Decomposing parabolic eigenvalue problem

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Abstract

A decomposition of a time periodic parabolic eigenvalue problem and an application to our effective diffusivity problem.

1 An Example

Consider the parabolic eigenvalue problem:

$$\psi_t - \Delta_x \psi + \cos(t)\, b(x)\, \psi = \lambda\, \psi, \tag{1.1}$$

subject to 2π periodic boundary conditions in x and t.

Substituting $\cos t = (e^{it} + e^{-it})/2$, and $\psi = \sum_\ell \psi_\ell(x) e^{i\ell t}$ in (1.1), we have:

$$\sum_\ell e^{i\ell t}\,(i\ell\,\psi_\ell\, - \Delta_x \psi_\ell) + \frac{1}{2} b(x)\, \sum_\ell (e^{i(\ell+1)t} + e^{i(\ell-1)t})\psi_\ell = \lambda \sum_\ell \psi_\ell e^{i\ell t},$$

or:

$$\sum_\ell e^{i\ell t}\,(i\ell\,\psi_\ell\, - \Delta_x \psi_\ell) + \frac{1}{2} b(x)\, \sum_\ell (\psi_{\ell-1} + \psi_{\ell+1})\, e^{i\ell t} = \lambda \sum_\ell \psi_\ell e^{i\ell t}.$$

Extracting the ℓ-th mode on both sides gives:

$$(-\Delta_x + i\,\ell)\,\psi_\ell + \frac{1}{2}\, b(x)\,(\psi_{\ell-1} + \psi_{\ell+1}) = \lambda\,\psi_\ell, \quad \ell \in Z, \tag{1.2}$$

which can be put in a tri-diagonal matrix acting on $[\cdots, \psi_{\ell-1}, \psi_\ell, \psi_{\ell+1}, \cdots]'$, with $b(x)/2$ on the off-diagonals, $-\Delta_x + i\,\ell$ on the diagonals.

Let us find a similar derivation for the eigenvalue problem of the advection-diffusion operator, with advection field being the time periodic cell flow:

$$\vec{u}(t, \vec{x}) = (\cos y, \cos x) + \delta \cos t\,(\sin y, \sin x) := \vec{u}_1(\vec{x}) + \delta \cos t\, \vec{u}_2(\vec{x}). \tag{1.3}$$

Question: Is it possible to carry out a similar decomposition in x and y and fully reduce the differential system like (1.2) into an algebraic system ?

1.1 An Application to our effective diffusivity problem

Consider the eigenvalue problem $A\psi_l = \imath\lambda_l\psi_l$, $\imath = \sqrt{-1}$, $\lambda_l \in \mathbb{R}$, $l = 1, 2, 3, \ldots$, involving the integro-differential operator $A = (-\Delta)^{-1}(\partial_t + \vec{u}\cdot\vec{\nabla})$, introduced in equation (45) of our (attached) effective-diffusivity paper, with $\vec{u} \mapsto -\vec{u}$. Here A is an anti-symmetric (normal) operator and the incompressible velocity field $\vec{u}(t, \vec{x})$ is given in equation (1.3) above. The equation $A\psi_l = \imath\lambda_l\psi_l$ may be rewritten as

$$(\partial_t + \vec{u}\cdot\vec{\nabla})\psi_l = -\imath\lambda_l\Delta\psi_l. \tag{1.4}$$

The eigenfunctions ψ_l satisfy the following orthogonality condition

$$\langle\psi_l, \psi_i\rangle_1 = \langle\vec{\nabla}\psi_l \cdot \vec{\nabla}\psi_i\rangle = \delta_{li}, \tag{1.5}$$

where δ_{li} is the Kronecker delta and $\langle\cdot\rangle$ denotes space-time averaging over the period cell $\mathcal{T} \times \mathcal{V}$, with $\mathcal{T} = [0, 2\pi]$ and $\mathcal{V} = [0, 2\pi] \times [0, 2\pi]$.

The eigenfunction ψ_l is $\mathcal{T} \times \mathcal{V}$ periodic, mean-zero, and $\psi_l \in \mathcal{A}(\mathcal{T}) \otimes \mathcal{H}^1(\mathcal{V})$, i.e. it is absolutely continuous in time for $t \in \mathcal{T}$, and is in the Sobolev space $\mathcal{H}^1(\mathcal{V})$ for $\vec{x} \in \mathcal{V}$. We denote the class of such functions by $\mathcal{F}$

$$\mathcal{F} = \{f \in \mathcal{A}(\mathcal{T}) \otimes \mathcal{H}^1(\mathcal{V}) \,|\, \langle f\rangle = 0 \text{ and is periodic on } \mathcal{T} \times \mathcal{V}\}. \tag{1.6}$$

Since the orthogonal set $\{\mathrm{e}^{\imath\ell t}\}$, $\ell \in \mathbb{Z}$, is dense in $\mathcal{A}(\mathcal{T})$, we may represent ψ_l by

$$\psi_l(t, \vec{x}) = \sum_\ell \psi_\ell^l(\vec{x})\,\mathrm{e}^{\imath\ell t}, \tag{1.7}$$

where $\psi_\ell^l \in \mathcal{H}^1(\mathcal{V})$. Write $\cos t = (\mathrm{e}^{\imath t} + \mathrm{e}^{-\imath t})/2$ and insert this and (1.7) into equation (1.4), yielding

$$\sum_\ell (\imath\ell + \vec{u}_1\cdot\vec{\nabla} + \imath\lambda_l\Delta)\psi_\ell^l(\vec{x})\mathrm{e}^{\imath\ell t} + \frac{\delta}{2}\sum_\ell (\mathrm{e}^{\imath(\ell+1)t} + \mathrm{e}^{\imath(\ell-1)t})\,\vec{u}_2\cdot\vec{\nabla}\psi_\ell^l(\vec{x}) = 0, \tag{1.8}$$

or:

$$\sum_\ell \left[(\imath\ell + \vec{u}_1\cdot\vec{\nabla} + \imath\lambda_l\Delta)\psi_\ell^l(\vec{x}) + \frac{\delta}{2}\vec{u}_2\cdot\vec{\nabla}(\psi_{\ell-1}^l(\vec{x}) + \psi_{\ell+1}^l(\vec{x}))\right]\mathrm{e}^{\imath\ell t} = 0. \tag{1.9}$$

By the completeness in $L^2(\mathcal{T})$ of the orthogonal set $\{\mathrm{e}^{\imath\ell t}\}$ we have, for all $\ell \in \mathbb{Z}$,

$$(\imath\ell + \vec{u}_1\cdot\vec{\nabla})\psi_\ell^l(\vec{x}) + \frac{\delta}{2}\vec{u}_2\cdot\vec{\nabla}(\psi_{\ell-1}^l(\vec{x}) + \psi_{\ell+1}^l(\vec{x})) = -\imath\lambda_l\Delta\psi_\ell^l(\vec{x}). \tag{1.10}$$

The system of partial differential equations in (1.10) can be reduced to a system of algebraic equations as follows. Recall that $\vec{u}_1(\vec{x}) = (\cos y, \cos x)$ and $\vec{u}_2(\vec{x}) = (\sin y, \sin x)$, which implies that

$$\begin{aligned}(\vec{u}_1\cdot\vec{\nabla})\psi_\ell^l(\vec{x}) &= \cos y\,\partial_x\psi_\ell^l(\vec{x}) + \cos x\,\partial_y\psi_\ell^l(\vec{x}) \\ (\vec{u}_2\cdot\vec{\nabla})\psi_\ell^l(\vec{x}) &= \sin y\,\partial_x\psi_\ell^l(\vec{x}) + \sin x\,\partial_y\psi_\ell^l(\vec{x})\end{aligned} \tag{1.11}$$

Since $\psi_\ell^l \in \mathcal{H}^1(\mathcal{V})$ and the orthogonal set $\{\mathrm{e}^{\imath(mx+ny)}\}$, $m, n \in \mathbb{Z}$, is dense in this space, we can represent $\psi_\ell^l(\vec{x})$ by

$$\psi_\ell^l(\vec{x}) = \sum_{m,n} a_{\ell,m,n}^l \, \mathrm{e}^{\imath(mx+ny)} \tag{1.12}$$

Write $\cos x = (\mathrm{e}^{\imath x} + \mathrm{e}^{-\imath x})/2$ and $\sin x = (\mathrm{e}^{\imath x} - \mathrm{e}^{-\imath x})/(2\imath)$, for example, and insert this and (1.12) into equation (1.11), yielding

$$\begin{aligned}
&(\vec{u}_1 \cdot \vec{\nabla})\psi_\ell^l \qquad\qquad (1.13)\\
&\quad = \frac{1}{2} \sum_{m,n} a_{\ell,m,n}^l \left[\imath m \, \mathrm{e}^{\imath mx}(\mathrm{e}^{\imath(n+1)y} + \mathrm{e}^{\imath(n-1)y}) + \imath n \, \mathrm{e}^{\imath ny}(\mathrm{e}^{\imath(m+1)x} + \mathrm{e}^{\imath(m-1)x})\right]\\
&(\vec{u}_2 \cdot \vec{\nabla})\psi_\ell^l\\
&\quad = \frac{1}{2\imath} \sum_{m,n} a_{\ell,m,n}^l \left[\imath m \, \mathrm{e}^{\imath mx}(\mathrm{e}^{\imath(n+1)y} - \mathrm{e}^{\imath(n-1)y}) + \imath n \, \mathrm{e}^{\imath ny}(\mathrm{e}^{\imath(m+1)x} - \mathrm{e}^{\imath(m-1)x})\right]
\end{aligned}$$

or:

$$\begin{aligned}
(\vec{u}_1 \cdot \vec{\nabla})\psi_\ell^l &= \frac{\imath}{2} \sum_{m,n} [m(a_{\ell,m,n-1}^l + a_{\ell,m,n+1}^l) + n(a_{\ell,m-1,n}^l + a_{\ell,m+1,n}^l)] \mathrm{e}^{\imath(mx+ny)}\\
(\vec{u}_2 \cdot \vec{\nabla})\psi_\ell^l &= \frac{1}{2} \sum_{m,n} [m(a_{\ell,m,n-1}^l - a_{\ell,m,n+1}^l) + n(a_{\ell,m-1,n}^l - a_{\ell,m+1,n}^l)] \mathrm{e}^{\imath(mx+ny)}.
\end{aligned} \tag{1.14}$$

We also have

$$-\Delta\psi_\ell^l = \sum_{m,n} a_{\ell,m,n}^l (m^2 + n^2) \mathrm{e}^{\imath(mx+ny)} \tag{1.15}$$

By the completeness of the orthogonal set $\{\mathrm{e}^{\imath(mx+ny)}\}$, inserting equations (1.14) and (1.15) into equation (1.10) yields

$$\begin{aligned}
\imath\ell \, a_{\ell,m,n}^l &+ \frac{\imath}{2}[m(a_{\ell,m,n-1}^l + a_{\ell,m,n+1}^l) + n(a_{\ell,m-1,n}^l + a_{\ell,m+1,n}^l)]\\
&\quad + \frac{\delta}{4}[m(a_{\ell-1,m,n-1}^l - a_{\ell-1,m,n+1}^l) + n(a_{\ell-1,m-1,n}^l - a_{\ell-1,m+1,n}^l)\\
&\qquad + m(a_{\ell+1,m,n-1}^l - a_{\ell+1,m,n+1}^l) + n(a_{\ell+1,m-1,n}^l - a_{\ell+1,m+1,n}^l)]\\
&= \imath\lambda_l (m^2 + n^2) a_{\ell,m,n}^l,
\end{aligned} \tag{1.16}$$

which is an infinite system of algebraic equations for the unknown Fourier coefficients $a_{\ell,m,n}^l$ associated with the eigenfunctions $\psi_l(t, \vec{x})$ and eigenvalues $\imath\lambda_l$, $l \in \mathbb{N}$ and $\ell, m, n \in \mathbb{Z}$. Recalling that ψ_l is mean-zero $\langle\psi_l\rangle = 0$, we have that $\ell^2 + m^2 + n^2 > 0$.

We now discuss how the orthogonality condition $\langle\psi_l, \psi_i\rangle_1 = \delta_{li}$ in (1.5) is transformed by the Fourier expansion of the eigenfunctions $\psi_l(t, \vec{x})$. This expansion of $\psi_l(t,\vec{x})$ implies that for $\vec{\nabla}\psi_l(t,\vec{x})$ as follows

$$\psi_l(t,\vec{x}) = \sum_{\ell,m,n} a^{\,l}_{\ell,m,n} \mathrm{e}^{\imath(\ell t+mx+ny)} \Rightarrow \vec{\nabla}\psi_l(t,\vec{x}) = \sum_{\ell,m,n} a^{\,l}_{\ell,m,n}\,(m,n)\,\mathrm{e}^{\imath(\ell t+mx+ny)}. \tag{1.17}$$

Therefore, by the orthogonality relation

$$\left\langle \mathrm{e}^{\imath(\ell t+mx+ny)},\, \mathrm{e}^{\imath(\ell' t+m'x+n'y)} \right\rangle_2 = (2\pi)^3 \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'}, \tag{1.18}$$

where $\langle\cdot,\cdot\rangle_2$ denotes the $L^2(\mathcal{T}\times\mathcal{V})$ inner-product, we have that the orthogonality relation in (1.5) is transformed to

$$\delta_{li} = \langle \vec{\nabla}\psi_l \cdot \vec{\nabla}\psi_i \rangle = (2\pi)^3 \sum_{m,n} (m^2+n^2)\overline{a^{\,l}_{0,m,n}} c^{\,i}_{0,m,n} \tag{1.19}$$

Recall that $\sum_i i^{-p}$ converges for all $p>1$. Consequently, from equation (1.19) we see that the square modulus of the Fourier coefficients $a^{\,l}_{0,m,n}$ must have the asymptotic behavior $|a^{\,l}_{0,m,n}|^2 \sim o((m^2+n^2)^{-3/2})$ as $m,n\to\pm\infty$. Since $\psi_l(\cdot,\vec{x})\in\mathcal{A}(\mathcal{T})$, i.e. $\partial_t\psi_l(\cdot,\vec{x})\in L^2(\mathcal{T})$, we also have $|a^{\,l}_{\ell,m,n}|^2\sim o(\ell^{-3})$ as $\ell\to\pm\infty$. Since $\partial_t\vec{\nabla}\psi_l \in L^2(\mathcal{T}\times\mathcal{V})$ we may generalize both of these statements by the following

$$|a^{\,l}_{\ell,m,n}|^2 \sim o(\ell^{-3}(m^2+n^2)^{-3/2}), \text{ as } \ell,m,n\to\pm\infty. \tag{1.20}$$

We now use the special nature of the velocity field in (1.3) and the Fourier expansion of the eigenfunctions in (1.17) to show that the Radon–Stieltjes integral representation for the symmetric $\boldsymbol{\kappa}^*$ and anti-symmetric $\boldsymbol{\alpha}^*$ parts of the effective diffusivity tensor $\boldsymbol{\mathcal{K}}^*$ depend only on the Fourier coefficients $a^{\,l}_{\ell,m,n}$ for $\ell,m,n\in\{-1,0,1\}$. Recall that the Hilbert space $\mathcal{H}$ underlying this problem is

$$\mathcal{H} = \{f\in L^2(\mathcal{T})\otimes\mathcal{H}^1(\mathcal{V}) \,|\, \langle f\rangle = 0 \text{ and is periodic on } \mathcal{T}\times\mathcal{V}\}, \tag{1.21}$$

with A maximal normal, i.e. $\imath A$ self-adjoint, on $\mathcal{F}\subset\mathcal{H}$ defined in (1.6). Since the eigenfunctions ψ_l and ψ_i of A associated with distinct eigenvalues, $\lambda_l\neq\lambda_i$, are orthonormal, $\langle\psi_l,\psi_i\rangle_1 = \langle\vec{\nabla}\psi_l\cdot\vec{\nabla}\psi_i\rangle = \delta_{li}$, the span of all eigenfunctions constitutes a closed linear manifold $\mathcal{E}$ in $\mathcal{F}$. Let $\mathcal{E}^\perp$ be its (closed) orthogonal complement in $\mathcal{H}$ so that [Stone]

$$\mathcal{H} = \mathcal{E}\oplus\mathcal{E}^\perp. \tag{1.22}$$

Recall the cell problem

$$(\varepsilon - A)\chi_j = g_j, \quad g_j = -\Delta^{-1}u_j, \tag{1.23}$$

where u_j is the j^{th} component of the velocity field $\vec{u}$. Since $\chi_j, g_j\in\mathcal{F}$ and $\mathcal{F}\subset\mathcal{E}\oplus\mathcal{E}^\perp$ they have the following representations

$$\chi_j = \sum_l \langle\psi_l,\chi_j\rangle_1\,\psi_l + \chi_j^\perp, \quad g_j = \sum_l \langle\psi_l,g_j\rangle_1\,\psi_l + g_j^\perp, \tag{1.24}$$

where $\psi_l \in \mathcal{E}$ and $\chi_j^\perp, g_j^\perp \in \mathcal{E}^\perp$. Using $A\psi_l = \imath\lambda_l\psi_l$ and the orthonormality of the set $\{\psi_l\}$, inserting (1.24) into the cell problem (1.23) yields

$$\sum_l [(\varepsilon - \imath\lambda_l)\langle\psi_l, \chi_j\rangle_1 - \langle\psi_l, g_j\rangle_1]\psi_l + (\varepsilon + A)\chi_j^\perp - g_j^\perp = 0. \tag{1.25}$$

By the orthonormality of the set $\{\psi_l\}$ and since $\langle A\chi_j^\perp, \psi_l\rangle_1 = -\langle\chi_j^\perp, A\psi_l\rangle_1 = -\imath\lambda_l\langle\chi_j^\perp, \psi_l\rangle_1 = 0$, taking the inner-product of both sides of (1.25) with ψ_l yields

$$\langle\psi_l, \chi_j\rangle_1 = \frac{\langle\psi_l, g_j\rangle_1}{(\varepsilon - \imath\lambda_l)}\,. \tag{1.26}$$

Recall that the components κ_{jk}^* and α_{jk}^* of $\boldsymbol{\kappa}^*$ and $\boldsymbol{\alpha}^*$ are given by

$$\kappa_{jk}^* = \varepsilon(\delta_{jk} + \langle\chi_j, \chi_k\rangle_1), \quad \alpha_{jk}^* = \langle A\chi_j, \chi_k\rangle_1. \tag{1.27}$$

From equations (1.24) and (1.26) and the orthonormality of the set $\{\psi_l\}$, we have

$$\begin{aligned}\langle\chi_j, \chi_k\rangle_1 - \langle\chi_j^\perp, \chi_k^\perp\rangle_1 &= \sum_l \overline{\langle\psi_l, \chi_j\rangle}_1\langle\psi_l, \chi_k\rangle_1 = \sum_l \frac{\overline{\langle\psi_l, g_j\rangle}_1\langle\psi_l, g_k\rangle_1}{\varepsilon^2 + \lambda_l^2}\\ \langle A\chi_j, \chi_k\rangle_1 - \langle A\chi_j^\perp, \chi_k^\perp\rangle_1 &= \sum_l (-\imath\lambda_l)\overline{\langle\psi_l, \chi_j\rangle}_1\langle\psi_l, \chi_k\rangle_1 = \sum_l \frac{\overline{\langle\psi_l, g_j\rangle}_1\langle\psi_l, g_k\rangle_1}{\varepsilon^2 + \lambda_l^2}\end{aligned} \tag{1.28}$$

The right hand sides of the formulas in equation (1.28) are Radon–Stieltjes integrals associated with a *discrete* measure. The terms $\langle\chi_j^\perp, \chi_k^\perp\rangle_1$ and $\langle A\chi_j^\perp, \chi_k^\perp\rangle_1$ also have Radon–Stieltjes integral representations with respect to *continuous* measures, and provides the standard decomposition of the *spectral measure* into its discrete and continuous components, in the general setting [Stone].

We now show that the special nature of the velocity field in (1.3) and the Fourier expansion of the eigenfunctions ψ_l in (1.17) allow the spectral weights $\langle\psi_l, g_j\rangle$ in equation (1.28) to be given in terms of the Fourier coefficients $a_{\ell,m,n}^l$ for the reduced index set $\ell, m, n \in \{-1, 0, 1\}$. First note that, since $u_j(t, \cdot) \in \mathcal{H}^1(\mathcal{V}) \subset L^2(\mathcal{V})$,

$$\langle\psi_l, g_j\rangle_1 = \langle\vec{\nabla}\psi_l \cdot \vec{\nabla}(-\Delta)^{-1}u_j\rangle = \langle\psi_l, (-\Delta)(-\Delta)^{-1}u_j\rangle_2 = \langle\psi_l, u_j\rangle_2. \tag{1.29}$$

Writing $\cos x = (\mathrm{e}^{\imath x} + \mathrm{e}^{-\imath x})/2$ and $\sin x = (\mathrm{e}^{\imath x} - \mathrm{e}^{-\imath x})/(2\imath)$, for example, from equation (1.3) we have that

$$u_1(t, x, y) = \cos y + \cos t \sin y \tag{1.30}$$

$$\begin{aligned}&= \frac{1}{2}\left(\mathrm{e}^{\imath y} + \mathrm{e}^{-\imath y}\right) + \frac{1}{4\imath}\left(\mathrm{e}^{\imath t} + \mathrm{e}^{-\imath t})(\mathrm{e}^{\imath y} - \mathrm{e}^{-\imath y}\right)\\ &= \frac{1}{2}\left(\mathrm{e}^{\imath y} + \mathrm{e}^{-\imath y}\right) + \frac{1}{4\imath}\left(\mathrm{e}^{\imath(t+y)} - \mathrm{e}^{\imath(t-y)} + \mathrm{e}^{\imath(-t+y)} - \mathrm{e}^{\imath(-t-y)}\right),\end{aligned} \tag{1.31}$$

and $u_2(t,x,y) = u_1(t,y,x)$. This, equation (1.29), and the orthogonality relation in (1.18) imply that

$$\langle \psi_l, g_1 \rangle_1 = (2\pi)^3 \left[\frac{1}{2} \left(a^l_{0,0,1} + a^l_{0,0,-1} \right) + \frac{1}{4\imath} \left(a^l_{1,0,1} - a^l_{1,0,-1} + a^l_{-1,0,1} - a^l_{-1,0,-1} \right) \right]$$
$$\langle \psi_l, g_2 \rangle_1 = (2\pi)^3 \left[\frac{1}{2} \left(a^l_{0,1,0} + a^l_{0,-1,0} \right) + \frac{1}{4\imath} \left(a^l_{1,1,0} - a^l_{1,-1,0} + a^l_{-1,1,0} - a^l_{-1,-1,0} \right) \right] \tag{1.32}$$

Since $\vec{u}_i$ is incompressible, there exists an anti-symmetric matrix $\mathbf{H}_i$ such that $\vec{u}_i = \vec{\nabla} \cdot \mathbf{H}_i$. This allows us to write $\vec{u}_i \cdot \vec{\nabla} = \vec{\nabla} \cdot \mathbf{H}_i \vec{\nabla}$, which is an anti-symmetric operator. When $\delta = 0$, the velocity field $\vec{u}$ is time-independent and the operator A, which arises from the cell problem, becomes $A = \Delta^{-1}(\vec{u}_1 \cdot \vec{\nabla})$. In this case, the eigenvalue problem in (1.4) becomes

$$\vec{\nabla} \cdot \mathbf{H}_1 \vec{\nabla} \psi = \lambda \Delta \psi. \tag{1.33}$$

Discretizing this equation leads to a generalized eigenvalue problem involving *sparse* matrices. This matrix formulation has all the desired properties of the associated abstract Hilbert space formulation. (I will be adding the details of this to our paper soon.) From this matrix problem, we obtain a discrete approximation of the Radon–Stieltjes integral representation for the symmetric $\boldsymbol{\kappa}^*$ and anti-symmetric $\boldsymbol{\alpha}^*$ parts of the effective diffusivity tensor $\boldsymbol{\mathcal{K}}^*$, displayed in equation (35) of our (attached) paper.

1.1.1 Matrix representation in the time-independent case

In the time independent case, where $\delta = 0$ in the velocity field of equation (1.3), the system of equations in (1.16) reduces to

$$m(a^l_{m,n-1} + a^l_{m,n+1}) + n(a^l_{m-1,n} + a^l_{m+1,n}) = 2\lambda_l(m^2 + n^2)a^l_{m,n} \tag{1.34}$$

for $m, n \in \mathbb{Z}$. For simplicity, we drop the super-script and sub-script and write $a_{m,n} = a^l_{m,n}$ and $\lambda = \lambda_l$. We will truncate the system by restricting the indices $-M \le m, n \le M$ so that equation (1.34) can be written in matrix form

$$B\vec{a} = 2\lambda C\vec{a} \tag{1.35}$$

where

$$\vec{a} = (\vec{a}_{-M}, \ldots, \vec{a}_{-1}, \vec{a}_0, \vec{a}_1, \ldots, \vec{a}_M) \tag{1.36}$$
$$\vec{a}_j = (a_{-M,j}, \ldots, a_{-1,j}, a_{0,j}, a_{1,j}, \ldots, a_{M,j}), \tag{1.37}$$

and B and C are $(2M+1)^2 \times (2M+1)^2$ matrices. To see the structure of these matrices for arbitrary M, we first set $M = 1$ and write out all 9 equations and the corresponding matrices B and C. We then set $M = 2$ and write out all 25 equations and the corresponding matrices B and C.

For $M = 1$, $n = -1$, and $-1 \leq m \leq 1$

$$
\begin{aligned}
1: &\ -1(a_{-1,-2} + a_{-1,+0}) - 1(a_{-2,-1} + a_{+0,-1}) = 2\lambda(1^2 + 1^2)a_{-1,-1} \\
2: &\ +0(a_{+0,-2} + a_{+0,+0}) - 1(a_{-1,-1} + a_{+1,-1}) = 2\lambda(0^2 + 1^2)a_{+0,-1} \\
3: &\ +1(a_{+1,-2} + a_{+1,+0}) - 1(a_{+0,-1} + a_{+2,-1}) = 2\lambda(1^2 + 1^2)a_{+1,-1}
\end{aligned}
$$

For $n = +0$ and $-1 \leq m \leq 1$

$$
\begin{aligned}
4: &\ -1(a_{-1,-1} + a_{-1,+1}) - 0(a_{-2,+0} + a_{+0,+0}) = 2\lambda(1^2 + 0^2)a_{-1,+0} \\
5: &\ +0(a_{+0,-1} + a_{+0,+1}) - 0(a_{-1,+0} + a_{+1,+0}) = 2\lambda(0^2 + 0^2)a_{+0,+0} \\
6: &\ +1(a_{+1,-1} + a_{+1,+1}) - 0(a_{+0,+0} + a_{+2,+0}) = 2\lambda(1^2 + 0^2)a_{+1,+0}
\end{aligned}
$$

For $n = +1$ and $-1 \leq m \leq 1$

$$
\begin{aligned}
7: &\ -1(a_{-1,+0} + a_{-1,+2}) + 1(a_{-2,+1} + a_{+0,+1}) = 2\lambda(1^2 + 1^2)a_{-1,+1} \\
8: &\ +0(a_{+0,+0} + a_{+0,+2}) + 1(a_{-1,+1} + a_{+1,+1}) = 2\lambda(0^2 + 1^2)a_{+0,+1} \\
9: &\ +1(a_{+1,+0} + a_{+1,+2}) + 1(a_{+0,+1} + a_{+2,+1}) = 2\lambda(1^2 + 1^2)a_{+1,+1}
\end{aligned}
$$

From these equations we see that

$$
B\vec{a} = \begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
+0 & -\mathbf{1} & +0 & -\mathbf{1} & +0 & +0 & +0 & +0 & +0 \\
-\mathbf{1} & +0 & -\mathbf{1} & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 \\
+0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 \\
-\mathbf{1} & +0 & +0 & +0 & +\mathbf{0} & +0 & -\mathbf{1} & +0 & +0 \\
+0 & +\mathbf{0} & +0 & +\mathbf{0} & +0 & +\mathbf{0} & +0 & +\mathbf{0} & +0 \\
+0 & +0 & +\mathbf{1} & +0 & +\mathbf{0} & +0 & +0 & +0 & +\mathbf{1} \\
+0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{1} & +0 \\
+0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +\mathbf{1} & +0 & +\mathbf{1} \\
+0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +\mathbf{1} & +0
\end{bmatrix}
\end{array}
\begin{array}{c}
a \\
\begin{bmatrix}
a_{-1,-1} \\ a_{+0,-1} \\ a_{+1,-1} \\ a_{-1,+0} \\ a_{+0,+0} \\ a_{+1,+0} \\ a_{-1,+1} \\ a_{+0,+1} \\ a_{+1,+1}
\end{bmatrix}
\end{array}
\begin{array}{c}
a \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9
\end{array}
$$

$$
C\vec{a} = \begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
+\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2}
\end{bmatrix}
\end{array}
\begin{array}{c}
a \\
\begin{bmatrix}
a_{-1,-1} \\ a_{+0,-1} \\ a_{+1,-1} \\ a_{-1,+0} \\ a_{+0,+0} \\ a_{+1,+0} \\ a_{-1,+1} \\ a_{+0,+1} \\ a_{+1,+1}
\end{bmatrix}
\end{array}
\begin{array}{c}
a \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9
\end{array}
$$

For $M = 2$, $n = -2$ and $-2 \leq m \leq 2$

$$\begin{aligned}
1: \; & -2(a_{-2,-3} + a_{-2,-1}) - 2(a_{-3,-2} + a_{-1,-2}) = 2\lambda(2^2 + 2^2)a_{-2,-2} \\
2: \; & -1(a_{-1,-3} + a_{-1,-1}) - 2(a_{-2,-2} + a_{+0,-2}) = 2\lambda(1^2 + 2^2)a_{-1,-2} \\
3: \; & +0(a_{+0,-3} + a_{+0,-1}) - 2(a_{-1,-2} + a_{+1,-2}) = 2\lambda(0^2 + 2^2)a_{+0,-2} \\
4: \; & +1(a_{+1,-3} + a_{+1,-1}) - 2(a_{+0,-2} + a_{+2,-2}) = 2\lambda(1^2 + 2^2)a_{+1,-2} \\
5: \; & +2(a_{+2,-3} + a_{+2,-1}) - 2(a_{+1,-2} + a_{+3,-2}) = 2\lambda(2^2 + 2^2)a_{+2,-2}
\end{aligned}$$

For $n = -1$ and $-2 \leq m \leq 2$

$$\begin{aligned}
6: \; & -2(a_{-2,-2} + a_{-2,+0}) - 1(a_{-3,-1} + a_{-1,-1}) = 2\lambda(2^2 + 1^2)a_{-2,-1} \\
7: \; & -1(a_{-1,-2} + a_{-1,+0}) - 1(a_{-2,-1} + a_{+0,-1}) = 2\lambda(1^2 + 1^2)a_{-1,-1} \\
8: \; & +0(a_{+0,-2} + a_{+0,+0}) - 1(a_{-1,-1} + a_{+1,-1}) = 2\lambda(0^2 + 1^2)a_{+0,-1} \\
9: \; & +1(a_{+1,-2} + a_{+1,+0}) - 1(a_{+0,-1} + a_{+2,-1}) = 2\lambda(1^2 + 1^2)a_{+1,-1} \\
10: \; & +2(a_{+2,-2} + a_{+2,+0}) - 1(a_{+1,-1} + a_{+3,-1}) = 2\lambda(2^2 + 1^2)a_{+2,-1}
\end{aligned}$$

For $n = +0$ and $-2 \leq m \leq 2$

$$\begin{aligned}
11: \; & -2(a_{-2,-1} + a_{-2,+1}) - 0(a_{-3,+0} + a_{-1,+0}) = 2\lambda(2^2 + 0^2)a_{-2,+0} \\
12: \; & -1(a_{-1,-1} + a_{-1,+1}) - 0(a_{-2,+0} + a_{+0,+0}) = 2\lambda(1^2 + 0^2)a_{-1,+0} \\
13: \; & +0(a_{+0,-1} + a_{+0,+1}) - 0(a_{-1,+0} + a_{+1,+0}) = 2\lambda(0^2 + 0^2)a_{+0,+0} \\
14: \; & +1(a_{+1,-1} + a_{+1,+1}) - 0(a_{+0,+0} + a_{+2,+0}) = 2\lambda(1^2 + 0^2)a_{+1,+0} \\
15: \; & +2(a_{+2,-1} + a_{+2,+1}) - 0(a_{+1,+0} + a_{+3,+0}) = 2\lambda(2^2 + 0^2)a_{+2,+0}
\end{aligned}$$

For $n = +1$ and $-2 \leq m \leq 2$

$$\begin{aligned}
16: \; & -2(a_{-2,+0} + a_{-2,+2}) + 1(a_{-3,+1} + a_{-1,+1}) = 2\lambda(2^2 + 1^2)a_{-2,+1} \\
17: \; & -1(a_{-1,+0} + a_{-1,+2}) + 1(a_{-2,+1} + a_{+0,+1}) = 2\lambda(1^2 + 1^2)a_{-1,+1} \\
18: \; & +0(a_{+0,+0} + a_{+0,+2}) + 1(a_{-1,+1} + a_{+1,+1}) = 2\lambda(0^2 + 1^2)a_{+0,+1} \\
19: \; & +1(a_{+1,+0} + a_{+1,+2}) + 1(a_{+0,+1} + a_{+2,+1}) = 2\lambda(1^2 + 1^2)a_{+1,+1} \\
20: \; & +2(a_{+2,+0} + a_{+2,+2}) + 1(a_{+1,+1} + a_{+3,+1}) = 2\lambda(2^2 + 1^2)a_{+2,+1}
\end{aligned}$$

For $n = +2$ and $-2 \leq m \leq 2$

$$\begin{aligned}
21: \; & -2(a_{-2,+1} + a_{-2,+3}) + 2(a_{-3,+2} + a_{-1,+2}) = 2\lambda(2^2 + 2^2)a_{-2,+2} \\
22: \; & -1(a_{-1,+1} + a_{-1,+3}) + 2(a_{-2,+2} + a_{+0,+2}) = 2\lambda(1^2 + 2^2)a_{-1,+2} \\
23: \; & +0(a_{+0,+1} + a_{+0,+3}) + 2(a_{-1,+2} + a_{+1,+2}) = 2\lambda(0^2 + 2^2)a_{+0,+2} \\
24: \; & +1(a_{+1,+1} + a_{+1,+3}) + 2(a_{+0,+2} + a_{+2,+2}) = 2\lambda(1^2 + 2^2)a_{+1,+2} \\
25: \; & +2(a_{+2,+1} + a_{+2,+3}) + 2(a_{+1,+2} + a_{+3,+2}) = 2\lambda(2^2 + 2^2)a_{+2,+2}
\end{aligned}$$

From these equations we see that

$$
B\vec{a} =
\begin{array}{c}
\begin{array}{ccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25
\end{array} \\
\begin{bmatrix}
+0 & -\mathbf{2} & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
-\mathbf{2} & +0 & -\mathbf{2} & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & -\mathbf{2} & +0 & -\mathbf{2} & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & -\mathbf{2} & +0 & -\mathbf{2} & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
-\mathbf{2} & +0 & +0 & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & -\mathbf{1} & +0 & +0 & +0 & -\mathbf{1} & +0 & -\mathbf{1} & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +\mathbf{0} & +0 & +0 & +0 & -\mathbf{1} & +0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & -\mathbf{1} & +0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{0} & +0 & +\mathbf{0} & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +\mathbf{0} & +0 & +\mathbf{0} & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +\mathbf{0} & +0 & +\mathbf{0} & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{1} & +0 & +\mathbf{1} & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +\mathbf{1} & +0 & +\mathbf{1} & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +\mathbf{1} & +0 & +\mathbf{1} & +0 & +0 & +0 & +\mathbf{1} & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & -\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & -\mathbf{1} & +0 & +0 & +0 & +\mathbf{2} & +0 & +\mathbf{2} & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +\mathbf{2} & +0 & +\mathbf{2} & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +\mathbf{2} & +0 & +\mathbf{2} \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +\mathbf{2} & +0
\end{bmatrix}
\end{array}
\begin{array}{c}
a \\
\begin{bmatrix}
a_{-2,-2} \\ a_{-1,-2} \\ a_{+0,-2} \\ a_{+1,-2} \\ a_{+2,-2} \\
a_{-2,-1} \\ a_{-1,-1} \\ a_{+0,-1} \\ a_{+1,-1} \\ a_{+2,-1} \\
a_{-2,+0} \\ a_{-1,+0} \\ a_{+0,+0} \\ a_{+1,+0} \\ a_{+2,+0} \\
a_{-2,+1} \\ a_{-1,+1} \\ a_{+0,+1} \\ a_{+1,+1} \\ a_{+2,+1} \\
a_{-2,+2} \\ a_{-1,+2} \\ a_{+0,+2} \\ a_{+1,+2} \\ a_{+2,+2}
\end{bmatrix}
\end{array}
\begin{array}{c}
a \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 25
\end{array}
$$

$$C\vec{a} = \begin{array}{ccc}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \end{matrix} & a & a \\
\left[\begin{matrix}
+\mathbf{8} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +\mathbf{5} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +\mathbf{4} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +\mathbf{5} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +\mathbf{8} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +\mathbf{5} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{5} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{4} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{0} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{4} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{5} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{1} & +0 & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{2} & +0 & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{5} & +0 & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{8} & +0 & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{5} & +0 & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{4} & +0 & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{5} & +0 \\
+0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +\mathbf{8}
\end{matrix}\right] &
\left[\begin{matrix} a_{-2,-2} \\ a_{-1,-2} \\ a_{+0,-2} \\ a_{+1,-2} \\ a_{+2,-2} \\ a_{-2,-1} \\ a_{-1,-1} \\ a_{+0,-1} \\ a_{+1,-1} \\ a_{+2,-1} \\ a_{-2,+0} \\ a_{-1,+0} \\ a_{+0,+0} \\ a_{+1,+0} \\ a_{+2,+0} \\ a_{-2,+1} \\ a_{-1,+1} \\ a_{+0,+1} \\ a_{+1,+1} \\ a_{+2,+1} \\ a_{-2,+2} \\ a_{-1,+2} \\ a_{+0,+2} \\ a_{+1,+2} \\ a_{+2,+2} \end{matrix}\right] &
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 25 \end{matrix}
\end{array}$$

2 Three-Dim Steady Cellular Flows

The 3 dimensional (3D) steady cellular flows are:

$$B = (\Phi_x(x,y)W'(z), \Phi_y(x,y)W'(z), k\Phi(x,y)W(z)), \tag{2.38}$$

with $-\Delta\Phi = k\Phi$.

A special case is $k = 2$, then

$$B(x,y,z) = (-\sin x \cos y \cos z, -\cos x \sin y \cos z, 2\cos x \cos y \sin z). \tag{2.39}$$

Question: Is the effective diffusivity problem easier for (2.39) than (1.3)?

Effective diffusivity in (2.38) is unknown, see [4] for related KPP problem, and [3] for an upper bound. Extrapolating the KPP scaling in [4] on the flow (2.39), effective diffusivity at small molecular diffusivity ϵ scales like $O(\epsilon^p)$, $p \approx 0.26$. In 2D, $p = 1/2$, see [2], also Fig.3 in [1].

References

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