

# Homomorphisms

**Definition 1** (Ring Homomorphism). *Let  $R$  and  $S$  be rings. A map  $\varphi : R \rightarrow S$  is called a ring homomorphism if the following are satisfied.*

- $\varphi(a + b) = \varphi(a) + \varphi(b)$  for all  $a, b \in R$ .
- $\varphi(ab) = \varphi(a)\varphi(b)$  for all  $a, b \in R$ .

*If  $R$  and  $S$  are both unital rings, we say that  $\varphi$  is unital if  $\varphi(1_R) = 1_S$ .*

**Proposition 1.**

1. *If  $R$  is a ring, then  $1 : R \rightarrow R$  given by  $1(x) = x$  is a ring homomorphism. If  $R$  is unital, then  $1$  is unital.*
2. *If  $\varphi : R \rightarrow S$  and  $\psi : S \rightarrow T$  are ring homomorphisms, then  $\psi \circ \varphi : R \rightarrow T$  is a homomorphism. If  $\varphi$  and  $\psi$  are unital, then  $\psi \circ \varphi$  is unital.*

Homomorphisms are *structure-preserving maps*. The arithmetic on a ring – the plus and times – are a kind of structure, and homomorphisms are the maps which “transport” this structure to another setting. If  $\varphi : R \rightarrow S$  is a ring homomorphism then in a concrete sense there is a “shadow” of  $R$  inside  $S$ . If  $R$  and  $S$  are both unital rings, then the one element is an extra bit of structure.

**Proposition 2.** *Suppose  $\varphi : R \rightarrow S$  is a ring homomorphism.*

- $\varphi(0_R) = 0_S$
- $\varphi(-a) = -\varphi(a)$  for all  $a \in R$ .
- $\varphi(a - b) = \varphi(a) - \varphi(b)$  for all  $a, b \in R$ .

## Examples

- The natural projection  $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/(n)$  is a unital ring homomorphism.
- If  $R$  is any ring, then there is exactly one ring homomorphism  $\varphi : R \rightarrow 0$ , and exactly one homomorphism  $\psi : 0 \rightarrow R$ . Neither of these is ever unital unless  $R = 0$ .
- Let  $R$  be any unital ring. Then  $\varphi : R \rightarrow \text{Mat}_2(R)$  given by

$$\varphi(r) = \begin{bmatrix} 0 & 0 \\ -r & r \end{bmatrix}$$

is a ring homomorphism. Although  $R$  (and hence  $\text{Mat}_2(R)$ ) is unital,  $\varphi$  is *not* a unital homomorphism. (Why?)

## Images and Kernels

**Definition 2** (Image and Kernel). Let  $\varphi : R \rightarrow S$  be a ring homomorphism. We define subsets of  $R$  and  $S$  as follows.

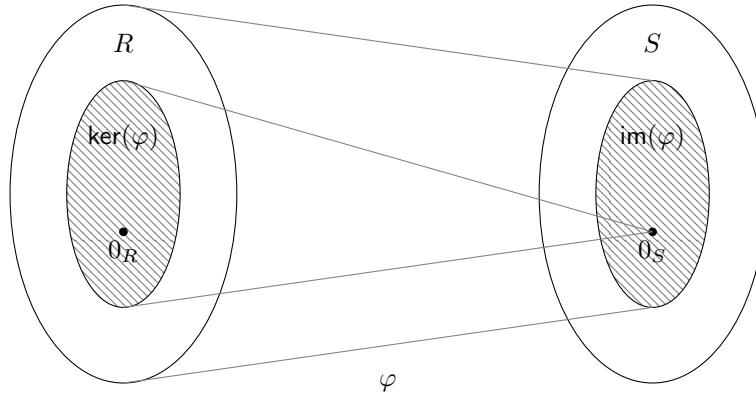
- The kernel of  $\varphi$ , denoted  $\ker(\varphi)$ , is the set

$$\ker(\varphi) = \{r \in R \mid \varphi(r) = 0\}.$$

- The image of  $\varphi$ , denoted  $\text{im}(\varphi)$ , is the set

$$\text{im}(\varphi) = \{s \in S \mid s = \varphi(r) \text{ for some } r \in R\}.$$

**Proposition 3.** If  $\varphi : R \rightarrow S$  is a ring homomorphism, then  $0_R \in \ker(\varphi)$  and  $0_S \in \text{im}(\varphi)$ .



The kernel measures how badly a homomorphism fails to be injective.

**Proposition 4.** A ring homomorphism  $\varphi$  is injective if and only if  $\ker(\varphi) = 0$ .

## Characteristic

**Proposition 5.** If  $R$  is a unital ring, then there is a unique unital homomorphism  $\varphi : \mathbb{Z} \rightarrow R$ .

We can think of the image of this map as a copy of the integers in  $R$ , with  $1 = 1_R$ ,  $2 = 1_R + 1_R$ , and so on.

**Definition 3** (Characteristic). Let  $R$  be a unital ring and  $\varphi : \mathbb{Z} \rightarrow R$  the unique unital homomorphism. If there is a positive integer  $n$  such that  $\varphi(n) = 0$ , then there is a smallest such integer. We call this the characteristic of  $R$ , denoted  $\text{char}(R)$ . That is,  $\text{char}(R)$  is the smallest positive natural number such that

$$\underbrace{1_R + 1_R + \cdots + 1_R}_{n \text{ times}} = 0_R.$$

If no such  $n$  exists, we say that  $\text{char}(R) = 0$ .