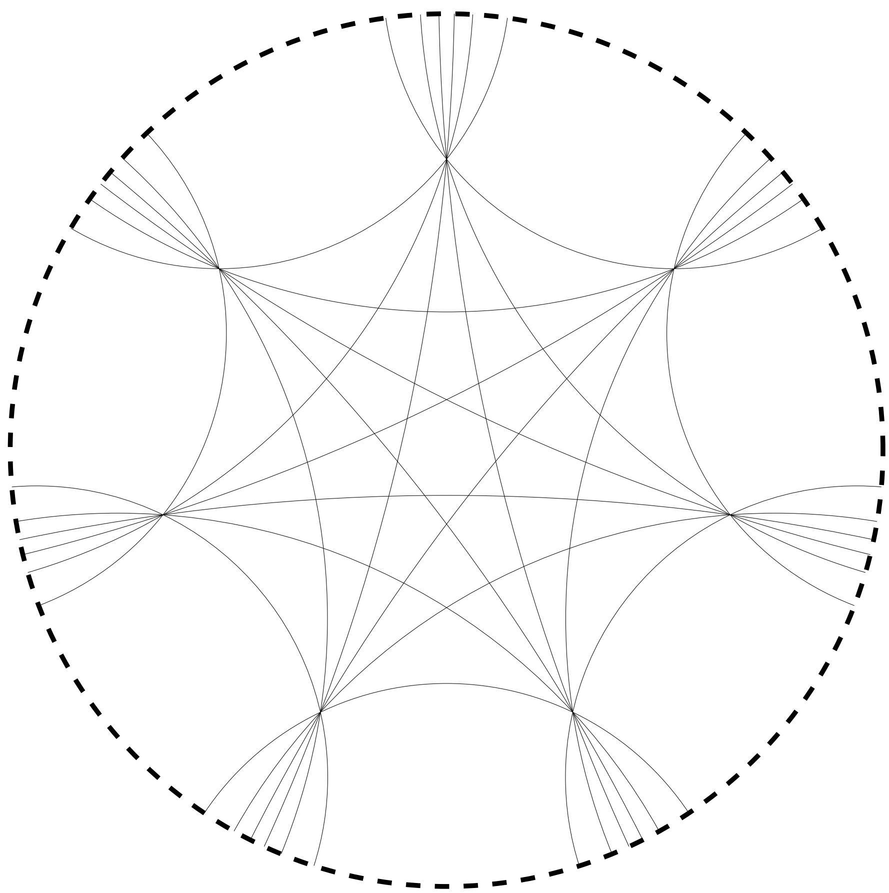
Elementary Geometrese

Nathan Bloomfield



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Incidence

1 Models and Theories

One of the goals of this class is to explore the classical theory of geometry as laid out in the famous *Elements of Geometry* by the Greek mathematician Euclid. Before we begin, though, a few words about the distinction between a *theory* and a *model* are in order.

- A **theory** consists of one or more *undefined terms*, which are used in one or more *axioms* or *definitions*, which are then the basis for a list of logical deductions called *theorems*. This may be how you think about geometry as you learned it in high school.
- A **model** is a concrete realization of a theory: a way to associate the undefined terms to "real" objects such that the axioms are satisfied. Models are where we perform calculations and draw pictures.

The correspondence between theories and models is not one-to-one (in either direction). There are theories which have many models, other theories with exactly one model, and yet other theories which have no models at all. Conversely, a given concrete object is likely a model of many different theories, though not all of them will be interesting.

The difference between theory and model turns out to have significant practical applications. In the real world we compute with concrete objects – things like numbers and sets. This is useful, but concrete objects have a tendency to get very messy very quickly. However if some aspects of a concrete object are a model for some theory, we can "throw away" unimportant details and compute more easily at an abstract level. For instance many important theorems about matrices are difficult and tedious to prove if we think of matrices as arrays of numbers but become simple if we think of matrices as linear transformations.

An example of a theory which you may have seen before is Euclid's postulates for geometry. These are a small number of statements which Euclid took to be obviously true (*axioms* in modern lingo) such as "two points determine a line" and so on. Euclid developed this theory of geometry in a book, called *The Elements*, which went on to become a standard mathematical textbook for many centuries.

The development and proliferation of Euclid's geometry predates the recognition of the need to carefully distinguish between a theory and its models, and early work did not make this distinction. Euclid seems to have written under the assumption that the universe comes equipped with exactly one geometry – that his theory has only one model. Unfortunately for us, this early confusion led to some problems. First, it turns out that there are many models of geometry, some of which are very strange. We will explore several models of geometry in this text.

The second problem we inherit from Euclid is more serious. Because he conflated his *theory* of geometry with only one particular *model*, and this confusion was not cleared up until much later, and because his book was so influential, Euclid left us with a language problem. The basic terms of geometry – point, line, circle, segment, angle – have multiple meanings. There is the meaning inside Euclid's *model* of geometry, which corresponds mostly to the bits of geometry we use in college algebra and calculus. But these words have another, more abstract meaning inside Euclid's *theory* of geometry, and when these terms are used inside other models we can easily get confused. We will look at models of geometry where lines are circles, where points are lines, and where circles are ellipses.

Theories as Languages

We can think of a theory as a kind of abstract *language*. The undefined terms are the words – nouns, verbs, and so on – while the axioms are the grammar, specifying how the words can be put together into meaningful phrases.

Unlike a natual language, logical theories are very well suited to implementation in software. So, for example, a valid "sentence" expressed in a geometric theory-language can be turned into a drawing by machine. This idea is not new, but it is very powerful.

$*$ $*$ Exercises $*$ $*$

1.1. Let $\alpha, \beta \in \mathbb{R}$ and $t \in (0,1)$.

(i) If $\alpha > 0$ and $\beta > 0$, then $\alpha + t(\beta - \alpha) > 0$.

(ii) If $\alpha > 0$ and $\beta > 0$, then $\alpha + t(\beta - \alpha) < 0$.

1.2. Let α, β, and γ be in $\mathbb{R}$ such that $\alpha < \beta < \gamma$. Show that either $\beta^2 < \alpha^2$ or $\beta^2 < \gamma^2$.

2 Incidence Geometry

Traditional plane geometry involves many different concepts, including *lines*, *angles*, *congruent*, and many others. In order to manage the complexity this entails, we will build up our geometries one piece at a time starting here with the idea of *collinearity*.

Def'n 2.1 (Incidence Geometry)**.** Let P be a set, whose elements we call *points*. A ternary relation $\langle *, *, * \rangle$ on P is called a *collinearity relation* if the following properties are satisfied.

IG1. If a, b, and c are points such that $\langle a, b, c\rangle$, then $\langle b, a, c\rangle$ and $\langle a, c, b\rangle$.

IG2. If a and b are points such that $a \neq b$, then $\langle a, b, b\rangle$.

IG3. $\langle a, a, a\rangle$ does not hold for any point a.

IG4. There exist distinct points a, b, and c such that $\langle a, b, c\rangle$ does not hold.

IG5. If a, b, u, and v are points such that $a \neq b$, $u \neq v$, $\langle a, b, u\rangle$, and $\langle a, b, v\rangle$, then $\langle a, u, v\rangle$.

If such a relation exists we say that $(P, \langle *, *, * \rangle)$ is an *incidence geometry*. In this case, when $\langle a, b, c\rangle$ we say that a, b, and c are *collinear*.

It is important to remember that the word "collinear" here is an undefined term, and we have to try very hard not to think of ordinary lines and points when using it. (This is part of the theory-model confusion we inherit from history.) The meaning of the word "collinear" is determined precisely by how it is used in the incidence geometry axioms, and only becomes concrete when we specify a particular model. In particular, it does not make sense to draw pictures of the points in an arbitrary incidence geometry!

Although "collinear" is an abstract, undefined term, we'd like for it to behave much like our intuition expects. We want this undefined term to *formalize* our intuition about collinearity. To that end, note that collinearity satisfies some additional basic properties.

Prop'n 2.2. Let P be an incidence geometry. Then we have the following for all points a, b, and c.

(i) If $\langle a, b, c\rangle$, then we also have $\langle b, c, a\rangle$, $\langle c, a, b\rangle$, and $\langle c, b, a\rangle$.

(ii) If $a \neq b$, then $\langle a, b, a\rangle$ and $\langle b, a, a\rangle$.

It may seem strange to define "collinear" before we define "line"; typically we think of points being collinear precisely when there is a unique line containing all of them. But we can just as easily define lines in terms of collinearity as follows.

Def'n 2.3 (Line)**.** Let P be an incidence geometry with distinct points a and b. We define the *line* through a and b to be the set

$$\overleftrightarrow{ab} = \{c \in P \mid \langle a, b, c\rangle\}.$$

If $c \in \overleftrightarrow{ab}$, we say that c *lies on* $\overleftrightarrow{ab}$.

That is, the line through a and b is precisely the set of points which are collinear with a and b. Remember: it is vital that we not think about drawings of points and lines here. In an arbitrary incidence geometry, "point" and "line" are just *words* which we assume have a particular relationship with one another. Thinking at this level of abstraction may seem unnecessarily difficult at first, but – and it is difficult to overstate this – the abstract way of thinking brings enormous power. Here are some basic properties of lines which can be derived from the properties of collinearity alone.

Prop'n 2.4. Let P be an incidence geometry. Then the following hold for all distinct points a and b in P.

(i) $a \in \overleftrightarrow{ab}$ and $b \in \overleftrightarrow{ab}$.

(ii) $\overleftrightarrow{ab} = \overleftrightarrow{ba}$.

(iii) If $c \in \overleftrightarrow{ab}$ and $c \neq a$, then $\overleftrightarrow{ac} = \overleftrightarrow{ab}$.

(iv) If $u, v \in \overleftrightarrow{ab}$ are distinct points, then $a, b \in \overleftrightarrow{uv}$.

Though we will eventually have several examples of incidence geometry, it is crucial when proving theorems that we not rely on any specific model. This is the power of abstraction: any theorem which depends only on properties common to *all* incidence geometries immediately holds in *any* incidence geometry. For example, consider the following theorem.

Prop'n 2.5 (Line Intersection)**.** Let P be an incidence geometry with lines ℓ_1 and ℓ_2. Then exactly one of the following holds.

(i) $\ell_1 = \ell_2$, and we say ℓ_1 and ℓ_2 are *coincident*,

(ii) $\ell_1 \cap \ell_2 = \varnothing$, and we say ℓ_1 and ℓ_2 are *disjoint*, or

(iii) $\ell_1 \cap \ell_2 = \{p\}$ for some point p, and we say ℓ_1 and ℓ_2 are *incident*.

In the first two cases (coincident or disjoint) we say that ℓ_1 and ℓ_2 are *parallel*.

Proof. Suppose $\ell_1 \cap \ell_2$ contains at least two points, say x and y. Then in fact $\ell_1 = \overleftrightarrow{xy} = \ell_2$. So if $\ell_1 \neq \ell_2$ then $\ell_1 \cap \ell_2$ contains either exactly one or zero points. ☐

This theorem holds in any model of incidence geometry. One problem: we don't have any models of incidence geometry yet! We'll fix this in the next section.

$*$ $*$ EXERCISES $*$ $*$

2.1. Let $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$ be in $\mathbb{R}^2$ such that $A \neq B$. Show that

$$\operatorname{det}\begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{bmatrix} = 0$$

if and only if $C = A + t(B - A)$ for some unique $t \in \mathbb{R}$. (Hint: Consider

$$\frac{c_x - a_x}{b_x - a_x} = \frac{c_y - a_y}{b_y - a_y}$$

.)

Def'n 2.6 (Collineation)**.** Suppose we have incidence geometries P and Q. A bijective mapping $\varphi : P \to Q$ is called a *collineation* if, for all points $x, y, z \in P$, whenever $\langle x, y, z\rangle$ in P, we also have $\langle \varphi(x), \varphi(y), \varphi(z)\rangle$ in Q.

2.2. Prove the following.

(i) If P is an incidence geometry, then the identity map $1 : P \to P$ is a collineation.

(ii) If $\varphi : P \to Q$ and $\psi : Q \to R$ are collineations, then $\psi \circ \varphi$ is a collineation.

(iii) If $\varphi : P \to Q$ is a collineation, then $\varphi^{-1} : Q \to P$ is a collineation.

3 $\mathbb{R}^2$ – The Cartesian Plane Model

Our definition of incidence geometry is a kind of **theory**, and a theory is only really useful if it has at least one **model**. So before we develop our theory of geometry further we'll spend the next few sections constructing some models. Remember that the words "point", "collinear", and "line" are context dependent – what they mean depends on the model – and so we may find ourselves using these words in unintuitive ways.

We'll start with a model of incidence geometry with which you are probably already familiar: the cartesian plane. To define this or any model it's enough to specify (1) what our points are and (2) what it means for three points to be collinear. At risk of giving away the punchline, in this model points are pairs of real numbers and lines are what you expect.

Prop'n 3.1 (Cartesian Plane)**.** Define a ternary relation on $\mathbb{R}^2$ as follows. Given $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$ in $\mathbb{R}^2$, we say that $\langle A, B, C\rangle$ if and only if A, B, and C are not all equal and

$$\mathsf{det}\begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{bmatrix} = 0.$$

This relation makes the set $\mathbb{R}^2$ an incidence geometry, which we call the *cartesian plane*.

Proof. IG1, IG2, and IG3 can be verified directly, and we can see that IG4 holds by considering the points $(0, 0)$, $(0, 1)$, and $(1, 0)$. So it suffices to show that IG5 holds. To this end suppose we have A, B, U, and V. Expanding and rearranging the known determinants, we have

$$(b_x - a_x)(u_y - a_y) = (b_y - a_y)(u_x - a_x)$$

and

$$(b_x - a_x)(v_y - a_y) = (b_y - a_y)(v_x - a_x).$$

If $a_x = b_x$, then we see that $u_x = a_x = v_x$ and thus $\langle A, U, V\rangle$. Similarly, if $a_y = b_y$, we see that $u_y = a_y = v_y$ and so $\langle A, U, V\rangle$. Finally, suppose we have $a_x \neq b_x$ and $a_y \neq b_y$. Now we have

$$\frac{u_y - a_y}{u_x - a_x} = \frac{b_y - a_y}{b_x - a_x} = \frac{v_y - a_y}{v_x - a_x}.$$

Equating the first and last of these expressions we see that $\langle A, U, V\rangle$. $\square$

This might seem like a strange way to define "collinearity", but it is easy to compute, and by expanding the determinant we can see that the lines in this geometry are precisely the solutions of linear equations.

Cor. 3.2 (Lines in $\mathbb{R}^2$)**.** Let $A = (a_x, a_y)$ and $B = (b_x, b_y)$ be distinct cartesian points. Then $\overleftrightarrow{AB}$ is the set of all points $X = (x, y)$ which satisfy the equation

$$(b_y - a_y)x - (b_x - a_x)y + a_y b_x - a_x b_y = 0.$$

That equation may look familiar as the standard form equation of a line. You might have noticed that our proof of **??** used nothing more than the arithmetic on $\mathbb{R}$. This means that the result still holds if we replace $\mathbb{R}$ by any object F where we have an arithmetic which behaves like that of $\mathbb{R}$. Such objects are called *fields*, and there are many examples, including the field $\mathbb{Q}$ of rational numbers and the field $\mathbb{C}$ of complex numbers. So we immediately get some additional models as well.

Cor. 3.3. The sets $\mathbb{Q}^2$ and $\mathbb{C}^2$ are incidence geometries, which we call the *rational plane* and the *complex plane*, respectively.

Note that lines in $\mathbb{Q}^2$ look much like lines in $\mathbb{R}^2$ except that they are filled with "holes"; any point on a line in $\mathbb{R}^2$ which has an irrational coordinate is not on the corresponding line in $\mathbb{Q}^2$. Lines in $\mathbb{C}^2$ are stranger still.

* * Exercises * *

3.1. Prove Corollary **??**.

3.2. **A parallel criterion in $\mathbb{R}^2$.** Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, and $D = (d_1, d_2)$ be points in the cartesian plane with $A \neq B$ and $C \neq D$. Show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel if and only if

$$\mathsf{det}\begin{bmatrix} b_1 - a_1 & d_1 - c_1 \\ b_2 - a_2 & d_2 - c_2 \end{bmatrix} = 0.$$

3.3. **A collinearity criterion in $\mathbb{R}^2$.** Let $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$ be points in $\mathbb{R}^2$ such that $A \neq C$ and $B \neq C$. Show that A, B, and C are collinear if and only if

$$\mathsf{det}\begin{bmatrix} a_x - c_x & b_x - c_x \\ a_y - c_y & b_y - c_y \end{bmatrix} = 0.$$

3.4. **The Unit Disc** Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$; these are points in the cartesian plane which are inside the unit circle. Given points A, B, and C in $\mathbb{D}$, we say they are collinear in $\mathbb{D}$ if they are collinear in $\mathbb{R}^2$. Verify that this relation makes $\mathbb{D}$ an incidence geometry, which we call the *unit disc*.

3.5. Fix $(\alpha, \beta) \in \mathbb{R}^2$, and define a mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x, y) = (x + \alpha, y + \beta)$. Show that φ is a collineation.

4 ℍ – The Hyperbolic Half Plane Model

We will now construct an odd model of incidence geometry. For this example, our set of points is $\mathbb{H} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y > 0\}$ – the "upper half plane". Suppose we have two such points $A = (a_x, a_y)$ and $B = (b_x, b_y)$. If $a_x \neq b_x$, then we define the *hyperbolic ideal point* of A and B is the number

$$H_{A,B} = \frac{b_x^2 + b_y^2 - a_x^2 - a_y^2}{2(b_x - a_x)}.$$

Intuitively, the ideal point is the x-coordinate of the point on the cartesian x-axis which is "equidistant" from A and B. More precisely, we have the following.

Lemma 4.1. Let $A = (a_x, a_y)$ and $B = (b_x, b_y)$ be points in $\mathbb{H}$ such that $a_x \neq b_x$. Then we have the following.

(i) The ideal point $H_{A,B}$ is the unique solution z of the following equation.
$$(a_x - z)^2 + a_y^2 = (b_x - z)^2 + b_y^2.$$

(ii) $H_{A,B} = H_{B,A}$.

Take care that $H_{A,B}$ only exists if A and B have distinct x-coordinates. We use the ideal point to define a collinearity relation on $\mathbb{H}$ as follows.

Prop'n 4.2. Define a ternary relation $\langle *, *, * \rangle$ on $\mathbb{H}$ as follows. Given points $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$ in $\mathbb{H}$, we say that $\langle A, B, C \rangle$ if one of the following holds.

(i) $a_x = b_x$ and $a_y = b_y$ and $C \neq A$.

(ii) $a_x = b_x$ and $a_y \neq b_y$ and $c_x = a_x$.

(iii) $a_x \neq b_x$ and $(c_x - H_{A,B})^2 + c_y^2 = (a_x - H_{A,B})^2 + a_y^2$.

This is a collinearity relation on the set $\mathbb{H}$, which we call the *hyperbolic half plane*.

Proof.

IG2. Suppose $A \neq B$. If $a_x = b_x$, then we must have $a_y \neq b_y$. Then $\langle A, B, B \rangle$ since $b_x = a_x$. If instead we have $a_x \neq b_x$, then $\langle A, B, B \rangle$ using **????**.

IG3. Note that $\langle A, A, A \rangle$ does not hold, because in this case we have $a_x = a_x$ and $a_y = a_y$ but $A = A$.

IG4. Let $A = (0, 1)$, $B = (0, 2)$, and $C = (1, 1)$. Now $\langle A, B, C \rangle$ does not hold, because we have $a_x = b_x$ and $a_y \neq b_y$ but $c_x \neq a_x$.

IG1. Suppose $\langle A, B, C\rangle$. First we will show that $\langle B, A, C\rangle$ by considering the three possibilities in the definition of $\langle *, *, *\rangle$. If $a_x = b_x$ and $a_y = b_y$ and $C \neq A$, then we have $b_x = a_x$ and $b_y = a_y$ and $C \neq B$, so that $\langle B, A, C\rangle$. If $a_x = b_x$ and $a_y \neq b_y$ and $c_x = a_x$, then we have $b_x = a_x$ and $b_y \neq a_y$ and $c_x = b_x$, so that $\langle B, A, C\rangle$. Suppose now that $a_x \neq b_x$ and

$$(c_x - H_{A,B})^2 + c_y^2 = (a_x - H_{A,B})^2 + a_y^2.$$

Now we have

$$\begin{aligned}(c_x - H_{B,A})^2 + c_y^2 &= (c_x - H_{A,B})^2 + c_y^2 \\ &= (a_x - H_{A,B})^2 + a_y^2 \\ &= (b_x - H_{A,B})^2 + b_y^2 \\ &= (b_x - H_{B,A})^2 + b_y^2\end{aligned}$$

so that $\langle B, A, C\rangle$.

Next we show that $\langle A, C, B\rangle$, again by considering the three possibilities. If $a_x = b_x$ and $a_y = b_y$ and $C \neq A$ then we have $\langle C, A, B\rangle$ by IG2, and in the previous paragraph we saw that we can swap the first two slots in $\langle *, *, *\rangle$, so that $\langle A, C, B\rangle$. Suppose $a_x = b_x$ and $a_y \neq b_y$ and $c_x = a_x$. Now if $c_y = a_y$, then we have $a_x = c_x$ and $a_y = c_y$ and $B \neq A$, so that $\langle A, C, B\rangle$. If instead we have $c_y \neq a_y$, then we have $a_x = c_x$ and $a_y \neq c_y$ and $b_x = a_x$, so that $\langle A, C, B\rangle$. Finally, suppose that $a_x \neq b_x$ and

$$(c_x - H_{A,B})^2 + c_y^2 = (a_x - H_{A,B})^2 + a_y^2.$$

If $a_x = c_x$, then we have $a_y = c_y$ since $a_y, c_y > 0$. Since $b_x \neq a_x$, we have $\langle A, C, B\rangle$. Suppose instead that $a_x \neq c_x$. By Lemma **??**, we have $H_{A,C} = H_{A,B}$. Thus

$$(b_x - H_{A,C})^2 + b_y^2 = (a_x - H_{A,C})^2 + a_y^2,$$

and thus $\langle A, C, B\rangle$.

IG5. Suppose we have points $A \neq B$ and $U \neq V$ such that $\langle A, B, U\rangle$ and $\langle A, B, V\rangle$. If $a_x = b_x$, then we have $u_x = a_x$ and $v_x = a_x$. Now $u_x = v_x$ and so $u_y \neq v_y$ and $a_x = u_x$, so that $\langle U, V, A\rangle$, and thus $\langle A, U, V\rangle$ by IG1. Suppose instead that $a_x \neq b_x$. Then we have

$$(u_x - H_{A,B})^2 + u_y^2 = (a_x - H_{A,B})^2 + a_y^2$$

and

$$(v_x - H_{A,B})^2 + v_y^2 = (a_x - H_{A,B})^2 + a_y^2.$$

If $a_x = u_x$, then $u_y = a_y$ and $V \neq A$, so that $\langle A, U, V\rangle$. If $a_x \neq u_x$ we have $H_{A,U} = H_{A,B}$, so that

$$(v_x - H_{A,U})^2 + v_y^2 = (a_x - H_{A,U})^2 + a_y^2$$

and thus $\langle A, U, V\rangle$. $\square$

Showing that $\mathbb{H}$ is an incidence geometry is a bit tedious, but it only has to be done once, and immediately all of our theorems about incidence geometries hold. A good question to ask is this: what are the lines in $\mathbb{H}$? It turns out that the lines in this model come in two flavors, which we will call Type I and Type II.

Cor. 4.3 (Lines in $\mathbb{H}$)**.** Let $A, B \in \mathbb{H}$ be distinct points.

(i) If $a_x = b_x$ we say $\overleftrightarrow{AB}$ is of *Type I*. In this case we have

$$\overleftrightarrow{AB} = \{(x, y) \in \mathbb{H} \mid x = a_x\}.$$

(ii) If $a_x \neq b_x$ we say $\overleftrightarrow{AB}$ is of *Type II*. In this case we have

$$\overleftrightarrow{AB} = \left\{(x, y) \in \mathbb{H} \mid (x - H_{A,B})^2 + y^2 = (a_x - H_{A,B})^2 + a_y^2\right\}.$$

Viewed as sets in the cartesian plane, the Type I lines in $\mathbb{H}$ are vertical half lines, and the Type II lines in $\mathbb{H}$ are semicircles centered on the x-axis. In fact the cartesian center of a Type II line is the point $(H_{A,B}, 0)$.

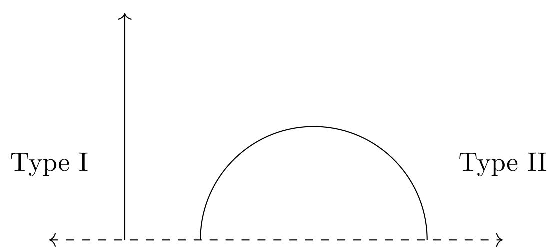


Figure 4.1: Lines in $\mathbb{H}$.

∗ ∗ Exercises ∗ ∗

4.1. Prove Lemma **??**.

4.2. **A parallel criterion in** $\mathbb{H}$ Suppose A, B, C, and D are points in $\mathbb{H}$ such that $a_x \neq b_x$ and $c_x \neq d_x$. Show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel if and only if $H_{A,B} = H_{C,D}$.

4.3. **A collinearity criterion in** $\mathbb{H}$ Suppose A, B, and C are points in $\mathbb{H}$ such that a_x, b_x, and c_x are distinct. Show that $\langle A, B, C\rangle$ if and only if $H_{A,C} = H_{A,B}$.

4.4. Let $A, B, U, V \in \mathbb{H}$ such that $A \neq B$ and $U \neq V$, and suppose we have

$$(u_x - H_{A,B})^2 + u_y^2 < (v_x - H_{A,B})^2 + v_y^2.$$

Show that $u_x < v_x$ if and only if $H_{A,B} < H_{U,V}$.

5 $\mathbb{P}$ – The Poincare Disc Model

We will now construct an alternative collinearity relation on the unit disc. For this example, our set of points is $\mathbb{P} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Given points $A = (a_x, a_y)$ and $B = (b_x, b_y)$ in $\mathbb{P}$, we define the abbreviation

$$D_{A,B} = \mathsf{det}\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}.$$

If $D_{A,B} \neq 0$, we define the *poincare ideal point* of A and B to be

$$P_{A,B} = \frac{1}{2}\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}^{-1}\begin{bmatrix} a_x^2 + a_y^2 + 1 \\ b_x^2 + b_y^2 + 1 \end{bmatrix}.$$

Lemma 5.1. Let $A, B \in \mathbb{P}$ such that $D_{A,B} \neq 0$. Then we have the following.

(i) $P_{B,A} = P_{A,B}$.

(ii) The ideal point $P_{A,B}$ is the unique solution (x, y) to the following system of equations.

$$\begin{cases} x^2 + y^2 = (a_x - x)^2 + (a_y - y)^2 + 1 \\ x^2 + y^2 = (b_x - x)^2 + (b_y - y)^2 + 1 \end{cases}$$

We can use the poincare ideal point to define a collinearity relation on $\mathbb{P}$ as follows.

Prop'n 5.2. Define a ternary relation on $\mathbb{P}$ as follows. Given $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$ in $\mathbb{P}$, we say that $\langle A, B, C\rangle$ if one of the following holds.

(i) Exactly two of A, B, and C are equal.

(ii) A, B, and C are distinct and $D_{A,B} = D_{A,C} = D_{B,C} = 0$.

(iii) A, B, and C are distinct and $D_{A,B}$, $D_{A,C}$, and $D_{B,C}$ are nonzero and $P_{A,B} = P_{A,C} = P_{B,C}$.

This is a collinearity relation on the set $\mathbb{P}$, which we call the *poincare disc*.

Proof.

IG2. Let $A, B \in \mathbb{P}$ be distinct. Then $\langle A, B, B\rangle$ holds by definition.

IG3. Note that $\langle A, A, A\rangle$ does not hold for any A by definition.

IG4. Let $A = (0,0)$, $B = (1,0)$, and $C = (0,1)$. Now A, B, and C are distinct, but we have $D_{A,B} = 0$ $D_{B,C} = 1 \neq 0$. So $\langle A, B, C\rangle$ does not hold.

IG1. Suppose we have $\langle A, B, C\rangle$. If exactly two of A, B, and C are equal, then we also have $\langle B, A, C\rangle$ since exactly two of B, A, and C are equal, and $\langle A, C, B\rangle$ since exactly two of A, C, and B are equal. Suppose then that A, B, and C are distinct. Certainly then B, A, and C are distinct, as are A, C, and B. If $D_{A,B} = D_{A,C} = D_{B,C} = 0$, then $D_{B,A} = D_{B,C} = D_{A,C} = 0$, so that $\langle B, A, C\rangle$, and $D_{A,C} = D_{C,B} = D_{C,B} = 0$ so that $\langle A, C, B\rangle$. Finally, suppose we have $D_{A,B}$, $D_{A,C}$, and $D_{B,C}$ nonzero and $P_{A,B} = P_{A,C} = P_{B,C}$. Then $D_{B,A}$, $D_{B,C}$, and $D_{A,C}$ are nonzero and $P_{B,A} = P_{B,C} = P_{A,C}$, so that $\langle B, A, C\rangle$, and likewise $D_{A,C}$, $D_{A,B}$, and $D_{C,B}$ are nonzero and $P_{A,C} = P_{A,B} = P_{C,B}$, so that $\langle A, C, B\rangle$ as claimed.

IG5. Suppose we have $A \neq B$ and $U \neq V$ such that $\langle A, B, U\rangle$ and $\langle A, B, V\rangle$. If $U = B$, then we have $\langle A, U, V\rangle$; similarly, if $V = B$ then $\langle A, V, U\rangle$, so that by IG1, $\langle A, U, V\rangle$. Suppose then that $U \neq B$ and $V \neq B$. If $U = A$, then $\langle A, U, V\rangle$ by definition; similarly, if $V = A$ then $\langle A, U, V\rangle$. Suppose then that $U \neq A$ and $V \neq A$. Now A, B, U, and V are all distinct. There are two possibilities for $D_{A,B}$: either zero or nonzero.

If $D_{A,B} = 0$, then in fact we have $D_{A,U} = D_{B,U} = 0$ and $D_{A,V} = D_{B,V} = 0$. Now since $A \neq B$, one of A or B must be nonzero; suppose without loss of generality it is A. Then either a_x or a_y must be nonzero; suppose it is a_x – the argument for a_y is similar. We have $a_x u_y = a_y u_x$, so that $v_x a_x u_y = v_x a_y u_x$, and thus $v_x a_x u_y = v_y a_x u_x$. Since $a_x \neq 0$, we have $v_x u_y = v_y u_x$, so that $D_{U,V} = 0$. Thus $\langle A, U, V\rangle$ as needed.

Suppose instead that $D_{A,B} \neq 0$. Then $D_{A,U}$, $D_{B,U}$, $D_{A,V}$, and $D_{B,V}$ are all nonzero, and we have $P_{A,B} = P_{A,U} = P_{B,U} = P_{A,V} = P_{B,V}$. Suppose first that $D_{U,V} = 0$. In this case, we have $V = kU$ for some constant k, since $U \neq 0$. Now we have

$$(u_x - x)^2 + (u_y - y)^2 = (ku_x - x)^2 + (ku_y - y)^2.$$

Expanding this equation and supposing $k \neq 1$ we solve for k as

$$k = 2\frac{u_x x + u_y y}{u_x^2 + u_y^2} - 1.$$

But note that $2(u_x x + u_y y) = u_x^2 + u_y^2 + 1$, so that in fact $k = 1/(u_x^2 + u_y^2)$. But now

$$v_x^2 + v_y^2 = 1/(u_x^2 + u_y^2) > 1,$$

a contradiction. So in fact we have $k = 1$, and thus $V = U$, a contradiction. So we must have $D_{U,V} \neq 0$. Now by **????**, we have $P_{U,V} = P_{A,U} = P_{A,V}$, and thus $\langle A, U, V\rangle$ as needed. $\square$

As with the hyperbolic half plane, showing that the poincare disc is an incidence geometry is a bit tedious. But this only has to be done once, and all our incidence geometry theorems immediately apply.

Prop'n 5.3 (Lines in $\mathbb{P}$)**.** Let $A, B \in \mathbb{P}$ be distinct points, and without loss of generality suppose $A \neq 0$.

(i) If $D_{A,B} = 0$, we say that $\overleftrightarrow{AB}$ is of *Type I*. In this case we have

$$\overleftrightarrow{AB} = \left\{ (x, y) \in \mathbb{P} \mid \mathsf{det} \begin{bmatrix} a_x & a_y \\ x & y \end{bmatrix} = 0 \right\}$$

(ii) If $D_{A,B} \neq 0$, we say that $\overleftrightarrow{AB}$ is of *Type II*. In this case we have

$$\overleftrightarrow{AB} = \left\{ (x, y) \in \mathbb{P} \mid (x - h)^2 + (y - k)^2 = t \right\},$$

where $(h, k) = P_{A,B}$ and $t = (a_x - h)^2 + (a_y - k)^2$.

∗ ∗ Exercises ∗ ∗

5.1. Consider the map φ on $\mathbb{H}$ given by

$$\varphi(x, y) = \left(\frac{2x}{x^2 + (y+1)^2}, \frac{x^2 + y^2 - 1}{x^2 + (y+1)^2} \right).$$

(i) Show that $\varphi : \mathbb{H} \to \mathbb{P}$.

(ii) Show that φ is bijective.

(iii) Show that φ is a collineation.

6 $\mathbb{K}$ – The Klein Disc Model

We define the *klein ideal point* of A and B to be

$$K_{A,B} = \frac{1}{2}\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}^{-1}\begin{bmatrix} a_x^2 + a_y^2 - 1 \\ b_x^2 + b_y^2 - 1 \end{bmatrix}.$$

7 $\mathbb{E}$ – The Euclidean Disc Model

8 Parallel Lines

Recall that two lines in an incidence geometry are called *parallel* if they do not meet at a single point. The following question about parallel lines turns out to be interesting.

Question

Suppose we have a line ℓ and a point p in an incidence geometry. How many lines exist which pass through p and are parallel to ℓ?

Our intuition says that the answer to this question is clearly 1, and Euclid agreed. It turns out that the parallel lines in our models so far behave in some very different ways.

Parallels in $\mathbb{R}^2$

In Exercise **??** we found a nice way to characterize whether the lines determined by four cartesian points are parallel: if $A = (a_x, a_y)$, $B = (b_x, b_y)$, $C = (c_x, c_y)$, and $D = (d_x, d_y)$ are points in $\mathbb{R}^2$ with $A \neq B$ and $C \neq D$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if

$$\mathsf{det}\begin{bmatrix} b_x - a_x & d_x - c_x \\ b_y - a_y & d_y - c_y \end{bmatrix} = 0.$$

With this, we can show the following.

Prop'n 8.1. If $\ell = \overleftrightarrow{AB}$ is a line and $C \notin \ell$ a point in $\mathbb{R}^2$, then there is exactly one line passing through C which is parallel to ℓ.

Proof. To see existence, let $D = C + B - A$. Now $D \neq C$ since $B \neq A$. Moreover, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ are parallel since

$$\mathsf{det}\begin{bmatrix} b_x - a_x & c_x + b_x - a_x - c_x \\ b_y - a_y & c_y + b_y - a_y - c_y \end{bmatrix} = \mathsf{det}\begin{bmatrix} b_x - a_x & b_x - a_x \\ b_y - a_y & b_y - a_y \end{bmatrix} = 0.$$

To see uniqueness, suppose $X = (x, y)$ is a point (different from C) such that $\overleftrightarrow{CX}$ is parallel to $\overleftrightarrow{AB}$. Then

$$0 = \mathsf{det}\begin{bmatrix} x - c_x & b_x - a_x \\ y - c_y & b_y - c_y \end{bmatrix} = \mathsf{det}\begin{bmatrix} x - c_x & c_x + b_x - a_x - c_x \\ y - c_y & c_y + b_y - a_y - c_y \end{bmatrix}.$$

So X, C, and D are collinear, and thus $\overleftrightarrow{CX} = \overleftrightarrow{CD}$. $\square$

This proof remains valid in the rational plane.

Parallels in $\mathbb{D}$

Suppose ℓ is a line and x a point in the unit disc. There are *infinitely many* lines passing through x which are parallel to ℓ. To see why, remember that ℓ is contained in a line $\ell_{A,B}$ in the Cartesian Plane. Choose any point y on this Cartesian line which is not in the unit disk, and let $\ell_{x,y}$ be the Cartesian line generated by x and y. Now $\ell' = \ell_{x,y} \cap \mathbb{D}$ is parallel to ℓ.

Parallels in $\mathbb{A}$

It is not too difficult to see that there are *no* pairs of parallel lines in the antipodal sphere; any two different great circles have to intersect.

Parallels in the Fano plane

In the Fano Plane, no two lines are parallel. In particular, if ℓ is a line and $x \notin \ell$ a point, there are *no* lines passing through x which are parallel to ℓ.

Considering these examples, there seem to be (at least) three qualitatively different possibilities for the answer to our Question about parallel lines. This observation is what motivates the following definition.

> **Def'n 8.2** (The Parallel Postulates)**.** We say that an incidence geometry P is
>
> **Elliptic** if there are *no* lines passing through x and parallel to ℓ, for all lines ℓ and points $x \notin \ell$.
>
> **Euclidean** if there is *exactly one* line passing through x and parallel to ℓ, for all lines ℓ and points $x \notin \ell$.
>
> **Hyperbolic** if there are *infinitely many* lines passing through x and parallel to ℓ, for all lines ℓ and points $x \notin \ell$.

With this definition, $\mathbb{R}^2$ and $\mathbb{Q}^2$ are Euclidean, $\mathcal{F}$ and $\mathbb{A}$ are Elliptic, and $\mathbb{D}$ and $\mathbb{H}$ are Hyperbolic. It is important to note that a given incidence geometry need not satisfy **any** of these properties! We will see a strange example of this in the exercises.

Transitivity of Parallelism

The kinds of "geometries" that arise from our three different Parallel Postulates will be different – perhaps drastically so – as illustrated by the following result.

> **Prop'n 8.3.** Suppose P is a Euclidean incidence geometry, with lines ℓ_1, ℓ_2, and ℓ_3. If $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$. That is, the relation "is parallel to" is transitive.

Proof. If $\ell_1 \cap \ell_2 = \varnothing$, then $\ell_1 \parallel \ell_3$ by definition. Suppose instead that ℓ_1 and ℓ_3 have *at least one* point in common, say p. Since ℓ_1 is parallel to ℓ_2, note that $p \notin \ell_2$. Since $\mathcal{P}$ is Euclidean, there is exactly one line passing through p which is parallel to ℓ_2; call this line ℓ. But now ℓ_1 is a line parallel to ℓ_2 which passes through p, so that $\ell_1 = \ell$. Likewise, $\ell_3 = \ell$. Hence $\ell_1 = \ell_3$, and so $\ell_1 \parallel \ell_3$ as claimed. □

Note that in a Hyperbolic incidence geometry, this need not be the case. If we have two lines ℓ_1 and ℓ_3 which pass through a point p and are parallel to a given line ℓ_2, then ℓ_1 and ℓ_3 are *not* parallel. And in an Elliptic incidence geometry the transitivity of parallelism is irrelevant: there are no pairs of distinct parallel lines to begin with.

∗ ∗ Exercises ∗ ∗

Note that although all of our example incidence geometries happen to be either elliptic, euclidean, or hyperbolic, any given incidence geometry need not be in any of these classes. The next two exercises construct an example.

> **Def'n 8.4** (The Two-Pointed Line)**.** Let $P = \mathbb{R} \cup \{A, B\}$, where we think of A and B simply as symbols not belonging to $\mathbb{R}$. We define a ternary relation on P as follows. Given points $x, y, z \in P$, not all equal, we say that $\langle x, y, z \rangle$ if one of the following holds.
>
> (i) $\{x, y, z\} \subseteq \mathbb{R}$.
>
> (ii) $\{x, y, z\} = \{A, \zeta\}$ for some $\zeta \in \mathbb{R}$.
>
> (iii) $\{x, y, z\} = \{B, \zeta\}$ for some $\zeta \in \mathbb{R}$.
>
> (iv) $\{x, y, z\} = \{A, B\}$.
>
> We refer to this structure as the Two-Pointed Line.

8.1. Show that the Two-Pointed Line is an incidence geometry.

8.2. Show that the Two-Pointed Line is an incidence geometry which is neither elliptic, nor euclidean, nor hyperbolic as follows.

(i) Find a line ℓ and a point x in the Two-Pointed Line such that there is exactly one line passing through x and parallel to ℓ.

(ii) Find a line ℓ and a point x in the Two-Pointed Line such that there are infinitely many lines passing through x and parallel to ℓ.

— II —

Order

9 Betweenness

In an incidence geometry we have the ability to detect whether three given points are collinear. However an arbitrary incidence geometry has no notion of "order" for the points on a given line, as we might intuitively expect. For instance, the points $(0,0)$, $(1,1)$, and $(2,2)$ are collinear in $\mathbb{R}^2$ and we think of $(1,1)$ as being "between" the other two. But in the Fano plane, does it make sense to order the points on a line? Presently we introduce another piece of technology to an incidence geometry which will allow us to formalize the concept of "betweenness".

Def'n 9.1 (Betweenness)**.** Let P be an incidence geometry. We say that a ternary relation $[***]$ on P is a *betweenness relation* if the following properties hold.

- B1. If $[xyz]$, then x, y, and z are distinct and $\langle x, y, z\rangle$.
- B2. If $[xzy]$, then $[yzx]$.
- B3. If x, y, and z are distinct points such that $\langle x, y, z\rangle$, then at least one of $[xyz]$, $[yzx]$, and $[zxy]$ is true.
- B4. If $[xyz]$ and $[xzw]$, then $[xyw]$ and $[yzw]$.
- B5. If $[xyz]$ and $[yzw]$, then $[xyw]$ and $[xzw]$.
- B6. **Interpolation Property:** If x and y are distinct points, then there exist points z_1, z_2, and z_3 such that $[z_1xy]$, $[xz_2y]$, and $[xyz_3]$.

If $[xyz]$, we say that z is *between* x and y. B1 says that "betweenness" is a refinement of "collinearity". B2 says that betweenness is symmetric (if we fix the middle point). B3 says that distinct collinear points must be in order somehow, outlawing the bizarre situation where points are collinear but not in order. B4 and B5 are a little more interesting; they are a kind of coherence condition for betweenness. Consider a situation like **??**. Axiom B4 says that if y is between x and z and z is between x and w, then y is between x and w and z is between y and w. That seems reasonable enough, and B5 says something similar.

As shorthand, if x, y, z, and w are distinct points, we will say $[xyzw]$ precisely when $[xyz]$, $[xyw]$, $[xzw]$, and $[yzw]$. More generally, if $x_1, \ldots, x_n$ are distinct points, then $[x_1x_2\ldots x_n]$ means that $[x_ix_jx_k]$ for all triples (i,j,k) with $1 \le i < j < k \le n$. The next definition will probably look familiar.

Def'n 9.2 (Segment, Ray)**.** Let x and y be distinct points in an incidence geometry P with a betweenness relation.

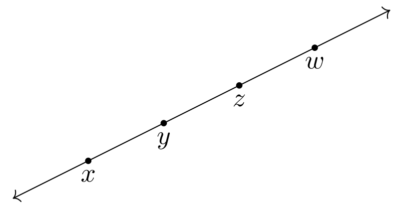


Figure 9.1: The coherence conditions B4 and B5.

(i) The set

$$\overline{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy]\}$$

is called the *segment* with *endpoints* x and y. If $z \in \overline{xy}$ and $z \neq x$ and $z \neq y$, we say that z is *interior to* $\overline{xy}$.

(ii) The set

$$\overrightarrow{xy} = \{z \in P \mid z = x \text{ or } z = y \text{ or } [xzy] \text{ or } [xyz]\}$$

is called the *ray* with *vertex x toward y*.

The following properties of segments and rays essentially fall out of the definition.

Prop'n 9.3. If P is an incidence geometry and $[\cdots]$ a betweenness relation on P, then the following hold.

1. $\overline{xy} = \overline{yx}$ for all distinct points x and y.
2. $\overline{xy} \subseteq \overrightarrow{xy} \subseteq \overleftrightarrow{xy}$ for all distinct points x and y.
3. If ℓ is a line and x and y distinct points, then $\overline{xy} \cap \ell$ is either $\overline{xy}$, $\varnothing$, or $\{p\}$ for some point p.
4. $\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$ for all distinct points x and y.

Prop'n 9.4 (Line Decomposition)**.** Suppose P is an incidence geometry with a betweenness relation, and let $x, y \in P$ be distinct points. Then

$$\overleftrightarrow{xy} = \{z \mid z = x \text{ or } z = y \text{ or } [zxy] \text{ or } [xzy] \text{ or } [xyz]\}.$$

Proposition **??** is a useful technical result: if we know that a given point z lies on a line $\ell_{x,y}$, then there are five possibilities. The next result is useful for the same reason.

Prop'n 9.5. Let P be an incidence geometry with a betweenness relation and suppose x, y, z, and w are points. If $[xzy]$ and $[xwy]$, then either $[xzw]$ or $[xwz]$ or $z = w$.

We can now define *convexity* in terms of betweenness as follows.

Def'n 9.6 (Convexity)**.** Let P be an incidence geometry with a betweenness relation. A non empty set S of points in P is called *convex* if it is closed under betweenness in the following sense: if $x, y \in S$ and $[xzy]$, then $z \in S$.

* * Exercises * *

9.1. Suppose P is an incidence geometry with a betweenness relation. If x, y, and z are distinct points such that $[xyz]$, then the following hold.

(i) $\overline{xy} \cup \overline{yz} = \overline{xz}$.

(ii) $\overline{xy} \cap \overline{yz} = \{y\}$.

(iii) $\overrightarrow{yx} \cap \overrightarrow{yz} = \{y\}$.

(iv) $\overrightarrow{xy} = \overrightarrow{xz}$.

(v) $\overrightarrow{xy} \cap \overrightarrow{yx} = \overline{xy}$.

(vi) $\overrightarrow{yx} \cup \overrightarrow{yz} = \overleftrightarrow{xz}$.

9.2. Let P be an incidence geometry with a betweenness relation, and let $S \subseteq P$. Show that S is convex if and only if $\overline{xy} \subseteq S$ for all distinct points $x, y \in S$.

9.3. Let P be an incidence geometry with a betweenness relation and $x, y \in P$ distinct points. Show that the following sets are convex.

(i) $\overleftrightarrow{xy}$

(ii) $\overline{xy}$

(iii) $\overrightarrow{xy}$

9.4. Let P be an incidence geometry with a betweenness relation. Show that every line in P contains infinitely many points.

10 Ordered Geometry

The existence of a betweenness relation on an incidence geometry says something very strong. Combined with an additional property – the Line Separation property – such geometries are called *ordered*.

Def'n 10.1 (Ordered Geometry)**.** Let P be an incidence geometry with a betweenness relation [$* * *$]. We say that P is an *ordered geometry* if it satisfies the following additional *Line Separation Property*.

LS. For every line ℓ there are two sets, $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$, which satisfy the following properties.

(i) $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$ are not empty.

(ii) ℓ, $\mathcal{H}_{1,\ell}$, and $\mathcal{H}_{2,\ell}$ partition P.

(iii) $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$ are convex.

(iv) If $x \in \mathcal{H}_{1,\ell}$ and $y \in \mathcal{H}_{2,\ell}$, then $\overline{xy} \cap \ell = \{p\}$ for some point p.

The sets $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$ are called *halfplanes*.

The Line Separation property essentially states that every line divides the plane into two "separate" pieces – we might call these pieces the *sides* of the line. Given a line ℓ and points $x, y \notin \ell$, we say that x and y are on the *same side* of ℓ if they are both in the same halfplane bounded by ℓ, and otherwise they are on *opposite sides*. See **??** for a picture. (But remember that a given ordered geometry might look very different!)

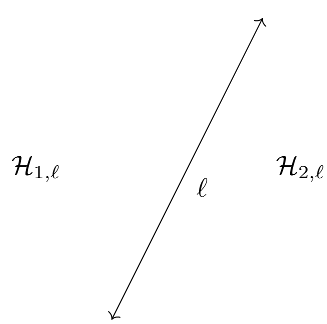


Figure 10.1: The halfplanes bounded by a line.

Def'n 10.2 (Triangle)**.** Let P be an incidence geometry, and let x, y, and z be distinct points. Then the set

$$\triangle xyz = \overline{xy} \cup \overline{yz} \cup \overline{zx}$$

is called the *triangle* with *vertices* x, y, and z. The segments $\overline{xy}$, $\overline{yz}$, and $\overline{zx}$ are called the *sides* of the triangle.

The next result seems intuitive, but must be proven; it states that if a line "enters" a triangle through one side and does not contain any of the triangle's vertices, then it must "exit" the triangle through one of the other sides. This is called *Pasch's Axiom* for historical reasons.

Prop'n 10.3 (Pasch's Axiom)**.** Let x, y, and z be distinct points in an ordered geometry, and let ℓ be a line such that $x, y, z \notin \ell$. Finally, suppose there is a point $w \in \ell$ such that $[xwy]$; that is, ℓ cuts the side $\overline{xy}$. Then precisely one of the following two things happens:

(i) ℓ cuts $\overline{yz}$ and does not cut $\overline{zx}$, or

(ii) ℓ cuts $\overline{zx}$ and does not cut $\overline{yz}$.

Proof. Since P is an ordered geometry, it satisfies the Plane Separation property. In particular, the points not on ℓ are partitioned into two convex, nonempty half-planes, $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$. Since x and y are not on ℓ, without loss of generality we have $x \in \mathcal{H}_{1,\ell}$ and $y \in \mathcal{H}_{2,\ell}$. Since $z \notin \ell$, there are two possibilities: either $z \in \mathcal{H}_{1,\ell}$ or $z \in \mathcal{H}_{2,\ell}$. In the first case, we see that ℓ cuts $\overline{yz}$ and does not cut $\overline{zx}$, and in the second case, ℓ cuts $\overline{zx}$ but not $\overline{yz}$. □

In other words, Pasch's Axiom states that if a line enters a triangle then it must also exit; see Figure **??**.

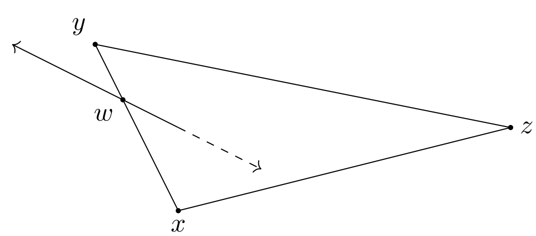


Figure 10.2: Pasch's Axiom

Lemma 10.4. Let ℓ be a line and $C \in \ell$ a point in an ordered geometry. Suppose A and B are points not on ℓ such that $[ABC]$. Then A and B are on the same side of ℓ.

Proof. Suppose otherwise that A and B are on opposite sides of ℓ. By the Line Separation property, and because A and B are not on ℓ, the segment $\overline{AB}$ cuts ℓ at a unique point D. That is, $D \in \ell$ and $[ADB]$. In particular, D and C

must be distinct since we have $[DBC]$. But note that $C, D \in \ell$, so $\overleftrightarrow{CD} = \ell$, and also $C, D \in \overleftrightarrow{AB}$, so that $\overleftrightarrow{CD} = \overleftrightarrow{AB}$. But then $\overleftrightarrow{AB} = \ell$, a contradiction. Thus A and B must be on the same side of ℓ. $\square$

$*$ $*$ Exercises $*$ $*$

11 Models of Ordered Geometry

In this section we will establish some models of ordered geometry. Remember that to show a given incidence geometry is also an ordered geometry, we need to specify (1) how to detect when one point is between two others and (2) how to determine when two points are on the same side of a line.

Betweenness in $\mathbb{D}$

The Unit Disc inherits a natural betweenness relation from the Cartesian Plane. Given points A, B, and C in the disc, say that $[ABC]$ if B is between A and C in $\mathbb{R}^2$. Similarly, if $A \neq B$ and ℓ is the line generated by A and B in RR^2, then the half-planes cut by $\overleftrightarrow{AB}$ in $\mathbb{D}$ are $H_1 \cap \mathbb{D}$ and $H_2 \cap \mathbb{D}$, where H_1 and H_2 are the half-planes cut by ℓ in $\mathbb{R}^2$.

Betweenness in the Fano Plane

Note that as a consequence of the Interpolation property, every line in an ordered geometry must contain infinitely many points. As a consequence, the Fano Plane cannot possibly be an ordered geometry – it has only 7 points.

12 Betweenness in $\mathbb{R}^2$

In this section we will establish that the cartesian plane is an ordered geometry. To do this, we need to specify (1) how to detect when one point is between two others, and (2) the halfplanes for each line.

Prop'n 12.1. Given points A, B, and C in $\mathbb{R}^2$, we say $[ACB]$ if $A \neq B$ and the equation $C = A + t(B - A)$ has a solution $t \in (0,1)$. Then this $[***]$ is a betweenness relation on $\mathbb{R}^2$.

Proof.

B1. Suppose $[ACB]$. Now $A \neq B$ by definition, and we have $C = A + t(B - A)$ for some $t \in (0,1)$. If $C = A$, then $(0,0) = t(B - A)$, so that $B = A$; a contradiction. If $C = B$, then $(0,0) = (t-1)(B - A)$, so that $B = A$; a contradiction. So A, B, and C are distinct. Now A, B, and C are collinear because

$$\det \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{bmatrix} = \det \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ a_x + t(b_x - a_x) & a_y + t(b_x - a_x) & 1 \end{bmatrix} = 0.$$

B2. If $C = A + t(B - A)$ where $t \in (0,1)$, then (rearranging) we also have $C = B + (1-t)(A - B)$ with $1 - t \in (0,1)$ as needed.

B3. Suppose A, B, and C are distinct such that $\langle A, B, C \rangle$. Using Exercise **??**, we have $C = A + t(B - A)$ for some real number t. Note that $t \neq 0$ and $t \neq 1$ since in the first case we would have $C = A$ and in the second, $C = B$. There are then three possibilities for t. If $t \in (0,1)$, then $[ACB]$ by definition. If $t > 1$, then "solving for B" we have $B = A + \frac{1}{t}(C - A)$, and since $1/t \in (0,1)$, we have $[ABC]$. If $t < 0$, then we have $A = C + \frac{-t}{1-t}(B - C)$, and since $\frac{-t}{1-t} \in (0,1)$, we have $[CAB]$.

B4. Suppose $[ABC]$ and $[ACD]$; say we have $B = A + t(C - A)$ and $C = A + s(D - A)$ where $s, t \in (0,1)$. Now $B = A + ts(D - A)$, and since $st \in (0,1)$, we have $[ABD]$. Similarly, we have $C = B + \frac{s - ts}{1 - ts}(D - B)$, and since $\frac{s-ts}{1-ts} \in (0,1)$, we have $[BCD]$ as needed.

B5. Suppose $[ABC]$ and $[BCD]$; say we have $B = t(C - A)$ and $C = B + s(D - B)$ where $s, t \in (0,1)$. Now $C = A + \frac{s}{1-t+st}(D - A)$, and we also have $\frac{s}{1-t+st} \in (0,1)$. (To see this, note that $0 < (1-s)(1-t)$ and rearrange to get $s < 1 - t + st$.) Thus $[BCD]$. Next note that $B = A + \frac{ts}{1-t+ts}(D - A)$, and since $\frac{ts}{1-t+ts} \in (0,1)$, we have $[ABD]$ as needed.

B6. Let A and B be distinct points. Given a real number t, let $C = A + t(B - A)$. If $t \in (0,1)$, then $[ACB]$ by definition. If $t > 1$, then $B = A + \frac{1}{t}(C - A)$ with $\frac{1}{t} \in (0,1)$ and we have $[ABC]$. If $t < 0$, then $A = C + \frac{-t}{1-t}(B - C)$ with $\frac{-t}{1-t} \in (0,1)$ and we have $[CAB]$. $\square$

Next, we'd like to show that $\mathbb{R}^2$ is an ordered geometry by showing that it has the Line Separation property. First, we need the following technical lemma about the intersection of a segment and a line in $\mathbb{R}^2$.

Lemma 12.2. Let $A = (a_x, a_y)$ and $B = (b_x, b_y)$ be distinct points in $\mathbb{R}^2$, and let $U = (u_1, u_2)$ and $V = (v_1, v_2)$ be distinct points not on $\overleftrightarrow{AB}$. Then $\overline{UV} \cap \overleftrightarrow{AB}$ consists of a single point if and only if

$$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ u_1 & u_2 & 1 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ v_1 & v_2 & 1 \end{bmatrix}$$

have opposite signs.

Proof. Note that $\overleftrightarrow{UV} \cap \overleftrightarrow{AB}$ contains exactly one point if and only if the equation $U + t(V - U) = A + u(B - A)$ has a unique solution (t, u). In fact, by comparing coordinates we can rewrite this equation in matrix form as

$$\begin{bmatrix} t \\ -u \end{bmatrix} = \begin{bmatrix} v_1 - u_1 & b_1 - a_1 \\ v_2 - u_2 & b_2 - a_2 \end{bmatrix}^{-1} \begin{bmatrix} a_1 - u_1 \\ a_2 - u_2 \end{bmatrix}.$$

(This matrix is invertible by Exercise **??**.) Comparing entries, we have

$$t = \frac{(b_2 - a_2)(a_1 - u_1) - (b_1 - a_1)(a_2 - u_2)}{(v_1 - u_1)(b_2 - a_2) - (v_2 - u_2)(b_1 - a_1)}.$$

Now by definition the unique point in $\overleftrightarrow{UV} \cap \overleftrightarrow{AB}$ is more specifically on the segment $\overline{UV}$ if and only if $t \in (0, 1)$.

There are now two possibilities, depending on whether the denominator of t is positive or negative. If the denominator of t is positive, we can see that $t \in (0, 1)$ if and only if

$$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ u_1 & u_2 & 1 \end{bmatrix} > 0 > \det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ v_1 & v_2 & 1 \end{bmatrix},$$

and if the denominator of t is negative, then $t \in (0, 1)$ if and only if

$$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ u_1 & u_2 & 1 \end{bmatrix} < 0 < \det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ v_1 & v_2 & 1 \end{bmatrix}$$

as needed. $\square$

We are now prepared to show the following.

Prop'n 12.3. For each line $\ell = \overleftrightarrow{AB}$ in $\mathbb{R}^2$, we define two halfplanes as follows.

$$\begin{aligned} \mathcal{H}_{1,\ell} &= \left\{ (x,y) \mid \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ x & y & 1 \end{bmatrix} > 0 \right\} \\ \mathcal{H}_{2,\ell} &= \left\{ (x,y) \mid \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ x & y & 1 \end{bmatrix} < 0 \right\}. \end{aligned}$$

With halfplanes defined this way for all lines, $\mathbb{R}^2$ satisfies the Line Separation Property and thus is an ordered geometry.

Proof. It is enough to show that the Line Separation property is satisfied. But first, we need to verify that our halfplanes are well-defined; that is, they do not depend on the choice of line generators. (@@@)

Certainly $\mathcal{H}_{1,\ell}$, $\mathcal{H}_{2,\ell}$, and ℓ partition $\mathbb{R}^2$ by the trichotomy property of $<$. To see that both halfplanes are nonempty we consider three cases: if $a_x = b_x$, then $(a_x+1, 0)$ and $(a_x-1, 0)$ are in opposite halfplanes; if $a_y = b_y$, then $(0, a_y+1)$ and $(0, a_y-1)$ are in opposite halfplanes; and if $a_x \neq b_x$ and $a_y \neq b_y$ then (a_x, b_y) and (b_x, a_y) are in opposite halfplanes. By Lemma **??**, if $U \in \mathcal{H}_{1,\ell}$ and $V \in \mathcal{H}_{2,\ell}$, then $\overline{UV} \cap \overleftrightarrow{AB}$ consists of a unique point. All that remains is to show that $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$ are convex.

To see that $\mathcal{H}_{1,\ell}$ is convex, let $U, V \in \mathcal{H}_{1,\ell}$. By definition, we have

$$\operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ u_x & u_y & 1 \end{bmatrix} > 0 \text{ and } \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ v_x & v_y & 1 \end{bmatrix} > 0.$$

Now suppose $[UWV]$; then again by definition we have $t \in (0,1)$ such that $W = U + t(V-U)$. Then

$$\begin{aligned} & \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ w_x & w_y & 1 \end{bmatrix} \\ = & \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ u_x + t(v_x - u_x) & u_y + t(v_y - u_y) & 1 + t(1-1) \end{bmatrix} \\ = & \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ u_x & u_y & 1 \end{bmatrix} + t \left(\operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ v_x & v_y & 1 \end{bmatrix} - \operatorname{det} \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ u_x & u_y & 1 \end{bmatrix} \right) \\ > & \ 0 \end{aligned}$$

using Exercise **??**, and because the determinant is multilinear. So we have $W \in \mathcal{H}_{1,\ell}$, and thus $\mathcal{H}_{1,\ell}$ is convex. A similar argument shows that $\mathcal{H}_{2,\ell}$ is convex. $\square$

The proofs of Propositions **??** and **??** remain valid if we replace $\mathbb{R}^2$ by $\mathbb{Q}^2$, so that the Rational Plane is also an ordered geometry. However we cannot replace $\mathbb{R}^2$ by $\mathbb{C}^2$, because the order relation $<$ does not make sense in the complex numbers.

13 Betweenness in $\mathbb{H}$

In this section we establish that the hyperbolic half plane is an ordered geometry.

Prop'n 13.1. Define a ternary relation $[***]$ on $\mathbb{H}$ as follows. Given $A = (a_x, a_y)$, $B = (b_x, b_y)$, and $C = (c_x, c_y)$, we say that $[ACB]$ if $\langle A, C, B\rangle$ and one of the following holds.

(i) $a_x = b_x$ and $\mathsf{min}(a_y, b_y) < c_y < \mathsf{max}(a_y, b_y)$.

(ii) $a_x \neq b_x$ and $\mathsf{min}(a_x, b_x) < c_x < \mathsf{max}(a_x, b_x)$.

This relation is a betweenness on $\mathbb{H}$.

Proof.

B1. Suppose $[ABC]$. Now $\langle A, B, C\rangle$ by definition. Moreover, either a_x, b_x, and c_x or a_y, b_y, and c_y are all distinct, so that A, B, and C are distinct.

B2. Follows because $=$ and $\neq$ are symmetric and we can permute the arguments of $\langle *, *, *\rangle$, $\mathsf{min}(*, *)$, and $\mathsf{max}(*, *)$.

B3. Suppose A, B, and C are distinct and $\langle A, B, C\rangle$. Suppose $a_x = b_x$. Then by the definition of collinearity in $\mathbb{H}$, we have $c_x = a_x$. Since A, B, and C are distinct, a_y, b_y, and c_y must be distinct. Since the order on $\mathbb{R}$ is total, there are six possibilities. If $a_y < b_y < c_y$, $a_y < c_y < b_y$, or $c_y < a_y < b_y$, we have $[ABC]$, $[ACB]$, or $[CAB]$, respectively. If $b_y < a_y < c_y$, $b_y < c_y < a_y$, or $c_y < b_y < a_y$, we have $[BAC]$, $[BCA]$, or $[CBA]$, respectively; using B2 we then have $[CAB]$, $[ACB]$, or $[ABC]$, respectively.

Suppose instead that $a_x \neq b_x$. We can see that because A, B, and C are distinct, a_x, b_x, and c_x must also be distinct. As in the previous paragraph, since the order on $\mathbb{R}$ is total there are six possibilities, and in each case either $[CAB]$, $[ACB]$, or $[ABC]$.

B4. Suppose we have $[ABC]$ and $[ACD]$. Note that $\langle A, B, C\rangle$ and $\langle A, C, D\rangle$, and that A, B, and C are distinct, and A, C, and D are distinct. If $a_x = b_x$, then we also have $c_x = d_x = a_x$. Now

$$\mathsf{min}(a_y, c_y) < b_y < \mathsf{max}(a_y, c_y) \text{ and } \mathsf{min}(a_y, d_y) < c_y < \mathsf{max}(a_y, d_y).$$

There are four possibilities. If $a_y < b_y < c_y$ and $a_y < c_y < d_y$, then we have $[ABD]$ and $[BCD]$. If $c_y < b_y < a_y$ and $d_y < c_y < a_y$, then we have $[DBA]$ and $[DCB]$, and by B2, $[ABD]$ and $[BCD]$. The other two possibilities lead to contradictions.

Suppose instead that $a_x \neq b_x$; a similar analysis of a_x, b_x, c_x, and d_x shows that $[ABD]$ and $[BCD]$.

B5. Similar to the proof for B4.

B6. (@@@)

$\square$

Prop'n 13.2. Given a line $\ell = \overleftrightarrow{AB}$ in $\mathbb{H}$, we define halfplanes $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$ as follows.

(i) If $a_x = b_x$, then

$$\begin{aligned} \mathcal{H}_{1,\ell} &= \{(x,y) \in \mathbb{H} \mid x < a_x\} \\ \mathcal{H}_{2,\ell} &= \{(x,y) \in \mathbb{H} \mid x > a_x\} \end{aligned}$$

(ii) If $a_x \neq b_x$, then

$$\begin{aligned} \mathcal{H}_{1,\ell} &= \{(x,y) \in \mathbb{H} \mid (x - H_{A,B})^2 + y^2 < (a_x - H_{A,B})^2 + a_y^2\} \\ \mathcal{H}_{2,\ell} &= \{(x,y) \in \mathbb{H} \mid (x - H_{A,B})^2 + y^2 > (a_x - H_{A,B})^2 + a_y^2\} \end{aligned}$$

With halfplanes defined this way for all lines, $\mathbb{H}$ satisfies the Line Separation Property and thus is an ordered geometry.

Proof. First we need to verify that these half planes are well-defined as functions of ℓ; that is, they do not depend on the points used to generate ℓ. To see this, suppose we have distinct points U and V such that $\ell = \overleftrightarrow{UV}$. If $a_x = b_x$, then in fact $u_x = a_x$, and we have $\{(x,y) \in \mathbb{H} \mid x < a_x\} = \{(x,y) \in \mathbb{H} \mid x < u_x\}$ and $\{(x,y) \in \mathbb{H} \mid x > a_x\} = \{(x,y) \in \mathbb{H} \mid x > u_x\}$ as needed. If $a_x \neq b_x$, then we have $(u_x - H_{A,B})^2 + u_y^2 = (a_x - H_{A,B})^2 + a_y^2$, and using Exercise **??** we have $H_{U,V} = H_{A,B}$. Thus

$$(u_x - H_{U,V})^2 + u_y^2 = (a_x - H_{A,B})^2 + a_y^2$$

as needed.

Certainly ℓ, $\mathcal{H}_{1,\ell}$, and $\mathcal{H}_{2,\ell}$ partition $\mathbb{H}$. To see that $\mathcal{H}_{1,\ell}$ and $\mathcal{H}_{2,\ell}$ are not empty, choose y_1 and y_2 such that $y_1^2 < (a_x - H_{A,B})^2 + a_y^2 < y_2^2$; then $(H_{A,B}, y_1) \in \mathcal{H}_{1,\ell}$ and $(H_{A,B}, y_2) \in \mathcal{H}_{2,\ell}$.

Next we show that $\mathcal{H}_{1,\ell}$ is convex; the proof that $\mathcal{H}_{2,\ell}$ is convex is very similar. To this end, let $U, V \in \mathcal{H}_{1,\ell}$ be distinct and suppose $[UWV]$. First suppose that $a_x = b_x$. If $u_x = v_x$ then we have $w_x = u_x < a_x$ so that $W \in \mathcal{H}_{1,\ell}$. If $u_x \neq v_x$ then we have $w_x < \mathsf{max}(u_x, v_x) < a_x$ and thus $W \in \mathcal{H}_{1,\ell}$. So $\mathcal{H}_{1,\ell}$ is convex in this case. Next suppose that $a_x \neq b_x$. If $u_x = v_x$ then we have $w_x = u_x = v_x$. Without loss of generality, suppose $\mathsf{max}(u_y, v_y) = u_y$; then we have

$$(w_x - H_{A,B})^2 + w_y^2 < (u_x - H_{A,B})^2 + u_y^2 < (a_x - H_{A,B})^2 + a_y^2,$$

and thus $W \in \mathcal{H}_{1,\ell}$. If $u_x \neq v_x$ and $H_{U,V} = H_{A,B}$, then we have

$$(w_x - H_{A,B})^2 + w_y^2 = (u_x - H_{A,B})^2 + u_y^2 < (a_x - H_{A,B})^2 + a_y^2,$$

and thus $W \in \mathcal{H}_{1,\ell}$. Suppose then that $u_x \neq v_x$ and $H_{U,V} \neq H_{A,B}$. Since $\langle U, V, W \rangle$, we have $(w_x - H_{U,V})^2 + w_y^2 = (u_x - H_{U,V})^2 + u_y^2$, which can be rearranged as

$$(w_x - H_{A,B})^2 + w_y^2 + 2(H_{U,V} - H_{A,B})(u_x - w_x) = (u_x - H_{A,B})^2 + u_y^2.$$

A similar equality holds for V. And since U and V are in $\mathcal{H}_{1,\ell}$, we have the system of inequalities

$$\begin{aligned}(w_x - H_{A,B})^2 + w_y^2 + 2(H_{U,V} - H_{A,B})(u_x - w_x) &< (a_x - H_{A,B})^2 + a_y^2 \\ (w_x - H_{A,B})^2 + w_y^2 + 2(H_{U,V} - H_{A,B})(v_x - w_x) &< (a_x - H_{A,B})^2 + a_y^2.\end{aligned}$$

Note that either $2(H_{U,V} - H_{A,B})(u_x - w_x)$ or $2(H_{U,V} - H_{A,B})(v_x - w_x)$ must be negative, since we have $w_x < \mathsf{max}(u_x, v_x)$. So $W \in \mathcal{H}_{1,\ell}$ as needed, and $\mathcal{H}_{1,\ell}$ is convex.

Finally, suppose we have $U \in \mathcal{H}_{1,\ell}$ and $V \in \mathcal{H}_{2,\ell}$. We need to show that there is a point $W \in \overline{UV} \cap \overleftrightarrow{AB}$. If $a_x = b_x$, then $u_x < a_x < v_x$. Note that by Exercise **??** we can say without loss of generality that $(a_x - H_{U,V})^2 < (u_x - H_{U,V})^2$. Now let $w_x = a_x$ and

$$w_y = \sqrt{(u_x - H_{U,V})^2 + u_y^2 - (a_x - H_{U,V})^2}.$$

We can see that $W = (w_x, w_y)$ is the unique point on both $\overline{UV}$ and $\overleftrightarrow{AB}$.

Suppose now that $a_x \neq b_x$. If $u_x = v_x$, then (@@@)

Finally suppose $u_x \neq v_x$; without loss of generality, say $u_x < v_x$ (the case $v_x < u_x$ is similar). By Exercise **??**, we have $H_{U,V} \neq H_{A,B}$. Note that $\overline{UV} \cap \overleftrightarrow{AB}$ is the solution(s) of the following system of equations:

$$\begin{cases} (x - H_{A,B})^2 + y^2 = (a_x - H_{A,B})^2 + a_y^2 \\ (x - H_{U,V})^2 + y^2 = (u_x - H_{U,V})^2 + u_y^2 \\ u_x < x < v_x. \end{cases}$$

We can solve the first two equations for x as

$$w_x = \frac{a_x^2 + a_y^2 - u_x^2 - u_y^2 + 2(u_x H_{U,V} - a_x H_{A,B})}{2(H_{U,V} - H_{A,B})}.$$

Note that

$$(u_x - H_{A,B})^2 + u_y^2 < (a_x - H_{A,B})^2 + a_y^2 < (v_x - H_{A,B})^2 + v_y^2$$

because $U \in \mathcal{H}_{1,\ell}$ and $V \in \mathcal{H}_{2,\ell}$. Now

$$\begin{aligned}(u_x - H_{A,B})^2 + u_y^2 < (a_x - H_{A,B})^2 + a_y^2 < \\ (v_x - H_{U,V})^2 + v_y^2 - H_{U,V}^2 + 2v_x(H_{U,V} - H_{A,B}) + H_{A,B}^2\end{aligned}$$

and thus

$$(u_x - H_{A,B})^2 + u_y^2 < (a_x - H_{A,B})^2 + a_y^2 < \\ (u_x - H_{U,V})^2 + u_y^2 - H_{U,V}^2 + 2v_x(H_{U,V} - H_{A,B}) + H_{A,B}^2$$

which rearranges as

$$u_x < \frac{a_x^2 + a_y^2 - u_x^2 - u_y^2 + 2(u_x H_{U,V} - a_x H_{A,B})}{2(H_{U,V} - H_{A,B})} < v_x.$$

(In the last step, we used Exercise **??**.) (@@@) $\square$

14 Angles

Def'n 14.1 (Angle)**.** Let P be an ordered geometry and x, o, and y distinct points in P. Then the set

$$\angle xoy = \overrightarrow{ox} \cup \overrightarrow{oy}$$

is called the *angle* with *vertex* o and *sides* $\overrightarrow{ox}$ and $\overrightarrow{oy}$. If $[xoy]$, then we say the angle is *straight*, and if $[oxy]$ or $[oyx]$, then we say the angle is *flat*.

Def'n 14.2 (Angle Pairs)**.** Suppose x, y, z, w, and o are distinct points in an ordered geometry.

(i) $\angle xoy$ and $\angle yoz$ are called an *adjacent pair*.

(ii) $\angle xoy$ and $\angle yoz$ are called a *linear pair* if $[xoz]$.

(iii) $\angle xoy$ and $\angle zow$ are called a *vertical pair* if $[xoz]$ and $[yow]$.

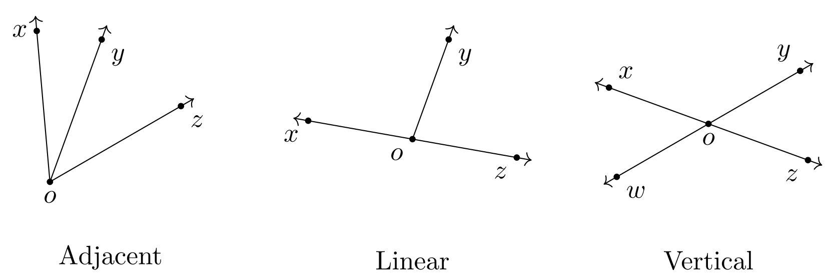


Figure 14.1: Some types of angle pairs.

Theorem 14.3 (Crossbar Theorem)**.** Suppose O, A, and B are noncollinear points in an ordered geometry, and that $C \in \mathsf{int}\angle AOB$. Then $\overrightarrow{OC}$ cuts $\overline{AB}$ at a unique point D. (See **??**.)

Proof. By the Interpolation property, there is a point P on $\overleftrightarrow{OA}$ such that $[POA]$. Note that A and P are on opposite sides of $\overleftrightarrow{OB}$, so that P and C are on opposite sides of $\overleftrightarrow{OB}$. (Since A and C are on the same side of $\overleftrightarrow{OB}$ by definition.) Consider now the triangle $\triangle PAB$. Note that the line $\overleftrightarrow{OC}$ does not contain A, B, or P, since C is not on $\overleftrightarrow{OA}$ or $\overleftrightarrow{OB}$ by hypothesis. Moreover, $\overleftrightarrow{OC}$ cuts $\overline{PA}$ at O. By Pasch's Axiom, $\overleftrightarrow{OC}$ must also cut either $\overline{PB}$ or $\overline{AB}$.

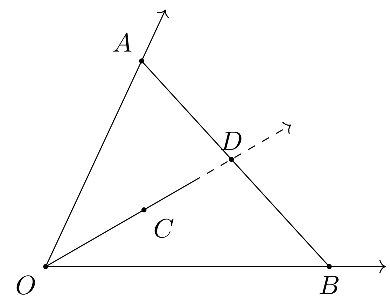


Figure 14.2: The Crossbar Theorem

Suppose $\overleftrightarrow{OC}$ cuts $\overline{PB}$ at a (necessarily unique) point Q. Note that $\overleftrightarrow{OC} = \overleftrightarrow{QC}$. Now P and Q are on the same side of $\overleftrightarrow{OB}$, so that Q and C are on *opposite* sides of $\overleftrightarrow{OB}$. Thus, there is a unique point R on $\overleftrightarrow{OB}$ such that $[QRC]$. In particular, $R \in \overleftrightarrow{OC}$. Now we have $O, R \in \overleftrightarrow{OC}$ and $O, R \in \overleftrightarrow{OB}$, so that $\overleftrightarrow{OC} = \overleftrightarrow{OB}$, a contradiction.

Hence $\overleftrightarrow{OC}$ must cut $\overline{AB}$ at a unique point; say D. Now D and A are on the same side of $\overleftrightarrow{OB}$, and so C and D are on the same side of $\overleftrightarrow{OB}$; in particular, we cannot have $[DOC]$. So in fact $\overrightarrow{OC}$ cuts $\overline{AB}$ at a unique point. $\square$

Def'n 14.4 (Angle Interior)**.** Suppose x, o, and y are noncollinear points in an ordered geometry. Note that the lines $\overleftrightarrow{ox}$ and $\overleftrightarrow{oy}$ each divide P into half-planes. Let H_1 be the y half-plane of $\overleftrightarrow{ox}$, and let K_1 be the x half-plane of $\overleftrightarrow{oy}$. We define the *interior* of $\angle xoy$ to be the set

$$\mathsf{int}\angle xoy = H_1 \cap K_1.$$

If x, y, and o are collinear, then the interior of $\angle xoy$ is not defined.

— III —

Congruence

15 Congruence

Intuitively, we want to say that two sets of points in a geometry are "congruent" if they have the same size and shape. Rather than defining congruence once and for all, we will define congruence in terms of primitive congruence relations on two special kinds of sets: segments and angles.

Def'n 15.1 (Segment Congruence)**.** Let P be an ordered geometry, and suppose we have an equivalence relation on pairs of points, denoted $(*,*) \cong_s (*,*)$. We call $(*,*) \cong_s (*,*)$ a *segment congruence* if the following properties are satisfied.

SC1. $(x,x) \cong_s (y,y)$ for all points x and y.

SC2. $(x,y) \cong_s (y,x)$ for all points x and y.

SC3. If $z \in \overrightarrow{xy}$ such that $(x,z) \cong_s (x,y)$, then $z = y$.

In this case, $(*,*) \cong_s (*,*)$ is well-defined on the set of *segments* in P, where we write $\overline{xy} \equiv \overline{ab}$ to mean $(x,y) \cong_s (a,b)$.

The first property handles the "trivial" case, the second makes the relation well-defined on segments, and the third ensures that it differentiates between segments on the same ray which share an endpoint.

Def'n 15.2 (Angle Congruence)**.** Let P be an ordered geometry, and suppose we have an equivalence relation on triples of points, denoted $(*,*,*) \cong_a (*,*,*)$. We call $(*,*,*) \cong_a (*,*,*)$ an *angle congruence* if the following properties are satisfied.

AC1. If $[xyz]$ and $[abc]$, then $(x,y,z) \cong_a (a,b,c)$ and $(y,x,z) \cong_a (b,a,c)$, and it is not the case that $(x,y,z) \cong_a (y,x,z)$.

AC2. If $x \in \overrightarrow{oa}$ and $y \in \overrightarrow{ob}$ and x, y, and o are distinct, then $(a,o,b) \cong_a (x,o,y)$.

AC3. $(a,o,b) \cong_a (b,o,a)$ for all points a, o, and b with $a \neq o$ and $b \neq o$.

AC4. If a, b, and o are noncollinear points and x is on the b-side of $\overleftrightarrow{oa}$ such that $(a,o,b) \cong_a (a,o,x)$, then $x \in \overrightarrow{ob}$.

In this case, $(*,*,*) \cong_a (*,*,*)$ is an equivalence relation on the set of *angles* in P, and we write $\angle aob \equiv \angle xpy$ to mean $(a,o,b) \cong_a (x,p,y)$.

Much like the properties of segment congruence, the first property handles the trivial cases, the second and third make the relation well-defined on angles, and the fourth ensures that it differentiates between angles on one half-plane which share a vertex.

Def'n 15.3 (Congruence Geometry)**.** Let P be an ordered geometry. If P has a segment congruence and an angle congruence, we say that P is an *ordered geometry*.

We can define congruence of many different kinds of figures in terms of segment and angle congruence. For instance...

Def'n 15.4 (Triangle Congruence)**.** Let a, b, and c be distinct points, and let x, y, and z be distinct points. We say that $\triangle abc$ is *congruent* to $\triangle xyz$, denoted $\triangle abc \equiv \triangle xyz$, if

$$\overline{ab} \equiv \overline{xy}, \quad \overline{bc} \equiv \overline{yz}, \quad \text{and} \quad \overline{ca} \equiv \overline{zx}$$

and

$$\angle abc \equiv \angle xyz, \quad \angle bca \equiv \angle yzx, \quad \text{and} \quad \angle cab \equiv \angle zxy.$$

Def'n 15.5. Let a, b, and c be distinct points.

- We say that the triangle $\triangle abc$ is *equilateral* if $\overline{ab} \equiv \overline{bc} \equiv \overline{ca}$.
- We say that the triangle $\triangle abc$ is *isoceles* if two of its sides are congruent to each other.

Def'n 15.6 (Supplementary Angles)**.** We say that angles $\angle aob$ and $\angle xpy$ are *supplementary* if there is a linear pair, $\angle uqv$ and $\angle vqw$, such that $\angle aob \equiv \angle uqv$ and $\angle xpy \equiv \angle vqw$. In this case we say that $\angle xpy$ is a *supplement* of $\angle aob$.

Prop'n 15.7. Let P be an ordered geometry with an angle congruence.

(i) If two angles form a linear pair, then they are supplementary.

(ii) Every angle has a supplement.

Def'n 15.8. An angle is called *right* if it is supplementary to itself.

$* \quad *$ Exercises $* \quad *$

15.1. Show that triangle congruence is an equivalence relation.

16 Models of Congruence Geometry

Remember: to show that an ordered geometry is a congruence geometry, we need to specify (1) how to detect when two segments are congruent and (2) how to detect when two angles are congruent.

Congruence in $\mathbb{R}^2$

Prop'n 16.1. Define a map $\delta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\delta(A, B) = (B-A)\cdot(B-A)$, where $\cdot$ denotes the usual dot product. Now $\mathbb{R}^2$ is a congruence geometry under the following relations.

(i) Given points A, B, X, and Y in $\mathbb{R}^2$, we say that $(A, B) \cong_s (X, Y)$ if

$$\delta(A, B) = \delta(X, Y).$$

(ii) Given points A, O, B, X, P, and Y in $\mathbb{R}^2$ such that $A \neq O$, $B \neq O$, $X \neq P$, and $Y \neq P$, we say that $(A, O, B) \cong_a (X, P, Y)$ if

$$\frac{((A-O)\cdot(B-O))^2}{\delta(A, O)\delta(B, O)} = \frac{((X-P)\cdot(Y-P))^2}{\delta(X, P)\delta(Y, P)}.$$

Proof. (@@@) □

Congruence in the Unit Disc

17 Circles

Def'n 17.1 (Circle)**.** Let P be a congruence geometry and let $o, a \in P$ be points. The set

$$\bigcirc oa = \{x \in P \mid \overline{ox} \equiv \overline{oa}\}$$

is called the *circle* with *center* o and *passing through* a.

Def'n 17.2 (Radius, Diameter, Chord)**.** (@@@)

— IV —

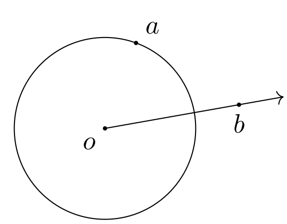
Neutral Plane Geometry

18 Plane Geometry

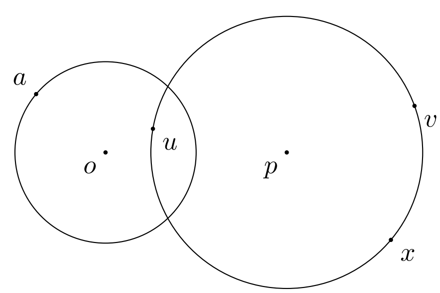
Def'n 18.1 (Plane Geometry)**.** Let P be an ordered geometry with a segment congruence and an angle congruence. We say that P is a *plane geometry* if the following properties are satisfied.

(i) **Right Angle Property.** Any two right angles are congruent.

(ii) **Circle-Ray Cut.** If o, a, and b are points such that $a \neq o$ and $b \neq o$, then there is a unique point $c \in \overrightarrow{ob}$ such that $\overline{oc} \equiv \overline{oa}$.



(iii) **Circle-Circle Cut.** Let o, a, P, and x be points, and suppose there are distinct points u and v on $\bigcirc px$ such that $u \in \mathsf{int} \bigcirc oa$ and $v \in \mathsf{ext} \bigcirc oa$. Then $\bigcirc oa \cap \bigcirc px$ contains two distinct points.



(iv) **Circle Cut Transfer.** Suppose a, b, c, d, x, y, z, and w are points such that $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\overline{cd} \equiv \overline{zw}$. If $\bigcirc ba \cap \bigcirc cd$ is not empty, then $\bigcirc yx \cap \bigcirc zw$ is not empty.

(v) **Angle-Side Congruence.** Suppose a, b, c, x, y, and z are points such that $\overline{ba} \equiv \overline{yx}$ and $\overline{bc} \equiv \overline{yz}$. Then $\overline{ac} \equiv \overline{xz}$ if and only if $\angle abc \equiv \angle xyz$.

The Circle Separation and Circle Cut properties allow us to construct points on the intersection of a circle with a central ray and of two circles, respectively. (Without these we have no way to construct points on circles!) The Circle Cut Transfer property says that our geometry is "uniform" in some sense, allowing us to shift points in the intersection of two circles. Angle-Side Congruence provides an essential link between segment congruence and angle congruence, which are otherwise unrelated.

In the remainder of this section, suppose P is a plane geometry.

Prop'n 18.2 (Circle Trichotomy)**.** Let o and a be distinct points. Then $\bigcirc oa$, $\mathsf{int} \bigcirc oa$, and $\mathsf{ext} \bigcirc oa$ partition the set of points in P. That is, every point is either on $\bigcirc oa$, interior to $\bigcirc oa$, or exterior to $\bigcirc oa$.

Prop'n 18.3 (SSS Theorem)**.** If two triangles can be labeled such that corresponding sides are congruent, then the triangles are congruent. More precisely, let a, b, and c be distinct points and x, y, and z be distinct points. If $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\overline{ca} \equiv \overline{zx}$, then $\triangle abc \equiv \triangle xyz$.

Proof. That $\angle abc \equiv \angle xyz$, $\angle bca \equiv \angle yzx$, and $\angle zxy \equiv \angle cab$ follows from three applications of the Angle-Side Congruence property. $\square$

Prop'n 18.4 (Uniqueness of Circle Cuts)**.** Let o, a, P, x, and h be points, with o and P distinct and with h not on $\overleftrightarrow{op}$. There is at most one point $u \in \bigcirc oa \cap \bigcirc px$ on the h-side of $\overleftrightarrow{op}$.

Proof. Suppose we have two such points, u and v. That is, both u and v are on the h-side of $\overleftrightarrow{op}$ and $u, v \in \bigcirc oa \cap \bigcirc px$. Note that $\overline{op} \equiv \overline{op}$, $\overline{pu} \equiv \overline{px} \equiv \overline{pv}$, and $\overline{uo} \equiv \overline{ao} \equiv \overline{vo}$. By the SSS Theorem, we have $\triangle uop \equiv \triangle vop$. In particular, we have $\angle uop \equiv \angle vop$ and $\angle upo \equiv \angle vpo$. Now by AC7, we have $v \in \overrightarrow{ou} \subseteq \overleftrightarrow{ou}$ and $u \in \overrightarrow{pv} \subseteq \overleftrightarrow{pv}$. That is, u and v are points in the intersection of the lines $\overleftrightarrow{ou}$ and $\overleftrightarrow{pv}$. Since o and P are distinct, these lines must be distinct, and so they intersect at a unique point. Hence $u = v$. $\square$

Prop'n 18.5 (SAS Theorem)**.** If two triangles can be labeled such that two corresponding sides, and the angles between, are congruent, then the triangles are congruent. More precisely, let a, b, and c be distinct points, and x, y, and z be distinct points. If $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\angle abc \equiv \angle xyz$, then $\triangle abc \equiv \triangle xyz$.

Prop'n 18.6 (Pons Asinorum (Bridge of Asses))**.** If $\triangle abc$ is isoceles with $\overline{ab} \equiv \overline{bc}$, then $\angle bac \equiv \angle bca$.

Proof. We have two triangles, $\triangle bac$ and $\triangle bca$, such that $\overline{bc} \equiv \overline{ba}$, $\overline{ba} \equiv \overline{bc}$, and $\angle cba \equiv \overline{abc}$. By the SAS Theorem, $\triangle bac \equiv \overline{bca}$, and thus $\angle bac \equiv \angle bca$. $\square$

Cor. 18.7. Every triangle which is equilateral is also equiangular; all three interior angles are congruent.

Construction 18.8 (equilateral triangle with a given side)**.** Given distinct points x and y, there exist points z_1 and z_2, on opposite sides of $\overleftrightarrow{xy}$, such that $\triangle xyz_1$ and $\triangle xyz_2$ are equilateral. In fact, we have $\triangle xyz_1 \equiv \triangle xyz_2$.

Proof. Consider the line $\overleftrightarrow{xy}$. By the Interpolation property, there exists a point u such that $[uxy]$. By the Circle Separation property, there is a point $w \in \bigcirc yx \cap \overrightarrow{xu}$. Note in particular that $[wxy]$, and hence w is exterior to the circle $\bigcirc yx$. Moreover, w is on $\bigcirc xy$. Now y is also on $\bigcirc xy$, and by definition, y is interior to $\bigcirc yx$. By the Circle Cut property, there exist two points in $\bigcirc xy \cap \bigcirc yx$, say z_1 and z_2, which must be on opposite sides of $\overleftrightarrow{xy}$ by the uniqueness of circle cuts. Now $\overline{xz_1} \equiv \overline{xy} \equiv \overline{yz_1}$ and $\overline{xz_2} \equiv \overline{xy} \equiv \overline{yz_2}$ by the definition of circles, so that $\triangle xyz_1$ and $\triangle xyz_2$ are equilateral by definition. Moreover, $\triangle xyz_1 \equiv \triangle xyz_2$ by the transitivity of segment congruence and the SSS Theorem. $\square$

Prop'n 18.9 (Segment Addition Theorem)**.** Suppose $[abc]$ and $[xyz]$. If any two of $\overline{ab} \equiv \overline{xy}$, $\overline{bc} \equiv \overline{yz}$, and $\overline{ac} \equiv \overline{xz}$ hold, then so does the third.

Proof. Note that $\angle abc \equiv \angle xyz$, $\angle bca \equiv \angle yzx$, and $\angle cab \equiv \angle zxy$ by AC4. The result then follows from the SAS Theorem. $\square$

Lemma 18.10. Suppose $[abc]$ and $y \in \overrightarrow{xz}$. If $\overline{ab} \equiv \overline{xy}$ and $\overline{ac} \equiv \overline{xz}$, then $[xyz]$.

Proof. Since $y \in \overrightarrow{xz}$, we have four possibilities: $y = x$, $[xyz]$, $y = z$, and $[xzy]$. If $y = x$, then we have $\overline{ab} \equiv \overline{xx}$, so that $b = a$, a contradiction. Similarly if $y = z$ then we have $\overline{xy} \equiv \overline{xz}$, so that $y = z$, also a contradiction. Now suppose that $[xzy]$. Note that $\angle cab \equiv \angle zxy$, $\overline{ac} \equiv \overline{xz}$, and $\overline{ab} \equiv \overline{xy}$; by the SAS Theorem, $\triangle abc \equiv \triangle xyz$. In particular, the flat angle $\angle acb$ is congruent to the straight angle $\angle xzy$, a contradiction. Thus $[xyz]$ as claimed. $\square$

Construction 18.11 (copy a segment onto a ray)**.** Let a and b be distinct points, and let o and t be distinct points. There exists a point x on $\overrightarrow{ot}$ such that $\overline{ox} \equiv \overline{ab}$.

Proof. First we construct a point z such that $\triangle aoz$ is equilateral; now $\overline{za} \equiv \overline{zo}$. Using the Interpolation property, construct a point h such that $[zah]$, and using the Circle Separation property, construct a point u on $\overrightarrow{ah}$ such that $\overline{au} \equiv \overline{ab}$. Again using Circle Separation, construct a point v on $\overrightarrow{zo}$ such that $\overline{zv} \equiv \overline{zu}$. By the previous proposition, $[zov]$. Now $\overline{za} \equiv \overline{zo}$ and $\overline{zu} \equiv \overline{zv}$, thus $\overline{au} \equiv \overline{ov}$.

Again using Circle Separation, construct a point x on $\overrightarrow{ot}$ such that $\overline{ox} \equiv \overline{ov}$. Then we have $\overline{ox} \equiv \overline{ov} \equiv \overline{au} \equiv \overline{ab}$ as needed. □

Construction 18.12 (copy an angle onto a ray)**.** Let a, o, b be distinct noncollinear points and let P and x be distinct points. There exist two points y_1 and y_2, on opposite sides of $\overleftrightarrow{px}$, such that $\angle xpy_1 \equiv \angle xpy_2 \equiv \angle aob$.

Proof. First copy segment $\overline{ob}$ onto $\overrightarrow{px}$ at the point u, then copy the segment $\overline{ba}$ onto the ray $\overrightarrow{up}$ at the point v. Now copy $\overline{oa}$ onto $\overrightarrow{px}$ at the point w. Note that $\overline{oa} \equiv \overline{pw}$, $\overline{ob} \equiv \overline{pu}$, and $\overline{ba} \equiv \overline{uv}$. Moreover, the intersection $\bigcirc oa \cap \bigcirc ba$ is nonempty, as it contains a. By the Circle Cut Transfer property, $\bigcirc pw \cap \bigcirc uv$ contains two points z_1 and z_2 on opposite sides of $\overleftrightarrow{px}$. By the SSS Theorem, we have $\triangle puz_1 \equiv \triangle oba \equiv \triangle puz_2$, and thus $\angle upz_1 \equiv \angle aob \equiv \angle upz_2$ as needed. □

> **Prop'n 18.13** (ASA Theorem)**.** Let a, b, c be distinct noncollinear points, and let x, y, z be distinct points. If $\angle abc \equiv \angle xyz$, $\overline{bc} \equiv \overline{yz}$, and $\angle bca \equiv \angle yzx$, then $\triangle abc \equiv \triangle xyz$.

Proof. Copy $\overline{yx}$ onto $\overrightarrow{ba}$ at d. Note that d and a are on the same side of $\overleftrightarrow{bc}$. Moreover, we have $\triangle dbc \equiv \triangle xyz$ by the SAS Theorem, and so $\angle bcd \equiv \angle yzx \equiv \angle bca$. By AC7, we have $d \in \overrightarrow{ca}$. Now d is on both $\overleftrightarrow{ba}$ and $\overleftrightarrow{cd}$, and since a, b, and c are not collinear, we must have $d = a$. So $\triangle abc \equiv \triangle xyz$ as claimed. □

> **Prop'n 18.14** (Angle Addition Theorem)**.** Suppose $B \in \mathsf{int}\angle AOC$ and $Y \in \mathsf{int}\angle XPZ$. If any two of $\angle AOC \equiv \angle XPZ$, $\angle AOB \equiv \angle XPY$, and $\angle BOC \equiv \angle YPZ$ holds, then so does the third.

Proof. (@@@ Uses SAS and segment addition.) □

19 Transversals

Prop'n 19.1 (Supplements are unique)**.**

- Suppose that $\angle AOB$ and $\angle BOC$ are a linear pair, and that $\angle XPY$ and $\angle YPZ$ are a linear pair. If $\angle AOB \equiv \angle XPY$, then $\angle BOC \equiv \angle YPZ$.
- Suppose $\angle ABC$ and $\angle XYZ$ are supplementary, and that $\angle ABC$ and $\angle HKL$ are supplementary. Then $\angle XYZ \equiv \angle HKL$.

Proof. Suppose we have two such linear pairs. Without loss of generality, we can suppose that

$$\overline{OA} \equiv \overline{OB} \equiv \overline{OC} \equiv \overline{PX} \equiv \overline{PY} \equiv \overline{PZ}.$$

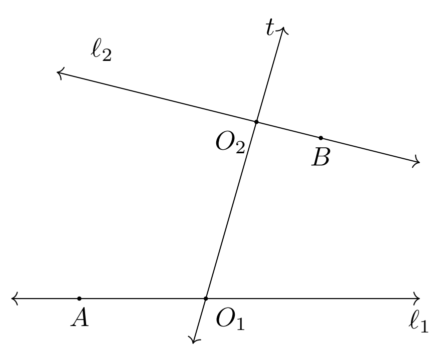
(If they aren't, we can use Circle Separation and the Segment Copy construction to find such points.) Now $\triangle BOA \equiv \triangle YPX$ by SAS, so that $\angle BAO \equiv \angle YXP$. Now $\overline{AC} \equiv \overline{XZ}$, so that $\triangle BAC \equiv \triangle YXZ$ by SAS. So $\overline{BC} \equiv \overline{YZ}$, and thus $\triangle BOC \equiv \triangle YPZ$ by SSS. Thus $\angle BOC \equiv \angle YPZ$.

The second statement follows easily. $\square$

Cor. 19.2. Vertical pairs of angles are congruent.

Def'n 19.3 (Transversal)**.** Suppose we have three lines ℓ_1, ℓ_2, and t in a plane geometry. We say that t is a *transversal* of ℓ_1 and ℓ_2 if t cuts both ℓ_1 and ℓ_2 at unique points, and these points are distinct.

Suppose t is a transversal of ℓ_1 and ℓ_2, cutting these lines at O_1 and O_2, respectively as shown.



If A is on ℓ_1 and B is on ℓ_2 such that A and B are on opposite sides of t, then we say that $\angle AO_1O_2$ and $\angle BO_2O_1$ are *alternate interior angles* of this transversal.

Prop'n 19.4 (Alternate Interior Angles)**.** If two lines ℓ_1 and ℓ_2 are cut by a transversal t so that a pair of alternate interior angles are congruent, then ℓ_1 and ℓ_2 are parallel.

Proof. Suppose t meets ℓ_1 and ℓ_2 at points O_1 and O_2 respectively, and that A and B are on ℓ_1 and ℓ_2, respectively, and on opposite sides of t. Let C be on ℓ_1 such that $[AO_1C]$. Suppose by way of contradiction that ℓ_1 and ℓ_2 are *not* parallel; rather, they meet at a point X which (WLOG) is on the A-side of t. Copy $\overline{O_1X}$ onto $\overrightarrow{O_2B}$ at the point Y. Now $\overline{O_1X} \equiv \overline{O_2Y}$, $\overline{O_1O_2} \equiv \overline{O_2O_1}$, and $\angle XO_1O_2 \equiv \angle YO_2O_1$, so by SAS we have $\triangle XO_1O_2 \equiv \triangle YO_2O_1$. In particular, $\angle O_2O_1Y \equiv \angle O_1O_2X$.

Now $\angle XO_2O_1$ and $\angle O_1O_2Y$ are supplementary, and $\angle O_1O_2Y \equiv \angle AO_1O_2$, so that $\angle AO_1O_2$ and $\angle XO_2O_1$ are supplementary. Since $\angle XO_2O_1 \equiv \angle YO_1O_2$, we have that $\angle AO_1O_2$ and $\angle YO_1O_2$ are supplementary. But also $\angle AO_1O_2$ and $\angle O_2O_1C$ are supplementary. Now $\angle O_2O_1Y \equiv \angle O_2O_1C$. By the uniqueness of congruent angles on a half-plane, we have that O_1, C, and Y are collinear, so that $Y \in \ell_1$. But now ℓ_1 and ell_2 have two points in common – X and Y – and thus must be equal, a contradiction.

So in fact ℓ_1 and ℓ_2 must be parallel. □

Prop'n 19.5 (AAS)**.** Suppose we have triangles $\triangle ABC$ and $\triangle XYZ$ such that $\angle CAB \equiv \angle ZXY$, $\angle ABC \equiv \angle XYZ$, and $\overline{BC} \equiv \overline{YZ}$. Then $\triangle ABC \equiv \triangle XYZ$.

Proof. Copy $\overline{BA}$ onto $\overrightarrow{YX}$ at the point W. Note that $\triangle WYZ \equiv \triangle ABC$ by SAS, so that $\angle BAC \equiv \triangle YWZ$. Suppose now that W and X are distinct points. In this case $\overleftrightarrow{XZ}$ and $\overleftrightarrow{WZ}$ are lines cut by a transversal $\overleftrightarrow{XY}$. Moreover, if we let U be a point such that $[UXZ]$, then $\angle UXW$ and $\angle YXZ$ are vertical, hence congruent, and so $\angle UXW \equiv \angle YXZ$. But now by the Alternate Interior Angles theorem $\overleftrightarrow{XZ}$ and $\overleftrightarrow{WZ}$ must be parallel, a contradiction since they meet at Z. So in fact X and W are the same point, and thus $\triangle ABC \equiv \triangle XYZ$ by SAS. □

Prop'n 19.6 (HL)**.** Let $\triangle ABC$ and $\triangle XYZ$ be triangles such that $\angle BCA$ and $\angle YZX$ are right and $\overline{AB} \equiv \overline{XY}$ and $\overline{BC} \equiv \overline{YZ}$. Then $\triangle ABC \equiv \triangle XYZ$.

Proof. Copy $\overline{ZX}$ onto the ray opposite $\overrightarrow{CA}$ at the point D. Now $\angle BCD$ is a right angle, since it is supplementary to $\angle ACB$. By SAS, we have $\triangle XYZ \equiv \triangle DCB$, and thus $\overline{BD} \equiv \overline{YX} \equiv \overline{BA}$. Now $\triangle ABD$ is isoceles with $\overline{BA} \equiv \overline{BD}$, so that $\angle BAC \equiv \angle BAD \equiv \angle BDA \equiv \angle YXZ$. By AAS, we have $\triangle ABC \equiv \triangle XYZ$. □

Prop'n 19.7. A triangle formed by three noncollinear points cannot have two interior angles which are both right.

Proof. Such a triangle would violate the Alternate Interior Angles theorem since right angles are self-supplementary, and any two right angles are congruent. □

Construction 19.8 (Angle Bisector)**.** Let A, O, and B be noncollinear points. There exists a unique line ℓ, containing O, such that if $U \in \ell$ is different from O then $\angle AOU \equiv \angle BOU$. This line is called the *bisector* of $\angle AOB$.

Proof. Note that we can assume WLOG that $\overline{OA} \equiv \overline{OB}$; if not, construct such a point on $\overrightarrow{OB}$ using the Circle Separation property. Since the intersection of $\bigcirc AO$ and $\bigcirc BO$ contains a point not on $\overleftrightarrow{AB}$, by Circle Cut Transfer there is a second point U on the opposite side of $\overleftrightarrow{AB}$ such that $\overline{AU} \equiv \overline{BU}$. Let $\ell = \overleftrightarrow{OU}$. Note that $\triangle AOU \equiv \triangle BOU$ by SSS, so that $\angle AOU \equiv \angle BOU$. Then if V is a point such that $[VOU]$, we have $\angle VOA \equiv \angle VOB$, since these are supplementary to congruent angles.

To see uniqueness, note that any such line must contain O and U. □

Cor. 19.9. A and B are on opposite sides of the bisector of $\angle AOB$. In particular, the bisector of $\angle AOB$ contains points which are interior to $\angle AOB$.

Proof. Suppose otherwise, and let $U \neq O$ be a point on the bisector. Then $\angle UOA$ and $\angle UOB$ are congruent angles on the same half-plane of a ray, so that A, B, and O are collinear – a contradiction. By the plane separation property there is a point W between A and B which is on the bisector; this point is interior to $\angle AOB$ as needed. □

Construction 19.10 (Segment Midpoint)**.** Let A and B be distinct points. There is a unique point M such that $[AMB]$ and $\overline{AM} \equiv \overline{BM}$. This point is called the *midpoint* of $\overline{AB}$.

Proof. Construct a point O such that $\triangle AOB$ is equilateral, and construct the bisector of $\angle AOB$. By the Crossbar theorem, this bisector must cut $\overline{AB}$ at an interior point, say M. Now $\triangle OAM \equiv \triangle OBM$ by SAS, and thus $\overline{AM} \equiv \overline{BM}$ as needed. Note that M is unique by the uniqueness of congruent segments on a ray. □

20 Perpendiculars and Tangents

We say that two lines are *perpendicular* if they form a right angle.

Def'n 20.1 (Foot)**.** Let ℓ be a line and p a point not on ℓ in a plane geometry. We say that a point $f \in \ell$ is a *foot* of p on ℓ if ℓ and $\overleftrightarrow{FP}$ are perpendicular.

Construction 20.2 (Foot of a point)**.** Let ℓ be a line and p a point not on ℓ in a plane geometry. Then p has a unique foot on ℓ.

Proof. To see existence, let x and y be distinct points on ℓ. Note that $\bigcirc xp \cap \bigcirc yp$ is not empty, and by Circle Cut Transfer there is a second point o in the intersection of these circles which is on the opposite side of ℓ. By the Plane Separation property, ℓ and $\overline{op}$ meet at a unique point f. Now $\triangle oxy \equiv \triangle pxy$ by SSS, so that $\angle pxf \equiv \angle oxf$. Then $\triangle pxf \equiv \triangle oxf$ by SAS. Then $\angle pfx \equiv \angle ofx$, so that ℓ and $\overleftrightarrow{op}$ meet at a right angle as needed.

To see uniqueness, note that if p has two distinct feet f_1 and f_2 on ℓ then p, f_1, and f_2 form a triangle with two internal right angles – a contradiction. $\square$

Construction 20.3 (Perpendicular at a point)**.** Let ℓ be a line and $p \in \ell$ a point in a plane geometry. There exists a unique line t containing p which is perpendicular to ℓ.

Proof. Let x be a point on ℓ different from p, and copy $\overline{px}$ to the opposite side of p at a point y by Circle Separation. Note that p is the midpoint of $\overline{xy}$. Construct a point z such that $\triangle xyz$ is equilateral. Now $\triangle zxp \equiv \triangle zyp$ by SSS, so that $\angle zpx \equiv \angle zpy$, and thus $\overleftrightarrow{pz}$ is perpendicular to ℓ.

Uniqueness follows from the uniqueness of angles on a half-plane. $\square$

Def'n 20.4 (Perpendicular Bisector)**.** If x and y are two points, then the (unique) line perpendicular to $\overleftrightarrow{xy}$ at the midpoint of $\overline{xy}$ is called the *perpendicular bisector* of $\overline{xy}$.

Intersections of Lines and Circles

Prop'n 20.5. In a plane geometry, a line and a circle can have at most two points in common.

Proof. Let ℓ be a line and $\bigcirc oa$ a circle which have at least three points in common; say x, y, and z. Suppose WLOG that $[xyz]$. Note that o cannot also be on ℓ, as in this case z cannot be distinct from both x and y by the uniqueness of congruent segments on rays. Now $\angle oyx \equiv \angle oxy$, $\angle oyz \equiv \angle ozy$, and $\angle oxz \equiv \angle ozx$ by Pons Asinorum. In particular, $\angle oyx$ is right, so that $\triangle oxy$ has two right interior angles – a contradiction. $\square$

Def'n 20.6 (Tangent, Secant)**.** Let ℓ be a line and C a circle in a plane geometry. We say that ℓ is *tangent to* C if ℓ and C have exactly one point in common. Suppose this point is t; in this case we say that ℓ is tangent to C *at* t. Similarly, we say that ℓ is a *secant* of C if ℓ and C have exactly two distinct points in common.

Prop'n 20.7. Let ℓ be a line and C a circle with center o in a plane geometry. Then ℓ is tangent to C if and only if o is not on ℓ and the foot of o on ℓ is on C.

Proof. Suppose ℓ is tangent to C at p. If $o \in \ell$, then $\ell \cap C$ contains a second point by Circle Separation; so in fact o is not on ℓ. Let f be the foot of o on ℓ. If $f \neq p$, then o, f, and p are noncollinear and form a triangle. Since $\overline{op} \equiv \overline{of}$ and $\angle ofp$ is right, $\angle opf$ is also right by Pons Asinorum. But no triangle can have two right interior angles.

Conversely, suppose ℓ does not contain o and that the foot f of o on ℓ is on C. Suppose BWOC that there is a second point $g \in \ell \cap C$. Now o, f, and g are noncollinear, and $\overline{of} \equiv \overline{og}$, and $\angle ofg$ is right (by the definition of foot). So $\angle ogf$ is right by Pons Asinorum, again a contradiction. So $C \cap \ell$ contains exactly one point as needed. □

Construction 20.8 (Tangent at a point)**.** Let C be a circle with center o and let p be a point on C. There exists a line ℓ which is tangent to C at p.

Proof. Construct the line ℓ which is perpendicular to $\overleftrightarrow{op}$ at p. Then o is not on ℓ, and p is the foot of o on ℓ. So ℓ is tangent to C at p. □

Construction 20.9 (Second cut of line and circle)**.** Let ℓ be a line and C a circle with center o in a plane geometry such that ℓ is not tangent to C. Suppose $p \in \ell \cap C$. We may construct the second point in $\ell \cap C$.

Proof. If o is on ℓ, use Circle Separation. If o not on ℓ, construct the foot f of o on ℓ. Using Circle Separation, copy $\overline{fp}$ onto the opposite side of f from p at the point q. Note that $\triangle ofp \equiv \triangle ofq$ by SAS, so that $\overline{op} \equiv \overline{oq}$; thus $q \in \ell \cap C$ as needed. □

Comparing Segments

Def'n 20.10. Let $\overline{ab}$ and $\overline{cd}$ be segments in a plane geometry. We say that $\overline{ab} \leq \overline{cd}$ if there is a point $x \in \overline{cd}$ such that $\overline{ab} \equiv \overline{cx}$.

Prop'n 20.11.

1. If $\overline{a_1b_1} \equiv \overline{a_2b_2}$, $\overline{c_1d_1} \equiv \overline{c_2d_2}$, and $\overline{a_1b_1} \leq \overline{c_1d_1}$, then $\overline{a_2b_2} \leq \overline{c_2d_2}$.

2. If $\overline{ab} \leq \overline{cd}$ and $\overline{cd} \leq \overline{ef}$, then $\overline{ab} \leq \overline{ef}$.
3. If $[abc]$, then $\overline{ab} \leq \overline{ac}$. If $[abcd]$, then $\overline{bc} \leq \overline{ad}$.
4. If $\overline{ab} \leq \overline{cd}$ and $\overline{cd} \leq \overline{ab}$, then $\overline{ab} \equiv \overline{cd}$.

Proof.

There is a point $x \in \overline{c_1d_1}$ such that $\overline{a_1b_1} \equiv \overline{c_1x}$. Now copy $\overline{c_1x}$ onto $\overrightarrow{c_2d_2}$ at the point y; note that $[c_2yd_2]$, so that $y \in \overline{c_2d_2}$. Now $\overline{a_2b_2} \equiv \overline{c_2y}$ as needed.

There exists a point $x \in \overline{cd}$ such that $\overline{ab} \equiv \overline{cx}$, and a point $y \in \overline{ef}$ such that $\overline{cd} \equiv \overline{ey}$. Now copy $\overline{cx}$ onto $\overrightarrow{ey}$ at the point z; note that $[ezy]$; in particular, $\overline{ab} \equiv \overline{ez}$.

Clear.

There exists a point $x \in \overline{cd}$ such that $\overline{cx} \equiv \overline{ab}$. Now either $x = c$, $x = d$, or $[cxd]$. If $x = c$, then $b = a$, and $d = c$, so that $\overline{ab} \equiv \overline{cd}$. Suppose $[cxd]$. There is a point $y \in \overline{ab}$ such that $\overline{cy} \equiv \overline{ab}$; but now $[aby]$, a contradiction. So we have $x = d$ as needed. $\square$

21 Incircles and Excircles

Prop'n 21.1. Let A, O, and B be distinct points. A point P in $\mathsf{int}\angle AOB$ is on the bisector of $\angle AOB$ if and only if $\overline{PX} \equiv \overline{PY}$, where X is the foot of P on $\overleftrightarrow{OA}$ and Y is the foot of P on $\overleftrightarrow{OB}$.

Proof. Suppose P has this property. Now $\triangle OPX$ and $\triangle OPY$ are right, with $\overline{PX} \equiv \overline{PY}$ and $\overline{OP} \equiv \overline{OP}$. By the HL Theorem, $\triangle OPX \equiv \triangle OPY$, and thus $\angle XOP \equiv \angle YOP$. So P is on the bisector of $\angle AOB$.

Conversely, suppose P is on the bisector of $\angle AOP$, and let X be the foot of P on $\overleftrightarrow{OA}$ and Y the foot of P on $\overleftrightarrow{OB}$. Now $\triangle XOP \equiv \triangle YOP$ by AAS, so that $\overline{PX} \equiv \overline{PY}$. $\square$

Construction 21.2 (Incircle Theorem)**.** Let A, B, and C be distinct points. Then we have the following.

1. The bisectors of the interior angles of $\triangle ABC$ are concurrent at a point O, called the *incenter* of the triangle.

2. The feet of O on the sides of $\triangle ABC$ lie on a circle, called the *incircle* of $\triangle ABC$, which is centered at O and tangent to the sides of $\triangle ABC$.

Proof. Let $\overrightarrow{AA'}$ be the bisector of $\angle BAC$. By the Crossbar Theorem this ray cuts $\overline{BC}$ at a point A''. Let $\overrightarrow{BB'}$ be the bisector of $\angle ABC$; again by the Crossbar Theorem this ray cuts $\overline{AA''}$ at a point O. Let X, Y, and Z be the feet of O on $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{BC}$, respectively. Since O is on the bisectors of $\angle BAC$ and $\angle ABC$, we have $\overline{OX} \equiv \overline{OY}$ and $\overline{OY} \equiv \overline{OZ}$; thus $\overline{OX} \equiv \overline{OZ}$, and so O is also on the bisector of $\angle BCA$. Thus the bisectors of the interior angles of $\triangle ABC$ are concurrent at O.

Now X, Y, and Z are the feet of O on the sides of $\triangle ABC$, and we've seen that $\overline{OX} \equiv \overline{OY} \equiv \overline{OZ}$. Thus the circle $\bigcirc OX$ contains X, Y, and Z, and moreover is tangent to the sides of $\triangle ABC$ at X, Y, and Z. $\square$

Construction 21.3 (Excircle Theorem)**.** Let A, B, and C be distinct points forming $\triangle ABC$. Then we have the following.

1. The bisector of the interior angle at A and the exterior angles at B and C are concurrent at a point O, called the *excenter* of $\triangle ABC$ at A.

2. The feet of O on the (extended) sides of $\triangle ABC$ lie on a circle, called the *excircle* of $\triangle ABC$ at A, which is centered at O and tangent to the sides of $\triangle ABC$.

Proof. Essentially the same as the proof of the Incircle Theorem. $\square$

To every triangle we can associate four special circles: the incircle, and one excircle for each vertex. These circles are tangent to all three (extended) sides of the circle.

Prop'n 21.4. Any circle which is tangent to all three (extended) sides of a triangle is either the incircle or one of the excircles.