# Angles

**Definition 1** (Angle)**.** *Let $\mathcal{P}$ be an ordered geometry and $x$, $o$, and $y$ distinct points.*

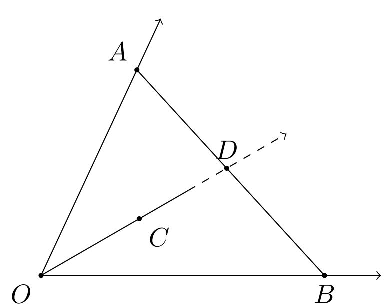
- *The set*
$$\angle xoy = \overrightarrow{ox} \cup \overrightarrow{oy}$$
*is called the* angle *with* vertex *$o$ and* sides *$\overrightarrow{ox}$ and $\overrightarrow{oy}$.*

- *Suppose further that $x$, $o$, and $y$ are not collinear. In this case, since $\mathcal{P}$ is an ordered geometry, the lines $\overleftrightarrow{ox}$ and $\overleftrightarrow{oy}$ divide $\mathcal{P}$ into half-planes. Let $H_1$ be the $y$ half-plane of $\overleftrightarrow{ox}$, and let $K_1$ be the $x$ half-plane of $\overleftrightarrow{oy}$. We define the* interior *of $\angle xoy$ to be the set*
$$\mathsf{int}\angle xoy = H_1 \cap K_1.$$
*If $x$, $y$, and $o$ are collinear, then the interior of $\angle xoy$ is not defined.*

**Definition 2** (Linear Pair, Vertial Pair)**.** *Suppose $x$, $y$, $z$, $w$, and $o$ are distinct points in an ordered geometry.*

- *$\angle xoy$ and $\angle yoz$ are called an* adjacent pair *if $y \in \mathsf{int}\angle xoz$.*
- *$\angle xoy$ and $\angle yoz$ are called a* linear pair *if $[xoz]$.*
- *$\angle xoy$ and $\angle zow$ are called a* vertical pair *if $[xoz]$ and $[yow]$.*

**Theorem 1** (Crossbar Theorem)**.** *Suppose $O$, $A$, and $B$ are noncollinear points in an ordered geometry, and that $C \in \mathsf{int}\angle AOB$. Then $\overrightarrow{OC}$ cuts $\overline{AB}$ at a unique point $D$.*




*Proof.* By the Interpolation property, there is a point $P$ on $\overleftrightarrow{OA}$ such that $[POA]$. Note that $A$ and $P$ are on opposite sides of $\overleftrightarrow{OB}$, so that $P$ and $C$ are on opposite sides of $\overleftrightarrow{OB}$. (Since $A$ and $C$ are on the same side of $\overleftrightarrow{OB}$ by definition.) Consider now the triangle $\triangle PAB$. Note that the line $\overleftrightarrow{OC}$ does not contain $A$, $B$, or $P$, since $C$ is not on $\overleftrightarrow{OA}$ or $\overleftrightarrow{OB}$ by hypothesis. Moreover, $\overleftrightarrow{OC}$ cuts $\overline{PA}$ at $O$. By Pasch's Axiom, $\overleftrightarrow{OC}$ must also cut either $\overline{PB}$ or $\overline{AB}$.

Suppose $\overleftrightarrow{OC}$ cuts $\overline{PB}$ at a (necessarily unique) point $Q$. Note that $\overleftrightarrow{OC} = \overleftrightarrow{QC}$. Now $P$ and $Q$ are on the same side of $\overleftrightarrow{OB}$, so that $Q$ and $C$ are on *opposite* sides of $\overleftrightarrow{OB}$. Thus, there is a unique point $R$ on $\overleftrightarrow{OB}$ such that $[QRC]$. In particular, $R \in \overleftrightarrow{OC}$. Now we have $O, R \in \overleftrightarrow{OC}$ and $O, R \in \overleftrightarrow{OB}$, so that $\overleftrightarrow{OC} = \overleftrightarrow{OB}$, a contradiction.

Hence $\overleftrightarrow{OC}$ must cut $\overline{AB}$ at a unique point; say $D$. Now $D$ and $A$ are on the same side of $\overleftrightarrow{OB}$, and so $C$ and $D$ are on the same side of $\overleftrightarrow{OB}$; in particular, we cannot have $[DOC]$. So in fact $\overrightarrow{OC}$ cuts $\overline{AB}$ at a unique point. $\square$