Ideal Arithmetic and Generating Sets

In the last section, we saw that special subsets of a ring called *ideals* can be used to construct new quotient rings. However it is not yet clear whether interesting examples of ideals even exist. In this section we will address this shortcoming.

Proposition 1. *Let R be a ring. Then R itself and the zero subset $0 = \{0_R\}$ are both ideals of R, called* trivial ideals. *In fact, $R/0 \cong R$ and $R/R \cong 0$.*

Proposition 2. *Let R be a ring and $I, J \subseteq R$ ideals. Define*

- $I + J = \{a + b \mid a \in I, b \in J\}$ *and*
- $IJ = \{\sum_{i=0}^{n} a_i b_i \mid n \in \mathbb{N}, a_i \in I, b_i \in J\}$.

Then $I + J$ and IJ are both ideals of R, with $IJ \subseteq I \cap J$ and $I, J \subseteq I + J$.

Proposition 3. *Let R be a ring and $I, J, K \subseteq R$ ideals. Then we have the following.*

1. $I + (J + K) = (I + J) + K$
2. $I + 0 = 0 + I = I$
3. $I + R = R + I = R$
4. $I(JK) = (IJ)K$
5. $I0 = 0I = 0$
6. $IR = RI = I$
7. $I(J + K) = IJ + IK$ *and* $(I + J)K = IK + JK$

Generating Sets

Proposition 4. *Let R be a ring, and let $\mathcal{I}$ be a collection of ideals of R. Then $\bigcap \mathcal{I}$ is an ideal of R.*

Not every subset of R is an ideal; in fact most aren't. However, *every subset of R is contained in a unique smallest ideal.*

Proposition 5 (Generated Ideals)**.** *Let R be a ring and $A \subseteq R$ a subset. We define the set (A) by*

$$(A) = \bigcap \{I \mid I \subseteq R \text{ is an ideal and } A \subseteq I\}.$$

1. *(A) is an ideal of R.*
2. *If A is an ideal of R, then $(A) = A$.*

3. *$A \subseteq (A)$*

4. *If I is an ideal of R and $A \subseteq R$, then $(A) \subseteq I$.*

We call (A) the ideal of R generated by *A. If I is an ideal and $I = (A)$, we say that A is a* generating set *for I.*

Proposition 6.

1. *$(A) + (B) = (A \cup B)$*

2. *$(A)(B) = (ab \mid a \in A, b \in B)$*

Definition 1. *Let I be an ideal of a ring.*

1. *We say I is* finitely generated *if there is a finite set A such that $I = (A)$.*

2. *We say a subset A is a* minimal generating set *of I if $I = (A)$ and whenever $B \subsetneq A$ is a proper subset, $(B) \subsetneq (A)$ is also a proper subset.*

Important note: "minimal" here does not mean "smallest size", but rather "contains no redundant elements". In general, an ideal has many minimal generating sets, and these may have very different cardinalities. For example, the ideal $(3) \subseteq \mathbb{Z}$ is minimally generated by the set $\{3\}$, but also by the set $\{6, 15\}$.

Proposition 7. *Let R be a commutative unital ring and A a subset of R. Then*

$$(A) = \{\sum_{i=0}^{n} r_i a_i \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}.$$

That is, in a commutative unital ring, the ideal generated by A consists of all finite R-linear combinations of elements of A.

Proof. (type this) □

Exercises

1. (N.B.: Find an example of an ideal with minimal generating sets of arbitrarily large cardinality.)