to 5} \left|x^2+5x+25\right| = |75| = 75$$
Now this function is continuous everywhere, so we can compute the limit by evaluating at $x=5$; this yields a limit of $\boxed{75}$.

11. Compute the limit of the difference quotient
$$\lim_{x\to t} \frac{f(x)-f(t)}{x-t}$$
when $f(x)=2x+1$ and $t=10$.

Solution: We have
$$\begin{aligned}
\lim_{x\to 10} \frac{f(x)-f(10)}{x-10} &= \lim_{x\to 10} \frac{(2x+1)-(2\cdot 10+1)}{x-10} \\
&= \lim_{x\to 10} \frac{2x-2\cdot 10}{x-10} \\
&= \lim_{x\to 10} \frac{2(x-10)}{x-10} \\
&= \lim_{x\to 10} 2 \\
&= 2
\end{aligned}$$

12. Compute the limit of the difference quotient

$$\lim_{x \to t} \frac{f(x) - f(t)}{x - t}$$

when $f(x) = 9x^2 + 8x + 3$ and $t = 7$.

Solution: We have

$$\begin{aligned}
\lim_{x \to 7} \frac{f(x) - f(7)}{x - 7} &= \lim_{x \to 7} \frac{(9x^2 + 8x + 3) - (9(7)^2 + 8(7) + 3)}{x - 7} \\
&= \lim_{x \to 7} \frac{9x^2 - 9(7)^2 + 8x - 8(7) + 3 - 3}{x - 7} \\
&= \lim_{x \to 7} \frac{9(x^2 - 7^2) + 8(x - 7)}{x - 7} \\
&= \lim_{x \to 7} \frac{9(x - 7)(x + 7) + 8(x - 7)}{x - 7} \\
&= \lim_{x \to 7} \frac{(x - 7)(9(x + 7) + 8)}{x - 7} \\
&= \lim_{x \to 7} (9(x + 7) + 8) \\
&= 9(7 + 7) + 8 \\
&= 134
\end{aligned}$$

13. Compute the limit of the difference quotient

$$\lim_{x \to t} \frac{f(x) - f(t)}{x - t}$$

when $f(x) = \sqrt{x + 2}$ and $t = 2$.

Solution: We have

$$\begin{aligned}
\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2} \frac{\sqrt{x + 2} - \sqrt{4}}{x - 2} \\
&= \lim_{x \to 2} \left(\frac{\sqrt{x + 2} - \sqrt{4}}{x - 2} \cdot \frac{\sqrt{x + 2} + \sqrt{4}}{\sqrt{x + 2} + \sqrt{4}} \right) \\
&= \lim_{x \to 2} \frac{(x + 2) - (4)}{(x - 2)\left(\sqrt{x + 2} + \sqrt{4}\right)} \\
&= \lim_{x \to 2} \frac{x - 2}{(x - 2)\left(\sqrt{x + 2} + \sqrt{4}\right)} \\
&= \lim_{x \to 2} \frac{1}{\sqrt{x + 2} + \sqrt{4}} \\
&= \frac{1}{2\sqrt{4}} \\
&= \frac{1}{4}
\end{aligned}$$

14. Compute the following limit.

$$\lim_{x \to 0} \frac{\sin(8x)}{x}$$

Solution: Note that

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(8x)}{x} &= \lim_{x\to 0}\frac{8\sin(8x)}{8x} \\
&= 8\lim_{x\to 0}\frac{\sin(8x)}{8x} \\
&= 8\lim_{x\to 0}\operatorname{sinc}(8x) \\
&= 8\operatorname{sinc}\left(\lim_{x\to 0}8x\right) \quad \text{(since sinc is continuous)} \\
&= 8\operatorname{sinc}(0) \\
&= 8.
\end{aligned}$$

15. Compute the following limit.

$$\lim_{x\to 0}\frac{3x^2+5x+\sin x}{x}$$

Solution: This expression is not defined if $x=0$. However, we can split this fraction like so:

$$\begin{aligned}
\lim_{x\to 0}\frac{3x^2+5x+\sin(x)}{x} &= \lim_{x\to 0}\left(\frac{3x^2+5x}{x}+\frac{\sin x}{x}\right) \\
&= \lim_{x\to 0}\left(3x+5+\frac{\sin x}{x}\right).
\end{aligned}$$

Recall that the limit of a sum is the sum of limits, *provided* the limit of each summand exists. In this case they do, and we have

$$\begin{aligned}
&= \lim_{x\to 0}(3x+5)+\lim_{x\to 0}\frac{\sin x}{x} \\
&= 5+1 \\
&= 6.
\end{aligned}$$

16. Let $f(x)$ be the function

$$f(x)=\begin{cases}\frac{x-b}{b+3} & \text{if } x<0 \\ x^2+b & \text{if } x\geq 0.\end{cases}$$

Find the value(s) of the constant b such that $f(x)$ is continuous everywhere.

Solution: Remember that $\lim_{x\to 0}f(x)$ exists precisely when the one-sided limits $\lim_{x\to 0^+}f(x)$ and $\lim_{x\to 0^-}f(x)$ exist and are equal to one another. In this case, we have

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x^2+b)=b$$

and

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}\frac{x-b}{b+3}=\frac{-b}{b+3}.$$

Setting these equal, we have

$$b=\frac{-b}{b+3}.$$

The values for b we want are precisely the solutions of this equation. Clearing denominators, we have

$$b^2 + 3b = -b,$$

and solving for zero we have

$$b^2 + 4b = 0 \qquad \text{which factors as} \qquad b(b+4) = 0.$$

So $b = 0$ or $b = -4$.

17. Compute the following derivative.

$$\frac{d}{dx}\left(6x^2 + 13x + 11\right)$$

Solution: Since this function is a polynomial, we can use the power rule on each term. The derivative is

$$\boxed{f'(x) = 12x + 13.}$$

18. Compute the derivative of the following function of t.

$$f(t) = \frac{7}{t^3} + \frac{4}{t} + 8t^5.$$

Solution: We start by moving all the variables to the numerators of fractions like so.

$$f(t) = 7t^{-3} + 4t^{-1} + 8t^5.$$

Now we can use the power rule on each term. In particular, we have

$$\boxed{f'(t) = -21t^{-4} - 4t^{-2} + 40t^4.}$$

19. Let $f(x) = x + \frac{4}{x}$.

(a) Compute the derivative of f.
(b) Find an equation for the line tangent to f at the point $(2, 4)$.

Solution: Note that

$$f(x) = x + 4x,$$

which we can differentiate term-by-term using the power rule to get

$$\boxed{f'(x) = -4x^{-2}.}$$

Recall that the line tangent to the graph of f at a point (u, v) has slope $f'(u)$ and passes through (u, v). Thus the slope of the tangent line to f at $x = 2$ is $f'(2) = -1$, and the line with this slope and passing through $(2, 4)$ is given by the equation

$$-1 = \frac{y-4}{x-2}$$

or, in slope-intercept form,

$$\boxed{y = -x + 6.}$$

20. Compute the derivative of the following function.

$$f(x) = \frac{x^2 + 5x + 3}{6x - 5}$$

Solution: We can use the quotient rule on this function as follows.

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} \frac{x^2 + 5x + 3}{6x - 5} \\
&= \frac{(6x-5) \cdot \frac{d}{dx}(x^2 + 5x + 3) - (x^2 + 5x + 3) \cdot \frac{d}{dx}(6x - 5)}{(6x-5)^2} \\
&= \frac{(6x-5)(2x+5) - (x^2 + 5x + 3)(6)}{(6x-5)^2} \\
&= \boxed{\frac{6x^2 - 10x - 43}{(6x-5)^2}}
\end{aligned}$$

21. Find the values of x at which the line tangent to

$$f(x) = x^3 + 11x^2 + 44x + 87$$

is horizontal.

Solution: Recall that a line is horizontal precisely when its slope is zero, and that the slope of the line tangent to f at c is given by $f'(c)$. So it suffices to find the solutions x of the equation $f'(x) = 0$. To this end, note that

$$f'(x) = 3x^2 + 23x + 44.$$

This is a quadratic, which we can solve using our favorite method to find that f has a horizontal tangent line precisely when x is $\boxed{\text{-4 or -11/3.}}$