Report on *Spectral analysis and computation of effective diffusivities in space-time periodic incompressible flows*, by N.B. Murphy, E. Cherkaev, J. Xin, J. Zhu and K.M. Golden

This manuscript is concerned with a time-dependent advection-diffusion problem, where the advection is driven by an incompressible velocity field, which is assumed to be periodic in time and space. The long time, large scale behavior of the solution is described by a homogenized problem, which turns out to be a purely diffusive problem with a constant homogenized diffusion tensor $D^\star$. This tensor is defined in terms of a corrector function. It may also be defined in terms of a spectral measure μ.

As recalled by the authors, this problem has already been considered in the literature in the time-independent case. The contribution of this manuscript is to extend the results to the time-dependent case. This involves several additional (and interesting) mathematical difficulties.

Before publication can be considered, I would like the following general remarks to be addressed (as obvious from this report, and as adviced by the AMSA team, I focused on the overall manuscript qualities rather than on a detailed technical review):

1. Overall, the manuscript is difficult to follow. The text would need more structure. The reader is sometimes lost in a very long section, and does not know what has been addressed, what remains to be done, ... It would be nice to split Sections 1 and 2 into several subsections, and to describe, at the beginning of each of them, what will be done in that subsection. The reader needs a better guideline.

2. In order to convey their message, I think that the authors should work out a simple example (for instance, a purely diffusive problem in a time-independent setting) very early in their manuscript, in an informal manner. The different quantities at hand in this work (the corrector problem, the eigenvalue problem (A) below, the spectral measure μ, the operators $Q(\lambda)$, ...) would appear, and their relation one to each other would be very clear. It would help the reader understand the overall strategy in a simple case.

3. The velocity field u is assumed to be divergence-free (see page 6). What remains true if this is not the case?

4. The fact that the authors have to call to spectral theory for *unbouded* operators seems to be related to the fact that the problem is time-dependent (see e.g. page 10, "Due to the time-dependence . . . "). However, this link is never clearly explained. In Appendix B page 21, it is explained that the operator ∂_t is unbounded. But this also holds for the operator Δ, which appears in the time-independent setting. I hence do not understand the particularity of the time-dependent setting.

5. Top of page 12, the authors introduce an eigenvalue problem,

$$A\varphi_l = i\lambda_l\varphi_l. \tag{A}$$

   The reason for introducing this problem is not explained. At that stage, the reader is concerned with a right-hand side problem, namely the corrector problem (9). Of course, when one knows the eigenelements of an operator, one knows how to solve a right-hand side problem (this seems to be the meaning of the line just above Eq. (80) page 38), but this is not the standard manner to proceed. Please comments on this choice and why the authors focus on eigenvalue problems.

   Furthermore, the link between this eigenvalue problem and the spectral measure μ is not clearly explained (or maybe it is in the Appendices, but this is somehow too late for the reader . . .).

6. In terms of mathematical content, Section 3 seems to be much simpler than the previous sections: the authors consider an eigenvalue problem, they have a basis for the Hilbert space, so they can write the problem as an algebraic eigenvalue problem, in a space of dimension infinite but countable. This infinite matrix problem is next truncated. Maybe it would be worth explaining at the beginning of Section 3 this overall strategy.

7. What is the reason for solving the eigenvalue problem introduced at the beginning of Section 3 *on a Fourier basis*? I understand that this eigenvalue problem is complemented by periodic boundary conditions, but it is unclear to me that a Fourier method is the method of choice.

8. In (15), the integral over λ seems to imply that the eigenvalues form a continuum. In contrast, the index l in the eigenvalue problem introduced at the beginning of Section 3 seems to imply that the eigenvalues are countable. Please clarify.

9. In Section 4, I understand that the numerical parameter M stands for the index at which the infinite matrix problem is truncated. It is however not precised how many eigenvalues are computed. Where do we stop in terms of the index l in Eq. (A) above? This is an important point as the homogenized quantities read (see Eqs. (11), (12) and (15)) as an integral over λ.

Here are also some additional comments:

1. In the abstract, the authors write that they consider *periodic* velocity-fields, in the time-dependent setting. Before that, there is a sentence on the state of the art in the time-independent setting. In that sentence, make precise that the velocity-field is also *periodic*.

2. Bottom of page 6: the authors define $\overline{\varphi}$, the complex conjugate of φ, and next write that all quantities considered in that section are real-valued. This definition could thus be postponed to a later location in the manuscript.

3. Page 8, three lines below Eq. (8): what do the authors mean by "For fixed $0 < \delta \ll 1, \ldots$", as δ is meant to go to 0 (to converge to the homogenized problem)?

4. Bottom of page 8: the authors consider the operator $(-\Delta)^{-1}$ but do not make precise the boundary conditions they consider.