

## Angles

**Definition 1** (Angle). Let  $\mathcal{P}$  be an ordered geometry and  $x$ ,  $o$ , and  $y$  distinct points.

- The set

$$\angle xoy = \overrightarrow{ox} \cup \overrightarrow{oy}$$

is called the angle with vertex  $o$  and sides  $\overrightarrow{ox}$  and  $\overrightarrow{oy}$ .

- Suppose further that  $x$ ,  $o$ , and  $y$  are not collinear. In this case, since  $\mathcal{P}$  is an ordered geometry, the lines  $\overleftrightarrow{ox}$  and  $\overleftrightarrow{oy}$  divide  $\mathcal{P}$  into half-planes. Let  $H_1$  be the  $y$  half-plane of  $\overleftrightarrow{ox}$ , and let  $K_1$  be the  $x$  half-plane of  $\overleftrightarrow{oy}$ . We define the interior of  $\angle xoy$  to be the set

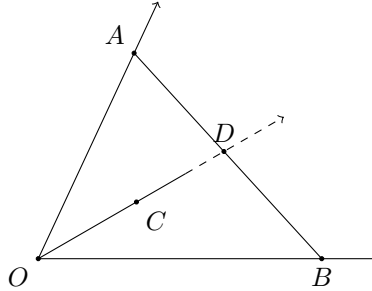
$$\text{int}\angle xoy = H_1 \cap K_1.$$

If  $x$ ,  $y$ , and  $o$  are collinear, then the interior of  $\angle xoy$  is not defined.

**Definition 2** (Linear Pair, Vertical Pair). Suppose  $x$ ,  $y$ ,  $z$ ,  $w$ , and  $o$  are distinct points in an ordered geometry.

- $\angle xoy$  and  $\angle yoz$  are called an adjacent pair if  $y \in \text{int}\angle xoz$ .
- $\angle xoy$  and  $\angle yoz$  are called a linear pair if  $[xoz]$ .
- $\angle xoy$  and  $\angle zow$  are called a vertical pair if  $[xoz]$  and  $[yow]$ .

**Theorem 1** (Crossbar Theorem). Suppose  $O$ ,  $A$ , and  $B$  are noncollinear points in an ordered geometry, and that  $C \in \text{int}\angle AOB$ . Then  $\overrightarrow{OC}$  cuts  $\overline{AB}$  at a unique point  $D$ .



*Proof.* By the Interpolation property, there is a point  $P$  on  $\overrightarrow{OA}$  such that  $[POA]$ . Note that  $A$  and  $P$  are on opposite sides of  $\overleftrightarrow{OB}$ , so that  $P$  and  $C$  are on opposite sides of  $\overleftrightarrow{OB}$ . (Since  $A$  and  $C$  are on the same side of  $\overleftrightarrow{OB}$  by definition.) Consider now the triangle  $\triangle PAB$ . Note that the line  $\overleftrightarrow{OC}$  does not contain  $A$ ,  $B$ , or  $P$ , since  $C$  is not on  $\overrightarrow{OA}$  or  $\overrightarrow{OB}$  by hypothesis. Moreover,  $\overleftrightarrow{OC}$  cuts  $\overline{PA}$  at  $O$ . By Pasch's Axiom,  $\overleftrightarrow{OC}$  must also cut either  $\overline{PB}$  or  $\overline{AB}$ .

Suppose  $\overleftrightarrow{OC}$  cuts  $\overline{PB}$  at a (necessarily unique) point  $Q$ . Note that  $\overleftrightarrow{OC} = \overleftrightarrow{QC}$ . Now  $P$  and  $Q$  are on the same side of  $\overleftrightarrow{OB}$ , so that  $Q$  and  $C$  are on *opposite* sides of  $\overleftrightarrow{OB}$ . Thus, there is a unique point  $R$  on  $\overleftrightarrow{OB}$  such that  $[QRC]$ . In particular,  $R \in \overleftrightarrow{OC}$ . Now we have  $O, R \in \overleftrightarrow{OC}$  and  $O, R \in \overleftrightarrow{OB}$ , so that  $\overleftrightarrow{OC} = \overleftrightarrow{OB}$ , a contradiction.

Hence  $\overleftrightarrow{OC}$  must cut  $\overline{AB}$  at a unique point; say  $D$ . Now  $D$  and  $A$  are on the same side of  $\overleftrightarrow{OB}$ , and so  $C$  and  $D$  are on the same side of  $\overleftrightarrow{OB}$ ; in particular, we cannot have  $[DOC]$ . So in fact  $\overleftrightarrow{OC}$  cuts  $\overline{AB}$  at a unique point.  $\square$