Parallel Lines

Definition 1 (Parallel). *Let ℓ_1 and ℓ_2 be lines in an incidence geometry. We say that ℓ_1 and ℓ_2 are* parallel*, denoted $\ell_1 \parallel \ell_2$, if either $\ell_1 \cap \ell_2 = \varnothing$ or $\ell_1 = \ell_2$.*

Question: Suppose we have a line ℓ and a point x in an incidence geometry. What are the lines which pass through p and are parallel to ℓ?

Examples

$\mathbb{R}^2$ Last time we gave a nice way to detect whether two lines intersect in a single point in terms of determinants. This criterion can be rephrased as follows: If $A = (a_1, b_1)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, and $D = (d_1, d_2)$ are points in $\mathbb{R}^2$ with $A \neq B$ and $C \neq D$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if

$$\mathsf{det}\begin{bmatrix} b_1 - a_1 & d_1 - c_1 \\ b_2 - a_2 & d_2 - c_2 \end{bmatrix} = 0.$$

With this, we can show the following.

Proposition 1. *If $\ell = \overleftrightarrow{AB}$ is a line and $C \notin \ell$ a point in $\mathbb{R}^2$, then there is exactly one line passing through C which is parallel to ℓ.*

Proof. To see existence, note that $\overleftrightarrow{C(C+B-A)} \parallel \overleftrightarrow{AB}$ since

$$\begin{aligned} \mathsf{det}\begin{bmatrix} b_1 - a_1 & c_1 + b_1 - a_1 - c_1 \\ b_2 - a_2 & c_2 + b_2 - a_2 - c_2 \end{bmatrix} &= \mathsf{det}\begin{bmatrix} b_1 - a_1 & b_1 - a_1 \\ b_2 - a_2 & b_2 - a_2 \end{bmatrix} \\ &= \mathsf{det}\begin{bmatrix} b_1 - a_1 & 0 \\ b_2 - a_2 & 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

To see uniqueness, suppose $X = (x_1, x_2)$ is a point (different from C) such that $\overleftrightarrow{CX} \parallel \overleftrightarrow{AB}$. Then

$$0 = \mathsf{det}\begin{bmatrix} x_1 - c_1 & b_1 - a_1 \\ x_2 - c_2 & b_2 - c_2 \end{bmatrix} = \mathsf{det}\begin{bmatrix} x_1 - c_1 & c_1 + b_1 - a_1 - c_1 \\ x_2 - c_2 & c_2 + b_2 - a_2 - c_2 \end{bmatrix}.$$

So X, C, and $C + B - A$ are collinear, and thus $\overleftrightarrow{CX} = \overleftrightarrow{C(C+B-A)}$. □

$\mathbb{Q}^2$ Similar to the Cartesian Plane, the Rational Plane has unique parallel lines through a given point.

$\mathbb{R}^3$ If ℓ is a line and $x \notin \ell$ a point in Three Space, then there are *infinitely many* lines through x which are parallel to ℓ. (Why?)

$\mathbb{D}$ Suppose ℓ is a line and x a point in the Unit Disk. There are *infinitely many* lines passing through x which are parallel to ℓ. To see why, remember that ℓ is contained in a line $\ell_{A,B}$ in the Cartesian Plane. Choose any point y on this Cartesian line which is not in the unit disk. Now $\ell' = \ell_{x,y} \cap \mathbb{D}$ is parallel to ℓ.

- $\mathcal{F}$ In the Fano Plane, no two lines are parallel. In particular, if ℓ is a line and $x \notin \ell$ a point, there are *no* lines passing through x which are parallel to ℓ.

Considering these examples, there seem to be three qualitatively different possibilities for the answer to our Question about parallel lines. This observation is what motivates the following definition.

Definition 2 (The Parallel Postulates)**.** *We say that an incidence geometry $\mathcal{P}$ is*

- ***Elliptic*** *if there are* no *lines passing through x and parallel to ℓ, for all lines ℓ and points $x \notin \ell$.*
- ***Euclidean*** *if there is* exactly one *line passing through x and parallel to ℓ, for all lines ℓ and points $x \notin \ell$.*
- ***Hyperbolic*** *if there are* infinitely many *lines passing through x and parallel to ℓ, for all lines ℓ and points $x \notin \ell$.*

With this definition, $\mathbb{R}^2$ and $\mathbb{Q}^2$ are Euclidean, $\mathcal{F}$ is Elliptic, and $\mathbb{D}$ and $\mathbb{R}^3$ are Hyperbolic. It is important to note that a given incidence geometry need not satisfy any of these properties!

Transitivity of Parallelism

The kinds of "geometries" that arise from our three different Parallel Postulates will be different - perhaps drastically so - as illustrated by the following result.

Proposition 2. *Suppose $\mathcal{P}$ is a Euclidean incidence geometry, with lines ℓ_1, ℓ_2, and ℓ_3. If $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$. That is, the relation "is parallel to" is transitive.*

Proof. If $\ell_1 \cap \ell_2 = \varnothing$, then $\ell_1 \parallel \ell_3$ by definition. Suppose instead that ℓ_1 and ℓ_3 have *at least one* point in common, say p. Since ℓ_1 is parallel to ℓ_2, note that $p \notin \ell_2$. Since $\mathcal{P}$ is Euclidean, there is exactly one line passing through p which is parallel to ℓ_2; call this line ℓ. But now ℓ_1 is a line parallel to ℓ_2 which passes through p, so that $\ell_1 = \ell$. Likewise, $\ell_3 = \ell$. Hence $\ell_1 = \ell_3$, and so $\ell_1 \parallel \ell_3$ as claimed. $\square$

Note that in a Hyperbolic incidence geometry, this need not be the case. If we have two lines ℓ_1 and ℓ_3 which pass through a point p and are parallel to a given line ℓ_2, then ℓ_1 and ℓ_3 are *not* parallel. And in an Elliptic incidence geometry the transitivity of parallelism is irrelevant: there are no pairs of parallel lines to begin with.

A Strange Example

To demonstrate that an incidence geometry need not be either Elliptic, Euclidean, or Hyperbolic, consider the following example, which we will call the *Two-Pointed Line*. Let $P = \mathbb{R} \cup \{A, B\}$. We define lines of four types:

- $\mathbb{R}$ is a line of Type 1;
- $\{x, A\}$, where $x \in \mathbb{R}$, is a line of Type 2;
- $\{x, B\}$, where $x \in \mathbb{R}$, is a line of Type 3; and
- $\{A, B\}$ is a line of Type 4.

Now consider the following.

1. Show that the Two-Pointed Line is an incidence geometry.
2. Find a line ℓ and a point x in the Two-Pointed Line such that there is exactly one line passing through x and parallel to ℓ.
3. Find a line ℓ and a point x in the Two-Pointed Line such that there are infinitely many lines passing through x and parallel to ℓ.

From these facts we can conclude that the Two-Pointed Line is an incidence geometry which is neither Elliptic, Euclidean, nor Hyperbolic. Can you think of a reason why this example is different from those we've seen so far?