Perpendiculars and Tangents

We say that two lines are *perpendicular* if they form a right angle.

Definition 1 (Foot)**.** *Let ℓ be a line and p a point not on ℓ in a plane geometry. We say that a point $f \in \ell$ is a* foot *of p on ℓ if ℓ and $\overleftrightarrow{FP}$ are perpendicular.*

Construction 1 (Foot of a point)**.** *Let ℓ be a line and p a point not on ℓ in a plane geometry. Then p has a unique foot on ℓ.*

Proof. To see existence, let x and y be distinct points on ℓ. Note that $\mathcal{C}_x(p) \cap \mathcal{C}_y(p)$ is not empty, and by Circle Cut Transfer there is a second point o in the intersection of these circles which is on the opposite side of ℓ. By the Plane Separation property, ℓ and $\overline{op}$ meet at a unique point f. Now $\triangle oxy \equiv \triangle pxy$ by SSS, so that $\angle pxf \equiv \angle oxf$. Then $\triangle pxf \equiv \triangle oxf$ by SAS. Then $\angle pfx \equiv \angle ofx$, so that ℓ and $\overleftrightarrow{op}$ meet at a right angle as needed.

To see uniqueness, note that if p has two distinct feet f_1 and f_2 on ℓ then p, f_1, and f_2 form a triangle with two internal right angles – a contradiction. $\square$

Construction 2 (Perpendicular at a point)**.** *Let ℓ be a line and $p \in \ell$ a point in a plane geometry. There exists a unique line t containing p which is perpendicular to ℓ.*

Proof. Let x be a point on ℓ different from p, and copy $\overline{px}$ to the opposite side of p at a point y by Circle Separation. Note that p is the midpoint of $\overline{xy}$. Construct a point z such that $\triangle xyz$ is equilateral. Now $\triangle zxp \equiv \triangle zyp$ by SSS, so that $\angle zpx \equiv \angle zpy$, and thus $\overleftrightarrow{pz}$ is perpendicular to ℓ.

Uniqueness follows from the uniqueness of angles on a half-plane. $\square$

Definition 2 (Perpendicular Bisector)**.** *If x and y are two points, then the (unique) line perpendicular to $\overleftrightarrow{xy}$ at the midpoint of $\overline{xy}$ is called the* perpendicular bisector *of $\overline{xy}$.*

Intersections of Lines and Circles

Proposition 3. *In a plane geometry, a line and a circle can have at most two points in common.*

Proof. Let ℓ be a line and $\mathcal{C}_o(a)$ a circle which have at least three points in common; say x, y, and z. Suppose WLOG that $[xyz]$. Note that o cannot also be on ℓ, as in this case z cannot be distinct from both x and y by the uniqueness of congruent segments on rays. Now $\angle oyx \equiv \angle oxy$, $\angle oyz \equiv \angle ozy$, and $\angle oxz \equiv \angle ozx$ by Pons Asinorum. In particular, $\angle oyx$ is right, so that $\triangle oxy$ has two right interior angles – a contradiction. $\square$

Definition 3 (Tangent, Secant)**.** *Let ℓ be a line and C a circle in a plane geometry. We say that ℓ is* tangent to C *if ℓ and C have exactly one point in common. Suppose this point is t; in this case we say that ℓ is tangent to C* at *t. Similarly, we say that ℓ is a* secant *of C if ℓ and C have exactly two distinct points in common.*

Proposition 4. *Let ℓ be a line and C a circle with center o in a plane geometry. Then ℓ is tangent to C if and only if o is not on ℓ and the foot of o on ℓ is on C.*

Proof. Suppose ℓ is tangent to C at p. If $o \in \ell$, then $\ell \cap C$ contains a second point by Circle Separation; so in fact o is not on ℓ. Let f be the foot of o on ℓ. If $f \neq p$, then o, f, and p are noncollinear and form a triangle. Since $\overline{op} \equiv \overline{of}$ and $\angle ofp$ is right, $\angle opf$ is also right by Pons Asinorum. But no triangle can have two right interior angles.

Conversely, suppose ℓ does not contain o and that the foot f of o on ℓ is on C. Suppose BWOC that there is a second point $g \in \ell \cap C$. Now o, f, and g are noncollinear, and $\overline{of} \equiv \overline{og}$, and $\angle ofg$ is right (by the definition of foot). So $\angle ogf$ is right by Pons Asinorum, again a contradiction. So $C \cap \ell$ contains exactly one point as needed. $\square$

Construction 5 (Tangent at a point)**.** *Let C be a circle with center o and let p be a point on C. There exists a line ℓ which is tangent to C at p.*

Proof. Construct the line ℓ which is perpendicular to $\overleftrightarrow{op}$ at p. Then o is not on ℓ, and p is the foot of o on ℓ. So ℓ is tangent to C at p. $\square$

Construction 6 (Second cut of line and circle)**.** *Let ℓ be a line and C a circle with center o in a plane geometry such that ℓ is not tangent to C. Suppose $p \in \ell \cap C$. We may construct the second point in $\ell \cap C$.*

Proof. If o is on ℓ, use Circle Separation. If o not on ℓ, construct the foot f of o on ℓ. Using Circle Separation, copy $\overline{fp}$ onto the opposite side of f from p at the point q. Note that $\triangle ofp \equiv \triangle ofq$ by SAS, so that $\overline{op} \equiv \overline{oq}$; thus $q \in \ell \cap C$ as needed. $\square$

Comparing Segments

Definition 4. *Let $\overline{ab}$ and $\overline{cd}$ be segments in a plane geometry. We say that $\overline{ab} \leq \overline{cd}$ if there is a point $x \in \overline{cd}$ such that $\overline{ab} \equiv \overline{cx}$.*

Proposition 7.

1. *If $\overline{a_1b_1} \equiv \overline{a_2b_2}$, $\overline{c_1d_1} \equiv \overline{c_2d_2}$, and $\overline{a_1b_1} \leq \overline{c_1d_1}$, then $\overline{a_2b_2} \leq \overline{c_2d_2}$.*
2. *If $\overline{ab} \leq \overline{cd}$ and $\overline{cd} \leq \overline{ef}$, then $\overline{ab} \leq \overline{ef}$.*
3. *If $[abc]$, then $\overline{ab} \leq \overline{ac}$. If $[abcd]$, then $\overline{bc} \leq \overline{ad}$.*
4. *If $\overline{ab} \leq \overline{cd}$ and $\overline{cd} \leq \overline{ab}$, then $\overline{ab} \equiv \overline{cd}$.*

Proof.

There is a point $x \in \overline{c_1d_1}$ such that $\overline{a_1b_1} \equiv \overline{c_1x}$. Now copy $\overline{c_1x}$ onto $\overrightarrow{c_2d_2}$ at the point y; note that $[c_2yd_2]$, so that $y \in \overline{c_2d_2}$. Now $\overline{a_2b_2} \equiv \overline{c_2y}$ as needed.

There exists a point $x \in \overline{cd}$ such that $\overline{ab} \equiv \overline{cx}$, and a point $y \in \overline{ef}$ such that $\overline{cd} \equiv \overline{ey}$. Now copy $\overline{cx}$ onto $\overrightarrow{ey}$ at the point z; note that $[ezy]$; in particular, $\overline{ab} \equiv \overline{ez}$.

Clear.

There exists a point $x \in \overline{cd}$ such that $\overline{cx} \equiv \overline{ab}$. Now either $x = c$, $x = d$, or $[cxd]$. If $x = c$, then $b = a$, and $d = c$, so that $\overline{ab} \equiv \overline{cd}$. Suppose $[cxd]$. There is a point $y \in \overline{ab}$ such that $\overline{cy} \equiv \overline{ab}$; but now $[aby]$, a contradiction. So we have $x = d$ as needed. $\square$