Domains and Fields

The integers have the following very nice “zero product property”:

$\boxed{\text{If } a \text{ and } b \text{ are integers and } ab = 0, \text{ then either } a = 0 \text{ or } b = 0.}$

Recall that $\mathbb{Z}/(n)$ does not necessarily have this property. For instance, in $\mathbb{Z}/(6)$ we have $2 \neq 0$ and $3 \neq 0$, but $2 \cdot 3 = 0$. In this case we say that 2 and 3 are zero divisors in $\mathbb{Z}/(6)$.

Definition 1 (Zero Divisor)**.** *Let R be a ring.*

- *We say that a nonzero element $r \in R$ is a* zero divisor *if there is a nonzero element $s \in R$ such that $rs = 0$.*
- *We say that R is an* integral domain*, or simply* domain*, if R is commutative and does not contain any zero divisors.*

Proposition 1 (Cancellation)**.** *Let R be a domain with $r, s, t \in R$. If $rs = rt$, then $s = t$.*

Units and Fields

Definition 2 (Unit)**.** *Let R be a unital ring.*

- *We say that $u \in R$ is a* unit *if there is an element $v \in R$ such that $uv = vu = 1_R$.*
- *We say that R is a* field *if R is commutative and every nonzero element of R is a unit.*

Proposition 2. *Every field is also an integral domain.*

Proposition 3. *$\mathbb{Z}/(n)$ is a field if and only if n is prime.*

Exercises

1. Ponder: Is the zero ring a domain?
2. Show that if R and S are nontrivial rings, then $R \oplus S$ is *not* a domain.
3. Show that every subring of a domain is a domain.
4. Show that every subring of a field is a domain.
5. **Nilpotence.** We say that an element r in a ring R is *nilpotent* if $r^n = 0$ for some natural number n.
6. **Skew fields.**