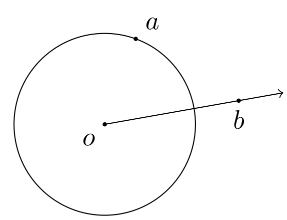
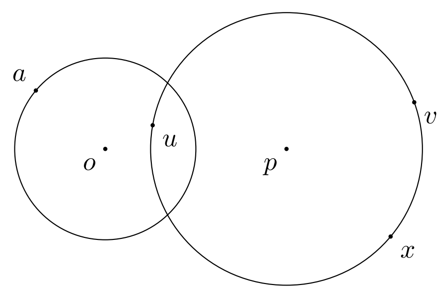
Plane Geometry

Definition 1 (Plane Geometry)**.** *Let $\mathcal{P}$ be an ordered geometry with a segment congruence and an angle congruence. We say that $\mathcal{P}$ is a* plane geometry *if the following properties are satisfied.*

- ***Circle Separation.*** *If o, a, and b are distinct points, then there is a unique point $c \in \mathcal{C}_o(a) \cap \overrightarrow{ob}$.*



- ***Circle Cut.*** *Let o, a, p, and x be points, and suppose there are distinct points u and v on $\mathcal{C}_p(x)$ such that $u \in \mathsf{int}\ \mathcal{C}_o(a)$ and $v \in \mathsf{ext}\ \mathcal{C}_o(a)$. Then $\mathcal{C}_o(a) \cap \mathcal{C}_p(x)$ contains two distinct points.*



- ***Circle Cut Transfer.*** *Suppose a, b, c, d, x, y, z, and w are points such that $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\overline{cd} \equiv \overline{zw}$. If $\mathcal{C}_b(a) \cap \mathcal{C}_c(d)$ is not empty, then $\mathcal{C}_y(x) \cap \mathcal{C}_z(w)$ is not empty.*

- ***Angle-Side Congruence.*** *Suppose a, b, c, x, y, and z are points such that $\overline{ba} \equiv \overline{yx}$ and $\overline{bc} \equiv \overline{yz}$. Then $\overline{ac} \equiv \overline{xz}$ if and only if $\angle abc \equiv \angle xyz$.*

The Circle Separation and Circle Cut properties allow us to construct points on the intersection of a circle with a central ray and of two circles, respectively. (Without these we have no way to construct points on circles!) The Circle Cut Transfer property says that our geometry is "uniform" in some sense, allowing us to shift points in the intersection of two circles. Angle-Side Congruence provides an essential link between segment congruence and angle congruence, which are otherwise unrelated.

Some Consequences

In the remainder of this section, suppose $\mathcal{P}$ is a plane geometry.

Proposition 1 (Circle Trichotomy)**.** *Let o and a be distinct points. Then $\mathcal{C}_o(a)$, $\mathsf{int}\ \mathcal{C}_o(a)$, and $\mathsf{ext}\ \mathcal{C}_o(a)$ partition the set of points in $\mathcal{P}$. That is, every point is either on $\mathcal{C}_o(a)$, interior to $\mathcal{C}_o(a)$, or exterior to $\mathcal{C}_o(a)$.*

Proposition 2 (SSS Theorem)**.** *If two triangles can be labeled such that corresponding sides are congruent, then the triangles are congruent. More precisely, let a, b, and c be distinct points and x, y, and z be distinct points. If $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\overline{ca} \equiv \overline{zx}$, then $\triangle abc \equiv \triangle xyz$.*

Proof. That $\angle abc \equiv \angle xyz$, $\angle bca \equiv \angle yzx$, and $\angle zxy \equiv \angle cab$ follows from three applications of the Angle-Side Congruence property. $\square$

Proposition 3 (Uniqueness of Circle Cuts)**.** *Let o, a, p, x, and h be points, with o and p distinct and with h not on $\overleftrightarrow{op}$. There is at most one point $u \in \mathcal{C}_o(a) \cap \mathcal{C}_p(x)$ on the h-side of $\overleftrightarrow{op}$.*

Proof. Suppose we have two such points, u and v. That is, both u and v are on the h-side of $\overleftrightarrow{op}$ and $u, v \in \mathcal{C}_o(a) \cap \mathcal{C}_p(x)$. Note that $\overline{op} \equiv \overline{op}$, $\overline{pu} \equiv \overline{px} \equiv \overline{pv}$, and $\overline{uo} \equiv \overline{ao} \equiv \overline{vo}$. By the SSS Theorem, we have $\triangle uop \equiv \triangle vop$. In particular, we have $\angle uop \equiv \angle vop$ and $\angle upo \equiv \angle vpo$. Now by AC7, we have $v \in \overrightarrow{ou} \subseteq \overleftrightarrow{ou}$ and $u \in \overrightarrow{pv} \subseteq \overleftrightarrow{pv}$. That is, u and v are points in the intersection of the lines $\overleftrightarrow{ou}$ and $\overleftrightarrow{pv}$. Since o and p are distinct, these lines must be distinct, and so they intersect at a unique point. Hence $u = v$. $\square$

Proposition 4 (SAS Theorem)**.** *If two triangles can be labeled such that two corresponding sides, and the angles between, are congruent, then the triangles are congruent. More precisely, let a, b, and c be distinct points, and x, y, and z be distinct points. If $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\angle abc \equiv \angle xyz$, then $\triangle abc \equiv \triangle xyz$.*

Construction 5 (equilateral triangle with a given side)**.** *Given distinct points x and y, there exist points z_1 and z_2, on opposite sides of $\overleftrightarrow{xy}$, such that $\triangle xyz_1$ and $\triangle xyz_2$ are equilateral. In fact, we have $\triangle xyz_1 \equiv \triangle xyz_2$.*

Proof. Consider the line $\overleftrightarrow{xy}$. By the Interpolation property, there exists a point u such that $[uxy]$. By the Circle Separation property, there is a point $w \in \mathcal{C}_y(x) \cap \overrightarrow{xu}$. Note in particular that $[wxy]$, and hence w is exterior to the circle $\mathcal{C}_y(x)$. Moreover, w is on $\mathcal{C}_x(y)$. Now y is also on $\mathcal{C}_x(y)$, and by definition, y is interior to $\mathcal{C}_y(x)$. By the Circle Cut property, there exist two points in $\mathcal{C}_x(y) \cap \mathcal{C}_y(x)$, say z_1 and z_2, which must be on opposite sides of $\overleftrightarrow{xy}$ by the uniqueness of circle cuts. Now $\overline{xz_1} \equiv \overline{xy} \equiv \overline{yz_1}$ and $\overline{xz_2} \equiv \overline{xy} \equiv \overline{yz_2}$ by the definition of circles, so that $\triangle xyz_1$ and $\triangle xyz_2$ are equilateral by definition. Moreover, $\triangle xyz_1 \equiv \triangle xyz_2$ by the transitivity of segment congruence and the SSS Theorem. $\square$