Over a GCD Domain – Part II

In this section we establish some important results about irreducibility and factorization for polynomials over a GCD domain.

Proposition 1 (Gauss' Lemma – Part II)**.** *Let R be a GCD domain with field of fractions F, and let $p(x) \in R[x]$ have positive degree. Then $p(x)$ is irreducible in $R[x]$ if and only if $p(x)$ is irreducible in $F[x]$ and primitive in $R[x]$.*

Proof. (type this) □

Corollary 2. *If $p(x) \in R[x]$ (R a GCD domain) is Eisenstein and primitive, then $p(x)$ is irreducible in $R[x]$.*

Proof. Suppose $p(x) = a(x)b(x)$ with $a, b \in R[x]$. Since p is Eisenstein, WLOG $a(x)$ is a constant; say $a(x) = a_0$. Now $a_0 | p$ in $R[x]$, so that $a_0 | \mathsf{content}(p)$ in R. Since $p(x)$ is primitive, a is a unit in R, hence a unit in $R[x]$. So $p(x)$ is irreducible in $R[x]$. □

Proposition 3 (Rational Root Theorem)**.** *Let R be a GCD domain with fraction field F. Suppose $p(x) \in R[x]$. Let $\frac{u}{v} \in F$ be a fraction in lowest terms; that is, $\mathsf{gcd}(u, v) = 1$ in R. If $\frac{u}{v}$ is a root of $p(x)$, then u divides the constant coefficient of p, and v divides the leading coefficient of p.*

Exercises

1. Let R be a GCD domain with $p(x), q(x) \in R[x]$ so that q is irreducible (hence prime), and let k be a natural number. Show that q^{k+1} divides p in $R[x]$ iff $q|p$ and $q^k|p'$ in $R[x]$. In particular, show that p is squarefree iff $\mathsf{gcd}(p, p') = 1$.