Homomorphisms

Definition 1 (Ring Homomorphism)**.** *Let R and S be rings. A map $\varphi : R \to S$ is called a* ring homomorphism *if the following are satisfied.*

- $\varphi(a+b) = \varphi(a) + \varphi(b)$ *for all* $a, b \in R$.
- $\varphi(ab) = \varphi(a)\varphi(b)$ *for all* $a, b \in R$.

If R and S are both unital rings, we say that φ is unital *if* $\varphi(1_R) = 1_S$.

Proposition 1.

1. *If R is a ring, then $\mathsf{id}_R : R \to R$ given by $\mathsf{id}_R(x) = x$ is a ring homomorphism. If R is unital, then id is unital.*
2. *If $\varphi : R \to S$ and $\psi : S \to T$ are ring homomorphisms, then $\psi \circ \varphi : R \to T$ is a homomorphism. If φ and ψ are unital, then $\psi \circ \varphi$ is unital.*

Homomorphisms are *structure-preserving maps*. The arithmetic on a ring – the plus and times – are a kind of structure, and homomorphisms are the maps which "transport" this structure to another setting. If $\varphi : R \to S$ is a ring homomorphism then in a concrete sense there is a "shadow" of R inside S. If R and S are both unital rings, then the one element is an extra bit of structure.

Proposition 2. *Suppose $\varphi : R \to S$ is a ring homomorphism.*

- $\varphi(0_R) = 0_S$
- $\varphi(-a) = -\varphi(a)$ *for all* $a \in R$.
- $\varphi(a-b) = \varphi(a) - \varphi(b)$ *for all* $a, b \in R$.

Examples

- The natural projection $\pi : \mathbb{Z} \to \mathbb{Z}/(n)$ is a unital ring homomorphism.
- If R is any ring, then there is exactly one ring homomorphism $\varphi : R \to 0$, and exactly one homomorphism $\psi : 0 \to R$. Neither of these is ever unital unless $R = 0$.
- Let R be any unital ring. Then $\varphi : R \to \mathsf{Mat}_2(R)$ given by

$$\varphi(r) = \begin{bmatrix} 0 & 0 \\ -r & r \end{bmatrix}$$

  is a ring homomorphism. Although R (and hence $\mathsf{Mat}_2(R)$) is unital, φ is *not* a unital homomorphism. (Why?)

Images and Kernels

Definition 2 (Image and Kernel)**.** *Let $\varphi : R \to S$ be a ring homomorphism. We define subsets of R and S as follows.*

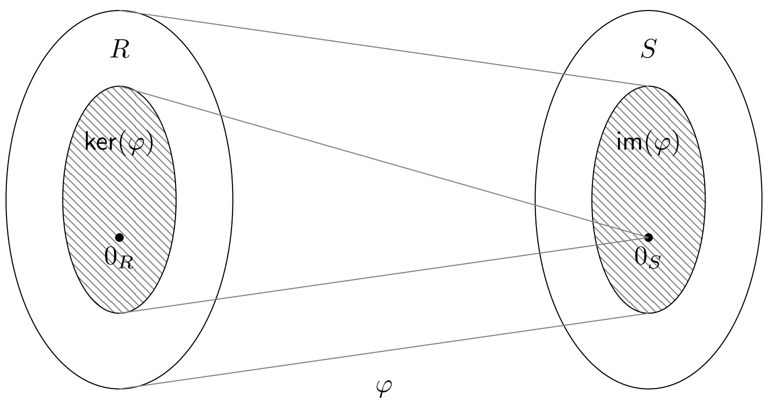
- *The* kernel *of φ, denoted $\mathsf{ker}(\varphi)$, is the set*

$$\mathsf{ker}(\varphi) = \{r \in R \mid \varphi(r) = 0\}.$$

- *The* image *of φ, denoted $\mathsf{im}(\varphi)$, is the set*

$$\mathsf{im}(\varphi) = \{s \in S \mid s = \varphi(r) \text{ for some } r \in R\}.$$

Proposition 3. *If $\varphi : R \to S$ is a ring homomorphism, then $0_R \in \mathsf{ker}(\varphi)$ and $0_S \in \mathsf{im}(\varphi)$.*



The kernel measures how badly a homomorphism fails to be injective.

Proposition 4. *A ring homomorphism φ is injective if and only if $\mathsf{ker}(\varphi) = 0$.*

Characteristic

Proposition 5. *If R is a unital ring, then there is a unique unital homomorphism $\varphi : \mathbb{Z} \to R$.*

We can think of the image of this map as a copy of the integers in R, with $1 = 1_R$, $2 = 1_R + 1_R$, and so on.

Definition 3 (Characteristic)**.** *Let R be a unital ring and $\varphi : \mathbb{Z} \to R$ the unique unital homomorphism. If there is a positive integer n such that $\varphi(n) = 0$, then there is a* smallest *such integer. We call this the* characteristic *of R, denoted $\mathsf{char}(R)$. That is, $\mathsf{char}(R)$ is the smallest positive natural number such that*

$$\underbrace{1_R + 1_R + \cdots + 1_R}_{n \text{ times}} = 0_R.$$

If no such n exists, we say that $\mathsf{char}(R) = 0$.

Isomorphisms

Definition 4. *If $\varphi : R \to S$ is a ring homomorphism which is also bijective as a mapping, we say φ is an* isomorphism*. In this case we say that R is* isomorphic *to S, denoted $R \cong S$.*

In a very concrete sense, if $R \cong S$, then R and S are really "the same" ring, with the elements relabeled.

Proposition 6. *For all rings R, S, and T, the following hold.*

1. *$R \cong R$.*
2. *If $R \cong S$ then $S \cong R$.*
3. *If $R \cong S$ and $S \cong T$ then $R \cong T$.*

$$\boxed{\text{Given rings } R \text{ and } S\text{, is it true that } R \cong S?}$$

To distinguish rings from each other, it is useful to have on hand several properties of rings which are *preserved* by isomorphisms. For instance, the property "contains the number 3 as an element" is *not* preserved by isomorphisms: the elements of isomorphic rings may have nothing to do with each other.

Proposition 7. *Let R and S be rings. If $R \cong S$, then the following hold.*

1. *R and S have the same cardinality.*
2. *R is commutative if and only if S is commutative.*
3. *R is unital if and only if S is unital.*
4. $\mathsf{char}(R) = \mathsf{char}(S)$.