# Betweenness

**Definition 1** (Betweenness)**.** *Let $\mathcal{P}$ be an incidence geometry. We say that a ternary relation $[\cdots]$ on the set of points of $\mathcal{P}$ is a* betweenness relation *if the following properties hold.*

*B1.* *If $[xyx]$, then $x = y$, for all points $x$ and $y$.*

*B2.* *If $x$ and $y$ are distinct points and $[xzy]$, then $[yzx]$ and $z \in \overleftrightarrow{xy}$.*

*B3.* *If $x$, $y$, and $z$ are distinct points, then at most one of $[xyz]$, $[yzx]$, and $[zxy]$ is true.*

**Definition 2** (Segment, Ray)**.** *Let $x$ and $y$ be distinct points in an incidence geometry $\mathcal{P} = (P, L)$.*

- *The set*
$$\overline{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy]\}$$
*is called the* segment *with* endpoints *$x$ and $y$. If $z \in \overline{xy}$ and $z \neq x$ and $z \neq y$, we say that $z$ is* interior to $\overline{xy}$.

- *The set*
$$\overrightarrow{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy] \text{ or } [xyz]\}$$
*is called the* ray *with* vertex *$x$* toward *$y$.*

**Proposition 1.** *If $\mathcal{P}$ is an incidence geometry and $[\cdots]$ a betweenness relation on $\mathcal{P}$, then the following hold.*

1. *$\overline{xy} = \overline{yx}$ for all distinct points $x$ and $y$.*
2. *$\overline{xy} \subseteq \overrightarrow{xy} \subseteq \overleftrightarrow{xy}$ for all distinct points $x$ and $y$.*
3. *If $\ell$ is a line and $x$ and $y$ distinct points, then $\overline{xy} \cap \ell$ is either $\overline{xy}$, $\varnothing$, or $\{p\}$ for some point $p$.*
4. *$\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ for all distinct points $x$ and $y$.*

## Examples

$\mathbb{R}^2$ Given points $A$, $B$, and $C$ in $\mathbb{R}^2$, we say $[ACB]$ if the equation $C = A + t(B - A)$ has a solution $t \in [0, 1]$. This is a betweenness relation.

B1. Suppose $[ABA]$. Now $B = A + t(A - A) = A$ as needed.

B2. Suppose $A$, $B$, and $C$ are distinct points such that $[ACB]$. By definition, we have $C = A + t(B - A)$ for some real number $t \in [0, 1]$. Certainly $C \in \overleftrightarrow{AB}$. Moreover, note that

$$\begin{aligned} B + (1-t)(A - B) &= B + A - B - t(A - B) \\ &= A + t(B - A) \\ &= C, \end{aligned}$$

so that $[BCA]$.

B3. Suppose we have distinct points $A$, $B$, and $C$ such that $[ABC]$ and $[BCA]$. Now $B = A + t(C - A)$ and $C = B + u(A - B)$ for some real numbers $u, t \in [0, 1]$ by definition. Substituting the second equation into the first, we see that $B = A + t(1-u)(B - A)$, so that $0 = (t(1-u) - 1)(B - A)$. Since $A$ and $B$ are distinct, we must have $t(1-u) = 1$. Similarly, substituting the first equation into the second, we have $u(1-t) = 1$. Then $t$ must be a root of the quadratic $t^2 - t + 1$, which has no real solutions.

We can say something a little stronger about the intersection of a segment and a line in $\mathbb{R}^2$; this next fact will become useful later, so we state and prove it now.

**Proposition 2.** *Let $A, B \in \mathbb{R}^2$ be distinct points and let $X = (x_1, x_2), Y = (y_1, y_2) \in \mathbb{R}^2$ be distinct points not in $\ell_{A,B}$. Then $\overline{XY} \cap \overleftrightarrow{AB}$ consists of a single point if and only if*

$$\mathsf{det}\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{det}\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ y_1 & y_2 & 1 \end{bmatrix}$$

*have opposite signs.*

*Proof.* Note that $\overleftrightarrow{XY} \cap \overleftrightarrow{AB}$ contains exactly one point if and only if the equation $X + t(Y - X) = A + u(B - A)$ has a unique solution $(t, u)$. In fact we have

$$\begin{bmatrix} t \\ -u \end{bmatrix} = \begin{bmatrix} y_1 - x_1 & b_1 - a_1 \\ y_2 - x_2 & b_2 - a_2 \end{bmatrix}^{-1} \begin{bmatrix} a_1 - x_1 \\ a_2 - x_2 \end{bmatrix}.$$

Comparing entries of this matrix, we see that

$$t = \frac{(b_2 - a_2)(a_1 - x_1) + (a_1 - b_1)(a_2 - x_2)}{(y_1 - x_1)(b_2 - a_2) - (y_2 - x_2)(b_1 - a_1)}.$$

Note that the unique point in $\overleftrightarrow{XY} \cap \overleftrightarrow{AB}$ is in fact in the segment $\overline{XY}$ if and only if $t \in [0, 1]$.

There are now two possibilities, depending on whether the denominator of $t$ is positive or negative. If the denominator of $t$ is positive, we can see that $t \in (0, 1)$ if and only if

$$\mathsf{det}\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} > 0 > \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ y_1 & y_2 & 1 \end{bmatrix}.$$

If the denominator of $t$ is negative, then $t \in (0, 1)$ if and only if

$$\mathsf{det}\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} < 0 < \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ y_1 & y_2 & 1 \end{bmatrix}.$$

$\square$

## The Trichotomy Property

**Definition 3.** *We say that a betweenness relation* $[\cdots]$ *on an incidence geometry* $\mathcal{P}$ *has the* Trichotomy Property *if, whenever* $x$*,* $y$*, and* $z$ *are distinct, collinear points, at least one of* $[xyz]$*,* $[yzx]$*, and* $[zxy]$ *is true. That is, given three collinear points, exactly one is between the other two.*

**Proposition 3.** *Suppose* $\mathcal{P}$ *is an incidence geometry and* $[\cdots]$ *a betweenness relation with the Trichotomy Property. Then the following hold.*

1. *For all distinct points* $x$ *and* $y$*,*

$$\overleftrightarrow{xy} = \{z \mid z = x \text{ or } z = y \text{ or } [zxy] \text{ or } [xzy] \text{ or } [xyz]\}.$$

2. $\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ *for all distinct points* $x$ *and* $y$*.*

## Examples

$\mathbb{R}^2$ The Cartesian Plane has the Trichotomy Property, as we show. Let $A$, $B$, and $C$ be distinct collinear points. Now $C \in \overleftrightarrow{AB}$, so that $C = A+t(B-A)$ for some real number $t$. If $t \in [0,1]$, then $[ACB]$. If $t > 1$, then $\frac{1}{t} \in (0,1)$, and we have $B = A + \frac{1}{t}(C-A)$ so that $[ABC]$. If $t < 0$, then $\frac{-t}{1-t} \in (0,1]$ and we have $A = C + \frac{-t}{1-t}(B-C)$, so that $[CAB]$.

## The 4-Point Property

First for some shorthand: if $x$, $y$, $z$, and $w$ are distinct points, we will say $[xyzw]$ precisely when $[xyz]$, $[xyw]$, $[xzw]$, and $[yzw]$. More generally, if $x_1, \ldots, x_n$ are distinct points, then $[x_1 x_2 \ldots x_n]$ means that $[x_i x_j x_k]$ for all triples $(i,j,k)$ with $1 \le i < j < k \le n$.

**Definition 4** (The 4-Point Property)**.** *We say that a betweenness relation* $[\cdots]$ *on an incidence geometry* $\mathcal{P}$ *has the* 4-Point Property *if the following hold for all distinct points* $x$*,* $y$*,* $z$*, and* $w$*.*

1. *If* $[xyz]$ *and* $[xzw]$*, then* $[xyw]$ *and* $[yzw]$*.*
2. *If* $[xyz]$ *and* $[yzw]$*, then* $[xyz]$ *and* $[xzw]$*.*

**Proposition 4.** *Suppose* $\mathcal{P}$ *is an incidence geometry and* $[\cdots]$ *a betweenness relation on* $\mathcal{P}$ *having the 4-Point Property. If* $x$*,* $y$*, and* $z$ *are distinct points such that* $[xyz]$*, then the following hold.*

1. $\overline{xy} \cup \overline{yz} = \overline{xz}$
2. $\overline{xy} \cap \overline{yz} = \{y\}$
3. $\overrightarrow{yx} \cap \overrightarrow{yz} = \{y\}$
4. $\overrightarrow{xy} = \overrightarrow{xz}$

**Proposition 5.** *If $\mathcal{P}$ is an incidence geometry with a betweenness relation having both the Trichotomy Property and the 4-Point Property, then the following hold.*

1. *If $[xzy]$ and $[xwy]$, then either $[xzw]$ or $[xwz]$ or $z = w$.*
2. *If $x$, $y$, and $z$ are distinct points such that $[xyz]$, then $\overrightarrow{yx} \cup \overrightarrow{yz} = \overleftrightarrow{xz}$.*

## The Interpolation Property

**Definition 5.** *We say that a betweenness relation $[\cdots]$ on an incidence geometry $\mathcal{P}$ has the* Interpolation Property *if for all distinct points $x$ and $y$ in $\mathcal{P}$, there exist points $z_1$, $z_2$, and $z_3$ such that $[z_1xy]$, $[xz_2y]$, and $[xyz_3]$.*

**Proposition 6.** *If $\mathcal{P}$ is an incidence geometry with a betweenness relation having both the Interpolation Property and the 4-Point Property, then every line in $\mathcal{P}$ has infinitely many points.*