Equivalence Relations

Recall:

Definition 1 (Equivalence Relation). *Let A be a set and let $\sigma \subseteq A \times A$ be a relation on A. We say that σ is an* equivalence relation *if the following hold.*

1. *$a\sigma a$ for all $a \in A$. (Reflexivity)*
2. *If $a\sigma b$ then $b\sigma a$ for all $a, b \in A$. (Symmetry)*
3. *If $a\sigma b$ and $b\sigma c$, then $a\sigma c$ for all $a, b, c \in A$. (Transitivity)*

If σ is an equivalence on A and $a \in A$, then the set

$$[a]_\sigma = \{b \in A \mid a\sigma b\}$$

is called the equivalence class *of a.*

Definition 2 (Partition). *Let A be a set, and let $P \subseteq 2^A$ be a collection of subsets of A. We say that P is a* partition *of A if the following hold.*

1. *$\bigcup P = A$ (P is collectively exhaustive)*
2. *If $C_1, C_2 \in P$ are distinct, then $C_1 \cap C_2 = \varnothing$.*

Equivalence relations and partitions are related in a fundamental way.

Theorem 1. *Let A be a set.*

1. *If σ is an equivalence relation, then the set*

$$A/\sigma = \{[a] \mid a \in A\}$$

   is a partition of A.

2. *If P is a partition of A, then the relation*

$$\sigma = \{(x, y) \mid y \in A, x \in [y]\}$$

   is an equivalence on A.

That is, given an equivalence we can build a partition, and given a partition we can build an equivalence relation.

Definition 3. *Let A be a set and σ an equivalence on A. Then the mapping $\pi_\sigma : A \to A/\sigma$ given by $\varphi(a) = [a]_\sigma$ is called the* natural projection *of A onto A/σ.*

Theorem 2.

- *If $\varphi : A \to B$ is a mapping, then $\mathsf{ker}(\varphi)$ is an equivalence relation.*

- *If σ is an equivalence relation on A and $\pi : A \to A/\sigma$ the natural projection, then $\mathsf{ker}(\pi) = \sigma$.*

Quotient sets, it turns out, are extremely important, and we use them frequently. Defining functions on a quotient set can be tricky.

Theorem 3 (First Isomorphism Theorem for Sets)**.** *Let $\varphi : A \to B$ be a function, and suppose we have an equivalence relation σ such that $\sigma \subseteq \mathsf{ker}(\varphi)$. Then there is a unique function $\overline{\varphi} : A/\sigma \to B$ such that $\varphi = \overline{\varphi} \circ \pi_\sigma$.*