

Modern Algebra: Rings

Final Exam

Fall 2015

Name: _____

Date: _____

READ THESE INSTRUCTIONS CAREFULLY!

- Circle or underline your final written answer.
- Justify your reasoning and show your work.
- If you run out of space, make a note and continue your work on the back of a page.

1. Solve the following system of congruences for x .

$$\begin{cases} x &\equiv 2 & (\text{mod } 45) \\ x &\equiv 5 & (\text{mod } 49) \\ x &\equiv 3 & (\text{mod } 121) \end{cases}$$

2. Let R be a ring. An element $x \in R$ is called *nilpotent* if $x^n = 0$ for some power n . For example, $\bar{2}$ is nilpotent in $\mathbb{Z}/(8)$ since $\bar{2}^3 = 0$.

Show that if R is commutative then the set $N \subseteq R$ consisting of all the nilpotent elements is an ideal. (Hint: the binomial theorem holds in any ring.)

3. A ring element x is called *idempotent* if $x^2 = x$. For example, 0 is idempotent in any ring since $0^2 = 0$.

(a) Determine which elements of $\mathbb{Z}/(30)$ are idempotent.

(b) Determine which elements of $\mathbb{F}_3[x]/(x^2 - x)$ are idempotent.