Angles

Definition 1 (Angle)**.** *Let $\mathcal{P}$ be an ordered geometry and x, o, and y distinct points.*

- *The set*
$$\angle xoy = \overrightarrow{ox} \cup \overrightarrow{oy}$$
is called the angle *with* vertex o *and* sides $\overrightarrow{ox}$ *and* $\overrightarrow{oy}$.

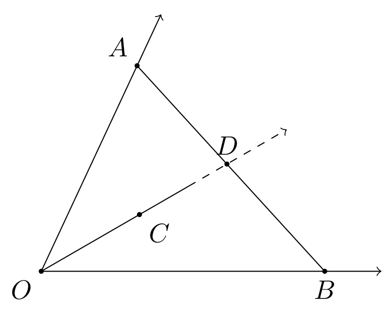
- *If* $[abc]$, *then* $\angle abc$ *is called a* straight *angle, and* $\angle bac$ *is called a* flat *angle.*

- *Suppose further that x, o, and y are not collinear. In this case, since $\mathcal{P}$ is an ordered geometry, the lines $\overleftrightarrow{ox}$ and $\overleftrightarrow{oy}$ divide $\mathcal{P}$ into half-planes. Let H_1 be the y half-plane of $\overleftrightarrow{ox}$, and let K_1 be the x half-plane of $\overleftrightarrow{oy}$. We define the* interior *of $\angle xoy$ to be the set*
$$\mathsf{int}\angle xoy = H_1 \cap K_1.$$
If x, y, and o are collinear, then the interior of $\angle xoy$ is not defined.

Definition 2 (Angle Pairs)**.** *Suppose x, y, z, w, and o are distinct points in an ordered geometry.*

- $\angle xoy$ *and* $\angle yoz$ *are called an* adjacent pair.
- $\angle xoy$ *and* $\angle yoz$ *are called a* linear pair *if* $[xoz]$.
- $\angle xoy$ *and* $\angle zow$ *are called a* vertical pair *if* $[xoz]$ *and* $[yow]$.

Theorem 1 (Crossbar Theorem)**.** *Suppose O, A, and B are noncollinear points in an ordered geometry, and that $C \in \mathsf{int}\angle AOB$. Then $\overrightarrow{OC}$ cuts $\overline{AB}$ at a unique point D.*



Proof. By the Interpolation property, there is a point P on $\overleftrightarrow{OA}$ such that $[POA]$. Note that A and P are on opposite sides of $\overleftrightarrow{OB}$, so that P and C are on opposite sides of $\overleftrightarrow{OB}$. (Since A and C are on the same side of $\overleftrightarrow{OB}$ by definition.) Consider now the triangle $\triangle PAB$. Note that the line $\overleftrightarrow{OC}$ does not contain A, B, or P, since C is not on $\overleftrightarrow{OA}$ or $\overleftrightarrow{OB}$ by hypothesis. Moreover, $\overleftrightarrow{OC}$ cuts $\overline{PA}$ at O. By Pasch's Axiom, $\overleftrightarrow{OC}$ must also cut either $\overline{PB}$ or $\overline{AB}$.

Suppose $\overleftrightarrow{OC}$ cuts $\overline{PB}$ at a (necessarily unique) point Q. Note that $\overleftrightarrow{OC} = \overleftrightarrow{QC}$. Now P and Q are on the same side of $\overleftrightarrow{OB}$, so that Q and C are on *opposite* sides of $\overleftrightarrow{OB}$. Thus, there is a unique point R on $\overleftrightarrow{OB}$ such that $[QRC]$. In particular, $R \in \overleftrightarrow{QC}$. Now we have $O, R \in \overleftrightarrow{OC}$ and $O, R \in \overleftrightarrow{OB}$, so that $\overleftrightarrow{OC} = \overleftrightarrow{OB}$, a contradiction.

Hence $\overleftrightarrow{OC}$ must cut $\overline{AB}$ at a unique point; say D. Now D and A are on the same side of $\overleftrightarrow{OB}$, and so C and D are on the same side of $\overleftrightarrow{OB}$; in particular, we cannot have $[DOC]$. So in fact $\overrightarrow{OC}$ cuts $\overline{AB}$ at a unique point. $\square$