Responses to Review 1 for "Spectral analysis and computation of effective diffusivities in space-time periodic incompressible flows"

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In order to make the substantial revision of our submitted manuscript more transparent, we now provide a synopsis of the modifications. The results of the manuscript remain the same. However, virtually the entire manuscript has been modified. The responses to the referee's questions/comments are given below.

Section 1 was split into three subsections. Section 2 was split into three subsections and the body of Sections 2.2 and 2.3 were significantly modified. Section 3 was streamlined and placed on more general footing by adding Appendix E.2. Section 4 has been substantially modified. A conclusion section was added in Section 5.

Appendices A and B are virtually the same. Appendix C has been almost completely rewritten. In particular, the Hilbert spaces and their dense subsets have been redefined to have elements that are spatially mean-zero. To precisely characterize the fluid velocity field, in Appendix C.1 a new Hilbert space $\mathscr{H}^{0,2}_{\mathcal{V}}$ was defined. Moreover, in Appendix C.1 the negative inverse Laplacian $(-\Delta)^{-1}$ and the associated Green's function are now discussed in detail, as its properties are needed in the proof of a new lemma, Lemma 2, and the proof of Theorem 1 etc. Appendix C.2 has been substantially modified. In particular, we have moved some of the proof of what is now Theorem 6 to a new lemma, Lemma 3, in Appendix C.2. Lemmas 2 and 3 are now used heavily in the proof of Corollary 4 in Appendix C.2. The proof of what is now Lemma 5 has been only slightly modified. In Appendix D a good portion of the old proof of Theorem 6 has been moved to Lemma 3, as mentioned above. We have added to Appendix E a subsection, Appendix E.2. However, the introductory paragraph of Appendix E and Appendix E.1, which comprised all of Appendix E in the submitted manuscript, are virtually the same. Appendix E.2 was added to streamline our results given in Section 3 and to place them on more general footing, as mentioned above.

We now address the reviewer's questions/comments.

1. Throughout the paper, the sentence the domain $D(M)$ of an operator M

is densely defined in the underlying Hilbert space is a bit confusing. I think something like the domain $D(M)$ is dense or M is densely defined is clearer. Or if these are not exactly what you mean, define the first sentence at the beginning.

- The phrase "densely defined" has been eliminated and the suggested changes have been utilized.

2. p.8, below eq.(8), can you comment a little bit more on the conditions to have uniform ($\sup_{x\in\mathbb{R}^d}$) convergence?

- The result in eq. (8) is stated in reference [55], *Majda and Kramer, Physics Reports 213, 237 (1999).* Therein, the proof of this result is referenced to the article *A.J. Majda, Lectures on turbulent diffusion, Lecture Notes at Princeton University, 1990.* We have tried to obtain an copy of this article without success, as it is not readily available to the public. We have not tried to reproduce the result and instead have referenced [55].

3. p.8, below eq.(10), make the definition of $\mathcal{T}$ and $\mathcal{V}$ explicit, namely $V = [0, 2\pi]^d$ and $\mathcal{T} = [0, 2\pi]$ or other choices. Make sure these are consistent with the Fourier series used in p.12 eq.(19).

- Below eq.(10), we have given the examples that you have suggested. These are consistent with the Fourier series used in Section 3.

4. p.9, line 3, in the definition of $(-\Delta)^{-1}$, specify which boundary condition is associated to the problem, namely periodic boundary condition with zero average. This operator is used throughout the paper; it worths to make this clarification.

- We have made the suggested clarification after the statement of Theorem 1 in Appendix C1 of the revised paper.

5. p.18, line 5 in paragraph 3, the sentence whenever $\{f_n\}$ and $\{\Phi f_n\}$ exist sounds strange.

- We have fixed this sentence.

6. p.19, eq.(33), it is better to refer to a specific theorem with specific definition of $Q(\lambda)$, and assumptions on Σ (or the operator Φ).

- We have made the reference to the spectral theorem, the operator Φ, and its spectrum Σ more specific.

7. p.19, eq.(34)(35) are defined for $\lambda \in \Sigma$ only, right? Are they extended somehow to the whole line/plane, or is $\lambda \in \Sigma$ assumed to be a line/plane?

- Yes, these equations are defined for $\lambda \in \Sigma$ only. We have made this more explicit in the paper. Also the spectrum Σ of a self-adjoint operator is real-valued, $\Sigma \subseteq \mathbb{R}$. We have made this more explicit in the paper as well.

8. p.23, in C.1., last line, $\langle \nabla\phi \cdot \nabla\varphi \rangle = \int_0^1 \int_{[0,1]^d} \nabla\phi \cdot \nabla\varphi dx dt$ does not define an inner product, since if φ is a function of the t variable only, then this product is zero but φ is not necessary the zero of $\mathscr{H}$. Also, for eq.(46), specify with respect to which norms of $\tilde{\mathscr{A}_\mathcal{T}}$ and $\mathscr{H}^1_\mathcal{V}$ are the completion done.

- The Sobolev space $\mathscr{H}^1_\mathcal{V}$ is now denoted by $\mathscr{H}^{1,2}_\mathcal{V}$. The problem that you pointed out has been fixed by defining $\mathscr{H}^{1,2}_\mathcal{V}$ to have spatially mean-zero elements. Otherwise, non-zero constant ψ satisfies $\int_\mathcal{V} |\boldsymbol{\nabla}\psi|^2 \,\mathrm{d}\boldsymbol{x} = 0$. With this change, $\langle \nabla\phi \cdot \nabla\varphi \rangle = \int_{\mathcal{T}\times\mathcal{V}} \nabla\phi \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$ *does* define an inner product on the function space $\tilde{\mathscr{A}_\mathcal{T}} \otimes \mathscr{H}^{1,2}_\mathcal{V}$. We have discussed this in detail in the revision. Also, the function space $\tilde{\mathscr{A}_\mathcal{T}}$ is *not* a complete Hilbert space and is instead an everywhere dense subset of the Hilbert space $\mathscr{H}_\mathcal{T}$. We have clarified this in the paper and also specified on which function space and with what norm the completion of the Sobolev space $\mathscr{H}^{1,2}_\mathcal{V}$ is done.

9. p.24, line 1, why is $\|\partial_t\psi\|_1 < \infty$?

- The norm $\|\cdot\|_1$ is now denoted $\|\cdot\|_{1,2}$. Since $\langle |\partial_t\psi|^2 \rangle_\mathcal{T} < \infty$ for all $\psi \in \tilde{\mathscr{A}_\mathcal{T}}$ and $\langle |\boldsymbol{\nabla}\psi|^2 \rangle_\mathcal{V} < \infty$ for all $\psi \in \mathscr{H}^{1,2}_\mathcal{V}$, we have $\|\partial_t\psi\|^2_{1,2} = \langle |\boldsymbol{\nabla}\partial_t\psi|^2 \rangle < \infty$ for all $\psi \in \mathscr{F}$, where $\mathscr{F} = \tilde{\mathscr{A}_\mathcal{T}} \otimes \mathscr{H}^{1,2}_\mathcal{V}$. To avoid confusion we have removed this sentence, as it is not used in the paper.

10. p.25, I am not sure about the derivation to the bound in eq.(49). $\mathscr{F}$ is defined through a completion, shouldn't it be the Sobolev space $H^1(\mathcal{T}\times\mathcal{V})$ of L^2 integrable functions on the torus with L^2 integrable (weak) derivatives? In that case, Sobolev embedding indicates that a L^∞ bound would fail in $d \geq 2$.

- The issue with the derivation to the bound in eq.(49) of the submitted paper has been fixed in the revision and the result has been generalized, now given in Lemma 2 of the revised paper.

11. The above estimate is used also in p.26, eq.(50), p.29, line 6 below eq.(56), etc. Please make sure the above question is clarified.

- The question in item 10 has been clarified in Lemma 2.

12. p.30, below eq.(62), the relation $\mathbf{J}_k = \boldsymbol{\sigma}\mathbf{E}_k$ is not local. In fact, this relation is a partial differential equation. It worths to mention this though the seemingly local formula is a nice analog with the corresponding constitutional equation in the Maxwells equations.

- This point has been clarified in the paper.

13. p.31, eq.(65), indeed $\mathbf{A} = \boldsymbol{\Gamma}\mathbf{S}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\mathbf{S}$ since on $\mathcal{H}$, the vector fields are already curl free. However, it is not clear to me that $\boldsymbol{\Gamma}\mathbf{S}\boldsymbol{\Gamma} = \mathbf{S}\boldsymbol{\Gamma}$ (and hence $\boldsymbol{\Gamma}\mathbf{S} = \mathbf{S}\boldsymbol{\Gamma}$), because it is not clear to me that the range of $\mathbf{S}$ is contained in $\mathcal{H}$. I don't see where $\boldsymbol{\Gamma}\mathbf{S} = \mathbf{S}\boldsymbol{\Gamma}$ is needed, in any case.

- These equalities were intended in a weak sense that has been clarified in equation (A-52) of the revised paper.

14. p.32, Lemma 4 and its proof, double check the indices in σ^*_{jk}. For instance, on p. 33, last sentence in paragraph 1, is $\langle \boldsymbol{J}_j \cdot \boldsymbol{E}_k \rangle = \langle \boldsymbol{J}_j \cdot \boldsymbol{e}_k \rangle$ equal to σ^*_{jk} or σ^*_{kj}? The same question for the fourth line in eq.(71). If the latter (instead the current version), then in eq.(70), there should be no "transpose."

- Since $\langle \boldsymbol{J}_j \rangle = \boldsymbol{\sigma}^* \langle \boldsymbol{E}_j \rangle = \boldsymbol{\sigma}^* \boldsymbol{e}_j$, we have $\langle \boldsymbol{J}_j \cdot \boldsymbol{E}_k \rangle = \langle \boldsymbol{J}_j \cdot \boldsymbol{e}_k \rangle = \boldsymbol{\sigma}^* \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \sigma^*_{jk}$. This and the formula $\mathsf{D}^*_{jk} = \varepsilon \delta_{jk} + \langle u_j \chi_k \rangle$ imply that $\sigma^*_{jk} = \langle \boldsymbol{J}_j \cdot \boldsymbol{e}_k \rangle = \langle u_k \chi_j \rangle + \varepsilon \delta_{jk} + \langle \mathsf{H}_{jk} \rangle = \mathsf{D}^*_{kj} + \langle \mathsf{H}_{jk} \rangle$, yielding $\boldsymbol{\sigma}^* = [\mathsf{D}^*]^T + \langle \mathsf{H} \rangle$.

15. p.34, last 3 lines in paragraph 2, for $f \in \mathscr{H}^1_{\mathcal{V}}$, $\partial f / \partial x_j$ could be defined as stated here, except that the limit of of $\{\partial f_n / \partial x_j\}$ should be sought for in $\|\cdot\|$ topology rather than $\|\cdot\|_1$ (which itself involves a derivative, according to the definition in eq.(45)). Also, it is not clear that $\{\partial^2 f_n / \partial x_j^2\}$ will have a limit, since $\mathscr{H}^1_{\mathcal{V}}$ is taken as the completion w.r.t. a norm that only concerns first order derivatives.

- We have updated these statements according to your suggestions and moved them to a more appropriate place below equation (A-15) of the revised paper, where the Sobolev space is defined.

16. Examine the consequence of item 8 above on the norms appearing in Appendix D.

- We have fixed the problem that you pointed out in item 8. Please see the corresponding response for details.

  Here are list of typos I found:

17. p.7, last line $\phi(0, x)$ should be ϕ^δ.

- We have fixed this typo.

18. p.23, line 4 in section C.1., in $\mathcal{V} = \times_{j=1}^d$, $\times$ should be $\otimes$?

- We have fixed this typo.

19. p.31, line 1, $\mathcal{H}$ should be $\mathscr{F}$?

- This line was removed in the revision.

20. p.32, line 3 from bottom of paragraph 1, "$\mathbf{S} = \ldots$ is self-adjoint" should be "$-\imath\mathbf{S} = \ldots$ is self-adjoint"?

- This line was removed in the revision.