Rings

So far we've studied two kinds of numbers: the integers, $\mathbb{Z}$, that we know and love, and the integers modulo n, $\mathbb{Z}/(n)$, which are a little strange. These kinds of numbers differ in some crucial ways. For example, $\mathbb{Z}$ comes with a useful order relation $\leq$ while $\mathbb{Z}/(n)$ does not, and in $\mathbb{Z}/(n)$ it may be possible to find "nonzero" numbers a and b such that $ab \equiv_n 0$, which cannot happen in $\mathbb{Z}$. However both $\mathbb{Z}$ and $\mathbb{Z}/(n)$ have an arithmetic – plus and times – which behave very similarly. Addition is associative and commutative, there is a zero element, and so on.

In mathematics, when different concrete objects have behavior in common it is frequently useful to "abstract out" the common behavior. This is what motivates the following definition.

Definition 1 (Ring)**.** *A* ring *is a set R equipped with two operations $+$ and $\cdot$, which satisfy the following properties.*

A1. $(a+b)+c = a+(b+c)$ *for all* $a, b, c \in R$.

A2. There is an element $0_R \in R$ *(called a* zero*) such that* $a + 0_R = 0_R + a = a$ *for all* $a \in R$.

A3. For every $a \in R$ *there is an element* $-a \in R$ *(called a* negative *of* a*) such that* $a + (-a) = (-a) + a = 0_R$.

A4. $a + b = b + a$ *for all* $a, b \in R$.

M. $(ab)c = a(bc)$ *for all* $a, b, c \in R$.

D. $a(b+c) = ab + ac$ *and* $(b+c)a = ba + ca$ *for all* $a, b, c \in R$.

Many of the usual properties of arithmetic in $\mathbb{Z}$ can be derived as properties of any ring.

Proposition 1. *Let R be a ring. Then we have the following.*

1. *The zero element of R is unique in the following sense: if $a, b \in R$ such that $a + b = a$, then $b = 0_R$.*
2. *Negative elements in R are unique in the following sense: If $a, b \in R$ such that $a + b = 0_R$, then $b = -a$.*
3. $-(-a) = a$ *for all* $a \in R$.
4. $0_R \cdot a = a \cdot 0_R = 0_R$ *for all* $a \in R$.
5. $(-a)b = a(-b) = -(ab)$ *for all* $a, b \in R$.
6. $(-a)(-b) = ab$ *for all* $a, b \in R$.

Proof.

1. Suppose $a+b=a$. Now $-a+(a+b)=-a+a$, and by A1 $(-a+a)+b=-a+a$. By A3 we have $0_R+b=0_R$, and by A2 we have $b=0_R$.

2. Suppose $a+b=0_R$. Now $-a+(a+b)=-a+0_R$, and by A1 we have $(-a+a)+b=-a+0_R$. By A3 we have $0_R+b=-a+0_R$, and using A2 (twice) we have $b=-a$.

3. By definition, $(-a)+a=0_R$, so by the uniqueness of negatives we have $a=-(-a)$.

4. Let $a \in R$. Now $a \cdot a + 0_R \cdot a = (a+0_R) \cdot a = a \cdot a$, and so $0_R \cdot a = 0_R$. The other equality is similar.

5. Let $a, b \in R$. Now $(-a)b + ab = (-a+a)b = 0_R \cdot b = 0_R$, so that $(-a)b = -(ab)$. The other equality is similar.

6. Using the previous statement, we have

$$(-a)(-b) = -(a(-b)) = -(-(ab)) = ab. \qquad \square$$

Examples

$\mathbb{Z}, \mathbb{Z}/(n)$ The integers are a ring by definition, and we showed that the integers mod n are a ring for any $n > 0$.

$\mathbb{Q}$ The rational numbers are a ring under the usual addition and multiplication; we will prove this later. (Actually we will define the rational numbers.)

$\mathbb{R}$, $\mathbb{C}$ The real and complex numbers are also rings, although even defining these sets of "numbers" is beyond the scope of this text.

0 The smallest possible ring must have at least one element, the zero. Suppose this is *all* we have. Now the arithmetic is pretty boring: $0+0=0$ and $0 \cdot 0 = 0$. It is straightforward to check that these operations make the set $\{0\}$ into a ring. This example isn't very interesting, so we call this the *trivial ring* or the *zero ring*.

R^A Let R be a ring, and let A be any nonempty set. Then the set

$$R^A = \{\varphi \mid \varphi : A \to R\}$$

is a ring under the "pointwise" operations

$$(\alpha+\beta)(x) = \alpha(x) + \beta(x) \quad \text{and} \quad (\alpha\beta)(x) = \alpha(x)\beta(x).$$

$\mathsf{Mat}_2(R)$ Let R be a ring, and consider the set

$$\mathsf{Mat}_2(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in R \right\}.$$

These are just the 2×2 matrices with entries in R. The usual matrix addition and multiplication make $\mathsf{Mat}_2(R)$ into a ring. Specifically, we define

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

$2\mathbb{Z}$ Consider the set

$$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}.$$

It is not too difficult to show that this set is a ring under the usual addition and multiplication of integers.

2^X Let X be any nonempty set. The powerset 2^X is a ring under the operations $A + B = (A \setminus B) \cup (B \setminus A)$ and $A \cdot B = A \cap B$. This is called a *ring of sets*.

Commutative and Unital Rings

Our list of examples is starting to get complicated, so we make two additional definitions to start drawing distinctions among them.

Definition 2. *Let R be a ring.*

- *We say that R is* commutative *if it satisfies the following property.*

  C. $ab = ba$ for all $a, b \in R$.

- *We say that R is* unital *if it satisfies the following property.*

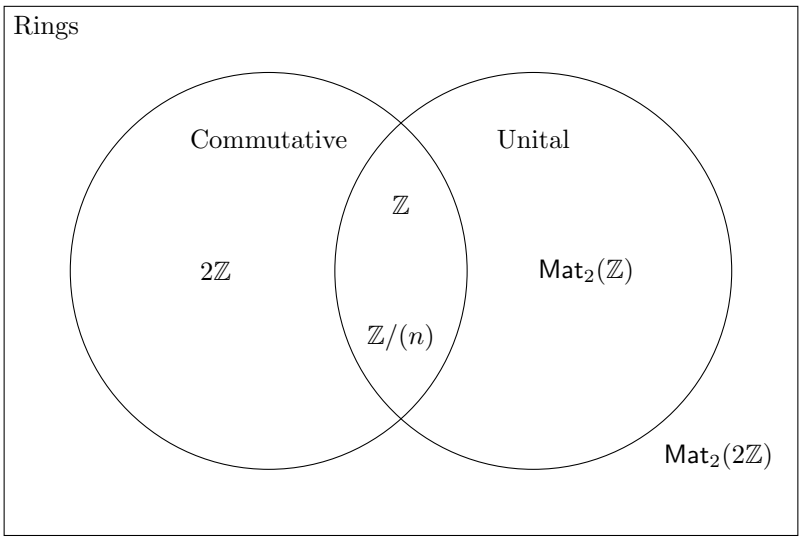
  U. There is an element $1 \in R$ (called a one*) such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.*

Examples

$\mathsf{Mat}_2(R)$ If R is unital, then $\mathsf{Mat}_2(R)$ is also unital, with

$$1_{\mathsf{Mat}_2(R)} = \begin{bmatrix} 1_R & 0_R \\ 0_R & 1_R \end{bmatrix}.$$

Is this ring ever commutative?



Proposition 2. *Let R be a unital ring.*

1. *The one element of R is unique in the following sense: if $u \in R$ such that $u \cdot a = a$ for all $a \in R$, then $u = 1$.*
2. *$-a = (-1) \cdot a$ for all $a \in R$.*

Proof.

1. Suppose u is such an element. In particular, $1 = u \cdot 1 = u$.
2. Let $a \in R$. Then $a + (-1)a = 1a + (-1)a = (1 + (-1))a = 0a = 0$, so that $(-1)a = -a$. □

Exercises

1. Let R be a ring. Show that $\mathsf{Mat}_2(R)$ is a ring by verifying that each of the properties in Definition 1 hold. This will be tedious.
2. Let X be a nonempty set. Show that the ring of sets 2^X is a ring by verifying that each of the properties in Definition 1 hold. This will be tedious.
3. Let R be a ring. Show that the ring $\mathsf{Mat}_2(R)$ is unital if and only if R is unital.
4. **Sigma Notation.** Let R be a ring and $a \leq b$ integers. If $\{r_i \mid a \leq i \leq b\}$ is a finite subset of R indexed by the integers from a to b, we define

$$\sum_{i=a}^{b} r_i = r_a + r_{a+1} + \cdots + r_{b-1} + r_b.$$

   Show that the following hold.

(a) $\left(\sum_{i=a}^{b} r_i\right) + (\sum_{i=b}^{c} r_i) = \sum_{i=a}^{c} r_i$ whenever $a \leq b \leq c$.

(b) $s \cdot \left(\sum_{i=a}^{b} r_i\right) = \sum_{i=a}^{b} sr_i$ and $\left(\sum_{i=a}^{b} r_i\right) \cdot s = \sum_{i=a}^{b} r_i s$.

5. A ring R is called *boolean* if for all $a \in R$, we have $aa = a$. Let R be a boolean ring.

   (a) Show that the ring of sets 2^X is boolean for any nonempy X.

   (b) Show that $-a = a$ for all $a \in R$. (Hint: consider $a + a$.)

   (c) Show that R is commutative. (Hint: let $a, b \in R$ and consider $a + b$.)

6. Find all the matrices $A \in \mathsf{Mat}_2(\mathbb{Z}/(5))$ such that $A^2 = 0$. This will be tedious.