Report for the paper:

Sepctral analysis and computation of effective diffusivities in space-time periodic incompressible flows

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The authors have revised their manuscript very carefully and answered all of my questions in the previous report. I believe the current manuscript is ready for publication, after the following points are clarified. The first one is a bit technical, and the second one is a very minor comment.

1. p.33, for eq. (A-27), I think if we apply (A-23) directly to the function $\boldsymbol{u} \cdot \boldsymbol{\nabla} f$, it will require the Hölder components of (A-27) to satisfy $1/p_1 + 1/q_1 = 1/2$, because in $\|\cdot\|_{1,2}$ the square of the function is integrated. More precisely,

$$\|(-\Delta)^{-1}(\boldsymbol{u}\cdot\nabla)f\|_{1,2}^2 = \int_{\mathcal{T}}\int_{\mathcal{V}} \left|\nabla((-\Delta)^{-1}(\boldsymbol{u}\cdot\nabla)f)\right|^2 \overset{\text{(A-23)}}{=} C\int_{\mathcal{T}} \|\boldsymbol{u}\cdot\nabla f\|_{L_x^2}^2$$

and if $1/p + 1/p' = 1/2$

$$\|\boldsymbol{u}\cdot\nabla f\|_{L_x^2} \leq \|\boldsymbol{u}\cdot\nabla f\|_{L_x^p}\|\boldsymbol{u}\cdot\nabla f\|_{L_x^{p'}}.$$

For this reason, the requirement (A-31) cannot be made since $1/p_1 + 1/q_1 = 1/2$ already forces $p_1 \geq 2$ and $q_1 \geq 2$.

I still have the feeling that a requirement of r where r is the component $\boldsymbol{u} \in L_t^r(L_x^r)$ necessarily depends on the dimension d. Essentially, (A-25) and (A-26) are $W_x^{1,2}$ estimates (I don't see how the time integral of absolute values of functions would help to improve the estimates) for m, the solution of the problem

$$-\Delta m = h := \boldsymbol{u}\cdot\nabla f.$$

We essentially require $\nabla f \in L_x^2$. So if $u \in L_x^r$, then $h \in L_x^p$ with $p = 2r/(r+2)$, i.e.

$$1/2 + 1/r = 1/p.$$

Then elliptic regularity (Calderón-Zygmund) says $m \in W^{2,p}$. In view of Sobolev embedding, if we want $m \in W^{1,2}$, a natural sufficient condition would be (assume

$d \geq 2$)

$$1/p^* = 1/p - 1/d \leq 1/2 \quad i.e. \quad p^* \geq 2,$$

which requires $r \geq d$.

2. Regarding the formula (A-60), I had a question about whether it should be

$$\sigma^*_{jk} = \langle \boldsymbol{J}_j \cdot \boldsymbol{e}_k \rangle \quad \text{or} \quad \sigma^*_{jk} = \langle \boldsymbol{J}_k \cdot \boldsymbol{e}_j \rangle.$$

According to (A-49), I would say σ^*_{jk}, being the element at the jth row and kth element, should be $\boldsymbol{e}_j \cdot (\boldsymbol{\sigma}^* \boldsymbol{e}_k)$, should be given by $\langle \boldsymbol{e}_j \cdot \boldsymbol{J}_k \rangle$ (the latter choice above). Then I will get

$$\text{LHS of (A-61)} = \sigma^*_{kj} = \cdots \stackrel{\text{4th row}}{=} \langle \chi_j u_k \rangle + \varepsilon \delta_{jk} + \langle H_{kj} \rangle = D^*_{kj} + \langle H_{kj} \rangle$$

(both the authors' and my choices of $A = (A_{jk})$ are consistent for $\boldsymbol{\sigma}^*$ and for H) and finally

$$\boldsymbol{\sigma}^* = [D^*] + \langle H \rangle.$$