# Decomposing parabolic eigenvalue problem

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**Abstract**

A decomposition of a time periodic parabolic eigenvalue problem and an application to our effective diffusivity problem.

## 1 An Example

Consider the parabolic eigenvalue problem:

$$\psi_t - \Delta_x \psi + \cos(t)\, b(x)\, \psi = \lambda\, \psi, \tag{1.1}$$

subject to $2\pi$ periodic boundary conditions in $x$ and $t$.

Substituting $\cos t = (e^{it} + e^{-it})/2$, and $\psi = \sum_\ell \psi_\ell(x) e^{i\ell t}$ in (1.1), we have:

$$\sum_\ell e^{i\ell t}\,(i\ell\,\psi_\ell\, - \Delta_x \psi_\ell) + \frac{1}{2} b(x)\, \sum_\ell (e^{i(\ell+1)t} + e^{i(\ell-1)t})\psi_\ell = \lambda \sum_\ell \psi_\ell e^{i\ell t},$$

or:

$$\sum_\ell e^{i\ell t}\,(i\ell\,\psi_\ell\, - \Delta_x \psi_\ell) + \frac{1}{2} b(x)\, \sum_\ell (\psi_{\ell-1} + \psi_{\ell+1})\, e^{i\ell t} = \lambda \sum_\ell \psi_\ell e^{i\ell t}.$$

Extracting the $\ell$-th mode on both sides gives:

$$(-\Delta_x + i\,\ell)\,\psi_\ell + \frac{1}{2}\, b(x)\,(\psi_{\ell-1} + \psi_{\ell+1}) = \lambda\,\psi_\ell, \;\; \ell \in Z, \tag{1.2}$$

which can be put in a tri-diagonal matrix acting on $[\cdots, \psi_{\ell-1}, \psi_\ell, \psi_{\ell+1}, \cdots]'$, with $b(x)/2$ on the off-diagonals, $-\Delta_x + i\,\ell$ on the diagonals.

Let us find a similar derivation for the eigenvalue problem of the advection-diffusion operator, with advection field being the time periodic cell flow:

$$\vec{u}(t, \vec{x}) = (\cos y, \cos x) + \delta \cos t\,(\sin y, \sin x) := \vec{u}_1(\vec{x}) + \delta \cos t\, \vec{u}_2(\vec{x}). \tag{1.3}$$

Question: Is it possible to carry out a similar decomposition in $x$ and $y$ and fully reduce the differential system like (1.2) into an algebraic system ?

## 1.1 An Application to our effective diffusivity problem

Consider the eigenvalue problem $A\psi_j = \lambda_j\psi_j$, $j = 1,2,3,\ldots$, involving the integro-differential operator $A = -\Delta^{-1}(\partial_t + \vec{u}\cdot\vec{\nabla})$, introduced in equation (45) of our (attached) effective-diffusivity paper, with $\vec{u} \mapsto -\vec{u}$. Here $A$ is an anti-symmetric (normal) operator and the incompressible velocity field $\vec{u}(t,\vec{x})$ is given in equation (1.3) above. The equation $A\psi_j = \lambda_j\psi_j$ may be rewritten as

$$(\partial_t + \vec{u}\cdot\vec{\nabla})\psi_j = -\lambda_j\Delta\psi_j. \tag{1.4}$$

The eigenfunctions $\psi_j$ satisfy the following orthogonality condition

$$\langle\vec{\nabla}\psi_j\cdot\vec{\nabla}\psi_k\rangle = \delta_{jk}, \tag{1.5}$$

where $\delta_{jk}$ is the Kronecker delta and $\langle\cdot\rangle$ denotes space-time averaging over the period cell $\mathcal{T}\times\mathcal{V}$, with $\mathcal{T} = [0,2\pi]$ and $\mathcal{V} = [0,2\pi]\times[0,2\pi]$.

The eigenfunction $\psi_j$ satisfies $\psi_j \in \mathcal{F} \subset \mathcal{A}(\mathcal{T})\otimes\mathcal{H}^1(\mathcal{V})$, i.e. it is mean-zero, absolutely continuous in time for $t\in\mathcal{T}$ and is in the Sobolev space $\mathcal{H}^1(\mathcal{V})$ for $\vec{x}\in\mathcal{V}$. Since the orthogonal set $\{\mathrm{e}^{\mathrm{i}\ell t}\}_{\ell\in\mathbb{Z}}$ is dense in $\mathcal{A}(\mathcal{T})$, we may represent $\psi_j$ by

$$\psi_j(t,\vec{x}) = \sum_\ell \psi_\ell^j(\vec{x})\,\mathrm{e}^{\mathrm{i}\ell t}, \tag{1.6}$$

where $\psi_\ell^j \in \mathcal{H}^1(\mathcal{V})$. Write $\cos t = (\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})/2$ and insert this and (1.6) into equation (1.4), yielding

$$\sum_\ell(\mathrm{i}\ell + \vec{u}_1\cdot\vec{\nabla} + \lambda_j\Delta)\psi_\ell^j(\vec{x})\mathrm{e}^{\mathrm{i}\ell t} + \frac{\delta}{2}\sum_\ell(\mathrm{e}^{\mathrm{i}(\ell+1)t} + \mathrm{e}^{\mathrm{i}(\ell-1)t})\,\vec{u}_2\cdot\vec{\nabla}\psi_\ell^j(\vec{x}) = 0, \tag{1.7}$$

or:

$$\sum_\ell\left[(\mathrm{i}\ell + \vec{u}_1\cdot\vec{\nabla} + \lambda_j\Delta)\psi_\ell^j(\vec{x}) + \frac{\delta}{2}\vec{u}_2\cdot\vec{\nabla}(\psi_{\ell-1}^j(\vec{x}) + \psi_{\ell+1}^j(\vec{x}))\right]\mathrm{e}^{\mathrm{i}\ell t} = 0. \tag{1.8}$$

By the completeness of the orthogonal set $\{\mathrm{e}^{\mathrm{i}\ell t}\}$ we have, for all $\ell\in\mathbb{Z}$, that

$$(\mathrm{i}\ell + \vec{u}_1\cdot\vec{\nabla})\psi_\ell^j(\vec{x}) + \frac{\delta}{2}\vec{u}_2\cdot\vec{\nabla}(\psi_{\ell-1}^j(\vec{x}) + \psi_{\ell+1}^j(\vec{x})) = -\lambda_j\Delta\psi_\ell^j(\vec{x}). \tag{1.9}$$

The system of partial differential equations in (1.9) can be reduced to a system of algebraic equations as follows. Recall that $\vec{u}_1(\vec{x}) = (\cos y, \cos x)$ and $\vec{u}_2(\vec{x}) = (\sin y, \sin x)$, which implies that

$$\begin{aligned}(\vec{u}_1\cdot\vec{\nabla})\psi_\ell^j(\vec{x}) &= \cos y\,\partial_x\psi_\ell^j(\vec{x}) + \cos x\,\partial_y\psi_\ell^j(\vec{x}) \\ (\vec{u}_2\cdot\vec{\nabla})\psi_\ell^j(\vec{x}) &= \sin y\,\partial_x\psi_\ell^j(\vec{x}) + \sin x\,\partial_y\psi_\ell^j(\vec{x})\end{aligned} \tag{1.10}$$

Since $\psi_\ell^j \in \mathcal{H}^1(\mathcal{V})$ and the orthogonal set $\{\mathrm{e}^{\mathrm{i}(mx+ny)}\}$, $m, n \in \mathbb{Z}$, is dense in this space, we can represent $\psi_\ell^j(\vec{x})$ by

$$\psi_\ell^j(\vec{x}) = \sum_{m,n} c_{\ell,m,n}^j \, \mathrm{e}^{\mathrm{i}(mx+ny)} \tag{1.11}$$

Write $\cos x = (\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x})/2$ and $\sin x = (\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x})/(2\mathrm{i})$, for example, and insert this and (1.11) into equation (1.10), yielding

$$\begin{aligned}
&(\vec{u}_1 \cdot \vec{\nabla})\psi_\ell^j \\
&\quad = \frac{1}{2}\sum_{m,n} c_{\ell,m,n}^j \left[\mathrm{i}m\,\mathrm{e}^{\mathrm{i}mx}(\mathrm{e}^{\mathrm{i}(n+1)y} + \mathrm{e}^{\mathrm{i}(n-1)y}) + \mathrm{i}n\,\mathrm{e}^{\mathrm{i}ny}(\mathrm{e}^{\mathrm{i}(m+1)x} + \mathrm{e}^{\mathrm{i}(m-1)x})\right] \\
&(\vec{u}_2 \cdot \vec{\nabla})\psi_\ell^j \\
&\quad = \frac{1}{2\mathrm{i}}\sum_{m,n} c_{\ell,m,n}^j \left[\mathrm{i}m\,\mathrm{e}^{\mathrm{i}mx}(\mathrm{e}^{\mathrm{i}(n+1)y} - \mathrm{e}^{\mathrm{i}(n-1)y}) + \mathrm{i}n\,\mathrm{e}^{\mathrm{i}ny}(\mathrm{e}^{\mathrm{i}(m+1)x} - \mathrm{e}^{\mathrm{i}(m-1)x})\right]
\end{aligned} \tag{1.12}$$

or:

$$\begin{aligned}
(\vec{u}_1 \cdot \vec{\nabla})\psi_\ell^j &= \frac{\mathrm{i}}{2}\sum_{m,n}[m(c_{\ell,m,n-1}^j + c_{\ell,m,n+1}^j) + n(c_{\ell,m-1,n}^j + c_{\ell,m+1,n}^j)]\mathrm{e}^{\mathrm{i}(mx+ny)} \\
(\vec{u}_2 \cdot \vec{\nabla})\psi_\ell^j &= \frac{1}{2}\sum_{m,n}[m(c_{\ell,m,n-1}^j - c_{\ell,m,n+1}^j) + n(c_{\ell,m-1,n}^j - c_{\ell,m+1,n}^j)]\mathrm{e}^{\mathrm{i}(mx+ny)}.
\end{aligned} \tag{1.13}$$

We also have

$$-\Delta\psi_\ell^j = \sum_{m,n} c_{\ell,m,n}^j (m^2+n^2)\mathrm{e}^{\mathrm{i}(mx+ny)} \tag{1.14}$$

By the completeness of the orthogonal set $\{\mathrm{e}^{\mathrm{i}(mx+ny)}\}$, inserting equations (1.13) and (1.14) into equation (1.9) yields

$$\begin{aligned}
&\mathrm{i}\ell + \frac{\mathrm{i}}{2}[m(c_{\ell,m,n-1}^j + c_{\ell,m,n+1}^j) + n(c_{\ell,m-1,n}^j + c_{\ell,m+1,n}^j)] \\
&\quad + \frac{\delta}{4}[m(c_{\ell-1,m,n-1}^j - c_{\ell-1,m,n+1}^j) + n(c_{\ell-1,m-1,n}^j - c_{\ell-1,m+1,n}^j \\
&\qquad + m(c_{\ell+1,m,n-1}^j - c_{\ell+1,m,n+1}^j) + n(c_{\ell+1,m-1,n}^j - c_{\ell+1,m+1,n}^j] \\
&\qquad = \lambda_j(m^2+n^2)c_{\ell,m,n}^j,
\end{aligned} \tag{1.15}$$

which is an infinite system of algebraic equations for the unknown Fourier coefficients $c_{\ell,m,n}^j$ associated with the eigenfunctions $\psi^j(t,\vec{x})$ and eigenvalues $\lambda_j$, $j \in \mathbb{N}$, $\ell, m, n \in \mathbb{Z}$.

We now discuss how the orthogonality condition $\langle \vec{\nabla}\psi_j \cdot \vec{\nabla}\psi_k \rangle = \delta_{jk}$ in (1.5) is transformed by the Fourier expansion of the eigenfunctions $\psi^j(t,\vec{x})$. This expansion implies the expansion of $\vec{\nabla}\psi^j(t,\vec{x})$ as follows

$$\psi^j(t,\vec{x}) = \sum_{\ell,m,n} c^j_{\ell,m,n} \mathrm{e}^{\mathrm{i}(\ell t+mx+ny)} \Rightarrow \vec{\nabla}\psi^j(t,\vec{x}) = \sum_{\ell,m,n} c^j_{\ell,m,n}\,(m,n)\,\mathrm{e}^{\mathrm{i}(\ell t+mx+ny)}. \tag{1.16}$$

Therefore, by the orthogonality relation

$$\left\langle \mathrm{e}^{\mathrm{i}(\ell t+mx+ny)}\,\mathrm{e}^{\mathrm{i}(\ell' t+m'x+n'y)} \right\rangle = (2\pi)^3 \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'} \tag{1.17}$$

we have that the orthogonality relation in (1.5) is transformed to

$$\delta_{jk} = \langle \vec{\nabla}\psi_j \cdot \vec{\nabla}\psi_k \rangle = (2\pi)^3 \sum_{\ell,m,n} (m^2+n^2) c^j_{\ell,m,n} c^k_{\ell,m,n} \tag{1.18}$$

Since $\vec{u}_i$ is incompressible, there exists an anti-symmetric matrix $\mathbf{H}_i$ such that $\vec{u}_i = \vec{\nabla}\cdot\mathbf{H}_i$. This allows us to write $\vec{u}_i\cdot\vec{\nabla} = \vec{\nabla}\cdot\mathbf{H}_i\vec{\nabla}$, which is an anti-symmetric operator. When $\delta = 0$, the velocity field $\vec{u}$ is time-independent and the operator $A$, which arises from the cell problem, becomes $A = \Delta^{-1}(\vec{u}_1\cdot\vec{\nabla})$. In this case, the eigenvalue problem in (1.4) becomes

$$\vec{\nabla}\cdot\mathbf{H}_1\vec{\nabla}\psi = \lambda\Delta\psi. \tag{1.19}$$

Discretizing this equation leads to a generalized eigenvalue problem involving *sparse* matrices. This matrix formulation has all the desired properties of the associated abstract Hilbert space formulation. (I will be adding the details of this to our paper soon.) From this matrix problem, we obtain a discrete approximation of the Stieltjes–Radon integral representation for the symmetric $\boldsymbol{\kappa}^*$ and anti-symmetric $\boldsymbol{\alpha}^*$ parts of the effective diffusivity tensor $\boldsymbol{\mathcal{K}}^*$, displayed in equation (35) of our (attached) paper.

# 2 Three-Dim Steady Cellular Flows

The 3 dimensional (3D) steady cellular flows are:

$$B = (\Phi_x(x,y)W'(z), \Phi_y(x,y)W'(z), k\Phi(x,y)W(z)), \tag{2.20}$$

with $-\Delta\Phi = k\Phi$.

A special case is $k = 2$, then

$$B(x,y,z) = (-\sin x\cos y\cos z, -\cos x\sin y\cos z, 2\cos x\cos y\sin z). \tag{2.21}$$

Question: Is the effective diffusivity problem easier for (2.21) than (1.3)?

Effective diffusivity in (2.20) is unknown, see [4] for related KPP problem, and [3] for an upper bound. Extrapolating the KPP scaling in [4] on the flow (2.21), effective diffusivity at small molecular diffusivity $\epsilon$ scales like $O(\epsilon^p)$, $p \approx 0.26$. In 2D, $p = 1/2$, see [2], also Fig.3 in [1].

## References


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