Relations and Functions

Definition 1 (Function)**.** *If A and B are sets, then the subsets of $A \times B$ are called* relations*. Suppose $\varphi \subseteq A \times B$ is a relation.*

- *We say that φ is* total *if for every $a \in A$, there is a $b \in B$ such that $(a, b) \in \varphi$.*
- *We say that φ is* well-defined *if whenever $a \in A$ and $b_1, b_2 \in B$ such that $(a, b_1), (a, b_2) \in \varphi$, in fact $b_1 = b_2$.*
- *We say that φ is a* function *or* mapping *from A to B if it is both total and well-defined. We denote this by $\varphi : A \to B$.*
- *If $\varphi : A \to B$ is a function and $a \in A$, then there is a unique $b \in B$ such that $(a, b) \in \varphi$; we call this b the* image *of a under φ, and denote it $\varphi(a)$.*

Proposition 1.

1. *If A is a set, then the relation $1_A = \{(x, x) \mid x \in A\}$ is a function, called the* identity *on A, and $1_A(x) = x$ for all $x \in A$.*
2. *If $\varphi : A \to B$ and $\psi : B \to C$ are functions, then the relation*

$$\psi \circ \varphi = \{(a, \psi(\varphi(a))) \mid a \in A\}$$

is a function $A \to C$.

Definition 2. *Let $\varphi : A \to B$ be a function.*

- *We say that φ is* injective *if whenever $\varphi(a_1) = \varphi(a_2)$, in fact $a_1 = a_2$.*
- *We say that φ is* surjective *if for every $b \in B$, there is an element $a \in A$ such that $\varphi(a) = b$.*
- *If φ is both injective and surjective, we say it is* bijective.

Definition 3 (Kernel)**.** *Let $\varphi : A \to B$ be a function. The relation*

$$\mathsf{ker}(\varphi) = \{(x, y) \mid \varphi(x) = \varphi(y)\}$$

is called the kernel *of φ.*