Euclidean Planes

Recall that an incidence geometry is called *euclidean* if, given any line ℓ and any point p not on ℓ, there is exactly one line passing through p which is parallel to ℓ. So far we have avoided using any assumptions about the uniqueness of parallel lines, and have been able to prove a good number of interesting results. We will now specialize to the Euclidean case for a while.

Proposition 1 (Converse of the Alternate Interior Angles Theorem). *In a Euclidean plane geometry, if two parallel lines are cut by a transversal, then alternate interior angles formed by the cut are congruent.*

Proof. (copy angle, use AIA, use uniqueness.) $\square$

Proposition 2. *If ℓ and m are parallel and m and t are parallel, then ℓ and t are parallel.*

Proof. We can assume that ℓ and t are distinct (if equal, they are parallel). Suppose BWOC that ℓ and t meet at the unique point x. Since ℓ and m are parallel, x is not on m. By the Euclidean property, there is a unique line s containing x which is parallel to m. But both ℓ and t satisfy this condition, and they are distinct - a contradiction. $\square$

Corollary 3. *If ℓ_1 and ℓ_2 are parallel and m is incident to ℓ_1, then m is incident to ℓ_2.*

Proposition 4. *If ℓ_1 and m are perpendicular, and if ℓ_1 and ℓ_2 are parallel, then ℓ_2 and m are perpendicular.*

Proof. If $\ell_1 = \ell_2$ there's nothing to do. Otherwise m is a transversal and the result follows from the converse of the AIA theorem. $\square$

Proposition 5. *If ℓ_1 and ℓ_2 are parallel, m_1 and ℓ_1 are perpendicular, and m_2 and ℓ_2 are perpendicular, then m_1 and m_2 are parallel.*

Proof. ℓ_2 and m_1 are perpendicular by the converse of AIA, and then m_1 and m_2 are parallel by AIA. $\square$

Construction 6. *Given 3 distinct noncollinear points A, B, and C, there is a unique circle which contains all of them. This circle is called the* circumcircle *of $\triangle ABC$, and its center is the* circumcenter.

Proof. Let ℓ be the perpendicular bisector of $\overline{AB}$ and let m be the perpendicular bisector of $\overline{BC}$. Now ℓ and m must meet, since otherwise $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are parallel (which they aren't, as they meet at the unique point B (since A, B, and C are not collinear)). Moreover they must meet at a *unique* point, say O, since otherwise we can show that $A = C$. Recall that points X on the perpendicular bisector of $\overline{AB}$ have the property that $\overline{AX} \equiv \overline{BX}$. So we have $\overline{AO} \equiv \overline{BO} \equiv \overline{CO}$, and thus $\mathcal{C}_O(A)$ contains A, B, and C. $\square$

Proposition 7.

1. *Opposite angles of a parallelogram are congruent.*
2. *Opposite sides of a parallelogram are congruent.*
3. *The diagonals of a parallelogram bisect each other.*

Proof. For the angles, use AIA and converse of AIA. For the sides, construct a diagonal and use AAS. For the diagonals, use converse of AIA and ASA. $\square$

Proposition 8 (Thales' Theorem)**.** *Suppose A and B are the opposite endpoints of a diameter of a circle centered at O, and that C is a point on this circle distinct from A and B. Then $\angle ACB$ is right.*