# A Unification of The Critical Theory of Transport in Binary Composite Media


N. B. Murphy[1]

(1) *University of Utah, Department of Mathematics,*

*155 S 1400 E RM 233, Salt Lake City, UT 84112-009, USA*

(K. M. Golden[1])



(Dated: August 28, 2011)



We demonstrate that G. A. Baker's critical theory of the Ising model may be adapted to provide a detailed description of percolation driven critical transitions in transport, exhibited by binary composite media. In a novel unified approach, we reproduce K. M. Golden's pioneering (static) results regarding an insulating/conductive medium, and produce the analogous static results regarding a conductive/superconductive medium, finding the (two–parameter) critical exponent scaling relations of each system. Moreover, we extend these results to the quasi–static regime, where the material contrast parameter becomes complex. Under a physically consistent symmetry assumption we link these two sets of scaling relations, so that the scaling relations of both systems are completely determined by the knowledge of only (two) exponents. We also provide a general proof of the fundamental assumption underlying Golden's original work. Namely, that critical transitions in transport are due to the formation of delta function components in the underlying spectral measures, at the spectral endpoints. More specifically, in percolation models we show that the onset of the critical transition (the formation of these delta components) occurs *precisely* at the percolation threshold.

## I. INTRODUCTION

Disordered composite materials have found a well rooted place in the sciences and engineering technology due to their countless applications. These materials are of great utility as they often *combine* the attributes of the underlying constituent materials [1]. There is even greater functionality when a composite has the ability to transition between the various underlying and/or combined attributes of the constituents. This is a key feature of disordered media whose effective behavior is dominated by the connectedness, or percolation properties, of a particular component. Examples include bone, doped semiconductors, radar absorbing composites, thin metal films, porous media such as sea ice and rock, and smart materials such as rheological suspensions, piezoresistors, and thermistors. The behavior of such media is particularly challenging to describe physically, and to predict mathematically.

Here, we construct a mathematical framework which unifies the critical theory of transport in binary composite media. We accomplish this by adapting techniques developed by G. A. Baker for the Ising model [2], to provide a detailed description of percolation driven critical transitions in transport. The most natural formulation of this problem is in terms of the conduction problem in the continuum $\mathbb{R}^d$, which includes the lattice $\mathbb{Z}^d$ as a special case [3, 4]. Although, the underlying symmetries [1] in the effective parameter problem of electrical conductivity and permittivity, magnetic permeability, and thermal conductivity, immediately generalize our results to all of these systems.

## II. BACKGROUND AND SUMMARY OF OUR RESULTS

In 1952 T. D. Lee and C. N. Yang showed that the root distribution of the Ising model partition function $Z$, a polynomial in an appropriate variable [2, 5–7], completely determines the associated equation of state [8]. Moreover, they demonstrated that the properties of the system, in relation to phase transitions, are governed by the behavior of these roots near the positive real axis. They did so by proving that the roots of $Z$ lie on the unit circle. This result is known as The Lee–Yang Theorem [5, 7].

In 1968 G. A. Baker used The Lee–Yang Theorem to represent the Gibbs free energy per spin $f = -\beta^{-1} \ln Z$ as a logarithmic potential [9], where $\beta = (kT)^{-1}$, $k$ is Boltzmann's constant [10], and $T$ is the absolute temperature. He used this special analytic structure to prove that the magnetization per spin $M(T,H) = -\partial f/\partial H$ [11] may be represented in terms of a Stieltjes function $G$ in the variable $\tau = \tanh \beta m H$ [2, 9],

$$\frac{M}{m} = \tau(1+(1-\tau^2)G(\tau^2)), \quad G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1+\tau^2 y}\,, \tag{1}$$

where $H$ is the applied magnetic field strength, $m$ is the (constant) magnetic dipole moment of each spin [12], and $\psi$ is a non–negative definite measure [9]. The integral representation (1) immediately leads to the inequalities

$$G \geq 0, \qquad \frac{\partial G}{\partial u} \leq 0, \qquad \frac{\partial^2 G}{\partial u^2} \geq 0, \tag{2}$$

where $u := \tau^2$. The last formula in equation (2) is the GHS inequality, which is an important tool in the study of the Ising model [3].

In 1970 D. Ruelle extended The Lee–Yang Theorem and proved that there exists a gap $\theta_0(T) > 0$ in the roots of $Z$ about the positive real axis for high temperatures [13]. Moreover, he proved that the gap collapses ($\theta_0(T) \to 0$) as $T$ decreases to a critical temperature $T_c > 0$. Consequently, the temperature driven phase transition (spontaneous magnetization) is unique, and is characterized by the pinching of the real axis by the roots of $Z$ [7].

In [2, 14] G. A. Baker exploited The Lee–Yang–Ruelle Theorem and defined a critical exponent $\Delta$ for the gap $\theta_0(T) \sim (T-T_c)^{\Delta}$, as $T \to T_c^+$, in the distribution of the Lee–Yang–Ruelle zeros. He showed that for $T > T_c$, the support $\Sigma_\psi$ of the measure $\psi$ is contained in the compact interval $[0, S(T)]$ and that $S(T) \sim (T-T_c)^{-2\Delta}$, as $T \to T_c^+$. Furthermore, he showed that the moments $\psi_n = \int_0^\infty y^n \, d\psi(y)$ of the measure $\psi$ diverge as $T \to T_c^+$, according to the power law $\psi_n \sim (T-T_c)^{-\gamma_n}$, $n \geq 0$. Using a Stieltjes function characterization theorem [2], he showed that the sequence $\gamma_n$ satisfies Baker's inequalities $\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0$, which imply that this sequence increases at least linearly with $n$. Later [2], he showed that this sequence *is* actually linear in $n$, $\gamma_n = \gamma + 2\Delta n$, with constant gap $\gamma_i - \gamma_{i-1} = 2\Delta$. The critical exponent $\gamma$ is defined via the magnetic susceptibility $\chi = \partial M/\partial H = -\partial^2 f/\partial H^2 \sim (T-T_c)^{-\gamma}$, as $T \to T_c^+$. The phase transition may be concisely described with two other critical exponents. When $H = 0$, $M(T,0) \sim (T-T_c)^{\beta}$, as $T \to T_c^-$, where the critical exponent $\beta$ is not to be confused with $(kT)^{-1}$. Along the critical isotherm $T = T_c$, $M(T_c, H) \sim H^{1/\delta}$, as $H \to 0$ [2, 15]. Using the integral representation (1), Baker obtained the (two–parameter) scaling relations between the critical exponents [2]

$$\beta = \Delta - \gamma, \qquad \delta = \Delta/(\Delta-\gamma), \qquad \gamma_n = \gamma + 2\Delta n. \tag{3}$$

The critical exponent $\gamma$, for example, is defined in terms of the following limit, and $\gamma$ exists when this limit exists [2]:

$$\gamma := \limsup_{T\to T_c^+,\ H=0} \left( \frac{-\ln \chi(T,H)}{\ln(T-T_c)} \right). \tag{4}$$

In 1997 K. M. Golden demonstrated that, in the static regime, Lee–Yang–Ruelle–Baker critical theory may be adapted to precisely describe percolation driven critical transitions in transport [16]. This deep and far reaching result puts these two classes of seemingly unrelated problems on an equal mathematical footing. He did so by considering percolation models of conductive binary composite media. There, the connectedness of the system is determined by the volume fraction $p$ of defect inclusions with conductance $\sigma_2$ in an otherwise homogeneous medium of conductance $\sigma_1$, whereby assumption $h = \sigma_1/\sigma_2 \in (0,1)$. He demonstrated that the function $m(p,h) = \sigma^*(p,h)/\sigma_2$ plays the role of the magnetization per spin $M(T,H)$ in the Ising model, where $\sigma^*(p,h)$ is the effective conductance of the random medium [4, 17, 18]. Moreover, he showed that the volume fraction $p$ mimics the temperature $T$ while the contrast ratio $h$ mimics the applied magnetic field $H$. More specifically, the critical insulator/conductor behavior in transport arises when $h = 0$ $(\sigma_1 = 0,\ 0 < \sigma_2 < \infty)$, as $p \to p_c^+$ [16], and the analogous non-magnetic/ferromagnetic critical behavior of the Ising model arises when $H = 0$, as $T \to T_c^+$ [15]. Using these mathematical parallels, Golden showed that the critical exponents of transport satisfy Baker's inequalities, Baker's two–parameter scaling relations (3), etc.

Here, using a novel unified approach, we reproduce Golden's *static* results ($h \in \mathbb{R}$) and produce the analogous static results associated with a conductive/superconductive medium in terms of $w(p,z) = \sigma^*(p,z)/\sigma_1$, where $z = 1/h$. Using Stieltjes function integral representations of $m(p,h;\mu)$ and $w(p,z;\alpha)$, where $\mu$ and $\alpha$ are underlying bounded positive measures, we determine the (two–parameter) critical exponent scaling relations of each system. We then extend these results to the quasi-static regime, where the contrast parameter becomes complex ($h \in \mathbb{C}$). Assuming a physically consistent symmetry in the properties of $\mu$ and $\alpha$, we link these two sets of scaling relations. Thereby showing that the scaling relations of both systems are determined by only (two) critical exponents. We also provide a general proof of the fundamental assumption underlying Golden's pioneering work, for finite lattice systems. More specifically, we explicitly show by construction that there are gaps in the supports of the measures $\alpha(d\lambda)$ and $\mu(d\lambda)$ about the spectral endpoints $\lambda = 0, 1$ for $p \ll 1$

and $1 - p \ll 1$, respectively, which collapse as $p$ tends towards $p_c$. Moreover, for infinite systems, we demonstrate that critical transitions in transport are due to the formation of delta function components of $\mu$ and $\alpha$ located at $\lambda = 0, 1$. We do so by constructing a measure $\varrho(d\lambda)$ that is supported on the set $\{0, 1\}$ which links the measures $\mu$ and $\alpha$. This result demonstrates, for percolation models, that the onset of the critical transition (the formation of these delta components) occurs *precisely* at the percolation threshold $p_c$.

## III. THE ANALYTIC CONTINUATION METHOD

We now formulate the effective parameter problem for two–component conductive media. Let $(\Omega, P)$ be a probability space and let $\boldsymbol{\sigma}(\vec{x}, \omega)$ be the local conductivity tensor and let $[\boldsymbol{\sigma}^{-1}](\vec{x}, \omega)$ be the local resistivity tensor, which are (spatially) stationary random fields in $\vec{x} \in \mathbb{R}^d$ and $\omega \in \Omega$. Here $\Omega$ is the set of all realizations of our random medium and $P(d\omega)$ is the underlying probability measure, which is compatible with stationarity [4]. Define the Hilbert space of stationary random fields $\mathscr{H}_s \subset L^2(\Omega, P)$, and the underlying Hilbert spaces of stationary curl free $\mathscr{H}_\times \subset \mathscr{H}_s$ and divergence free $\mathscr{H}_\bullet \subset \mathscr{H}_s$ random fields [4]:

$$\begin{aligned} \mathscr{H}_\times &:= \{\vec{Y}(\omega) \in \mathscr{H}_s \mid \vec{\nabla} \times \vec{Y} = 0 \text{ weakly and } \langle \vec{Y} \rangle = 0\}, \\ \mathscr{H}_\bullet &:= \{\vec{Y}(\omega) \in \mathscr{H}_s \mid \vec{\nabla} \cdot \vec{Y} = 0 \text{ weakly and } \langle \vec{Y} \rangle = 0\}, \end{aligned} \tag{5}$$

where $\vec{Y} : \Omega \mapsto \mathbb{R}^d$ and $\langle \cdot \rangle$ means ensemble average over $\Omega$, or by an ergodic theorem [4] spatial average over all of $\mathbb{R}^d$.

Consider the following variational problems: find $\vec{E}_f \in \mathscr{H}_\times$ and $\vec{J}_f \in \mathscr{H}_\bullet$ such that

$$\langle \boldsymbol{\sigma}(\vec{E}_0 + \vec{E}_f) \cdot \vec{Y} \rangle = 0 \quad \forall\, \vec{Y} \in \mathscr{H}_\times, \qquad \langle \boldsymbol{\sigma}^{-1}(\vec{J}_0 + \vec{J}_f) \cdot \vec{Y} \rangle = 0 \quad \forall\, \vec{Y} \in \mathscr{H}_\bullet, \tag{6}$$

respectively [4]. Under the assumption that the bilinear forms $a(\vec{u}, \vec{v}) = \vec{u}^T \boldsymbol{\sigma}(\vec{x}, \omega) \vec{v}$ and $\hat{a}(\vec{u}, \vec{v}) = \vec{u}^T [\boldsymbol{\sigma}^{-1}](\vec{x}, \omega) \vec{v}$ are bounded and coercive, where $\vec{u}, \vec{v} \in \mathbb{R}^d$, these problems have unique solutions satisfying [4]

$$\begin{array}{lllll} \vec{\nabla} \times \vec{E} = 0, & \vec{\nabla} \cdot \vec{J} = 0, & \vec{J} = \boldsymbol{\sigma} \vec{E}, & \vec{E} = \vec{E}_0 + \vec{E}_f, & \langle \vec{E} \rangle = \vec{E}_0, \\ \vec{\nabla} \times \vec{E} = 0, & \vec{\nabla} \cdot \vec{J} = 0, & \vec{E} = \boldsymbol{\sigma}^{-1} \vec{J}, & \vec{J} = \vec{J}_0 + \vec{J}_f, & \langle \vec{J} \rangle = \vec{J}_0, \end{array} \tag{7}$$

respectively. Here $\vec{E}_f$ and $\vec{J}_f$ are the fluctuating electric field and current density of mean zero, respectively, about the (constant) averages $\vec{E}_0$ and $\vec{J}_0$, respectively.

We assume that the tensor $\boldsymbol{\sigma}(\vec{x},\omega)$ takes the values $\sigma_1$ and $\sigma_2$, and that the tensor $[\boldsymbol{\sigma}^{-1}](\vec{x},\omega)$ takes the values $1/\sigma_1$ and $1/\sigma_2$, and write $\boldsymbol{\sigma}(\vec{x},\omega) := \sigma_1\chi_1(\vec{x},\omega)+\sigma_2\chi_2(\vec{x},\omega)$ and $[\boldsymbol{\sigma}^{-1}](\vec{x},\omega) := \chi_1(\vec{x},\omega)/\sigma_1 + \chi_2(\vec{x},\omega)/\sigma_2$, where $\chi_j$ is the characteristic function of medium $j = 1,2$, which equals one for all $\omega \in \Omega$ having medium $j$ at $\vec{x}$, and zero otherwise [4]. As $\vec{E}_f \in \mathscr{H}_\times$ and $\vec{J}_f \in \mathscr{H}_\bullet$, equation (6) yields the energy (power density) constraints $\langle \vec{J}\cdot\vec{E}_f\rangle = \langle \vec{E}\cdot\vec{J}_f\rangle = 0$, which lead to the following reduced energy representations:

$$\langle \vec{J}\cdot\vec{E}\rangle = \langle \vec{J}\rangle\cdot\vec{E}_0, \qquad \langle \vec{E}\cdot\vec{J}\rangle = \langle \vec{E}\rangle\cdot\vec{J}_0\,. \tag{8}$$

In light of (8), we define the effective complex conductivity tensor $\boldsymbol{\sigma}^*$ and the effective complex resistivity tensor $[\boldsymbol{\sigma}^{-1}]^*$ by

$$\langle \vec{J}\,\rangle = \boldsymbol{\sigma}^*\vec{E}_0, \qquad \langle \vec{E}\,\rangle = [\boldsymbol{\sigma}^{-1}]^*\vec{J}_0, \tag{9}$$

respectively. For simplicity, we focus on one diagonal component of these symmetric tensors: $\sigma^* := \boldsymbol{\sigma}^*_{kk}$ and $[\sigma^{-1}]^* := [\boldsymbol{\sigma}^{-1}]^*_{kk}$.

Due to the homogeneity of these functions, e.g. $\sigma^*(a\sigma_1, a\sigma_2) = a\sigma^*(\sigma_1,\sigma_2)$ for any complex number $a$, $\sigma^*$ and $[\sigma^{-1}]^*$ depend only on the ratio $h := \sigma_1/\sigma_2$, and we define the dimensionless functions $m(h) := \sigma^*/\sigma_2$, $w(z) := \sigma^*/\sigma_1$, $\tilde{m}(h) := \sigma_1[\sigma^{-1}]^*$, and $\tilde{w}(z) := \sigma_2[\sigma^{-1}]^*$, where $z = z(h) := 1/h$. The functions $m(h)$ and $\tilde{m}(h)$ are analytic off the negative real axis in the $h$–plane, taking the upper half plane to the upper half plane, and the functions $w(z)$ and $\tilde{w}(z)$ are analytic off the negative real axis in the $z$–plane, taking the upper half plane to the upper half plane, so that they are examples of Herglotz functions [4]. We assume that $|h| < 1$, i.e. $0 < |\sigma_1| < |\sigma_2| < \infty$, and we further restrict $h$ in the complex plane to the set

$$\mathcal{U} := \{h := h_r + \mathrm{i}h_i \in \mathbb{C} : |h| < 1 \text{ and } h \notin (-1,0]\}, \tag{10}$$

where $m(h)$, $w(z(h))$, $\tilde{m}(h)$, and $\tilde{w}(z(h))$ are analytic functions of $h$ [4]. In order to illuminate the symmetries between these functions, we will henceforth focus on the variable $h$.

The key step in the method is obtaining integral representations for $\sigma^*$ and $[\sigma^{-1}]^*$. These integral representations are given in terms of the following resolvent representations of the electric field $\vec{E}$ and the current density $\vec{J}$:

$$\vec{E} = s(s+\Gamma\chi_1)^{-1}\vec{E}_0 = t(t+\Gamma\chi_2)^{-1}\vec{E}_0, \qquad \vec{J} = s(s-\Upsilon\chi_2)^{-1}\vec{J}_0 = t(t-\Upsilon\chi_1)^{-1}\vec{J}_0, \tag{11}$$

where we have defined $s := 1/(1-h)$ and $t := 1/(1-z) = 1-s$. These formulas follow from manipulations of equation (7). The operator $-\Gamma := -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$ is a projection onto curl-free fields, based on convolution with the free-space Green's function for the Laplacian $-\Delta = -\nabla^2$ [4]. More specifically, $-\Gamma : \mathscr{H}_s \mapsto \mathscr{H}_\times$ and for every $\vec{\zeta} \in \mathscr{H}_\times$, we have $-\Gamma\vec{\zeta} = \vec{\zeta}$. To the authors knowledge, the operator $\Upsilon := \vec{\nabla} \times (-\Delta)^{-1}\vec{\nabla}\times$ is being introduced here for the first time. For the convenience of the reader, we recall a few vector calculus facts. For every $\vec{\zeta} \in \mathscr{H}_\bullet$ we have $\vec{\zeta} = \vec{\nabla} \times (\vec{A} + \vec{C})$ weakly, where $\vec{\nabla} \times \vec{C} = 0$ weakly [19]. The arbitrary vector $\vec{C}$ can be chosen so that $\vec{\nabla} \cdot \vec{A} = 0$ weakly [19]. Hence, $\vec{\nabla} \times \vec{\zeta} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta\vec{A} = -\Delta\vec{A}$ weakly. The vector $\vec{C}$ chosen in this manner gives the Coulomb (or transverse) gauge of $\vec{\zeta}$ [19]. Let $\mathscr{C}_\bullet \subset \mathscr{H}_\bullet$ denote the *closure* of the space of stationary divergence free random fields of Coulomb gauge. On the Hilbert space $\mathscr{C}_\bullet$, one can show that the operator $\Upsilon$ is a projector, based on convolution with the free-space Green's function for the Laplacian $-\Delta$. More specifically, $\Upsilon : \mathscr{H}_s \mapsto \mathscr{H}_\bullet$ and for every $\vec{\zeta} \in \mathscr{C}_\bullet$, we have $\Upsilon\vec{\zeta} = \vec{\zeta}$.

It is more convenient to consider the functions $F(s) := 1 - m(h)$ and $E(s) := 1 - \tilde{m}(h)$, which are analytic off $[0,1]$ in the $s$–plane, and $G(t) := 1 - w(z(h))$ and $H(t) := 1 - \tilde{w}(z(h))$, which are analytic off $[0,1]$ in the $t$–plane [4, 17], and satisfy

$$0 < |F(s)|, |E(s)| < 1, \quad 0 < |G(t)|, |H(t)| < \infty, \tag{12}$$

where $G(t)$ and $H(t)$ are not to be confused with the Stieltjes function in (1) and the magnetic field strength in the Ising model, respectively. We write $\vec{E}_0 = E_0\,\vec{e}_k$ and $\vec{J}_0 = J_0\vec{j}_k$, where $\vec{e}_k$ and $\vec{j}_k$ are unit vectors, for some $k = 1, \ldots, d$. Using $\vec{J} = \sigma\vec{E}$, $\vec{E} = [\sigma^{-1}]\,\vec{J}$, $\langle\vec{E}\,\rangle = \vec{E}_0$, $\langle\vec{J}\rangle = \vec{J}_0$, $\chi_1 = 1 - \chi_2$ (which leads to the identities $\sigma = \sigma_2(1 - \chi_1/s) = \sigma_1(1 - \chi_2/t)$ and $[\sigma^{-1}] = (1 - \chi_2/s)/\sigma_1 = (1 - \chi_1/t)/\sigma_2$), equations (9) and (11), and The Spectral Theorem [20], we have [4, 17]

$$F(s) = \langle\chi_1(s + \Gamma\chi_1)^{-1}\vec{e}_k \cdot \vec{e}_k\rangle := \int_{\lambda_0}^{\lambda_1} \frac{d\mu(\lambda)}{s - \lambda}\,, \quad E(s) = \langle\chi_2(s - \Upsilon\chi_2)^{-1}\vec{j}_k \cdot \vec{j}_k\rangle := \int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1} \frac{d\eta(\lambda)}{s - \lambda}\,, \tag{13}$$

$$G(t) = \langle\chi_2(t + \Gamma\chi_2)^{-1}\vec{e}_k \cdot \vec{e}_k\rangle := \int_{\hat{\lambda}_0}^{\hat{\lambda}_1} \frac{d\alpha(\lambda)}{t - \lambda}\,, \quad H(t) = \langle\chi_1(t - \Upsilon\chi_1)^{-1}\vec{j}_k \cdot \vec{j}_k\rangle := \int_{\check{\lambda}_0}^{\check{\lambda}_1} \frac{d\kappa(\lambda)}{t - \lambda}\,.$$

In order to illuminate the symmetries between the integral representations of (13), in the last two formulas of equation (13) we will henceforth make the change of variables $t(s) = 1 - s$ and $\lambda \mapsto 1 - \lambda$, so that $G(t(s)) = -\int_{1-\hat{\lambda}_1}^{1-\hat{\lambda}_0}[-d\alpha(1-\lambda)]/(s - \lambda)$, for example.

In equation (13), $\mu$, $\eta$, $\alpha$, and $\kappa$ are bounded positive measures which depend only on the geometry of the medium, and are supported on $\Sigma_\mu, \Sigma_\eta, \Sigma_\alpha, \Sigma_\kappa \subseteq [0,1]$, respectively

[4, 21], where the supremum and infimum of these sets are defined to be the respective upper and lower limits of integration in (13). The underlying Integro-differential operators $\mathbf{M}_1 := \chi_1(-\Gamma)\chi_1$, $\mathbf{M}_2 := \chi_2(-\Gamma)\chi_2$, $\mathbf{K}_1 := \chi_1\Upsilon\chi_1$, and $\mathbf{K}_2 := \chi_2\Upsilon\chi_2$ are self–adjoint on the Hilbert space $L^2(\Omega, P)$ [4]. They are compositions of projection operators on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{C}_\bullet$, and are consequently bounded by 1 in the underlying operator norm [22, 23]. Equation (13) involves spectral representations of resolvents involving the self adjoint operators, $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$, since, for example, in the Hilbert space $L^2(\Omega, P)$ with weight $\chi_2$ in the inner product, $\Gamma\chi_2$ is a bounded self–adjoint operator [4]. The measures $\mu$, $\eta$, $\alpha$, and $\kappa$ are spectral measures of the family of projections of these operators in the respective $\langle\vec{e}_k, \vec{e}_k\rangle$ or $\langle\vec{j}_k, \vec{j}_k\rangle$ state [4, 20].

A key feature of equations (8) and (13) is that the parameter information in $s$ and $E_0$ is *separated* from the geometry of the composite, which is encapsulated in the measures $\mu$, $\eta$, $\alpha$, and $\kappa$ through their moments $\mu_n$, $\eta_n$, $\alpha_n$, and $\kappa_n$, $n \geq 0$, which depend on the correlation functions of the medium [4]. For example, $\alpha_0 = \eta_0 = p$ and $\mu_0 = \kappa_0 = 1 - p$. A principal application of the analytic continuation method is to derive *forward bounds* on $\sigma^*$ and $[\sigma^{-1}]^*$, given partial information on the microgeometry [4, 18, 21, 24]. One can also use the integral representations (13) to obtain *inverse bounds*, allowing one to use data about the electromagnetic response of a sample to bound its structural parameters such as $p$ [25].

By applying The Spectral Theorem to the energy constraints $\langle\vec{J} \cdot \vec{E}_f\rangle = \langle\vec{E} \cdot \vec{J}_f\rangle = 0$, we have obtained detailed decompositions of the system energy in terms of the measures $\mu$, $\eta$, $\alpha$, and $\kappa$. For example, $\langle\vec{J} \cdot \vec{E}_f\rangle = 0$, $\vec{E} = \vec{E}_0 + \vec{E}_f$, $\langle\vec{E}_f\rangle = 0$, and $\sigma = \sigma_2(1 - \chi_1/s)$ imply that $0 = \langle\sigma\vec{E} \cdot \vec{E}_f\rangle = \langle\sigma_2(1 - \chi_1/s)(\vec{E}_f \cdot \vec{E}_0 + E_f^2)\rangle = \sigma_2 \left[\langle E_f^2\rangle - (\langle\chi_1\vec{E}_f \cdot \vec{E}_0\rangle + \langle\chi_1 E_f^2\rangle)/s\right]$. Therefore, by The Spectral Theorem [20] and the symmetries in equation (13), we have

$$\frac{\langle E_f^2\rangle}{E_0^2} = \int_0^1 \frac{\lambda\, d\mu(\lambda)}{(s-\lambda)^2} = \int_0^1 \frac{\lambda\, d\alpha(\lambda)}{(1-s-\lambda)^2}\,, \quad \frac{\langle J_f^2\rangle}{J_0^2} = \int_0^1 \frac{\lambda\, d\eta(\lambda)}{(s-\lambda)^2} = \int_0^1 \frac{\lambda\, d\kappa(\lambda)}{(1-s-\lambda)^2}\,. \tag{14}$$

Equation (14) then leads to a *complete* decomposition of the system energy in terms of Herglotz functions involving the spectral measures $\mu$, $\eta$, $\alpha$, and $\kappa$.

The Stieltjes transforms (13) of the measures $\mu$, $\eta$, $\alpha$, and $\kappa$ may be represented in terms of Stieltjes functions [2] of $h$ via the following change of variables: $s(h) = 1/(1-h)$ and $\lambda(y) = y/(1+y) \iff y(\lambda) = \lambda/(1-\lambda)$. For example,

$$F(s) = (1-h)\int_{S_0}^{S} \frac{(1+y)d\mu(\frac{y}{1+y})}{1+hy}\,, \qquad G(t(s)) = (h-1)\int_{\hat{S}_0}^{\hat{S}} \frac{(1+y)[-d\alpha(\frac{1}{1+y})]}{1+hy}\,. \tag{15}$$

Here $S_0 := \lambda_0/(1-\lambda_0)$, $S := \lambda_1/(1-\lambda_1)$, $\hat{S}_0 := (1-\hat{\lambda}_1)/\hat{\lambda}_1$, and $\hat{S} := (1-\hat{\lambda}_0)/\hat{\lambda}_0$, so that $\lim_{\lambda_0\to 0} S_0 = 0$, $\lim_{\lambda_1\to 1} S = \infty$, $\lim_{\hat{\lambda}_1\to 1}\hat{S}_0 = 0$, $\lim_{\hat{\lambda}_0\to 0}\hat{S} = \infty$. Therefore, by equation (15) and the underlying symmetries in equation (13), the Stieltjes function representations of the formulas in (13) are given by

$$
\begin{aligned}
m(h) &= 1 + (h-1)g(h), \quad g(h) := \int_0^\infty \frac{d\phi(y)}{1+hy}, \quad d\phi(y) := (1+y)d\mu\left(\frac{y}{1+y}\right), \\
\tilde{m}(h) &= 1 + (h-1)\tilde{g}(h), \quad \tilde{g}(h) := \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy}, \quad d\tilde{\phi}(y) := (1+y)d\eta\left(\frac{y}{1+y}\right), \\
w(z(h)) &= 1 - (h-1)\hat{g}(h), \quad \hat{g}(h) := \int_0^\infty \frac{d\hat{\phi}(y)}{1+hy}, \quad d\hat{\phi}(y) := (1+y)\left[-d\alpha\left(\frac{1}{1+y}\right)\right], \\
\tilde{w}(z(h)) &= 1 - (h-1)\check{g}(h), \quad \check{g}(h) := \int_0^\infty \frac{d\check{\phi}(y)}{1+hy}, \quad d\check{\phi}(y) := (1+y)\left[-d\kappa\left(\frac{1}{1+y}\right)\right].
\end{aligned} \tag{16}
$$

Equation (16) should be compared to equation (1) regarding the Ising model. The Stieltjes functions $g(h), \tilde{g}(h)$, $\hat{g}(h)$, and $\check{g}(h)$ are analytic for all $h \in \mathcal{U}$ [4]. As $\mu$, $\eta$, $\alpha$, and $\kappa$ are bounded positive measures on $[0,1]$, $\phi$, $\tilde{\phi}$, $\hat{\phi}$, and $\check{\phi}$ are positive measures on $[0,\infty]$, and are also bounded if the supports are bounded. Consequently, the following inequalities hold

$$
\frac{\partial^{2n}\zeta}{\partial h^{2n}} > 0, \quad \frac{\partial^{2n-1}\zeta}{\partial h^{2n-1}} < 0, \quad \left|\frac{\partial^n\zeta}{\partial h^n}\right| > 0, \quad \zeta = g(h), \tilde{g}(h), \hat{g}(h), \check{g}(h), \quad h \in \mathcal{U}, \tag{17}
$$

where $n \geq 0$. The first two inequalities in (17) hold for $h \in \mathcal{U} \cap \mathbb{R}$, and the last inequality holds for $h \in \mathcal{U}$ such that $h_i \neq 0$. Equation (17) is the analogue of equation (2) in the Ising model. The formula $\partial^2 m(h)/\partial h^2 > 0$ in (17), for example, is a macroscopic version of the fact that the effective resistance of a finite network is a concave downward function of the resistances of the individual network elements [3].

By equation (16), the moments $\phi_n$ of $\phi$ satisfy

$$
\phi_n = \int_0^\infty y^n d\phi(y) = \int_0^\infty y^n(1+y)d\mu\left(\frac{y}{1+y}\right) = \int_0^1 \frac{\lambda^n d\mu(\lambda)}{(1-\lambda)^{n+1}}\,. \tag{18}
$$

A partial fraction expansion of $\lambda^n/(1-\lambda)^{n+1}$ then shows that

$$
\frac{(-1)^n}{n!}\lim_{s\to 1}\frac{\partial^n F(s)}{\partial s^n} = \int_0^1 \frac{d\mu(\lambda)}{(1-\lambda)^{n+1}} = \sum_{j=0}^n \binom{n}{j}\phi_j\,, \tag{19}
$$

demonstrating that $\phi_n$ depends on $\int_0^1 d\mu(\lambda)/(1-\lambda)^{n+1}$ *and* all the lower moments of $\phi$: $\phi_j$, $j = 0, 1, \ldots, n-1$. From equations (13)–(14), we see that the first two moments of $\phi$ are identified with energy components:

$$
\phi_0 = \lim_{s\to 1}\frac{\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle}{E_0^2}, \quad \phi_1 = \lim_{s\to 1}\frac{\langle E_f^2\rangle}{E_0^2}. \tag{20}
$$

Thereby equation (19), *all* of the moments $\phi_j$, $j \geq 2$ depend on these energy components. Equation (18) suggests that the moments $\phi_n$, $n \geq 0$, become singular as $\lambda_1 := \sup\{\Sigma_\mu\} \to 1$. However, we will show that this is only true for the moments of order $n \geq 1$, and that $\lambda = 1$ is a removable *simple* singularity under $\mu$.

Similarly, the moments $\hat{\phi}_n$ of $\hat{\phi}$ satisfy

$$\hat{\phi}_n = \int_0^1 \frac{(1-\lambda)^n d\alpha(\lambda)}{\lambda^{n+1}}, \qquad \frac{(-1)^{n+1}}{n!} \lim_{s\to 1} \frac{\partial^n G(t(s))}{\partial^n t} = \int_0^1 \frac{d\alpha(\lambda)}{\lambda^{n+1}} = \sum_{j=0}^{n} \binom{n}{j} \hat{\phi}_j \,. \tag{21}$$

Equations (13)–(14) similarly identify the first two moments, $\hat{\phi}_0$ and $\hat{\phi}_1$, of $\hat{\phi}$ with energy components. Equation (21) then implies that all the higher moments $\hat{\phi}_j$, $j \geq 2$, depend on these energy components. Equation (21) suggests, and we will show that *all* the moments $\hat{\phi}_n$, $n \geq 0$, become singular as $\hat{\lambda}_0 := \inf\{\Sigma_\alpha\} \to 0$. By the symmetries in equations (13) and (16), equations (18)–(19) hold for $\tilde{\phi}$ with $E(s)$ and $\eta$ in lieu of $F(s)$ and $\mu$, respectively, and equation (21) holds for $\check{\phi}$ with $H(t(s))$ and $\kappa$ in lieu of $G(t(s))$ and $\alpha$, respectively.

By equations (8)–(9), we have the following two energy representations of $\sigma^*$ and $[\sigma^{-1}]^*$, $\langle \vec{J}\cdot\vec{E}\rangle = \sigma_2 m(h) E_0^2 = \sigma_1 w(z(h)) E_0^2$ and $\langle \vec{E}\cdot\vec{J}\rangle = \tilde{m}(h) E_0^2/\sigma_1 = \tilde{w}(z(h)) E_0^2/\sigma_2$, which imply

$$\begin{aligned} m(h) = h w(z(h)) \iff 1 - F(s) = (1-1/s)(1-G(t(s))), \\ \tilde{m}(h) = h \tilde{w}(z(h)) \iff 1 - E(s) = (1-1/s)(1-H(t(s))). \end{aligned} \tag{22}$$

Using equation (16), minor algebraic manipulation in equation (22) implies that

$$g(h) + h\hat{g}(h) = 1, \qquad \tilde{g}(h) + h\check{g}(h) = 1, \quad h \in \mathcal{U}. \tag{23}$$

For $h \in \mathcal{U}$, the functions $g(h)$, $\hat{g}(h)$, $\tilde{g}(h)$, and $\check{g}(h)$ are analytic [4], and have bounded $h$ derivatives of all orders [22]. An inductive argument applied to equation (23) yields

$$\frac{\partial^n g}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}}{\partial h^{n-1}} + h\frac{\partial^n \hat{g}}{\partial h^n} = 0, \qquad \frac{\partial^n \tilde{g}}{\partial h^n} + n\frac{\partial^{n-1}\check{g}}{\partial h^{n-1}} + h\frac{\partial^n \check{g}}{\partial h^n} = 0, \quad n \geq 1. \tag{24}$$

In the complex quasi–static case, where $h \in \mathcal{U}$ such that $h_i \neq 0$, the complex representation of equation (24) is, for example,

$$\frac{\partial^n g_r}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}_r}{\partial h^{n-1}} + h_r\frac{\partial^n \hat{g}_r}{\partial h^n} - h_i\frac{\partial^n \hat{g}_i}{\partial h^n} = 0, \quad \frac{\partial^n g_i}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}_i}{\partial h^{n-1}} + h_r\frac{\partial^n \hat{g}_i}{\partial h^n} + h_i\frac{\partial^n \hat{g}_r}{\partial h^n} = 0, \tag{25}$$

where we have used the following definitions:

$$\frac{\partial^n g_r}{\partial h^n} := \mathrm{Re}\frac{\partial^n g}{\partial h^n}, \quad \frac{\partial^n g_i}{\partial h^n} := \mathrm{Im}\frac{\partial^n g}{\partial h^n}, \qquad \frac{\partial^n \hat{g}_r}{\partial h^n} := \mathrm{Re}\frac{\partial^n \hat{g}}{\partial h^n}, \quad \frac{\partial^n \hat{g}_i}{\partial h^n} := \mathrm{Im}\frac{\partial^n \hat{g}}{\partial h^n}\,.$$

The formulas, analogous to (25), associated with $\tilde{g}$ and $\check{g}$ follow from the substitutions $g \mapsto \tilde{g}$ and $\hat{g} \mapsto \check{g}$.

The integral representations of equations (24)–(25) follow from Lemma III.1 below. We focus on the measures $\phi$ and $\hat{\phi}$, as the analogous results involving $\tilde{\phi}$ and $\check{\phi}$ follow from the symmetries in equations (13) and (16).

**Lemma III.1** *Set $Y_{i,j}(h,y) := y^i/(1+hy)^j$. Then for all $h \in \mathcal{U}$ and $i,j \in \mathbb{R}$ satisfying $0 < i \leq j-1$, we have $Y_{i,j}(h,y) \in L^1(\hat{\phi}(dy))$, and for $0 < i \leq j$, $Y_{i,j}(h,y) \in L^1(\phi(dy))$. Consequently ([23] Theorem 2.27), the Stieltjes functions $g(h)$ and $\hat{g}(h)$ may be differentiated under the integral sign, i.e. for $n \geq 0$ we have*

$$\frac{\partial^n g(h)}{\partial h^n} = (-1)^n n! \int_0^\infty \frac{y^n d\phi(y)}{(1+hy)^{n+1}}\,, \qquad \frac{\partial^n \hat{g}(h)}{\partial h^n} = (-1)^n n! \int_0^\infty \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}\,. \tag{26}$$

Before we prove Lemma III.1, we note that equations (24) and (26) imply that

$$\int_0^S \frac{y^n d\phi(y)}{(1+hy)^{n+1}} = \int_0^\infty \frac{y^{n-1} d\hat{\phi}(y)}{(1+hy)^n} - h\int_0^\infty \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}\,, \quad n \geq 1,\ h \in \mathcal{U}. \tag{27}$$

Moreover, Lemma III.1 and (27) yield the integral representations of (25) using, for example,

$$\frac{(-1)^n}{n!}\frac{\partial^n g(h)}{\partial h^n} = \int_0^\infty \frac{y^n d\phi(y)}{|1+hy|^{2(n+1)}}(1+\bar{h}y)^{n+1} = \sum_{j=0}^{n+1}\binom{n+1}{j}\bar{h}^j \int_0^\infty \frac{y^{n+j} d\phi(y)}{|1+hy|^{2(n+1)}}\,, \tag{28}$$

where $\bar{h}$ denotes complex conjugation of the complex variable $h$.

**Proof of Lemma III.1**: The support of $\phi$ and $\hat{\phi}$ are $\Sigma_\phi := [S_0, S\,]$ and $\Sigma_{\hat{\phi}} := [\hat{S}_0, \hat{S}\,]$, respectively, which are defined in terms $\Sigma_\mu$ and $\Sigma_\alpha$, respectively, directly below equation (15). For every $h \in \mathcal{U}$, it is clear that there exists real, strictly positive $S_h$ such that

$$1 \ll |h| S_h < \infty. \tag{29}$$

Set $h \in \mathcal{U}$ and $0 \ll S_h < \infty$ satisfying (29), and write $\Sigma_\phi := [S_0, S_h] \cup (S_h, S\,]$ and $\Sigma_\phi := [\hat{S}_0, S_h] \cup (S_h, \hat{S}\,]$. Equations (12) and (18) imply that $0 \leq \lim_{h\to 0}|m(h)| = 1 - \phi_0 < 1$, which implies that the mass $\phi_0$ of $\phi$ is uniformly bounded. Therefore, for all $h \in \mathcal{U}$,

$$\int_{S_0}^{S_h} |Y_{i,j}(h,y)| d\phi(y) \leq \frac{S_h^i\,\phi([S_0,S_h])}{|1+hS_0|^j} < \infty, \quad \int_{\hat{S}_0}^{S_h} |Y_{i,j}(h,y)| d\phi(y) \leq \frac{S_h^i\,\hat{\phi}([\hat{S}_0,S_h])}{|1+h\hat{S}_0|^j} < \infty, \tag{30}$$

where $\phi([S_0,S_h])$ is the *bounded* $\phi$ measure of the set $[S_0,S_h]$. The boundedness of the second formula in equation (30) follows from equations (15)–(16), showing that the $\hat{\phi}$ measure of the compact interval $[\hat{S}_0, S_h]$ is bounded. More specifically, in terms of $\Sigma_\alpha$ we have $\hat{\lambda}_1 = 1 - \hat{S}_0/(1+\hat{S}_0)$ and $\hat{\lambda}_h := 1 - S_h/(1+S_h) > 0$. Thus equations (15)–(16) imply that

$$\hat{\phi}([\hat{S}_0, S_h]) = \int_{\hat{S}_0}^{S_h} d\hat{\phi}(y) = \int_{\hat{S}_0}^{S_h} (1+y)\left[-d\alpha\left(\frac{1}{1+y}\right)\right] = \int_{\hat{\lambda}_h}^{\hat{\lambda}_1} \frac{d\alpha(\lambda)}{\lambda} \leq \frac{\alpha_0}{\hat{\lambda}_h} < \infty.$$

If $\Sigma_\phi$ and $\Sigma_{\hat{\phi}}$ are compact intervals, we are done. Otherwise set $S = \hat{S} = \infty$. In terms of $\Sigma_\mu$ and $\Sigma_\alpha$, we have $\lambda_h := S_h/(1+S_h)$ and $\lambda_1 = S/(1+S) \equiv 1$, and $\hat{\lambda}_0 = 1 - \hat{S}/(1+\hat{S}) \equiv 0$ and $\hat{\lambda}_h = 1 - S_h/(1+S_h)$, respectively, where $0 \ll \lambda_h < 1$ and $0 < \hat{\lambda}_h \ll 1$. When $0 < i \leq j-1$, equations (16) and (29) imply that, for all $h \in \mathcal{U}$,

$$\begin{aligned}|h|^j \int_{S_h}^{\hat{S}} |Y_{i,j}(h,y)| d\hat{\phi}(y) &\sim \int_{S_h}^{\hat{S}} \frac{1+y}{y^{j-i}} d\alpha\left(\frac{1}{1+y}\right) = \int_{1-\hat{\lambda}_h}^{1-\hat{\lambda}_0} \frac{(1-\lambda)^{j-i-1}\left[-d\alpha(1-\lambda)\right]}{\lambda^{j-i}} \\ &= \int_{\hat{\lambda}_0}^{\hat{\lambda}_h} \frac{\lambda^{j-i-1}\, d\alpha(\lambda)}{(1-\lambda)^{j-i}} \leq \frac{\hat{\lambda}_h^{j-i-1}\,\alpha_0}{(1-\hat{\lambda}_h)^{j-i}} < \infty.\end{aligned}$$

When $0 < i \leq j$, equations (12), (16), and (29) imply that, for all $h \in \mathcal{U}$,

$$|h|^j \int_{S_h}^{S} |Y_{i,j}(h,y)| d\phi(y) \sim \int_{\lambda_h}^{\lambda_1} \frac{(1-\lambda)^{j-i-1}}{\lambda^{j-i}} d\mu(\lambda) \leq \frac{(1-\lambda_h)^{j-i}}{\lambda_h^{j-i}} \int_{\lambda_h}^{1} \frac{d\mu(\lambda)}{1-\lambda} < \infty,$$

as $0 < F(1) = \int_0^1 d\mu(\lambda)/(1-\lambda) \leq 1$. This concludes the proof of Lemma III.1 □.

All the equations given in this section display general formulas holding for two–component stationary random media in lattice and continuum settings [16]. In section III A below, we demonstrate that equations (22)–(23) and The Stieltjes-Perron Inversion Theorem [26] allow us to construct a measure $\varrho$, supported on the set $\{0,1\}$, which links the measures $\mu$ and $\alpha$. Moreover, the properties of $\varrho$ imply that critical transitions in the transport properties of $\sigma^*$ and $[\sigma^{-1}]^*$ are caused by the formation of delta function components in the spectral measures $\mu(d\lambda)$ and $\alpha(d\lambda)$, at the spectral endpoints $\lambda = 0, 1$.

### A. Measure Equivalences in Transport

In this section, we show that the symmetries underlying the analytic continuation method allow one to construct precise relations between the measures $\mu$ and $\alpha$, and $\eta$ and $\kappa$. We already noted that the formulas in equation (13) are Stieltjes transforms of the measures $\mu$, $\eta$, $\alpha$, and $\kappa$. Conversely, given the Stieltjes transform of a measure, The Stieltjes-Perron Inversion Theorem [1, 26, 27] allows one to recover the underlying measure. For example,

$$\mu(\upsilon) = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0} \mathrm{Im}(F(\upsilon + \mathrm{i}\epsilon)), \quad \upsilon \in \Sigma_\mu. \tag{31}$$

To evoke this theorem directly, in equation (13) we define $d\tilde{\alpha}(\lambda) := [-d\alpha(1-\lambda)]$ and $d\tilde{\kappa}(\lambda) = [-d\kappa(1-\lambda)]$, and write $G(t(s)) = -\int_0^1 d\tilde{\alpha}(\lambda)/(s-\lambda)$ and $H(t(s)) = -\int_0^1 d\tilde{\kappa}(\lambda)/(s-\lambda)$. Setting $s = \upsilon + \mathrm{i}\epsilon$ for $\upsilon \in \Sigma_\mu \cap \Sigma_\alpha$, equations (22) and (31) imply that

$$\upsilon\mu(\upsilon) = (1-\upsilon)[-\alpha(1-\upsilon)] - \upsilon\varrho(\upsilon), \qquad \upsilon\eta(\upsilon) = (1-\upsilon)[-\kappa(1-\upsilon)] - \upsilon\tilde{\varrho}(\upsilon), \quad (32)$$
$$\varrho(\upsilon) = \lim_{\epsilon\downarrow 0} \frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2}\int_0^1 \frac{(\upsilon+\lambda-1)\,d\alpha(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2}, \quad \tilde{\varrho}(\upsilon) = \lim_{\epsilon\downarrow 0} \frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2}\int_0^1 \frac{(\upsilon+\lambda-1)\,d\kappa(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2}.$$

Here, we show that (22)–(23) and (32) explicitly determine the measures $\varrho$ and $\tilde{\varrho}$.

The integral representations of equation (23) follow from equation (16), and are given by

$$\int_0^\infty \frac{d\phi(y)}{1+hy} + h\int_0^\infty \frac{d\hat{\phi}(y)}{1+hy} = 1, \qquad \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy} + h\int_0^\infty \frac{d\check{\phi}(y)}{1+hy} = 1. \quad (33)$$

Due to the underlying symmetries of this framework, without loss of generality, we henceforth focus on $F(s;\mu)$, $G(t(s);\alpha)$, $g(h;\phi)$, and $\hat{g}(h;\hat{\phi})$. We wish to re-express the first formula in equation (33) in a more suggestive form, by adding and subtracting the quantity $h\int_0^\infty y\,d\phi(y)/(1+hy)$. This is permissible if the modulus of this quantity is finite for all $h \in \mathcal{U}$ [22, 23]. The affirmation of this fact is given by Lemma IV.1, and we may therefore add and subtract it in equation (33), yielding

$$h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} \equiv 1-\phi_0 \equiv m(0), \quad d\Phi_0(y) := d\hat{\phi}(y) - y\,d\phi(y), \quad \forall\ h \in \mathcal{U}, \quad (34)$$

as $1-\phi_0 \equiv 1 - F(s)|_{s=1} = m(h)|_{h=0}$. Equation (34) gives an alternate representation of $m(0) = \lim_{h\to 0} hw(h)$, and shows that the transform of the signed measure [22] $\Phi_0$, $h\int_0^\infty d\Phi_0(y)/(1+hy)$, is independent of $h$ for all $h \in \mathcal{U}$. Using equation (16) and the variable identity $y = \lambda/(1-\lambda) \iff \lambda = y/(1+y)$, we may relate this representation of $m(0)$ to the measure $\varrho$ found in equation (32):

$$d\Phi_0(y) == \frac{1}{(1-\lambda)^2}((1-\lambda)\,[-d\alpha(1-\lambda)] - \lambda\,d\mu(\lambda)) = \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2} = y(1+y)\,d\varrho\left(\frac{y}{1+y}\right).$$

We may therefore express equation (34) in terms of $\varrho(d\lambda)$ as follows:

$$m(0) = h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} = h\int_0^\infty \frac{y(1+y)d\varrho(\frac{y}{1+y})}{1+hy} = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h + \lambda(1-\lambda)}. \quad (35)$$

**Remark III.1** *Define the transform $\mathcal{D}(h;\varrho)$ of the measure $\varrho$ by*

$$\mathcal{D}(h;\varrho) = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h + \lambda(1-\lambda)}. \quad (36)$$

*Equations* (12) *and* (34)–(35) *show that $\mathcal{D}(h;\varrho)$ satisfies the following properties:* ***(1)*** *$\mathcal{D}(h;\varrho)$ is independent of $h$,* ***(2)*** $0 < \mathcal{D}(h;\varrho) < 1$, *and* ***(3)*** $\mathcal{D}(h;\varrho) \equiv m(0)$.

The Lemma III.2 is the key result of this section.

**Lemma III.2** *Let $\mathcal{D}(h;\varrho)$ be defined as in equation (36), where $h \in \mathcal{U}$, and let the Herglotz functions $m(0) := m(h)|_{h=0} = 1 - F(s)|_{s=1}$ and $w(0) := w(z)|_{z=0} = 1 - G(t)|_{t=1}$ be defined as in equation (13), which satisfy $0 \leq m(0), w(0) < 1$. If $\mathcal{D}(h;\varrho)$ satisfies the properties of Remark III.1, then*

$$\begin{aligned} \varrho(d\lambda) &= -w(0)\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda), \\ \tilde{\varrho}(d\lambda) &= -\tilde{w}(0)\delta_0(d\lambda) + \tilde{m}(0)(1-\lambda)\delta_1(d\lambda), \end{aligned} \tag{37}$$

*where $\delta_{\lambda_0}(d\lambda)$ is the Dirac measure centered at $\lambda_0$.*

**Proof**: The proof of the second formula in equation (37) follows directly from the proof of the first formula in (37) and the underlying symmetries of this mathematical framework. Let $\mathcal{D}(h;\varrho)$, defined in equation (36), satisfy the properties of Remark III.1. The measure $\varrho$ is independent of $h$ [4]. If the support $\Sigma_\varrho$ of the measure $\varrho$ is over continuous spectrum [20] then $\mathcal{D}(h;\varrho)$ depends on $h$, contradicting property (1). Therefore the measure $\varrho$ is defined over pure point spectrum $\Sigma_\varrho \subset \sigma_{pp}$ [20]. Moreover, in order for properties (1) and (3) to be satisfied, we must have $\Sigma_\varrho \equiv \{0,1\}$. This implies that the measure $\varrho$ is of the form

$$\varrho(d\lambda) = W_0(\lambda)\delta_0(d\lambda) + W_1(\lambda)\delta_1(d\lambda),$$

where the $W_j(\lambda)$, $j = 0,1$, are functions of $\lambda \in [0,1]$ which are to be determined. In view of the numerator of the integrand in equation (36), we may assume that the function $W_0(\lambda) \equiv W_0(0) := W_0 \not\equiv 0$ is independent of $\lambda$. In order for property (2) to be satisfied we must have $W_1(\lambda) \sim 1-\lambda$ as $\lambda \to 1$ (any other power of $1-\lambda$ would contradict property (2)). Therefore, with out loss of generality, we may set $W_1(\lambda) = \mathrm{w}_1(1-\lambda)$, where $\mathrm{w}_1$ is independent of $\lambda$. Property (3) then implies that $\mathrm{w}_1 = m(0)$.

We have shown that $\varrho(d\lambda) = W_0\,\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda)$, $W_0 \not\equiv 0$. By plugging this formula into equation (32) ($\lambda d\mu(\lambda) = (1-\lambda)[-d\alpha(1-\lambda)] - \lambda d\varrho(\lambda)$), we are able determine $W_0$. Indeed, using the definition of $F(s)$ (13), equation (22) ($F(s) - (1-1/s)G(t(s)) = 1/s$), and $(1-\lambda)/(\lambda(s-\lambda)) = -(1-1/s)/(s-\lambda) + 1/(s\lambda)$, we find that

$$\begin{aligned} F(s) &= -\left(1-\frac{1}{s}\right)\int_0^1 \frac{[-d\alpha(1-\lambda)]}{s-\lambda} + \frac{1}{s}\int_0^1 \frac{[-d\alpha(1-\lambda)]}{\lambda} - \int_0^1 \frac{d\varrho(\lambda)}{s-\lambda} \\ &= \left(1-\frac{1}{s}\right)G(t(s)) + \frac{1}{s}\int_0^1 \frac{d\alpha(\lambda)}{1-\lambda} - \frac{W_0}{s} - m(0)\lim_{\lambda\to 1}\frac{1-\lambda}{s-\lambda}, \quad \forall\ |s| > 1 \end{aligned} \tag{38}$$

which implies that $W_0 = -w(0)$. This concludes the proof of Lemma III.2 $\square$.

The equations of this section display general formulas holding for two–component stationary random media in the lattice and continuum settings [16]. It is worth mentioning that equation (27) can be written in terms of the measure $d\Phi_{n-1}(y) := y^{n-1} d\Phi_0(y)$: $\int_0^\infty d\Phi_{n-1}(y)/(1+hy)^{n+1} \equiv 0$, for all $n \geq 1$, $h \in \mathcal{U}$. By Lemma III.1, this integral involving the signed measure $\Phi_0(dy)$ (34) is defined. Furthermore, in equation (25) for $n = 1$ , equation (28) implies that $\int_0^\infty d\Phi_1(y)/|1+hy|^4 \equiv 0$, for all $h \in \mathcal{U}$ such that $h_i \neq 0$. These formulas are easily seen to be consistent with Lemma III.2.

The formulas in equation (37) demonstrate that $\lambda = 1$ is a removable *simple* singularity under $\mu$, $\alpha$, $\eta$, and $\kappa$, and illustrate how the relations (12) $0 < |F(s)|, |E(s)| \leq 1$ can hold even when $s = 1$ ($h = 0$) and the spectra extends all the way to $\lambda = 1$. Moreover, this shows that $\phi$ and $\tilde{\phi}$ are bounded measures with mass (18) $0 < \phi_0, \tilde{\phi}_0 \leq 1$, and that the higher moments $\phi_j$ and $\tilde{\phi}_j$, $j \geq 2$, become singular when the spectra extends all the way to $\lambda = 1$. Furthermore, these formulas (37) illustrate that singular behavior can develop in $G(t(s))$ and $H(t(s))$ when the strength $w(0)$ of the delta component at $\lambda = 0$ becomes non–zero. Moreover, this shows that $\hat{\phi}$ and $\check{\phi}$ (21) are unbounded measures when $w(0) \neq 0$. How these features relate to percolation models will be discussed in more detail in Section IV.

## IV. CRITICAL BEHAVIOR OF TRANSPORT IN LATTICE AND CONTINUUM PERCOLATION MODELS

We now formulate the problem of percolation driven critical transitions in transport, exhibited by two–component conductive media. For percolation models such as the random bond network (RBN) [28, 29], the connectedness of the system is determined by the volume fraction $p$ of type two inclusions in an otherwise homogeneous type one medium. The average cluster size of these inclusions grows as $p$ increases, and there is a critical volume fraction $p_c$, $0 < p_c < 1$, called the *percolation threshold*, where an infinite cluster of the inclusions first appears.

Consider transport through a RBN [16] where bonds are assigned electrical conductivities $\sigma_2$ with probability $p$, and $\sigma_1$ with probability $1-p$. In the limit $h \to 0$, the composite may be interpreted as a conductor/insulator system ($\sigma_1 \to 0$ while $0 < |\sigma_2| < \infty$) or a conductor/superconductor system ($\sigma_2 \to \infty$ while $0 < |\sigma_1| < \infty$). In this limit, the transport properties of the system exhibit critical behavior. As $h \to 0$ ($\sigma_1 \to 0$ and $0 < |\sigma_2| < \infty$), the effective complex conductivity $\sigma^*(p,h) := \sigma_2\, m(p,h)$ and the effective complex resistance $[\sigma^{-1}]^*(p,h) := \sigma_2^{-1}\tilde{w}(p,z(h))$ undergo a conductor/insulator critical transition:

$$|\sigma^*(p,0)| := |\sigma_2\, m(p,0)| = \begin{cases} 0, & \text{for } p < p_c \\ 0 < |\sigma_1| < |\sigma^*(p)| < |\sigma_2|, & \text{for } p > p_c \end{cases}, \tag{39}$$

$$|[\sigma^{-1}]^*(p,z(0))| := |\sigma_2^{-1}\tilde{w}(p,z(0))| = \begin{cases} \infty, & \text{for } p < p_c \\ |\sigma_2|^{-1} < |[\sigma^{-1}]^*(p)| < |\sigma_1|^{-1}, & \text{for } p > p_c \end{cases}.$$

While, as $h \to 0$ ($\sigma_2 \to \infty$ and $0 < |\sigma_1| < \infty$), the effective complex conductivity $\sigma^*(p,h) := \sigma_1 w(p,z(h))$ and the effective complex resistance $[\sigma^{-1}]^*(p,h) := \sigma_1^{-1}\tilde{m}(p,h)$ undergo a conductor/superconductor critical transition:

$$|\sigma^*(p,z(0))| := |\sigma_1 w(p,z(0))| = \begin{cases} 0 < |\sigma^*(p)| < \infty, & \text{for } p < p_c \\ \infty, & \text{for } p > p_c \end{cases}, \tag{40}$$

$$|[\sigma^{-1}]^*(p,0)| := |\sigma_1^{-1}\tilde{m}(p,0)| = \begin{cases} 0 < |[\sigma^{-1}]^*(p)| < \infty, & \text{for } p < p_c \\ 0, & \text{for } p > p_c \end{cases}.$$

We will focus on the conductor/insulator critical transition of the effective complex conductivity $\sigma^*(p,h) = \sigma_2\, m(p,h)$ and the conductor/superconductor critical transition of the effective complex conductivity $\sigma^*(p,h) = \sigma_1 w(p,z(h))$. It is clear from equations (16) and (39)–(40), that our results immediately generalize to $[\sigma^{-1}]^*(p,h) = \sigma_1^{-1}\tilde{m}(p,h)$ and $[\sigma^{-1}]^*(p,h) = \sigma_2^{-1}\tilde{w}(p,z(h))$, respectively, with $p \mapsto 1-p$.

The critical behavior of binary conductors is made more precise through the definition of critical exponents. For $h \in \mathbb{R} \cap \mathcal{U}$, as $h \to 0$ the effective conductivity $\sigma^*(p,h) = \sigma_2\, m(p,h)$ exhibits the following critical conductor/insulator behavior near the percolation threshold, $\sigma^*(p,0) \sim (p-p_c)^t$ as $p \to p_c^+$, and at $p = p_c$, $\sigma^*(p_c,h) \sim h^{1/\delta}$ as $h \to 0$, where the critical exponent $t$ is not to be confused with the contrast parameter. We assume the existence (4) of the critical exponents $t$ and $\delta$, as well as $\gamma$, defined via a conductive susceptibility $\chi(p,0) := \partial m(p,0)/\partial h \sim (p-p_c)^{-\gamma}$ as $p \to p_c^+$. Furthermore, for $p > p_c$, we assume that there is a gap $\theta_\mu \sim (p-p_c)^\Delta$ in the support of $\mu$ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^+$, or that any spectrum in this region does not affect power law behavior [16]. Therefore, for our percolation models with $p > p_c$, the support of $\phi$ is contained in the compact interval $[0, S(p)]$, where $S(p) \sim (p-p_c)^{-\Delta}$ as $p \to p_c^+$. As the moments of $\phi$ become singular as $\theta_\mu \to 0$ (18), we also assume that there exist (4) critical exponents $\gamma_n$ such that $\phi_n(p) \sim (p-p_c)^{-\gamma_n}$ as $p \to p_c^+$, $n \geq 0$. When $h \in \mathcal{U}$ such that $h_i \neq 0$, we assume the existence (4) of critical exponents $t_r$, $\delta_r$, $t_i$ and $\delta_i$ corresponding to $m_r(p,h) := \mathrm{Re}(m(p,h))$ and $m_i(p,h) := \mathrm{Im}(m(p,h))$. In summary:

$$\begin{aligned}
&m(p,0) \sim (p-p_c)^t, && m_r(p,0) \sim (p-p_c)^{t_r}, && m_i(p,0) \sim (p-p_c)^{t_i}, && \text{as } p \to p_c^+ \\
&m(p_c,h) \sim h^{1/\delta}, && m_r(p_c,h) \sim |h|^{1/\delta_r}, && m_i(p_c,h) \sim |h|^{1/\delta_i}, && \text{as } |h| \to 0, \\
&\chi(p,0) \sim (p-p_c)^{-\gamma}, && \phi_n \sim (p-p_c)^{-\gamma_n}, && S(p) \sim (p-p_c)^{-\Delta}, && \text{as } p \to p_c^+.
\end{aligned} \tag{41}$$

We also assume the existence of critical exponents associated with the left hand limit $p \to p_c^-$: $\gamma'$, $\gamma_n'$, and $\Delta'$. The critical exponents $\gamma$, $\delta$, $\Delta$, and $\gamma_n$ for transport are different from those defined in section II for the Ising model (3).

For $h \in \mathbb{R} \cap \mathcal{U}$, as $h \to 0$ the effective conductivity $\sigma^*(p,z(h)) = \sigma_1 w(p,z(h))$ exhibits critical conductor/superconductor behavior near $p_c$, $\sigma^*(p,z(0)) \sim (p-p_c)^{-s}$ as $p \to p_c^-$, and at $p = p_c$, $\sigma^*(p_c,z(h)) \sim h^{-1/\hat{\delta}}$ as $h \to 0$, where the superconductor critical exponent $s$ is not to be confused with the contrast parameter. We assume the existence (4) of the critical exponents $s$ and $\hat{\delta}$, as well as $\hat{\gamma}'$, defined via a conductive susceptibility $\hat{\chi}(p) := \partial w(p,z(0))/\partial h \sim (p-p_c)^{-\hat{\gamma}'}$ as $p \to p_c^-$. Furthermore, for $p < p_c$, we assume that there is a gap $\theta_\alpha \sim (p-p_c)^{\hat{\Delta}'}$ in the support of $[-d\alpha(1-\lambda)]$ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^-$, so that the support of $\hat{\phi}$ is contained in the compact interval $[0, \hat{S}(p)]$, where $\hat{S}(p) \sim (p-p_c)^{-\Delta}$ as $p \to p_c^+$. As the moments of $\hat{\phi}$ become singular as $\theta_\alpha \to 0$ (21), we also assume that there exist (4) critical exponents $\hat{\gamma}_n'$ such that $\hat{\phi}_n(p) \sim (p-p_c)^{-\hat{\gamma}_n'}$ as $p \to p_c^-$, $n \geq 0$. When $h \in \mathcal{U}$ such that $h_i \neq 0$, we assume the existence (4) of critical exponents $s_r$, $s_i$, $\hat{\delta}_r$, and $\hat{\delta}_i$ corresponding to $w_r(p,z(h)) := \mathrm{Re}(w(p,z(h)))$ and $w_i(p,z(h)) := \mathrm{Im}(w(p,z(h)))$. In summary:

$$\begin{aligned}
&w(p,z(0)) \sim (p-p_c)^{-s}, && w_r(p,z(0)) \sim (p-p_c)^{-s_r}, && w_i(p,z(0)) \sim (p-p_c)^{-s_i}, && \text{as } p \to p_c^- \\
&w(p_c,z(h)) \sim h^{-1/\hat{\delta}}, && w_r(p_c,z(h)) \sim |h|^{-1/\hat{\delta}_r}, && w_i(p_c,z(h)) \sim |h|^{-1/\hat{\delta}_i}, && \text{as } |h| \to 0, \\
&\hat{\chi}(p) \sim (p-p_c)^{-\hat{\gamma}'}, && \hat{\phi}_n \sim (p-p_c)^{-\hat{\gamma}_n'}, && \hat{S}(p) \sim (p-p_c)^{-\hat{\Delta}'}, && \text{as } p \to p_c^-.
\end{aligned} \tag{42}$$

We also assume the existence of critical exponents associated with the right hand limit $p \to p_c^+$: $\hat{\gamma}$, $\hat{\gamma}_n$, and $\hat{\Delta}$.

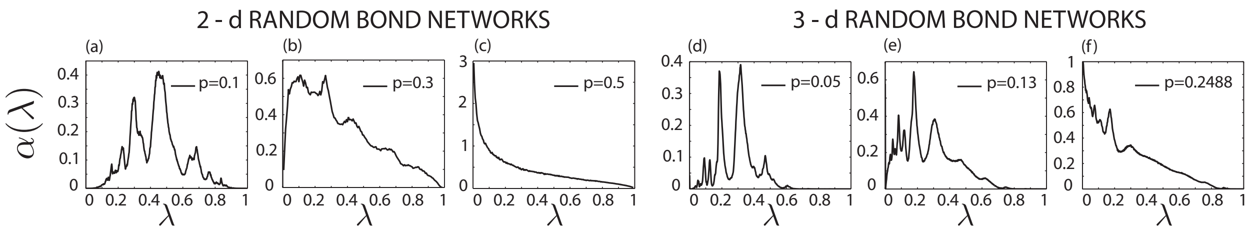


FIG. 1. *The spectral function for the 2–d and 3–d square random bond networks.* As the volume fraction $p$ of defect bonds increases, from left to right, the width of the gaps in the spectrum near $\lambda = 0, 1$ shrink to 0 with increasing connectedness as the percolation thresholds $p_c = 0.5$ and $p_c \approx 0.2488$ are approached.

### A. Spectral Characterization of Critical Transitions in Transport

We now discuss the gaps $\theta_\alpha$ and $\theta_\eta$ (for $p < p_c$), and $\theta_\mu$ and $\theta_\kappa$ (for $p > p_c$). As the operators $-\Gamma$ and $\Upsilon$ are projectors on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{C}_\bullet$, respectively, the eigenvalues thereof are confined to the set $\{0, 1\}$ [20]. The associated operators $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$ are positive definite compositions of projection operators, thus the eigenvalues thereof are confined to the set $[0, 1]$ [30].

While, in general, the spectra actually extends all the way to the spectral endpoints $\lambda = 0, 1$, the part close to $\lambda = 0, 1$ corresponds to very large, but very rare connected regions of the defect inclusions (Lifshitz phenomenon), and is believed to give exponentially small contributions to the effective complex conductivity (resistivity), and not affect power law behavior [16]. In [31] O. Bruno has proven the existence of spectral gaps in matrix/particle systems with polygonal inclusions, and studied how the gaps vanishes as the inclusions touch (like $p \to p_c$). In Figure IV A, we give a graphical representation of the spectral measure $\alpha(d\lambda)$ for finite 2–$d$ and 3–$d$ RBNs [25]. This figure shows that, as $p \to p_c^+$, the width of the gaps in the spectrum near $\lambda = 0, 1$ vanish. Our simulations also show that, as $p$ increases beyond $p_c$, the spectrum piles up at the spectral endpoints $\lambda = 0, 1$ until $\Sigma_\alpha = \{0, 1\}$, when $p = 1$. This behavior is predicted by our result (37) in section III A, with weights $m(0) = m(p, 0)$ and $w(0) = w(p, 0)$. From equation (39) we see that the onset of the critical transition (the increase of these weights from zero) occurs *precisely* at the percolation threshold $p = p_c$, causing delta function components of the measures $\mu(d\lambda)$ and $\alpha(d\lambda)$ to be present at $\lambda = 0, 1$ for all $p > p_c$.

We now provide a proof, for large but finite lattice systems, of the existence of spectral gaps in these measures which collapse as $p$ tends towards $p_c$. For lattice systems with a finite number $n$ of lattice sites, the differential equations in (7) become difference equations (Kirchoff's laws) [3]. Consequently, the operators $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$ are given by $N \times N$ matrices, say [3, 25]. We focus on $\mathbf{M}_2 = \chi_2(-\Gamma)\chi_2$, as our results extend to the other operators by symmetry. In this lattice setting, $-\Gamma$ is a real symmetric projection matrix and can therefore be diagonalized: $-\Gamma = \mathbf{QDQ}^T$, where $\mathbf{D}$ is a diagonal matrix of zeros and ones and $\mathbf{Q}$ is a real orthogonal matrix. More specifically,

$$-\Gamma = \begin{bmatrix} - \vec{q}_1 - \\ \vdots \\ - \vec{q}_N - \end{bmatrix} \begin{bmatrix} \mathbf{I}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} - \vec{q}_1 - \\ \vdots \\ - \vec{q}_N - \end{bmatrix}^T = \begin{bmatrix} (\vec{q}_1 \cdot \vec{q}_1)_L & (\vec{q}_1 \cdot \vec{q}_2)_L & \cdots & (\vec{q}_1 \cdot \vec{q}_N)_L \\ \vdots & \vdots & \ddots & \vdots \\ (\vec{q}_N \cdot \vec{q}_1)_L & (\vec{q}_N \cdot \vec{q}_2)_L & \cdots & (\vec{q}_N \cdot \vec{q}_N)_L \end{bmatrix}, \quad (43)$$

where $0 < L < N$ when $N \gg 1$, $\mathbf{I}_L$ is the $L \times L$ identity matrix, $\mathbf{0}$ is a matrix of zeros of arbitrary dimension, $(\vec{q}_i \cdot \vec{q}_j)_L := \sum_{l=1}^{L} (\vec{q}_i)_l (\vec{q}_j)_l$, and $(\vec{q}_i)_l$ is the $l^{\text{th}}$ component of the vector $\vec{q}_i \in \mathbb{R}^N$. Here, we consider the case where $N \gg 1$ so that $1 \ll L < N$.

The spectral measure $\alpha(d\lambda)$ of the matrix $\mathbf{M}_2$ is given by a sum of "Dirac $\delta$ functions,"

$$\alpha(d\lambda) = \left[ \sum_{j=1}^{N} m_j \delta_{\lambda_j}(d\lambda) \right] d\lambda := \alpha(\lambda) d\lambda, \quad (44)$$

where $\delta_{\lambda_j}(d\lambda)$ is the Dirac delta measure centered at $\lambda_j$, $m_j = \langle \vec{e}_k^{\,T} [\vec{v}_j \vec{v}_j^{\,T}] \, \vec{e}_k \rangle$, $\vec{e}_k$ is a $N$–dimensional vector of ones, and $\lambda_j$ and $\vec{v}_j$ are the eigenvalues and eigenvectors of $\mathbf{M}_2$, respectively [25]. In this matrix case, the associated Stieltjes transformation of the measure $\alpha(d\lambda)$ (13) is given by the sum $G(t(s)) = \sum_{j=1}^{n} m_j/(1 - s - \lambda_j)$, and $\alpha(\lambda)$ in equation (44) is called "*the spectral function*," which is defined only pointwise on the set of eigenvalues $\{\lambda_j\}$. In Figure IV A we give a graphical representation of the spectral measure for finite 2–$d$ and 3–$d$ RBNs. It displays linearly connected peaks of histograms with bin sizes on the order of $10^{-2}$. The apparent smoothness of the spectral function graphs in this figure is due to the large number ($\sim 10^6$) of eigenvalues and eigenvectors calculated, and ensemble averaged.

In the matrix case, the action of $\chi_2$ is given by that of a square diagonal matrix of zeros and ones [25]. The action of $\chi_2$ in the matrix $\chi_2(-\Gamma)\chi_2$ introduces a row and column of zeros in the matrix $-\Gamma$, corresponding to every diagonal entry of $\chi_2$ with value 0. When there is only one defect inclusion, $p = 1/n$, located at the $j^{\text{th}}$ bond, $\chi_2$ has all zero entries except at the $j^{\text{th}}$ diagonal: $\chi_2 = \text{diag}(0, \cdots, 0, 1, 0, \cdots, 0) := \text{diag}(\vec{v}_j)$. Therefore, the only non-trivial eigenvalue is given by $\lambda_0 = (\vec{q}_j \cdot \vec{q}_j)_L = \sum_{l=1}^{L} (\vec{q}_j)_l^2 = 1 - \sum_{l=L+1}^{N} (\vec{q}_j)_l^2$, with eigenvector $\vec{\mathrm{v}}_j$ and weight $m_0 = 1/n$. This implies that there is a gap at $\lambda = 0$, $\theta_0 := \sum_{l=1}^{L} (\vec{q}_j)_l^2 > 0$, and a gap at $\lambda = 1$, $\theta_1 := \sum_{l=L+1}^{N} (\vec{q}_j)_l^2 > 0$. It is clear that these bounds hold for all $\omega \in \Omega$ such that $p = 1/n$ when $L \gg 1$. We have already mentioned that the eigenvalues of $\mathbf{M}_1$ are restricted to the set $\{0, 1\}$ when $p = 1$ ($\chi_2 \equiv \mathbf{I}_N$). Therefore, there exists $0 < p_0 < 1$ such that, for all $p \geq p_0$, there exists a $\omega \in \Omega$ such that $\theta_0(\omega) = 0$ and/or $\theta_1(\omega) = 0$.

### B. Baker's Critical Theory for Transport in Binary Composite Media

Baker's critical theory characterizes phase transitions of a given system via the asymptotic behaviors of underlying Stieltjes functions, near a critical point. This powerful method has been very successful in the Ising model, precisely characterizing the phase transition (spontaneous magnetization) [2]. We now show how this method may be adapted to provide a detailed description of phase transitions in transport, exhibited by binary composite media. The following theorem characterizes Stieltjes functions (series of Stieltjes) [2].

**Theorem IV.1** *Let $D(i,j)$ denote the following determinant*

$$D(i,j) = \begin{vmatrix} \xi_i & \xi_{i+1} & \cdots & \xi_{i+j} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{i+j} & \xi_{i+j+1} & \cdots & \xi_{i+2j} \end{vmatrix}. \tag{45}$$

*The $\xi_n$ form a series of Stieltjes if and only if $D(i,j) \geq 0$ for all $i, j = 0, 1, 2, \ldots$*

Baker's inequalities for the sequences $\gamma_n$ (41) and $\hat{\gamma}_n$ (42) of transport follow from Theorem IV.1. Indeed, for example, $\phi_n \sim (p - p_c)^{-\gamma_n}$ and Theorem IV.1 with $\phi_i = \xi_i$, $i = n$, and $j = 1$, imply that, for $|p - p_c| \ll 1$,

$$\begin{aligned} &(p-p_c)^{-\gamma_n - \gamma_{n+2}} - (p-p_c)^{-2\gamma_{n+1}} \geq 0 \iff (p-p_c)^{-\gamma_n - \gamma_{n+2} + 2\gamma_{n+1}} \geq 1 \\ &\iff -\gamma_n - \gamma_{n+2} + 2\gamma_{n+1} \leq 0 \iff \boxed{\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0}. \end{aligned} \tag{46}$$

The sequence of inequalities (46) are *Baker's inequalities* for transport, corresponding to $m(p,h)$, and they imply that the sequence $\gamma_n$ increases at least linearly with $n$. The symmetries in equations (16) and (41)–(42) imply that Baker's inequalities also hold for the sequences $\gamma_n'$, $\hat{\gamma}_n$, and $\hat{\gamma}_n'$.

**Lemma IV.1** *Let $0 < h \ll 1$ and $|p - p_c| \ll 1$. Then the integrals in equation* (26) *have the following asymptotics for $n \geq 0$*

$$\frac{\partial^n g(p,h)}{\partial h^n} \sim \phi_n, \qquad \frac{\partial^n \hat{g}(p,h)}{\partial h^n} \sim \hat{\phi}_n. \tag{47}$$

**Proof**: The asymptotic behaviors in equation (47) follow from equations (18)–(19), (21), Baker's inequalities (46), and equation (16) ($g(p,h) = sF(p,s)$ and $\hat{g}(p,h) = -s\, G(p,t(s))$). They imply that, for $c_j, b_j \in \mathbb{Z}$,

$$\lim_{h\to 0} \frac{\partial^n g(p,h)}{\partial h^n} = \sum_{j=0}^{n} c_j \lim_{s\to 1} \frac{\partial^j F(p,s)}{\partial s^j} \sim \phi_n\,, \quad \lim_{h\to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} = \sum_{j=0}^{n} b_j \lim_{s\to 1} \frac{\partial^j G(p,t(s))}{\partial t^j} \sim \hat{\phi}_n \;\; \square.$$

The key results of this section are the two–parameter scaling relations between the critical exponents in the conductor/insulator system, defined in equations (41), and that of the conductor/superconductor system, defined in equations (42). By equation (22) we know that $m(p,h)$ and $w(p,z(h))$ are related, therefore the Stieltjes functions $g(p,h)$ and $\hat{g}(p,h)$ are related. Moreover by equation (32), we know that the measures $\mu$ and $\alpha$ are related, therefore the measures $\phi$ and $\hat{\phi}$ are related. We therefore anticipate that these two sets of critical exponents are also related. This is indeed the case, and the resultant relationship between the critical exponents $t$ and $s$ is in agreement with the seminal paper by A. L. Efros and B. I. Shklovskii [32]. These results are summarized in Theorem IV.2 below.

**Theorem IV.2** *Let $t$, $t_r$, $t_i$, $\delta$, $\delta_r$, $\delta_i$, $\gamma$, $\gamma_n$, $\Delta$, $\gamma'_n$, and $\Delta'$ be defined as in equations* (41), *and $s$, $s_r$, $s_i$, $\hat{\delta}$, $\hat{\delta}_r$, $\hat{\delta}_i$, $\hat{\gamma}'$, $\hat{\gamma}'_n$, $\hat{\Delta}'$, $\hat{\gamma}_n$, and $\hat{\Delta}$ be defined as in equations* (42)*. Then the following scaling relations hold:*

**1**) $\gamma_1 = \gamma,\ \gamma'_1 = \gamma',\ \hat{\gamma}_1 = \hat{\gamma},$ *and* $\hat{\gamma}'_1 = \hat{\gamma}'$. **2**) $\gamma'_0 = 0,\ \gamma_0 < 0,\ \gamma'_n > 0$ *and* $\gamma_n >,\ n \geq 1$.

**3**) $\hat{\gamma}'_n > 0$ *for* $n \geq 0$. **4**) $\gamma = \hat{\gamma}_0$ *and* $\Delta = \hat{\Delta}$. **5**) $\gamma' = \hat{\gamma}'_0$ *and* $\Delta' = \hat{\Delta}'$.

**6**) $\gamma_n = \gamma + \Delta(n-1)$ *for* $n \geq 1$. **7**) $\hat{\gamma}'_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ *for* $n \geq 0$.

**8**) $t = \Delta - \gamma$. **9**) $s = \hat{\gamma}'_0 = \hat{\gamma}' - \hat{\Delta}'$. **10**) $\delta = \dfrac{\Delta}{\Delta - \gamma}$. **11**) $\hat{\delta}\,' = \dfrac{\hat{\Delta}'}{\hat{\gamma}'_0} = \dfrac{\hat{\Delta}'}{\hat{\gamma}' - \hat{\Delta}'}$.

**12**) $t_r = t_i = t$. **13**) $s_r = s_i = s$. **14**) $\delta_r = \delta_i = \delta$. **15**) $\hat{\delta}_r = \hat{\delta}_i = \hat{\delta}$.

**16**) *If* $\Delta = \Delta'$ *and* $\gamma = \gamma'$, *then* $t + s = \Delta$ *and* $1/\delta + 1/\hat{\delta}\,' = 1$.

Theorem IV.2 will be proven via a sequence of lemmas as we collect some important properties of $m(p,h)$, $g(p,h)$, $w(p,z(h))$, and $\hat{g}(p,h)$, and how they are related.

**Lemma IV.2** *$\gamma_1 = \gamma$, $\gamma_1' = \gamma'$, $\hat{\gamma}_1 = \hat{\gamma}$, and $\hat{\gamma}_1' = \hat{\gamma}'$*

**Proof**: Set $0 < p - p_c \ll 1$. By equations (16) ($g(p,h) = sF(p,s)$), (19), (41), and (46)

$$(p - p_c)^{-\gamma} \sim \chi(p,0) := \frac{\partial m(p,0)}{\partial h} = \lim_{s\to 1}\left[-\frac{\partial F(p,s)}{\partial s}\right] = \phi_0 + \phi_1 \sim \phi_1 \sim (p-p_c)^{-\gamma_1}, \quad (48)$$

hence $\gamma_1 = \gamma$. Similarly for $0 < p_c - p \ll 1$, we have $\gamma_1' = \gamma'$. By equation (48), the symmetries between $m$ and $w$ (16) and the critical exponent definitions (41)–(42), we also have $\hat{\gamma}_1 = \hat{\gamma}$ and $\hat{\gamma}_1' = \hat{\gamma}'$ $\Box$.

Equation (22) is consistent with, and provides a link between equations (39) and (40). We will see that the fundamental asymmetry between $m(p,h)$ and $w(p,z(h))$ ($\gamma_0' = 0$ and $\hat{\gamma}_0' > 0$), given in Theorem IV.2.2-3, is a direct and essential consequence of equation (22), and has deep and far reaching implications.

**Lemma IV.3** *Let the sequences $\gamma_n$ and $\gamma_n'$, $n \geq 0$, be defined as in equation* (41)*. Then*

$$\mathbf{1)} \quad \gamma_0' = 0,\ \gamma_0 < 0,\ \gamma_n' > 0,\ \textit{and}\ \ \gamma_n > 0, \quad \textit{for}\ \ n \geq 1.$$
$$\mathbf{2)} \quad 0 < \lim_{h\to 0}\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle / E_0^2 < 1 \ \ \textit{for all}\ \ p \in [0,1],\ h \in \mathcal{U}.$$

**Proof**: By equation (40) $|w(p,z(0))|$ is bounded for all $p < p_c$. Thus for all $p < p_c$, equations (19), (22), and (41) imply that

$$0 = \lim_{h\to 0} hw(p,z(h)) = \lim_{h\to 0} m(p,h) = \lim_{s\to 1}(1 - F(p,s)) = 1 - \phi_0(p) \sim 1 - (p_c - p)^{-\gamma_0'},$$

where the rightmost relation holds for $0 < p_c - p \ll 1$ and the leftmost relation is consistent with equation (39). Therefore, $\gamma_0' = 0$ and $\phi$ is a probability measure for all $p < p_c$. The strict positivity of the $\gamma_n'$, for $n \geq 1$, follows from Baker's inequalities (46). Thus, from equation (48) we have

$$\infty = \lim_{p\to p_c^-} \phi_1(p) = -\lim_{p\to p_c^-} \frac{\partial m(p,0)}{\partial h}\,. \quad (49)$$

For $p > p_c$, equations (19) and (39) imply that $0 < \lim_{h\to 0}|m(p,h)| = 1 - \phi_0 < 1$. Therefore, $(p - p_c)^{-\gamma_0} \sim \phi_0 < 1$ for all $0 < p - p_c \ll 1$, hence $\gamma_0 < 0$. The strict positivity of $\gamma_1$ follows from equation (49), and the strict positivity of the $\gamma_n$ for $n \geq 2$ follows from Baker's inequalities (46). Equation (20) and the inequality $0 < \lim_{h\to 0}|m(p,h)| = 1 - \phi_0 < 1$ imply that $0 < \lim_{h\to 0}\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle / E_0^2 < 1$ for all $p \in [0,1]$ $\Box$.

**Lemma IV.4** *Let the sequence $\hat{\gamma}'_n$, $n \geq 0$, be defined as in equation* (42). *Then*

$$\mathbf{1)} \quad \hat{\gamma}'_n > 0 \quad \textit{for all} \quad n \geq 0.$$
$$\mathbf{2)} \quad \lim_{h\to 0} \langle E_f^2 \rangle = \infty \quad \textit{for all} \quad p > p_c\,.$$

**Proof**: By equation (39) we have $0 < \lim_{h\to 0} |m(p,h)| < 1$, for all $p > p_c$. Therefore equation (22) implies that $\lim_{h\to 0} w(p, z(h)) = \lim_{h\to 0} m(p,h)/h = \infty$, for all $p > p_c$, which is consistent with equation (40). More specifically, for all $p > p_c$, equations (22) and (39) imply that $0 \leq \lim_{h\to 0} |m(p,h)| = \lim_{h\to 0} |hw(p, z(h))| := L(p) < 1$, where $L(p) = 0$ for all $p < p_c$. Therefore, by equation (16), we have

$$\begin{aligned} &\lim_{h\to 0} |h\, w(p, z(h))| = \lim_{h\to 0} |h\, \hat{g}(p,h)| \in (0,1), \text{ for all } p > p_c, \\ &\lim_{h\to 0} |h\, w(p, z(h))| = \lim_{h\to 0} |h\, \hat{g}(p,h)| = 0, \text{ for all } p < p_c\,. \end{aligned} \tag{50}$$

By equations (21), (40), and (42) we have, for all $p > p_c$,

$$\infty = \lim_{p\to p_c^-} \lim_{h\to 0} w(p, z(h)) = \lim_{p\to p_c^-} \lim_{s\to 1} (1 - G(p, t(s))) = 1 + \lim_{p\to p_c^-} \hat{\phi}_0(p) \sim 1 + \lim_{p\to p_c^-} (p_c - p)^{-\hat{\gamma}'_0},$$

hence $\hat{\gamma}'_0 > 0$. Baker's inequalities (46) then imply that $\hat{\gamma}'_n > 0$ for all $n \geq 0$. Equations (20) and (50), and $\hat{\gamma}'_0 > 0$ imply that $\lim_{h\to 0} \langle E_f^2 \rangle = \infty$ for all $p > p_c$ $\Box$.

The asymptotic behavior of $\hat{g}(p,h)$ in equation (47), and Lemma IV.4 motivates the following fundamental homogenization assumption of this section [2]:

**Remark IV.1** *Near the critical point $(p,h) = (p_c, 0)$, the asymptotic behavior of the Stieltjes function $\hat{g}(p,h)$ is determined primarily by the mass $\hat{\phi}_0(p)$ of the measure $\hat{\phi}$ and the rate of collapse of the spectral gap $\theta_\alpha$.*

By remark IV.1, and in light of Lemmas IV.2–IV.4, we make the following variable changes:

$$\begin{aligned} &\hat{q} := y(p_c - p)^{\hat{\Delta}'}, \qquad &&\hat{Q}(p) := \hat{S}(p)(p_c - p)^{\hat{\Delta}'}, \qquad &&d\hat{\pi}(\hat{q}) := (p_c - p)^{\hat{\gamma}'_0}\, d\hat{\phi}(y), \\ &q := y(p - p_c)^{\Delta}, &&Q(p) := S(p)(p - p_c)^{\Delta}, &&d\pi(q) := (p - p_c)^{\gamma}\, y\, d\hat{\phi}(y), \end{aligned} \tag{51}$$

so that, by equations (41)–(42), $\hat{Q}(p), Q(p) \sim 1$ and the masses $\hat{\pi}_0$ and $\pi_0$ of the measures $\hat{\pi}$ and $\pi$, respectively, satisfy $\hat{\pi}_0, \pi_0 \sim 1$ as $p \to p_c$.

Equation (51) defines the following scaling functions $G_{n-1}(x)$, $\hat{G}_n(\hat{x})$, $\mathcal{G}_{n-1,j}(x)$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$ as follows. For $h \in \mathcal{U} \cap \mathbb{R}$, equations (26) and (51) imply, for $n \geq 0$, that

$$\begin{aligned} &\frac{\partial^n g}{\partial h^n} \propto (p-p_c)^{-(\gamma+\Delta(n-1))} G_{n-1}(x), && \frac{\partial^n \hat{g}}{\partial h^n} \propto (p_c-p)^{-(\hat{\gamma}_0'+\hat{\Delta}' n)} \hat{G}_n(\hat{x}), \\ &G_{n-1}(x) := \int_0^{Q(p)} \frac{q^{n-1} d\pi(q)}{(1+xq)^{n+1}}, && \hat{G}_n(\hat{x}) := \int_0^{\hat{Q}(p)} \frac{\hat{q}^n d\hat{\pi}(\hat{q})}{(1+\hat{x}\hat{q})^{n+1}}, \\ &x := h(p-p_c)^{-\Delta}, \quad 0 < p-p_c \ll 1, && \hat{x} := h(p_c-p)^{-\hat{\Delta}'}, \quad 0 < p_c - p \ll 1. \end{aligned} \tag{52}$$

Analogous formulas are defined for the opposite limits involving $\hat{\Delta}$, $\hat{\gamma}_0$, $\Delta'$, and $\gamma'$.

For $h \in \mathcal{U}$ such that $h_i \neq 0$, we define the scaling functions $\mathcal{R}_{n-1}(x)$, $\mathcal{I}_{n-1}(x)$, $\hat{\mathcal{R}}_n(\hat{x})$, and $\hat{\mathcal{I}}_n(\hat{x})$ as follows. Using equations (28) and (51) we have, for $0 < p - p_c \ll 1$,

$$\begin{aligned} \frac{\partial^n g}{\partial h^n} &= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{h}^j \int_0^{S(p)} \frac{y^{n+j} d\phi(y)}{|1+hy|^{2(n+1)}} \\ &:= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} [\bar{x}(p-p_c)^{\Delta}]^j (p-p_c)^{-(\gamma+\Delta(n-1+j))} \mathcal{G}_{n-1,j}(x) \\ &:= (-1)^n n! (p-p_c)^{-(\gamma+\Delta(n-1))} \mathcal{K}_{n-1}(x), \quad \mathcal{K}_{n-1}(x) := \mathcal{R}_{n-1}(x) + \mathrm{i}\,\mathcal{I}_{n-1}(x), \\ \frac{\partial^n \hat{g}}{\partial h^n} &:= (-1)^n n! (p-p_c)^{-(\hat{\gamma}_0+\hat{\Delta} n)} \hat{\mathcal{K}}_n(\hat{x}), \quad \hat{\mathcal{K}}_n(\hat{x}) := \hat{\mathcal{R}}_n(\hat{x}) + \mathrm{i}\,\hat{\mathcal{I}}_n(\hat{x}). \end{aligned} \tag{53}$$

Here, $x$ and $\hat{x}$ are defined in equation (52) and

$$\begin{aligned} &\mathcal{G}_{n-1,j}(x) := \int_0^{Q(p)} \frac{q^{n-1+j} d\pi(q)}{|1+xq|^{2(n+1)}}, && \hat{\mathcal{G}}_{n,j}(\hat{x}) := \int_0^{\hat{Q}(p)} \frac{\hat{q}^{n+j} d\hat{\pi}(\hat{q})}{|1+\hat{x}\hat{q}|^{2(n+1)}}, \\ &\mathcal{K}_{n-1}(x) := \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{x}^j \mathcal{G}_{n-1,j}(x), && \hat{\mathcal{K}}_n(\hat{x}) := \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{\hat{x}}^j \hat{\mathcal{G}}_{n,j}(\hat{x}), \end{aligned} \tag{54}$$

where we have made the definitions $\mathcal{R}_{n-1}(x) := \mathrm{Re}(\mathcal{K}_{n-1}(x))$, $\mathcal{I}_{n-1}(\hat{x}) := \mathrm{Im}(\mathcal{K}_{n-1}(x))$, $\hat{\mathcal{R}}_n(\hat{x}) := \mathrm{Re}(\hat{\mathcal{K}}_n(\hat{x}))$, and $\hat{\mathcal{I}}_n(\hat{x}) := \mathrm{Im}(\hat{\mathcal{K}}_n(\hat{x}))$. Analogous formulas are defined for the opposite limit, $0 < p_c - p \ll 1$, involving $\hat{\Delta}'$, $\hat{\gamma}_0'$, $\Delta'$, and $\gamma'$.

From equation (17) we have, for $h \in \mathcal{U}$, $p \in [0,1]$, and $n \geq 0$,

$$G_{n-1}(x) > 0, \quad \mathcal{G}_{n-1,j}(x) > 0, \qquad \hat{G}_n(\hat{x}) > 0, \quad \hat{\mathcal{G}}_{n,j}(\hat{x}) > 0. \tag{55}$$

By our gap hypothesis, the $h$ derivatives of $g(p,h)$ and $\hat{g}(p,h)$, of all orders, are bounded at $h = 0$ for $p > p_c$ and $p < p_c$, respectively. Therefore,

$$\begin{aligned} &\lim_{h\to 0} G_{n-1}(x) < \infty, && \lim_{h\to 0} \mathcal{G}_{n-1,j}(x) < \infty, && \text{for all} \;\; p > p_c, \;\; n \geq 0 \\ &\lim_{h\to 0} \hat{G}_n(\hat{x}) < \infty, && \lim_{h\to 0} \hat{\mathcal{G}}_{n,j}(\hat{x}) < \infty, && \text{for all} \;\; p < p_c, \;\; n \geq 0. \end{aligned} \tag{56}$$

**Lemma IV.5** *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (52), *for $p > p_c$. Then*

1) $G_{n-1}(x) \sim 1$ *as* $x \to 0$ $(h \to 0$ *and* $0 < p - p_c \ll 1)$ *for all* $n \geq 1$.

2) $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ *as* $\hat{x} \to 0$ $(h \to 0$ *and* $0 < p - p_c \ll 1)$ *for all* $n \geq 1$.

3) $\gamma = \hat{\gamma}_0$ .

4) $\Delta = \hat{\Delta}$ .

**Proof**: Let $h \in \mathcal{U} \cap \mathbb{R}$ and $p > p_c$. Equations (27), (52), and (55)–(56) imply that we have, for all $n \geq 1$, $0 < p - p_c \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p - p_c)^{-(\gamma+\Delta(n-1))} G_{n-1}(x) = (p - p_c)^{-(\hat{\gamma}_0+\hat{\Delta}(n-1))} [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})]. \tag{57}$$

Equations (55)–(56) imply that $G_{n-1}(x) \sim 1$ as $x \to 0$, for all $n \geq 1$. Equation (57) then implies that $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ as $\hat{x} \to 0$, for all $n \geq 1$ (a competition in sign between two diverging terms). Or equivalently, generalizing (50), $\hat{G}_0(\hat{x}) - \hat{x}^n\hat{G}_n(\hat{x}) \sim 1$. Therefore,

$$\gamma + \Delta(n - 1) = \hat{\gamma}_0 + \hat{\Delta}(n - 1), \quad n \geq 1. \tag{58}$$

Which in turn, implies that $\gamma = \hat{\gamma}_0$ and $\Delta = \hat{\Delta}$ □.

**Lemma IV.6** *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (52), *for $p < p_c$. Then*

1) $\hat{G}_{n-1}(\hat{x}) \sim 1$ *as* $\hat{x} \to 0$ $(h \to 0$ *and* $0 < p_c - p \ll 1)$, *for all* $n \geq 1$.

2) $G_{n-1}(x) \sim 1$ *as* $x \to 0$ $(h \to 0$ *and* $0 < p_c - p \ll 1$, *for all* $n \geq 1$.

3) $\gamma' = \hat{\gamma}'_0$.

4) $\Delta' = \hat{\Delta}'$.

**Proof**: Let $h \in \mathcal{U} \cap \mathbb{R}$ and $p < p_c$. By equations (26) and (56) we have the following: $\lim_{h\to 0} h \int_0^{S(p)} y^n d\hat{\phi}(y)/(1 + hy)^{n+1} = 0$. Therefore, equations (27), (52), and (55)–(56) imply that, for all $n \geq 1$, $0 < p_c - p \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p_c - p)^{-(\hat{\gamma}'+\hat{\Delta}'(n-1))} \hat{G}_{n-1}(\hat{x}) \sim (p_c - p)^{-(\gamma'+\Delta'(n-1))} G_{n-1}(x). \tag{59}$$

Equations (55)–(56) imply that $\hat{G}_{n-1}(\hat{x}) \sim 1$ as $\hat{x} \to 0$ for all $n \geq 1$. Equation (59) then implies that $G_{n-1}(x) \sim 1$ as $x \to 0$ for all $n \geq 1$. Therefore,

$$\gamma' + \Delta'(n - 1) = \hat{\gamma}'_0 + \hat{\Delta}'(n - 1), \quad n \geq 1.$$

Which in turn, implies that $\gamma' = \hat{\gamma}'_0$ and $\Delta' = \hat{\Delta}'$ □.

**Lemma IV.7** *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (52). *Then*

$$\begin{aligned}
&\mathbf{1)}\quad \gamma_n = \gamma + \Delta(n-1), \quad \textit{for all} \;\; n \geq 1.\\
&\mathbf{2)}\quad \hat{\gamma}'_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1), \quad \textit{for all} \;\; n \geq 0.\\
&\mathbf{3)}\quad t = \Delta - \gamma.\\
&\mathbf{4)}\quad s = \hat{\gamma}'_0 = \hat{\gamma}' - \hat{\Delta}'.
\end{aligned}$$

**Proof**: Let $0 < p - p_c \ll 1$. By equations (41), (47), and (52), and Lemma IV.5 we have, for all $n \geq 1$,

$$(p-p_c)^{-\gamma_n} \sim \phi_n \sim \lim_{h\to 0} \frac{\partial^n g(p,h)}{\partial h^n} \sim (p-p_c)^{-(\gamma+\Delta(n-1))} \lim_{x\to 0} G_{n-1}(x) \sim (p-p_c)^{-(\gamma+\Delta(n-1))}.$$

Therefore $\gamma_n = \gamma + \Delta(n-1)$ for all $n \geq 1$, with constant gap $\gamma_i - \gamma_{i-1} = \Delta$, which is consistent with the absence of multifractal behavior for the bulk conductivity [28].

Now let $0 < p_c - p \ll 1$. By equations (42), (47), and (52), and Lemma IV.6 we have, for all $n \geq 1$,

$$(p_c-p)^{-\hat{\gamma}_n} \sim \hat{\phi}_n \sim \lim_{h\to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} \propto (p_c-p)^{-(\hat{\gamma}'_0+\hat{\Delta}' n)} \lim_{\hat{x}\to 0} \hat{G}_n(\hat{x}) \sim (p_c-p)^{-(\hat{\gamma}'_0+\hat{\Delta}' n)}.$$

Therefore, by Lemma IV.2, we have $\hat{\gamma}_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ for all $n \geq 0$, with constant gap $\hat{\gamma}'_i - \hat{\gamma}'_{i-1} = \hat{\Delta}$, which is consistent with the absence of multifractal behavior for the bulk conductivity [28].

Again let $0 < p - p_c \ll 1$. Equations (16), (23), (41), (50), and (52) yield

$$\begin{aligned}
(p-p_c)^t &\sim \lim_{h\to 0} m(p,h) = 1 - \lim_{h\to 0} g(p,h) = \lim_{h\to 0} h\hat{g}(p,h) = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{\hat{x}\to 0} \hat{x}\hat{G}_0(\hat{x})\\
&\sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}.
\end{aligned} \tag{60}$$

Therefore, by Lemma IV.5 we have $t = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma$.

Finally let $0 < p_c - p \ll 1$. By equations (16), (42), and (52), and Lemmas IV.4 and IV.6, we have

$$(p_c-p)^{-s} \sim \lim_{h\to 0} w(p,z(h)) \sim \lim_{h\to 0} \hat{g}(p,h) = (p_c-p)^{-\hat{\gamma}'_0} \lim_{\hat{x}\to 0} \hat{G}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}'_0}.$$

Therefore, by Lemma IV.7.2, we have $s = \hat{\gamma}'_0 = \hat{\gamma}' - \hat{\Delta}'$ $\square$.

**Lemma IV.8** *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (52)*, for $p > p_c$ and $p < p_c$. Then for all $n \geq 1$*

**1)** $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma+\Delta(n-1))/\Delta}$, *as $\hat{x} \to \infty$ ($p \to p_c^+$ and $0 < h \ll 1$).*

**2)** $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}$, *as $x \to \infty$ ($p \to p_c^-$ and $0 < h \ll 1$).*

**3)** $\delta = \Delta/(\Delta - \gamma)$.

**4)** $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}' - \hat{\Delta}')$.

**Proof**: Let $0 < h \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0,1]$ [4]. The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that all orders of $h$ derivatives of these functions are bounded as $p \to p_c$, from the left or the right. Therefore, equation (57) holds for $0 < p - p_c \ll 1$, and

$$(0,\infty) \ni (p_c - p)^{-(\gamma'+\Delta'(n-1))} G_{n-1}(x) = (p_c - p)^{-(\hat{\gamma}_0'+\hat{\Delta}'(n-1))}[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \tag{61}$$

holds for $0 < p_c - p \ll 1$. Moreover, in order to cancel the diverging $p$ dependent prefactors in equations (57) and (61) we must have, for all $n \geq 1$,

$$\begin{aligned} G_{n-1}(x) &\sim x^{-(\gamma+\Delta(n-1))/\Delta}, \quad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}_0+\hat{\Delta}(n-1))/\hat{\Delta}}, \quad \text{as } \ p \to p_c^+, \\ G_{n-1}(x) &\sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}, \quad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}_0'+\hat{\Delta}'(n-1))/\hat{\Delta}'}, \quad \text{as } \ p \to p_c^-. \end{aligned} \tag{62}$$

Lemma IV.8.1-2 follows from equation (62) and Lemmas IV.5–IV.6.

Now by equations (16), (22), (41), (52), and (62) for $n = 1$, we have

$$\begin{aligned} h^{1/\delta} &\sim \lim_{p\to p_c^+} m(p,h) = \lim_{p\to p_c^+} hw(p,z(h)) \sim \lim_{p\to p_c^+} h\hat{g}(p,h) = h \lim_{p\to p_c^+} (p - p_c)^{-\hat{\gamma}_0}\hat{G}_0(\hat{x}) \\ &\sim h(p-p_c)^{-\hat{\gamma}_0} h^{-\hat{\gamma}_0/\hat{\Delta}}(p-p_c)^{-\hat{\Delta}(-\hat{\gamma}_0/\hat{\Delta})} = h^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}. \end{aligned} \tag{63}$$

Therefore by Lemma IV.6, we have $\delta = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0) = \Delta/(\Delta - \gamma)$. Similarly by equations (16), (42), (52), and (62) for $n = 1$, and Lemma IV.4, we have

$$h^{-1/\hat{\delta}'} \sim \lim_{p\to p_c^-} w(p,z(h)) \sim \lim_{p\to p_c^-} \hat{g}(p,h) = \lim_{p\to p_c^-} (p-p_c)^{-\hat{\gamma}_0'}\hat{G}_0(\hat{x}) = h^{-\hat{\gamma}_0'/\hat{\Delta}\,'}. \tag{64}$$

Therefore, by Lemma IV.7 we have $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}' - \hat{\Delta}')$ □.

**Lemma IV.9** *Let $h \in \mathcal{U}$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in equations* (53)–(54) *for $p > p_c$ and $p < p_c$. Furthermore,* *let $s_r$, $s_i$, $t_r$, and $t_i$ be defined as in equations* (41)–(42)*. Then,*

**1)** $[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r\hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \hat{x}_i\hat{\mathcal{G}}_{0,1}(\hat{x}) \sim 1$ *as* $\hat{x} \to 0$ ($h \to 0$ *and* $0 < p_c - p \ll 1$).

**2)** $\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}|^2\hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \lim_{\hat{x}\to 0}[\hat{x}_i\hat{\mathcal{G}}_{0,0}(\hat{x})] \sim 1$ *for* $0 < p - p_c \ll 1$.

**3)** $s_r = s_i = \hat{\gamma}_0' = s$.

**4)** $t_r = t_i = \Delta - \gamma = t$.

**Proof**: Let $0 < p_c - p \ll 1$, $h \in \mathcal{U}$ such that $h_i \neq 0$, and $0 < |h| \ll 1$. By equations (53)–(54), for $n = 0$, we have

$$\hat{g}(p,h) = \int_0^{\hat{S}(p)} \frac{d\hat{\phi}(y)}{|1+hy|^2} + \bar{h}\int_0^{\hat{S}(p)} \frac{y\,d\hat{\phi}(y)}{|1+hy|^2} = (p_c - p)^{-\hat{\gamma}_0'}[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \bar{\hat{x}}\hat{\mathcal{G}}_{0,1}(\hat{x})], \tag{65}$$

so that

$$\begin{aligned}\hat{g}_r &= (p_c - p)^{-\hat{\gamma}_0'}\hat{\mathcal{R}}_0(\hat{x}) = (p_c - p)^{-\hat{\gamma}_0'}[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r\hat{\mathcal{G}}_{0,1}(\hat{x})] \\ \hat{g}_i &= (p_c - p)^{-\hat{\gamma}_0'}\hat{\mathcal{I}}_0(\hat{x}) = -(p_c - p)^{-\hat{\gamma}_0'}\hat{x}_i\hat{\mathcal{G}}_{0,1}(\hat{x}).\end{aligned} \tag{66}$$

Equations (50) and (55) imply that $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim 1$ as $\hat{x} \to 0$ ($h \to 0$ and $0 < p_c - p \ll 1$). Therefore, equations (16), (42), (66) and Lemma IV.4 imply that

$$\begin{aligned}(p_c - p)^{-s_r} &\sim w_r(p,0) \sim \hat{g}_r(p,0) \sim (p_c - p)^{-\hat{\gamma}_0'}\lim_{\hat{x}\to 0}\hat{\mathcal{R}}_0(\hat{x}) \sim (p_c - p)^{-\hat{\gamma}_0'}, \\ (p_c - p)^{-s_i} &\sim w_i(p,0) \sim \hat{g}_i(p,0) \sim (p_c - p)^{-\hat{\gamma}_0'}\lim_{\hat{x}\to 0}\hat{\mathcal{I}}_0(\hat{x}) \sim (p_c - p)^{-\hat{\gamma}_0'}.\end{aligned} \tag{67}$$

Equation (67) and Lemma IV.7 imply that $s_r = s_i = \hat{\gamma}_0' = s$. It's worth noting that these scaling relations are independent of the path of the limit $h \to 0$.

Now let $0 < p - p_c \ll 1$ with $h$ as before. In equation (60) we demonstrated that $m(p,0) = \lim_{h\to 0} h\hat{g}(p,h)$. Therefore equation (66), for $p > p_c$, implies that

$$\begin{aligned}m_r(p,0) &\sim \lim_{h\to 0}[h_r\hat{g}_r(p,h) - h_i\hat{g}_i(p,h)] = (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0}\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2\hat{\mathcal{G}}_{0,1}(\hat{x})] \\ m_i(p,0) &\sim \lim_{h\to 0}[h_i\hat{g}_r(p,h) + h_r\hat{g}_i(p,h)] = (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0}\lim_{\hat{x}\to 0}[\hat{x}_i\hat{\mathcal{G}}_{0,0}(\hat{x})]\end{aligned} \tag{68}$$

By equation (50) we have $\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}|^2\hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \lim_{\hat{x}\to 0}[\hat{x}_i\hat{\mathcal{G}}_{0,0}(\hat{x})] \sim 1$ for all $0 < p - p_c \ll 1$. Therefore, equations (41) and (68) imply that

$$(p - p_c)^{t_r} \sim m_r(p,0) \sim (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0}, \qquad (p - p_c)^{t_i} \sim m_i(p,0) \sim (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0}. \tag{69}$$

Equation (69) and Lemmas IV.5 and IV.7 imply that $t_r = t_i = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma = t$. Again, these scaling relations are independent of the path of the limit $h \to 0$ $\Box$.

**Lemma IV.10** *Let $h \in \mathcal{U}$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in equations* (53)–(54) *for $p > p_c$ and $p < p_c$. Furthermore, let $\hat{\delta}_r$, $\hat{\delta}_i$, $\delta_r$, and $\delta_i$ be defined as in equations* (41)–(42)*. Then,*

**1)** $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |\hat{x}|^{-\hat{\gamma}_0'/\hat{\Delta}'}$, *as* $\hat{x} \to \infty$ $(p \to p_c^-$ *and* $0 < |h| \ll 1)$.

**2)** $[\hat{x}_r \hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i \hat{\mathcal{I}}_0(\hat{x})] \sim [\hat{x}_r \hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i \hat{\mathcal{R}}_0(\hat{x})] \sim |\hat{x}|^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}$, *as* $\hat{x} \to \infty$.

**3)** $\hat{\delta}_r{}' = \hat{\delta}_i{}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}$.

**4)** $\delta_r = \delta_i = \Delta/(\Delta - \gamma) = \delta$.

**Proof**: Let $h \in \mathcal{U}$ such that $h_i \neq 0$ and $0 < |h| \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0,1]$ [4]. Equations (16), (42), (66) and Lemma IV.4 imply that

$$|h|^{-1/\hat{\delta}_r'} \sim w_r(p_c,h) \sim \hat{g}_r(p_c,h) \sim \lim_{p \to p_c^-} (p_c - p)^{-\hat{\gamma}_0'} \hat{\mathcal{R}}_0(\hat{x}), \tag{70}$$
$$|h|^{-1/\hat{\delta}_i'} \sim w_i(p_c,h) \sim \hat{g}_i(p_c,h) \sim \lim_{p \to p_c^-} (p_c - p)^{-\hat{\gamma}_0'} \hat{\mathcal{I}}_0(\hat{x}).$$

The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that they are bounded for all $p \in [0,1]$. Therefore, in order to cancel the diverging $p$ dependent prefactors in equations (70), we must have $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |x|^{-\hat{\gamma}_0'/\hat{\Delta}'}$ as $\hat{x} \to \infty$ $(p \to p_c^-$ and $0 < h \ll 1)$. Equation (70) then implies that

$$|h|^{-1/\hat{\delta}_r'} \sim (p_c - p)^{-\hat{\gamma}_0'} |h|^{-\hat{\gamma}_0'/\hat{\Delta}'} (p_c - p)^{-\hat{\Delta}'(-\hat{\gamma}_0'/\hat{\Delta}')} = |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}, \quad |h|^{-1/\hat{\delta}_i'} \sim |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}. \tag{71}$$

Therefore, by Lemma IV.8, $\hat{\delta}_r{}' = \hat{\delta}_i{}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}\,'$.

Equations (16) and (22) imply that $m(p_c,h) \sim \lim_{p \to p_c^+} h\hat{g}(p,h)$, for $0 < |h| \ll 1$. Therefore equations (41) and (68) implies that

$$|h|^{1/\delta_r} \sim m_r(p_c,h) = (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{p \to p_c^+} [\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})], \tag{72}$$
$$|h|^{1/\delta_i} \sim m_i(p_c,h) = (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{p \to p_c^+} [\hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x})].$$

The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that they are bounded for all $p \in [0,1]$. Therefore, in order to cancel the diverging $p$ dependent prefactors in equations (72), we must have $[\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x}) \sim |x|^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}$ as $\hat{x} \to \infty$ $(p \to p_c^+$ and $0 < h \ll 1)$. Therefore equation (72), and Lemmas IV.5 and IV.8 imply that $\delta_r = \delta_i = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0) = \Delta/(\Delta - \gamma) = \delta$ $\square$.

**Lemma IV.11** *The measure $y\,d\phi(y)$ has the symmetry property ($\Delta=\Delta'$ and $\gamma=\gamma'$) if and only if the measure $d\hat{\phi}(y)$ has the symmetry property ($\hat{\Delta}=\hat{\Delta}'$ and $\hat{\gamma}_0=\hat{\gamma}_0'$). If either measure this symmetry, then*

**1**) $s+t=\Delta\,.$ **2**) $1/\delta+1/\hat{\delta}\,'=1.$ **3**) $\Delta=\hat{\Delta}=\Delta'=\hat{\Delta}'.$ **4**) $\gamma=\gamma'=\hat{\gamma}_0=\hat{\gamma}_0'.$

**Proof**: We have shown in Lemmas IV.5–IV.6 that $\gamma=\hat{\gamma}_0$, $\Delta=\hat{\Delta}$, $\gamma'=\hat{\gamma}_0'$, and $\Delta'=\hat{\Delta}'$. Therefore, it is clear that, ($\Delta=\Delta'$ and $\gamma=\gamma'$) $\iff$ ($\hat{\Delta}=\hat{\Delta}'$ and $\hat{\gamma}_0=\hat{\gamma}_0'$). Assume that either of the measures, $d\hat{\phi}(y)$ or $y\,d\phi(y)$, has this symmetry. Thus, $\Delta=\hat{\Delta}=\hat{\Delta}'=\Delta'$ and $\gamma=\hat{\gamma}_0=\hat{\gamma}_0'=\gamma'$. By Lemma IV.7 we have $t=\Delta-\gamma$ and $s=\hat{\gamma}_0'$, and by Lemma IV.8 we have $\delta=\Delta/(\Delta-\gamma)$ and $\hat{\delta}\,'=\hat{\Delta}'/\hat{\gamma}_0'$. Therefore,

$$s+t=\hat{\gamma}_0'+\Delta-\gamma=\hat{\gamma}_0+\Delta-\gamma=\Delta\,.$$

$$\delta=\Delta/(\Delta-\gamma)=1/(1-\gamma/\Delta)=1/(1-\hat{\gamma}_0/\hat{\Delta})=1/(1-\hat{\gamma}_0'/\hat{\Delta}')=1/(1-1/\hat{\delta}\,')\quad\Box.$$

This concludes the proof of Theorem IV.2 □.

## V. CONCLUDING REMARKS

We have constructed a mathematical framework which unifies the critical theory of transport for binary composite media, in the continuum and lattice settings. We have focused on critical transitions exhibited by the effective complex conductivity $\sigma^*=\sigma_2 m(h)=\sigma_1 w(z)$, as the symmetries underlying this framework extend our results to that regarding the effective complex resistivity $[\sigma^{-1}]^*=\sigma_1^{-1}\tilde{m}(h)=\sigma_2^{-1}\tilde{w}(z)$. Moreover, the underlying symmetries in the effective parameter problem of electrical conductivity and permittivity, magnetic permeability, and thermal conductivity, extend our results to all of these systems! We have shown that critical transitions in transport properties are, in general, characterized by the formation of delta function components in the underlying spectral measures, at the spectral endpoints. Moreover, for percolation models, we have shown that the onset of the critical transition (the formation of these delta components) occurs *precisely* at the percolation threshold.

The mathematical transport properties of such systems, displayed in section III, hold for general two–component stationary random media in lattice and continuum settings [4]. Moreover, the critical exponent scaling relations and the various transport properties, displayed in Lemmas IV.2–IV.11, hold for general percolation models regarding this class of composite media [16]. This type of critical behavior has been studied before in the lattice [32–34], and alternate methods have shown that $\Delta = s + t$, $\delta = (s+t)/t$, and $\gamma = s$ [16]. These are precisely the relations that we have shown to hold for general lattice and continuum percolation models, under the symmetry condition of Lemma IV.11. There is no apparent mathematical necessity for this spectral symmetry. Although, it leads to the well known two dimensional duality relation $s = t$ in the lattice [32–34].

The scaling relations derived here are independent of the limiting path $h \to 0$, which is opposed to the results of other workers [32–34] which use heuristic scaling forms. The starting point for our critical theory is equation (13), which displays *exact* formulas for infinite systems [16]. Away from the critical point $(p, h) = (p_c, 0)$, the behavior of the system, as a function of $p \in [0, 1]$, is highly dependent on the location of $h \in \mathcal{U}$ and is governed by equation (24), or when $h \in \mathbb{C}$ such that $h_i \neq 0$, by the system of coupled partial differential equations (25).

Our relations are satisfied by the exponents in effective medium theory for the lattice [27], $t = \gamma = 1$ and $\delta = \Delta = 2$. These relations then imply that $t = t_r = t_i = s = s_r = s_i = \gamma = \gamma' = \hat{\gamma}_0 = \hat{\gamma}_0' = 1$, $\Delta = \Delta' = \hat{\Delta} = \hat{\Delta}' = \delta = \delta_r = \delta_i = \hat{\delta} = \hat{\delta}_r = \hat{\delta}_i = \delta' = \delta_r' = \delta_i' = \hat{\delta}' = \hat{\delta}_r' = \hat{\delta}_i' = 2$. For the $d = 2$ lattice, where $\delta = 2$ and $t = 1.3$ [28, 32], these relations imply that $t = \gamma = s = \Delta/2$, so that the numerical values of the exponents are $t = t_r = t_i = s = s_r = s_i = \gamma = \gamma' = \hat{\gamma}_0 = \hat{\gamma}_0' = 1.3$, $\Delta = \Delta' = \hat{\Delta} = \hat{\Delta}' = 2.6$, and $\delta = \delta_r = \delta_i = \hat{\delta} = \hat{\delta}_r = \hat{\delta}_i = \delta' = \delta_r' = \delta_i' = \hat{\delta}' = \hat{\delta}_r' = \hat{\delta}_i' = 2$. For the $d = 2$ checkerboard, where it is believed that $\delta = 4$, these relations imply $t = \gamma/3 = \Delta/4$ and $s = \gamma = 3\Delta/4$, and so forth. Unfortunately, it appears as if not enough is known numerically or experimentally at this point, about the critical exponents other than $t$, to directly test the validity of our relations.


[1] G. W. Milton, *Theory of Composites* (Cambridge University Press, Cambridge, 2002).

[2] G. A. Baker, *Quantitative Theory of Critical Phenomena* (Academic Press, New York, 1990).

[3] K. M. Golden, J. Math. Phys., **36**, 5627 (1995).

[4] K. M. Golden and G. Papanicolaou, Comm. Math. Phys., **90**, 473 (1983).

[5] T. D. Lee and C. N. Yang, Phys. Rev, **87**, 410 (1952).

[6] D. Ruelle, Ann. Math., **171**, 589 (2010).

[7] D. Ruelle, *Statistical Mechanics: Rigorous Results* (W. A. Benjamin, New York, 1969).

[8] C. N. Yang and T. D. Lee, Phys. Rev., **87**, 404 (1952).

[9] G. A. Baker, Phys. Rev. Lett., **20**, 990 (1968).

[10] C. J. Thompson, *Classical Equilibrium Statistical Mechanics* (Oxford University Press, Oxford, 1988).

[11] H. S. Robertson, *Statistical Thermophysics* (Prentice Hall, Englewood Cliffs, NJ, 1993).

[12] D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, NJ, 1999).

[13] D. Ruelle, Phys. Rev. Lett., **26**, 303 (1971).

[14] J. G. A. Baker and D. S. Gaunt, Phys. Rev. B, **1**, 1184 (1970).

[15] K. Christensen and N. R. Moloney, *Complexity and Criticality* (Imperial College Press, Covent Garden, London, 2005).

[16] K. M. Golden, Phys. Rev. Lett., **78**, 3935 (1997).

[17] D. J. Bergman, Phys. Rep. C, **43**, 377 (1978).

[18] G. W. Milton, Appl. Phys. Lett., **37**, 300 (1980).

[19] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, Inc., New York, 1999).

[20] M. C. Reed and B. Simon, *Functional Analysis* (Academic Press, Scan Diego CA, 1980).

[21] D. J. Bergman, Ann. Phys., **138**, 78 (1982).

[22] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, Inc., New York, NY, 1987).

[23] G. B. Folland, *Introduction to Partial Differential Equations* (Princeton University Press, Princeton, NJ, 1995).

[24] D. J. Bergman, Phys. Rev. Lett., **44**, 1285 (1980).

[25] K. M. Golden, N. B. Murphy, and E. Cherkaev, J. Biomech, **44**, 337 (2011).

[26] P. Henrici, *Applied and Computational Complex Analysis. Volume 3. Discrete Fourier analysis–Cauchy integrals–construction of conformal maps–univalent functions.* (John Wiley & Sons Inc., New York, 1974).

[27] A. R. Day and M. F. Thorpe, J. Phys.: Cond. Matt., **8**, 4389 (1996).

[28] D. Stauffer and A. Aharony, *Introduction to Percolation Theory, Second Edition* (Taylor and Francis, London, 1992).

[29] S. Torquato, *Random Heterogeneous Materials: Microstructure and Macroscopic Properties* (Springer-Verlag, New York, 2002).

[30] K. M. Golden, Comm. Math. Phys., **143**, 467 (1992).

[31] O. Bruno, Proc. R. Soc. London A, **433**, 353 (1991).

[32] A. L. Efros and B. I. Shklovskii, Phys. Stat. Sol. (B), **76**, 303 (1976).

[33] J. P. Clerc, G. Giraud, J. M. Laugier, and J. M. Luck, Adv. Phys., **39**, 191 (1990).

[34] D. J. Bergman and D. Stroud, Solid State Phys., **46**, 147 (1992).