Names: ______________________ ______________________ ______________________

Activity #2: Continuity (Solutions) **Calculus I**

1. Compute the following limit.
$$\lim_{x\to 0} \left|2x^2 - x - 3\right|$$

Solution: The function which we are computing the limit of here is continuous everywhere; so to find the limit, we can simply evaluate this function at $x = 0$. Thus

$$\lim_{x\to 0} \left|2x^2 - x - 3\right| = |-3| = 3$$

So the limit is $\boxed{3}$.

2. Compute the following limit.
$$\lim_{x\to 3} \left|\frac{x^3 - 27}{x - 3}\right|$$

Solution: Note that this expression is not defined if $x = 3$ because of the $x-3$ in the denominator. However, we can factor the numerator as a difference of cubes and cancel.

$$\lim_{x\to 3} \left|\frac{x^3 - 27}{x - 3}\right| = \lim_{x\to 3} \left|\frac{(x-3)(x^2+3x+9)}{x - 3}\right| = \lim_{x\to 3} \left|x^2 + 3x + 9\right| = |27| = 27$$

Now this function is continuous everywhere, so we can compute the limit by evaluating at $x = 3$; this yields a limit of $\boxed{27}$.

3. Compute the limit of the difference quotient

$$\lim_{x\to t} \frac{f(x) - f(t)}{x - t}$$

when $f(x) = 10x + 8$ and $t = 2$.

Solution: We have

$$\begin{aligned}
\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x\to 2} \frac{(10x+8) - (10\cdot 2 + 8)}{x - 2} \\
&= \lim_{x\to 2} \frac{10x - 10\cdot 2}{x - 2} \\
&= \lim_{x\to 2} \frac{10(x-2)}{x - 2} \\
&= \lim_{x\to 2} 10 \\
&= 10
\end{aligned}$$

4. Compute the limit of the difference quotient

$$\lim_{x\to t} \frac{f(x) - f(t)}{x - t}$$

when $f(x) = 2x^2 + 4x + 9$ and $t = 3$.

Solution: We have

$$\begin{aligned}
\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} &= \lim_{x\to 3} \frac{(2x^2 + 4x + 9) - (2(3)^2 + 4(3) + 9)}{x - 3} \\
&= \lim_{x\to 3} \frac{2x^2 - 2(3)^2 + 4x - 4(3) + 9 - 9}{x - 3} \\
&= \lim_{x\to 3} \frac{2(x^2 - 3^2) + 4(x - 3)}{x - 3} \\
&= \lim_{x\to 3} \frac{2(x - 3)(x + 3) + 4(x - 3)}{x - 3} \\
&= \lim_{x\to 3} \frac{(x - 3)(2(x + 3) + 4)}{x - 3} \\
&= \lim_{x\to 3} (2(x + 3) + 4) \\
&= 2(3 + 3) + 4 \\
&= 16
\end{aligned}$$

5. Compute the limit of the difference quotient

$$\lim_{x\to t} \frac{f(x) - f(t)}{x - t}$$

when $f(x) = \sqrt{x + 7}$ and $t = 2$.

Solution: We have

$$\begin{aligned}
\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x\to 2} \frac{\sqrt{x + 7} - \sqrt{9}}{x - 2} \\
&= \lim_{x\to 2} \left(\frac{\sqrt{x + 7} - \sqrt{9}}{x - 2} \cdot \frac{\sqrt{x + 7} + \sqrt{9}}{\sqrt{x + 7} + \sqrt{9}} \right) \\
&= \lim_{x\to 2} \frac{(x + 7) - (9)}{(x - 2)\left(\sqrt{x + 7} + \sqrt{9}\right)} \\
&= \lim_{x\to 2} \frac{x - 2}{(x - 2)\left(\sqrt{x + 7} + \sqrt{9}\right)} \\
&= \lim_{x\to 2} \frac{1}{\sqrt{x + 7} + \sqrt{9}} \\
&= \frac{1}{2\sqrt{9}} \\
&= \frac{1}{6}
\end{aligned}$$

6. Compute the following limit.

$$\lim_{x\to 0} \frac{\sin(3x)}{x}$$

Solution: Note that

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(3x)}{x} &= \lim_{x\to 0}\frac{3\sin(3x)}{3x}\\
&= 3\lim_{x\to 0}\frac{\sin(3x)}{3x}\\
&= 3\lim_{x\to 0}\operatorname{sinc}(3x)\\
&= 3\operatorname{sinc}\left(\lim_{x\to 0}3x\right) \quad \text{(since sinc is continuous)}\\
&= 3\operatorname{sinc}(0)\\
&= 3.
\end{aligned}$$

7. Compute the following limit.

$$\lim_{x\to 0}\frac{6x^2+11x+\sin x}{x}$$

Solution: This expression is not defined if $x = 0$. However, we can split this fraction like so:

$$\begin{aligned}
\lim_{x\to 0}\frac{6x^2+11x+\sin(x)}{x} &= \lim_{x\to 0}\left(\frac{6x^2+11x}{x}+\frac{\sin x}{x}\right)\\
&= \lim_{x\to 0}\left(6x+11+\frac{\sin x}{x}\right).
\end{aligned}$$

Recall that the limit of a sum is the sum of limits, *provided* the limit of each summand exists. In this case they do, and we have

$$\begin{aligned}
&= \lim_{x\to 0}(6x+11)+\lim_{x\to 0}\frac{\sin x}{x}\\
&= 11+1\\
&= 12.
\end{aligned}$$

8. Let $f(x)$ be the function

$$f(x)=\begin{cases}\frac{x-b}{b+6} & \text{if } x<0\\[2ex] x^2+b & \text{if } x\geq 0.\end{cases}$$

Find the value(s) of the constant b such that $f(x)$ is continuous everywhere.

Solution: Remember that $\lim_{x\to 0}f(x)$ exists precisely when the one-sided limits $\lim_{x\to 0^+}f(x)$ and $\lim_{x\to 0^-}f(x)$ exist and are equal to one another. In this case, we have

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x^2+b)=b$$

and

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}\frac{x-b}{b+6}=\frac{-b}{b+6}.$$

Setting these equal, we have

$$b=\frac{-b}{b+6}.$$

The values for b we want are precisely the solutions of this equation. Clearing denominators, we have

$$b^2 + 6b = -b,$$

and solving for zero we have

$$b^2 + 7b = 0 \qquad \text{which factors as} \qquad b(b+7) = 0.$$

So $b = 0$ or $b = -7$.