Incidence Structures

Definition 1 (Incidence Structure)**.** *Let P be a set and let $L \subseteq 2^P$ be any collection of subsets of P. Then $\mathcal{P} = (P, L)$ is called an* incidence structure*.*

If (P, L) is an incidence structure, then we also make the following definitions.

- Elements of P are called *points*.
- Elements of L are called *lines*.
- If p is a point and ℓ a line such that $p \in \ell$, we say that p *lies on* ℓ or that ℓ *contains* p.
- Given a set $S \subseteq P$ of points, we say that S is *collinear* if there is a line ℓ such that $S \subseteq \ell$.

Examples

To define an incidence structure, we need to specify two things: the set of points, and which sets of points are considered to be lines. It is important to remember that "point" and "line" here are just words! Exactly what they mean depends on context. Here are some incidence structures that we will use as examples.

2^P **Trivial Incidence Structures.** Let P be *any* set, and let $L = 2^P$ be the collection of all subsets of P. Certainly $(P, 2^P)$ is an incidence structure. This example is not terribly interesting, however, since every set of points is collinear. To be really useful, collinearity should say something very strong about a set of points – this structure has too many lines.

$\mathbb{R}^2$ **The Cartesian Plane.** Let $P = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. Now elements of $\mathbb{R}^2$ have a kind of arithmetic as follows.

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ t(x, y) &= (tx, ty) \text{ if } t \in \mathbb{R} \end{aligned}$$

(This may look familiar as the vector space structure on $\mathbb{R}^2$.) Given distinct points $A, B \in \mathbb{R}^2$, we define the *line generated by* A and B as follows.

$$\ell_{A,B} = \{A + t(B - A) \mid t \in \mathbb{R}\}.$$

Finally, let $L = \{\ell_{A,B} \mid A, B \in \mathbb{R}^2, A \neq B\}$ be the set of all such lines. We will call the incidence structure $\mathbb{R}^2 = (P, L)$ the *Cartesian plane*.

As an exercise, can you find a succinct description of the line generated by $(0, 1)$ and $(1, 2)$?

More generally, we can give a succinct description of any such line.

Proposition 1. *Given distinct points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the Cartesian plane, the line $\ell_{A,B}$ is precisely the set of points $X = (x, y)$ which satisfy the equation*

$$x(b_2 - a_2) - y(b_1 - a_1) = a_1 b_2 - a_2 b_1.$$

Proof. If $X \in \ell_{A,B}$, then $X = A + t(B - A)$ for some real number t. Comparing entries, we have $x = a_1 + t(b_1 - a_1)$ and $y = a_2 + t(b_2 - a_2)$. Solving each equation for t, we see that

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2}.$$

We can rearrange this equation as needed.

Conversely, suppose (x, y) is a solution of the equation above; we can rearrange so that

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = t;$$

it is straightforward to show that $A + t(B - A) = X$, so that $X \in \ell_{A,B}$. □

In fact, this leads to another useful characterization of the points on $\ell_{A,B}$.

Corollary 2. *If A and B are distinct points, then $(x_1, x_2) \in \ell_{A,B}$ if and only if*

$$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} = 0.$$

$\mathbb{D}$ **The Unit Disk.** Let $P = \{(x, y) \mid x, y \in \mathbb{R}, x^2 + y^2 < 1\}$; these are points in the Cartesian plane whose distance from the origin is less than 1. Given distinct points $A, B \in \mathbb{D}$, we define the *line generated by* A and B as follows.

$$\ell^{\mathbb{D}}_{A,B} = \ell_{A,B} \cap \mathbb{D}$$

That is, a "line" is an ordinary Cartesian line intersected with the unit disk. Then let L be the set of all such lines. We will call the incidence structure $\mathbb{D} = (P, L)$ the *Unit Disk*.

F **The Fano Plane.** Let $P = \{1, 2, 3, 4, 5, 6, 7\}$, and then let $L = \{\{1, 2, 3\}, \{2, 4, 6\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \{3, 6, 7\}\}$. We call the incidence structure $\mathcal{F} = (P, L)$ the *Fano plane*.

$\mathbb{Q}^2$ **The Rational Plane.** Similar to the Cartesian plane, let $P = \mathbb{Q}^2$, and given two distinct rational points A and B define the line generated by A and B to be

$$\ell_{A,B} = \{A + t(B - A) \mid t \in \mathbb{Q}\}.$$

Let L be the set of all such lines. Then the incidence structure $\mathbb{Q}^2 = (P, L)$ is called the *rational plane*. Note that lines in $\mathbb{Q}^2$ look much like lines in $\mathbb{R}^2$ except that they are filled with "holes"; any point on a line in $\mathbb{R}^2$ which has an irrational coordinate is not on the corresponding line in $\mathbb{Q}^2$.

$\mathbb{R}^3$ **Three-Space.** Also similar to the Cartesian plane, let $P = \mathbb{R}^3$, and given two distinct triples A and B define the line generated by A and B to be

$$\ell_{A,B} = \{A + t(B - A) \mid t \in \mathbb{Q}\}.$$

Let L be the set of all such lines. Then the incidence structure $\mathbb{R}^3 = (P, L)$ is called *Three-Space.*

$\widehat{\mathcal{P}}$ **Dual Incidence Structures.** This example is a little different from the others in that it is not a single example, but rather a way to make new incidence structures out of old ones. Suppose we have an incidence structure $\mathcal{P} = (P, L)$. Given a point $x \in P$, we define $M_x = \{\ell \in L \mid x \in \ell\}$. That is, M_x is the set of all lines in $\mathcal{P}$ which contain the point x. If we let $M = \{M_x \mid x \in P\}$, then certainly $\widehat{\mathcal{P}} = (L, M)$ is an incidence structure, which we call the *dual* of $\mathcal{P}$.