

Hyperbolic Half-Planes

We've seen that every plane geometry must be either Euclidean or Hyperbolic, and we have a concrete model of Euclidean plane geometry – the Cartesian plane. In this section we will construct a concrete model of Hyperbolic plane geometry called the Poincare Half-Plane.

Let \mathcal{P} be a Euclidean plane geometry. In this section, we will refer to the lines of \mathcal{P} as \mathcal{P} -lines. Let \mathcal{L} be a \mathcal{P} -line, and let \mathcal{H} be one of the half-planes induced by \mathcal{L} via the plane separation property.

We will consider subsets of \mathcal{H} , called \mathcal{H} -lines, of the following two types.

- A *Type I* \mathcal{H} -line is a subset of the form $\mathcal{H} \cap \ell$, where ℓ is a \mathcal{P} -line which is perpendicular to \mathcal{L} .
- A *Type II* \mathcal{H} -line is a subset of the form $\mathcal{H} \cap \mathcal{C}_o(x)$, where $\mathcal{C}_o(x)$ is a \mathcal{P} -circle whose center o is on \mathcal{L} . We will call o the *phantom center* of this \mathcal{H} -line.

Certainly \mathcal{H} , together with the family of all possible \mathcal{H} -lines described above, is an incidence structure.

Proposition 1. *\mathcal{H} is an incidence geometry.*

Proof.

- IG1. Let $x, y \in \mathcal{H}$ be distinct points. If the \mathcal{P} -line \overleftrightarrow{xy} is perpendicular to the \mathcal{P} -line \mathcal{L} , then there is exactly one \mathcal{H} -line of type I containing x and y , and there are no \mathcal{H} -lines of type II containing x and y . If the \mathcal{P} -line \overleftrightarrow{xy} is not perpendicular to the \mathcal{P} -line \mathcal{L} , then there is exactly one type II \mathcal{H} -line containing x and y , and there are no type I \mathcal{H} -lines containing x and y .
- IG2. (Every \mathcal{H} -line contains at least two distinct points.)
- IG3. Note that \mathcal{H} is not empty by the Plane Separation property; let $x \in \mathcal{H}$. Construct the foot f of x on \mathcal{L} , and by interpolation let y be a point such that $[f y x]$. Now construct a point g on \mathcal{L} different from f by interpolation, and again let z be a point such that $[g z x]$ by interpolation. Note that both y and z are in \mathcal{H} . Moreover, there is a unique Type I \mathcal{H} -line containing x and y , which does not contain z , and there are no Type II \mathcal{H} -lines containing x and y . So x, y , and z are not collinear. \square

Note that two \mathcal{H} -lines intersect if and only if they intersect as sets in \mathcal{P} .

Proposition 2. *\mathcal{H} is Hyperbolic.*

Proof. Recall that an incidence geometry is called *hyperbolic* if, given a line ℓ and a point p not on ℓ , there are infinitely many lines t which pass through p and do not intersect ℓ . Since \mathcal{H} has lines of two types, we need to consider each one separately.

- Let $\mathcal{H} \cap \ell$ be a Type I \mathcal{H} -line, where ℓ is a \mathcal{P} -line which is perpendicular to \mathcal{L} at o . Let $p \in \mathcal{H}$ be a point, and let f be the foot of p on \mathcal{L} (in \mathcal{P}). Let q be any point such that $[oqf]$ in \mathcal{P} . There is a unique point a on \mathcal{L} such that $\overline{ap} \equiv \overline{aq}$, and $\mathcal{H} \cap \mathcal{C}_a(p)$ is a Type II \mathcal{H} -line containing p . Note that the foot of a on ℓ is exterior to $\mathcal{C}_a(p)$, and thus $\mathcal{H} \cap \mathcal{C}_a(p)$ and $\mathcal{H} \cap \ell$ are parallel. There are infinitely many possible qs , all of which yield distinct \mathcal{H} -lines. So there are infinitely many \mathcal{H} -lines which pass through p and are parallel to $\mathcal{H} \cap \ell$.
- (finish this) □

A Concrete Model

We've seen that *any* line in *any* Euclidean plane geometry can be used to construct a new hyperbolic incidence geometry, which we've called a half-plane model. Let's now choose a specific Euclidean plane and a specific line: consider the line $\mathcal{L} = \overleftrightarrow{(0,0)(1,0)}$ in the Cartesian plane, and let \mathcal{H} be the $(0,1)$ -side of \mathcal{L} . That is, $\mathcal{H} = \{(x, y) \mid x, y \in \mathbb{R}, y > 0\}$.

If X and Y are distinct points in \mathcal{H} , then there is a unique line \overleftrightarrow{XY} which contains both. Presently, we will construct a *parameterization* of \overleftrightarrow{XY} ; that is, a function $\Phi_{X,Y} : \mathbb{R} \rightarrow \overleftrightarrow{XY}$ having the property that $\Phi_{X,Y}(0) = X$ and $\Phi_{X,Y}(1) = Y$. You may recall that we did something similar in the Cartesian plane, although there we *defined* lines in terms of their parameterization.

Suppose X and Y generate a Type I line in \mathcal{H} . In this case, we have $X = (z, x_2)$ and $Y = (z, y_2)$ for some z , and both x_2 and y_2 are strictly positive. Let $\alpha = \ln(x_2)$ and $\beta = \ln(y_2)$, and define $\Phi_{X,Y}(t) = (z, e^{\alpha+t(\beta-\alpha)})$. Certainly $\Psi_{X,Y} : \mathbb{R} \rightarrow \overleftrightarrow{XY}$ is bijective, and we have $\Phi_{X,Y}(0) = X$ and $\Phi_{X,Y}(1) = Y$. Note also that $\Phi_{Y,X}(t) = \Phi_{X,Y}(1-t)$.

Now suppose X and Y generate a Type II line in \mathcal{H} . Let $O = (c, 0)$ be the phantom center of X and Y , and let $r = \sqrt{(X-O) \cdot (X-O)}$. Now define a mapping $\varphi : \mathbb{R} \rightarrow \overleftrightarrow{XY}$ by

$$\varphi(t) = \left(c - \frac{rt}{\sqrt{t^2 + r^2}}, \frac{r^2}{\sqrt{t^2 + r^2}} \right).$$

Note that φ is bijective, with inverse $\psi(a, b) = r(c - a)/b$. Let $\alpha = \psi(X)$ and $\beta = \psi(Y)$. Finally, define $\Phi_{X,Y} : \mathbb{R} \rightarrow \overleftrightarrow{XY}$ by $\Phi_{X,Y}(t) = \varphi(\alpha + t(\beta - \alpha))$. Certainly $\Phi_{X,Y}$ is a bijection, and we have $\Phi_{X,Y}(0) = X$ and $\Phi_{X,Y}(1) = Y$. Note also that $\Phi_{Y,X}(t) = \Phi_{X,Y}(1-t)$.

Proposition 3. *The ternary relation $[\cdot \cdot \cdot]$ given by $[xzy]$ iff there is a $t \in (0, 1)$ such that $Z = \Phi_{X,Y}(t)$. is a betweenness relation.*

Proof.

B2. Suppose $[XZY]$. By definition, $Z = \Phi_{X,Y}(t)$ where $t \in (0, 1)$. Now $1-t \in (0, 1)$, and we have $Z = \Phi_{Y,X}(1-t)$ as needed. Certainly $Z \in \overleftrightarrow{XY}$.

B3. Let X, Y , and Z be distinct points in \mathcal{H} , and suppose WLOG that $[XYZ]$ and $[XZY]$. Now X, Y , and Z are collinear.

- Suppose X, Y , and Z are on a Type I line; say $X = (u, x)$, $Y = (u, y)$, and $Z = (u, z)$. Now $z = e^{\ln(x)+t(\ln(y)-\ln(x))}$ and $y = e^{\ln(x)+v(\ln(z)-\ln(x))}$ for some $t, v \in (0, 1)$. Substituting, we see that $z = e^{\ln(x)+tv(\ln(z)-\ln(x))}$, and since the exponential map is injective, $\ln(x) = \frac{\ln(y)-v\ln(z)}{1-v}$. Similarly, $\ln(x) = \frac{\ln(z)-t\ln(y)}{1-t}$. From here we can show that $\ln(y) = \ln(z)$, a contradiction.
- Suppose X, Y , and Z are on a Type II line. Let $\alpha = \psi(X)$, $\beta = \psi(Y)$, and $\gamma = \psi(Z)$. Now $Z = \varphi(\alpha + t(\beta - \alpha))$ and $Y = \varphi(\alpha + v(\gamma - \alpha))$ for some $t, v \in (0, 1)$. Hitting both of these equations with ψ , we see $\beta - \alpha = v(\gamma - \alpha)$ and $\gamma - \alpha = t(\beta - \alpha)$, so that $tv = 1$, a contradiction.

□

Note that $[XYZ]$ iff X, Y , and Z are collinear and either $\psi(X) < \psi(Y) < \psi(Z)$ or $\psi(X) > \psi(Y) > \psi(Z)$.

Proposition 4. \mathcal{H} has the Trichotomy property, the Four-Point property, and the Interpolation property.

Proof.

- **Trichotomy.** Let X and Y be distinct points in \mathcal{H} , and let $Z \in \overleftrightarrow{XY}$ be distinct from both. Now $Z = \Phi_{X,Y}(t)$ for some $t \in \mathbb{R}$. Let $\alpha = \psi(X)$, $\beta = \psi(Y)$, and $\gamma = \psi(Z)$. Now $Z = \Phi_{X,Y}(t)$ for some $t \in \mathbb{R}$, and hitting both sides of this equation with ψ , we have $\gamma = \alpha + t(\beta - \alpha)$. Solving this equation for α and then β , we see that $\alpha = \beta + \frac{1}{1-t}(\gamma - \beta)$ and $\beta = \alpha + \frac{1}{t}(\gamma - \alpha)$. Note that one of t , $1/t$, and $1/(1-t)$ must be in the interval $(0, 1)$.
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Proposition 5. \mathcal{H} has the Plane-Separation property.

Proof. Use plane separation in \mathbb{R}^2 for Type I lines, and interior and exterior of circles for Type II lines. □