Responses to the Referee Review for
(now entitled) Spectral Analysis and Computation for Homogenization
of Advection Diffusion Processes in Steady Flows (Fourth Revision)

Manuscript #16-0766R2:

Editor Decision: Major Revision - required

Associate Editor Decision: Major Revision - required

Associate Editor Comments to the Author:

The referee is still not satisfied with the revised version and corrections. Therefore, I strongly suggest to the authors to address all the issues raised by the referee in a clear way.

Referee Recommendation: Publish with revision required

Quality of Research: Good

Quality of Presentation: Good

Referee (Comments to the Author):

As I said in the previous review, I can recommend the manuscript for publication, but three local issues should be fixed. The first, which was introduced in the revision, is important; the second and third are minor and optional.

Unfortunately, the unsolicited revisions in Section II have added some confusion, though they are easily fixed. Before Eq. (1), I assume what is intended is $\langle \mathbf{u} \rangle = 0$. A larger concern is the new discussion of the nondimensionalization, which I find too ambiguous for a mathematical journal. Equation (6) is standard; we understand the $\mapsto$ symbol means the transformation under the rescaling $t \mapsto t/\tau$ and $\mathbf{x} \mapsto \mathbf{x}/\ell$. Now if everything is to be consistent with what is said two paragraphs later, namely $\mathbf{u} \to \mathbf{v}$, then we must have $\tau/\ell = u_0$. If we also want $\epsilon \mapsto \epsilon/u_0 l$ then again consistency requires $\tau/\ell^2 = 1/u_0 l$. This can all be said more simply by simply *choosing* reference scales in (6) to be $\ell = l$ and $\tau = l/u_0$. That would be a completely standard setup for nondimensionalization. What is written makes it look, frankly, like a careless reassignment of parameters for convenience that raises the question of whether it makes assumptions and how the transformed equations relate to the original equations. By simplying making an appropriate choice of the reference scales τ and ℓ, it's all transparent.

Other than that, my criticism of the limit point discussion on p. 35 was not fully interpreted. I was simply saying that a limit point of the real line cannot have eigenvalues; operators or matrices have eigenvalues. This is just an issue of language; I know what is meant but the wording is awkward.

On the restored Figure 4, it would help in the caption to remind the reader of the relation between m and μ because otherwise it is hard to interpret the last sentence saying that m_{kk} can be determined by a relation involving μ.

General response to the Referee's comments:

First, we are very grateful to the referee for such thoughtful, detailed, and effective comments throughout the various revisions of the manuscript. We believe that the modifications made to the current draft of the manuscript address the points raised by the referee, and improve the paper.

In addition to changes made to address the issues brought up by the referee (detailed below) we have also made many minor changes and type-o corrections. We do hope that these changes will make the revised manuscript appropriate for publication in the Journal of Mathematical Physics.

Thanks very much for your consideration.

Sincerely yours,

The Authors

I. RESPONSES TO REFEREE COMMENTS

1. **Referee comment:**

   Before Eq. (1), I assume what is intended is $\langle \mathbf{u} \rangle = 0$.

   Response:

   This type-o has been fixed.

2. **Referee comment:**

   A larger concern is the new discussion of the nondimensionalization, which I find too ambiguous for a mathematical journal. Equation (6) is standard; we understand the $\mapsto$ symbol means the transformation under the rescaling $t \mapsto t/\tau$ and $\mathbf{x} \mapsto \mathbf{x}/\ell$. Now if everything is to be consistent with what is said two paragraphs later, namely $\mathbf{u} \to \mathbf{v}$, then we must have $\tau/\ell = u_0$. If we also want $\epsilon \mapsto \epsilon/u_0 l$ then again consistency requires $\tau/\ell^2 = 1/u_0 l$. This can all be said more simply by simply *choosing* reference scales in (6) to be $\ell = l$ and $\tau = l/u_0$. That would be a completely standard setup for nondimensionalization. What is written makes it look, frankly, like a careless reassignment of parameters for convenience that raises the question of whether it makes assumptions and how the transformed equations relate to the original equations. By simplying making an appropriate choice of the reference scales τ and ℓ, it's all transparent.

Response:

We have addressed this issue by rewriting the paragraph that defines our choice of non-dimensionalation of the advection diffusion equation. In particular, we define the dimensional fluid velocity field by $\boldsymbol{u} = u_0\boldsymbol{v}$, where the parameter u_0 has dimensions of velocity and represents the "*flow strength*" of $\boldsymbol{u}$ which is independent of the geometry of the flow; the flow geometry is encapsulated in the non-dimensional vector field $\boldsymbol{v}$. With these definitions, we choose reference scales τ and ℓ in equation (6) to satisfy $u_0 = \ell/\tau$ so that $\boldsymbol{u} \mapsto \boldsymbol{v}$ and $\varepsilon \mapsto \varepsilon/u_0\,\ell\,.$

3. **Referee comment:**

   Other than that, my criticism of the limit point discussion on p. 35 was not fully interpreted. I was simply saying that a limit point of the real line cannot have eigenvalues; operators or matrices have eigenvalues. This is just an issue of language; I know what is meant but the wording is awkward.

   Response:

   We have replaced the term "limit point" with the term "accumulation point."

4. **Referee comment:**

   On the restored Figure 4, it would help in the caption to remind the reader of the relation between m and μ because otherwise it is hard to interpret the last sentence saying that m_{kk} can be determined by a relation involving μ.

   Response:

   In the updated manuscript, we have reminded the reader of the relationship between the spectral weights m_{jk} and the spectral measure $\mathrm{d}\mu_{jk}(\lambda) = \sum_n \langle m_{jk}(n)\, \delta_{\lambda_n^1}(\mathrm{d}\lambda)\rangle$, both in the caption of Fig. 4 and in the associated explanatory paragraph.