# Direct Sums

**Theorem 1** (Direct Sum)**.** *Given rings $R$ and $S$, we define "componentwise" operations on the cartesian product $R \times S$ as follows.*

$$\begin{aligned} (r_1, s_1) + (r_2, s_2) &= (r_1 + r_2, s_1 + s_2) \\ (r_1, s_1) \cdot (r_2, s_2) &= (r_1 \cdot r_2, s_1 \cdot s_2) \end{aligned}$$

*Then we have the following.*

- *These operations make $R \times S$ into a ring, which we call the* direct sum *of $R$ and $S$ and denote by $R \oplus S$.*
- *$R \oplus S$ is commutative iff $R$ and $S$ are commutative.*
- *$R \oplus S$ is unital iff $R$ and $S$ are unital, and in this case $1_{R \oplus S} = (1_R, 1_S)$.*
- *The coordinate projections $\pi_1 : R \oplus S \to R$ and $\pi_2 : R \oplus S \to S$, given by $\pi_1(r, s) = r$ and $\pi_2(r, s) = s$, are surjective ring homomorphisms. If $R$ and $S$ are unital, then $\pi_1$ and $\pi_2$ are unital.*

**Proposition 2.**

1. *If $R_1 \cong R_2$ and $S_1 \cong S_2$, then $R_1 \oplus S_1 \cong R_2 \oplus S_2$.*
2. *$R \oplus 0 \cong R$*
3. *$R \oplus S \cong S \oplus R$*
4. *$(R \oplus S) \oplus T \cong R \oplus (S \oplus T)$*