

Abstract Algebra

Day 1: The \mathbb{Z} Axioms

Nathan Bloomfield

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The \mathbb{Z} Axioms

There is a set \mathbb{Z} , whose elements are called *integers*, which is equipped with two operations $+$ and \cdot and a binary relation \leq which satisfy the following properties.

The \mathbb{Z} Axioms: Arithmetic

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- M. $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{Z}$.
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- U. There is an integer 1 such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{Z}$.
- Z. If $ab = 0$, then either $a = 0$ or $b = 0$ for all $a, b \in \mathbb{Z}$.

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- O3. $0 < 1$.

The \mathbb{Z} Axioms: Well-Ordering Property

We call

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WOP. Every nonempty subset of \mathbb{N} has a \leq -least element.

That is, if $S \subseteq \mathbb{N}$ is not empty, there is a natural number $m \in S$ such that $m \leq s$ for all $s \in S$.

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- ... etc.

Principle of Mathematical Induction

Theorem (Induction)

Suppose $B \subseteq \mathbb{N}$ is a subset such that

- $0 \in B$ (the Base Case) and
- If $n \in B$, then $n + 1 \in B$ (the Inductive Step).

Then $B = \mathbb{N}$.

Theorem (Strong Induction)

Suppose $B \subseteq \mathbb{N}$ is a subset such that

- $0 \in B$ and
- If $k \in B$ for all $0 \leq k \leq n$, then $n + 1 \in B$.

Then $B = \mathbb{N}$.

Proof: Use WOP. These two statements are equivalent in power, but sometimes Strong Induction is convenient.

Principle of Mathematical Induction: Examples

Proposition

For all natural numbers n , we have

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For all natural numbers n , we have

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(Hint: Use two base cases, 0 and 1.)