Subrings

We've seen several examples of rings: the integers, modular integers, matrices, and more. As we will see there is quite a bit of technology that makes sense in any ring, or in lots of different rings. And so a natural question to ask is this:

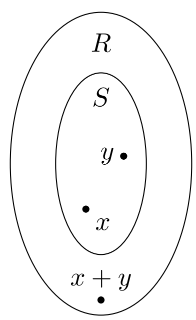
$$\boxed{\text{Where can we find rings?}}$$

Of course the difficult part of "building" a ring is coming up with the arithmetic – the plus and times – so that the ring axioms are satisfied. Well, suppose we have a particular ring R lying around. Perhaps we can use the fact that R already has a nice plus and times to build new rings.

$$\boxed{\text{Given a ring } R\text{, how can we construct new rings out of the "parts" of } R\text{?}}$$

It turns out there are several ways to do exactly this. The simplest is to take *subsets* of R, and restrict the arithmetic on R to that subset.

There is a potential obstacle to making this work, though; given a subset $S \subseteq R$ and two elements $x, y \in S$, *a priori* we expect their sum $x + y$ to be in R but not necessarily in S. This is a problem! To avoid this, we single out the subsets of R for which precisely this does not happen. That is, the subsets which are closed under the arithmetic on R.



Definition 1 (Subring)**.** *Let R be a ring and $S \subseteq R$ a subset. We say that S is a* subring *of R if S is closed under the operations in R. Specifically,*

1. $0_R \in S$,
2. *If $x, y \in S$ then $x + y \in S$,*
3. *If $x \in S$ then $-x \in S$, and*
4. *If $x, y \in S$ then $xy \in S$.*

If R is unital, we say that a subring S is unital *if in addition $1_R \in S$.*

Proposition 1. *If R is a ring and $S \subseteq R$ a subring, then S is itself a ring under the restricted operations on R.*

Proposition 2 (Subring Criterion)**.** *Let $S \subseteq R$ be a subset. Then S is a subring of R if and only if S is not empty and is closed under subtraction and multiplication. That is, S is a subring of R iff the following hold.*

- $S \neq \varnothing$.
- *If* $x, y \in S$ *then* $x - y \in S$.
- *If* $x, y \in S$ *then* $xy \in S$.

We have a slightly easier way to characterize *unital* subrings.

Proposition 3 (Unital Subring Criterion)**.** *Let R be a unital ring. Then $A \subseteq R$ is a unital subring if and only if $1_R \in A$ and for all $x, y, z \in A$, $x - yz \in A$.*

Examples

0 Let R be any ring. The subset $0 = \{0_R\} \subseteq R$ is a subring. (Show it!)

$k\mathbb{Z}$ Let k be a positive integer, and define $k\mathbb{Z} = \{kt \mid t \in \mathbb{Z}\}$. Then $k\mathbb{Z} \subseteq \mathbb{Z}$ is a subring, but is *not* a unital subring.

aR More generally, let R be any ring and $a \in R$. Then $aR = \{ar \mid r \in R\}$ is a subring of R; similarly, $Ra = \{ra \mid r \in R\}$ is a subring of R.

$Z(R)$ Let R be a ring. We define a subset of R called the *center* as follows.

$$Z(R) = \{a \in R \mid ax = xa \text{ for all } x \in R\}$$

That is, the center is the set of all ring elements which commute with every other element of R. For example, $0_R \in Z(R)$, since if $x \in R$ we have $0 \cdot x = 0 = x \cdot 0$. Then $Z(R)$ is a subring of R. If R is unital, then $Z(R)$ is a unital subring.

$S_1 \cap S_2$ Suppose $S_1, S_2 \subseteq R$ are (unital) subrings of R. Then $S_1 \cap S_2 \subseteq R$ is also a (unital) subring of R.

Exercises

1. Let $R = \mathbb{Z}$ and $S \subseteq R$ the set of all prime integers. Show that S is *not* a subring of R.

2. Let R be a ring, and let $e \in R$ be idempotent (that is, $e^2 = e$). Show that

$$eRe = \{ere \mid r \in R\}$$

is a subring of R. Show that as a ring, $S = eRe$ is unital with $1_S = e$. In particular, if R is a unital ring and $e \neq 1_R$, then S is not a *unital subring*, even though it is a *subring which is unital*, since in a unital subring S we have $1_S = 1_R$.