Direct Sums

Theorem 1 (Direct Sum)**.** *Given rings R and S, we define "componentwise" operations on the cartesian product $R \times S$ as follows.*

$$\begin{aligned}(r_1, s_1) + (r_2, s_2) &= (r_1 + r_2, s_1 + s_2) \\ (r_1, s_1) \cdot (r_2, s_2) &= (r_1 \cdot r_2, s_1 \cdot s_2)\end{aligned}$$

Then we have the following.

- *These operations make $R \times S$ into a ring, which we call the* direct sum *of R and S and denote by $R \oplus S$.*
- *$R \oplus S$ is commutative iff R and S are commutative.*
- *$R \oplus S$ is unital iff R and S are unital, and in this case $1_{R \oplus S} = (1_R, 1_S)$.*
- *The coordinate projections $\pi_1 : R \oplus S \to R$ and $\pi_2 : R \oplus S \to S$, given by $\pi_1(r, s) = r$ and $\pi_2(r, s) = s$, are surjective ring homomorphisms. If R and S are unital, then π_1 and π_2 are unital.*

Proposition 2.

1. *If $R_1 \cong R_2$ and $S_1 \cong S_2$, then $R_1 \oplus S_1 \cong R_2 \oplus S_2$.*
2. *$R \oplus 0 \cong R$*
3. *$R \oplus S \cong S \oplus R$*
4. *$(R \oplus S) \oplus T \cong R \oplus (S \oplus T)$*

Proposition 3. *Let $a, b > 1$ be coprime integers. Then $\mathbb{Z}/(a) \oplus \mathbb{Z}/(b) \cong \mathbb{Z}/(ab)$.*

(Lemma: If $m|n$ and $x \equiv y \mod n$, then $x \equiv y \mod m$.)