Hyperbolic Half-Planes

We've seen that every plane geometry must be either Euclidean or Hyperbolic, and we have a concrete model of Euclidean plane geometry – the Cartesian plane. In this section we will construct a concrete model of Hyperbolic plane geometry called the Poincare Half-Plane.

Let $\mathcal{P}$ be a Euclidean plane geometry. In this section, we will refer to the lines of $\mathcal{P}$ as $\mathcal{P}$-lines. Let $\mathcal{L}$ be a $\mathcal{P}$-line, and let $\mathcal{H}$ be one of the half-planes induced by $\mathcal{L}$ via the plane separation property.

We will consider subsets of $\mathcal{H}$, called $\mathcal{H}$-lines, of the following two types.

- A *Type I* $\mathcal{H}$-line is a subset of the form $\mathcal{H} \cap \ell$, where ℓ is a $\mathcal{P}$-line which is perpendicular to $\mathcal{L}$.
- A *Type II* $\mathcal{H}$-line is a subset of the form $\mathcal{H} \cap \mathcal{C}_o(x)$, where $\mathcal{C}_o(x)$ is a $\mathcal{P}$-circle whose center o is on $\mathcal{L}$. We will call o the *phantom center* of this $\mathcal{H}$-line.

Certainly $\mathcal{H}$, together with the family of all possible $\mathcal{H}$-lines described above, is an incidence structure.

Proposition 1. *$\mathcal{H}$ is an incidence geometry.*

Proof.

IG1. Let $x, y \in \mathcal{H}$ be distinct points. If the $\mathcal{P}$-line $\overleftrightarrow{xy}$ is perpendicular to the $\mathcal{P}$-line $\mathcal{L}$, then there is exactly one $\mathcal{H}$-line of type I containing x and y, and there are no $\mathcal{H}$-lines of type II containing x and y. If the $\mathcal{P}$-line $\overleftrightarrow{xy}$ is not perpendicular to the $\mathcal{P}$-line $\mathcal{L}$, then there is exactly one type II $\mathcal{H}$-line containing x and y, and there are no type I $\mathcal{H}$-lines containing x and y.

IG2. (Every $\mathcal{H}$-line contains at least two distinct points.)

IG3. Note that $\mathcal{H}$ is not empty by the Plane Separation property; let $x \in \mathcal{H}$. Construct the foot f of x on $\mathcal{L}$, and by interpolation let y be a point such that $[fyx]$. Now construct a point g on $\mathcal{L}$ different from f by interpolation, and again let z be a point such that $[gzx]$ by interpolation. Note that both y and z are in $\mathcal{H}$. Moreover, there is a unique Type I $\mathcal{H}$-line containing x and y, which does not contain z, and there are no Type II $\mathcal{H}$-lines containing x and y. So x, y, and z are not collinear. $\square$

Note that two $\mathcal{H}$-lines intersect if and only if they intersect as sets in $\mathcal{P}$.

Proposition 2. *$\mathcal{H}$ is Hyperbolic.*

Proof. Recall that an incidence geometry is called *hyperbolic* if, given a line ℓ and a point p not on ℓ, there are infinitely many lines t which pass through p and do not intersect ℓ. Since $\mathcal{H}$ has lines of two types, we need to consider each one separately.

- Let $\mathcal{H} \cap \ell$ be a Type I $\mathcal{H}$-line, where ℓ is a $\mathcal{P}$-line which is perpendicular to $\mathcal{L}$ at o. Let $p \in \mathcal{H}$ be a point, and let f be the foot of p on $\mathcal{L}$ (in $\mathcal{P}$). Let q be any point such that $[oqf]$ in $\mathcal{P}$. There is a unique point a on $\mathcal{L}$ such that $\overline{ap} \equiv \overline{aq}$, and $\mathcal{H} \cap \mathcal{C}_a(p)$ is a Type II $\mathcal{H}$-line containing p. Note that the foot of a on ℓ is exterior to $\mathcal{C}_a(p)$, and thus $\mathcal{H} \cap \mathcal{C}_a(p)$ and $\mathcal{H} \cap \ell$ are parallel. There are infinitely many possible qs, all of which yield distinct $\mathcal{H}$-lines. So there are infinitely many $\mathcal{H}$-lines which pass through p and are parallel to $\mathcal{H} \cap \ell$.
- (finish this) $\square$

A Concrete Model

We've seen that *any* line in *any* Euclidean plane geometry can be used to construct a new hyperbolic incidence geometry, which we've called a half-plane model. Let's now choose a specific Euclidean plane and a specific line: consider the line $\mathcal{L} = \overleftrightarrow{(0,0)(1,0)}$ in the Cartesian plane, and let $\mathcal{H}$ be the $(0,1)$-side of $\mathcal{L}$. That is, $\mathcal{H} = \{(x,y) \mid x, y \in \mathbb{R}, y > 0\}$.

If X and Y are distinct points in $\mathcal{H}$, then there is a unique line $\overleftrightarrow{XY}$ which contains both. Presently, we will construct a *parameterization* of $\overleftrightarrow{XY}$; that is, a function $\Phi_{X,Y} : \mathbb{R} \to \overleftrightarrow{XY}$ having the property that $\Phi_{X,Y}(0) = X$ and $\Phi_{X,Y}(1) = Y$. You may recall that we did something similar in the Cartesian plane, although there we *defined* lines in terms of their parameterization.

Suppose X and Y generate a Type I line in $\mathcal{H}$. In this case, we have $X = (z, x_2)$ and $Y = (z, y_2)$ for some z, and both x_2 and y_2 are strictly positive. Let $\alpha = \ln(x_2)$ and $\beta = \ln(y_2)$, and define $\Phi_{X,Y}(t) = (z, e^{\alpha + t(\beta - \alpha)})$. Certainly $\Psi_{X,Y} : \mathbb{R} \to \overleftrightarrow{XY}$ is bijective, and we have $\Phi_{X,Y}(0) = X$ and $\Phi_{X,Y}(1) = Y$. Note also that $\Phi_{Y,X}(t) = \Phi_{X,Y}(1-t)$.

Now suppose X and Y generate a Type II line in $\mathcal{H}$. Let $O = (c, 0)$ be the phantom center of X and Y, and let $r = \sqrt{(X-O)\cdot(X-O)}$. Now define a mapping $\varphi : \mathbb{R} \to \overleftrightarrow{XY}$ by

$$\varphi(t) = \left(c - \frac{rt}{\sqrt{t^2 + r^2}}, \frac{r^2}{\sqrt{t^2 + r^2}} \right).$$

Note that φ is bijective, with inverse $\psi(a,b) = r(c-a)/b$. Let $\alpha = \psi(X)$ and $\beta = \psi(Y)$. Finally, define $\Phi_{X,Y} : \mathbb{R} \to \overleftrightarrow{XY}$ by $\Phi_{X,Y}(t) = \varphi(\alpha + t(\beta - \alpha))$. Certainly $\Phi_{X,Y}$ is a bijection, and we have $\Phi_{X,Y}(0) = X$ and $\Phi_{X,Y}(1) = Y$. Note also that $\Phi_{Y,X}(t) = \Phi_{X,Y}(1-t)$.

Proposition 3. *The ternary relation $[\cdots]$ given by $[xzy]$ iff there is a $t \in (0,1)$ such that $Z = \Phi_{X,Y}(t)$. is a betweenness relation.*

Proof.

B2. Suppose $[XZY]$. By definition, $Z = \Phi_{X,Y}(t)$ where $t \in (0,1)$. Now $1-t \in (0,1)$, and we have $Z = \Phi_{Y,X}(1-t)$ as needed. Certainly $Z \in \overleftrightarrow{XY}$.

B3. Let X, Y, and Z be distinct points in $\mathcal{H}$, and suppose WLOG that $[XYZ]$ and $[XZY]$. Now X, Y, and Z are collinear.

   – Suppose X, Y, and Z are on a Type I line; say $X = (u, x)$, $Y = (u, y)$, and $Z = (u, z)$. Now $z = e^{\ln(x)+t(\ln(y)-\ln(x))}$ and $y = e^{\ln(x)+v(\ln(z)-\ln(x))}$ for some $t, v \in (0, 1)$. Substituting, we see that $z = e^{\ln(x)+tv(\ln(z)-\ln(x))}$, and since the exponential map is injective, $\ln(x) = \frac{\ln(y)-v\ln(z)}{1-v}$. Similarly, $\ln(x) = \frac{\ln(z)-t\ln(y)}{1-t}$. From here we can show that $\ln(y) = \ln(z)$, a contradiction.

   – Suppose X, Y, and Z are on a Type II line. Let $\alpha = \psi(X)$, $\beta = \psi(Y)$, and $\gamma = \psi(Z)$. Now $Z = \varphi(\alpha + t(\beta - \alpha))$ and $Y = \varphi(\alpha + v(\gamma - \alpha))$ for some $t, v \in (0, 1)$. Hitting both of these equations with ψ, we see $\beta - \alpha = v(\gamma - \alpha)$ and $\gamma - \alpha = t(\beta - \alpha)$, so that $tv = 1$, a contradiction.

$\square$

Note that $[XYZ]$ iff X, Y, and Z are collinear and either $\psi(X) < \psi(Y) < \psi(Z)$ or $\psi(X) > \psi(Y) > \psi(Z)$.

Proposition 4. *$\mathcal{H}$ has the Trichotomy property, the Four-Point property, and the Interpolation property.*

Proof.

- **Trichotomy.** Let X and Y be distinct points in $\mathcal{H}$, and let $Z \in \overleftrightarrow{XY}$ be distinct from both. Now $Z = \Phi_{X,Y}(t)$ for some $t \in \mathbb{R}$. Let $\alpha = \psi(X)$, $\beta = \psi(Y)$, and $\gamma = \psi(Z)$. Now $Z = \Phi_{X,Y}(t)$ for some $t \in \mathbb{R}$, and hitting both sides of this equation with ψ, we have $\gamma = \alpha + t(\beta - \alpha)$. Solving this equation for α and then β, we see that $\alpha = \beta + \frac{1}{1-t}(\gamma - \beta)$ and $\beta = \alpha + \frac{1}{t}(\gamma - \alpha)$. Note that one of t, $1/t$, and $1/(1-t)$ must be in the interval $(0, 1)$.
- (finish this)
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$\square$

Proposition 5. *$\mathcal{H}$ has the Plane-Separation property.*

Proof. Use plane separation in $\mathbb{R}^2$ for Type I lines, and interior and exterior of circles for Type II lines. $\square$