

## Incidence Geometries

**Definition 1** (Incidence Geometry). Let  $\mathcal{P} = (P, L)$  be an incidence structure. We say  $\mathcal{P}$  is an incidence geometry if the following properties are satisfied.

IG1. If  $x, y \in P$  are distinct points, then there is a unique line  $\ell \in L$  such that  $x, y \in \ell$ . We denote this line  $\overleftrightarrow{xy}$ .

IG2. If  $\ell \in L$  is a line, then there are at least two distinct points  $x, y \in \ell$ .

IG3. There is a set of three distinct points which is noncollinear.

**Proposition 1.** Let  $\mathcal{P} = (P, L)$  be an incidence geometry.

1. If  $x, y \in P$ , then  $\overleftrightarrow{xy} = \overleftrightarrow{yx}$ .
2. If  $x, y, z \in P$ , then the set  $\{x, y, z\}$  is collinear if and only if  $z \in \overleftrightarrow{xy}$ .
3. If  $z \in \overleftrightarrow{xy}$ , then  $\overleftrightarrow{xz} = \overleftrightarrow{xy}$ .

### Examples

$2^P$  If  $P$  is a nonempty set, then the trivial incidence structure  $2^P$  is *not* an incidence geometry since it includes lines with only one point.

$\mathbb{R}^2$  The Cartesian Plane is an incidence geometry, as we show.

IG1. Let  $A, B \in \mathbb{R}^2$  be distinct points; we need to show that there is exactly one line containing  $A$  and  $B$ . First note that  $A, B \in \ell_{A,B}$  (since  $A = A + 0(B - A)$  and  $B = A + 1(B - A)$ ), so there is at least one such line. Suppose that  $\ell = \ell_{P,Q}$  is a line such that  $A, B \in \ell$ ; say  $A = P + t_A(Q - P)$  and  $B = P + t_B(Q - P)$ . (Since  $A$  and  $B$  are distinct, we have  $t_A \neq t_B$ .) We claim that  $\ell_{A,B} = \ell_{P,Q}$ . To this end, if  $X \in \ell_{A,B}$ , say with  $X = A + t(B - A)$ , then we have

$$X = A + t(B - A) = P + tt_A(t_B - t_A)(Q - P) \in \ell_{P,Q}.$$

Thus we have  $\ell_{A,B} \subseteq \ell_{P,Q}$ . Now suppose  $X \in \ell_{P,Q}$ ; say  $X = P + t(Q - P)$ . We have

$$A + \frac{t - t_A}{t_B - t_A}(B - A) = X,$$

so that  $X \in \ell_{A,B}$  as needed. So we have  $\ell_{A,B} = \ell_{P,Q}$ ; in particular, any line containing  $A$  and  $B$  is equal to  $\ell_{A,B}$ .

IG2. By definition, since  $A = A + 0(B - A)$ ,  $B = A + 1(B - A) \in \ell_{A,B}$ .

IG3. The point  $(0, 1)$  is not on  $\ell_{(0,0),(1,0)}$ .

$\mathbb{D}$  The Unit Disk is an incidence geometry; to show this, use the fact that  $\mathbb{R}^2$  is an incidence geometry.

$\mathbb{Q}^2$  The Rational Plane is an incidence geometry; the proof of this is similar to that for  $\mathbb{R}^2$ .

$\mathbb{R}^3$  Three Space is an incidence geometry; the proof of this is similar to that for  $\mathbb{R}^2$ .

## Intersecting Lines

**Proposition 2.** *Let  $\mathcal{P} = (P, L)$  be an incidence geometry, with  $\ell_1, \ell_2 \in L$  lines. Then exactly one of the following holds.*

- $\ell_1 = \ell_2$ ,
- $\ell_1 \cap \ell_2 = \emptyset$ , and
- $\ell_1 \cap \ell_2 = \{p\}$ .

*Proof.* Suppose  $\ell_1 \cap \ell_2$  contains at least two points, say  $x$  and  $y$ . Then in fact  $\ell_1 = \overleftrightarrow{xy} = \ell_2$ . So  $\ell_1 \cap \ell_2$  contains either exactly one or zero points.  $\square$

**Corollary 3.** *In an incidence geometry, three points  $x$ ,  $y$ , and  $z$  are not collinear if and only if  $\overleftrightarrow{xy} \cap \overleftrightarrow{xz} = \{x\}$ .*

## Examples

In  $\mathbb{R}^2$ , we have a nice criterion which detects pairs of lines which intersect at a single point.

**Proposition 4.** *Let  $A = (a_1, a_2)$ ,  $B = (b_1, b_2)$ ,  $C = (c_1, c_2)$ , and  $D = (d_1, d_2)$  be points in the Cartesian Plane,  $\mathbb{R}^2$ . Then  $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{p\}$  is a singleton if and only if*

$$\det \begin{bmatrix} b_1 - a_1 & d_1 - c_1 \\ b_2 - a_2 & d_2 - c_2 \end{bmatrix} \neq 0.$$

**Corollary 5.** *In  $\mathbb{R}^2$ , three points  $A$ ,  $B$ , and  $C$  are not collinear if and only if*

$$\det \begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix} \neq 0.$$

**Corollary 6.** *This statement is also true in the Rational Plane,  $\mathbb{Q}^2$ .*