Congruence

Definition 1 (Segment Congruence)**.** *Let $\mathcal{P}$ be an ordered geometry, and suppose we have an equivalence relation on pairs of points, denoted $\cong_s$. We call $\cong_s$ a* segment congruence *if the following properties are satisfied.*

SC1. $(x, y) \cong_s (x, y)$ *for all points* x *and* y.

SC2. *If* $(x, y) \cong_s (z, w)$ *then* $(z, w) \cong_s (x, y)$ *for all points* x, y, z, *and* w.

SC3. *If* $(x, y) \cong_s (z, w)$ *and* $(z, w) \cong_s (u, v)$, *then* $(x, y) \cong_s (u, v)$ *for all* x, y, z, w, u, *and* v.

SC4. $(x, x) \cong_s (y, y)$ *for all points* x *and* y.

SC5. $(x, y) \cong_s (y, x)$ *for all points* x *and* y.

SC6. *If* $z \in \overrightarrow{xy}$ *such that* $(x, z) \cong_s (x, y)$, *then* $z = y$.

In this case, $\cong_s$ *is and equivalence relation on the set of* segments *in* $\mathcal{P}$, *and we write* $\overline{xy} \equiv \overline{ab}$ *to mean* $(x, y) \cong_s (a, b)$.

The first three properties ensure that $\cong_s$ is an equivalence relation; the fourth handles the "trivial" case, the fifth makes $\cong_s$ well-defined on segments, and the sixth ensures that $\cong_s$ differentiates between segments on the same ray which share an endpoint.

Examples

$\mathbb{R}^2$ Given points A, B, X, and Y in the Cartesian plane, we say that $\overline{AB} \equiv \overline{XY}$ if $(B - A) \cdot (B - A) = (Y - X) \cdot (Y - X)$, where $\cdot$ is the usual dot product of vectors. It is straightforward to show that this is a segment congruence.

Angle Congruence

Definition 2 (Angle Congruence)**.** *Let $\mathcal{P}$ be an ordered geometry, and suppose we have an equivalence relation on triples of points, denoted $\cong_a$. We call $\cong_a$ an* angle congruence *if the following properties are satisfied.*

AC1. $(a, o, b) \cong_a (a, o, b)$ *for all points* a, o, *and* b.

AC2. *If* $(a, o, b) \cong_a (x, p, y)$, *then* $(x, p, y) \cong_a (a, o, b)$ *for all points* a, o, b, x, p, *and* y.

AC3. *If* $(a, o, b) \cong_a (x, p, y)$ *and* $(x, p, y) \cong_a (h, q, k)$, *then* $(a, o, b) \cong_a (h, q, k)$ *for all points* a, o, b, x, p, y, h, q, *and* k.

AC4. *If* $[xyz]$ *and* $[abc]$, *then* $(x, y, z) \cong_a (a, b, c)$ *and* $(y, x, z) \cong_a (b, a, c)$, *and* $(x, y, z) \ncong_a (y, x, z)$.

AC5. *If $x \in \overrightarrow{oa}$ and $y \in \overrightarrow{ob}$ and x, y, and o are distinct, then $(a, o, b) \cong_a (x, o, y)$.*

AC6. *$(a, o, b) \cong_a (b, o, a)$ and $(a, o, b) \cong_a (a, o, b)$ for all points a, o, and b.*

AC7. *If a, b, and o are noncollinear points and x is on the b-side of $\overleftrightarrow{oa}$ such that $(a, o, b) \cong_a (a, o, x)$, then $x \in \overrightarrow{ob}$.*

In this case, $\cong_a$ is an equivalence relation on the set of angles *in $\mathcal{P}$, and we write $\angle aob \equiv \angle xpy$ to mean $(x, o, y) \cong_a (x, p, y)$.*

Again, the first three properties make $\cong_a$ an equivalence, the fourth handles the trivial case, the fifth and sixth make $\cong_a$ well-defined on angles, and the seventh ensures that $\cong_a$ differentiates between angles on one half-plane which share a vertex.

Definition 3 (Triangle Congruence)**.** *Let a, b, and c be distinct points, and let x, y, and z be distinct points. We say that $\triangle abc$ is* congruent *to $\triangle xyz$, denoted $\triangle abc \equiv \triangle xyz$, if*

$$\overline{ab} \equiv \overline{xy}, \quad \overline{bc} \equiv \overline{yz}, \quad \text{and} \quad \overline{ca} \equiv \overline{zx}$$

and

$$\angle abc \equiv \angle xyz, \quad \angle bca \equiv \angle yzx, \quad \text{and} \quad \angle cab \equiv \angle zxy.$$

Proposition 1.

1. *$\triangle abc \equiv \triangle abc$.*
2. *If $\triangle abc \equiv \triangle xyz$, then $\triangle xyz \equiv \triangle abc$.*
3. *If $\triangle abc \equiv \triangle xyz$ and $\triangle xyz \equiv \triangle hk\ell$, then $\triangle abc \equiv \triangle hk\ell$.*
4. *If $\triangle abc \equiv \triangle xyz$, then $\triangle bca \equiv \triangle yzx$.*

Definition 4. *Let a, b, and c be distinct points.*

- *We say that the triangle $\triangle abc$ is* equilateral *if $\overline{ab} \equiv \overline{bc} \equiv \overline{ca}$.*
- *We say that the triangle $\triangle abc$ is* isoceles *if two of its sides are congruent to each other.*

Supplementary and Right Angles

Definition 5 (Supplementary Angles)**.** *We say that angles $\angle aob$ and $\angle xpy$ are* supplementary *if there is a linear pair, $\angle uqv$ and $\angle vqw$, such that $\angle aob \equiv \angle uqv$ and $\angle xpy \equiv \angle vqw$. In this case we say that $\angle xpy$ is a* supplement *of $\angle aob$.*

Proposition 2. *Let $\mathcal{P}$ be an ordered geometry with an angle congruence.*

1. *If two angles form a linear pair, then they are supplementary.*
2. *Every angle has a supplement.*

Definition 6. *An angle is called* right *if it is supplementary to itself.*