Abstract Algebra
Day 1: The $\mathbb{Z}$ Axioms

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The $\mathbb{Z}$ Axioms

There is a set $\mathbb{Z}$, whose elements are called *integers*, which is equipped with two operations $+$ and $\cdot$ and a binary relation $\leq$ which satisfy the following properties.

A1. $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{Z}$.

The $\mathbb{Z}$ Axioms: Arithmetic

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D. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbb{Z}$.

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Z. If $ab = 0$, then either $a = 0$ or $b = 0$ for all $a, b \in \mathbb{Z}$.

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O3. $0 < 1$.

The $\mathbb{Z}$ Axioms: Well-Ordering Property

We call

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That is, if $S \subseteq \mathbb{N}$ is not empty, there is a natural number $m \in S$ such that $m \leq s$ for all $s \in S$.

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- ... etc.

Theorem (Induction)

Suppose $B \subseteq \mathbb{N}$ is a subset such that

- $0 \in B$ *(the Base Case) and*
- *If $n \in B$, then $n+1 \in B$ (the Inductive Step).*

Then $B = \mathbb{N}$.

Theorem (Strong Induction)

Suppose $B \subseteq \mathbb{N}$ is a subset such that

- $0 \in B$ *and*
- *If $k \in B$ for all $0 \leq k \leq n$, then $n+1 \in B$.*

Then $B = \mathbb{N}$.

Proof: Use WOP. These two statements are equivalent in power, but sometimes Strong Induction is convenient.

Principle of Mathematical Induction: Examples

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For all natural numbers n, we have

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$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(Hint: Use two base cases, 0 and 1.)