The Plane Separation Property

Definition 1 (Convexity)**.** *Let $\mathcal{P}$ be an incidence geometry with a betweenness relation $[\cdots]$. A non empty set S of points in $\mathcal{P}$ is called* convex *if whenever $x, y \in S$ are distinct points, $\overline{xy} \subseteq S$.*

Definition 2 (Plane Separation Property)**.** *Let $\mathcal{P}$ be an incidence geometry with a betweenness relation $[\cdots]$. We say that this geometry has the* Plane Separation Property *if every line ℓ partitions the set of points not on ℓ into two nonempty, disjoint, convex sets, H_1 and H_2, with the property that if $x \in H_1$ and $y \in H_2$ then $\overline{xy} \cap \ell = \{p\}$ for some point p. The sets H_1 and H_2 are called* half-planes.

Examples

To show that a particular incidence geometry has the plane separation property, given any line we must specify the half-planes H_1 and H_2 and *show* that they are nonempty, disjoint, convex sets, which have the intersection property.

$\mathbb{R}^2$ Given a line $\ell = \overleftrightarrow{AB}$, we define two half-planes as follows:

$$H_1 = \left\{ X = (x_1, x_2) \mid \mathsf{det} \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} > 0 \right\}$$

and

$$H_2 = \left\{ X = (x_1, x_2) \mid \mathsf{det} \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ x_1 & x_2 & 1 \end{bmatrix} < 0 \right\}.$$

Certainly both H_1 and H_2 are not empty, and they are disjoint by construction.

To see that H_1 is convex, suppose BWOC that we have points $X, Y \in H_1$ and a point $Z = (z_1, z_2)$ such that $[XZY]$ and $Z \notin H_1$. Now

$$m = \mathsf{det} \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ z_1 & z_2 & 1 \end{bmatrix}$$

is either 0 or negative. If $m = 0$, then in fact $Z \in \overleftrightarrow{AB}$. Since $X, Y \notin \overleftrightarrow{AB}$, we have that $\overline{XY}$ and $\overleftrightarrow{AB}$ meet at a single point Z; but we've seen this can only happen if $X \in H_1$ and $Y \in H_2$ (or vice versa). Suppose instead that $m < 0$; that is, $Z \in H_2$. Now we have that $\overline{XZ}$ and $\overline{YZ}$ each intersect $\overleftrightarrow{AB}$ at unique points, say W and V, respectively. Note that $[XWZ]$ and $[YVZ]$. Since $[XZY]$, we have that X, Y, Z, W, and V are all collinear. If W and V are distinct points, then in fact $X, Y \in \overleftrightarrow{WV} = \overleftrightarrow{AB}$, a contradiction. If $W = V$, then we have $[XWZ]$ and $[YWZ]$, so by the

4-point axiom, $[WZY]$, a contradiction. So we must have $Z \in H_1$, and thus H_1 is convex. A similar argument shows that H_2 is convex.

Finally, we need to show that if $X \in H_1$ and $Y \in H_2$, then $\overline{XY} \cap \overleftrightarrow{AB}$ consists of a unique point. We showed precisely this previously.

Ordered Geometries

Definition 3 (Ordered Geometry)**.** *Let $\mathcal{P}$ be an incidence geometry with a betweenness relation $[\cdots]$. We say that $\mathcal{P}$ (with this betweenness relation) is an* Ordered Geometry *if it has the Trichotomy Property, the 4-Point Property, the Interpolation Property, and the Line Separation Property.*

For example, both $\mathbb{R}^2$ and $\mathbb{Q}^2$ are ordered geometries.

Definition 4 (Triangle)**.** *Let $\mathcal{P}$ be an incidence geometry, and let x, y, and z be distinct points. Then the set*

$$\triangle xyz = \overline{xy} \cup \overline{yz} \cup \overline{zx}$$

is called the triangle *with* vertices *x, y, and z. The segments $\overline{xy}$, $\overline{yz}$, and $\overline{zx}$ are called the* sides *of the triangle.*

Theorem 1 (Pasch's Axiom)**.** *Let x, y, and z be distinct points in an ordered geometry, and let ℓ be a line such that $x, y, z \notin \ell$. Finally, suppose there is a point $w \in \ell$ such that $[xwy]$; that is, ℓ cuts the side $\overline{xy}$.*

Then precisely one of the following two things happens:

1. *ℓ cuts $\overline{yz}$ and does not cut $\overline{zx}$, or*
2. *ℓ cuts $\overline{zx}$ and does not cut $\overline{yz}$.*

Proof. Since $\mathcal{P}$ is an ordered geometry, it satisfies the Plane Separation property. In particular, the points not on ℓ are partitioned into two convex, nonempty half-planes, H_1 and H_2. Since $\overline{xy} \cap \ell = \{w\}$ is not empty, without loss of generality we have $x \in H_1$ and $y \in H_2$. Since $z \notin \ell$, there are two possibilities: either $z \in H_1$ or $z \in H_2$. In the first case, we see that ℓ cuts $\overline{yz}$ and does not cut $\overline{zx}$, and in the second case, ℓ cuts $\overline{zx}$ but not $\overline{yz}$. □

In other words, Pasch's Axiom states that if a line enters a triangle then it must also exit.

Lemma 2. *Let ℓ be a line and $C \in \ell$ a point in an ordered geometry. Suppose A and B are points not on ℓ such that $[ABC]$. Then A and B are on the same side of ℓ.*

Proof. Suppose otherwise that A and B are on opposite sides of ℓ. By the Plane Separation property, and because A and B are not on ℓ, the segment $\overline{AB}$ cuts ℓ at a unique point D. That is, $D \in \ell$ and $[ADB]$. But note that $C, D \in \ell$, so $\overleftrightarrow{CD} = \ell$, and also $C, D \in \overleftrightarrow{AB}$, so that $\overleftrightarrow{CD} = \overleftrightarrow{AB}$. But then $\overleftrightarrow{AB} = \ell$, a contradiction. Thus A and B must be on the same side of ℓ. □