Name: _____

College Geometry Homework #2: Poincaré Half-Plane

In class we began to explore a strange new model of geometry called the Poincaré Half-Plane (pronounced "pwan kar ay"). Even though the half-plane is non-euclidean, and behaves very strangely compared to our geometric intuition, the primitive notions of "line", "between", and "congruent" can be understood in terms of their Euclidean counterparts, allowing us to visualize the Half-Plane on top of the Euclidean plane.

In this activity, you will build SketchPad (or Geogebra) tools to represent the primitive constructions of lines, segments, rays, and circles in this model. Open a fresh Sketchpad or Geogebra file and save it as "(Your Name)'s Poincare Tools". After completing the following constructions, **send me a copy to grade**.

1. Remember: All constructions in the Half-Plane are carried out on a fixed side of a fixed (Euclidean) line, $\mathcal{L}$. Start by constructing two points, called P and Q, and the line $\mathcal{L}$ through them. We will call this line the "ideal axis".

2. **Type II Lines.** Lines in the Half-Plane come in two flavors, called Type I and Type II (these are not very creative names). The Type I lines are Euclidean half-lines which are perpendicular to the ideal axis; "almost none" of the lines are Type I, so we can safely ignore them for now.

   Construct two points, A and B, on the same side of the ideal axis. Construct the perpendicular bisector of $\overline{AB}$, and construct the point O where this line cuts the ideal axis; we will call this O the *ideal center* of the line generated by A and B. Now construct the circle centered at O and passing through A, and construct the two points H and K where this circle cuts the ideal axis. Finally, select the points H, A, K (in this order) and select "arc through 3 points" from the Construct menu in Sketchpad (Geogebra has a similar tool). **This arc α is the Type II line generated by A and B.**

   To avoid having to carry out all these steps whenever we want to make a Type II line, we can make a tool. Select P, Q, A, B, and α (in this order!) and choose "Create New Tool". Call this tool "Type II line from two points". We will also create a tool to construct the point O by selecting P, Q, A, B, and O (in this order!); call this tool "Ideal center of two points".

   After creating these two tools, you can delete A and B. Test out your new tools to make sure they work as expected.

3. **Type II Segments.** Segments in the Half-Plane also come in two flavors, depending on whether their endpoints generate a Type I or a Type II line. We will only consider the Type II segments.

   Construct two points, A and B, on the same side of the ideal axis. Then construct the Type II line ℓ generated by A and B, the ideal center O of A and B, and the Euclidean midpoint M of $\overline{AB}$. Now construct the Euclidean ray $\overrightarrow{OM}$, and construct the point T where this ray cuts the Type II line ℓ. Finally, select the points A, T, and B (in this order) and construct the arc through these 3 points. **This arc α is the Type II segment generated by A and B.**

   Make a tool to construct Type II segments by selecting P, Q, A, B, and α in this order. Call your tool "Type II segment from endpoints".

   After making the segment tool, delete A and B. Test the new tool to make sure it works as expected.

4. **Type II Rays.** Constructing rays will require us to trick Sketchpad (or Geogebra) a little bit.

   Construct two points A and B on the same side of the ideal axis. Construct the ideal center O of A and B and the Euclidean midpoint M of A and B, and construct the Euclidean line $\overleftrightarrow{OM}$. Now construct the (unique!) Euclidean line passing through B and parallel to $\overleftrightarrow{OM}$. Construct the point T where this line cuts the ideal axis. Now construct the Euclidean circle c centered at O and passing through A, and the Euclidean ray $\overrightarrow{OT}$. Construct the point U where these two meet. (Make sure you intersect with the ray $\overrightarrow{OT}$, not the line $\overleftrightarrow{OT}$! This is the trick.) Finally, select the points A, B, and U (in this order) and construct the arc through these 3 points. **This arc α is the Type II ray from A toward B.**

   Make a tool by selecting P, Q, A, B, and α (in this order), and call it "Type II ray from two points".

   After making the ray tool, delete A and B. Test the new tool to make sure it works as expected.

5. **Type II Circles.** Like lines, the construction of the circle centered at O and passing through A depends on whether or not the Euclidean line $\overleftrightarrow{OA}$ is perpendicular to the ideal axis; we distinguish these cases as Type I and Type II circles. Almost all pairs of points are in the Type II case, so we can safely ignore Type I.

Construct two points, O and A, on the same side of the ideal axis. O will be the center of the Type II circle, and A will be a point on the circle. Construct the ideal center X of A and O, and construct the line ℓ which is perpendicular to the Euclidean line $\overleftrightarrow{XA}$ at A. Now construct the Euclidean line m which is perpendicular to the ideal axis and passes through O. Let Y be the point where ℓ and m meet. Finally, construct the Euclidean circle centered at Y and passing through A. **This circle c is the Type II circle centered at O and passing through A.**

Make a new tool by selecting P, Q, O, A, and c (in this order) and call it "Type II circle from center and point".

After making the circle tool, delete O and A. Test the new tool to make sure it works as expected.

Using these four tools, all of the neutral plane geometry constructions we've seen can be carried out **without modification**: copying segments, copying angles, and so on. This is the power of a well-chosen abstraction. We will be doing this in the next homework assignment, so be sure to save your tools!