# Domains and Fields

The integers have the following very nice "zero product property":

$$\boxed{\text{If } a \text{ and } b \text{ are integers and } ab = 0, \text{ then either } a = 0 \text{ or } b = 0.}$$

Recall that $\mathbb{Z}/(n)$ does not necessarily have this property. For instance, in $\mathbb{Z}/(6)$ we have $2 \neq 0$ and $3 \neq 0$, but $2 \cdot 3 = 0$. In this case we say that 2 and 3 are zero divisors in $\mathbb{Z}/(6)$.

**Definition 1** (Zero Divisor)**.** *Let $R$ be a ring.*

- *We say that a nonzero element $r \in R$ is a* zero divisor *if there is a nonzero element $s \in R$ such that $rs = 0$.*
- *We say that $R$ is an* integral domain*, or simply* domain*, if $R$ is commutative and does not contain any zero divisors.*

**Proposition 1** (Cancellation)**.** *Let $R$ be a domain with $r, s, t \in R$. If $rs = rt$, then $s = t$.*

## Units and Fields

**Definition 2** (Unit)**.** *Let $R$ be a unital ring.*

- *We say that $u \in R$ is a* unit *if there is an element $v \in R$ such that $uv = vu = 1_R$.*
- *We say that $R$ is a* field *if $R$ is commutative and every nonzero element of $R$ is a unit.*

**Proposition 2.** *Every field is also an integral domain.*

**Proposition 3.** *$\mathbb{Z}/(n)$ is a field if and only if $n$ is prime.*

## Exercises

1. Ponder: Is the zero ring a domain?
2. Show that if $R$ and $S$ are nontrivial rings, then $R \oplus S$ is *not* a domain.
3. Show that every subring of a domain is a domain.
4. Show that every subring of a field is a domain.
5. **Nilpotence.** We say that an element $r$ in a ring $R$ is *nilpotent* if $r^n = 0$ for some natural number $n$.
6. **Skew fields.**
7. **Every finite domain is a field.** Let $R$ be a *finite* integral domain. In this exercise we will show that $R$ must be a field.

(a) Let $r \in R$ be a nonzero element and define a mapping $\varphi_r : R \to R$ by $\varphi_r(x) = rx$. Show that $\varphi_r$ must be injective.

(b) Deduce that $\varphi_r$ must be bijective.

(c) Deduce that $r$ must be a unit in $R$, and then that $R$ must be a field.