The Ising Model and a General Theory of Critical Transport Transitions in Binary Composite Media

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We present a general critical theory for transport in binary composite media holding for lattice and continuum percolation models in the static, and frequency dependent (complex parameter) quasi-static regimes. Through a direct, analytic correspondence between the magnetization of the Ising model and the effective parameter problem of two–phase random media, we show that the critical exponents of transport satisfy the standard scaling relations for phase transitions in statistical mechanics. Our work also shows that delta components form in the underlying spectral measures at the spectral endpoints precisely at the percolation threshold. This is analogous to the Lee-Yang-Ruelle characterization of criticality in the Ising model, and identifies these critical conduction transitions with the collapse of spectral gaps in these measures.

I. INTRODUCTION

Lattice and continuum percolation models have been used to study a broad range of disordered composite materials including semiconductors [1], radar absorbing coatings [2], bone [3, 4], rocks [5, 6], glacial ice [7], polycrystalline metals [8], carbon nanotube composites [9], and sea ice [10]. A key feature of these materials is the critical dependence of the effective transport properties on the connectedness, or percolation properties, of a particular component. The behavior of such composite media is particularly challenging to describe physically, and to predict mathematically.

Here we construct a mathematical framework which unifies the critical theory of transport in two–phase random media. By adapting techniques developed by G. A. Baker for the Ising model [11], we provide a detailed description of percolation–driven critical transitions in transport exhibited by such media. The most natural formulation of this problem is in terms of the conduction problem in the continuum $\mathbb{R}^d$, which includes the lattice $\mathbb{Z}^d$ as a special case [12, 13]. Although, the underlying symmetries [14] in the effective parameter problem of electrical conductivity and permittivity, magnetic permeability, and thermal conductivity, generalize our results to all of these systems.

II. BACKGROUND AND SUMMARY OF OUR RESULTS

In 1952 T. D. Lee and C. N. Yang showed that the root distribution of the Ising model partition function Z, a polynomial in the activity variable [11, 15–17], completely determines the associated equation of state [18]. Moreover, they demonstrated that the properties of the system, in relation to phase transitions, are governed by the behavior of these roots near the positive real axis. They did so by proving that the roots of Z lie on the unit circle. This result is known as the Lee–Yang Theorem [15, 16].

In 1968 G. A. Baker used the Lee–Yang Theorem to represent the Gibbs free energy per spin $f = -(N\beta)^{-1} \ln Z$ as a logarithmic potential [19], where N is the number of spins, $\beta = (kT)^{-1}$, k is Boltzmann's constant, and T is the absolute temperature [20]. He used this special analytic structure to prove that the magnetization per spin $M(T, H) = -\partial f/\partial H$ [21] may be represented in terms of a Stieltjes function G in the variable $\tau = \tanh \beta m H$,

$$\frac{M}{m} = \tau(1 + (1 - \tau^2)G(\tau^2)), \quad G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1 + \tau^2 y}\,, \tag{1}$$

where H is the applied magnetic field strength, m is the (constant) magnetic dipole moment of each spin [22], and ψ is a non–negative definite measure [11, 20]. The integral representation (1) immediately leads to the inequalities

$$G \geq 0, \qquad \frac{\partial G}{\partial u} \leq 0, \qquad \frac{\partial^2 G}{\partial u^2} \geq 0, \tag{2}$$

where $u = \tau^2$. The last formula in equation (2) is the GHS inequality, which is an important tool in the study of the Ising model [12].

In 1970 D. Ruelle extended the Lee–Yang Theorem and proved that there exists a *gap* $\theta_0(T) > 0$ in the roots of Z about the positive real axis for high temperatures [23]. Moreover, he proved that the gap collapses ($\theta_0(T) \to 0$) as T decreases to a critical temperature $T_c > 0$. Consequently, the temperature–driven phase transition (spontaneous magnetization) is unique, and is characterized by the pinching of the real axis by the roots of Z [16].

In [11, 24] G. A. Baker exploited the Lee–Yang–Ruelle Theorem and provided a detailed description of the percolation aspects of the phase transition [25]. He defined a critical exponent Δ for the gap in the distribution of the Lee–Yang–Ruelle zeros, $\theta_0(T) \sim (T - T_c)^{\Delta}$, as $T \to T_c^+$, and proved that the measure ψ is supported on the compact interval $[0, S(T)]$ for $T > T_c$, with $S(T) \sim (T - T_c)^{-2\Delta}$ as $T \to T_c^+$. He demonstrated that the moments $\psi_n = \int_0^\infty y^n \, d\psi(y)$ of ψ diverge as $T \to T_c^+$ according to the power law $\psi_n \sim (T - T_c)^{-\gamma_n}$, $n \geq 0$, by proving that the sequence γ_n satisfies Baker's inequalities $\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0$, which imply that this sequence increases at least linearly with n. He later proved that this sequence is actually linear in n, $\gamma_n = \gamma + 2\Delta n$, with constant gap $\gamma_i - \gamma_{i-1} = 2\Delta$ [11]. The critical exponent γ is defined via the magnetic susceptibility per spin $\chi = \partial M / \partial H = -\partial^2 f / \partial H^2 \sim (T - T_c)^{-\gamma}$, as $T \to T_c^+$. The phase transition may be concisely described with two other critical exponents. When $H = 0$, $M(T, 0) \sim (T - T_c)^{\beta}$, as $T \to T_c^-$, where the critical exponent β is not to be confused with $(kT)^{-1}$, and along the critical isotherm $T = T_c$, $M(T_c, H) \sim H^{1/\delta}$, as $H \to 0$ [11, 25]. Using the integral representation (1), Baker obtained (two–parameter) scaling relations for these critical exponents [11]

$$\beta = \Delta - \gamma, \qquad \delta = \Delta / (\Delta - \gamma), \qquad \gamma_n = \gamma + 2\Delta n. \tag{3}$$

The critical exponent γ, for example, is defined in terms of the following limit, and γ exists when this limit exists [11],

$$\gamma := \limsup_{T \to T_c^+,\, H = 0} \left(\frac{- \ln \chi(T, H)}{\ln(T - T_c)} \right). \tag{4}$$

In 1997 K. M. Golden demonstrated that Baker's critical theory may be adapted to provide a precise description of percolation–driven critical transitions in transport, exhibited by two–phase random media in the static regime [26]. This deep and far reaching result puts these two classes of seemingly unrelated problems on an equal mathematical footing. He did so by considering percolation models of classical conductive two–phase composite media, where the connectedness of the system is determined by the volume fraction p of defect inclusions with conductance σ_2 in an otherwise homogeneous medium of conductance σ_1, whereby assumption $h = \sigma_1/\sigma_2 \in [0, 1)$. He demonstrated that the function $m(p, h) = \sigma^*(p, h)/\sigma_2$ plays the role of the magnetization $M(T, H)$, where σ^* is the effective conductance of the medium [13, 27, 28]. Moreover, he showed that the volume fraction p mimics the temperature T while the contrast ratio h mimics the applied magnetic field strength H. More specifically, critical insulator/conductor behavior in transport arises when $h = 0$ $(\sigma_1 = 0,\ 0 < \sigma_2 < \infty)$, as $p \to p_c^+$ [26], and non-magnetic/ferromagnetic critical behavior of the Ising model arises when $H = 0$, as $T \to T_c^+$ [25]. Using these mathematical parallels, Golden showed that the critical exponents of transport satisfy an analogue of Baker's scaling relations (3).

Here, using a novel unified approach, we reproduce Golden's static results ($h \in \mathbb{R}$) and produce the analogous static results associated with a conductive–superconductive medium in terms of $w(p, z) = \sigma^*(p, z)/\sigma_1$, where $z = 1/h$. Using Stieltjes function integral representations of $m(p, h; \mu)$ and $w(p, z; \alpha)$, where μ and α are each spectral measures of a self–adjoint random operator, we determine the (two–parameter) critical exponent scaling relations of each system. We then extend these results to the frequency dependent quasi-static regime ($h \in \mathbb{C}$). We link these two sets of critical exponents, showing that they are all, in general, determined by only three critical exponents, and are determined by only two critical exponents under a physically consistent symmetry in the properties of μ and α. In arbitrary, finite lattice systems we explicitly show that there are *gaps* in the supports of the measures $\alpha(d\lambda)$ and $\mu(d\lambda)$ about the spectral endpoints $\lambda = 0, 1$ for $p \ll 1$ and $1 - p \ll 1$, respectively, which collapse as p tends towards p_c. Moreover in infinite lattice or continuum composite systems, we demonstrate that critical transitions in transport are due to the formation of delta function components in μ and α located at $\lambda = 0, 1$. We do so by constructing a measure $\varrho(d\lambda)$ which is supported on the set $\{0, 1\}$ that links the measures μ and α. This general result demonstrates that, for percolation models, the onset of criticality (the formation of these delta components) occurs *precisely* at the percolation threshold p_c.

III. THE ANALYTIC CONTINUATION METHOD

We now formulate the effective parameter problem for two–component conductive media. Let (Ω, P) be a probability space, and let $\boldsymbol{\sigma}(\vec{x},\omega)$ and $[\boldsymbol{\sigma}^{-1}](\vec{x},\omega)$ be the local conductivity and resistivity tensors, respectively, which are (spatially) stationary random fields in $\vec{x}\in\mathbb{R}^d$ and $\omega\in\Omega$. Here Ω is the set of all geometric realizations of our random medium and $P(d\omega)$ is the underlying probability measure, which is compatible with stationarity [13]. Define the Hilbert space of stationary random fields $\mathscr{H}_s\subset L^2(\Omega,P)$, and the underlying Hilbert spaces of stationary curl free $\mathscr{H}_\times\subset\mathscr{H}_s$ and divergence free $\mathscr{H}_\bullet\subset\mathscr{H}_s$ random fields [13]

$$\begin{aligned}\mathscr{H}_\times &:= \{\vec{Y}(\omega)\in\mathscr{H}_s \mid \vec{\nabla}\times\vec{Y}=0 \text{ weakly and } \langle\vec{Y}\rangle=0\},\\ \mathscr{H}_\bullet &:= \{\vec{Y}(\omega)\in\mathscr{H}_s \mid \vec{\nabla}\cdot\vec{Y}=0 \text{ weakly and } \langle\vec{Y}\rangle=0\},\end{aligned} \tag{5}$$

where $\vec{Y}:\Omega\mapsto\mathbb{R}^d$ and $\langle\cdot\rangle$ means ensemble average over Ω, or by an ergodic theorem [13] spatial average over all of $\mathbb{R}^d$.

Consider the following variational problems: find $\vec{E}_f\in\mathscr{H}_\times$ and $\vec{J}_f\in\mathscr{H}_\bullet$ such that

$$\langle\boldsymbol{\sigma}(\vec{E}_0+\vec{E}_f)\cdot\vec{Y}\rangle=0\quad\forall\,\vec{Y}\in\mathscr{H}_\times\qquad\text{and}\qquad\langle\boldsymbol{\sigma}^{-1}(\vec{J}_0+\vec{J}_f)\cdot\vec{Y}\rangle=0\quad\forall\,\vec{Y}\in\mathscr{H}_\bullet\,, \tag{6}$$

respectively [13]. Under the assumption that the bilinear forms $a(\vec{u},\vec{v})=\vec{u}^T\boldsymbol{\sigma}(\vec{x},\omega)\vec{v}$ and $\tilde{a}(\vec{u},\vec{v})=\vec{u}^T[\boldsymbol{\sigma}^{-1}](\vec{x},\omega)\vec{v}$ are bounded and coercive, where $\vec{u},\vec{v}\in\mathbb{R}^d$, these problems have unique solutions satisfying [13]

$$\begin{aligned}&\vec{\nabla}\times\vec{E}=0, &&\vec{\nabla}\cdot\vec{J}=0, &&\vec{J}=\boldsymbol{\sigma}\vec{E}, &&\vec{E}=\vec{E}_0+\vec{E}_f, &&\langle\vec{E}\,\rangle=\vec{E}_0,\\ &\vec{\nabla}\times\vec{E}=0, &&\vec{\nabla}\cdot\vec{J}=0, &&\vec{E}=\boldsymbol{\sigma}^{-1}\vec{J}, &&\vec{J}=\vec{J}_0+\vec{J}_f, &&\langle\vec{J}\,\rangle=\vec{J}_0,\end{aligned} \tag{7}$$

respectively. Here $\vec{E}_f$ and $\vec{J}_f$ are the fluctuating electric field and current density of mean zero, respectively, about the (constant) averages $\vec{E}_0$ and $\vec{J}_0$, respectively.

We assume that $\boldsymbol{\sigma}(\vec{x},\omega)$, locally, takes the values σ_1 and σ_2, and that $[\boldsymbol{\sigma}^{-1}](\vec{x},\omega)$, locally, takes the values $1/\sigma_1$ and $1/\sigma_2$, and write $\boldsymbol{\sigma}(\vec{x},\omega):=\sigma_1\chi_1(\vec{x},\omega)+\sigma_2\chi_2(\vec{x},\omega)$ and $[\boldsymbol{\sigma}^{-1}](\vec{x},\omega):=\chi_1(\vec{x},\omega)/\sigma_1+\chi_2(\vec{x},\omega)/\sigma_2$. Here χ_j is the characteristic function of medium $j=1,2$, which equals one for all $\omega\in\Omega$ having medium j at $\vec{x}$, and zero otherwise [13]. As $\vec{E}_f\in\mathscr{H}_\times$ and $\vec{J}_f\in\mathscr{H}_\bullet$, equation (6) yields the energy (power density) constraints $\langle\vec{J}\cdot\vec{E}_f\rangle=\langle\vec{E}\cdot\vec{J}_f\rangle=0$, which lead to the reduced energy representations

$$\langle\vec{J}\cdot\vec{E}\rangle=\langle\vec{J}\rangle\cdot\vec{E}_0\quad\text{and}\quad\langle\vec{E}\cdot\vec{J}\rangle=\langle\vec{E}\rangle\cdot\vec{J}_0\,. \tag{8}$$

The effective complex conductivity and resistivity tensors, $\boldsymbol{\sigma}^*$ and $[\boldsymbol{\sigma}^{-1}]^*$, are defined by

$$\langle \vec{J}\,\rangle = \boldsymbol{\sigma}^* \vec{E}_0 \quad \text{and} \quad \langle \vec{E}\,\rangle = [\boldsymbol{\sigma}^{-1}]^* \vec{J}_0, \tag{9}$$

respectively. For simplicity, we focus on one diagonal component of these symmetric tensors: $\sigma^* := \boldsymbol{\sigma}^*_{kk}$ and $[\sigma^{-1}]^* := [\boldsymbol{\sigma}^{-1}]^*_{kk}$, for some $k = 1, \ldots, d$.

Due to the homogeneity of these functions, e.g. $\sigma^*(a\sigma_1, a\sigma_2) = a\sigma^*(\sigma_1, \sigma_2)$ for any complex number a, they depend only on the ratio $h := \sigma_1/\sigma_2$, and we define the dimensionless functions $m(h) := \sigma^*/\sigma_2$, $w(z) := \sigma^*/\sigma_1$, $\tilde{m}(h) := \sigma_1[\sigma^{-1}]^*$, and $\tilde{w}(z) := \sigma_2[\sigma^{-1}]^*$, where $z = z(h) := 1/h$. The functions $m(h)$ and $\tilde{m}(h)$ are analytic off the negative real axis in the h–plane, and the functions $w(z)$ and $\tilde{w}(z)$ are analytic off the negative real axis in the z–plane [13]. Each take the corresponding upper half plane to the upper half plane, so that they are examples of Herglotz functions [13]. We assume that $0 < |h| < 1$, i.e. $0 < |\sigma_1| < |\sigma_2| < \infty$, and we further restrict h in the complex plane to the set

$$\mathcal{U} := \{h \in \mathbb{C} : |h| < 1 \text{ and } h \notin (-1, 0]\}. \tag{10}$$

In order to illuminate the symmetries between these functions, we will henceforth focus on the complex variable $h := h_r + \mathrm{i}h_i$, where $h_r := \mathrm{Re}(h)$ and $h_i := \mathrm{Im}(h)$.

The key step in the method is obtaining integral representations for σ^* and $[\sigma^{-1}]^*$. These integral representations are given in terms of resolvent representations of the electric field $\vec{E}$, and the current density $\vec{J}$,

$$\vec{E} = s(s + \Gamma\chi_1)^{-1}\vec{E}_0 = t(t + \Gamma\chi_2)^{-1}\vec{E}_0 \quad \text{and} \quad \vec{J} = s(s - \Upsilon\chi_2)^{-1}\vec{J}_0 = t(t - \Upsilon\chi_1)^{-1}\vec{J}_0, \tag{11}$$

where we have defined $s := 1/(1-h)$ and $t := 1/(1-z) = 1 - s$. These formulas follow from manipulations of equation (7). The operator $-\Gamma := -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$ is a projection onto curl-free fields, based on convolution with the free-space Green's function for the Laplacian $-\Delta = -\nabla^2$ [13]. More specifically $-\Gamma : \mathscr{H}_s \mapsto \mathscr{H}_\times$, and for every $\vec{\zeta} \in \mathscr{H}_\times$ we have $-\Gamma\vec{\zeta} = \vec{\zeta}$. To the authors knowledge, the operator $\Upsilon := \vec{\nabla} \times (-\Delta)^{-1}\vec{\nabla}\times$ is being introduced here for the first time. For the convenience of the reader, we recall a few vector calculus facts. For every $\vec{\zeta} \in \mathscr{H}_\bullet$ we have $\vec{\zeta} = \vec{\nabla} \times (\vec{A} + \vec{C})$ weakly, where $\vec{\nabla} \times \vec{C} = 0$ weakly [29, 30]. The arbitrary vector $\vec{C}$ can be chosen so that $\vec{\nabla} \cdot \vec{A} = 0$ weakly [29]. Hence, $\vec{\nabla} \times \vec{\zeta} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta\vec{A} = -\Delta\vec{A}$ weakly. The vector $\vec{C}$ chosen in this manner gives the Coulomb (or transverse) gauge of $\vec{\zeta}$ [29]. Let $\mathscr{C}_\bullet \subset \mathscr{H}_\bullet$ denote the *closure* of the space of stationary divergence free random fields, of Coulomb gauge. On the Hilbert space $\mathscr{C}_\bullet$ one can show that the operator Υ is a projector, based on convolution with the free-space Green's function for the Laplacian $-\Delta$. More specifically $\Upsilon : \mathscr{H}_s \mapsto \mathscr{H}_\bullet$, and for every $\vec{\zeta} \in \mathscr{C}_\bullet$ we have $\Upsilon\vec{\zeta} = \vec{\zeta}$.

It is more convenient to consider the functions $F(s) := 1 - m(h)$ and $E(s) := 1 - \tilde{m}(h)$, which are analytic off $[0,1]$ in the s–plane, and $G(t) := 1 - w(z(h))$ and $H(t) := 1 - \tilde{w}(z(h))$, which are analytic off $[0,1]$ in the t–plane [13, 27], and satisfy

$$0 < |F(s)|, |E(s)| < 1, \quad 0 < |G(t)|, |H(t)| < \infty, \quad h \in \mathcal{U}, \tag{12}$$

where $G(t)$ and $H(t)$ are not to be confused with the Stieltjes function in (1) and the magnetic field strength in the Ising model, respectively. We write $\vec{E}_0 = E_0\,\vec{e}_k$ and $\vec{J}_0 = J_0\,\vec{j}_k$, where $\vec{e}_k$ and $\vec{j}_k$ are unit vectors, for some $k = 1, \ldots, d$. Using $\chi_1 = 1 - \chi_2$ and equations (7), (9), and (11), the Spectral Theorem [31] yields [13, 27]

$$F(s) = \langle \chi_1 (s + \Gamma\chi_1)^{-1} \vec{e}_k \cdot \vec{e}_k \rangle := \int_{\lambda_0}^{\lambda_1} \frac{d\mu(\lambda)}{s - \lambda}\,, \quad E(s) = \langle \chi_2 (s - \Upsilon\chi_2)^{-1} \vec{j}_k \cdot \vec{j}_k \rangle := \int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1} \frac{d\eta(\lambda)}{s - \lambda}\,, \tag{13}$$

$$G(t) = \langle \chi_2 (t + \Gamma\chi_2)^{-1} \vec{e}_k \cdot \vec{e}_k \rangle := \int_{\hat{\lambda}_0}^{\hat{\lambda}_1} \frac{d\alpha(\lambda)}{t - \lambda}\,, \quad H(t) = \langle \chi_1 (t - \Upsilon\chi_1)^{-1} \vec{j}_k \cdot \vec{j}_k \rangle := \int_{\check{\lambda}_0}^{\check{\lambda}_1} \frac{d\kappa(\lambda)}{t - \lambda}\,.$$

In order to illuminate the symmetries between the integral representations of (13), in the last two formulas of equation (13) we will henceforth make the change of variables $t(s) = 1 - s$ and $\lambda \mapsto 1 - \lambda$, so that $G(t(s)) = -\int_{1-\hat{\lambda}_1}^{1-\hat{\lambda}_0} [-d\alpha(1-\lambda)]/(s - \lambda)$, for example.

In equation (13), μ, η, α, and κ are bounded positive measures which depend only on the geometry of the medium, and are supported on $\Sigma_\mu, \Sigma_\eta, \Sigma_\alpha, \Sigma_\kappa \subseteq [0,1]$, respectively [13, 32]. The supremum and infimum of these sets are defined to be the respective upper and lower limits of integration in (13). The Integro-differential operators $\mathbf{M}_j := \chi_j(-\Gamma)\chi_j$ and $\mathbf{K}_j := \chi_j \Upsilon \chi_j$, $j = 1, 2$, are compositions of projection operators on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{C}_\bullet$, respectively, and are consequently bounded by 1 in the underlying operator norm [30, 33]. They are self–adjoint on $L^2(\Omega, P)$ [13]. Consequently, in the Hilbert space $L^2(\Omega, P)$ with weight χ_2 in the inner product, for example, $\Gamma\chi_2$ is a bounded self–adjoint operator [13]. Equation (13) involves spectral representations of resolvents involving these self adjoint operators. The measures μ, η, α, and κ are spectral measures of the family of projections of these operators in the respective $\langle \vec{e}_k, \vec{e}_k \rangle$ or $\langle \vec{j}_k, \vec{j}_k \rangle$ state [13, 31].

A key feature of equations (8)–(9) and (13) is that the parameter information in s and E_0 is *separated* from the geometry of the composite, which is encapsulated in the measures μ, η, α, and κ through their moments μ_n, η_n, α_n, and κ_n, $n \geq 0$, which depend on the correlation functions of the medium [13]. For example, $\alpha_0 = \eta_0 = p$ and $\mu_0 = \kappa_0 = 1 - p$. A principal application of the analytic continuation method is to derive *forward bounds* on σ^* and $[\sigma^{-1}]^*$, given partial information on the microgeometry [13, 28, 32, 34]. One can also use the integral representations (13) to obtain *inverse bounds*, allowing one to use data about the electromagnetic response of a sample to bound its structural parameters such as p [4].

Applying the Spectral Theorem to the energy constraints $\langle \vec{J} \cdot \vec{E}_f \rangle = \langle \vec{E} \cdot \vec{J}_f \rangle = 0$, we have obtained detailed decompositions of the system energy in terms of the measures μ, η, α, and κ. For example, $\langle \vec{J} \cdot \vec{E}_f \rangle = 0$, $\vec{E} = \vec{E}_0 + \vec{E}_f$, $\langle \vec{E}_f \rangle = 0$, and $\sigma = \sigma_2(1 - \chi_1/s)$ imply that $0 = \langle \sigma \vec{E} \cdot \vec{E}_f \rangle = \langle \sigma_2(1 - \chi_1/s)(\vec{E}_f \cdot \vec{E}_0 + E_f^2) \rangle = \sigma_2 \left[\langle E_f^2 \rangle - (\langle \chi_1 \vec{E}_f \cdot \vec{E}_0 \rangle + \langle \chi_1 E_f^2 \rangle)/s \right]$. By the Spectral Theorem [31] and the symmetries in equation (13), we therefore have

$$\frac{\langle E_f^2 \rangle}{E_0^2} = \int_0^1 \frac{\lambda\, d\mu(\lambda)}{(s-\lambda)^2} = \int_0^1 \frac{\lambda\, d\alpha(\lambda)}{(1-s-\lambda)^2}\,, \quad \frac{\langle J_f^2 \rangle}{J_0^2} = \int_0^1 \frac{\lambda\, d\eta(\lambda)}{(s-\lambda)^2} = \int_0^1 \frac{\lambda\, d\kappa(\lambda)}{(1-s-\lambda)^2}\,. \tag{14}$$

Equation (14) then leads to Herglotz function representations of all such energy components involving the spectral measures μ, η, α, and κ, e.g. $\langle \chi_1 \vec{E}_f \cdot \vec{E}_0 \rangle / E_0^2 = \int_0^1 \lambda d\mu(\lambda)/(s-\lambda)$.

The Stieltjes transforms (13) of the measures μ, η, α, and κ may be represented in terms of Stieltjes functions [11] of h via the change of variables $s(h) = 1/(1-h)$ and $\lambda(y) = y/(1+y) \iff y(\lambda) = \lambda/(1-\lambda)$. For example,

$$F(s) = (1-h) \int_{S_0}^{S} \frac{(1+y) d\mu(\frac{y}{1+y})}{1+hy}\,, \qquad G(t(s)) = (h-1) \int_{\hat{S}_0}^{\hat{S}} \frac{(1+y)[-d\alpha(\frac{1}{1+y})]}{1+hy}\,. \tag{15}$$

Here $S_0 := \lambda_0/(1-\lambda_0)$, $S := \lambda_1/(1-\lambda_1)$, $\hat{S}_0 := (1-\hat{\lambda}_1)/\hat{\lambda}_1$, and $\hat{S} := (1-\hat{\lambda}_0)/\hat{\lambda}_0$, so that $\lim_{\lambda_0 \to 0} S_0 = 0$, $\lim_{\lambda_1 \to 1} S = \infty$, $\lim_{\hat{\lambda}_1 \to 1} \hat{S}_0 = 0$, $\lim_{\hat{\lambda}_0 \to 0} \hat{S} = \infty$. By equation (15) and the underlying symmetries in equation (13), the Stieltjes function representations of $m(h)$, $\tilde{m}(h)$, $w(z(h))$, and $\tilde{w}(z(h))$ are given by

$$\begin{aligned}
m(h) &= 1 + (h-1) g(h), \quad g(h) := \int_0^\infty \frac{d\phi(y)}{1+hy}\,, \quad d\phi(y) := (1+y) d\mu\left(\frac{y}{1+y}\right), \\
\tilde{m}(h) &= 1 + (h-1) \tilde{g}(h), \quad \tilde{g}(h) := \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy}\,, \quad d\tilde{\phi}(y) := (1+y) d\eta\left(\frac{y}{1+y}\right), \\
w(z(h)) &= 1 - (h-1) \hat{g}(h), \quad \hat{g}(h) := \int_0^\infty \frac{d\hat{\phi}(y)}{1+hy}\,, \quad d\hat{\phi}(y) := (1+y) \left[-d\alpha\left(\frac{1}{1+y}\right)\right], \\
\tilde{w}(z(h)) &= 1 - (h-1) \check{g}(h), \quad \check{g}(h) := \int_0^\infty \frac{d\check{\phi}(y)}{1+hy}\,, \quad d\check{\phi}(y) := (1+y) \left[-d\kappa\left(\frac{1}{1+y}\right)\right].
\end{aligned} \tag{16}$$

Equation (16) should be compared to equation (1) regarding the Ising model. The Stieltjes functions $g(h), \tilde{g}(h)$, $\hat{g}(h)$, and $\check{g}(h)$ are analytic for all $h \in \mathcal{U}$ [13]. As μ, η, α, and κ are positive measures on $[0,1]$, ϕ, $\tilde{\phi}$, $\hat{\phi}$, and $\check{\phi}$ are positive measures on $[0,\infty]$. Consequently, the following inequalities hold

$$\frac{\partial^{2n}\zeta}{\partial h^{2n}} > 0, \quad \frac{\partial^{2n-1}\zeta}{\partial h^{2n-1}} < 0, \qquad \left|\frac{\partial^{n}\zeta}{\partial h^{n}}\right| > 0, \qquad \zeta = g(h), \tilde{g}(h), \hat{g}(h), \check{g}(h), \quad h \in \mathcal{U}, \tag{17}$$

for $n \geq 0$, which are analogs of equation (2) in the Ising model [12]. The first two inequalities in (17) hold for $h \in \mathcal{U} \cap \mathbb{R}$, and the last inequality holds for $h \in \mathcal{U}$ such that $h_i \neq 0$.

By equation (16), the moments ϕ_n of ϕ satisfy

$$\phi_n = \int_0^\infty y^n d\phi(y) = \int_0^\infty y^n (1+y) d\mu\left(\frac{y}{1+y}\right) = \int_0^1 \frac{\lambda^n d\mu(\lambda)}{(1-\lambda)^{n+1}}\,. \tag{18}$$

A partial fraction expansion of $\lambda^n/(1-\lambda)^{n+1}$ then shows that

$$\frac{(-1)^n}{n!} \lim_{s\to 1} \frac{\partial^n F(s)}{\partial s^n} = \int_0^1 \frac{d\mu(\lambda)}{(1-\lambda)^{n+1}} = \sum_{j=0}^{n} \binom{n}{j} \phi_j\,. \tag{19}$$

Equation (19) demonstrates that ϕ_n depends on $\int_0^1 d\mu(\lambda)/(1-\lambda)^{n+1}$ *and* all the lower moments ϕ_j, $j = 0, 1, \ldots, n-1$, of ϕ. From equations (13)–(14), we see that the first two moments of ϕ are identified with energy components:

$$\phi_0 = \lim_{s\to 1} \frac{\langle \chi_1 \vec{E} \cdot \vec{E}_0 \rangle}{E_0^2}, \quad \phi_1 = \lim_{s\to 1} \frac{\langle E_f^2 \rangle}{E_0^2}. \tag{20}$$

Thereby equation (19), *all* of the higher moments ϕ_j, $j \geq 2$, depend on these energy components. Equation (18) suggests that the moments ϕ_n, $n \geq 0$, become singular as $\lambda_1 := \sup\{\Sigma_\mu\} \to 1$. However, we will show that this is only true for the moments of order $n \geq 1$, and that $\lambda = 1$ is a removable *simple* singularity under μ.

Similarly, the moments $\hat{\phi}_n$ of $\hat{\phi}$ satisfy

$$\hat{\phi}_n = \int_0^1 \frac{(1-\lambda)^n d\alpha(\lambda)}{\lambda^{n+1}}, \qquad \frac{(-1)^{n+1}}{n!} \lim_{s\to 1} \frac{\partial^n G(t(s))}{\partial^n t} = \int_0^1 \frac{d\alpha(\lambda)}{\lambda^{n+1}} = \sum_{j=0}^{n} \binom{n}{j} \hat{\phi}_j\,. \tag{21}$$

Equations (13)–(14) also identify the first two moments, $\hat{\phi}_0$ and $\hat{\phi}_1$, of $\hat{\phi}$ with energy components. Equation (21) then implies that all of the higher moments $\hat{\phi}_j$, $j \geq 2$, depend on these energy components. Equation (21) suggests, and we will show that *all* the moments $\hat{\phi}_n$, $n \geq 0$, become singular as $\hat{\lambda}_0 := \inf\{\Sigma_\alpha\} \to 0$. By the symmetries in equations (13) and

(16), equations (18)–(19) hold for $\tilde{\phi}$ with $E(s)$ and η in lieu of $F(s)$ and μ, respectively, and equation (21) holds for $\check{\phi}$ with $H(t(s))$ and κ in lieu of $G(t(s))$ and α, respectively.

Equations (8)–(9) yield the energy representations $\langle \vec{J}\cdot\vec{E}\rangle = \sigma_2 m(h)E_0^2 = \sigma_1 w(z(h))E_0^2$ and $\langle \vec{E}\cdot\vec{J}\rangle = \tilde{m}(h)E_0^2/\sigma_1 = \tilde{w}(z(h))E_0^2/\sigma_2$ involving σ^* and $[\sigma^{-1}]^*$, which imply that

$$\begin{aligned} m(h) = hw(z(h)) &\iff 1-F(s) = (1-1/s)(1-G(t(s))), \\ \tilde{m}(h) = h\tilde{w}(z(h)) &\iff 1-E(s) = (1-1/s)(1-H(t(s))). \end{aligned} \tag{22}$$

Using equation (16), minor algebraic manipulation in equation (22) implies that

$$g(h)+h\hat{g}(h)=1, \qquad \tilde{g}(h)+h\check{g}(h)=1, \quad h\in\mathcal{U}. \tag{23}$$

For $h\in\mathcal{U}$, the functions $g(h)$, $\hat{g}(h)$, $\tilde{g}(h)$, and $\check{g}(h)$ are analytic [13] and have bounded h derivatives of all orders [33]. An inductive argument applied to equation (23) yields

$$\frac{\partial^n g}{\partial h^n}+n\frac{\partial^{n-1}\hat{g}}{\partial h^{n-1}}+h\frac{\partial^n \hat{g}}{\partial h^n}=0, \qquad \frac{\partial^n \tilde{g}}{\partial h^n}+n\frac{\partial^{n-1}\check{g}}{\partial h^{n-1}}+h\frac{\partial^n \check{g}}{\partial h^n}=0, \quad n\geq 1. \tag{24}$$

In the complex quasi–static case, where $h\in\mathcal{U}$ such that $h_i\neq 0$, the complex representation of equation (24) is, for example,

$$\frac{\partial^n g_r}{\partial h^n}+n\frac{\partial^{n-1}\hat{g}_r}{\partial h^{n-1}}+h_r\frac{\partial^n \hat{g}_r}{\partial h^n}-h_i\frac{\partial^n \hat{g}_i}{\partial h^n}=0, \quad \frac{\partial^n g_i}{\partial h^n}+n\frac{\partial^{n-1}\hat{g}_i}{\partial h^{n-1}}+h_r\frac{\partial^n \hat{g}_i}{\partial h^n}+h_i\frac{\partial^n \hat{g}_r}{\partial h^n}=0, \tag{25}$$

where we have used the definitions

$$\frac{\partial^n g_r}{\partial h^n}:=\mathrm{Re}\frac{\partial^n g}{\partial h^n}\,, \quad \frac{\partial^n g_i}{\partial h^n}:=\mathrm{Im}\frac{\partial^n g}{\partial h^n}\,, \qquad \frac{\partial^n \hat{g}_r}{\partial h^n}:=\mathrm{Re}\frac{\partial^n \hat{g}}{\partial h^n}\,, \quad \frac{\partial^n \hat{g}_i}{\partial h^n}:=\mathrm{Im}\frac{\partial^n \hat{g}}{\partial h^n}\,.$$

The analog of (25) involving $\tilde{g}$ and $\check{g}$ follows from the substitutions $g\mapsto\tilde{g}$ and $\hat{g}\mapsto\check{g}$. The integral representations of equations (24)–(25) follow from Lemma III.1 below. We focus on the measures ϕ and $\hat{\phi}$, as the analogous results involving $\tilde{\phi}$ and $\check{\phi}$ follow from the symmetries in equations (13) and (16).

Lemma III.1 *Set $Y_{i,j}(h,y):=y^i/(1+hy)^j$. Then for all $h\in\mathcal{U}$ and $i,j\in\mathbb{R}$ satisfying $0<i\leq j-1$, we have $Y_{i,j}(h,y)\in L^1(\hat{\phi}(dy))$, and for $0<i\leq j$, $Y_{i,j}(h,y)\in L^1(\phi(dy))$. Consequently ([30] Theorem 2.27), the Stieltjes functions $g(h)$ and $\hat{g}(h)$ may be repeatedly differentiated under the integral sign, i.e. for all $n=0,1,2,\ldots$ we have*

$$\frac{\partial^n g(h)}{\partial h^n}=(-1)^n n!\int_0^\infty\frac{y^n d\phi(y)}{(1+hy)^{n+1}}\,, \qquad \frac{\partial^n \hat{g}(h)}{\partial h^n}=(-1)^n n!\int_0^\infty\frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}\,. \tag{26}$$

Before we prove Lemma III.1, we note that equations (24) and (26) imply that

$$\int_0^\infty \frac{y^n d\phi(y)}{(1+hy)^{n+1}} = \int_0^\infty \frac{y^{n-1} d\hat{\phi}(y)}{(1+hy)^n} - h\int_0^\infty \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}\,, \quad n \geq 1,\; h \in \mathcal{U}. \tag{27}$$

Moreover, Lemma III.1 and (27) yield the integral representations of (25) using, for example,

$$\frac{(-1)^n}{n!}\frac{\partial^n g(h)}{\partial h^n} = \int_0^\infty \frac{y^n d\phi(y)}{|1+hy|^{2(n+1)}}(1+\bar{h}y)^{n+1} = \sum_{j=0}^{n+1}\binom{n+1}{j}\bar{h}^j \int_0^\infty \frac{y^{n+j} d\phi(y)}{|1+hy|^{2(n+1)}}\,, \tag{28}$$

where $\bar{h}$ denotes complex conjugation of the complex variable h.

Proof of Lemma III.1: The supports of ϕ and $\hat{\phi}$ are $\Sigma_\phi := [S_0, S\,]$ and $\Sigma_{\hat{\phi}} := [\hat{S}_0, \hat{S}\,]$, respectively, which are defined in terms of Σ_μ and Σ_α, respectively, directly below equation (15). For every $h \in \mathcal{U}$, it is clear that there exists real, strictly positive S_h such that

$$1 \ll |h| S_h < \infty. \tag{29}$$

Set $h \in \mathcal{U}$ and $0 \ll S_h < \infty$ satisfying (29), and write $\Sigma_\phi := [S_0, S_h] \cup (S_h, S\,]$ and $\Sigma_\phi := [\hat{S}_0, S_h] \cup (S_h, \hat{S}\,]$. Equations (12) and (18) imply that $0 \leq \lim_{h\to 0}|m(h)| = 1 - \phi_0 < 1$, which implies that the mass ϕ_0 of ϕ is uniformly bounded. Therefore for all $h \in \mathcal{U}$,

$$\int_{S_0}^{S_h} |Y_{i,j}(h,y)| d\phi(y) \leq \frac{S_h^i\,\phi([S_0,S_h])}{|1+hS_0|^j} < \infty, \quad \int_{\hat{S}_0}^{S_h} |Y_{i,j}(h,y)| d\phi(y) \leq \frac{S_h^i\,\hat{\phi}([\hat{S}_0,S_h])}{|1+h\hat{S}_0|^j} < \infty,$$

Here $\phi([S_0, S_h])$ is the *bounded* ϕ measure of the set $[S_0, S_h]$, and the boundedness of the second formula follows from equations (15)–(16), showing that the $\hat{\phi}$ measure of the compact interval $[\hat{S}_0, S_h]$ is bounded. More specifically, in terms of Σ_α we have $\hat{\lambda}_1 = 1 - \hat{S}_0/(1+\hat{S}_0)$ and $\hat{\lambda}_h := 1 - S_h/(1+S_h) > 0$. Thus equations (15)–(16) imply that

$$\hat{\phi}([\hat{S}_0, S_h]) = \int_{\hat{S}_0}^{S_h} d\hat{\phi}(y) = \int_{\hat{S}_0}^{S_h} (1+y)\left[-d\alpha\left(\frac{1}{1+y}\right)\right] = \int_{\hat{\lambda}_h}^{\hat{\lambda}_1} \frac{d\alpha(\lambda)}{\lambda} \leq \frac{\alpha_0}{\hat{\lambda}_h} < \infty.$$

If Σ_ϕ and $\Sigma_{\hat{\phi}}$ are compact intervals, we are done. Otherwise set $S = \hat{S} = \infty$. In terms of Σ_μ and Σ_α, we have $\lambda_h := S_h/(1+S_h)$ and $\lambda_1 = S/(1+S) \equiv 1$, and $\hat{\lambda}_0 = 1 - \hat{S}/(1+\hat{S}) \equiv 0$ and $\hat{\lambda}_h = 1 - S_h/(1+S_h)$, respectively, where $0 \ll \lambda_h < 1$ and $0 < \hat{\lambda}_h \ll 1$. When $0 < i \leq j-1$, equations (16) and (29) imply that, for all $h \in \mathcal{U}$,

$$\begin{aligned}|h|^j \int_{S_h}^{\hat{S}} |Y_{i,j}(h,y)| d\hat{\phi}(y) &\sim \int_{S_h}^{\hat{S}} \frac{1+y}{y^{j-i}} d\alpha\left(\frac{1}{1+y}\right) = \int_{1-\hat{\lambda}_h}^{1-\hat{\lambda}_0} \frac{(1-\lambda)^{j-i-1}\,[-d\alpha(1-\lambda)]}{\lambda^{j-i}} \\ &= \int_{\hat{\lambda}_0}^{\hat{\lambda}_h} \frac{\lambda^{j-i-1}\,d\alpha(\lambda)}{(1-\lambda)^{j-i}} \leq \frac{\hat{\lambda}_h^{j-i-1}\,\alpha_0}{(1-\hat{\lambda}_h)^{j-i}} < \infty.\end{aligned}$$

When $0 < i \leq j$, equations (12), (16), and (29) imply that, for all $h \in \mathcal{U}$,

$$|h|^j \int_{S_h}^{S} |Y_{i,j}(h,y)| d\phi(y) \sim \int_{\lambda_h}^{\lambda_1} \frac{(1-\lambda)^{j-i-1}}{\lambda^{j-i}} d\mu(\lambda) \leq \frac{(1-\lambda_h)^{j-i}}{\lambda_h^{j-i}} \int_{\lambda_h}^{1} \frac{d\mu(\lambda)}{1-\lambda} < \infty,$$

as $0 < F(1) = \int_0^1 d\mu(\lambda)/(1-\lambda) \leq 1$. This concludes the proof of Lemma III.1 $\square$.

All the equations given in this section display general formulas holding for two–component stationary random media in the lattice and continuum settings [26]. In section III A below, we demonstrate that equations (22)–(23) and the Stieltjes-Perron Inversion Theorem [35] allow us to construct measures ϱ and $\tilde{\varrho}$, supported on the set $\{0, 1\}$, which link the measures μ and α and the measures η and κ, respectively. The properties of ϱ and $\tilde{\varrho}$ imply that critical transitions in the transport properties of σ^* and $[\sigma^{-1}]^*$ are caused by the formation of delta function components in the spectral measures $\mu(d\lambda)$, $\alpha(d\lambda)$, $\eta(d\lambda)$, and $\kappa(d\lambda)$ at $\lambda = 0, 1$.

A. Spectral Characterization of Criticality in Transport

In this section we show that the symmetries underlying the analytic continuation method allow one to construct precise relations between the measures μ and α, and the measures η and κ. The geometric characteristics of a random medium determine these measures and the Stieltjes transforms thereof (13) in turn, determine the effective transport properties of the medium. Conversely, given the Stieltjes transform of a measure, the Stieltjes-Perron Inversion Theorem [14, 35, 36] allows one to recover the underlying measure. For example,

$$\mu(\upsilon) = -\frac{1}{\pi} \lim_{\epsilon \downarrow 0} \mathrm{Im}(F(\upsilon + \mathrm{i}\epsilon)) \ , \quad \upsilon \in \Sigma_\mu. \tag{30}$$

To evoke this theorem directly, in equation (13) we define $d\tilde{\alpha}(\lambda) := [-d\alpha(1-\lambda)]$ and $d\tilde{\kappa}(\lambda) = [-d\kappa(1-\lambda)]$, and write $G(t(s)) = -\int_0^1 d\tilde{\alpha}(\lambda)/(s-\lambda)$ and $H(t(s)) = -\int_0^1 d\tilde{\kappa}(\lambda)/(s-\lambda)$. Setting $s = \upsilon + \mathrm{i}\epsilon$, equations (22) and (30) imply that

$$\upsilon\mu(\upsilon) = (1-\upsilon)[-\alpha(1-\upsilon)] - \upsilon\varrho(\upsilon), \qquad \upsilon\eta(\upsilon) = (1-\upsilon)[-\kappa(1-\upsilon)] - \upsilon\tilde{\varrho}(\upsilon), \tag{31}$$
$$\varrho(\upsilon) = \lim_{\epsilon\downarrow 0} \frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2} \int_0^1 \frac{(\upsilon+\lambda-1)\, d\alpha(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2} \, , \quad \tilde{\varrho}(\upsilon) = \lim_{\epsilon\downarrow 0} \frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2} \int_0^1 \frac{(\upsilon+\lambda-1)\, d\kappa(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2} \, .$$

We now demonstrate that (22)–(23) and (31) explicitly determine the measures ϱ and $\tilde{\varrho}$. The integral representations of equation (23) follow from equation (16), and are given by

$$\int_0^\infty \frac{d\phi(y)}{1+hy} + h\int_0^\infty \frac{d\hat{\phi}(y)}{1+hy} = 1, \qquad \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy} + h\int_0^\infty \frac{d\check{\phi}(y)}{1+hy} = 1. \tag{32}$$

Due to the underlying symmetries of this framework, without loss of generality, we henceforth focus on $F(s;\mu)$, $G(t(s);\alpha)$, $g(h;\phi)$, and $\hat{g}(h;\hat{\phi})$. We wish to re-express the first formula in equation (32) in a more suggestive form by adding and subtracting the quantity $h\int_0^\infty y\,d\phi(y)/(1+hy)$. This is permissible if the modulus of this quantity is finite for all $h\in\mathcal{U}$ [30, 33]. The affirmation of this fact is given by Lemma III.1 and we may therefore add and subtract the said quantity in equation (32), yielding

$$h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} \equiv 1-\phi_0 = m(0), \quad d\Phi_0(y) := d\hat{\phi}(y) - y\,d\phi(y), \quad h\in\mathcal{U}, \tag{33}$$

as $1-\phi_0 = 1-F(s)|_{s=1} = m(h)|_{h=0}$ (18). Equation (33) provides another representation for the quantity $m(0)$ and shows that the transform $h\int_0^\infty d\Phi_0(y)/(1+hy)$ of Φ_0, a signed measure [33], is independent of h for all $h\in\mathcal{U}$. Equation (16) and the identity $y=\lambda/(1-\lambda) \iff \lambda = y/(1+y)$ relates this representation of $m(0)$ to the measure ϱ in equation (31):

$$d\Phi_0(y) = \frac{1}{(1-\lambda)^2}((1-\lambda)\,[-d\alpha(1-\lambda)] - \lambda\,d\mu(\lambda)) = \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2} = y(1+y)\,d\varrho\left(\frac{y}{1+y}\right).$$

We may now express equation (33) in terms of $\varrho(d\lambda)$ as follows:

$$m(0) = h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} = h\int_0^\infty \frac{y(1+y)d\varrho(\frac{y}{1+y})}{1+hy} = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h + \lambda(1-\lambda)}. \tag{34}$$

Remark III.1 *Define the transform $\mathcal{D}(h;\varrho)$ of the measure ϱ by*

$$\mathcal{D}(h;\varrho) = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h + \lambda(1-\lambda)}. \tag{35}$$

Equations (12) *and* (34) *show that $\mathcal{D}(h;\varrho)$ has the following properties for all $h\in\mathcal{U}$:* ***(1)*** *$\mathcal{D}(h;\varrho)$ is independent of h,* ***(2)*** *$0\le|\mathcal{D}(h;\varrho)|<1$, and* ***(3)*** *$\mathcal{D}(h;\varrho) = m(0)\not\equiv 0$.*

Lemma III.2 *Let the quantities $m(0) := m(h)|_{h=0} = 1-F(s)|_{s=1}$ and $w(0) := w(z)|_{z=0} = 1-G(t)|_{t=1}$ be defined as in equation* (13)*, which satisfy $0\le m(0), w(0) < 1$. If $\mathcal{D}(h;\varrho)$, defined in equation* (35)*, satisfies the properties of Remark III.1 for all $h\in\mathcal{U}$, then*

$$\begin{aligned} \varrho(d\lambda) &= -w(0)\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda), \\ \tilde{\varrho}(d\lambda) &= -\tilde{w}(0)\delta_0(d\lambda) + \tilde{m}(0)(1-\lambda)\delta_1(d\lambda), \end{aligned} \tag{36}$$

where $\delta_{\lambda_0}(d\lambda)$ is the Dirac measure centered at λ_0.

Proof: The proof of the second formula in equation (36) follows directly from the proof of the first formula in (36) and the underlying symmetries of this mathematical framework. Let

$\mathcal{D}(h;\varrho)$, defined in equation (35), satisfy properties (**1**)–(**3**) of Remark III.1. The measure ϱ is independent of h [13]. If the support Σ_ϱ of the measure ϱ is over continuous spectrum [31] then $\mathcal{D}(h;\varrho)$ depends on h, contradicting property (**1**). Therefore the measure ϱ is defined over pure point spectrum [31]. The most general pure point set Σ_ϱ which satisfies properties (**1**) and (**3**) is given by $\Sigma_\varrho = \{0,1\}$. This implies that the measure ϱ is of the form

$$\varrho(d\lambda) = W_0(\lambda)\delta_0(d\lambda) + W_1(\lambda)\delta_1(d\lambda),$$

where the $W_j(\lambda)$, $j = 0, 1$, are bounded functions of $\lambda \in [0,1]$ which are to be determined. In view of the numerator of the integrand in equation (35), we may assume that the function $W_0(\lambda) \equiv W_0(0) := W_0 \not\equiv 0$ is independent of λ. In order for properties (**2**) and (**3**) to be satisfied we must have $W_1(\lambda) \sim (1-\lambda)^1$ as $\lambda \to 1$ (any other power of $1-\lambda$ would contradict one of these two properties). Therefore with out loss of generality, we may set $W_1(\lambda) = \mathrm{w}_1\,(1-\lambda)$, where w_1 is independent of λ. Property (**3**) now yields $\mathrm{w}_1 = m(0)$.

We have shown that $\varrho(d\lambda) = W_0\,\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda)$, $W_0 \not\equiv 0$. By plugging this formula into equation (31) ($\lambda d\mu(\lambda) = (1-\lambda)[-d\alpha(1-\lambda)] - \lambda d\varrho(\lambda)$), we are able determine W_0. Indeed using the definition of $F(s)$ (13), equation (22) ($F(s) - (1-1/s)G(t(s)) = 1/s$), and $(1-\lambda)/(\lambda(s-\lambda)) = -(1-1/s)/(s-\lambda) + 1/(s\lambda)$, we find that

$$\begin{aligned} F(s) &= -\left(1-\frac{1}{s}\right)\int_0^1 \frac{[-d\alpha(1-\lambda)]}{s-\lambda} + \frac{1}{s}\int_0^1 \frac{[-d\alpha(1-\lambda)]}{\lambda} - \int_0^1 \frac{d\varrho(\lambda)}{s-\lambda} \\ &= \left(1-\frac{1}{s}\right)G(t(s)) + \frac{1}{s}\int_0^1 \frac{d\alpha(\lambda)}{1-\lambda} - \frac{W_0}{s} - m(0)\lim_{\lambda\to 1}\frac{1-\lambda}{s-\lambda}, \quad \forall\ |s|>1, \end{aligned} \tag{37}$$

which implies that $-W_0 = 1 - \int_0^1 d\alpha(\lambda)/(1-\lambda) = w(0)$ $\Box$.

Corollary III.1 *If we instead focus on the contrast variables z and t in lieu of h and s, respectively, equations* (31) *and* (36) *become*

$$\begin{aligned} &\upsilon\alpha(\upsilon) = (1-\upsilon)[-\mu(1-\upsilon)] - \upsilon\varrho(\upsilon), \quad \varrho(d\lambda) = -m(0)\delta_0(d\lambda) + w(0)(1-\lambda)\delta_1(d\lambda), \\ &\upsilon\kappa(\upsilon) = (1-\upsilon)[-\eta(1-\upsilon)] - \upsilon\tilde{\varrho}(\upsilon), \quad \tilde{\varrho}(d\lambda) = -\tilde{m}(0)\delta_0(d\lambda) + \tilde{w}(0)(1-\lambda)\delta_1(d\lambda). \end{aligned} \tag{38}$$

Lemma III.2 gives a rigorous proof of the key result of this section, generalizing that in [36] found by heuristic means. It is worth mentioning that equation (27) can be written as $\int_0^\infty d\Phi_{n-1}(y)/(1+hy)^{n+1} \equiv 0$, for all $n \geq 1$ and $h \in \mathcal{U}$, in terms of the signed measure $d\Phi_{n-1}(y) := y^{n-1} d\Phi_0(y)$. By Lemma III.1, this integral involving $\Phi_{n-1}(dy)$ is defined.

Furthermore in equation (25) for $n = 1$, equation (28) implies that $\int_0^\infty d\Phi_1(y)/|1+hy|^4 \equiv 0$, for all $h \in \mathcal{U}$ such that $h_i \neq 0$. These formulas are consistent with (36) of Lemma III.2.

The formulas in equation (36) demonstrate that $\lambda = 1$ is a removable *simple* singularity under μ, α, η, and κ, and illustrate how the relations (12), $0 < |F(s)|, |E(s)| \leq 1$, can hold even when $s = 1$ ($h = 0$) and the spectra extends all the way to $\lambda = 1$. These formulas and (18) also show that ϕ and $\tilde{\phi}$ are bounded measures with mass $0 < \phi_0, \tilde{\phi}_0 \leq 1$, and that the higher moments ϕ_j and $\tilde{\phi}_j$, $j \geq 1$, become singular when the spectra extends all the way to $\lambda = 1$ and $m(0) > 0$. Furthermore, the formulas in equation (21) and (36) demonstrate that the masses $\hat{\phi}_0 = G(0)$ and $\check{\phi}_0 = H(0)$ of the measures $\hat{\phi}$ and $\check{\phi}$, respectively, become unbounded when the spectra extends all the way to $\lambda = 0$ and $w(0) > 0$. In Section IV below, we will discuss how these general features relate to percolation models of binary composite media.

IV. CRITICAL BEHAVIOR OF TRANSPORT IN LATTICE AND CONTINUUM PERCOLATION MODELS

We now formulate the problem of percolation–driven critical transitions in transport exhibited by two–component conductive media. For percolation models, the connectedness of the system is determined by the volume fraction p of defect inclusions in an otherwise homogeneous medium. In the simplest case of the two dimensional square lattice [37, 38], the bonds are open with probability p and closed with probability $1 - p$. Connected sets of open bonds are called open clusters. The average cluster size grows as p increases, and there is a critical volume fraction p_c, $0 < p_c < 1$, called the *percolation threshold*, where an infinite cluster of open bonds first appears. Now consider transport through a random resistor network (RRN), where bonds are assigned electrical conductivities σ_2 with probability p, and σ_1 with probability $1 - p$. In the limit $h = \sigma_1/\sigma_2 \to 0$, the system exhibits two types of critical behavior. First as $h \to 0$ ($\sigma_1 \to 0$ and $0 < |\sigma_2| < \infty$), the effective complex conductivity $\sigma^*(p, h) := \sigma_2\, m(p, h)$ and the effective complex resistivity $[\sigma^{-1}]^*(p, z) := \sigma_2^{-1}\tilde{w}(p, z(h))$ undergo a conductor–insulator critical transition [39]

$$
\begin{aligned}
&|\sigma^*(p,0)| = 0, \quad \text{for} \;\; p < p_c, \;\; \text{and} \;\; 0 = |\sigma_1| < |\sigma^*(p,0)| < |\sigma_2| < \infty, \quad \text{for} \;\; p > p_c, \qquad (39)\\
&\lim_{p \to p_c^+} |[\sigma^{-1}]^*(p, z(0))| = \infty, \;\; \text{and} \;\; 0 < |\sigma_2|^{-1} < \left|[\sigma^{-1}]^*(p, z(0))\right| < |\sigma_1|^{-1} = \infty, \quad \text{for} \;\; p > p_c.
\end{aligned}
$$

Second as $h \to 0$ ($\sigma_2 \to \infty$ and $0 < |\sigma_1| < \infty$), the effective complex conductivity $\sigma^*(p,z) := \sigma_1 w(p,z(h))$ and the effective complex resistivity $[\sigma^{-1}]^*(p,h) := \sigma_1^{-1}\tilde{m}(p,h)$ undergo a conductor–superconductor critical transition [39]:

$$0 < |\sigma_1| < |\sigma^*(p,z(0))| < |\sigma_2| = \infty, \quad \text{for} \;\; p < p_c, \;\; \text{and} \quad \lim_{p \to p_c^-} |\sigma^*(p,z(0))| = \infty. \tag{40}$$

$$0 = |\sigma_2|^{-1} < \left|[\sigma^{-1}]^*(p,0)\right| < |\sigma_1|^{-1} < \infty, \quad \text{for} \;\; p < p_c, \;\; \text{and} \quad \left|[\sigma^{-1}]^*(p,0)\right| = 0, \quad \text{for} \;\; p > p_c.$$

We will focus on the conductor–insulator critical transition of the effective complex conductivity $\sigma^*(p,h) = \sigma_2\, m(p,h)$ and the conductor–superconductor critical transition of the effective complex conductivity $\sigma^*(p,z(h)) = \sigma_1 w(p,z(h))$. It is clear from equations (16) and (39)–(40) that our results immediately generalize to $[\sigma^{-1}]^*(p,h) = \sigma_1^{-1}\tilde{m}(p,h)$ and $[\sigma^{-1}]^*(p,z(h)) = \sigma_2^{-1}\tilde{w}(p,z(h))$, respectively, with $p \mapsto 1-p$. These critical behaviors in transport are made more precise through the definition of critical exponents.

In the static limit, $h \in \mathcal{U} \cap \mathbb{R}$, as $h \to 0$ the effective conductivity $\sigma^*(p,h) = \sigma_2\, m(p,h)$ exhibits critical behavior near the percolation threshold $\sigma^*(p,0) \sim (p-p_c)^t$, as $p \to p_c^+$. Here, the critical exponent t, not to be confused with the contrast parameter, is believed to be *universal* for lattices, depending only on dimension [26]. At $p = p_c$, $\sigma^*(p_c,h) \sim h^{1/\delta}$ as $h \to 0$. We assume the existence (4) of the critical exponents t and δ, as well as γ, defined via a conductive susceptibility $\chi(p,0) := \partial m(p,0)/\partial h \sim (p-p_c)^{-\gamma}$, as $p \to p_c^+$. For $p > p_c$, we assume that there is a gap $\theta_\mu \sim (p-p_c)^\Delta$ in the support of μ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^+$, or that any spectrum in this region does not affect power law behavior [26]. Consequently, for $p > p_c$ the support of ϕ is contained in the interval $[0, S(p)]$, with $S(p) \sim (p-p_c)^{-\Delta}$ as $p \to p_c^+$. As the moments of ϕ become singular as $\theta_\mu \to 0$ (18), we also assume the existence of critical exponents γ_n such that $\phi_n(p) \sim (p-p_c)^{-\gamma_n}$ as $p \to p_c^+$, $n \geq 0$. When $h \in \mathcal{U}$ such that $h_i \neq 0$, we also assume the existence of critical exponents t_r, δ_r, t_i and δ_i corresponding to $m_r := \mathrm{Re}(m)$ and $m_i := \mathrm{Im}(m)$. In summary:

$$\begin{aligned}
&m(p,0) \sim (p-p_c)^t, && m_r(p,0) \sim (p-p_c)^{t_r}, && m_i(p,0) \sim (p-p_c)^{t_i}, && \text{as} \;\; p \to p_c^+ \\
&m(p_c,h) \sim h^{1/\delta}, && m_r(p_c,h) \sim |h|^{1/\delta_r}, && m_i(p_c,h) \sim |h|^{1/\delta_i}, && \text{as} \;\; |h| \to 0, \\
&\chi(p,0) \sim (p-p_c)^{-\gamma}, && \phi_n \sim (p-p_c)^{-\gamma_n}, && S(p) \sim (p-p_c)^{-\Delta}, && \text{as} \;\; p \to p_c^+.
\end{aligned} \tag{41}$$

We also assume the existence of critical exponents γ', γ_n', and Δ' associated with the left hand limit $p \to p_c^-$. The critical exponents γ, δ, Δ, and γ_n for transport are different from those defined in section II for the Ising model (3).

Similarly, in the static limit, as $h \to 0$ the effective conductivity $\sigma^*(p, z(h)) = \sigma_1 w(p, z(h))$ exhibits critical conductor–superconductor behavior near p_c, $\sigma^*(p, z(0)) \sim (p - p_c)^{-s}$ as $p \to p_c^-$, where the superconductor critical exponent s is not to be confused with the contrast parameter. At $p = p_c$, $\sigma^*(p_c, z(h)) \sim h^{-1/\hat{\delta}}$ as $h \to 0$. We assume the existence (4) of the critical exponents s and $\hat{\delta}$, as well as $\hat{\gamma}'$, defined via $\hat{\chi}(p) := \partial w(p, z(0))/\partial h \sim (p - p_c)^{-\hat{\gamma}'}$, as $p \to p_c^-$. For $p < p_c$ we assume that there is a gap $\theta_\alpha \sim (p - p_c)^{\hat{\Delta}'}$ in the support of $[-d\alpha(1 - \lambda)]$ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^-$, so that $\hat{\phi}$ is supported on the interval $[0, \hat{S}(p)]$, with $\hat{S}(p) \sim (p - p_c)^{-\Delta}$ as $p \to p_c^+$. We also assume that there exist critical exponents $\hat{\gamma}_n'$ such that $\hat{\phi}_n(p) \sim (p - p_c)^{-\hat{\gamma}_n'}$ as $p \to p_c^-$ (21), $n \geq 0$. When $h \in \mathcal{U}$ such that $h_i \neq 0$, we assume the existence of critical exponents s_r, s_i, $\hat{\delta}_r$, and $\hat{\delta}_i$ corresponding to $w_r := \mathrm{Re}(w)$ and $w_i := \mathrm{Im}(w)$. In summary:

$$\begin{array}{llll} w(p, z(0)) \sim (p - p_c)^{-s}, & w_r(p, z(0)) \sim (p - p_c)^{-s_r}, & w_i(p, z(0)) \sim (p - p_c)^{-s_i}, & \text{as } p \to p_c^- \\ w(p_c, z(h)) \sim h^{-1/\hat{\delta}}, & w_r(p_c, z(h)) \sim |h|^{-1/\hat{\delta}_r}, & w_i(p_c, z(h)) \sim |h|^{-1/\hat{\delta}_i}, & \text{as } |h| \to 0, \\ \hat{\chi}(p) \sim (p - p_c)^{-\hat{\gamma}'}, & \hat{\phi}_n \sim (p - p_c)^{-\hat{\gamma}_n'}, & \hat{S}(p) \sim (p - p_c)^{-\hat{\Delta}'}, & \text{as } p \to p_c^- . \end{array} \tag{42}$$

We also assume the existence of critical exponents $\hat{\gamma}$, $\hat{\gamma}_n$, and $\hat{\Delta}$, associated with the right hand limit $p \to p_c^+$.

Theorem IV.1 *Let t, t_r, t_i, δ, δ_r, δ_i, γ, γ_n, Δ, γ', γ_n', and Δ' be defined as in equation* (41)*, and s, s_r, s_i, $\hat{\delta}$, $\hat{\delta}_r$, $\hat{\delta}_i$, $\hat{\gamma}'$, $\hat{\gamma}_n'$, $\hat{\Delta}'$, $\hat{\gamma}$, $\hat{\gamma}_n$, and $\hat{\Delta}$ be defined as in equation* (42)*. Then the following scaling relations hold:*

1) $\gamma_1 = \gamma,\ \gamma_1' = \gamma',\ \hat{\gamma}_1 = \hat{\gamma},$ *and* $\hat{\gamma}_1' = \hat{\gamma}'$. **2**) $\gamma_0' = 0,\ \gamma_0 < 0,\ \gamma_n' > 0$ *and* $\gamma_n >,\ n \geq 1$.

3) $\hat{\gamma}_n' > 0$ *for* $n \geq 0$. **4**) $\gamma = \hat{\gamma}_0$ *and* $\Delta = \hat{\Delta}$. **5**) $\gamma' = \hat{\gamma}_0'$ *and* $\Delta' = \hat{\Delta}'$.

6) $\gamma_n = \gamma + \Delta(n-1)$ *for* $n \geq 1$. **7**) $\hat{\gamma}_n' = \hat{\gamma}_0' + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ *for* $n \geq 0$.

8) $t = \Delta - \gamma$. **9**) $s = \hat{\gamma}_0' = \hat{\gamma}' - \hat{\Delta}'$. **10**) $\delta = \dfrac{\Delta}{\Delta - \gamma}$. **11**) $\hat{\delta}\,' = \dfrac{\hat{\Delta}'}{\hat{\gamma}_0'} = \dfrac{\hat{\Delta}'}{\hat{\gamma}' - \hat{\Delta}'}$.

12) $t_r = t_i = t$. **13**) $s_r = s_i = s$. **14**) $\delta_r = \delta_i = \delta$. **15**) $\hat{\delta}_r = \hat{\delta}_i = \hat{\delta}$.

16) *If* $\Delta = \Delta'$ *and* $\gamma = \gamma'$, *then* $t + s = \Delta$ *and* $1/\delta + 1/\hat{\delta}\,' = 1$.

17) *In general* $t/\Delta + s/\hat{\Delta}' = 1$, *and* $\Delta = \hat{\Delta}' \iff \gamma = \hat{\gamma}_0'$.

Before we present the proof of Theorem IV.1, which is given in section IV B below, we first demonstrate that the scaling relations therein are exactly satisfied by effective–medium theory (EMT). This verification is essential, as there exists a binary composite medium which realizes the effective parameter of EMT [14]. Through our exploration of EMT, we will also uncover features of this system which illuminate general features of critical transport transitions exhibited by binary composite media, discussed in Section IV C.

A. Effective Medium Theory

An EMT for the effective parameter problem may be constructed from dilute limits [36]. The EMT approximation for σ^* with percolation threshold p_c is given by [36]

$$p\,\frac{\sigma_2-\sigma^*}{1+p_c\,(\sigma_2/\sigma^*-1)}+(1-p)\,\frac{\sigma_1-\sigma^*}{1+p_c\,(\sigma_1/\sigma^*-1)}=0. \tag{43}$$

Equation (43) leads to quadratic formulas involving $m(p,h)=\sigma^*/\sigma_2$ and $w(p,z(h))=\sigma^*/\sigma_1$. The quadratic equation demonstrates that the relation $m(p,h)=h\,w(p,z(h))$ (22) is exactly satisfied and that

$$\begin{aligned} m(p,h(s)) &= \frac{-b\,(s,p,p_c)+\sqrt{-\zeta(s,p)}}{2s(1-p_c)}, \quad \zeta(\lambda,p):=-\lambda^2+2(1-\varphi)\lambda+\nu^2-(1-\varphi)^2, \\ w(p,z(t)) &= \frac{-b\,(s,1-p,p_c)+\sqrt{-\zeta(t,1-p)}}{2t(1-p_c)}, \quad \zeta(\lambda,1-p):=-\lambda^2+2\varphi\lambda+\nu^2-\varphi^2, \end{aligned} \tag{44}$$

where $b(\lambda,p,p_c)=(2p_c-1)\lambda+(1-p-p_c)$, $\varphi=\varphi(p,p_c)=p\,(1-p_c)+p_c(1-p)$, and $\nu=\nu(p,p_c)=2\sqrt{p\,(1-p)\,p_c(1-p_c)}$.

The spectral measures μ and α (13) may be extracted from equation (44) using the Stieltjes–Perron Inversion Theorem (30). These measures are absolutely continuous, i.e. there exist density functions such that $\mu(d\lambda)=\mu(\lambda)d\lambda$ and $\alpha(d\lambda)=\alpha(\lambda)d\lambda$. Direct calculation shows that, for $p\neq p_c, 1-p_c$, these measures have gaps in the spectrum about $\lambda=0,1$: $\mu(\lambda)=0 \iff \zeta(\lambda,p)\le 0 \iff |\lambda-(1-\varphi)|\ge\nu$ and $\alpha(\lambda)=0 \iff \zeta(\lambda,1-p)\le 0 \iff |\lambda-\varphi|\ge\nu$. The Stieltjes transformations (13) of μ and α are given by

$$F(p,s)=\int_{\lambda_0}^{1-\theta}\frac{\sqrt{\zeta(\lambda,p)}\,d\lambda}{2\pi(1-p_c)\,\lambda(s-\lambda)}\,, \qquad G(p,t)=\int_{\theta}^{\hat\lambda_1}\frac{\sqrt{\zeta(\lambda,1-p)}\,d\lambda}{2\pi(1-p_c)\,\lambda(t-\lambda)}\,, \tag{45}$$

where $\theta=\theta(p,p_c)=\varphi-\nu$ and $\hat\lambda_1=1-\lambda_0=\varphi+\nu$ define *spectral gaps*, which satisfy $\lim_{p\to 1-p_c}\lambda_0=0$, $lim_{p\to p_c}\theta=0$, and $\lim_{p\to 1-p_c}\hat\lambda_1=1$.

Define a critical exponent Δ for the spectral gap $\theta(p)\sim|p-p_c|^{\Delta}$, as $p\to p_c$, in $\mu(d\lambda)$ about $\lambda=1$ and $\alpha(d\lambda)$ about $\lambda=0$. Using the definition (4) of Δ and L'Hôpital's rule we have shown that $\Delta = 2$. Moreover $\lambda_0 = 1 - \hat{\lambda}_1 \sim |p - (1 - p_c)|^{\Delta}$, as $p \to 1 - p_c$, with the same critical exponent. The absolutely continuous nature of the measures μ and α in EMT imply that critical indices are the same for $p \to p_c^+$ and $p \to p_c^-$. Therefore the spectral symmetry properties in the hypothesis of Lemma IV.11 hold for EMT.

We have explicitly calculated the integrals in equation (45) for real and complex h using the symbolic mathematics software Maple 15. Using the exact representation (45) of $G(p, t(h))$, as a function of $0 \le \theta \ll 1$ and $0 \le |h| \ll 1$, we have calculated the critical exponents (42) s, $\hat{\delta}$, $\hat{\delta}_r$, $\hat{\delta}_i$, and $\hat{\gamma}_n$, for $n = 0, 1, 2, \ldots$ These results are in agreement with our general theory. With $h = 0$ and $0 < \theta \ll 1$, we found that $w(p, z(0)) \sim \theta^{-1/2}$ which yields $s = \Delta/2 = 1$. When $\theta = 0$ and $0 < h \ll 1$, one must split up the integration domain, $\Sigma_\alpha \supset (0, h - \epsilon) \cup (h + \epsilon, \hat{\lambda}_1)$, and take the principal value of the integral as $\epsilon \to 0$. Doing so yields $\hat{\delta} = \hat{\delta}_r = \hat{\delta}_i = 2$. As in our general theory, the values of the exponents are independent of the path of h to zero. More specifically, these relations hold for $0 < |h_r| = |ah_i| \ll 1$ with arbitrary $a \in \mathbb{R}$, and for independent h_r and h_i satisfying $0 < |h_r|, |h_i| \ll 1$. The critical exponents $\hat{\gamma}_n$ associated with the moments $\hat{\phi}_n$ of the measure $\hat{\phi}$ satisfy our general relation $\hat{\gamma}_n = \hat{\gamma}_0 + \Delta n$ with $\hat{\gamma}_0 = \Delta = 2$ so that $\hat{\gamma}_n = \Delta(n + 1)$.

Using the exact representation (45) of $F(p, h)$, as a function of $0 \le \theta \ll 1$ and $0 \le |h| \ll 1$, we have also calculated the critical exponents (41) t, δ, δ_r, δ_i, and γ_n, for $n = 0, 1, 2, \ldots$ These results are also in agreement with our general theory. In accordance with [36], we obtain $t = \Delta/2 = 1$, so that the relation $s + t = \Delta = 2$ is satisfied. By direct calculation we have obtained $\delta = \delta_r = \delta_i = 2$. We have also obtained these values by use of the relation $m(p, h) = hw(p, h)$ and the associated relations for complex h, $m_r = h_r w_r - h_i w_i$ and $m_i = h_r w_i + h_i w_r$, with $\hat{\delta} = \hat{\delta}_r = \hat{\delta}_i$ and $1/\delta + 1/\hat{\delta} = 1$. The mass $\phi_0(p) = F(p, 1)$ of the measure ϕ behaves logarithmically as $\theta \to 0$, yielding $\gamma_0 = 0$. The critical exponents of the higher moments satisfy our general relation $\gamma_n = \gamma_0 + \Delta n = \gamma + \Delta(n - 1)$, so that $\gamma_n = \Delta n$.

In summary, we have extended EMT to the complex quasi–static regime and shown that the critical exponents thereof exactly satisfy our scaling relations displayed in Theorem IV.1. Moreover, in EMT there are gaps in the spectral measures μ and α for $p \ne p_c, 1 - p_c$. The gaps in μ and α about $\lambda = 1$ and $\lambda = 0$, respectively, collapse as $p \to p_c$, and the gaps in μ and α about $\lambda = 0$ and $\lambda = 1$, respectively, collapse as $p \to 1 - p_c$. This is precisely the behavior displayed in Lemma III.2 and Corollary III.1 which hold for general percolation models of stationary two–phase random media, with $m(0) = m(p, 0)$ and $w(0) = w(p, 0)$. In this way the spectral measures μ and α truly are independent of, and how we define, the material contrast ratio. For example, here we have focused on the contrast ratio $h = \sigma_1/\sigma_2$ and defined an insulator–conductor system by letting $\sigma_1 \to 0$, resulting in critical behavior (the formation of a delta component in μ at $\lambda = 1$ with weight $m(p, 0)$) as p surpasses p_c, where $p = \langle \chi_2 \rangle$. We could have instead focused on $z = \sigma_2/\sigma_1$ and defined an insulator–conductor system by letting $\sigma_2 \to 0$, resulting in critical behavior (the formation of a delta component in α at $\lambda = 1$ with weight $w(p, 0)$) as p surpasses $1 - p_c$. Lemma III.2 and Corollary III.1 demonstrate, through spectral means, the equivalence of these two systems.

B. Proof of Theorem IV.1

Baker's critical theory characterizes phase transitions of a given system via the asymptotic behaviors of underlying Stieltjes functions, near a critical point. This powerful method has been very successful in the Ising model, precisely characterizing the phase transition (spontaneous magnetization) [11]. We will now show how this method may be adapted to provide a detailed description of phase transitions in transport, exhibited by binary composite media. Theorem IV.1 will be proven via a sequence of lemmas as we collect some important properties of $m(p, h)$, $g(p, h)$, $w(p, z(h))$, and $\hat{g}(p, h)$, and how they are related. The following theorem characterizes Stieltjes functions (series of Stieltjes) [11].

Theorem IV.2 *Let $D(i, j)$ denote the determinant*

$$D(i,j) = \begin{vmatrix} \xi_i & \xi_{i+1} & \cdots & \xi_{i+j} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{i+j} & \xi_{i+j+1} & \cdots & \xi_{i+2j} \end{vmatrix}. \tag{46}$$

The ξ_n form a series of Stieltjes if and only if $D(i, j) \geq 0$ for all $i, j = 0, 1, 2, \ldots$

Baker's inequalities for the sequences γ_n (41) and $\hat{\gamma}_n$ (42) of transport follow directly from Theorem IV.2. For example, $\phi_n \sim (p - p_c)^{-\gamma_n}$ and Theorem IV.2 with $\phi_i = \xi_i$, $i = n$, and $j = 1$, imply that, for $|p - p_c| \ll 1$,

$$\begin{aligned} &(p-p_c)^{-\gamma_n-\gamma_{n+2}} - (p-p_c)^{-2\gamma_{n+1}} \geq 0 \iff (p-p_c)^{-\gamma_n-\gamma_{n+2}+2\gamma_{n+1}} \geq 1 \\ &\iff -\gamma_n - \gamma_{n+2} + 2\gamma_{n+1} \leq 0 \iff \boxed{\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0}. \end{aligned} \tag{47}$$

The sequence of inequalities (47) are *Baker's inequalities* for transport, corresponding to $m(p,h)$, and they imply that the sequence γ_n increases at least linearly with n. The symmetries in equations (16) and (41)–(42) imply that Baker's inequalities also hold for the sequences γ'_n, $\hat{\gamma}_n$, and $\hat{\gamma}'_n$.

The following lemma provides the asymptotic behaviors of h derivatives of $g(p,h)$ and $\hat{g}(p,h)$, which will be used extensively in this section.

Lemma IV.1 *Let $0 < |h| \ll 1$ and $|p - p_c| \ll 1$. Then the integrals in equation* (26) *have the following asymptotics for $n \geq 0$*

$$\frac{\partial^n g(p,h)}{\partial h^n} \sim \phi_n, \qquad \frac{\partial^n \hat{g}(p,h)}{\partial h^n} \sim \hat{\phi}_n. \tag{48}$$

Proof: The asymptotic behaviors in equation (48) follow from equations (18)–(19), (21), Baker's inequalities (47), and equation (16) ($g(p,h) = sF(p,s)$ and $\hat{g}(p,h) = -s\,G(p,t(s))$). They imply that, for $c_j, b_j \in \mathbb{Z}$,

$$\lim_{h\to 0} \frac{\partial^n g(p,h)}{\partial h^n} = \sum_{j=0}^{n} c_j \lim_{s\to 1} \frac{\partial^j F(p,s)}{\partial s^j} \sim \phi_n\,, \quad \lim_{h\to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} = \sum_{j=0}^{n} b_j \lim_{s\to 1} \frac{\partial^j G(p,t(s))}{\partial t^j} \sim \hat{\phi}_n \ \ \square.$$

Lemma IV.2 $\gamma_1 = \gamma$*,* $\gamma'_1 = \gamma'$*,* $\hat{\gamma}_1 = \hat{\gamma}$*, and* $\hat{\gamma}'_1 = \hat{\gamma}'$

Proof: Set $0 < p - p_c \ll 1$. By equations (16) ($g(p,h) = sF(p,s)$), (19), (41), and (47)

$$(p - p_c)^{-\gamma} \sim \chi(p,0) := \frac{\partial m(p,0)}{\partial h} = \lim_{s\to 1}\left[-\frac{\partial F(p,s)}{\partial s}\right] = \phi_0 + \phi_1 \sim \phi_1 \sim (p - p_c)^{-\gamma_1}, \tag{49}$$

hence $\gamma_1 = \gamma$. Similarly for $0 < p_c - p \ll 1$, we have $\gamma'_1 = \gamma'$. By equation (49), the symmetries between m and w (16) and the critical exponent definitions (41)–(42), we also have $\hat{\gamma}_1 = \hat{\gamma}$ and $\hat{\gamma}'_1 = \hat{\gamma}'$ $\square$.

Equation (22) is consistent with, and provides a link between equations (39) and (40). We will see that the fundamental asymmetry between $m(p,h)$ and $w(p,z(h))$ ($\gamma'_0 = 0$ and $\hat{\gamma}'_0 > 0$), given in Theorem IV.1.2-3, is a direct and essential consequence of equation (22), and has deep and far reaching implications.

Lemma IV.3 *Let the sequences γ_n and γ'_n, $n \geq 0$, be defined as in equation* (41)*. Then*

1) $\gamma'_0 = 0,\ \gamma_0 < 0,\ \gamma'_n > 0,$ *and* $\gamma_n > 0,$ *for* $n \geq 1.$

2) $0 < \lim_{h\to 0}\langle \chi_1 \vec{E} \cdot \vec{E}_0\rangle / E_0^2 < 1$ *for all* $p \in [0,1],\ h \in \mathcal{U}.$

Proof: By equation (40) $|w(p,z(0))|$ is bounded for all $p<p_c$. Thus for all $p<p_c$, equations (19), (22), and (41) imply that

$$0=\lim_{h\to 0} hw(p,z(h))=\lim_{h\to 0} m(p,h)=\lim_{s\to 1}(1-F(p,s))=1-\phi_0(p)\sim 1-(p_c-p)^{-\gamma_0'},$$

where the rightmost relation holds for $0<p_c-p\ll 1$ and the leftmost relation is consistent with equation (39). Therefore, $\gamma_0'=0$ and ϕ is a probability measure for all $p<p_c$. The strict positivity of the γ_n', for $n\geq 1$, follows from Baker's inequalities (47). Thus, from equation (49) we have

$$\infty=\lim_{p\to p_c^-}\phi_1(p)=-\lim_{p\to p_c^-}\frac{\partial m(p,0)}{\partial h}\,. \tag{50}$$

For $p>p_c$, equations (19) and (39) imply that $0<\lim_{h\to 0}|m(p,h)|=1-\phi_0<1$. Therefore, $(p-p_c)^{-\gamma_0}\sim\phi_0<1$ for all $0<p-p_c\ll 1$, hence $\gamma_0<0$. The strict positivity of γ_1 follows from equation (50), and the strict positivity of the γ_n for $n\geq 2$ follows from Baker's inequalities (47). Equation (20) and the inequality $0<\lim_{h\to 0}|m(p,h)|=1-\phi_0<1$ imply that $0<\lim_{h\to 0}\langle\chi_1\vec{E}\cdot\vec{E}_0\rangle/E_0^2<1$ for all $p\in[0,1]$ $\square$.

Lemma IV.4 *Let the sequence $\hat{\gamma}_n'$, $n\geq 0$, be defined as in equation* (42)*. Then*

$$\textbf{1)}\quad \hat{\gamma}_n'>0\ \ \textit{for all}\ \ n\geq 0.$$
$$\textbf{2)}\quad \lim_{h\to 0}\langle E_f^2\rangle=\infty\ \ \textit{for all}\ \ p>p_c\,.$$

Proof: By equation (39) we have $0<\lim_{h\to 0}|m(p,h)|<1$, for all $p>p_c$. Therefore equation (22) implies that $\lim_{h\to 0}w(p,z(h))=\lim_{h\to 0}m(p,h)/h=\infty$, for all $p>p_c$, which is consistent with equation (40). More specifically, for all $p>p_c$, equations (22) and (39) imply that $0\leq\lim_{h\to 0}|m(p,h)|=\lim_{h\to 0}|hw(p,z(h))|:=L(p)<1$, where $L(p)=0$ for all $p<p_c$. Therefore, by equation (16), we have

$$\begin{aligned}
&\lim_{h\to 0}|h\,w(p,z(h))|=\lim_{h\to 0}|h\,\hat{g}(p,h)|\in(0,1),\ \text{for all}\ p>p_c,\\
&\lim_{h\to 0}|h\,w(p,z(h))|=\lim_{h\to 0}|h\,\hat{g}(p,h)|=0,\ \text{for all}\ p<p_c\,.
\end{aligned} \tag{51}$$

By equations (21), (40), and (42) we have, for all $p>p_c$,

$$\infty=\lim_{p\to p_c^-}\lim_{h\to 0}w(p,z(h))=\lim_{p\to p_c^-}\lim_{s\to 1}(1-G(p,t(s)))=1+\lim_{p\to p_c^-}\hat{\phi}_0(p)\sim 1+\lim_{p\to p_c^-}(p_c-p)^{-\hat{\gamma}_0'},$$

hence $\hat{\gamma}_0' > 0$. Baker's inequalities (47) then imply that $\hat{\gamma}_n' > 0$ for all $n \geq 0$. Equations (20) and (51), and $\hat{\gamma}_0' > 0$ imply that $\lim_{h\to 0}\langle E_f^2\rangle = \infty$ for all $p > p_c$ $\square$.

The asymptotic behavior of $\hat{g}(p,h)$ in equation (48), and Lemma IV.4 motivates the following fundamental homogenization assumption of this section [11]:

Remark IV.1 *Near the critical point $(p,h) = (p_c, 0)$, the asymptotic behavior of the Stieltjes function $\hat{g}(p,h)$ is determined primarily by the mass $\hat{\phi}_0(p)$ of the measure $\hat{\phi}$ and the rate of collapse of the spectral gap θ_α.*

By remark IV.1, and in light of Lemmas IV.2–IV.4, we make the following variable changes:

$$\begin{aligned}
&\hat{q} := y(p_c - p)^{\hat{\Delta}'}, \qquad && \hat{Q}(p) := \hat{S}(p)(p_c - p)^{\hat{\Delta}'}, \qquad && d\hat{\pi}(\hat{q}) := (p_c - p)^{\hat{\gamma}_0'}\, d\hat{\phi}(y),\\
&q := y(p - p_c)^{\Delta}, && Q(p) := S(p)(p - p_c)^{\Delta}, && d\pi(q) := (p - p_c)^{\gamma}\, y\, d\hat{\phi}(y),
\end{aligned} \tag{52}$$

so that, by equations (41)–(42), $\hat{Q}(p), Q(p) \sim 1$ and the masses $\hat{\pi}_0$ and π_0 of the measures $\hat{\pi}$ and π, respectively, satisfy $\hat{\pi}_0, \pi_0 \sim 1$ as $p \to p_c$.

Equation (52) defines the following scaling functions $G_{n-1}(x)$, $\hat{G}_n(\hat{x})$, $\mathcal{G}_{n-1,j}(x)$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$ as follows. For $h \in \mathcal{U} \cap \mathbb{R}$, equations (26) and (52) imply, for $n \geq 0$, that

$$\begin{aligned}
&\frac{\partial^n g}{\partial h^n} \propto (p - p_c)^{-(\gamma + \Delta(n-1))} G_{n-1}(x), \qquad && \frac{\partial^n \hat{g}}{\partial h^n} \propto (p_c - p)^{-(\hat{\gamma}_0' + \hat{\Delta}' n)} \hat{G}_n(\hat{x}),\\
&G_{n-1}(x) := \int_0^{Q(p)} \frac{q^{n-1} d\pi(q)}{(1 + xq)^{n+1}}, && \hat{G}_n(\hat{x}) := \int_0^{\hat{Q}(p)} \frac{\hat{q}^{\,n} d\hat{\pi}(\hat{q})}{(1 + \hat{x}\hat{q})^{n+1}},\\
&x := h(p - p_c)^{-\Delta}, \quad 0 < p - p_c \ll 1, && \hat{x} := h(p_c - p)^{-\hat{\Delta}'}, \quad 0 < p_c - p \ll 1.
\end{aligned} \tag{53}$$

Analogous formulas are defined for the opposite limits involving $\hat{\Delta}$, $\hat{\gamma}_0$, Δ', and γ'.

For $h \in \mathcal{U}$ such that $h_i \neq 0$, we define the scaling functions $\mathcal{R}_{n-1}(x)$, $\mathcal{I}_{n-1}(x)$, $\hat{\mathcal{R}}_n(\hat{x})$, and $\hat{\mathcal{I}}_n(\hat{x})$ as follows. Using equations (28) and (52) we have, for $0 < p - p_c \ll 1$,

$$\begin{aligned}
\frac{\partial^n g}{\partial h^n} &= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{h}^j \int_0^{S(p)} \frac{y^{n+j} d\phi(y)}{|1 + hy|^{2(n+1)}} \\
&:= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} [\bar{x}(p - p_c)^{\Delta}]^j (p - p_c)^{-(\gamma + \Delta(n-1+j))} \mathcal{G}_{n-1,j}(x) \\
&:= (-1)^n n! (p - p_c)^{-(\gamma + \Delta(n-1))} \mathcal{K}_{n-1}(x), \quad \mathcal{K}_{n-1}(x) := \mathcal{R}_{n-1}(x) + \mathrm{i}\, \mathcal{I}_{n-1}(x), \\
\frac{\partial^n \hat{g}}{\partial h^n} &:= (-1)^n n! (p - p_c)^{-(\hat{\gamma}_0 + \hat{\Delta} n)} \hat{\mathcal{K}}_n(\hat{x}), \quad \hat{\mathcal{K}}_n(\hat{x}) := \hat{\mathcal{R}}_n(\hat{x}) + \mathrm{i}\, \hat{\mathcal{I}}_n(\hat{x}).
\end{aligned} \tag{54}$$

Here, x and $\hat{x}$ are defined in equation (53) and

$$\mathcal{G}_{n-1,j}(x) := \int_0^{Q(p)} \frac{q^{n-1+j} d\pi(q)}{|1+xq|^{2(n+1)}}\,, \qquad \hat{\mathcal{G}}_{n,j}(\hat{x}) := \int_0^{\hat{Q}(p)} \frac{\hat{q}^{n+j} d\hat{\pi}(\hat{q})}{|1+\hat{x}\hat{q}|^{2(n+1)}}\,, \tag{55}$$
$$\mathcal{K}_{n-1}(x) := \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{x}^j \mathcal{G}_{n-1,j}(x), \qquad \hat{\mathcal{K}}_n(\hat{x}) := \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{\hat{x}}^j \hat{\mathcal{G}}_{n,j}(\hat{x}),$$

where we have made the definitions $\mathcal{R}_{n-1}(x) := \mathrm{Re}(\mathcal{K}_{n-1}(x))$, $\mathcal{I}_{n-1}(\hat{x}) := \mathrm{Im}(\mathcal{K}_{n-1}(x))$, $\hat{\mathcal{R}}_n(\hat{x}) := \mathrm{Re}(\hat{\mathcal{K}}_n(\hat{x}))$, and $\hat{\mathcal{I}}_n(\hat{x}) := \mathrm{Im}(\hat{\mathcal{K}}_n(\hat{x}))$. Analogous formulas are defined for the opposite limit, $0 < p_c - p \ll 1$, involving $\hat{\Delta}'$, $\hat{\gamma}_0'$, Δ', and γ'.

From equation (17) we have, for $h \in \mathcal{U}$, $p \in [0, 1]$, and $n \geq 0$,

$$G_{n-1}(x) > 0, \quad \mathcal{G}_{n-1,j}(x) > 0, \qquad \hat{G}_n(\hat{x}) > 0, \quad \hat{\mathcal{G}}_{n,j}(\hat{x}) > 0. \tag{56}$$

By our gap hypothesis the h derivatives of $g(p, h)$ and $\hat{g}(p, h)$, of all orders, are bounded at $h = 0$ for $p > p_c$ and $p < p_c$, respectively. Therefore,

$$\lim_{h\to 0} G_{n-1}(x) < \infty, \qquad \lim_{h\to 0} \mathcal{G}_{n-1,j}(x) < \infty, \qquad \text{for all } \; p > p_c, \quad n \geq 0 \tag{57}$$
$$\lim_{h\to 0} \hat{G}_n(\hat{x}) < \infty, \qquad \lim_{h\to 0} \hat{\mathcal{G}}_{n,j}(\hat{x}) < \infty, \qquad \text{for all } \; p < p_c, \quad n \geq 0.$$

Lemma IV.5 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (53)*, for $p > p_c$. Then*

- **1**) $G_{n-1}(x) \sim 1$ *as* $x \to 0$ ($h \to 0$ *and* $0 < p - p_c \ll 1$) *for all* $n \geq 1$.
- **2**) $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ *as* $\hat{x} \to 0$ ($h \to 0$ *and* $0 < p - p_c \ll 1$) *for all* $n \geq 1$.
- **3**) $\gamma = \hat{\gamma}_0$.
- **4**) $\Delta = \hat{\Delta}$.

Proof: Let $h \in \mathcal{U} \cap \mathbb{R}$ and $p > p_c$. Equations (27), (53), and (56)–(57) imply that we have, for all $n \geq 1$, $0 < p - p_c \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p - p_c)^{-(\gamma + \Delta(n-1))} G_{n-1}(x) = (p - p_c)^{-(\hat{\gamma}_0 + \hat{\Delta}(n-1))} [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})]. \tag{58}$$

Equations (56)–(57) imply that $G_{n-1}(x) \sim 1$ as $x \to 0$, for all $n \geq 1$. Equation (58) then implies that $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ as $\hat{x} \to 0$, for all $n \geq 1$ (a competition in sign between two diverging terms). Or equivalently, generalizing (51), $\hat{G}_0(\hat{x}) - \hat{x}^n \hat{G}_n(\hat{x}) \sim 1$. Therefore,

$$\gamma + \Delta(n - 1) = \hat{\gamma}_0 + \hat{\Delta}(n - 1), \quad n \geq 1. \tag{59}$$

Which in turn, implies that $\gamma = \hat{\gamma}_0$ and $\Delta = \hat{\Delta}$ □.

Lemma IV.6 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (53)*, for $p < p_c$. Then*

1) $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ *as* $\hat{x} \to 0$ $(h \to 0$ *and* $0 < p_c - p \ll 1)$, *for all* $n \geq 1$.

2) $G_{n-1}(x) \sim 1$ *as* $x \to 0$ $(h \to 0$ *and* $0 < p_c - p \ll 1$, *for all* $n \geq 1$.

3) $\gamma' = \hat{\gamma}'_0$.

4) $\Delta' = \hat{\Delta}'$.

Proof: Let $h \in \mathcal{U} \cap \mathbb{R}$ and $p < p_c$. Equations (27), (53), and (56)–(57) imply that, for all $n \geq 1$, $0 < p_c - p \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p_c - p)^{-(\hat{\gamma}'_0 + \hat{\Delta}'(n-1))}[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] = (p_c - p)^{-(\gamma' + \Delta'(n-1))}G_{n-1}(x) \qquad (60)$$

Equations (56)–(57) imply that $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ as $\hat{x} \to 0$ for all $n \geq 1$. Equation (60) then implies that $G_{n-1}(x) \sim 1$ as $x \to 0$ for all $n \geq 1$. Therefore,

$$\gamma' + \Delta'(n-1) = \hat{\gamma}'_0 + \hat{\Delta}'(n-1), \quad n \geq 1.$$

Which in turn, implies that $\gamma' = \hat{\gamma}'_0$ and $\Delta' = \hat{\Delta}'$ $\square$.

Lemma IV.7 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (53)*. Then*

1) $\gamma_n = \gamma + \Delta(n-1)$, *for all* $n \geq 1$.

2) $\hat{\gamma}'_n = \hat{\gamma}'_0 + \hat{\Delta}'n = \hat{\gamma}' + \hat{\Delta}'(n-1)$, *for all* $n \geq 0$.

3) $t = \Delta - \gamma$.

4) $s = \hat{\gamma}'_0 = \hat{\gamma}' - \hat{\Delta}'$.

Proof: Let $0 < p - p_c \ll 1$. By equations (41), (48), and (53), and Lemma IV.5 we have, for all $n \geq 1$,

$$(p - p_c)^{-\gamma_n} \sim \phi_n \sim \lim_{h \to 0} \frac{\partial^n g(p,h)}{\partial h^n} \sim (p - p_c)^{-(\gamma + \Delta(n-1))} \lim_{x \to 0} G_{n-1}(x) \sim (p - p_c)^{-(\gamma + \Delta(n-1))}.$$

Therefore $\gamma_n = \gamma + \Delta(n-1)$ for all $n \geq 1$, with constant gap $\gamma_i - \gamma_{i-1} = \Delta$, which is consistent with the absence of multifractal behavior for the bulk conductivity [37].

Now let $0 < p_c - p \ll 1$. By equations (42), (48), and (53), and Lemma IV.6 we have, for all $n \geq 1$,

$$(p_c - p)^{-\hat{\gamma}_n} \sim \hat{\phi}_n \sim \lim_{h\to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} \propto (p_c - p)^{-(\hat{\gamma}'_0 + \hat{\Delta}' n)} \lim_{\hat{x}\to 0} \hat{G}_n(\hat{x}) \sim (p_c - p)^{-(\hat{\gamma}'_0 + \hat{\Delta}' n)}.$$

Therefore, by Lemma IV.2, we have $\hat{\gamma}_n = \hat{\gamma}'_0 + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ for all $n \geq 0$, with constant gap $\hat{\gamma}'_i - \hat{\gamma}'_{i-1} = \hat{\Delta}$, which is consistent with the absence of multifractal behavior for the bulk conductivity [37].

Again let $0 < p - p_c \ll 1$. Equations (16), (23), (41), (51), and (53) yield

$$\begin{aligned}(p - p_c)^t &\sim \lim_{h\to 0} m(p,h) = 1 - \lim_{h\to 0} g(p,h) = \lim_{h\to 0} h\hat{g}(p,h) = (p-p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{\hat{x}\to 0} \hat{x}\hat{G}_0(\hat{x}) \\ &\sim (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0}. \end{aligned} \tag{61}$$

Therefore, by Lemma IV.5 we have $t = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma$.

Finally let $0 < p_c - p \ll 1$. By equations (16), (42), and (53), and Lemmas IV.4 and IV.6, we have

$$(p_c - p)^{-s} \sim \lim_{h\to 0} w(p, z(h)) \sim \lim_{h\to 0} \hat{g}(p,h) = (p_c - p)^{-\hat{\gamma}'_0} \lim_{\hat{x}\to 0} \hat{G}_0(\hat{x}) \sim (p_c - p)^{-\hat{\gamma}'_0}.$$

Therefore, by Lemma IV.7.2, we have $s = \hat{\gamma}'_0 = \hat{\gamma}' - \hat{\Delta}'$ $\Box$.

Lemma IV.8 *Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in equation* (53), *for $p > p_c$ and $p < p_c$. Then for all $n \geq 1$*

1) $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma + \Delta(n-1))/\Delta}$, *as $\hat{x} \to \infty$ ($p \to p_c^+$ and $0 < h \ll 1$)*.

2) $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma' + \Delta'(n-1))/\Delta'}$, *as $x \to \infty$ ($p \to p_c^-$ and $0 < h \ll 1$)*.

3) $\delta = \Delta/(\Delta - \gamma)$.

4) $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}'_0 = \hat{\Delta}'/(\hat{\gamma}' - \hat{\Delta}')$.

Proof: Let $0 < h \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0,1]$ [13]. The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that all orders of h derivatives of these functions are bounded as $p \to p_c$, from the left or the right. Therefore, equation (58) holds for $0 < p - p_c \ll 1$, and equation (60) holds for $0 < p_c - p \ll 1$. Moreover, in order to cancel the diverging p dependent prefactors in equations (58) and (60) we must have, for all $n \geq 1$,

$$\begin{aligned} &G_{n-1}(x) \sim x^{-(\gamma+\Delta(n-1))/\Delta}, \quad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}_0 + \hat{\Delta}(n-1))/\hat{\Delta}}, \quad \text{as } p \to p_c^+, \\ &G_{n-1}(x) \sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}, \quad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}'_0 + \hat{\Delta}'(n-1))/\hat{\Delta}'}, \quad \text{as } p \to p_c^-. \end{aligned} \tag{62}$$

Lemma IV.8.1-2 follows from equation (62) and Lemmas IV.5–IV.6.

Now by equations (16), (22), (41), (53), and (62) for $n = 1$, we have

$$h^{1/\delta} \sim \lim_{p\to p_c^+} m(p,h) = \lim_{p\to p_c^+} hw(p, z(h)) \sim \lim_{p\to p_c^+} h\hat{g}(p,h) = h \lim_{p\to p_c^+} (p-p_c)^{-\hat{\gamma}_0}\hat{G}_0(\hat{x}) \qquad (63)$$
$$\sim h(p-p_c)^{-\hat{\gamma}_0} h^{-\hat{\gamma}_0/\hat{\Delta}}(p-p_c)^{-\hat{\Delta}(-\hat{\gamma}_0/\hat{\Delta})} = h^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}.$$

Therefore by Lemma IV.6, we have $\delta = \hat{\Delta}/(\hat{\Delta}-\hat{\gamma}_0) = \Delta/(\Delta-\gamma)$. Similarly by equations (16), (42), (53), and (62) for $n = 1$, and Lemma IV.4, we have

$$h^{-1/\hat{\delta}'} \sim \lim_{p\to p_c^-} w(p, z(h)) \sim \lim_{p\to p_c^-} \hat{g}(p,h) = \lim_{p\to p_c^-} (p-p_c)^{-\hat{\gamma}_0'}\hat{G}_0(\hat{x}) = h^{-\hat{\gamma}_0'/\hat{\Delta}'}. \qquad (64)$$

Therefore, by Lemma IV.7 we have $\hat{\delta}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}'-\hat{\Delta}')$ □.

Lemma IV.9 *Let $h \in \mathcal{U}$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in equations* (54)–(55) *for $p > p_c$ and $p < p_c$. Furthermore, let s_r, s_i, t_r, and t_i be defined as in equations* (41)–(42)*. Then,*

1) $[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r\hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \hat{x}_i\hat{\mathcal{G}}_{0,1}(\hat{x}) \sim 1$ *as* $\hat{x} \to 0$ ($h \to 0$ *and* $0 < p_c - p \ll 1$).

2) $\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}|^2\hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \lim_{\hat{x}\to 0}[\hat{x}_i\hat{\mathcal{G}}_{0,0}(\hat{x})] \sim 1$ *for* $0 < p - p_c \ll 1$.

3) $s_r = s_i = \hat{\gamma}_0' = s$.

4) $t_r = t_i = \Delta - \gamma = t$.

Proof: Let $0 < p_c - p \ll 1$, $h \in \mathcal{U}$ such that $h_i \neq 0$, and $0 < |h| \ll 1$. By equations (54)–(55), for $n = 0$, we have

$$\hat{g}(p,h) = \int_0^{\hat{S}(p)} \frac{d\hat{\phi}(y)}{|1+hy|^2} + \bar{h}\int_0^{\hat{S}(p)} \frac{y\, d\hat{\phi}(y)}{|1+hy|^2} = (p_c-p)^{-\hat{\gamma}_0'}[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \bar{\hat{x}}\hat{\mathcal{G}}_{0,1}(\hat{x})], \qquad (65)$$

so that

$$\hat{g}_r = (p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{R}}_0(\hat{x}) = (p_c-p)^{-\hat{\gamma}_0'}[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r\hat{\mathcal{G}}_{0,1}(\hat{x})] \qquad (66)$$
$$\hat{g}_i = (p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{I}}_0(\hat{x}) = -(p_c-p)^{-\hat{\gamma}_0'}\hat{x}_i\hat{\mathcal{G}}_{0,1}(\hat{x}).$$

Equations (51) and (56) imply that $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim 1$ as $\hat{x} \to 0$ ($h \to 0$ and $0 < p_c - p \ll 1$). Therefore, equations (16), (42), (66) and Lemma IV.4 imply that

$$(p_c-p)^{-s_r} \sim w_r(p,0) \sim \hat{g}_r(p,0) \sim (p_c-p)^{-\hat{\gamma}_0'}\lim_{\hat{x}\to 0}\hat{\mathcal{R}}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}_0'}, \qquad (67)$$
$$(p_c-p)^{-s_i} \sim w_i(p,0) \sim \hat{g}_i(p,0) \sim (p_c-p)^{-\hat{\gamma}_0'}\lim_{\hat{x}\to 0}\hat{\mathcal{I}}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}_0'}.$$

Equation (67) and Lemma IV.7 imply that $s_r = s_i = \hat{\gamma}_0' = s$. It's worth noting that these scaling relations are independent of the path of the limit $h \to 0$.

Now let $0 < p - p_c \ll 1$ with h as before. In equation (61) we demonstrated that $m(p,0) = \lim_{h\to 0} h\hat{g}(p,h)$. Therefore equation (66), for $p > p_c$, implies that

$$\begin{aligned} m_r(p,0) &\sim \lim_{h\to 0}[h_r\hat{g}_r(p,h) - h_i\hat{g}_i(p,h)] = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2\hat{\mathcal{G}}_{0,1}(\hat{x})] \\ m_i(p,0) &\sim \lim_{h\to 0}[h_i\hat{g}_r(p,h) + h_r\hat{g}_i(p,h)] = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{\hat{x}\to 0}[\hat{x}_i\hat{\mathcal{G}}_{0,0}(\hat{x})] \end{aligned} \tag{68}$$

By equation (51) we have $\lim_{\hat{x}\to 0}[\hat{x}_r\hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}|^2\hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \lim_{\hat{x}\to 0}[\hat{x}_i\hat{\mathcal{G}}_{0,0}(\hat{x})] \sim 1$ for all $0 < p - p_c \ll 1$. Therefore, equations (41) and (68) imply that

$$(p-p_c)^{t_r} \sim m_r(p,0) \sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}, \qquad (p-p_c)^{t_i} \sim m_i(p,0) \sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}. \tag{69}$$

Equation (69) and Lemmas IV.5 and IV.7 imply that $t_r = t_i = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma = t$. Again, these scaling relations are independent of the path of the limit $h \to 0$ □.

Lemma IV.10 *Let $h \in \mathcal{U}$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in equations* (54)–(55) *for $p > p_c$ and $p < p_c$. Furthermore, let $\hat{\delta}_r$, $\hat{\delta}_i$, δ_r, and δ_i be defined as in equations* (41)–(42). *Then,*

1) $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |\hat{x}|^{-\hat{\gamma}_0'/\hat{\Delta}'}$, *as $\hat{x} \to \infty$ ($p \to p_c^-$ and $0 < |h| \ll 1$).*

2) $[\hat{x}_r\hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i\hat{\mathcal{I}}_0(\hat{x})] \sim [\hat{x}_r\hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i\hat{\mathcal{R}}_0(\hat{x})] \sim |\hat{x}|^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}$, *as $\hat{x} \to \infty$.*

3) $\hat{\delta}_r{}' = \hat{\delta}_i{}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}$.

4) $\delta_r = \delta_i = \Delta/(\Delta - \gamma) = \delta$.

Proof: Let $h \in \mathcal{U}$ such that $h_i \neq 0$ and $0 < |h| \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0,1]$ [13]. Equations (16), (42), (66) and Lemma IV.4 imply that

$$\begin{aligned} |h|^{-1/\hat{\delta}_r'} &\sim w_r(p_c,h) \sim \hat{g}_r(p_c,h) \sim \lim_{p\to p_c^-}(p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{R}}_0(\hat{x}), \\ |h|^{-1/\hat{\delta}_i'} &\sim w_i(p_c,h) \sim \hat{g}_i(p_c,h) \sim \lim_{p\to p_c^-}(p_c-p)^{-\hat{\gamma}_0'}\hat{\mathcal{I}}_0(\hat{x}). \end{aligned} \tag{70}$$

The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that they are bounded for all $p \in [0,1]$. Therefore, in order to cancel the diverging p dependent prefactors in equations (70), we must have $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |x|^{-\hat{\gamma}_0'/\hat{\Delta}'}$ as $\hat{x} \to \infty$ ($p \to p_c^-$ and $0 < h \ll 1$). Equation (70) then implies that

$$|h|^{-1/\hat{\delta}_r'} \sim (p_c-p)^{-\hat{\gamma}_0'}|h|^{-\hat{\gamma}_0'/\hat{\Delta}'}(p_c-p)^{-\hat{\Delta}'(-\hat{\gamma}_0'/\hat{\Delta}')} = |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}, \quad |h|^{-1/\hat{\delta}_i'} \sim |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}. \tag{71}$$

Therefore, by Lemma IV.8, $\hat{\delta}_r{}' = \hat{\delta}_i{}' = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}\,'$.

Equations (16) and (22) imply that $m(p_c, h) \sim \lim_{p\to p_c^+} h\hat{g}(p,h)$, for $0 < |h| \ll 1$. Therefore equations (41) and (68) implies that

$$|h|^{1/\delta_r} \sim m_r(p_c, h) = (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{p\to p_c^+} [\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})], \qquad (72)$$
$$|h|^{1/\delta_i} \sim m_i(p_c, h) = (p - p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{p\to p_c^+} [\hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x})].$$

The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that they are bounded for all $p \in [0,1]$. Therefore, in order to cancel the diverging p dependent prefactors in equations (72), we must have $[\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x}) \sim |x|^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}$ as $\hat{x} \to \infty$ ($p \to p_c^+$ and $0 < h \ll 1$). Therefore equation (72), and Lemmas IV.5 and IV.8 imply that $\delta_r = \delta_i = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0) = \Delta/(\Delta - \gamma) = \delta$ □.

Lemma IV.11 *The measure $y\,d\phi(y)$ has the symmetry property ($\Delta = \Delta'$ and $\gamma = \gamma'$) if and only if the measure $d\hat{\phi}(y)$ has the symmetry property ($\hat{\Delta} = \hat{\Delta}'$ and $\hat{\gamma}_0 = \hat{\gamma}_0'$). If either measure has this symmetry, then*

1) $s + t = \Delta$. **2**) $1/\delta + 1/\hat{\delta}\,' = 1$. **3**) $\Delta = \hat{\Delta} = \Delta' = \hat{\Delta}'$. **4**) $\gamma = \gamma' = \hat{\gamma}_0 = \hat{\gamma}_0'$.

Proof: We have shown in Lemmas IV.5–IV.6 that $\gamma = \hat{\gamma}_0$, $\Delta = \hat{\Delta}$, $\gamma' = \hat{\gamma}_0'$, and $\Delta' = \hat{\Delta}'$. Therefore, it is clear that, ($\Delta = \Delta'$ and $\gamma = \gamma'$) $\iff$ ($\hat{\Delta} = \hat{\Delta}'$ and $\hat{\gamma}_0 = \hat{\gamma}_0'$). Assume that either of the measures, $d\hat{\phi}(y)$ or $y\,d\phi(y)$, has this symmetry. Thus, $\Delta = \hat{\Delta} = \hat{\Delta}' = \Delta'$ and $\gamma = \hat{\gamma}_0 = \hat{\gamma}_0' = \gamma'$. By Lemma IV.7 we have $t = \Delta - \gamma$ and $s = \hat{\gamma}_0'$, and by Lemma IV.8 we have $\delta = \Delta/(\Delta - \gamma)$ and $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}_0'$. Therefore,

$$s + t = \hat{\gamma}_0' + \Delta - \gamma = \hat{\gamma}_0 + \Delta - \gamma = \Delta\,.$$
$$\delta = \Delta/(\Delta - \gamma) = 1/(1 - \gamma/\Delta) = 1/(1 - \hat{\gamma}_0/\hat{\Delta}) = 1/(1 - \hat{\gamma}_0'/\hat{\Delta}') = 1/(1 - 1/\hat{\delta}\,') \quad \square.$$

In Lemma IV.9 we proved that $t_r = t_i = t$ and $s_r = s_i = s$. These scaling relations are a fundamental identity, as these sets of critical exponents are defined in terms of $m(p,0)$ and $w(p, z(0))$, where $h = 0 \in \mathbb{R}$. The calculation of these scaling relations serves as a consistency check of this mathematical framework. Another consistency check was given in Lemma IV.11, where we proved that $1/\delta + 1/\hat{\delta} = 1$. This is another fundamental identity which follows from the relation $m(p,h) = h\,w(p, z(h))$ (22) and the definition of these critical exponents (41)–(42): $h^{1/\delta} \sim m(p_c, h) = h\,w(p_c, h) \sim h\,h^{-1/\hat{\delta}} \sim h^{1-1/\hat{\delta}}$, for $0 < |h| \ll 1$. It

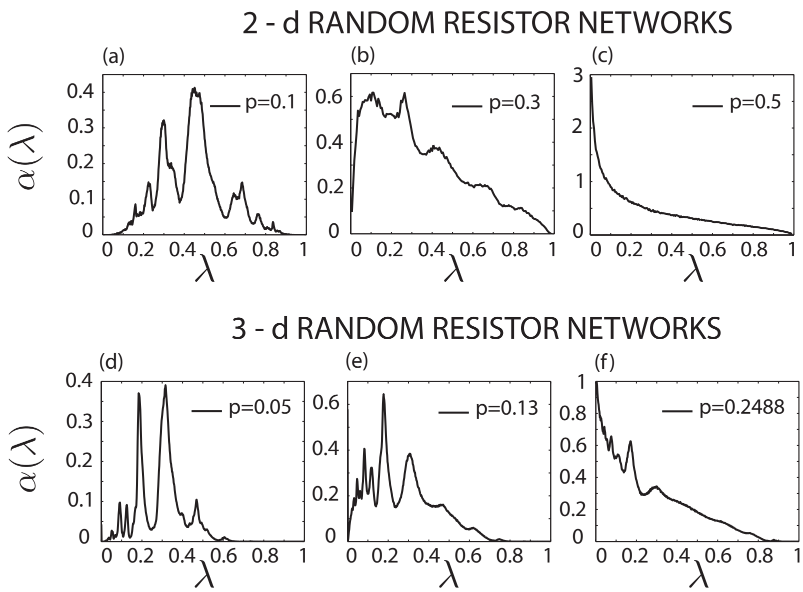

FIG. 1. *The spectral function for the 2–d and 3–d square random bond networks.* As the volume fraction p of defect bonds increases, from left to right, the width of the gaps in the spectrum near $\lambda = 0, 1$ shrink to 0 with increasing connectedness as the percolation thresholds $p_c = 0.5$ and $p_c \approx 0.2488$ are approached.

follows that relation (22) provides a partial converse to the assumption underlying lemma IV.11. Indeed as $1/\delta + 1/\hat{\delta} = 1$, where $\delta = \Delta/(\Delta - \gamma)$ and $\hat{\delta}\,' = \hat{\Delta}'/\hat{\gamma}_0'$, then $1 - \gamma/\Delta = 1/\delta = 1 - 1/\hat{\delta} = 1 - \hat{\gamma}_0'/\hat{\Delta}'$. Which implies that, in general,

$$t/\Delta + s/\hat{\Delta}' = 1, \qquad \Delta = \hat{\Delta}' \iff \gamma = \hat{\gamma}_0'. \tag{73}$$

This concludes the proof of Theorem IV.1 $\square$.

C. Spectral Characterization of Critical Transitions in Transport

We now discuss the gaps θ_α and θ_η (for $p < p_c$), and θ_μ and θ_κ (for $p > p_c$). As the operators $-\Gamma$ and Υ are projectors on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{C}_\bullet$, respectively, the eigenvalues thereof are confined to the set $\{0, 1\}$ [31]. The associated operators $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$ are positive definite compositions of projection operators, thus the eigenvalues thereof are confined to the set $[0, 1]$ [40].

While in general, the spectra actually extends all the way to the spectral endpoints $\lambda = 0, 1$, the part close to $\lambda = 0, 1$ corresponds to very large, but very rare connected regions of the defect inclusions (Lifshitz phenomenon). It is believed that this phenomenon gives exponentially small contributions to the effective complex conductivity (resistivity), and does not affect the power law behavior thereof [26]. In [41] O. Bruno has proven the existence of spectral gaps in matrix/particle systems with polygonal inclusions, and studied how the gaps vanishes as the inclusions touch (like $p \to p_c$). In Section IV A we explicitly showed that there are gaps in the measures μ and α of EMT, and discussed the details of how the gaps collapse as a function of p. In Figure IV C we give a graphical representation of the spectral measure α for finite 2–d and 3–d RRN [4]. The gaps in these simulations behave precisely as those in EMT, and as p surpasses p_c and $1 - p_c$ the spectra pile up at $\lambda = 0$ and $\lambda = 1$, respectively, forming delta function–like components in the measure. In the conclusion of Section IV A we also gave a discussion regarding Lemma III.2 and Corollary III.1, which hold for general percolation models of stationary two–phase random media, and prove that criticality is characterized by the formation of delta function components in μ and α at $\lambda = 0, 1$ *precisely* at $p = p_c$ and $p = 1 - p_c$. We now provide an analytical proof of the existence of spectral gaps in α for arbitrary lattice systems, which collapse as p increases.

For lattice systems with a finite number n of lattice sites, the differential equations in (7) become difference equations (Kirchoff's laws) [12]. Consequently, the operators $\mathbf{M}_j$ and $\mathbf{K}_j$, $j = 1, 2$ are given by $N \times N$ matrices, say [4, 12]. We focus on $\mathbf{M}_2 = \chi_2(-\Gamma)\chi_2$, as our results extend to the other operators by symmetry. In this lattice setting, $-\Gamma$ is a real symmetric projection matrix and can therefore be diagonalized: $-\Gamma = \mathbf{QDQ}^T$, where $\mathbf{D}$ is a diagonal matrix of zeros and ones and $\mathbf{Q}$ is a real orthogonal matrix. More specifically,

$$-\Gamma = \begin{bmatrix} \text{—}\ \vec{q}_1\ \text{—} \\ \vdots \\ \text{—}\ \vec{q}_N\ \text{—} \end{bmatrix} \begin{bmatrix} \mathbf{I}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \text{—}\ \vec{q}_1\ \text{—} \\ \vdots \\ \text{—}\ \vec{q}_N\ \text{—} \end{bmatrix}^T = \begin{bmatrix} (\vec{q}_1 \cdot \vec{q}_1)_L & (\vec{q}_1 \cdot \vec{q}_2)_L & \cdots & (\vec{q}_1 \cdot \vec{q}_N)_L \\ \vdots & \vdots & \ddots & \vdots \\ (\vec{q}_N \cdot \vec{q}_1)_L & (\vec{q}_N \cdot \vec{q}_2)_L & \cdots & (\vec{q}_N \cdot \vec{q}_N)_L \end{bmatrix}, \tag{74}$$

where $0 < L < N$ when $N \gg 1$, $\mathbf{I}_L$ is the $L \times L$ identity matrix, $\mathbf{0}$ is a matrix of zeros of arbitrary dimension, $(\vec{q}_i \cdot \vec{q}_j)_L := \sum_{l=1}^{L} (\vec{q}_i)_l (\vec{q}_j)_l$, and $(\vec{q}_i)_l$ is the l^{th} component of the vector $\vec{q}_i \in \mathbb{R}^N$. Here, we consider the case where $N \gg 1$ so that $1 \ll L < N$.

The spectral measure $\alpha(d\lambda)$ of the matrix $\mathbf{M}_2$ is given by a sum of "Dirac δ functions,"

$$\alpha(d\lambda) = \left[\sum_{j=1}^{N} m_j \delta_{\lambda_j}(d\lambda) \right] d\lambda := \alpha(\lambda) d\lambda, \tag{75}$$

where $\delta_{\lambda_j}(d\lambda)$ is the Dirac delta measure centered at λ_j, $m_j = \langle \vec{e}_k^{\,T}[\vec{v}_j\vec{v}_j^{\,T}]\,\vec{e}_k\rangle$, $\vec{e}_k$ is a N–dimensional vector of ones, and λ_j and $\vec{v}_j$ are the eigenvalues and eigenvectors of $\mathbf{M}_2$, respectively [4]. In this matrix case, the associated Stieltjes transformation of the measure $\alpha(d\lambda)$ (13) is given by the sum $G(t) = \sum_{j=1}^{n} m_j/(t-\lambda_j)$, and $\alpha(\lambda)$ in equation (75) is called "*the spectral function*," which is defined only pointwise on the set of eigenvalues $\{\lambda_j\}$. In Figure IV C we give a graphical representation of the spectral measure for finite 2–d and 3–d RRN. It displays linearly connected peaks of histograms with bin sizes on the order of 10^{-2}. The apparent smoothness of the spectral function graphs in this figure is due to the large number ($\sim 10^6$) of eigenvalues and eigenvectors calculated, and ensemble averaged.

In the matrix case, the action of χ_2 is given by that of a square diagonal matrix of zeros and ones [4]. The action of χ_2 in the matrix $\chi_2(-\Gamma)\chi_2$ introduces a row and column of zeros in the matrix $-\Gamma$, corresponding to every diagonal entry of χ_2 with value 0. When there is only one defect inclusion ($p = 1/n$) located at the j^{th} bond, χ_2 has all zero entries except at the j^{th} diagonal: $\chi_2 = \operatorname{diag}(0,\cdots,0,1,0,\cdots,0) := \operatorname{diag}(\vec{\mathrm{v}}_j)$. Therefore, the only non-trivial eigenvalue is given by $\lambda_0 = (\vec{q}_j\cdot\vec{q}_j)_L = \sum_{l=1}^{L}(\vec{q}_j)_l^2 = 1 - \sum_{l=L+1}^{N}(\vec{q}_j)_l^2$, with eigenvector $\vec{\mathrm{v}}_j$ and weight $m_0 = 1/n$. This implies that there is a gap at $\lambda = 0$, $\theta_0 := \sum_{l=1}^{L}(\vec{q}_j)_l^2 > 0$, and a gap at $\lambda = 1$, $\theta_1 := \sum_{l=L+1}^{N}(\vec{q}_j)_l^2 > 0$. It is clear that these bounds hold for all $\omega \in \Omega$ such that $p = 1/n$ when $L \gg 1$. We have already mentioned that the eigenvalues of $\mathbf{M}_1$ are restricted to the set $\{0,1\}$ when $p = 1$ ($\chi_2 \equiv \mathbf{I}_N$). Therefore, there exists $0 < p_0 < 1$ such that, for all $p \geq p_0$, there exists a $\omega \in \Omega$ such that $\theta_0(\omega) = 0$ and/or $\theta_1(\omega) = 0$.

V. CONCLUDING REMARKS

We have constructed a mathematical framework which unifies the critical theory of transport for binary composite media, in the continuum and lattice settings. We have focused on critical transitions exhibited by the effective complex conductivity $\sigma^* = \sigma_2 m(h) = \sigma_1 w(z)$, as the symmetries underlying this framework extend our results to that regarding the effective complex resistivity $[\sigma^{-1}]^* = \sigma_1^{-1}\tilde{m}(h) = \sigma_2^{-1}\tilde{w}(z)$. We have shown that critical transitions in transport properties are, in general, characterized by the formation of delta function components in the underlying spectral measures, at the spectral endpoints. Moreover, for percolation models, we have shown that the onset of the critical transition (the formation of these delta components) occurs *precisely* at the percolation threshold.

The mathematical transport properties of such systems, displayed in section III, hold for general two–component stationary random media in lattice and continuum settings [13]. Moreover, the critical exponent scaling relations and the various transport properties, displayed in Lemmas IV.2–IV.11, hold for general percolation models regarding this class of composite media [26]. This type of critical behavior has been studied before in the lattice [39, 42, 43], and alternate methods have shown that $\Delta = s + t$, $\delta = (s + t)/t$, and $\gamma = s$ [26]. These are precisely the relations that we have shown to hold for general lattice and continuum percolation models, under the symmetry condition of Lemma IV.11. There is no apparent mathematical necessity for this spectral symmetry, in general. Although, it leads to the well known two dimensional duality relation $s = t$ in the lattice [39, 42, 43].

As in EMT, our general scaling relations involving $|h|$ are independent of the limiting path as $h \to 0$. This is opposed to the results of other workers [39, 42, 43] which use heuristic scaling forms as a starting point. The starting point for our critical theory is equation (13), which displays *exact* formulas for infinite systems [26]. We have verified the validity of our framework using several consistency checks including the verification that our relations are satisfied directly by the exponents in EMT.

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