Betweenness

Definition 1 (Betweenness)**.** *Let $\mathcal{P}$ be an incidence geometry. We say that a ternary relation $[\cdots]$ on the set of points of $\mathcal{P}$ is a* betweenness relation *if the following properties hold.*

B1. If $[xyx]$, then $x = y$, for all points x and y.

B2. If x and y are distinct points and $[xzy]$, then $[yzx]$ and $z \in \overleftrightarrow{xy}$.

B3. If x, y, and z are distinct points, then at most one of $[xyz]$, $[yzx]$, and $[zxy]$ is true.

Definition 2 (Segment, Ray)**.** *Let x and y be distinct points in an incidence geometry $\mathcal{P} = (P, L)$.*

- *The set*
$$\overline{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy]\}$$
is called the segment *with* endpoints *x and y. If $z \in \overline{xy}$ and $z \neq x$ and $z \neq y$, we say that z is* interior to $\overline{xy}$.

- *The set*
$$\overrightarrow{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy] \text{ or } [xyz]\}$$
is called the ray *with* vertex *x* toward *y.*

Proposition 1. *If $\mathcal{P}$ is an incidence geometry and $[\cdots]$ a betweenness relation on $\mathcal{P}$, then the following hold.*

1. *$\overline{xy} = \overline{yx}$ for all distinct points x and y.*
2. *$\overline{xy} \subseteq \overrightarrow{xy} \subseteq \overleftrightarrow{xy}$ for all distinct points x and y.*
3. *If ℓ is a line and x and y distinct points, then $\overline{xy} \cap \ell$ is either $\overline{xy}$, $\varnothing$, or $\{p\}$ for some point p.*
4. *$\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ for all distinct points x and y.*

Examples

$\mathbb{R}^2$ Given points A, B, and C in $\mathbb{R}^2$, we say $[ACB]$ if the equation $C = A + t(B - A)$ has a solution $t \in [0, 1]$. This is a betweenness relation.

B1. Suppose $[ABA]$. Now $B = A + t(A - A) = A$ as needed.

B2. Suppose A, B, and C are distinct points such that $[ACB]$. By definition, we have $C = A + t(B - A)$ for some real number $t \in [0, 1]$. Certainly $C \in \overleftrightarrow{AB}$. Moreover, note that

$$\begin{aligned} B + (1 - t)(A - B) &= B + A - B - t(A - B) \\ &= A + t(B - A) \\ &= C, \end{aligned}$$

so that $[BCA]$.

B3. Suppose we have distinct points A, B, and C such that $[ABC]$ and $[BCA]$. Now $B = A + t(C - A)$ and $C = B + u(A - B)$ for some real numbers $u, t \in [0, 1]$ by definition. Substituting the second equation into the first, we see that $B = A + t(1 - u)(B - A)$, so that $0 = (t(1 - u) - 1)(B - A)$. Since A and B are distinct, we must have $t(1-u) = 1$. Similarly, substituting the first equation into the second, we have $u(1-t) = 1$. Then t must be a root of the quadratic $t^2 - t + 1$, which has no real solutions.

The Trichotomy Property

Definition 3. *We say that a betweenness relation $[\cdots]$ on an incidence geometry $\mathcal{P}$ has the* Trichotomy Property *if, whenever x, y, and z are distinct, collinear points, at least one of $[xyz]$, $[yzx]$, and $[zxy]$ is true. That is, given three collinear points, exactly one is between the other two.*

Proposition 2. *Suppose $\mathcal{P}$ is an incidence geometry and $[\cdots]$ a betweenness relation with the Trichotomy Property. Then the following hold.*

1. *For all distinct points x and y,*

$$\overleftrightarrow{xy} = \{z \mid z = x \text{ or } z = y \text{ or } [zxy] \text{ or } [xzy] \text{ or } [xyz]\}.$$

2. $\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ *for all distinct points x and y.*

Examples

$\mathbb{R}^2$ The Cartesian Plane has the Trichotomy Property, as we show. Let A, B, and C be distinct collinear points. Now $C \in \overleftrightarrow{AB}$, so that $C = A + t(B - A)$ for some real number t. If $t \in [0, 1]$, then $[ACB]$. If $t > 1$, then $\frac{1}{t} \in (0, 1)$, and we have $B = A + \frac{1}{t}(C - A)$ so that $[ABC]$. If $t < 0$, then $\frac{-t}{1-t} \in (0, 1]$ and we have $A = C + \frac{-t}{1-t}(B - C)$, so that $[CAB]$.

The 4-Point Property

First for some shorthand: if x, y, z, and w are distinct points, we will say $[xyzw]$ precisely when $[xyz]$, $[xyw]$, $[xzw]$, and $[yzw]$. More generally, if $x_1, \ldots, x_n$ are distinct points, then $[x_1 x_2 \ldots x_n]$ means that $[x_i x_j x_k]$ for all triples (i, j, k) with $1 \le i < j < k \le n$.

Definition 4 (The 4-Point Property)**.** *We say that a betweenness relation $[\cdots]$ on an incidence geometry $\mathcal{P}$ has the* 4-Point Property *if the following hold for all distinct points x, y, z, and w.*

1. *If $[xyz]$ and $[xzw]$, then $[xyw]$ and $[yzw]$.*
2. *If $[xyz]$ and $[yzw]$, then $[xyz]$ and $[xzw]$.*

Proposition 3. *Suppose $\mathcal{P}$ is an incidence geometry and $[\cdot\cdot\cdot]$ a betweenness relation on $\mathcal{P}$ having the 4-Point Property. If x, y, and z are distinct points such that $[xyz]$, then the following hold.*

1. $\overline{xy} \cup \overline{yz} = \overline{xz}$
2. $\overline{xy} \cap \overline{yz} = \{y\}$
3. $\overrightarrow{yx} \cap \overrightarrow{yz} = \{y\}$
4. $\overrightarrow{xy} = \overrightarrow{xz}$

Proposition 4. *If $\mathcal{P}$ is an incidence geometry with a betweenness relation having both the Trichotomy Property and the 4-Point Property, then the following hold.*

1. *If $[xzy]$ and $[xwy]$, then either $[xzw]$ or $[xwz]$ or $z = w$.*
2. *If x, y, and z are distinct points such that $[xyz]$, then $\overrightarrow{yx} \cup \overrightarrow{yz} = \overleftrightarrow{xz}$.*

The Interpolation Property

Definition 5. *We say that a betweenness relation $[\cdot\cdot\cdot]$ on an incidence geometry $\mathcal{P}$ has the* Interpolation Property *if for all distinct points x and y in $\mathcal{P}$, there exist points z_1, z_2, and z_3 such that $[z_1xy]$, $[xz_2y]$, and $[xyz_3]$.*

Proposition 5. *If $\mathcal{P}$ is an incidence geometry with a betweenness relation having both the Interpolation Property and the 4-Point Property, then every line in $\mathcal{P}$ has infinitely many points.*