Rings

Definition 1 (Ring)**.** *A* ring *is a set R equipped with two operations $+$ and $\cdot$, which satisfy the following properties.*

A1. $(a+b)+c = a+(b+c)$ *for all* $a, b, c \in R$.

A2. *There is an element* $0_R \in R$ *(called a* zero*) such that* $a + 0_R = 0_R + a = a$ *for all* $a \in R$.

A3. *For every* $a \in R$ *there is an element* $-a \in R$ *(called a* negative *of* a*) such that* $a + (-a) = (-a) + a = 0_R$.

A4. $a + b = b + a$ *for all* $a, b \in R$.

M. $(ab)c = a(bc)$ *for all* $a, b, c \in R$.

D. $a(b+c) = ab + ac$ *and* $(b+c)a = ba + ca$ *for all* $a, b, c \in R$.

Proposition 1. *Let R be a ring.*

1. *The zero element of R is unique in the following sense: if $a, b \in R$ such that $a + b = a$, then $b = 0_R$.*
2. *Negative elements in R are unique in the following sense: If $a, b \in R$ such that $a + b = 0_R$, then $b = -a$.*
3. $-(-a) = a$ *for all* $a \in R$.
4. $0_R \cdot a = a \cdot 0_R = 0_R$ *for all* $a \in R$.
5. $(-a)b = a(-b) = -(ab)$ *for all* $a, b \in R$.
6. $(-a)(-b) = ab$ *for all* $a, b \in R$.

Proof.

1. Suppose $a + b = a$. Now $-a + (a+b) = -a + a$, and by A1 $(-a+a)+b = -a + a$. By A3 we have $0_R + b = 0_R$, and by A2 we have $b = 0_R$.
2. Suppose $a + b = 0_R$. Now $-a + (a+b) = -a + 0_R$, and by A1 we have $(-a+a)+b = -a+0_R$. By A3 we have $0_R + b = -a + 0_R$, and using A2 (twice) we have $b = -a$.
3. By definition, $(-a) + a = 0_R$, so by the uniqueness of negatives we have $a = -(-a)$.
4. Let $a \in R$. Now $a \cdot a + 0_R \cdot a = (a + 0_R) \cdot a = a \cdot a$, and so $0_R \cdot a = 0_R$. The other equality is similar.
5. Let $a, b \in R$. Now $(-a)b + ab = (-a+a)b = 0_R \cdot b = 0_R$, so that $(-a)b = -(ab)$. The other equality is similar.
6. Using the previous statement, we have $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$.

□

Examples

$\mathbb{Z}, \mathbb{Z}/(n)$ The integers are a ring by definition, and we showed that the integers mod n are a ring for any $n > 0$.

$\mathbb{Q}$ The rational numbers are a ring under the usual addition and multiplication; we will prove this later. (Actually we will define the rational numbers.)

$\mathbb{R}$, $\mathbb{C}$ The real and complex numbers are also rings, although even defining these sets of "numbers" is beyond the scope of this text.

0 The smallest possible ring must have at least one element, the zero. Suppose this is *all* we have. Now the arithmetic is pretty boring: $0 + 0 = 0$ and $0 \cdot 0 = 0$. It is straightforward to check that these operations make the set $\{0\}$ into a ring. This example isn't very interesting, so we call this the *trivial ring* or the *zero ring*.

R^A Let R be a ring, and let A be any nonempty set. Then the set

$$R^A = \{\varphi \mid \varphi : A \to R\}$$

is a ring under the "pointwise" operations

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x) \quad \text{and} \quad (\alpha\beta)(x) = \alpha(x)\beta(x).$$

$\mathsf{Mat}_2(R)$ Let R be a ring, and consider the set

$$\mathsf{Mat}_2(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in R \right\}.$$

These are just the 2×2 matrices with entries in R. The usual matrix addition and multiplication make $\mathsf{Mat}_2(R)$ into a ring. Specifically, we define

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

$2\mathbb{Z}$ Consider the set

$$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}.$$

It is not too difficult to show that this set is a ring under the usual addition and multiplication of integers.

2^X Let X be any nonempty set. The powerset 2^X is a ring under the operations $A + B = (A \setminus B) \cup (B \setminus A)$ and $A \cdot B = A \cap B$. This is called a *ring of sets*.

Commutative and Unital Rings

Our list of examples is starting to get complicated, so we make two additional definitions to start drawing distinctions among them.

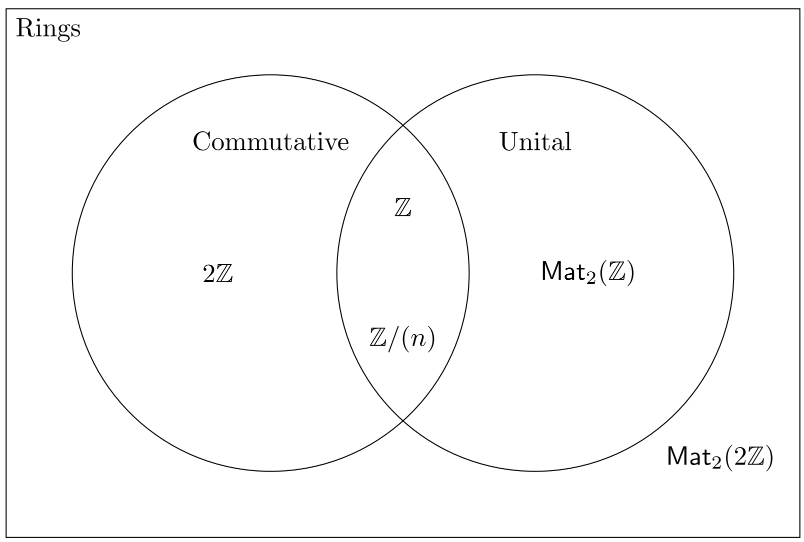
Definition 2. *Let R be a ring.*

- *We say that R is* commutative *if it satisfies the following property.*

  C. $ab = ba$ for all $a, b \in R$.

- *We say that R is* unital *if it satisfies the following property.*

  U. There is an element $1 \in R$ (called a one*) such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.*



Proposition 2. *Let R be a unital ring.*

1. *The one element of R is unique in the following sense: if $u \in R$ such that $u \cdot a = a$ for all $a \in R$, then $u = 1$.*

2. *$-a = (-1) \cdot a$ for all $a \in R$.*

Proof.

1. Suppose u is such an element. In particular, $1 = u \cdot 1 = u$.

2. Let $a \in R$. Then $a + (-1)a = 1a + (-1)a = (1 + (-1))a = 0a = 0$, so that $(-1)a = -a$. $\square$