A Unification of the Critical Theory of Transport in Binary Composite Media

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We demonstrate that Lee–Yang–Ruelle–Baker critical theory of the Ising model may be adapted to provide a detailed characterization of percolation driven critical transitions in transport. In a novel unified approach, we reproduce Golden's pioneering, static, results regarding insulating/conducting binary composite media, and produce the analogous static results for conducting/superconducting binary composite media, finding the (two–parameter) scaling relations of each system. Moreover, we extend the results pertaining to each system to the quasi–static limit, where the material contrast parameter becomes complex. Under a physically consistent symmetry assumption, we link these two sets of scaling relations so that, under this assumption, the scaling relations of both systems are determined by only (two) parameters. We also provide a general proof of the fundamental assumption underlying Golden's original work: the existence of spectral gaps which collapse as the volume fraction of defect inclusions approaches the percolation threshold. The proof thereof identifies the phase transition with the appearance of a delta function component in the underlying spectral measures at the right spectral endpoint, and the divergence of the weight associated with the essential delta function at the left spectral endpoint, *precisely* at the percolation threshold.

I. INTRODUCTION

Disordered composite materials have found a well rooted place in the sciences and engineering technology due to their countless applications. These materials are of great utility, as they often *combine* the attributes of the underlying constituent materials [1]. They have even greater functionality when the composite has the ability to transition between the various underlying and/or combined attributes of the constituents. This is a key feature of disordered media whose effective behavior is dominated by the connectedness, or percolation properties, of a particular component. Examples include bone, doped semiconductors, radar absorbing composites, thin metal films, porous media such as sea ice and rock, and smart materials such as rheological suspensions, piezoresistors, and thermistors. The behavior of such media is particularly challenging to describe physically, and to predict mathematically.

Here we construct a mathematical framework which unifies the critical theory of transport in binary composite media. The most natural formulation of this problem is in terms of the conduction problem in the continuum $\mathbb{R}^d$, which includes the lattice $\mathbb{Z}^d$ as a special case [2, 3]. Although, the underlying symmetries in the effective parameter problem of electrical conductivity and permittivity, magnetic permeability, and thermal conductivity, immediately generalize the results of this work to all of these systems [1]. We accomplish this by adapting techniques developed by G. A. Baker, in order to characterize the phase transition in the Ising model, to characterize percolation driven critical transitions in transport, exhibited by binary composite media [4].

II. BACKGROUND

In 1952 T. D. Lee and C. N. Yang showed that the root distribution of the Ising model partition function, a polynomial in an appropriate complex variable, completely determines the associated equation of state [5]. They demonstrated that the properties of the system, in relation to phase transitions, are determined by the behavior of the roots near the positive real axis. They did so by proving that the roots of the partition function lie on the unit circle (The Lee–Yang Theorem) [6, 7]. Thus, the Gibbs free energy may be represented as a logarithmic potential.

In 1968 G. A. Baker used the Lee–Yang theorem and the logarithmic representation of the Gibbs free energy, to prove that the magnetization per spin $M(T,H)$ has the special analytic structure in the variable $\tau = \tanh \beta m H$ [8],

$$\frac{M}{m} = \tau(1+(1-\tau^2)G(\tau^2)), \quad G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1+\tau^2 y} , \tag{1}$$

where H is the applied magnetic field strength, m is the (constant) magnetic dipole moment of each spin [9], $\beta = (kT)^{-1}$, k is Boltzmann's constant [10], and T is the absolute temperature. The function G is a Stieltjes (or Herglotz) function of τ^2 and $\psi(dy)$ is a non–negative definite measure [8]. The integral representation (1) immediately leads to the inequalities

$$G \geq 0, \qquad \frac{\partial G}{\partial u} \leq 0, \qquad \frac{\partial^2 G}{\partial u^2} \geq 0, \tag{2}$$

where $u := \tau^2$. The last equation in (2) is the GHS inequality, which is an important tool in the study of the Ising model [2].

In 1970 D. Ruelle extended The Lee–Yang Theorem and proved that there exists a gap $\theta_0(T)$ in the partition function zeros about the positive real axis for high temperatures T. Moreover, he proved that the gap collapses ($\theta_0(T) \to 0$) as T decreases to a critical temperature $T_c > 0$ [11]. Consequently, the temperature driven phase transition (spontaneous magnetization) is unique, and is characterized by the pinching of the real axis by the roots of the partition function [7, 12].

In [4, 13] G. A. Baker exploited The Lee–Yang–Ruelle Theorem and defined a critical exponent Δ for the gap, $\theta_0(T) \sim (T-T_c)^\Delta$, as $T \to T_c^+$, in the distribution of the Lee–Yang–Ruelle zeros. He showed that for $T > T_c$, the support Σ_ψ of the measure ψ is given by the compact interval $[0, S(T)]$, and that $S(T) \sim (T-T_c)^{-2\Delta}$, as $T \to T_c^+$. Furthermore, he showed that the moments $\psi_n = \int_0^\infty y \, d\psi(y)$ of the measure ψ diverge as $T \to T_c^+$ according to the power law $\psi_n \sim (T-T_c)^{-\gamma_n}$. Using a Stieltjes function characterization theorem [4], he showed that the sequence $\{\gamma_n\}$ satisfies Baker's inequalities $\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0$, which implies that this sequence increases at least linearly with n [8]. Later [4], he showed that this sequence *is* actually linear in n, $\gamma_n = \gamma + 2\Delta n$, $n \geq 0$, with constant gap $\gamma_i - \gamma_{i-1} = 2\Delta$. The critical exponent γ is defined via the magnetic susceptibility $\chi = \partial M/\partial H = -\partial^2 f/\partial H^2 \sim (T-T_c)^{-\gamma}$, as $T \to T_c^+$, where f is the Gibbs free energy per spin. The phase transition is concisely described by two other critical exponents. When $H = 0$, the magnetization per spin $M(T,0) \sim (T-T_c)^\beta$, as $T \to T_c^-$, where the critical exponent β is not to be confused with $(kT)^{-1}$. Along the critical isotherm $T = T_c$, $M(T_c, H) \sim H^{1/\delta}$, as $H \to 0$, [4, 14]. Using the integral representation (1), he obtained the (two–parameter) scaling relations $\beta = \Delta - \gamma$ and $\delta = \Delta/(\Delta - \gamma)$ [4]. The critical exponent γ, for example, is defined in terms of the following limit [4]:

$$\gamma := \limsup_{T \to T_c^+,\, H=0} \left(\frac{-\ln \chi(T,H)}{\ln(T - T_c)} \right). \tag{3}$$

In 1997 K. M. Golden proved that, in the static limit, Lee–Yang–Ruelle–Baker critical theory may be adapted to characterize percolation driven critical transitions in transport [15]. This deep and far reaching result puts these two classes of seemingly unrelated problems on an equal mathematical footing. He did so by considering percolation models of conductive binary composite media. There, the connectedness of the system is determined by the volume fraction p of defect inclusions with conductance σ_2 in an otherwise homogeneous medium of conductance σ_1, whereby assumption $h = \sigma_1/\sigma_2 \in (0,1)$. He demonstrated that the function $m(p,h) = \sigma^*(p,h)/\sigma_2$ plays the role of the magnetization per spin $M(T,H)$ in the Ising model, where $\sigma^*(p,h)$ is the effective conductance of the random medium. Moreover, the volume fraction p mimics the temperature T while the contrast ratio h mimics the applied magnetic field H. More specifically, the critical insulator/conductor behavior in transport arises when $h = 0$ ($\sigma_1 = 0,\ 0 < \sigma_2 < \infty$), as $p \to p_c^+$ [15], and in the Ising model the analogous non-magnetic/ferromagnetic critical behavior arises when $H = 0$, as $T \to T_c^+$ [14]. Using these mathematical parallels, K. M. Golden showed that the critical exponents of transport satisfy Baker's inequalities, Baker's (two–parameter) scaling relations, etc.

Here, using a novel unified approach, we reproduce Golden's *static* results ($h \in \mathbb{R}$) and produce the analogous static results associated with a conductive/superconductive medium in terms of $w(p,z) = \sigma^*(p,z)/\sigma_1$, where $z = 1/h$. Using Stieltjes integral representations of $m(p,h;\mu)$ and $w(p,z;\alpha)$, where μ and α are underlying bounded positive measures, we determine the two–parameter scaling relations of each system. We then extend these results to the quasi-static limit where the contrast parameter becomes complex ($h \in \mathbb{C}$). Assuming a physically consistent symmetry in the spectral properties of μ and α, we link these two sets of scaling relations. Showing that, under this assumption, the scaling relations of both of these systems are determined by only (two) parameters. We also provide a proof of the assumption underlying Golden's pioneering work, that there exists gaps in the support of the measures μ and α at the spectral endpoints. We do so by constructing a measure $\varrho(d\lambda)$ supported on the set $\{0,1\}$, which links the measures $\mu(d\lambda)$ and $\alpha(d\lambda)$. We demonstrate that, *precisely* at the percolation threshold, the weight at $\lambda = 1$ increases from zero and the weight at $\lambda = 0$ diverges. Identifying the percolation threshold with the collapse in spectral gaps at the spectral endpoints.

III. THE ANALYTIC CONTINUATION METHOD

We now formulate the effective parameter problem for two–component conductive media. Let (Ω, P) be a probability space and let $\boldsymbol{\sigma}(\vec{x}, \omega)$ be the local conductivity tensor with inverse $[\boldsymbol{\sigma}^{-1}](\vec{x}, \omega)$, which are (spatially) stationary random fields in $\vec{x} \in \mathbb{R}^d$ and $\omega \in \Omega$. Here Ω is the set of all realizations of our random medium and $P(d\omega)$ is the underlying probability measure, which is compatible with stationarity [3]. Define the Hilbert space of stationary random fields $\mathscr{H}_s \subset L^2(\Omega, P)$, and the associated Hilbert spaces of stationary curl free random fields $\mathscr{H}_\times \subset \mathscr{H}_s$ and stationary divergence free random fields $\mathscr{H}_\bullet \subset \mathscr{H}_s$ [3]:

$$
\begin{aligned}
\mathscr{H}_\times &:= \{\vec{Y}(\omega) \in \mathscr{H}_s \mid \vec{\nabla} \times \vec{Y} = 0 \text{ weakly and } \langle \vec{Y} \rangle = 0\}, \\
\mathscr{H}_\bullet &:= \{\vec{Y}(\omega) \in \mathscr{H}_s \mid \vec{\nabla} \cdot \vec{Y} = 0 \text{ weakly and } \langle \vec{Y} \rangle = 0\},
\end{aligned}
\tag{4}
$$

where $\vec{Y} : \Omega \mapsto \mathbb{R}^d$ and $\langle \cdot \rangle$ means ensemble average over Ω, or by an ergodic theorem [3] spatial average over all of $\mathbb{R}^d$.

Consider the following variational problems: find $\vec{E}_f \in \mathscr{H}_\times$ and $\vec{J}_f \in \mathscr{H}_\bullet$ such that

$$
\langle \boldsymbol{\sigma}(\vec{E}_0 + \vec{E}_f) \cdot \vec{Y} \rangle = 0 \quad \forall\, \vec{Y} \in \mathscr{H}_\times, \qquad \langle \boldsymbol{\sigma}^{-1}(\vec{J}_0 + \vec{J}_f) \cdot \vec{Y} \rangle = 0 \quad \forall\, \vec{Y} \in \mathscr{H}_\bullet, \tag{5}
$$

respectively [3]. Under the assumption that the bilinear forms $a(\vec{u}, \vec{v}) = \vec{u}^T \boldsymbol{\sigma}(\vec{x}, \omega) \vec{v}$ and $\hat{a}(\vec{u}, \vec{v}) = \vec{u}^T [\boldsymbol{\sigma}^{-1}](\vec{x}, \omega) \vec{v}$ are bounded and coercive, where $\vec{u}, \vec{v} \in \mathbb{R}^d$, these problems have unique solutions satisfying [3]

$$
\vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{J} = 0, \qquad \vec{J} = \boldsymbol{\sigma} \vec{E}, \quad \vec{E} = \vec{E}_0 + \vec{E}_f, \quad \langle \vec{E} \rangle = \vec{E}_0, \tag{6}
$$

$$
\vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{J} = 0, \qquad \vec{E} = \boldsymbol{\sigma}^{-1} \vec{J}, \quad \vec{J} = \vec{J}_0 + \vec{J}_f, \quad \langle \vec{J} \rangle = \vec{J}_0, \tag{7}
$$

respectively. Here $\vec{E}_f$ and $\vec{J}_f$ are the fluctuating electric field and current density of mean zero, respectively, about the (constant) averages $\vec{E}_0$ and $\vec{J}_0$, respectively.

We assume that the tensor $\boldsymbol{\sigma}(\vec{x}, \omega)$ takes the values σ_1 and σ_2, and that the tensor $[\boldsymbol{\sigma}^{-1}](\vec{x}, \omega)$ takes the values $1/\sigma_1$ and $1/\sigma_2$, and write $\boldsymbol{\sigma}(\vec{x}, \omega) := \sigma_1 \chi_1(\vec{x}, \omega) + \sigma_2 \chi_2(\vec{x}, \omega)$ and $[\boldsymbol{\sigma}^{-1}](\vec{x}, \omega) := \chi_1(\vec{x}, \omega)/\sigma_1 + \chi_2(\vec{x}, \omega)/\sigma_2$, where χ_j is the characteristic function of medium

$j = 1, 2$, which equals one for all $\omega \in \Omega$ having medium j at $\vec{x}$, and zero otherwise [3]. As $\vec{E}_f \in \mathscr{H}_\times$ and $\vec{J}_f \in \mathscr{H}_\bullet$, equation (5) yields the energy (power density) constraints $\langle \vec{J} \cdot \vec{E}_f \rangle = \langle \vec{E} \cdot \vec{J}_f \rangle = 0$, which lead to the following reduced energy representations:

$$\langle \vec{J} \cdot \vec{E} \rangle = \langle \vec{J} \rangle \cdot \vec{E}_0, \qquad \langle \vec{E} \cdot \vec{J} \rangle = \langle \vec{E} \rangle \cdot \vec{J}_0 \,. \tag{8}$$

Therefore, we define the effective complex permittivity tensor $\boldsymbol{\sigma}^*$, and $[\boldsymbol{\sigma}^{-1}]^*$ by

$$\langle \vec{J} \rangle = \boldsymbol{\sigma}^* \vec{E}_0, \qquad \langle \vec{E}\, \rangle = [\boldsymbol{\sigma}^{-1}]^* \vec{J}_0, \tag{9}$$

respectively. For simplicity, we focus on one diagonal component of these symmetric tensors: $\sigma^* := \boldsymbol{\sigma}^*_{kk}$ and $[\sigma^{-1}]^* := [\boldsymbol{\sigma}^{-1}]^*_{kk}$.

Due to the homogeneity of these functions, $\sigma^*(a\sigma_1, a\sigma_2) = a\sigma^*(\sigma_1, \sigma_2)$, for any complex number a, σ^* and $[\sigma^{-1}]^*$ depend only on the ratio $h := \sigma_1/\sigma_2$, and we define the dimensionless functions $m(h) := \sigma^*/\sigma_2$, $w(z) := \sigma^*/\sigma_1$, $\tilde{m}(h) := \sigma_1[\sigma^{-1}]^*$, and $\tilde{w}(z) := \sigma_2[\sigma^{-1}]^*$, where $z = z(h) := 1/h$. The functions $m(h)$ and $\tilde{m}(h)$ are analytic off the negative real axis in the h–plane, taking the upper half plane to the upper half plane, and the functions $w(z)$ and $\tilde{w}(z)$ are analytic off the negative real axis in the z–plane, taking the upper half plane to the upper half plane, so that they are examples of Herglotz, or Stieltjes functions [3]. We assume that $|h| < 1$, i.e. $0 < |\sigma_1| < |\sigma_2| < \infty$, and we further restrict h in the complex plane to the set

$$\mathcal{U} := \{h := h_r + \mathrm{i}h_i \in \mathbb{C} : |h| < 1 \text{ and } h \notin (-1, 0]\}, \tag{10}$$

where $m(h)$, $w(z(h))$, $\tilde{m}(h)$, and $\tilde{w}(z(h))$ are analytic functions of h [3]. In order to illuminate the symmetries between these functions, we will henceforth focus on the variable h.

The key step in the method is obtaining integral representations for σ^* and $[\sigma^{-1}]^*$. These integral representations are given in terms of the following resolvent representations of the electric field $\vec{E}$, and the current density $\vec{J}$:

$$\vec{E} = s(s + \Gamma\chi_1)^{-1}\vec{E}_0 = t(t + \Gamma\chi_2)^{-1}\vec{E}_0, \qquad \vec{J} = s(s - \Upsilon\chi_2)^{-1}\vec{J}_0 = t(t - \Upsilon\chi_1)^{-1}\vec{J}_0. \tag{11}$$

These formulas follow from manipulations of equations (6)–(7), where $s := 1/(1-h)$ and $t := 1/(1-z) = 1 - s$. The operator $-\Gamma = -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$ is a projection onto curl-free fields, based on convolution with the free-space Green's function for the Laplacian $-\Delta = -\nabla^2$ [3], i.e. $-\Gamma : \mathscr{H}_s \mapsto \mathscr{H}_\times$ and for every $\vec{\zeta} \in \mathscr{H}_\times$, we have $-\Gamma\vec{\zeta} = \vec{\zeta}$. To the authors knowledge, the operator $\Upsilon = \vec{\nabla} \times (-\Delta)^{-1} \vec{\nabla} \times$ is being introduced here for the first time. For the convenience of the reader, we recall a few vector calculus facts. For every $\vec{\zeta} \in \mathscr{H}_\bullet$ we have $\vec{\zeta} = \vec{\nabla} \times (\vec{A} + \vec{C})$ weakly, where $\vec{\nabla} \times \vec{C} = 0$, weakly [16]. The arbitrary vector $\vec{C}$ can be chosen so that $\vec{\nabla} \cdot \vec{A} = 0$, weakly [16]. Hence, $\vec{\nabla} \times \vec{\zeta} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = -\Delta \vec{A}$, weakly. The vector $\vec{C}$ chosen in this manner gives the Coulomb (or transverse) gauge of $\vec{\zeta}$ [16]. Let $\mathscr{C}_\bullet \subset \mathscr{H}_\bullet$ denote the *closure* of the space of stationary divergence free random fields of Coulomb gauge. On the Hilbert space $\mathscr{C}_\bullet$, one can show that the operator Υ is a projector, based on convolution with the free-space Green's function for $-\Delta$, i.e. $\Upsilon : \mathscr{H}_s \mapsto \mathscr{H}_\bullet$ and for every $\vec{\zeta} \in \mathscr{C}_\bullet$, we have $\Upsilon \vec{\zeta} = \vec{\zeta}$.

It is more convenient to consider the functions $F(s) := 1 - m(h)$ and $E(s) = 1 - \tilde{m}(h)$, $s = 1/(1-h)$, which are analytic off $[0,1]$ in the s–plane, and $G(t) = 1 - w(z(h))$ and $H(t) = 1 - \tilde{w}(z(h))$, which are analytic off $[0,1]$ in the t–plane [3, 17], and satisfy

$$0 < |F(s)|, |E(s)| < 1, \quad 0 < |G(t(s))|, |H(t(s))| < \infty. \tag{12}$$

We write $\vec{E}_0 = E_0\, \vec{e}_k$ and $\vec{J}_0 = J_0 \vec{j}_k$, where $\vec{e}_k$ and $\vec{j}_k$ are unit vectors, for some $k = 1, \ldots, d$. Using $\vec{J} = \sigma \vec{E}$, $\vec{E} = [\sigma^{-1}]\, \vec{J}$, $\langle \vec{E}\, \rangle = \vec{E}_0$, $\langle \vec{J}\, \rangle = \vec{J}_0$, and $\chi_1 = 1 - \chi_2$ leading to the identities $\sigma = \sigma_2(1 - \chi_1/s) = \sigma_1(1 - \chi_2/t)$ and $[\sigma^{-1}] = (1 - \chi_2/s)/\sigma_1 = (1 - \chi_1/t)/\sigma_2$, equations (9) and (11), and The Spectral Theorem [18], we have [3, 17]

$$F(s) = \langle \chi_1 (s + \Gamma \chi_1)^{-1} \vec{e}_k \cdot \vec{e}_k \rangle = \int_{\lambda_0}^{\lambda_1} \frac{d\mu(\lambda)}{s - \lambda}, \quad E(s) = \langle \chi_2 (s - \Upsilon \chi_2)^{-1} \vec{j}_k \cdot \vec{j}_k \rangle = \int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1} \frac{d\eta(\lambda)}{s - \lambda}, \tag{13}$$

$$G(t) = \langle \chi_2 (t + \Gamma \chi_2)^{-1} \vec{e}_k \cdot \vec{e}_k \rangle = \int_{\hat{\lambda}_0}^{\hat{\lambda}_1} \frac{d\alpha(\lambda)}{t - \lambda}, \quad H(t) = \langle \chi_1 (t - \Upsilon \chi_1)^{-1} \vec{j}_k \cdot \vec{j}_k \rangle = \int_{\check{\lambda}_0}^{\check{\lambda}_1} \frac{d\kappa(\lambda)}{t - \lambda}.$$

In order to illuminate the symmetries between the integral representations of (13), in the last two formulas of equation (13) we will henceforth make the change of variables $t(s) = 1 - s$ and $\lambda \mapsto 1 - \lambda$. Equation (13) displays general formulas holding for two–component stationary random media in lattice and continuum settings [15].

In equation (13), μ, η, α, and κ are bounded positive measures which depend only on the geometry of the medium, and are supported on $\Sigma_\mu, \Sigma_\eta, \Sigma_\alpha, \Sigma_\kappa \subseteq [0,1]$, respectively [3, 19], with $\lambda_0 := \inf(\Sigma_\mu)$, $\lambda_1 := \sup(\Sigma_\mu)$, $\tilde{\lambda}_0 := \inf(\Sigma_\eta)$, $\tilde{\lambda}_1 := \sup(\Sigma_\eta)$, $\hat{\lambda}_0 := \inf(\Sigma_\alpha)$, $\hat{\lambda}_1 := \sup(\Sigma_\alpha)$, $\check{\lambda}_0 := \inf(\Sigma_\kappa)$, and $\check{\lambda}_1 := \sup(\Sigma_\kappa)$. The underlying Integro-differential operators $\mathbf{M}_1 := \chi_1(-\Gamma)\chi_1$, $\mathbf{M}_2 := \chi_2(-\Gamma)\chi_2$, $\mathbf{K}_1 := \chi_1 \Upsilon \chi_1$, and $\mathbf{K}_2 := \chi_2 \Upsilon \chi_2$ are self adjoint on the Hilbert space $L^2(\Omega, P)$ [3]. They are compositions of projection operators on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{C}_\bullet$, and are hence are bounded by 1 in the underlying operator norm [3, 20]. Equation (13) involves spectral representations of resolvents involving the self adjoint operators, $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$, as, for example, in the Hilbert space $L^2(\Omega, P)$ with weight χ_2 in the inner product, $\Gamma\chi_2$ is a bounded self–adjoint operator [3]. The measures μ, η, α, and κ are spectral measures of the family of projections of these operators in the $\langle \vec{e}_k, \vec{e}_k \rangle$, or $\langle \vec{j}_k, \vec{j}_k \rangle$, state [3, 18].

A key feature of the formulas (13) is that the parameter information in s and E_0 is *separated* from the geometry of the composite encapsulated in the spectral measures, through their moments, which depend on the correlation functions of the medium [3]. For example, $\mu_0 := \int_0^1 \lambda^0 \mu(\lambda) = \langle \chi_1 \rangle = 1 - p$, the volume fraction of material component 1, and $\alpha_0 = p$. A principal application of the analytic continuation method is to derive *forward bounds* on σ^*, given partial information on the microgeometry [3, 19, 21, 22]. One can also use the integral representations (13) to obtain *inverse bounds*, allowing one to use data about the electromagnetic response of a sample to bound its structural parameters such as p [23] (and references therein).

By applying The Spectral Theorem to the energy constraints $\langle \vec{J} \cdot \vec{E}_f \rangle = \langle \vec{E} \cdot \vec{J}_f \rangle = 0$, we have obtained detailed decompositions of the system energy in terms of the measures μ, η, α, and κ. For example, $\langle \vec{J} \cdot \vec{E}_f \rangle = 0$, $\vec{E} = \vec{E}_0 + \vec{E}_f$, $\langle \vec{E} \rangle = \vec{E}_0$, and $\sigma = \sigma_2(1 - \chi_1/s)$ imply

$$0 = \langle \sigma \vec{E} \cdot \vec{E}_f \rangle = \langle \sigma_2 (1 - \chi_1/s)(\vec{E}_0 \cdot \vec{E}_f + E_f^2) \rangle = \sigma_2 \left(\langle E_f^2 \rangle - \frac{1}{s} \left(\langle \chi_1 \vec{E}_0 \cdot \vec{E}_f \rangle + \langle \chi_1 E_f^2 \rangle \right) \right).$$

Therefore, by The Spectral Theorem [18] and the symmetries in equation (13), we have

$$\frac{\langle E_f^2 \rangle}{E_0^2} = \int_0^1 \frac{\lambda \, d\mu(\lambda)}{(s-\lambda)^2} = \int_0^1 \frac{\lambda \, d\alpha(\lambda)}{(1-s-\lambda)^2}, \quad \frac{\langle J_f^2 \rangle}{J_0^2} = \int_0^1 \frac{\lambda \, d\eta(\lambda)}{(s-\lambda)^2} = \int_0^1 \frac{\lambda \, d\kappa(\lambda)}{(1-s-\lambda)^2}. \tag{14}$$

Equation (14) then leads to a *complete* decomposition of the system energy in terms of Herglotz functions involving the spectral measures μ, η, α, and κ. These decompositions of the system energy hold for general two–component stationary random media in lattice and continuum settings [15].

The formulas in equation (13) define Stieltjes transforms of the bounded positive measures μ, η, α, and κ. These can be transformed in terms of Stieltjes functions [4] of h via the change of variables $s = 1/(1-h)$ and $\lambda(y) = y/(y+1) \iff y(\lambda) = \lambda/(1-\lambda)$ so that, for example,

$$F(s) = (1-h)\int_{S_0}^{S} \frac{(y+1)d\mu(\frac{y}{y+1})}{1+hy}, \qquad G(t(s)) = (h-1)\int_{\hat{S}_0}^{\hat{S}} \frac{(y+1)[-d\alpha(\frac{1}{y+1})]}{1+hy}. \tag{15}$$

Here $S_0 := \lambda_0/(1-\lambda_0)$, $S := \lambda_1/(1-\lambda_1)$, $\hat{S}_0 := (1-\hat{\lambda}_1)/\hat{\lambda}_1$, and $\hat{S} := (1-\hat{\lambda}_0)/\hat{\lambda}_0$, so that $\lim_{\lambda_0 \to 0} S_0 = 0$, $\lim_{\lambda_1 \to 1} S = \infty$, $\lim_{\hat{\lambda}_1 \to 1} \hat{S}_0 = 0$, $\lim_{\hat{\lambda}_0 \to 0} \hat{S} = \infty$. Therefore, by equation (15) and the underlying symmetries in equation (13), the Stieltjes function representations of the formulas in (13) are given by

$$\begin{aligned}
m(h) &= 1+(h-1)g(h), \quad g(h) := \int_0^\infty \frac{d\phi(y)}{1+hy}, \quad d\phi(y) := (y+1)d\mu\left(\frac{y}{y+1}\right), \\
\tilde{m}(h) &= 1+(h-1)\tilde{g}(h), \quad \tilde{g}(h) := \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy}, \quad d\tilde{\phi}(y) := (y+1)d\eta\left(\frac{y}{y+1}\right), \\
w(z(h)) &= 1-(h-1)\hat{g}(h), \quad \hat{g}(h) := \int_0^\infty \frac{d\hat{\phi}(y)}{1+hy}, \quad d\hat{\phi}(y) := (y+1)\left[-d\alpha\left(\frac{1}{y+1}\right)\right], \\
\tilde{w}(z(h)) &= 1-(h-1)\check{g}(h), \quad \check{g}(h) := \int_0^\infty \frac{d\check{\phi}(y)}{1+hy}, \quad d\check{\phi}(y) := (y+1)\left[-d\kappa\left(\frac{1}{y+1}\right)\right].
\end{aligned} \tag{16}$$

Equation (16) displays general formulas holding for two component stationary random media in lattice and continuum settings [15], and should be compared to equation (1) regarding the Ising model. As μ, η, α, and κ are bounded positive measures on $[0,1]$, ϕ, $\tilde{\phi}$, $\hat{\phi}$, and $\check{\phi}$ are positive measures on $[0,\infty]$, and are also bounded if the supremum of the support of these measures is finite. Consequently the following inequalities hold for all $h \in \mathcal{U}$:

$$\frac{\partial^{2n}\zeta}{\partial h^{2n}} > 0, \quad \frac{\partial^{2n-1}\zeta}{\partial h^{2n-1}} < 0, \qquad \left|\frac{\partial^n \zeta}{\partial h^n}\right| > 0, \qquad \zeta = g(h), \tilde{g}(h), \hat{g}(h), \check{g}(h) \tag{17}$$

where $n \geq 0$, the first two inequalities hold for $h \in \mathbb{R}$, and the last inequality holds for $h \in \mathcal{U}$ such that $h_i \neq 0$. The formula $\partial^2 m/\partial h^2 > 0$ in (17), for example, is a macroscopic version of the fact that the effective resistance of a finite network is a concave downward function of the resistances of the individual network elements [2]. Equation (17) is the analogue of equation (2) in the Ising model.

By equation (16), the moments ϕ_n of ϕ satisfy

$$\phi_n = \int_0^\infty y^n d\phi(y) = \int_0^\infty y^n (y+1) d\mu\left(\frac{y}{y+1}\right) = \int_0^1 \frac{\lambda^n d\mu(\lambda)}{(1-\lambda)^{n+1}}. \tag{18}$$

A partial fraction expansion of $\lambda^n/(1-\lambda)^{n+1}$ then shows that

$$\frac{(-1)^n}{n!}\lim_{s\to 1}\frac{\partial^n F(s)}{\partial s^n} = \int_0^1 \frac{d\mu(\lambda)}{(1-\lambda)^{n+1}} = \sum_{j=0}^{n}\binom{n}{j}\phi_j, \tag{19}$$

demonstrating that ϕ_n depends on $\int_0^1 d\mu(\lambda)/(1-\lambda)^{n+1}$ (and) all the lower moments of ϕ: ϕ_j, $j=0,1,\ldots,n-1$. From equations (13) and (14), we see that the first two moments of ϕ are identified with energy components:

$$\phi_0 = \lim_{s\to 1} \frac{\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle}{E_0^2}, \quad \phi_1 = \lim_{s\to 1} \frac{\langle E_f^2\rangle}{E_0^2}. \tag{20}$$

Therefore by equation (19), *all* of the moments ϕ_j, $j \geq 2$ depend on these energy components. Equation (18) suggests that the moments ϕ_n, $n \geq 0$, become singular as $\sup\{\Sigma_\mu\} \to 1$. However, we will show that this is only true for the moments of order $n \geq 1$, and that $\lambda = 1$ is a removable simple singularity under μ.

Similarly, the moments $\hat{\phi}_n$ of $\hat{\phi}$ satisfy

$$\hat{\phi}_n = \int_0^1 \frac{(1-\lambda)^n d\alpha(\lambda)}{\lambda^{n+1}}, \qquad \frac{(-1)^n}{n!} \lim_{s\to 1} \frac{\partial^n G(t(s))}{\partial^n t} = \int_0^1 \frac{d\alpha(\lambda)}{\lambda^{n+1}} = \sum_{j=0}^{n} \binom{n}{j} \hat{\phi}_j\,. \tag{21}$$

Equations (13) and (14) similarly identify the first two moments, $\hat{\phi}_0$ and $\hat{\phi}_1$, of $\hat{\phi}$ with energy components. Equation (21) then implies that all the higher moments $\hat{\phi}_j$, $j \geq 2$, depend on these energy components. Equation (21) suggests, and we will show, that *all* the moments $\hat{\phi}_n$, $n \geq 0$, become singular as $\inf\{\Sigma_\alpha\} \to 0$. By the symmetries in equations (13) and (16), equations (18)–(19) hold for $\tilde{\phi}$ with $E(s)$ and η in lieu of $F(s)$ and μ, and equation (21) holds for $\check{\phi}$ with $H(s)$ and κ in lieu of $G(t(s))$ and α. In order to make connections to $F(s)$ and $G(t(s))$ in the representation of equations (19) and (21), we have assumed that $F(s)$ and $G(t(s))$ may be differentiated under the integral sign with respect to s. This is warranted by Lemma **??** below.

By equations (8)–(9), we have the following two energy representations of σ^* and $[\sigma^{-1}]^*$, $\langle \vec{J}\cdot\vec{E}\rangle = \sigma_2 m(h) E_0^2 = \sigma_1 w(z(h)) E_0^2$ and $\langle \vec{E}\cdot\vec{J}\rangle = \tilde{m}(h) E_0^2/\sigma_1 = \tilde{w}(z(h)) E_0^2/\sigma_2$, which imply

$$\begin{aligned} m(h) = h w(z(h)) \iff 1 - F(s) = (1-1/s)(1-G(t(s))), \\ \tilde{m}(h) = h\tilde{w}(z(h)) \iff 1 - E(s) = (1-1/s)(1-H(t(s))). \end{aligned} \tag{22}$$

Using equation (16), minor algebraic manipulation in equation (22) implies that

$$g(h) + h\hat{g}(h) = 1, \qquad \tilde{g}(h) + h\check{g}(h) = 1, \quad h \in \mathcal{U}. \tag{23}$$

For all $h \in \mathcal{U}$, the functions $g(h)$, $\hat{g}(h)$, $\tilde{g}(h)$, and $\check{g}(h)$ are analytic [3], and therefore have bounded h derivatives of all orders [24]. An inductive argument applied to equation (23) yields,

$$\frac{\partial^n g}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}}{\partial h^{n-1}} + h\frac{\partial^n \hat{g}}{\partial h^n} = 0, \qquad \frac{\partial^n \tilde{g}}{\partial h^n} + n\frac{\partial^{n-1}\check{g}}{\partial h^{n-1}} + h\frac{\partial^n \check{g}}{\partial h^n} = 0, \quad n \geq 1. \tag{24}$$

In the complex quasi–static case, where $h \in \mathcal{U}$ such that $h_i \neq 0$, the complex representation of equation (24) is, for example,

$$\frac{\partial^n g_r}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}_r}{\partial h^{n-1}} + h_r\frac{\partial^n \hat{g}_r}{\partial h^n} - h_i\frac{\partial^n \hat{g}_i}{\partial h^n} = 0, \quad \frac{\partial^n g_i}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}_i}{\partial h^{n-1}} + h_r\frac{\partial^n \hat{g}_i}{\partial h^n} + h_i\frac{\partial^n \hat{g}_r}{\partial h^n} = 0, \tag{25}$$

where we have used the following definitions:

$$\frac{\partial^n g_r}{\partial h^n} := \mathrm{Re}\frac{\partial^n g}{\partial h^n}, \quad \frac{\partial^n g_i}{\partial h^n} := \mathrm{Im}\frac{\partial^n g}{\partial h^n}, \qquad \frac{\partial^n \hat{g}_r}{\partial h^n} := \mathrm{Re}\frac{\partial^n \hat{g}}{\partial h^n}, \quad \frac{\partial^n \hat{g}_i}{\partial h^n} := \mathrm{Im}\frac{\partial^n \hat{g}}{\partial h^n}.$$

The respective formulas associated with $\tilde{g}$ and $\check{g}$ in (24) follow from the substitutions $g \mapsto \tilde{g}$ and $\hat{g} \mapsto \check{g}$. Equations (22)–(25) are general formulas holding for two component stationary random media in the lattice and continuum settings [15].

A. Measure Equivalences in Transport

In this section we show that the symmetries underlying the analytic continuation method, allow one to construct precise relations between the measures μ and α, and η and κ. We already noted that the formulas in equation (13) are Stieltjes transforms of the measures μ, η, α, and κ. Conversely, given the Stieltjes transform of a measure, The Stieltjes-Perron Inversion Theorem [1, 25, 26] allows one to recover the underlying measure. For example,

$$\mu(\upsilon) = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\mathrm{Im}(F(\upsilon + \mathrm{i}\epsilon)) \;, \quad \upsilon \in \Sigma_\mu. \tag{26}$$

To evoke this theorem directly, in equation (13) we define $d\tilde{\alpha}(\lambda) := [-d\alpha(1-\lambda)]$ and $d\tilde{\kappa}(\lambda) = [-d\kappa(1-\lambda)]$ so that $G(t(s)) = -\int_0^1 d\tilde{\alpha}(\lambda)/(s-\lambda)$ and $H(t(s)) = -\int_0^1 d\tilde{\kappa}(\lambda)/(s-\lambda)$. Therefore, by setting $s = \upsilon + \mathrm{i}\epsilon$ for $\upsilon \in \Sigma_\mu \cap \Sigma_\alpha$, equations (22) and (26) imply that

$$\begin{gathered}\upsilon\mu(\upsilon) = (1-\upsilon)[-\alpha(1-\upsilon)] - \upsilon\varrho(\upsilon), \qquad \upsilon\eta(\upsilon) = (1-\upsilon)[-\tau(1-\upsilon)] - \upsilon\varrho_0(\upsilon), \\ \varrho(\upsilon) = \lim_{\epsilon\downarrow 0}\frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2}\int_0^1\frac{(\upsilon+\lambda-1)d\alpha(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2}\,, \quad \varrho_0(\upsilon) = \lim_{\epsilon\downarrow 0}\frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2}\int_0^1\frac{(\upsilon+\lambda-1)d\kappa(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2}\,.\end{gathered} \tag{27}$$

Equation (27) displays general formulas holding for two component stationary random media in the lattice and continuum settings [15].

In this section, we will show that equations (22)–(23) and (27) completely determine the measures ϱ and ϱ_0. The integral representations of equation (23) follow from equation (16), and are given by

$$\int_0^\infty \frac{d\phi(y)}{1+hy} + h\int_0^\infty \frac{d\hat{\phi}(y)}{1+hy} = 1, \qquad \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy} + h\int_0^\infty \frac{d\check{\phi}(y)}{1+hy} = 1. \tag{28}$$

Due to the underlying symmetries of this framework, without loss of generality, we henceforth focus on $F(s;\mu)$, $G(t(s);\alpha)$, $g(h;\phi)$, and $\hat{g}(h,\hat{\phi})$. We wish to re-express the first formula in equation (28) in a more suggestive form, by adding and subtracting the quantity $h\int_0^{S(p)} y\,d\phi(y)/(1+hy)$. This is permissible if the modulus of this quantity is finite for all $h\in\mathcal{U}$ [20, 24]. The affirmation of this fact is given in Lemma **??** below, and we may therefore add and subtract it in equation (28), yielding

$$h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} \equiv 1-\phi_0 \equiv m(0), \quad d\Phi_0(y) := d\hat{\phi}(y) - y\,d\phi(y), \quad \forall\ h\in\mathcal{U}, \tag{29}$$

as $1-\phi_0 \equiv 1-F(s)|_{s=1} := m(h)|_{h=0}$. Equation (29) gives an alternate representation of the function $m(0)=\lim_{h\to 0} hw(h)$, and shows that the transform of the signed measure [24] Φ_0, $h\int_0^\infty d\Phi_0(y)/(1+hy)$, is independent of h for all $h\in\mathcal{U}$. We may relate this representation of $m(0)$ to the measure ϱ found in equation (27) using equation (16) and the variable identity $y=\lambda/(1-\lambda) \iff \lambda = y/(1+y)$:

$$d\Phi_0(y) = = \frac{1}{(1-\lambda)^2}((1-\lambda)\,[-d\alpha(1-\lambda)] - \lambda\,d\mu(\lambda)) = \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2} = y(1+y)\,d\varrho\left(\frac{y}{1+y}\right).$$

We may therefore express equation (29) in terms of $\varrho(d\lambda)$ as follows:

$$h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} = h\int_0^\infty \frac{y(1+y)d\varrho(\frac{y}{1+y})}{1+hy} = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h+\lambda(1-\lambda)}\,. \tag{30}$$

Equations (29)–(30) are general formulas holding for two–component stationary random media in the lattice and continuum settings [15].

Remark III.1 *Define the transform $\mathcal{D}(h;\varrho)$ of the measure ϱ by*

$$\mathcal{D}(h;\varrho) = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h+\lambda(1-\lambda)}\,. \tag{31}$$

Equations (12) *and* (29)–(30) *show that $\mathcal{D}(h;\varrho)$ satisfies the following properties:*
(1) *$\mathcal{D}(h;\varrho)$ is independent of h,* ***(2)*** $0<\mathcal{D}(h;\varrho)<1$, *and* ***(3)*** $\mathcal{D}(h;\varrho)\equiv m(0)$.

The following lemma is the key result of this section.

Lemma III.1 *Let $\mathcal{D}(h;\varrho)$ be defined as in equation* (31)*, where $h \in \mathcal{U}$, and let the Herglotz functions $m(0) := m(h)|_{h=0} = 1 - F(s)|_{s=1}$ and $w(0) := w(z)|_{z=0} = 1 - G(t)|_{t=1}$ be defined as in equation* (13)*. If $\mathcal{D}(h;\varrho)$ satisfies the properties of Remark III.1, then*

$$\varrho(d\lambda) = -w(0)\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda), \qquad \varrho_0(d\lambda) = -\tilde{w}(0)\delta_0(d\lambda) + \tilde{m}(0)(1-\lambda)\delta_1(d\lambda), \tag{32}$$

where $\delta_{\lambda_0}(d\lambda)$ is the Dirac measure centered at λ_0, the representation of $\varrho_0(d\lambda)$ follows from symmetry, and $0 \le m(h)|_{h=0}, w(z)|_{z=0} \le 1$.

Proof: The proof of the second formula in equation (32) follows directly from the proof of the first formula in (32) and the underlying symmetries of this mathematical framework. Let $\mathcal{D}(h;\varrho)$, defined in equation (31), satisfy the properties of Remark III.1. The measure ϱ is independent of h [3]. If the support Σ_ϱ of the measure ϱ is over continuous spectrum [18] then $\mathcal{D}(h;\varrho)$ depends on h, contradicting property (1). Therefore the measure ϱ is defined over pure point spectrum: $\Sigma_\varrho \subset \sigma_{pp}$ [18]. Moreover, in order for properties (1) and (3) to be satisfied we must have $\Sigma_\varrho \equiv \{0,1\}$, so that the measure ϱ is of the form

$$\varrho(d\lambda) = W_0(\lambda)\delta_0(d\lambda) + W_1(\lambda)\delta_1(d\lambda),$$

where the $W_j(\lambda)$, $j = 0,1$, are functions of $\lambda \in [0,1]$ which are to be determined. In view of the numerator of the integrand in equation (31), we may assume that the function $W_0(\lambda) = W_0(0) := W_0 \not\equiv 0$ is independent of λ. In order for property (2) to be satisfied we must have $W_1(\lambda) \sim 1-\lambda$ as $\lambda \to 1$ (any other power of $1-\lambda$ would contradict property (2)). Therefore, with out loss of generality, we may set $W_1(\lambda) = (1-\lambda)W_1$, where W_1 is independent of λ. Property (3) then yields $W_1 = m(0)$.

We have shown that $\varrho(d\lambda) = W_0\,\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda)$, $W_0 \not\equiv 0$. Plugging this formula into equation (27), $(\lambda d\mu(\lambda) = (1-\lambda)[-d\alpha(1-\lambda)] - \lambda d\varrho(\lambda))$, we are able determine W_0. Indeed, using the definition of $F(s)$ (13), equation (22) $(F(s) - (1-1/s)G(t(s)) = 1/s)$, and $(1-\lambda)/(\lambda(s-\lambda)) = -(1-1/s)/(s-\lambda) + 1/(s\lambda)$ we find that

$$\begin{aligned} F(s) &= -\left(1-\frac{1}{s}\right)\int_0^1 \frac{[-d\alpha(1-\lambda)]}{s-\lambda} + \frac{1}{s}\int_0^1 \frac{[-d\alpha(1-\lambda)]}{\lambda} - \int_0^1 \frac{d\varrho(\lambda)}{s-\lambda} \\ &= \left(1-\frac{1}{s}\right)G(t(s)) + \frac{1}{s}\int_0^1 \frac{d\alpha(\lambda)}{1-\lambda} - \frac{W_0(p)}{s} - m(p,0)\lim_{\lambda\to 1}\frac{1-\lambda}{s-\lambda}, \quad \forall\ |s|>1 \end{aligned} \tag{33}$$

which implies that $W_0 = -w(0)$. Therefore $F(s)$ $\Box$.

The formulas in equation (32) demonstrate that $\lambda = 1$ is a removable simple singularity under μ, α, η, and κ, and illustrate how the relations $0 < |F(s)|, |E(s)| < 1$ can hold even when $s = 1$ ($h = 0$) and there is spectra at $\lambda = 1$. Moreover, this shows that ϕ and $\tilde{\phi}$ are bounded measures with mass (18) $0 \leq \phi_0, \tilde{\phi}_0 \leq 1$, and that the higher moments ϕ_j and $\tilde{\phi}_j$, $j \geq 2$ become singular when there is spectra at $\lambda = 1$. Furthermore, these formulas (32) illustrate that singular behavior can develop in $G(t(s))$ and $H(t(s))$ when the strength $w(0)$ of the delta component at $\lambda = 0$ becomes non–zero. Moreover, this shows that $\hat{\phi}$ and $\check{\phi}$ are unbounded measures (21). These features of the model will be discussed in more detail in Section IV below.

IV. CRITICAL BEHAVIOR OF TRANSPORT IN LATTICE AND CONTINUUM PERCOLATION MODELS

We now formulate the problem of percolation driven critical transitions in transport, exhibited by two–component conductive media. For percolation models such as the random resistor network (RRN) [27, 28], the connectedness of the system is determined by the volume fraction p of type two inclusions in an otherwise homogeneous type one medium. The average cluster size of these inclusions grows as p increases, and there is a critical volume fraction p_c, $0 < p_c < 1$, called the *percolation threshold*, where an infinite cluster of the inclusions first appears.

Consider transport through a RRN [15] where bonds are assigned electrical conductivities σ_2 with probability p, and σ_1 with probability $1 - p$. In the limit $h \to 0$, the composite may be interpreted as a conductor/insulator system ($\sigma_1 \to 0$ while $0 < |\sigma_2| < \infty$) or a conductor/superconductor system ($\sigma_2 \to \infty$ while $0 < |\sigma_1| < \infty$). As $h \to 0$ ($\sigma_1 \to 0$ and $0 < |\sigma_2| < \infty$), the effective complex conductivity $\sigma^*(p,h) := \sigma_2\, m(p,h)$ and the effective complex resistance $[\sigma^{-1}]^*(p,h) := \sigma_2^{-1}\tilde{w}(p, z(h))$ undergo a conductor/insulator critical transition:

$$|\sigma^*(p,0)| := |\sigma_2\, m(p,0)| = \begin{cases} 0, & \text{for } p < p_c \\ 0 < |\sigma_1| < |\sigma^*(p)| < |\sigma_2|, & \text{for } p > p_c \end{cases}, \tag{34}$$

$$|[\sigma^{-1}]^*(p,0)| := |\sigma_2^{-1}\tilde{w}(p,0)| = \begin{cases} \infty, & \text{for } p < p_c \\ |\sigma_2|^{-1} < |[\sigma^{-1}]^*(p)| < |\sigma_1|^{-1}, & \text{for } p > p_c \end{cases}.$$

While, as $h \to 0$ ($\sigma_2 \to \infty$ and $0 < |\sigma_1| < \infty$), the effective complex conductivity $\sigma^*(p,h) := \sigma_1 w(p,z(h))$ and the effective complex resistance $[\sigma^{-1}]^*(p,h) := \sigma_1^{-1}\tilde{m}(p,h)$ undergo a conductor/superconductor critical transition:

$$|\sigma^*(p,0)| := |\sigma_1 w(p,0)| = \begin{cases} 0 < |\sigma^*(p)| < \infty, & \text{for } p < p_c \\ \infty, & \text{for } p > p_c \end{cases}, \tag{35}$$

$$|[\sigma^{-1}]^*(p,0)| := |\sigma_1^{-1}\tilde{m}(p,0)| = \begin{cases} 0 < |[\sigma^{-1}]^*(p)| < \infty, & \text{for } p < p_c \\ 0, & \text{for } p > p_c \end{cases}.$$

We will focus on the conductor/insulator critical transition of the effective complex conductivity $\sigma^*(p,h) = \sigma_2\, m(p,h)$ and the conductor/superconductor critical transition of the effective complex conductivity $\sigma^*(p,h) = \sigma_1 w(p,z(h))$. It is clear from equations (16) and (34)–(35), that the corresponding results immediately generalize to $[\sigma^{-1}]^*(p,h) = \sigma_1^{-1}\tilde{m}(p,h)$ and $[\sigma^{-1}]^*(p,h) = \sigma_2^{-1}\tilde{w}(p,z(h))$, respectively, with $p \mapsto 1-p$.

The critical behavior of binary conductors is made more precise through the definition of critical exponents. For $h \in \mathbb{R} \cap \mathcal{U}$, as $h \to 0$ the effective complex conductivity $\sigma^*(p,h) = \sigma_2\, m(p,h)$ exhibits critical conductor/insulator behavior near the percolation threshold p_c, $\sigma^*(p,0) \sim (p-p_c)^t$ as $p \to p_c^+$, moreover at $p = p_c$, $\sigma^*(p_c,h) \sim h^{1/\delta}$ as $h \to 0$. We assume the existence of the critical exponents t and δ, as well as γ, defined via a conductive susceptibility $\chi(p,0) := \partial m(p,0)/\partial h \sim (p-p_c)^{-\gamma}$ as $p \to p_c^+$. Furthermore, for $p > p_c$, we assume that there is a gap $\theta_\mu \sim (p-p_c)^{\Delta}$ in the support of μ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^+$, or that any spectrum in this region does not affect power law behavior [15]. Therefore, for our percolation models with $p > p_c$, the support of ϕ is contained in the compact interval $[0, S(p)] \subset\subset \mathbb{R}^+$, where $S(p) \sim (p-p_c)^{-\Delta}$ as $p \to p_c^+$. As the moments of ϕ become singular as $\theta_\mu \to 0$ (18), we also assume that there exist critical exponents γ_n such that $\phi_n(p) \sim (p-p_c)^{-\gamma_n}$ as $p \to p_c^+$, $n \geq 0$. When $h \in \mathcal{U}$ such that $h_i \neq 0$, we assume the existence of critical exponents t_r, δ_r, t_i and δ_i corresponding to $m_r(p,h) := \mathrm{Re}(m(p,h))$ and $m_i(p,h) := \mathrm{Im}(m(p,h))$. The critical exponents, γ_n and Δ, associated with the measure ϕ are independent of h and are thus unaffected. In summary:

$$\begin{array}{llll} m(p,0) \sim (p-p_c)^t, & m_r(p,0) \sim (p-p_c)^{t_r}, & m_i(p,0) \sim (p-p_c)^{t_i}, & \text{as } p \to p_c^+ \\ m(p_c,h) \sim h^{1/\delta}, & m_r(p_c,h) \sim h^{1/\delta_r}, & m_i(p_c,h) \sim h^{1/\delta_i}, & \text{as } |h| \to 0, \\ \chi(p,0) \sim (p-p_c)^{-\gamma}, & \phi_n \sim (p-p_c)^{-\gamma_n}, & S(p) \sim (p-p_c)^{-\Delta}, & \text{as } p \to p_c^+. \end{array} \tag{36}$$

We also assume the existence of critical exponents associated with the left hand limit $p \to p_c^-$: γ', γ_n', and Δ'. The critical exponents γ, δ, Δ, and γ_n for transport are different from those defined in section II for the Ising model.

For $h \in \mathbb{R} \cap \mathcal{U}$, as $h \to 0$ the effective conductivity $\sigma^*(p,h) = \sigma_1 w(p, z(h))$ exhibits critical conductor/superconductor behavior near p_c, $\sigma^*(p,0) \sim (p-p_c)^{-s}$ as $p \to p_c^-$, and at $p = p_c$, $\sigma^*(p_c,h) \sim h^{-1/\hat{\delta}}$ as $h \to 0$, where the superconductor critical exponent s is not to be confused with the contrast parameter. We assume the existence of the critical exponents s and $\hat{\delta}$, as well as $\hat{\gamma}$, defined via a conductive susceptibility $\hat{\chi}(p) := \partial w(p,0)/\partial h \sim (p-p_c)^{-\hat{\gamma}}$ as $p \to p_c^-$. Furthermore, for $p < p_c$, we assume that there is a gap $\theta_\alpha \sim (p-p_c)^{\hat{\Delta}'}$ in the support of $[-d\alpha(1-\lambda)]$ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^-$, so that the support of $\hat{\phi}$ is contained in the compact interval $[0, \hat{S}(p)] \subset\subset \mathbb{R}^+$, where $\hat{S}(p) \sim (p-p_c)^{-\Delta}$ as $p \to p_c^+$. As the moments of $\hat{\phi}$ become singular as $\theta_\alpha \to 0$ (21), we also assume that there exist critical exponents $\hat{\gamma}_n'$ such that $\hat{\phi}_n(p) \sim (p-p_c)^{-\hat{\gamma}_n'}$ as $p \to p_c^-$, $n \geq 0$. When $h \in \mathcal{U}$ such that $h_i \neq 0$, we assume the existence of critical exponents s_r, s_i, $\hat{\delta}_r$, and $\hat{\delta}_i$ corresponding to $w_r(p,h) := \mathrm{Re}(w(p,z(h)))$ and $w_i(p,h) := \mathrm{Im}(w(p,z(h)))$. The critical exponents, γ_n' and Δ', associated with the measure $\hat{\phi}$ are independent of h and are thus unaffected. In summary:

$$
\begin{aligned}
&\hat{w}(p,0) \sim (p-p_c)^s, && \hat{w}_r(p,0) \sim (p-p_c)^{s_r}, && \hat{w}_i(p,0) \sim (p-p_c)^{s_i}, && \text{as } p \to p_c^- \\
&\hat{w}(p_c,h) \sim h^{1/\hat{\delta}}, && \hat{w}_i(p_c,h) \sim h^{1/\hat{\delta}_i}, && \hat{w}_i(p_c,h) \sim h^{1/\hat{\delta}_i}, && \text{as } |h| \to 0, \\
&\hat{\chi}(p,0) \sim (p-p_c)^{-\hat{\gamma}'}, && \hat{\phi}_n \sim (p-p_c)^{-\hat{\gamma}_n'}, && \hat{S}(p) \sim (p-p_c)^{-\hat{\Delta}'}, && \text{as } p \to p_c^-.
\end{aligned}
\tag{37}
$$

We also assume the existence of critical exponents associated with the right hand limit $p \to p_c^+$: $\hat{\gamma}$, $\hat{\gamma}_n$, and $\hat{\Delta}$. To be more precise, when we assume the existence of a critical exponent, we assume the existence of the corresponding limit (3) [4].

A. Spectral Characterization of the Critical Transitions in Transport

We now discuss the gaps θ_α and θ_η (for $p < p_c$), and θ_μ and θ_κ (for $p > p_c$). As the operators $-\Gamma$ and Υ are projectors on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{C}_\bullet$, respectively, the eigenvalues thereof are confined to the set $\{0,1\}$ [18]. The associated operators $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1,2$ are positive definite compositions of projection operators, thus the eigenvalues thereof are confined to the set $[0,1]$ [29].

While, in general, the spectra actually extends all the way to the spectral endpoints $\lambda = 0, 1$, the part close to $\lambda = 0, 1$ corresponds to very large, but very rare connected regions of the defect inclusions (Lifshitz phenomenon), and is believed to give exponentially small contributions to the effective complex conductance (resistance), and not affect power law behavior [15]. In [30] O. Bruno has proven the existence of spectral gaps in matrix/particle systems with polygonal inclusions, and studied how the gaps vanishes as the inclusions touch (like $p \to p_c$). In Figure IV A below, we give a graphical representation of the spectral measure $\alpha(d\lambda)$ for finite 2–d and 3–d RRNs [23]. This figure shows that, as $p \to p_c$, the width of the gaps in the spectrum near $\lambda = 0, 1$ vanish, leading to a divergence in the measure at $\lambda = 0$. Our simulations show that, as p increases beyond p_c, the spectrum piles up at the spectral endpoints $\lambda = 0, 1$ until $\Sigma_\alpha \subset \{0, 1\}$, when $p = 1$. This behavior is predicted by our result in section III A. There we proved that, *precisely* at $p = p_c$, a delta function component of the measure $\alpha(d\lambda)$ appears at $\lambda = 1$, and that the weight of the essental delta function at $\lambda = 0$ diverges.

We now provide a proof, for large lattice systems, of the existence of spectral gaps which colapse with increasing p. For lattice systems with a finite number n of lattice sites, the differential equations in (6)–(7) become difference equations (Kirchoff's laws) [2]. Consequently, the operators $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$ are given by $N \times N$ matrices, say [2, 23]. We focus on $\mathbf{M}_2 = \chi_2(-\Gamma)\chi_2$, as our results extend to the other operators by symmetry. In this lattice setting, $-\Gamma$ is a real symmetric projection matrix and can therefore be diagonalized: $-\Gamma = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$, where $\mathbf{D}$ is a diagonal matrix of zeros and ones and $\mathbf{Q}$ is a real orthogonal matrix. More specifically,

$$-\Gamma = \begin{bmatrix} - \vec{q}_1 - \\ \vdots \\ - \vec{q}_N - \end{bmatrix} \begin{bmatrix} \mathbf{I}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} - \vec{q}_1 - \\ \vdots \\ - \vec{q}_N - \end{bmatrix}^T = \begin{bmatrix} (\vec{q}_1 \cdot \vec{q}_1)_L & (\vec{q}_1 \cdot \vec{q}_2)_L & \cdots & (\vec{q}_1 \cdot \vec{q}_N)_L \\ \vdots & \vdots & \ddots & \vdots \\ (\vec{q}_N \cdot \vec{q}_1)_L & (\vec{q}_N \cdot \vec{q}_2)_L & \cdots & (\vec{q}_N \cdot \vec{q}_N)_L \end{bmatrix}, \tag{38}$$

where $0 < L < N$ when $N \gg 1$, $\mathbf{I}_L$ is the $L \times L$ identity matrix, $\mathbf{0}$ is a matrix of zeros of arbitrary dimension, $(\vec{q}_i \cdot \vec{q}_j)_L := \sum_{l=1}^{L} (\vec{q}_i)_l (\vec{q}_j)_l$, and $(\vec{q}_i)_l$ is the l^{th} component of the vector $\vec{q}_i \in \mathbb{R}^N$. Here, we consider the case where $1 \ll L < N$.

The spectral measure $\alpha(d\lambda)$ of the matrix $\mathbf{M}_2$ is given by a sum of "Dirac δ functions,"

$$\alpha(d\lambda) = \left[\sum_{j=1}^{N} m_j \delta_{\lambda_j}(d\lambda) \right] d\lambda := \alpha(\lambda) d\lambda, \tag{39}$$

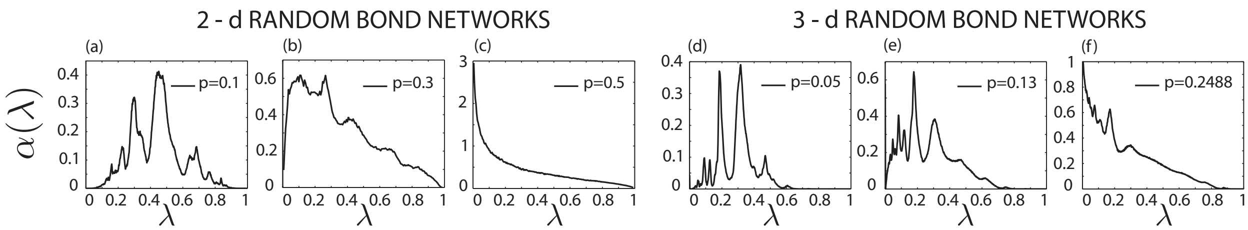

FIG. 1. *The spectral function for the 2–d and 3–d random bond network.* As the volume fraction p of defect bonds increases, from left to right, the width of the gaps in the spectrum near $\lambda = 0, 1$ shrink to 0 with increasing connectedness as the percolation thresholds $p_c = 0.5$ and $p_c \approx 0.2488$ are approached.

where $\delta_{\lambda_j}(d\lambda)$ is the Dirac delta measure centered at λ_j, $m_j = \langle \vec{e}_k^{\,T} [\vec{v}_j \vec{v}_j^{\,T}] \, \vec{e}_k \rangle$, $\vec{e}_k$ is a N–dimensional vector of ones, and λ_j and $\vec{v}_j$ are the eigenvalues and eigenvectors of $\mathbf{M}_2$, respectively. In this matrix case, the associated Stieltjes transformation of the measure $\alpha(d\lambda)$ (13) is given by the sum $G(t(s)) = \sum_{j=1}^{n} m_j/(1-s-\lambda_j)$, and $\alpha(\lambda)$ in equation (39) is called "*the spectral function*," which is defined only pointwise on the set of eigenvalues $\{\lambda_j\}$. In Figure IV A we give a graphical representation of the spectral measure for finite 2–d and 3–d RRNs. It displays linearly connected peaks of histograms with bin sizes on the order of 10^{-2}. The apparent smoothness of the spectral function graphs in this figure is due to the large number ($\sim 10^6$) of eigenvalues and eigenvectors calculated, and ensemble averaged [23].

In the matrix case, the action of χ_2 is given by that of a square diagional matrix of zeros and ones [23]. Thus the action of χ_2 in the matrix $\chi_2(-\Gamma)\chi_2$ is to introduce a row and column of zeros in the matrix $-\Gamma$, corresponding to every diagional entry of χ_2 with value 0. This causes the matrix $\chi_2(-\Gamma)\chi_2$ to have a large null space, and is the source of the essental δ component of the measure $\alpha(d\lambda)$ at $\lambda = 0$, as $n \to \infty$.

When there is only one defect inclusion, $p = 1/n$, located at the j^{th} bond, χ_2 has all zero entries except at the j^{th} diagonal: $\chi_2 = \text{diag}(0, \cdots, 0, 1, 0, \cdots, 0) := \text{diag}(\vec{\text{v}}_j)$. Therefore, the only non-trivial eigenvalue is given by $\lambda = (\vec{q}_j \cdot \vec{q}_j)_L = \sum_{l=1}^{L} (\vec{q}_j)_l^2 = 1 - \sum_{l=L+1}^{N} (\vec{q}_j)_l^2$, with eigenvector $\vec{\text{v}}_j$ and weight $m_0 = 1/n$. This implies that there is a gap at $\lambda = 0$, $\theta_0 := \sum_{l=1}^{L} (\vec{q}_j)_l^2 > 0$, and a gap at $\lambda = 1$, $\theta_1 := \sum_{l=L+1}^{N} (\vec{q}_j)_l^2 > 0$. It is clear that these bounds hold for all $\omega \in \Omega$ such that $p = 1/n$ when $L \gg 1$. We have already mentioned that the eigenvalues of $\mathbf{M}_1$ are restricted to the set $\{0, 1\}$ when $p = 1$ ($\chi_2 \equiv \mathbf{I}_N$). Therefore, there exists a $0 < p_0 < 1$ such that, for all $p \geq p_0$, there exists a $\omega \in \Omega$ such that $\theta_0(\omega) = 0$ and/or $\theta_1(\omega) = 0$.

B. Baker's Critical Theory for Transport in Binary Composite Media

Baker's critical theory characterizes phase transitions of a given system via the asymptotic behaviors of underlying Stieltjes functions, near a critical point. This powerfull method has been very successfull in the Ising model, precisely characterizing the phase transition (spontaneous magnetization) [4]. We will show that this method has far reaching utility in the characterization of phase transitions in transport, exhibited by a wide variety of two–component composites. The following theorem characterizes Stieltjes functions [4].

Theorem IV.1 *Let $D(i,j)$ denote the following determinant*

$$\mathcal{D}(i,j) = \begin{vmatrix} \xi_i & \xi_{i+1} & \cdots & \xi_{i+j} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{i+j} & \xi_{i+j+1} & \cdots & \xi_{i+2j} \end{vmatrix}. \tag{40}$$

The ξ_n form a series of Stieltjes if and only if $\mathcal{D}(i,j) \geq 0$ for all $i, j = 0, 1, 2, \ldots$

Baker's inequalities for the sequences γ_n (36) and $\hat{\gamma}_n$ (37) of transport follow from Theorem IV.1. Indeed, for example, $\phi_n \sim (p - p_c)^{-\gamma_n}$ and Theorem IV.1 with $\phi_i = \xi_i$, $i = n$, and $j = 1$, imply that,

$$\begin{aligned} &(p-p_c)^{-\gamma_n-\gamma_{n+2}} - (p-p_c)^{-2\gamma_{n+1}} \geq 0 \iff (p-p_c)^{-\gamma_n-\gamma_{n+2}+2\gamma_{n+1}} \geq 1 \\ &\iff -\gamma_n - \gamma_{n+2} + 2\gamma_{n+1} \leq 0 \iff \boxed{\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0}, \text{ for } |p - p_c| \ll 1. \end{aligned} \tag{41}$$

The sequence of inequalities (41) are *Baker's inequalities* for transport, corresponding to $m(p,h)$, and they imply that the sequence γ_n increases at least linearly with n. The symmetries in equations (16) and (36)–(37) imply that Baker's inequalities also hold for the sequences γ'_n, $\hat{\gamma}_n$, and $\hat{\gamma}'_n$.

The key results of this section are the two–parameter scaling relations between the critical exponents in the conductor/insulator system, defined in equations (36), and that of the conductor/superconductor system, defined in equations (37). By equation (22) we know that $m(p,h)$ and $w(p,z(h))$ are related, therefore the Stieltjes functions $g(p,h)$ and $\hat{g}(p,h)$

are related. Moreover by equation (27), we know that the measures μ and α are related, therefore the measures ϕ and $\hat{\phi}$ are related. We therefore anticipate that these sets of critical exponents are also related. This is indeed the case, and the resultant relationship between the insulation critical exponent t and the superconduction critical exponent s is in agreement with the seminal paper by A. L. Efros and B. I. Shklovskii [31]. These results are summarized in Theorem IV.2 below.

Theorem IV.2 *Let t, t_r, t_i, δ, δ_r, δ_i, γ, γ_n, Δ, γ'_n, and Δ' be defined as in equations* (36), *and s, s_r, s_i, $\hat{\delta}$, $\hat{\delta}_r$, $\hat{\delta}_i$, $\hat{\gamma}$, $\hat{\gamma}'_n$, $\hat{\Delta}'$, $\hat{\gamma}_n$, and $\hat{\Delta}$ be defined as in equations* (37). *Then the following scaling relations hold:*

1) $\gamma_1 = \gamma,\ \gamma'_1 = \gamma',\ \hat{\gamma}_1 = \hat{\gamma},\ \textit{and}\ \hat{\gamma}'_1 = \hat{\gamma}'$, **2**) $\gamma'_0 = 0,\ \gamma_0 < 0,\ \gamma'_n > 0\ \textit{and}\ \gamma_n > 0\ \textit{for}\ n \geq 1$

3) $\hat{\gamma}'_n > 0\ \textit{for}\ n \geq 0$, **4**) $\gamma_1 = \hat{\gamma}_0\ \textit{and}\ \Delta = \hat{\Delta}$, **5**) $\gamma'_1 = \hat{\gamma}'_0\ \textit{and}\ \Delta' = \hat{\Delta}'$,

6) $\gamma_n = \gamma + \Delta(n-1)\ \textit{for}\ n \geq 1$, **7**) $\hat{\gamma}'_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma} + \hat{\Delta}'(n-1)\ \textit{for}\ n \geq 0$,

8) $t = \Delta - \gamma$, **9**) $s = \hat{\gamma}'_0 = \hat{\gamma} - \hat{\Delta}'$, **10**) $\delta = \dfrac{\Delta}{\Delta - \gamma}$, **11**) $\hat{\delta}\,' = \dfrac{\hat{\Delta}'}{\hat{\gamma}'_0} = \dfrac{\hat{\Delta}'}{\hat{\gamma} - \hat{\Delta}'}$,

12) $t_r = t_i = t$, **13**) $s_r = s_i = s$, **14**) $\delta_r = \delta_i = \delta\ \textit{and}\ \hat{\delta}_r = \hat{\delta}_i = \hat{\delta}$,

15) *If* $\Delta = \Delta'$ *and* $\gamma = \gamma'$, *then* $t + s = \Delta$ *and* $\delta = 1/(1 - 1/\hat{\delta}\,')$

Theorem IV.2 will be proven via a sequence of lemmas as we collect some important properties of $m(p,h)$, $g(p,h)$, $w(p,z(h))$, and $\hat{g}(p,h)$, and how they are related.

Lemma IV.1 *Set $Y_{i,j}(h,y) := y^i/(1+hy)^j$. Then for all $h \in \mathcal{U}$, we have $Y_{i,j}(h,y) \in L^1(\phi(dy)), L^1(\hat{\phi}(dy))$, for all $i < j$.*

Proof: For general $p \in [0,1]$, the support of ϕ and $\hat{\phi}$ are given by $\Sigma_\phi := [S_0(p), S(p)]$ and $\Sigma_{\hat{\phi}} := [\hat{S}_0(p), \hat{S}(p)]$, respectively, and are defined in terms of the support of μ and α, respectively, directly below equation (15). By hypothesis, the extremum of these sets satisfy $\lim_{p\to p_c^+} S_0(p) = \lim_{p\to p_c^-} \hat{S}_0(p) = 0$ and $\lim_{p\to p_c^+} S(p) = \lim_{p\to p_c^-} \hat{S}(p) = \infty$. For every $h \in \mathcal{U}$, it is clear that there exists real, strictly positive, S_h such that

$$1 \ll |h| S_h < \infty. \tag{42}$$

Set $h \in \mathcal{U}$ and $0 \ll S_h < \infty$ satisfying (42), and write $\Sigma_\phi := [S_0(p), S_h] \cup (S_h, S(P)]$ and $\Sigma_\phi := [\hat{S}_0(p), S_h] \cup (S_h, \hat{S}(P)]$. For all $p \in [0,1]$, equations (19) and (34) imply that

$0 \le \lim_{h\to 0} |m(p,h)| = 1-\phi_0 < 1$, which implies that the mass ϕ_0 of ϕ is uniformly bounded for all $p \in [0,1]$. Therefore, for all $h \in \mathcal{U}$,

$$\int_{S_0}^{S_h} |Y_{i,j}(h,y)| d\phi(y) \le \frac{S_h^i\, \phi([S_0, S_h])}{|1+hS_0|^j} < \infty, \quad \int_{\hat{S}_0}^{S_h} |Y_{i,j}(h,y)| d\phi(y) \le \frac{S_h^i\, \hat{\phi}([\hat{S}_0, S_h])}{|1+h\hat{S}_0|^j} < \infty, \tag{43}$$

where $\phi([S_0(p), S_h])$ is the *bounded* ϕ measure of the set $[S_0(p), S_h]$. The boundedness of the second formula in equation (43) follows from equations (15)–(16), showing that the $\hat{\phi}$ measure of the compact interval $[\hat{S}_0(p), S_h]$ is bounded. More specifically, in terms of Σ_α, we have $\hat{\lambda}_1(p) = 1 - \hat{S}_0(p)/(1+\hat{S}_0(p))$ and $\hat{\lambda}_h := 1 - S_h/(1+S_h) > 0$. Thus equations (15)–(16) imply

$$\hat{\phi}([\hat{S}_0(p), S_h]) = \int_{\hat{S}_0(p)}^{S_h} d\hat{\phi}(y) = \int_{\hat{S}_0(p)}^{S_h} (1+y)\left[-d\alpha\left(\frac{1}{1+y}\right)\right] = \int_{\hat{\lambda}_h}^{\hat{\lambda}_1(p)} \frac{d\alpha(\lambda)}{\lambda} \le \frac{\alpha_0}{\hat{\lambda}_h} < \infty.$$

If the sets Σ_ϕ and $\Sigma_{\hat{\phi}}$ are compact intervals, we are done. Otherwise set $S(p) = \hat{S}(p) = \infty$. In terms of Σ_μ and Σ_α, we have $\lambda_h := S_h/(1+S_h)$ and $\lambda_1(p) := S(p)/(1+S(p)) \equiv 1$, and $\hat{\lambda}_0(p) = 1 - \hat{S}(p)/(1+\hat{S}(p)) \equiv 0$ and $\hat{\lambda}_h := 1 - S_h/(1+S_h)$, respectively, where $0 \ll \lambda_h < 1$ and $0 < \hat{\lambda}_h \ll 1$. Equations (16) and (42) imply, for all $h \in \mathcal{U}$,

$$\begin{aligned}\int_{S_h}^{S(p)} |Y_{i,j}(h,y)| d\phi(y) &\sim \frac{1}{|h|^j} \int_{S_h}^{S(p)} \frac{d\phi(y)}{y^{j-i}} = \frac{1}{|h|^j} \int_{S_h}^{S(p)} \frac{1+y}{y^{j-i}} d\mu\left(\frac{y}{1+y}\right) \\ &= \frac{1}{|h|^j} \int_{\lambda_h}^{\lambda_1(p)} \frac{(1-\lambda)^{j-i-1}}{\lambda^{j-i}} d\mu(\lambda) \le \frac{1}{|h|^j} \frac{(1-\lambda_h)^{j-i-1}\mu_0}{\lambda_h^{j-i}} < \infty, \\ \int_{S_h}^{\hat{S}(p)} |Y_{i,j}(h,y)| d\hat{\phi}(y) &= \frac{1}{|h|^j} \int_{\hat{\lambda}_0(p)}^{\hat{\lambda}_h} \frac{\lambda^{j-i-1}\, d\alpha(\lambda)}{(1-\lambda)^{j-i}} \le \frac{1}{|h|^j} \frac{\hat{\lambda}_h^{j-i-1}\alpha_0}{(1-\hat{\lambda}_h)^{j-i}} < \infty. \ \Box\end{aligned} \tag{44}$$

Lemma IV.2 *For all $h \in \mathcal{U}$ and $p \in [0,1]$, the Stieltjes functions $g(p,h)$ and $\hat{g}(p,h)$ may be differentiated under the integral sign:*

$$\begin{aligned}\frac{\partial^n g(p,h)}{\partial h^n} &= \frac{\partial^n}{\partial h^n} \int_0^\infty \frac{d\phi(y)}{1+hy} = (-1)^n n! \int_0^\infty \frac{y^n d\phi(y)}{(1+hy)^{n+1}} \sim \phi_n\,, \\ \frac{\partial^n \hat{g}(p,h)}{\partial h^n} &= \frac{\partial^n}{\partial h^n} \int_0^\infty \frac{d\hat{\phi}(y)}{1+hy} = (-1)^n n! \int_0^\infty \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}} \sim \hat{\phi}_n,\end{aligned} \tag{45}$$

where $n \ge 1$ and the asymptotics in equation (45) hold when $0 < |h| \ll 1$ and $|p - p_c| \ll 1$.

Proof: By Lemma IV.1 the Stieltjes functions $g(p,h)$ and $\hat{g}(p,h)$ may be differentiated under the integral sign ([20] Theorem 2.27). The asymptotic behaviors in equation (45) follow from equations (18)–(19), (21), Baker's inequalities (41), and equation (16) ($g(p,h) = sF(p,s)$ and $\hat{g}(p,h) = -s\,G(p,t(s))$) which implies that, for $c_j, b_j \in \mathbb{Z}$,

$$\lim_{h\to 0} \frac{\partial^n g(p,h)}{\partial h^n} = \sum_{j=0}^{n} c_j \lim_{s\to 1} \frac{\partial^j F(p,s)}{\partial s^j} \sim \phi_n\,, \quad \lim_{h\to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} = \sum_{j=0}^{n} b_j \lim_{s\to 1} \frac{\partial^j G(p,t(s))}{\partial t^j} \sim \hat{\phi}_n\,. \;\square$$

Equation (24) and Lemma IV.2 imply that

$$\int_0^{S(p)} \frac{y^n d\phi(y)}{(1+hy)^{n+1}} = \int_0^{\hat{S}(p)} \frac{y^{n-1} d\hat{\phi}(y)}{(1+hy)^{n}} - h\int_0^{\hat{S}(p)} \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}\,, \tag{46}$$

which holds, by Lemma IV.1, for all $n \geq 1$, $p \in [0,1]$, and $h \in \mathcal{U}$. Moreover, the integral representations of equations (25) may be obtained by equation (45) as follows:

$$\begin{aligned} \frac{\partial^n g}{\partial h^n} &= (-1)^n n! \int_0^{S(p)} \frac{y^n d\phi(y)}{|1+hy|^{2(n+1)}} (1+\bar{h}y)^{n+1} \\ &= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{h}^j \int_0^{S(p)} \frac{y^{n+j} d\phi(y)}{|1+hy|^{2(n+1)}}\,, \end{aligned} \tag{47}$$

which holds, by Lemma IV.1, for all $n \geq 1$, $p \in [0,1]$, and $h \in \mathcal{U}$. It is worth mentioning that equation (46) can be written in terms of the measure $d\Phi_{n-1}(y) := y^{n-1} d\Phi_0(y)$: $\int_0^\infty d\Phi_{n-1}(y)/(1+hy)^{n+1} \equiv 0$, for all $n \geq 1$, $h \in \mathcal{U}$, and $p \in [0,1]$, where the signed measure $\Phi_0(dy)$ is defined in equation (29). Moreover, in equation (25) for $n = 1$, equation (47) implies that $\int_0^\infty d\Phi_1(y)/|1+hy|^4 \equiv 0$, for all $h \in \mathcal{U}$ such that $h_i \neq 0$ and $p \in [0,1]$. These formulas are easily seen to be consistent with Lemma III.1.

Lemma IV.3 *$\gamma_1 = \gamma$, $\gamma_1' = \gamma'$, $\hat{\gamma}_1 = \hat{\gamma}$, and $\hat{\gamma}_1' = \hat{\gamma}'$*

Proof: Set $0 < p - p_c \ll 1$, by equation (16) ($g(h) = sF(s)$), (19), and (36) we have

$$(p-p_c)^{-\gamma} \sim \chi(p,0) := \frac{\partial m(p,0)}{\partial h} = \lim_{s\to 1}\left(F(p,s) + \frac{\partial F(p,s)}{\partial s}\right) = -\phi_1 \sim (p-p_c)^{-\gamma_1}, \tag{48}$$

hence $\gamma_1 = \gamma$. Similarly for $0 < p_c - p \ll 1$, we have $\gamma_1' = \gamma'$. By equation (48), and the symmetries between m and w (16) and the critical exponent definitions (36)–(37), we also have that $\hat{\gamma}_1 = \hat{\gamma}$ and $\hat{\gamma}_1' = \hat{\gamma}'$ $\square$.

Equation (22) is consistent with, and provides a link between equations (34) and (35). We will see that the fundamental asymmetry between $m(p,h)$ and $w(p,z(h))$ ($\gamma_0' = 0$ and $\hat{\gamma}_0' > 0$), given by Theorem IV.2.2-3, is a direct and essential consequence of equation (22), and has deep and far reaching implications.

Lemma IV.4 *Let the sequences* γ_n *and* γ_n', $n \geq 0$, *be defined as in equation* (36). *Then*

$$\begin{aligned} &1) \quad \gamma_0' = 0,\ \gamma_0 < 0,\ \gamma_n' > 0,\ \textit{and}\ \ \gamma_n > 0\ \ \textit{for}\ \ n \geq 1 \\ &2) \quad 0 < \lim_{h\to 0} \langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle / E_0^2 < 1\ \ \textit{for all}\ \ p \in [0,1] \end{aligned}$$

Proof: By equation (35) $|w(p,0)|$ is bounded for all $p < p_c$. Thus for all $p < p_c$, equations (19), (36), and (22) imply that

$$0 = \lim_{h\to 0} hw(p,z(h)) = \lim_{h\to 0} m(p,h) = \lim_{s\to 1}(1 - F(p,s)) = 1 - \phi_0(p) \sim 1 - (p_c - p)^{-\gamma_0'},$$

for $0 < p_c - p \ll 1$, which is consistent with equation (34). Therefore, $\gamma_0' = 0$ and ϕ is a probability measure for all $p < p_c$. The strict positivity of the γ_n', for $n \geq 1$, follows from Baker's inequalities (41). Therefore, from equation (48) we have that

$$\infty = \lim_{p\to p_c^-} \phi_1(p) = -\lim_{p\to p_c^-} \frac{\partial m(p,0)}{\partial h}. \tag{49}$$

For $p > p_c$, equations (19) and (34) imply that $0 < \lim_{h\to 0} |m(p,h)| = 1 - \phi_0 < 1$. Therefore, $(p - p_c)^{-\gamma_0} \sim \phi_0 < 1$ for all $0 < p - p_c \ll 1$, hence $\gamma_0 < 0$. The strict positivity of γ_1 follows from equation (49), and the strict positivity of the γ_n for $n \geq 2$ follows from Baker's inequalities (41). Equation (20) and the inequality $0 < \lim_{h\to 0} |m(p,h)| = 1 - \phi_0 < 1$ implies that $0 < \lim_{h\to 0}\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle / E_0^2 < 1$ for all $p \in [0,1]$ $\Box$.

Lemma IV.5 *Let the sequence* $\hat{\gamma}_n'$, $n \geq 0$, *be defined as in equation* (37). *Then*

$$\begin{aligned} &1) \quad \hat{\gamma}_n' > 0\ \ \textit{for all}\ \ n \geq 0 \\ &2) \quad \lim_{h\to 0}\langle E_f^2\rangle = \infty\ \ \textit{for all}\ \ p > p_c. \end{aligned}$$

Proof: By equation (34) we have $0 < \lim_{h\to 0} |m(p,h)| < 1$, for all $p > p_c$. Therefore equation (22) implies that $\lim_{h\to 0} w(p,z(h)) = \lim_{h\to 0} m(p,h)/h = \infty$, for all $p > p_c$, which is consistent with equation (35). More specifically, equations (34) and (22) imply that $0 < \lim_{h\to 0} |m(p,h)| = \lim_{h\to 0} |hw(p,z(h))| := L(p) < 1$, where $\lim_{p\to p_c^+} L(p) = 0$. Therefore, by equation (16), we have

$$\begin{aligned} &\lim_{h\to 0} |hw(p,z(h))| = \lim_{h\to 0} |h\hat{g}(p,h)| \in (0,1),\ \text{for all}\ p > p_c, \\ &\lim_{h\to 0} |hw(p,z(h))| = \lim_{h\to 0} |h\hat{g}(p_c,h)| = 0\ \text{for all}\ p < p_c. \end{aligned} \tag{50}$$

As will be shown below, equation (50) has very important consequences. By equations (21), (35), and (37) we have,

$$\infty = \lim_{p\to p_c^-} \lim_{h\to 0} w(p, z(h)) = \lim_{p\to p_c^-} \lim_{s\to 1} (1 - G(p, s)) = 1 + \lim_{p\to p_c^-} \hat{\phi}_0(p) \sim 1 + \lim_{p\to p_c^-} (p_c - p)^{-\hat{\gamma}_0'},$$

hence $\hat{\gamma}_0' > 0$. Baker's inequalities (41) then imply that $\hat{\gamma}_n' > 0$ for all $n \geq 0$. Equation (20) and $\hat{\gamma}_0' > 0$ implies that

$$\lim_{h\to 0} \langle E_f^2 \rangle = \infty \quad \forall\, p > p_c, \quad \Box.$$

The asymptotic behavior of $\hat{g}(p, h)$ in equation (45) and Lemma IV.5 motivates the following fundamental homogenization assumption of this section [4]:

Remark IV.1 *Near the critical point $(p, h) = (p_c, 0)$, the asymptotic behavior of the Stieltjes function $\hat{g}(p, h)$ is determined primarily by the mass $\hat{\phi}_0(p)$ of the measure $\hat{\phi}$ and the rate of collapse of the spectral gap θ_α.*

By remark IV.1, and in light of Lemmas IV.3–IV.5, we make the following changes in variables

$$\hat{q} := y(p_c - p)^{\hat{\Delta}'}, \quad \hat{Q}(p) := \hat{S}(p)(p_c - p)^{\hat{\Delta}'}, \quad d\hat{\pi}(\hat{q}) := (p_c - p)^{\hat{\gamma}_0'}\, d\hat{\phi}(y), \tag{51}$$

$$q := y(p - p_c)^{\Delta}, \quad Q(p) := S(p)(p - p_c)^{\Delta}, \quad d\pi(q) := (p - p_c)^{\gamma}\, y\, d\hat{\phi}(y), \tag{52}$$

so that $\hat{Q}(p), Q(p) \sim 1$ and the masses $\hat{\pi}_0$ and π_0 of the measures $\hat{\pi}$ and π, respectively, satisfy $\hat{\pi}_0, \pi_0 \sim 1$ as $p \to p_c$. Equations (51)–(52) define the following scaling functions $G_{n-1}(x)$, $\hat{G}_n(\hat{x})$, $\mathcal{G}_{n-1,j}(x)$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$ as follows.

For $h \in \mathcal{U} \cap \mathbb{R}$, equation (45) of Lemma **??** and equations (51)–(52) imply, for $n \geq 0$, that

$$\begin{aligned}
&\frac{\partial^n g}{\partial h^n} \propto (p - p_c)^{-(\gamma + \Delta(n-1))} G_{n-1}(x), &&\frac{\partial^n \hat{g}}{\partial h^n} \propto (p_c - p)^{-(\hat{\gamma}_0' + \hat{\Delta}' n)} \hat{G}_n(\hat{x}), \\
&G_{n-1}(x) := \int_0^{Q(p)} \frac{q^{n-1} d\pi(q)}{(1 + xq)^{n+1}}, &&\hat{G}_n(\hat{x}) := \int_0^{\hat{Q}(p)} \frac{\hat{q}^n d\hat{\pi}(\hat{q})}{(1 + \hat{x}\hat{q})^{n+1}}, \\
&x := h(p - p_c)^{-\Delta}, \quad 0 < p - p_c \ll 1, &&\hat{x} := h(p_c - p)^{-\hat{\Delta}'}, \quad 0 < p_c - p \ll 1.
\end{aligned} \tag{53}$$

Analogous formulas are defined for the opposite limits, involving $\hat{\Delta}$, $\hat{\gamma}_0$, Δ', and γ'.

For $h \in \mathcal{U}$ such that $h_i \neq 0$, we analogously define scaling functions $\mathcal{R}_{n-1}(x)$, $\mathcal{I}_{n-1}(x)$, $\hat{\mathcal{R}}_n(\hat{x})$, and $\hat{\mathcal{I}}_n(\hat{x})$ as follows. Using equations (47) (which follows from Remark **??**) and

(51)–(52) we have, for $0 < p - p_c \ll 1$,

$$
\begin{aligned}
\frac{\partial^n g}{\partial h^n} &= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{h}^j \int_0^{S(p)} \frac{y^{n+j} d\phi(y)}{|1+hy|^{2(n+1)}} \\
&:= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} [\bar{x}(p-p_c)^{\Delta}]^j (p-p_c)^{-(\gamma+\Delta(n-1+j))} \mathcal{G}_{n-1,j}(x) \\
&:= (-1)^n n! (p-p_c)^{-(\gamma+\Delta(n-1))} \mathcal{K}_{n-1}(x) \\
&:= (-1)^n n! (p-p_c)^{-(\gamma+\Delta(n-1))} \left[\mathcal{R}_{n-1}(x) + \mathrm{i}\, \mathcal{I}_{n-1}(x)\right], \text{ and similarly,} \\
\frac{\partial^n \hat{g}}{\partial h^n} &:= (-1)^n n! (p-p_c)^{-(\hat{\gamma}_0+\hat{\Delta} n)} \left[\hat{\mathcal{R}}_n(\hat{x}) + \mathrm{i}\, \hat{\mathcal{I}}_n(\hat{x})\right],
\end{aligned}
\tag{54}
$$

where x and $\hat{x}$ are defined in equation (53) and

$$
\begin{aligned}
\mathcal{G}_{n-1,j}(x) &:= \int_0^{Q(p)} \frac{q^{n-1+j} d\pi(q)}{|1+xq|^{2(n+1)}} & \hat{\mathcal{G}}_{n,j}(\hat{x}) &:= \int_0^{\hat{Q}(p)} \frac{\hat{q}^{n+j} d\hat{\pi}(\hat{q})}{|1+\hat{x}\hat{q}|^{2(n+1)}} \\
\mathcal{K}_{n-1}(x) &:= \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{x}^j \mathcal{G}_{n-1,j}(x) & \hat{\mathcal{K}}_n(\hat{x}) &:= \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{\hat{x}}^j \hat{\mathcal{G}}_{n,j}(\hat{x}) \\
\mathcal{R}_{n-1}(x) &:= \mathrm{Re}(\mathcal{K}_{n-1}(x)), & \hat{\mathcal{R}}_n(\hat{x}) &:= \mathrm{Re}(\hat{\mathcal{K}}_n(\hat{x})), \\
\mathcal{I}_{n-1}(\hat{x}) &:= \mathrm{Im}(\mathcal{K}_{n-1}(x)), & \hat{\mathcal{I}}_n(\hat{x}) &:= \mathrm{Im}(\hat{\mathcal{K}}_n(\hat{x})).
\end{aligned}
\tag{55}
$$

Analogous formulas are defined for the opposite limit, $0 < p_c - p \ll 1$ involving $\hat{\Delta}'$, $\hat{\gamma}_0'$, Δ', and γ'.

From equations (17)–(**??**) we have, for $h \in \mathcal{U}$, $p \in [0,1]$, and $n \geq 0$,

$$
\begin{aligned}
&G_{n-1}(x) > 0, & &\mathcal{G}_{n-1,j}(x) > 0, \\
&\hat{G}_n(\hat{x}) > 0, & &\hat{\mathcal{G}}_{n,j}(\hat{x}) > 0.
\end{aligned}
\tag{56}
$$

We asume that, for all $p < p_c$, $w(p,0)$ is analytic, and for all $p > p_c$, $m(p,0)$ is analytic (or that h derivatives of $g(p,h)$ and $\hat{g}(p,h)$ of all orders are bounded at $h = 0$ for $p > p_c$ and $p < p_c$ respectively ([3, 29, 32] and refrences therein)). Therefore we have, for $n \geq 0$,

$$
\begin{aligned}
&\lim_{h\to 0} G_{n-1}(x) < \infty, & &\lim_{h\to 0} \mathcal{G}_{n-1,j}(x) < \infty, \quad p > p_c, \\
&\lim_{h\to 0} \hat{G}_n(\hat{x}) < \infty, & &\lim_{h\to 0} \hat{\mathcal{G}}_{n,j}(\hat{x}) < \infty, \quad p < p_c.
\end{aligned}
\tag{57}
$$

Lemma IV.6 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (53)*, for $p > p_c$. Then*

1) $G_{n-1}(x) \sim 1$ *as* $x \to 0$ ($h \to 0$ *and* $0 < p - p_c \ll 1$) *for all* $n \geq 1$

2) $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ *as* $\hat{x} \to 0$ ($h \to 0$ *and* $0 < p - p_c \ll 1$) *for all* $n \geq 1$

3) $\gamma = \hat{\gamma}_0$

4) $\Delta = \hat{\Delta}$

Proof: Let $h \in \mathcal{U} \cap \mathbb{R}$, so that $g(p, h)$ and $\hat{g}(p, h)$ are real analytic [3], and $p > p_c$ so that, by equation (57), all h derivatives of $g(p, h)$ are bounded for $h = 0$. Therefore, equations (46), (53), and (56)–(57) imply that, for all $n \geq 1$, $0 < p - p_c \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p - p_c)^{-(\gamma + \Delta(n-1))} G_{n-1}(x) = (p - p_c)^{-(\hat{\gamma} + \hat{\Delta}(n-1))} [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})]. \tag{58}$$

Equations (56)–(57) imply that $G_{n-1}(x) \sim 1$ as $x \to 0$, for all $n \geq 1$. Equation (58) then implies that $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ as $\hat{x} \to 0$, for all $n \geq 1$, a competition in sign between two diverging terms. Therefore,

$$\gamma + \Delta(n-1) = \hat{\gamma}_0 + \hat{\Delta}(n-1), \quad n \geq 1. \tag{59}$$

Which in turn, implies that $\gamma = \hat{\gamma}_0$ and $\Delta = \hat{\Delta}$ $\square$.

Lemma IV.7 *Let* $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, *and the associated critical exponents be defined as in equation* (53), *for* $p < p_c$. *Then*

1) $\hat{G}_{n-1}(\hat{x}) \sim 1$ *as* $\hat{x} \to 0$ ($h \to 0$ *and* $0 < p_c - p \ll 1$) *for all* $n \geq 1$

2) $G_{n-1}(x) \sim 1$ *as* $x \to 0$ ($h \to 0$ *and* $0 < p_c - p \ll 1$ *for all* $n \geq 1$

3) $\gamma' = \hat{\gamma}'_0$

4) $\Delta' = \hat{\Delta}'$

Proof: Let $h \in \mathcal{U} \cap \mathbb{R}$, so that $g(p, h)$ and $\hat{g}(p, h)$ are real analytic [3]. Moreover let $p < p_c$ so that, by equation (57), all h derivatives of $\hat{g}(p, h)$ are bounded for $h = 0$. Thus by equation (45) we have

$$\lim_{h \to 0} h \int_0^{S(p)} \frac{y^n d\hat{\phi}(y)}{(1 + hy)^{n+1}} = 0.$$

Therefore, equations (46), (53), and (56)–(57) imply that, for all $n \geq 1$, $0 < p_c - p \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p_c - p)^{-(\hat{\gamma}' + \hat{\Delta}'(n-1))} \hat{G}_{n-1}(\hat{x}) \sim (p_c - p)^{-(\gamma' + \Delta'(n-1))} G_{n-1}(x). \tag{60}$$

Equations (56)–(57) imply that $\hat{G}_{n-1}(\hat{x}) \sim 1$ as $\hat{x} \to 0$ for all $n \geq 1$. Equation (60) then implies that $G_{n-1}(x) \sim 1$ as $x \to 0$ for all $n \geq 1$. Therefore,

$$\gamma' + \Delta'(n-1) = \hat{\gamma}'_0 + \hat{\Delta}'(n-1), \quad n \geq 1.$$

Which in turn, implies that $\gamma' = \hat{\gamma}'_0$ and $\Delta' = \hat{\Delta}'$ $\square$.

Lemma IV.8 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (53)*. Then*

1) $\gamma_n = \gamma + \Delta(n-1)$ *for all* $n \geq 1$

2) $\hat{\gamma}'_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ *for all* $n \geq 0$

3) $t = \Delta - \gamma$

4) $s = \hat{\gamma}'_0 = \hat{\gamma}' - \hat{\Delta}'$

Proof: Let $0 < p - p_c \ll 1$. By equations (45) of Remark **??**, (36), (53), and Lemma IV.6 we have, for all $n \geq 1$,

$$\begin{aligned}(p - p_c)^{-\gamma_n} &\sim \phi_n \sim \lim_{h \to 0} \frac{\partial^n g(p,h)}{\partial h^n} \sim (p - p_c)^{-(\gamma + \Delta(n-1))} \lim_{x \to 0} G_{n-1}(x) \\ &\sim (p - p_c)^{-(\gamma + \Delta(n-1))}.\end{aligned}$$

Therefore $\gamma_n = \gamma + \Delta(n-1)$ for all $n \geq 1$, with constant gap $\gamma_i - \gamma_{i-1} = \Delta$, which is consistent with the absence of multifractal behavior for the bulk conductivity [27].

Let $0 < p_c - p \ll 1$. Similarly, by equations (45) of Remark **??**, (37), (53), and Lemma IV.7 and we have, for all $n \geq 1$,

$$(p_c - p)^{-\hat{\gamma}_n} \sim \hat{\phi}_n \sim \lim_{h \to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} \propto (p_c - p)^{-(\hat{\gamma}'_0 + \hat{\Delta}' n)} \lim_{\hat{x} \to 0} \hat{G}_n(\hat{x}) \sim (p_c - p)^{-(\hat{\gamma}'_0 + \hat{\Delta}' n)}.$$

Therefore, by Lemma IV.3, we have $\hat{\gamma}_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ for all $n \geq 0$, with constant gap $\hat{\gamma}'_i - \hat{\gamma}'_{i-1} = \hat{\Delta}$, which is consistent with the absence of multifractal behavior for the bulk conductivity [27].

Let $0 < p - p_c \ll 1$. Equations (16), (36), (23), (50), and (53) yield

$$\begin{aligned}(p-p_c)^t &\sim \lim_{h\to 0} m(p,h) = 1 - \lim_{h\to 0} g(p,h) = \lim_{h\to 0} h\hat{g}(p,h) = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{\hat{x}\to 0} \hat{x}\hat{G}_0(\hat{x}) \\ &\sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}. \end{aligned} \tag{61}$$

Therefore, by Lemma IV.6 we have $t = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma$.

Let $0 < p_c - p \ll 1$. By equations (16), (37), (53), and Lemmas IV.5 and IV.7,

$$(p_c-p)^{-s} \sim \lim_{h\to 0} w(p, z(h)) \sim \lim_{h\to 0} \hat{g}(p,h) = (p_c-p)^{-\hat{\gamma}_0'} \lim_{\hat{x}\to 0} \hat{G}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}_0'}.$$

Therefore, by Lemma IV.8.2, we have $s = \hat{\gamma}_0' = \hat{\gamma}' - \hat{\Delta}'$ □.

Lemma IV.9 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation (53), for $p > p_c$ and $p < p_c$. Then for all $n \geq 1$*

1) $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma+\Delta(n-1))/\Delta}$ *as* $x \to \infty$ ($p \to p_c^+$ *and* $0 < h \ll 1$)

2) $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}$ *as* $x \to \infty$ ($p \to p_c^-$ *and* $0 < h \ll 1$)

3) $\delta = \Delta/(\Delta - \gamma)$

4) $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}' - \hat{\Delta}')$

Proof: Let $0 < h \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0,1]$ [3]. The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that all orders of h derivatives of these functions are bounded as $p \to p_c$, from the left or the right. Therefore, equation (58) holds for $0 < p - p_c \ll 1$, and

$$(0,\infty) \ni (p_c-p)^{-(\gamma'+\Delta'(n-1))} G_{n-1}(x) = (p_c-p)^{-(\hat{\gamma}'+\hat{\Delta}'(n-1))}[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \tag{62}$$

holds for $0 < p_c - p \ll 1$. Moreover, in order to cancel the diverging p dependent prefactors in equations (58) and (62) we must have, for all $n \geq 1$,

$$\begin{aligned} G_{n-1}(x) &\sim x^{-(\gamma+\Delta(n-1))/\Delta}, \qquad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}+\hat{\Delta}(n-1))/\hat{\Delta}}, \quad \text{as } p \to p_c^+, \\ G_{n-1}(x) &\sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}, \qquad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}'+\hat{\Delta}'(n-1))/\hat{\Delta}'}, \quad \text{as } p \to p_c^-. \end{aligned} \tag{63}$$

Lemma IV.9.1-2 follows from equation (63) and Lemmas IV.6–IV.7.

Now by equations (16), (36), (22), (53), and (63) for $n = 1$, we have

$$h^{1/\delta} \sim \lim_{p\to p_c^+} m(p,h) \sim \lim_{p\to p_c^+} h\hat{g}(p,h) = h \lim_{p\to p_c^+} (p-p_c)^{-\hat{\gamma}_0}\hat{G}_0(\hat{x}) \tag{64}$$
$$\sim h(p-p_c)^{-\hat{\gamma}_0} h^{-\hat{\gamma}_0/\hat{\Delta}}(p-p_c)^{-\hat{\Delta}(-\hat{\gamma}_0/\hat{\Delta})} = h^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}, \text{ as } h \to 0.$$

for $0 < h \ll 1$. Therefore by Lemma IV.7, we have $\delta = \hat{\Delta}/(\hat{\Delta}-\hat{\gamma}_0) = \Delta/(\Delta-\gamma)$. Similarly by equations (16), (37), (53), and (63) for $n = 1$, and Lemma IV.5

$$h^{-1/\hat{\delta}'} \sim \lim_{p\to p_c^-} w(p,z(h)) \sim \lim_{p\to p_c^-} \hat{g}(p,h) = \lim_{p\to p_c^-} (p-p_c)^{-\hat{\gamma}_0'}\hat{G}_0(\hat{x}) = h^{-\hat{\gamma}_0'/\hat{\Delta}'}, \tag{65}$$

for $0 < h \ll 1$. Therefore, by Lemma IV.8 we have $\hat{\delta}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}'-\hat{\Delta}')$ □.

Lemma IV.10 *Let $h \in \mathcal{U}$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in equations* (54)–(55) *for $p > p_c$ and $p < p_c$. Furthermore, let s_r, s_i, t_r, and t_i be defined as in equations* (36)–(37)*. Then,*

1) $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim 1$ as $\hat{x} \to 0$ ($h \to 0$ and $0 < p_c - p \ll 1$)

2) $\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i\hat{\mathcal{I}}_0(\hat{x})] \sim \lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i\hat{\mathcal{R}}_0(\hat{x})] \sim 1$ for $0 < p - p_c \ll 1$

3) $s_r = s_i = \hat{\gamma}_0' = s$

4) $t_r = t_i = \Delta - \gamma = t$

Proof: Let $0 < p_c - p \ll 1$, $h \in \mathcal{U}$ such that $h_i \neq 0$, and $0 < |h| \ll 1$. By equation (54)–(55), for $n = 0$, we have

$$\hat{g}(p,h) = \int_0^{\hat{S}(p)} \frac{d\hat{\phi}(y)}{|1+hy|^2} + \bar{h}\int_0^{\hat{S}(p)} \frac{y\, d\hat{\phi}(y)}{|1+hy|^2} = (p-p_c)^{-\hat{\gamma}_0'}[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \bar{\hat{x}}\hat{\mathcal{G}}_{0,1}(\hat{x})], \tag{66}$$

so that

$$\hat{g}_r = (p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{R}}_0(\hat{x}) = (p_c-p)^{-\hat{\gamma}_0'}[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r\hat{\mathcal{G}}_{0,1}(\hat{x})] \tag{67}$$
$$\hat{g}_i = (p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{I}}_0(\hat{x}) = -(p_c-p)^{-\hat{\gamma}_0'}\hat{x}_i\hat{\mathcal{G}}_{0,1}(\hat{x}).$$

Equations (56)–(57) imply that $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim 1$ as $\hat{x} \to 0$ ($h \to 0$ and $0 < p_c - p \ll 1$). Therefore, equations (16), (37), (67) and Lemma IV.5 imply that

$$(p_c-p)^{-s_r} \sim w_r(p,0) \sim \hat{g}_r(p,0) \sim (p_c-p)^{-\hat{\gamma}_0'} \lim_{\hat{x}\to 0}\hat{\mathcal{R}}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}_0'}, \tag{68}$$
$$(p_c-p)^{-s_i} \sim w_i(p,0) \sim \hat{g}_i(p,0) \sim (p_c-p)^{-\hat{\gamma}_0'} \lim_{\hat{x}\to 0}\hat{\mathcal{I}}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}_0'}.$$

Equation (68) and Lemma IV.8 imply that $s_r = s_i = \hat{\gamma}_0' = s$, which generalizes the result involving s in Lemma IV.8. It's worth noting that these scaling relations are independent of the path of the limit $h \to 0$. The above proof does not preclude that either $s_r = 0$ or $s_i = 0$, but not both. Although, this is not physically consistent [31].

Let $0 < p - p_c \ll 1$ and $0 < |h| \ll 1$. Equation (61) shows that we have $m(p, 0) = \lim_{h\to 0} h\hat{g}(p, h)$. Therefore equation (67), for $p > p_c$, implies that

$$\begin{aligned} m_r(p,0) &\sim \lim_{h\to 0}[h_r\hat{g}_r(p,h) - h_i\hat{g}_i(p,h)] = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i\hat{\mathcal{I}}_0(\hat{x})], \\ m_i(p,0) &\sim \lim_{h\to 0}[h_r\hat{g}_i(p,h) + h_i\hat{g}_r(p,h)] = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i\hat{\mathcal{R}}_0(\hat{x})]. \end{aligned} \tag{69}$$

By equation (50), $\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i\hat{\mathcal{I}}_0(\hat{x})] \sim \lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{I}}_0(\hat{x}) - \hat{x}_i\hat{\mathcal{R}}_0(\hat{x})] \sim 1$ for $0 < p - p_c \ll 1$. Therefore, equations (36) and (69) imply that

$$(p-p_c)^{t_r} \sim m_r(p,0) \sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}, \quad (p-p_c)^{t_i} \sim m_i(p,0) \sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0} \tag{70}$$

Equation (70) and Lemmas IV.6 and IV.8 imply that $t_r = t_i = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma = t$, which generalizes the result involving t in Lemma IV.8. It's worth noting that these scaling relations are independent of the path of the limit $h \to 0$. The above proof does not preclude that either $t_r = 0$ or $t_i = 0$, but not both. Although, this is not physically consistent [31] $\square$.

Lemma IV.11 *Let $h \in \mathcal{U}$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in equations* (54)–(55) *for $p > p_c$ and $p < p_c$. Furthermore, let $\hat{\delta}_r$, $\hat{\delta}_i$, δ_r, and δ_i be defined as in equations* (36)–(37). *Then,*

1) $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |\hat{x}|^{-\hat{\gamma}_0'/\hat{\Delta}'}$ *as* $\hat{x} \to \infty$ $(p \to p_c^-$ *and* $0 < |h| \ll 1)$

2) $[\hat{x}_r\hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i\hat{\mathcal{I}}_0(\hat{x})] \sim [\hat{x}_r\hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i\hat{\mathcal{R}}_0(\hat{x})] \sim |\hat{x}|^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}$ *as* $\hat{x} \to \infty$

3) $\hat{\delta}_r{}' = \hat{\delta}_i{}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}$

4) $\delta_r = \delta_i = \Delta/(\Delta - \gamma) = \delta$

Proof: Let $0 < h \ll 1$, so that $g(p, h)$ and $\hat{g}(p, h)$ are analytic for all $p \in [0, 1]$ [3]. Equations (16), (37), (67) and Lemma IV.5 imply that

$$\begin{aligned} |h|^{-1/\hat{\delta}_r'} &\sim w_r(p_c,h) \sim \hat{g}_r(p_c,h) \sim \lim_{p\to p_c^-}(p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{R}}_0(\hat{x}), \\ |h|^{-1/\hat{\delta}_i'} &\sim w_i(p_c,h) \sim \hat{g}_i(p_c,h) \sim \lim_{p\to p_c^-}(p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{I}}_0(\hat{x}). \end{aligned} \tag{71}$$

The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that they are bounded for all $p \in [0,1]$. Therefore, in order to cancel the diverging p dependent prefactors in equations (71), we must have $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |x|^{-\hat{\gamma}_0'/\hat{\Delta}'}$ as $\hat{x} \to \infty$ ($p \to p_c^-$ and $0 < h \ll 1$). Equation (71) then implies that

$$
\begin{aligned}
&|h|^{-1/\hat{\delta}_r'} \sim (p_c - p)^{-\hat{\gamma}_0'} |h|^{-\hat{\gamma}_0'/\hat{\Delta}'} (p_c - p)^{-\hat{\Delta}'(-\hat{\gamma}_0'/\hat{\Delta}')} = |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}, \\
&|h|^{-1/\hat{\delta}_i'} \sim |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}.
\end{aligned} \tag{72}
$$

Therefore, by Lemma IV.9, $\hat{\delta}_r{}' = \hat{\delta}_i{}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}\,'$.

Let $0 < h \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0,1]$ [3]. Equations (16) and (22) imply that shows that $m(p_c,h) \sim \lim_{p \to p_c^+} h\hat{g}(p,h)$. Therefore equations (36) and (67) implies that

$$
\begin{aligned}
&|h|^{1/\delta_r} \sim m_r(p_c,0) = (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{p \to p_c^+} [\hat{x}_r \hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i \hat{\mathcal{I}}_0(\hat{x})], \\
&|h|^{1/\delta_i} \sim m_i(p,0) = (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{p \to p_c^+} [\hat{x}_r \hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i \hat{\mathcal{R}}_0(\hat{x})].
\end{aligned} \tag{73}
$$

The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that they are bounded for all $p \in [0,1]$. Therefore, in order to cancel the diverging p dependent prefactors in equations (73), we must have $[\hat{x}_r \hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i \hat{\mathcal{I}}_0(\hat{x})] \sim [\hat{x}_r \hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i \hat{\mathcal{R}}_0(\hat{x})] \sim |x|^{(\hat{\Delta} - \hat{\gamma}_0)/\hat{\Delta}}$ as $\hat{x} \to \infty$ ($p \to p_c^+$ and $0 < h \ll 1$). Therefore equation (73), and Lemmas IV.6 and IV.9 imply that $\delta_r = \delta_i = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0) = \Delta/(\Delta - \gamma) = \delta$ $\Box$.

Lemma IV.12 *If $\Delta = \Delta'$, $\gamma = \gamma'$, $\hat{\Delta} = \hat{\Delta}'$, and $\hat{\gamma}_0 = \hat{\gamma}_0'$. Then*

1) $s + t = \Delta$

2) $\delta = 1/(1 - 1/\hat{\delta}\,')$

Proof: Assume that the spectral properties of the measures, $d\hat{\phi}(y)$ and $y\,d\phi(y)$, have the symmetry $\Delta = \Delta'$, $\gamma = \gamma'$, $\hat{\Delta} = \hat{\Delta}'$, and $\hat{\gamma}_0 = \hat{\gamma}_0'$. By Lemma IV.8 we have $t = \Delta - \gamma$ and $s = \hat{\gamma}_0'$, and by Lemma IV.9 we have $\delta = \Delta/(\Delta - \gamma)$ and $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}_0'$. Lemmas IV.6–IV.7 show that $\gamma = \hat{\gamma}_0$, $\Delta = \hat{\Delta}$, $\gamma' = \hat{\gamma}_0'$, and $\Delta' = \hat{\Delta}'$. Therefore,

$$
\begin{aligned}
&s + t = \hat{\gamma}_0' + \Delta - \gamma = \hat{\gamma}_0 + \Delta - \gamma = \Delta \\
&\delta = \Delta/(\Delta - \gamma) = 1/(1 - \gamma/\Delta) = 1/(1 - \hat{\gamma}_0/\hat{\Delta}) = 1/(1 - \hat{\gamma}_0'/\hat{\Delta}') = 1/(1 - 1/\hat{\delta}\,') \quad \Box.
\end{aligned}
$$

In this section we derived the (two–parameter) scaling relations regarding the conductor/insulator critical transition (34) and that of the conductor/superconductor critical transition (35). Assuming that the measures, $d\hat{\phi}(y)$ and $y\,d\phi(y)$, have the spectral symmetry property $\Delta = \Delta'$, $\gamma = \gamma'$, $\hat{\Delta} = \hat{\Delta}'$, and $\hat{\gamma}_0 = \hat{\gamma}_0'$, we also showed that there are (two–parameter) scaling relations between these two sets of critical exponents. There is no apparent mathematical necessity for this spectral symmetry. Although, the relation $s + t = \Delta$ is consistent with equation (4) in [31], $s = t(\delta - 1)$, which was derived under a physical scaling hypothesis and leads to the two dimensional duality relation $s = t$ [33, 34], as $\delta = 2$ for $d = 2$ [31]. The scaling relations are independent of the path of the limit $h \to 0$. Although the behavior of the system, as a function of $p \in [0, 1]$, is highly dependent on the location of $h \in \mathcal{U}$ and is governed by equation (24) (or formally by equation (46)) or equivalently by the system of coupled partial differential equations (25).

We have shown how the symmetries between the integral representations of σ^* and $[\sigma^{-1}]^*$ may be used to generalize the results of this section in terms of $[\sigma^{-1}]^*$. This beautiful mathematical framework, regarding the geometric critical transitions of percolating binary composites, will be extended further to our statistical mechanics description of electrically/thermally driven critical transitions of binary dielectrics and metal/dielectric composites in section **??**. In section **??** we discuss some of the more subtle measure theoretic details regarding the underlying symmmetries between $m(p, h)$ and $w(p, z(h))$. We will show that this leads to a generalization of a result [25] which characterizes the measure ϱ found in equation (**??**), in terms of the symmetries between the measures $d\hat{\phi}(y)$ and $y\,d\phi(y)$.

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