Transversals

Proposition 1 (Supplements are unique)**.**

- *Suppose that $\angle AOB$ and $\angle BOC$ are a linear pair, and that $\angle XPY$ and $\angle YPZ$ are a linear pair. If $\angle AOB \equiv \angle XPY$, then $\angle BOC \equiv \angle YPZ$.*
- *Suppose $\angle ABC$ and $\angle XYZ$ are supplementary, and that $\angle ABC$ and $\angle HKL$ are supplementary. Then $\angle XYZ \equiv \angle HKL$.*

Proof. Suppose we have two such linear pairs. Without loss of generality, we can suppose that

$$\overline{OA} \equiv \overline{OB} \equiv \overline{OC} \equiv \overline{PX} \equiv \overline{PY} \equiv \overline{PZ}.$$

(If they aren't, we can use Circle Separation and the Segment Copy construction to find such points.) Now $\triangle BOA \equiv \triangle YPX$ by SAS, so that $\angle BAO \equiv \angle YXP$. Now $\overline{AC} \equiv \overline{XZ}$, so that $\triangle BAC \equiv \triangle YXZ$ by SAS. So $\overline{BC} \equiv \overline{YZ}$, and thus $\triangle BOC \equiv \triangle YPZ$ by SSS. Thus $\angle BOC \equiv \angle YPZ$.

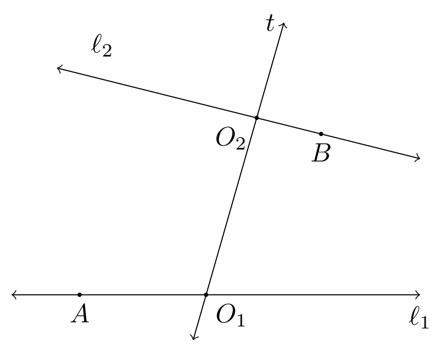
The second statement follows easily. $\square$

Corollary 2. *Vertical pairs of angles are congruent.*

Transversals

Definition 1 (Transversal)**.** *Suppose we have three lines ℓ_1, ℓ_2, and t in a plane geometry. We say that t is a* transversal *of ℓ_1 and ℓ_2 if t cuts both ℓ_1 and ℓ_2 at unique points, and these points are distinct.*

Suppose t is a transversal of ℓ_1 and ℓ_2, cutting these lines at O_1 and O_2, respectively as shown.



If A is on ℓ_1 and B is on ℓ_2 such that A and B are on opposite sides of t, then we say that $\angle AO_1O_2$ and $\angle BO_2O_1$ are *alternate interior angles* of this transversal.

Proposition 3 (Alternate Interior Angles)**.** *If two lines ℓ_1 and ℓ_2 are cut by a transversal t so that a pair of alternate interior angles are congruent, then ℓ_1 and ℓ_2 are parallel.*

Proof. Suppose t meets ℓ_1 and ℓ_2 at points O_1 and O_2 respectively, and that A and B are on ℓ_1 and ℓ_2, respectively, and on opposite sides of t. Let C be on ℓ_1 such that $[AO_1C]$. Suppose by way of contradiction that ℓ_1 and ℓ_2 are *not* parallel; rather, they meet at a point X which (WLOG) is on the A-side of t. Copy $\overline{O_1X}$ onto $\overrightarrow{O_2B}$ at the point Y. Now $\overline{O_1X} \equiv \overline{O_2Y}$, $\overline{O_1O_2} \equiv \overline{O_2O_1}$, and $\angle XO_1O_2 \equiv \angle YO_2O_1$, so by SAS we have $\triangle XO_1O_2 \equiv \triangle YO_2O_1$. In particular, $\angle O_2O_1Y \equiv \angle O_1O_2X$.

Now $\angle XO_2O_1$ and $\angle O_1O_2Y$ are supplementary, and $\angle O_1O_2Y \equiv \angle AO_1O_2$, so that $\angle AO_1O_2$ and $\angle XO_2O_1$ are supplementary. Since $\angle XO_2O_1 \equiv \angle YO_1O_2$, we have that $\angle AO_1O_2$ and $\angle YO_1O_2$ are supplementary. But also $\angle AO_1O_2$ and $\angle O_2O_1C$ are supplementary. Now $\angle O_2O_1Y \equiv \angle O_2O_1C$. By the uniqueness of congruent angles on a half-plane, we have that O_1, C, and Y are collinear, so that $Y \in \ell_1$. But now ℓ_1 and ell_2 have two points in common – X and Y – and thus must be equal, a contradiction.

So in fact ℓ_1 and ℓ_2 must be parallel. □

Proposition 4 (AAS)**.** *Suppose we have triangles $\triangle ABC$ and $\triangle XYZ$ such that $\angle CAB \equiv \angle ZXY$, $\angle ABC \equiv \angle XYZ$, and $\overline{BC} \equiv \overline{YZ}$. Then $\triangle ABC \equiv \triangle XYZ$.*

Proof. Copy $\overline{BA}$ onto $\overrightarrow{YX}$ at the point W. Note that $\triangle WYZ \equiv \triangle ABC$ by SAS, so that $\angle BAC \equiv \triangle YWZ$. Suppose now that W and X are distinct points. In this case $\overleftrightarrow{XZ}$ and $\overleftrightarrow{WZ}$ are lines cut by a transversal $\overleftrightarrow{XY}$. Moreover, if we let U be a point such that $[UXZ]$, then $\angle UXW$ and $\angle YXZ$ are vertical, hence congruent, and so $\angle UXW \equiv \angle YXZ$. But now by the Alternate Interior Angles theorem $\overleftrightarrow{XZ}$ and $\overleftrightarrow{WZ}$ must be parallel, a contradiction since they meet at Z.

So in fact X and W are the same point, and thus $\triangle ABC \equiv \triangle XYZ$ by SAS. □

Proposition 5 (HL)**.** *Let $\triangle ABC$ and $\triangle XYZ$ be triangles such that $\angle BCA$ and $\angle YZX$ are right and $\overline{AB} \equiv \overline{XY}$ and $\overline{BC} \equiv \overline{YZ}$. Then $\triangle ABC \equiv \triangle XYZ$.*

Proof. Copy $\overline{ZX}$ onto the ray opposite $\overrightarrow{CA}$ at the point D. Now $\angle BCD$ is a right angle, since it is supplementary to $\angle ACB$. By SAS, we have $\triangle XYZ \equiv \triangle DCB$, and thus $\overline{BD} \equiv \overline{YX} \equiv \overline{BA}$. Now $\triangle ABD$ is isoceles with $\overline{BA} \equiv \overline{BD}$, so that $\angle BAC \equiv \angle BAD \equiv \angle BDA \equiv \angle YXZ$. By AAS, we have $\triangle ABC \equiv \triangle XYZ$. □

Proposition 6. *A triangle formed by three noncollinear points cannot have two interior angles which are both right.*

Proof. Such a triangle would violate the Alternate Interior Angles theorem since right angles are self-supplementary, and any two right angles are congruent. □

Bisection

Construction 7 (Angle Bisector)**.** *Let A, O, and B be noncollinear points. There exists a unique line ℓ, containing O, such that if $U \in \ell$ is different from O then $\angle AOU \equiv \angle BOU$. This line is called the* bisector *of $\angle AOB$.*

Proof. Note that we can assume WLOG that $\overline{OA} \equiv \overline{OB}$; if not, construct such a point on $\overrightarrow{OB}$ using the Circle Separation property. Since the intersection of $\mathcal{C}_A(O)$ and $\mathcal{C}_B(O)$ contains a point not on $\overleftrightarrow{AB}$, by Circle Cut Transfer there is a second point U on the opposite side of $\overleftrightarrow{AB}$ such that $\overline{AU} \equiv \overline{BU}$. Let $\ell = \overleftrightarrow{OU}$. Note that $\triangle AOU \equiv \triangle BOU$ by SSS, so that $\angle AOU \equiv \angle BOU$. Then if V is a point such that $[VOU]$, we have $\angle VOA \equiv \angle VOB$, since these are supplementary to congruent angles.

To see uniqueness, note that any such line must contain O and U. □

Corollary 8. *A and B are on opposite sides of the bisector of $\angle AOB$. In particular, the bisector of $\angle AOB$ contains points which are interior to $\angle AOB$.*

Proof. Suppose otherwise, and let $U \neq O$ be a point on the bisector. Then $\angle UOA$ and $\angle UOB$ are congruent angles on the same half-plane of a ray, so that A, B, and O are collinear – a contradiction. By the plane separation property there is a point W between A and B which is on the bisector; this point is interior to $\angle AOB$ as needed. □

Construction 9 (Segment Midpoint)**.** *Let A and B be distinct points. There is a unique point M such that $[AMB]$ and $\overline{AM} \equiv \overline{BM}$. This point is called the* midpoint *of $\overline{AB}$.*

Proof. Construct a point O such that $\triangle AOB$ is equilateral, and construct the bisector of $\angle AOB$. By the Crossbar theorem, this bisector must cut $\overline{AB}$ at an interior point, say M. Now $\triangle OAM \equiv \triangle OBM$ by SAS, and thus $\overline{AM} \equiv \overline{BM}$ as needed. Note that M is unique by the uniqueness of congruent segments on a ray. □