Congruence

Definition 1 (Segment Congruence)**.** *Let $\mathcal{P}$ be an ordered geometry, and suppose we have an equivalence relation on pairs of points, denoted $\cong_s$. We call $\cong_s$ a* segment congruence *if the following properties are satisfied.*

SC1. $(x, y) \cong_s (x, y)$ *for all points x and y.*

SC2. *If $(x, y) \cong_s (z, w)$ then $(z, w) \cong_s (x, y)$ for all points x, y, z, and w.*

SC3. *If $(x, y) \cong_s (z, w)$ and $(z, w) \cong_s (u, v)$, then $(x, y) \cong_s (u, v)$ for all x, y, z, w, u, and v.*

SC4. $(x, x) \cong_s (y, y)$ *for all points x and y.*

SC5. $(x, y) \cong_s (y, x)$ *for all points x and y.*

SC6. *If $z \in \overrightarrow{xy}$ such that $(x, z) \cong_s (x, y)$, then $z = y$.*

In this case, $\cong_s$ is and equivalence relation on the set of segments *in $\mathcal{P}$, and we write $\overline{xy} \equiv \overline{ab}$ to mean $(x, y) \cong_s (a, b)$.*

The first three properties ensure that $\cong_s$ is an equivalence relation; the fourth handles the "trivial" case, the fifth makes $\cong_s$ well-defined on segments, and the sixth ensures that $\cong_s$ differentiates between segments on the same ray which share an endpoint.

Examples

$\mathbb{R}^2$ Given points A, B, X, and Y in the Cartesian plane, we say that $\overline{AB} \equiv \overline{XY}$ if $(B - A) \cdot (B - A) = (Y - X) \cdot (Y - X)$, where $\cdot$ is the usual dot product of vectors. It is straightforward to show that this is a segment congruence.

Angle Congruence

Definition 2 (Angle Congruence)**.** *Let $\mathcal{P}$ be an ordered geometry, and suppose we have an equivalence relation on triples of points, denoted $\cong_a$. We call $\cong_a$ an* angle congruence *if the following properties are satisfied.*

AC1. $(a, o, b) \cong_a (a, o, b)$ *for all points a, o, and b.*

AC2. *If $(a, o, b) \cong_a (x, p, y)$, then $(x, p, y) \cong_a (a, o, b)$ for all points a, o, b, x, p, and y.*

AC3. *If $(a, o, b) \cong_a (x, p, y)$ and $(x, p, y) \cong_a (h, q, k)$, then $(a, o, b) \cong_a (h, q, k)$ for all points a, o, b, x, p, y, h, q, and k.*

AC4. *If $[xyz]$ and $[abc]$, then $(x, y, z) \cong_a (a, b, c)$ and $(y, x, z) \cong_a (b, a, c)$.*

AC5. *If $x \in \overrightarrow{oa}$ and $y \in \overrightarrow{ob}$ and x, y, and o are distinct, then $(a, o, b) \cong_a (x, o, y)$.*

AC6. $(a,o,b) \cong_a (b,o,a)$ *and* $(a,o,b) \cong_a (a,o,b)$ *for all points* a, o, *and* b.

AC7. If a, b, *and* o *are noncollinear points and* x *is on the* b*-side of* $\overleftrightarrow{oa}$ *such that* $(a,o,b) \cong_a (a,o,x)$, *then* $x \in \overrightarrow{ob}$.

In this case, $\cong_a$ *is an equivalence relation on the set of* angles *in* $\mathcal{P}$, *and we write* $\angle aob \equiv \angle xpy$ *to mean* $(x,o,y) \cong_a (x,p,y)$.

Again, the first three properties make $\cong_a$ an equivalence, the fourth handles the trivial case, the fifth and sixth make $\cong_a$ well-defined on angles, and the seventh ensures that $\cong_a$ differentiates between angles on one half-plane which share a vertex.

Definition 3 (Triangle Congruence)**.** *Let* a, b, *and* c *be distinct points, and let* x, y, *and* z *be distinct points. We say that* $\triangle abc$ *is* congruent *to* $\triangle xyz$, *denoted* $\triangle abc \equiv \triangle xyz$, *if*

$$\overline{ab} \equiv \overline{xy}, \quad \overline{bc} \equiv \overline{yz}, \quad \text{and} \quad \overline{ca} \equiv \overline{zx}$$

and

$$\angle abc \equiv \angle xyz, \quad \angle bca \equiv \angle yzx, \quad \text{and} \quad \angle cab \equiv \angle zxy.$$

Proposition 1.

1. $\triangle abc \equiv \triangle abc$.
2. *If* $\triangle abc \equiv \triangle xyz$, *then* $\triangle xyz \equiv \triangle abc$.
3. *If* $\triangle abc \equiv \triangle xyz$ *and* $\triangle xyz \equiv \triangle hk\ell$, *then* $\triangle abc \equiv \triangle hk\ell$.

Definition 4. *Let* a, b, *and* c *be distinct points.*

- *We say that the triangle* $\triangle abc$ *is* equilateral *if* $\overline{ab} \equiv \overline{bc} \equiv \overline{ca}$.
- *We say that the triangle* $\triangle abc$ *is* isoceles *if two of its sides are congruent to each other.*

Supplementary and Right Angles

Definition 5 (Supplementary Angles)**.** *We say that angles* $\angle aob$ *and* $\angle xpy$ *are* supplementary *if there is a linear pair,* $\angle uqv$ *and* $\angle vqw$, *such that* $\angle aob \equiv \angle uqv$ *and* $\angle xpy \equiv \angle vqw$. *In this case we say that* $\angle xpy$ *is a* supplement *of* $\angle aob$.

Proposition 2. *Let* $\mathcal{P}$ *be an ordered geometry with an angle congruence.*

1. *If two angles form a linear pair, then they are supplementary.*
2. *Every angle has a supplement.*