Localization

In a general ring with 1, or even a general domain, elements typically do not have multiplicative inverses. Those which do are called units and are very special. In this section we will see how a domain can be "extended" to a larger ring so that any given element can be made into a unit.

Definition 1 (Multiplicative subset)**.** *Let R be a domain and $S \subseteq R$. We say that S is a* multiplicative subset *of R if $0 \notin R$ and if S is closed under multiplication.*

Domains have plenty of multiplicative sets. For instance, the set of all nonzero elements is multiplicative. If $a \in R$ is not zero, then the set $S = \{1, a, a^2, a^3, \ldots\}$ of powers of a is multiplicative.

Here is the punch line of this section.

> If $S \subseteq R$ is a multiplicative subset, then we can construct a new ring, T, which contains R as a subset, but in which the elements of S are units.

Proposition 1. *Let R be a domain and $S \subseteq R$ a multiplicative subset. We define a relation Φ on the cartesian product $S \times R$ as follows:*

$$(s_1, r_1)\Phi(s_2, r_2) \quad \text{iff} \quad r_1 s_2 = r_2 s_1.$$

This relation Φ is an equivalence.

Proof.

- $rs = rs$ for all $r \in R$ and $s \in S$, so that $(s, r)\Phi(s, r)$.
- Suppose $(s_1, r_1)\Phi(s_2, r_2)$. Then $r_1 s_2 = r_2 s_1$, so that $r_2 s_1 = r_1 s_2$, and thus $(s_2, r_1)\Phi(s_1, r_1)$.
- Suppose $(s_1, r_1)\Phi(s_2, r_2)$ and $(s_2, r_2)\Phi(s_3, r_3)$. Now $r_1 s_2 = r_2 s_1$ and $r_2 s_3 = r_3 s_2$. We then have $r_1 s_2 r_2 s_3 = r_2 s_1 r_3 s_2$; rearranging (since R is commutative) and using cancellation, we have $r_1 s_3 = r_3 s_1$. So $(s_1, r_1)\Phi(s_3, r_3)$ as needed. $\square$

Since Φ is an equivalence, it induces a partition on the set $S \times R$. We will denote this quotient set $S^{-1}R = (S \times R)/\Phi$, and denote the equivalence class of (s, r) by $\frac{r}{s}$.

Proposition 2. *Let R be a domain with $S \subseteq R$ a multiplicative subset. Define operations $+$ and $\cdot$ on $S^{-1}R$ as follows.*

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

Then we have the following.

1. *$+$ and $\cdot$ are well-defined.*

2. *$S^{-1}R$, with these operations, is an integral domain, which we call the* localization *of R* at *S.*

3. *If $t \in S$, then the mapping $\iota : R \to S^{-1}R$ given by $\iota(r) = \frac{rt}{t}$ is an injective ring homomorphism, and $\iota(t)$ is a unit in $S^{-1}R$.*

Proof. (super tedious) $\square$

So $S^{-1}R$ is a new ring which contains a "copy" (homomorphic image) of R, within which the elements of S become units.

Definition 2. *Let R be a domain and let $D = \{x \in R \mid x \neq 0\}$ be the multiplicative subset of all nonzero elements of R. Then the localization $D^{-1}R$ is a field, called the* field of fractions *of R.*

For example, $\mathbb{Q}$ is properly defined as the field of fractions of $\mathbb{Z}$.

Special things

Proposition 3. *If R is a UFD and $S \subseteq R$ any multiplicative set, then $S^{-1}R$ is also a UFD.*

Proof. (type this) $\square$

Proposition 4. *If R is a Euclidean domain and $S \subseteq R$ any multiplicative set, then $S^{-1}R$ is a Euclidean domain.*

Proof. (type this) $\square$

Exercises

1. (do stuff in $\mathbb{Z}[\frac{1}{2}]$)

2. (universal property of localization)

3. Let R be a GCD domain, with F its field of fractions. An element $\frac{a}{b} \in F$ is said to be *reduced* (or in *lowest terms*) if $\mathsf{gcd}\,(a, b) = 1$.

   (a) Show that every element of F has a reduced representative.

   (b) Show that reduced fractions are unique in the following sense: If $\frac{a}{b} = \frac{c}{d}$ are both reduced, then $c = au$ and $d = bu$ for some unit u.