Responses to Review 2 for "Spectral analysis and computation of effective diffusivities in space-time periodic incompressible flows"

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In order to make the substantial revision of our submitted manuscript more transparent, we now provide a synopsis of the modifications. The results of the manuscript remain the same. However, virtually the entire manuscript has been modified. The responses to the referee's questions/comments are given below.

Section 1 was split into three subsections. Section 2 was split into three subsections and the body of Sections 2.2 and 2.3 were significantly modified. Section 3 was streamlined and placed on more general footing by adding Appendix E.2. Section 4 has been substantially modified. A conclusion section was added in Section 5.

Appendices A and B are virtually the same. Appendix C has been almost completely rewritten. In particular, the Hilbert spaces and their dense subsets have been redefined to have elements that are spatially mean-zero. To precisely characterize the fluid velocity field, in Appendix C.1 a new Hilbert space $\mathscr{H}_{\mathcal{V}}^{0,2}$ was defined. Moreover, in Appendix C.1 the negative inverse Laplacian $(-\Delta)^{-1}$ and the associated Green's function are now discussed in detail, as its properties are needed in the proof of a new lemma, Lemma 2, and the proof of Theorem 1 etc. Appendix C.2 has been substantially modified. In particular, we have moved some of the proof of what is now Theorem 6 to a new lemma, Lemma 3, in Appendix C.2. Lemmas 2 and 3 are now used heavily in the proof of Corollary 4 in Appendix C.2. The proof of what is now Lemma 5 has been only slightly modified. In Appendix D a good portion of the old proof of Theorem 6 has been moved to Lemma 3, as mentioned above. We have added to Appendix E a subsection, Appendix E.2. However, the introductory paragraph of Appendix E and Appendix E.1, which comprised all of Appendix E in the submitted manuscript, are virtually the same. Appendix E.2 was added to streamline our results given in Section 3 and to place them on more general footing, as mentioned above.

We now address the reviewer's questions/comments.

1. Overall, the manuscript is difficult to follow. The text would need more

structure. The reader is sometimes lost in a very long section, and does not know what has been addressed, what remains to be done,... It would be nice to split Sections 1 and 2 into several subsections, and to describe, at the beginning of each of them, what will be done in that subsection. The reader needs a better guideline.

- In order to give the reader a better guide to the paper, in the revised manuscript we have split various sections into subsections with introductory paragraphs. We have also given more prompts about where the current section is leading.

2. In order to convey their message, I think that the authors should work out a simple example (for instance, a purely diffusive problem in a time-independent setting) very early in their manuscript, in an informal manner. The different quantities at hand in this work (the corrector problem, the eigenvalue problem (A) below, the spectral measure μ, the operators $Q(\lambda), \ldots$) would appear, and their relation one to each other would be very clear. It would help the reader understand the overall strategy in a simple case.

- We have worked out a specific case in Appendix E.1 and also made its connection between the eigenvalue problem, the spectral measure, and the operator $Q(\lambda)$ clearer in the paragraph containing equations (22) and (23), early on in the revised manuscript.

3. The velocity field u is assumed to be divergence-free (see page 6). What remains true if this is not the case?

- In the introduction, we have alerted the reader to two papers that treat the problem of homogenizing an advection-diffusion equation with a compressible flow. 1) R. M. McLaughlin and M. G. Forest, Phys. Fluids, 11(4):880–892, 1999. 2) G. C. Papanicolaou, Surveys in Applied Mathematics, chapter Diffusion in Random Media, pages 205-253, 1995.

4. The fact that the authors have to call to spectral theory for *unbounded* operators seems to be related to the fact that the problem is time-dependent (see e.g. page 10, Due to the time-dependence . . .). However, this link is never clearly explained. In Appendix B page 21, it is explained that the operator ∂_t is unbounded. But this also holds for the operator Δ, which appears in the time-independent setting. I hence do not understand the particularity of the time-dependent setting.

- It is true that both of the operators ∂_t and Δ are unbounded on $L^2(\mathcal{T})$ and $L^2(\mathcal{V})$, respectively. However, it is the operator $(-\Delta)^{-1}$ that appears in the spectral theory of effective diffusivity, which is smoothing and compact. For this reason, only the unboundedness of the operator ∂_t must be addressed in the theory.

5. Top of page 12, the authors introduce an eigenvalue problem,

$$A\varphi_l = \imath\lambda_l\varphi_l. \tag{1}$$

The reason for introducing this problem is not explained. At that stage, the reader is concerned with a right-hand side problem, namely the corrector problem (9). Of course, when one knows the eigenelements of an operator, one knows how to solve a right-hand side problem (this seems to be the meaning of the line just above Eq. (80) page 38), but this is not the standard manner to proceed. Please comments on this choice and why the authors focus on eigenvalue problems. Furthermore, the link between this eigenvalue problem and the spectral measure μ is not clearly explained (or maybe it is in the Appendices, but this is somehow too late for the reader . . .).

- This information was in the Appendices. We have moved this information to equations (19)–(23) in the revised manuscript, which clearly explains the link between this eigenvalue problem and the spectral measure μ.

6. In terms of mathematical content, Section 3 seems to be much simpler than the previous sections: the authors consider an eigenvalue problem, they have a basis for the Hilbert space, so they can write the problem as an algebraic eigenvalue problem, in a space of dimension infinite but countable. This infinite matrix problem is next truncated. Maybe it would be worth explaining at the beginning of Section 3 this overall strategy.

- We have added an introductory paragraph to Section 3 that explains this overall strategy.

7. What is the reason for solving the eigenvalue problem introduced at the beginning of Section 3 on a Fourier basis? I understand that this eigenvalue problem is complemented by periodic boundary conditions, but it is unclear to me that a Fourier method is the method of choice.

- We have made the following comment below equation (30) in the revised manuscript. Using the orthonormal trigonometric basis functions $\phi_{\ell,\boldsymbol{k}}(t,\boldsymbol{x}) = \exp[\imath(\ell t + \boldsymbol{k}\cdot\boldsymbol{x})]$ leads to an exact representation of the spectral measure weights $m_{jk}(l) = \langle u_j, \varphi_l\rangle\,\overline{\langle u_k, \varphi_l\rangle}$ which involves only a finite number of terms. Of course, we could have used a different orthonormal basis. However, the spectral weights would then be given by an infinite series. In other words, the velocity field is represented by six Fourier terms in equation (28) of the revised manuscript, which would require infinitely many terms in other orthogonal basis.

8. In (15), the integral over λ seems to imply that the eigenvalues form a continuum. In contrast, the index l in the eigenvalue problem introduced at the beginning of Section 3 seems to imply that the eigenvalues are countable. Please clarify.

- We have clarified this in equations (19)–(23) in the revised manuscript.

9. In Section 4, I understand that the numerical parameter M stands for the index at which the infinite matrix problem is truncated. It is however not precised how many eigenvalues are computed. Where do we stop in terms of the index l in Eq. (A) above? This is an important point as the homogenized quantities read (see Eqs. (11), (12) and (15)) as an integral over λ.

- In the revised manuscript we have replaced the numerical parameter M with N, as M also represents the self-adjoint operator $M = -\imath A$. We compute *all* of the eigenvalues of the symmetric matrix $\mathsf{C}^{-1/2}\mathsf{B}\mathsf{C}^{-1/2}$. In our computations, we used for the steady case $N = 150$, yielding matrices of size $(2N+1)^2 - 1 = 90,600$, while in the dynamic case we used $N = 20$, yielding matrices of size $(2N+1)[(2N+1)^2 - 1] = 68,880$. This information is given in the two paragraphs below equation (37) of the revised manuscript.

Here are also some additional comments:

1. In the abstract, the authors write that they consider *periodic* velocity fields, in the time-dependent setting. Before that, there is a sentence on the state of the art in the time-independent setting. In that sentence, make precise that the velocity-field is also *periodic*.

- The abstract mentions “Two different formulations for integral representations for D^* were developed of the case of *time-independent* fluid velocity fields...” The abstract also mentions “Here, we extend both of these approaches to the case of *space-time periodic* velocity fields...” The two different formulations are presented in 1) M. Avellaneda and A. Majda. Comm. Math. Phys., 138:339–391, 1991. and 2) G. A. Pavliotis, PhD thesis, Rensselaer Polytechnic Institute Troy, New York, 2002. The first paper provides integral representations for D^* involving a *stochastic* time-independent fluid velocity field. The second paper provides integral representations for D^* involving a *periodic* time-independent fluid velocity field. For the sake of brevity in the abstract, this distinction is not made. Although, this distinction is made in the introduction.

2. Bottom of page 6: the authors define $\overline{\varphi}$, the complex conjugate of φ and next write that all quantities considered in that section are real-valued. This definition could thus be postponed to a later location in the manuscript.

- We have mentioned this definition throughout the revised manuscript. Although, the authors feel that it is also appropriate to provide the definition used throughout the paper at its first appearance. We have made this clear below equation (2) in the revised manuscript to avoid confusion.

3. Page 8, three lines below Eq. (8): what do the authors mean by "For fixed $0 < \delta \ll 1$ ..."as δ is meant to go to 0 (to converge to the homogenized problem)?

- We have removed this part of the sentence to avoid confusion.

4. Bottom of page 8: the authors consider the operator $(-\Delta)^{-1}$ but do not make precise the boundary conditions they consider.

- We made the boundary conditions precise below the statement of Theorem 1 in the revised manuscript.