Incidence Geometries

Definition 1 (Incidence Geometry)**.** *Let $\mathcal{P} = (P, L)$ be an incidence structure. We say $\mathcal{P}$ is an* incidence geometry *if the following properties are satisfied.*

IG1. If $x, y \in P$ are distinct points, then there is a unique line $\ell \in L$ such that $x, y \in \ell$. We denote this line $\overleftrightarrow{xy}$.

IG2. If $\ell \in L$ is a line, then there are at least two distinct points $x, y \in \ell$.

IG3. There is a set of three distinct points which is noncollinear.

Proposition 1. *Let $\mathcal{P} = (P, L)$ be an incidence geometry.*

1. *If $x, y \in P$, then $\overleftrightarrow{xy} = \overleftrightarrow{yx}$.*
2. *If $x, y, z \in P$, then the set $\{x, y, z\}$ is collinear if and only if $z \in \overleftrightarrow{xy}$.*
3. *If $z \in \overleftrightarrow{xy}$, then $\overleftrightarrow{xz} = \overleftrightarrow{xy}$.*

Examples

2^P If P is a nonempty set, then the trivial incidence structure 2^P is *not* an incidence geometry since it includes lines with only one point.

$\mathbb{R}^2$ The Cartesian Plane is an incidence geometry, as we show.

IG1. Let $A, B \in \mathbb{R}^2$ be distinct points; we need to show that there is exactly one line containing A and B. First note that $A, B \in \ell_{A,B}$ (since $A = A + 0(B - A)$ and $B = A - 1(B - A)$), so there is at least one such line. Suppose that $\ell = \ell_{P,Q}$ is a line such that $A, B \in \ell$; say $A = P + t_A(Q - P)$ and $B = P + t_B(Q - P)$. (Since A and B are distinct, we have $t_A \neq t_B$.) We claim that $\ell_{A,B} = \ell_{P,Q}$. To this end, if $X \in \ell_{A,B}$, say with $X = A + t(B - A)$, then we have

$$X = A + t(B - A) = P + (t_A + t(t_B - t_A))(Q - P) \in \ell_{P,Q}.$$

Thus we have $\ell_{A,B} \subseteq \ell_{P,Q}$. Now suppose $X \in \ell_{P,Q}$; say $X = P + t(Q - P)$. We have

$$A + \frac{t - t_A}{t_B - t_A}(B - A) = X,$$

so that $X \in \ell_{A,B}$ as needed. So we have $\ell_{A,B} = \ell_{P,Q}$; in particular, any line containing A and B is equal to $\ell_{A,B}$.

IG2. By definition, since $A = A + 0(B - A), B = A + 1(B - A) \in \ell_{A,B}$.

IG3. The point $(0, 1)$ is not on $\ell_{(0,0),(1,0)}$.

$\mathbb{D}$ The Unit Disk is an incidence geometry; to show this, use the fact that $\mathbb{R}^2$ is an incidence geometry.

$\mathbb{Q}^2$ The Rational Plane is an incidence geometry; the proof of this is similar to that for $\mathbb{R}^2$.

$\mathbb{R}^3$ Three Space is an incidence geometry; the proof of this is similar to that for $\mathbb{R}^2$.

Intersecting Lines

Proposition 2. *Let $\mathcal{P} = (P, L)$ be an incidence geometry, with $\ell_1, \ell_2 \in L$ lines. Then exactly one of the following holds.*

- $\ell_1 = \ell_2$,
- $\ell_1 \cap \ell_2 = \varnothing$, *and*
- $\ell_1 \cap \ell_2 = \{p\}$.

Proof. Suppose $\ell_1 \cap \ell_2$ contains at least two points, say x and y. Then in fact $\ell_1 = \overleftrightarrow{xy} = \ell_2$. So $\ell_1 \cap \ell_2$ contains either exactly one or zero points. $\square$

Corollary 3. *In an incidence geometry, three points x, y, and z are not collinear if and only if $\overleftrightarrow{xy} \cap \overleftrightarrow{xz} = \{x\}$.*

Examples

In $\mathbb{R}^2$, we have a nice criterion which detects pairs of lines which intersect at a single point.

Proposition 4. *Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, and $D = (d_1, d_2)$ be points in the Cartesian Plane with $A \neq B$ and $C \neq D$. Then $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{p\}$ is a singleton if and only if*

$$\mathsf{det}\begin{bmatrix} b_1 - a_1 & d_1 - c_1 \\ b_2 - a_2 & d_2 - c_2 \end{bmatrix} \neq 0.$$

Proof. Note that

$$\begin{aligned}
& \overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{p\} \\
\Leftrightarrow \quad & A + t(B - A) = C + u(D - C) \text{ has a unique solution } (t, u) \\
\Leftrightarrow \quad & (B - A)t - (D - C)u = C - A \text{ has a unique solution } (t, u) \\
\Leftrightarrow \quad & \begin{bmatrix} b_1 - a_1 & d_1 - c_1 \\ b_2 - a_2 & d_2 - c_2 \end{bmatrix} \begin{bmatrix} t \\ -u \end{bmatrix} = \begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \end{bmatrix} \text{ has a unique solution } (t, u) \\
\Leftrightarrow \quad & \mathsf{det}\begin{bmatrix} b_1 - a_1 & d_1 - c_1 \\ b_2 - a_2 & d_2 - c_2 \end{bmatrix} \neq 0.
\end{aligned}$$

$\square$

Corollary 5. *In $\mathbb{R}^2$, three points A, B, and C are not collinear if and only if*

$$\mathsf{det}\begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix} \neq 0.$$

Corollary 6. *This statement is also true in the Rational Plane, $\mathbb{Q}^2$.*