(xii) – (xv) of Fact 1.3.2, which show how to negate conjunctions, disjunctions, conditionals and biconditionals. For example, what is the negation of the statement "it is raining and I am happy"? We could write "it is not the case that it is raining and I am happy," but that is cumbersome, and slightly ambiguous (does the phrase "it is not the case that" apply only to "it is raining," or also to "I am happy"?) A common error would be to say "it is not raining and I am unhappy." Note that the original statement "it is raining and I am happy" is true if and only if both "it is raining" is true and if "I am happy" is true. If either of these two component statements is false, then the whole original statement is false. Thus, to negate "it is raining and I am happy," it is not necessary to negate both component statements, but only to know that at least one of them is false. Hence the correct negation of "it is raining and I am happy" is "it is not raining or I am unhappy." A similar phenomenon occurs when negating a statement with "or" in it. The precise formulation of these ideas, known as De Morgan's laws, are Fact 1.3.2 (xii) and (xiii).

What is the negation of the statement "if it snows, I will go outside"? As before, we could write "it is not the case that if it snows, I will go outside," and again that would be cumbersome. A common error would be to say "if it snows, I will not go outside." To see that this latter statement is not the negation of the original statement, suppose that "it snows" is false, and "I will go outside" is true. Then both "if it snows, I will go outside" and "if it snows, I will not go outside" are true, so the latter is not the negation of the former. The original statement "if it snows, I will go outside" is true if and only if "I will go outside" is true whenever "it snows" is true. The negation of the original statement thus holds whenever "it snows" is true and "I will go outside" is false; that is, whenever the statement "it snows and I will not go outside" is true. The precise formulation of this observations is given in Fact 1.3.2 (xiv).

Exercises

1.3.1. Let P, Q, R and S be statements. Show that the following are true.

(1) $\neg(P \to Q) \Rightarrow P$.

(2) $(P \to Q) \wedge (P \to \neg Q) \Rightarrow \neg P$.

(3) $P \to Q \Rightarrow (P \wedge R) \to (Q \wedge R)$.

(4) $P \wedge (Q \leftrightarrow R) \Rightarrow (P \wedge Q) \leftrightarrow R$.

(5) $P \to (Q \wedge R) \Rightarrow (P \wedge Q) \leftrightarrow (P \wedge R)$.

(6) $(P \leftrightarrow R) \wedge (Q \leftrightarrow S) \Rightarrow (P \vee Q) \leftrightarrow (R \vee S)$.

1.3.2. [Used in Section 2.4.] Let P, Q, A and B be statements. Show that the following are true.

(1) $P \iff P \vee (P \wedge Q)$.

(2) $P \iff P \wedge (P \vee Q)$.

(3) $P \leftrightarrow Q \iff (P \to Q) \wedge (\neg P \to \neg Q)$.

(4) $P \to (A \wedge B) \iff (P \to A) \wedge (P \to B)$.

(5) $P \to (A \vee B) \iff (P \wedge \neg A) \to B$.

(6) $(A \vee B) \to Q \iff (A \to Q) \wedge (B \to Q)$.

(7) $(A \wedge B) \to Q \iff (A \to Q) \vee (B \to Q)$.

(8) $(A \wedge B) \to Q \iff A \to (B \to Q)$.

1.3.3. Let P be a statement, let TA be a tautology, and let CO be a contradiction.

(1) Show that $P \wedge TA \iff P$.

(2) Show that $P \vee CO \iff P$.

1.3.4. For each pair of statements, determine whether or not the first implies the second.

(1) "If you will kiss me I will dance a jig, and I will dance a jig;" and "you will kiss me."

(2) "Yolanda has a cat and a dog, and Yolanda has a python;" and "Yolanda has a dog."

(3) "If cars pollute then we are in trouble, and cars pollute;" and "we are in trouble."

(4) "Our time is short or the end is near, and doom is impending;" and "the end is near."

(5) "Vermeer was a musician or a painter, and he was not a musician;" and "Vermeer was a painter."

(6) "If I eat frogs' legs I will get sick, or if I eat snails I will get sick;" and "if I eat frogs' legs or snails I will get sick."

1.3.5. For each pair of statements, determine whether or not the two statements are equivalent.

(1) "If it rains, then I will see a movie;" and "it is not raining or I will see a movie."

(2) "This shirt has stripes, and it has short sleeves or a band collar;" and "this shirt has stripes and it has short sleeves, or it has a band collar."

(3) "It is not true that I like apples and oranges;" and "I do not like apples and I do not like oranges."

(4) "The cat is gray, or it has stripes and speckles;" and "the cat is gray or it has stripes, and the cat is gray or it has speckles."

(5) "It is not the case that: melons are ripe iff they are soft to the touch;" and "melons are ripe and soft to the touch, or they are not ripe or not soft to the touch."

1.3.6. [Used in Section 1.3.] Prove Fact 1.3.1 (ii) – (xiii).

1.3.7. [Used in Section 1.3.] Prove Fact 1.3.2 (i) – (vi), (viii), (x) – (xv).

1.3.8. State the inverse, converse and contrapositive of each of the following statements.

(1) If it's Tuesday, it must be Belgium.

(2) I will go home if it is after midnight.

(3) Good fences make good neighbors.

(4) Lousy food is sufficient for a quick meal.

(5) If you like him, give him a hug.

1.3.9. For each of the following pair of statements, determine whether the second statement is the inverse, converse, or contrapositive of the first statements, or none of these?

(1) "If I buy a new book, I will be happy;" and "If I do not buy a new book, I will be unhappy."

(2) "I will be cold if I do not wear a jacket;" and "I will not be cold if I do not wear a jacket."

(3) "If you smile a lot, your mouth will hurt;" and "If your mouth hurts, you will smile a lot."

(4) "A warm house implies a warm bathroom;" and "A cold bathroom implies a cold house."

(5) "Eating corn implies that I will have to floss my teeth;" and "Not having to floss my teeth implies that I will eat corn."

(6) "Going to the beach is sufficient for me to have fun;" and "Not going to the beach is sufficient for me not to have fun."

1.3.10. Negate each of the following statements.

(1) $e^5 > 0$.

(2) $3 < 5$ or $7 \geq 8$.