Betweenness

Definition 1 (Betweenness)**.** *Let $\mathcal{P}$ be an incidence geometry. We say that a ternary relation $[\cdots]$ on the set of points of $\mathcal{P}$ is a* betweenness relation *if the following properties hold.*

B1. If $[xyx]$, then $x = y$, for all points x and y.

B2. If x and y are distinct points and $[xzy]$, then $[yzx]$ and $z \in \overleftrightarrow{xy}$.

B3. If x, y, and z are distinct points, then at most one of $[xyz]$, $[yzx]$, and $[zxy]$ is true.

Definition 2 (Segment, Ray)**.** *Let x and y be distinct points in an incidence geometry $\mathcal{P} = (P, L)$.*

- *The set*
$$\overline{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy]\}$$
is called the segment *with* endpoints *x and y. If $z \in \overline{xy}$ and $z \neq x$ and $z \neq y$, we say that z is* interior to *$\overline{xy}$.*

- *The set*
$$\overrightarrow{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy] \text{ or } [xyz]\}$$
is called the ray *with* vertex *x* toward *y.*

Proposition 1. *If $\mathcal{P}$ is an incidence geometry and $[\cdots]$ a betweenness relation on $\mathcal{P}$, then the following hold.*

1. *$\overline{xy} = \overline{yx}$ for all distinct points x and y.*
2. *$\overline{xy} \subseteq \overrightarrow{xy} \subseteq \overleftrightarrow{xy}$ for all distinct points x and y.*
3. *If ℓ is a line and x and y distinct points, then $\overline{xy} \cap \ell$ is either $\overline{xy}$, $\varnothing$, or $\{p\}$ for some point p.*
4. *$\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ for all distinct points x and y.*

Examples

$\mathbb{R}^2$

$\mathcal{A}$

The Trichotomy Property

Definition 3. *We say that a betweenness relation $[\cdots]$ on an incidence geometry $\mathcal{P}$ has the* Trichotomy Property *if, whenever x, y, and z are distinct, collinear points, exactly one of $[xyz]$, $[yzx]$, and $[zxy]$ is true. That is, given three collinear points, exactly one is between the other two.*

Proposition 2. *Suppose $\mathcal{P}$ is an incidence geometry and $[\cdots]$ a betweenness relation with the Trichotomy Property. Then the following hold.*

1. *For all distinct points x and y,*

$$\overleftrightarrow{xy} = \{z \mid z = x \text{ or } z = y \text{ or } [zxy] \text{ or } [xzy] \text{ or } [xyz]\}.$$

2. $\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ *for all distinct points x and y.*

The 4-Point Property

First for some shorthand: if x, y, z, and w are distinct points, we will say $[xyzw]$ precisely when $[xyz]$, $[xyw]$, $[xzw]$, and $[yzw]$. More generally, if $x_1, \ldots, x_n$ are distinct points, then $[x_1 x_2 \ldots x_n]$ means that $[x_i x_j x_k]$ for all triples (i, j, k) with $1 \le i < j < k \le n$.

Definition 4 (The 4-Point Property)**.** *We say that a betweenness relation $[\cdots]$ on an incidence geometry $\mathcal{P}$ has the* 4-Point Property *if the following hold for all distinct points x, y, z, and w.*

1. *If $[xyz]$ and $[xzw]$, then $[xyw]$ and $[yzw]$.*

2. *If $[xyz]$ and $[yzw]$, then $[xyz]$ and $[xzw]$.*

Proposition 3. *Suppose $\mathcal{P}$ is an incidence geometry and $[\cdots]$ a betweenness relation on $\mathcal{P}$ having the 4-Point Property. If x, y, and z are distinct points such that $[xyz]$, then the following hold.*

1. $\overline{xy} \cup \overline{yz} = \overline{xz}$

2. $\overline{xy} \cap \overline{yz} = \{y\}$

3. $\overrightarrow{yx} \cap \overrightarrow{yz} = \{y\}$

4. $\overrightarrow{xy} = \overrightarrow{xz}$

Proposition 4. *If $\mathcal{P}$ is an incidence geometry with a betweenness relation having both the Trichotomy Property and the 4-Point Property, and if x, y, and z are distinct points such that $[xyz]$, then $\overrightarrow{yx} \cup \overrightarrow{yz} = \overleftrightarrow{xz}$.*

The Interpolation Property

Definition 5. *We say that a betweenness relation* $[\cdot\cdot\cdot]$ *on an incidence geometry* $\mathcal{P}$ *has the* Interpolation Property *if for all distinct points* x *and* y *in* $\mathcal{P}$, *there exist points* z_1, z_2, *and* z_3 *such that* $[z_1xy]$, $[xz_2y]$, *and* $[xyz_3]$.

Proposition 5. *If* $\mathcal{P}$ *is an incidence geometry with a betweenness relation having both the Interpolation Property and the 4-Point Property, then every line in* $\mathcal{P}$ *has infinitely many points.*