Incircles and Excircles

Proposition 1. *Let A, O, and B be distinct points. A point P in $\mathsf{int}\angle AOB$ is on the bisector of $\angle AOB$ if and only if $\overline{PX} \equiv \overline{PY}$, where X is the foot of P on $\overleftrightarrow{OA}$ and Y is the foot of P on $\overleftrightarrow{OB}$.*

Proof. Suppose P has this property. Now $\triangle OPX$ and $\triangle OPY$ are right, with $\overline{PX} \equiv \overline{PY}$ and $\overline{OP} \equiv \overline{OP}$. By the HL Theorem, $\triangle OPX \equiv \triangle OPY$, and thus $\angle XOP \equiv \angle YOP$. So P is on the bisector of $\angle AOB$.

Conversely, suppose P is on the bisector of $\angle AOP$, and let X be the foot of P on $\overleftrightarrow{OA}$ and Y the foot of P on $\overleftrightarrow{OB}$. Now $\triangle XOP \equiv \triangle YOP$ by AAS, so that $\overline{PX} \equiv \overline{PY}$. □

Construction 2 (Incircle Theorem)**.** *Let A, B, and C be distinct points. Then we have the following.*

1. *The bisectors of the interior angles of $\triangle ABC$ are concurrent at a point O, called the* incenter *of the triangle.*

2. *The feet of O on the sides of $\triangle ABC$ lie on a circle, called the* incircle *of $\triangle ABC$, which is centered at O and tangent to the sides of $\triangle ABC$.*

Proof. Let $\overrightarrow{AA'}$ be the bisector of $\angle BAC$. By the Crossbar Theorem this ray cuts $\overline{BC}$ at a point A''. Let $\overrightarrow{BB'}$ be the bisector of $\angle ABC$; again by the Crossbar Theorem this ray cuts $\overline{AA''}$ at a point O. Let X, Y, and Z be the feet of O on $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{BC}$, respectively. Since O is on the bisectors of $\angle BAC$ and $\angle ABC$, we have $\overline{OX} \equiv \overline{OY}$ and $\overline{OY} \equiv \overline{OZ}$; thus $\overline{OX} \equiv \overline{OZ}$, and so O is also on the bisector of $\angle BCA$. Thus the bisectors of the interior angles of $\triangle ABC$ are concurrent at O.

Now X, Y, and Z are the feet of O on the sides of $\triangle ABC$, and we've seen that $\overline{OX} \equiv \overline{OY} \equiv \overline{OZ}$. Thus the circle $\mathcal{C}_O(X)$ contains X, Y, and Z, and moreover is tangent to the sides of $\triangle ABC$ at X, Y, and Z. □

Construction 3 (Excircle Theorem)**.** *Let A, B, and C be distinct points forming $\triangle ABC$. Then we have the following.*

1. *The bisector of the interior angle at A and the exterior angles at B and C are concurrent at a point O, called the* excenter *of $\triangle ABC$ at A.*

2. *The feet of O on the (extended) sides of $\triangle ABC$ lie on a circle, called the* excircle *of $\triangle ABC$ at A, which is centered at O and tangent to the sides of $\triangle ABC$.*

Proof. Essentially the same as the proof of the Incircle Theorem. □

To every triangle we can associate four special circles: the incircle, and one excircle for each vertex. These circles are tangent to all three (extended) sides of the circle.

Proposition 4. *Any circle which is tangent to all three (extended) sides of a triangle is either the incircle or one of the excircles.*