Report for the paper:

Sepctral analysis and computation of effective diffusivities in space-time periodic incompressible flows

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In this paper, the authors studied the effective diffusivity in the large scale long time dynamics of the advection-diffusion equation

$$\begin{cases} \partial_t \phi(t,x) = u(t,x)\cdot\nabla\phi(t,x) + \epsilon\Delta\phi(t,x) & (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ \phi(0,x) = \phi_0(\delta x) & (t,x)\in\{0\}\times\mathbb{R}^d, \end{cases}$$

where $u(t,x)$ is a given space-time periodic incompressible vector field, and ϵ is a fixed constant diffusion coefficient. In the long time large spatial scaling $t\mapsto t/\delta^2$ and $x\mapsto x/\delta$, the equation reads

$$\begin{cases} \partial_t \phi^\delta(t,x) = \frac{1}{\delta}u(t/\delta^2, x/\delta)\cdot\nabla\phi^\delta(t,x) + \epsilon\Delta\phi^\delta(t,x) & (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ \phi^\delta(0,x) = \phi_0(x) & (t,x)\in\{0\}\times\mathbb{R}^d. \end{cases}$$

The authors reviewed the derivation of the effective diffusivity D^* so that $\phi^\delta(t,x)$ converges uniformly in space and locally uniformly in time to $\overline{\phi}$ which solves the effective diffusive equation

$$\begin{cases} \partial_t \overline{\phi}(t,x) = \nabla\cdot D^*\nabla\overline{\phi}(t,x) & (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ \overline{\phi}(0,x) = \phi_0(x) & (t,x)\in\{0\}\times\mathbb{R}^d. \end{cases}$$

Moreover, the authors investigated the spectral representation of the effective diffusive matrix (more precisely, its symmetric and anti-symmetric parts) using spectral measures of certain natural operators arising from the homogenization analysis. The main results and novelties of the paper are:

(1) They study (in general) space-time periodic flow $u(t,x)$. The cell problem (corrector equation), for each coordinate index k, is then a time-dependent problem of the form

$$\partial_\tau\chi_k(\tau,\xi) - \epsilon\Delta_\xi\chi_k(\tau,\xi) - u(\tau,\xi)\cdot\nabla_\xi\chi_k(\tau,\xi) = u_k(\tau,\xi)$$

defined on the space-time periodic cell, with normalization condition $\int_{\mathbb{T}^{d+1}}\chi_k = 0$. This problem is naturally recast as $(\epsilon + (-\Delta)^{-1})(\partial_\tau - u\cdot\nabla))\chi_k = (-\Delta)^{-1}u_k$ where $(-\Delta)^{-1}$ is the solution operator to the Laplacian on the periodic cell (with periodic boundary condition and with the normalization condition). A natural representation of the solution is given by $\chi_k = (\epsilon + A)^{-1} g_k$ where $A = (-\Delta)^{-1}(\partial_\tau - u \cdot \nabla)$ and $g_k = (-\Delta)^{-1} u_k$. For the spectral representation of D^*, the natural operator to consider involves A, which is unbounded in view of the differentiation in time. One of the novelty of this paper is to study the spectral measure associated to unbounded operators on proper (subsets of) Hilbert spaces.

(2) The authors also studied a parallel formulation of the scalar cell problem above, which involves vector fields J_k, E_k and is given by

$$\nabla \cdot J_k = 0, \quad \nabla \times E_k = 0, \quad J_k = (\epsilon + (-\Delta)^{-1}\partial_\tau + H)E_k$$

where H is a skew-symmetric matrix such that $u = \nabla \cdot H$, $(-\Delta)^{-1}$ and ∂_τ are operated component wise on the vector E_k. The effective conductivity (constitutive relation) σ^* is a matrix so that $\langle J_k \rangle = \sigma^* \langle E_k \rangle = \sigma^* e_k$ holds. The author derived a parallel spectral analysis of the involved (unbounded) operator, and the spectral representation of σ^*.

(3) The authors showed that the two formulations are equivalent. In fact, there are isometric correspondences between the involved operators, the underlying Hilbert spaces, their spectral measures, the correctors, etc.

(4) For some special space-time periodic flow, the authors derived a Fourier method for computing D^* and computed the spectral measure and the effective diffusivity.

The results presented in this paper and the observations made along the proof are very interesting. The paper is also written carefully. I would like to recommend the paper to be published in *Annals of Mathematical Sciences and Applications*, provided that the following minor comments are taken into account. (Item 10 below is my main concern.)

1. Throughout the paper, the sentence "the domain $D(M)$ of an operator M is densely defined in the underlying Hilbert space" is a bit confusing. I think something like "the domain $D(M)$ is dense" or "M is densely defined" is clearer. Or if these are not exactly what you mean, define the first sentence at the beginning.

2. p.8, below eq.(8), can you comment a little bit more on the conditions to have uniform ($\sup_{\boldsymbol{x} \in \mathbb{R}^d}$) convergence?

3. p.8, below eq.(10), make the definition of $\mathcal{T}$ and $\mathcal{V}$ explicit, namely $\mathcal{V} = [0, 2\pi]^d$ and $\mathcal{T} = [0, 2\pi]$ or other choices. Make sure these are consistent with the fourier series used in p.12 eq.(19).

4. p.9, line 3, in the definition of $(-\Delta)^{-1}$, specify which boundary condition is associated to the Δ problem, namely periodic boundary condition with zero average. This operator is used throughout the paper; it worths to make this clarification.

5. p.18, line 5 in paragraph 3, the sentence "whenever $\{f_n\}$ and $\{\Phi f_n\}$ exist" sounds strange.

6. p.19, eq.(33), it is better to refer to a specific theorem with specific definition of $Q(\lambda)$, and assumptions on Σ (or the operator Φ).

7. p.19, eq.(34)(35) are defined for $\lambda \in \Sigma$ only, right? Are they extended somehow to the whole line/plane, or is Σ assumed to be a line/plane?

8. p.23, in C.1., last line, $\langle \nabla\psi \cdot \nabla\varphi \rangle = \int_0^1 \int_{[0,1]^d} \nabla\psi \cdot \nabla\varphi dxdt$ does not define an inner product, since if φ is a function of the t variable only, then this product is zero but φ is not necessary the zero of $\mathscr{H}$. Also, for eq.(46), specify with respect to which norms of $\tilde{\mathscr{A}}_{\mathcal{T}}$ and $\mathscr{H}^1_{\mathcal{V}}$ are the completion done.

9. p.24, line 1, why is $\|\partial_t \psi\|_1 < \infty$?

10. p.25, I am not sure about the derivation to the bound in eq.(49). $\mathscr{F}$ is defined through a completion, shouldn't it be the Sobolev space $H^1(\mathcal{T} \times \mathcal{V})$ of L^2 integrable functions on the torus with L^2 integrable (weak) derivatives? In that case, Sobolev embedding indicates that a L^∞ bound would fail in $d \geq 2$.

11. The above estimate is used also in p.26, eq.(50), p.29, line 6 below eq.(56), etc. Please make sure the above question is clarified.

12. p.30, below eq.(62), the relation $\boldsymbol{J}_k = \boldsymbol{\sigma} \boldsymbol{E}_k$ is not "local". In fact, this relation is a partial differential equation. It worths to mention this though the seemingly "local" formula is a nice analog with the corresponding constitutional equation in the Maxwell's equations.

13. p.31, eq.(65), indeed $\mathbf{A} = \mathbf{\Gamma S \Gamma} = \mathbf{\Gamma S}$ since on $\mathcal{H}$, the vector fields are already curl free. However, it is not clear to me that $\mathbf{\Gamma S \Gamma} = \mathbf{S \Gamma}$ (and hence $\mathbf{\Gamma S} = \mathbf{S \Gamma}$), because it is not clear to me that the range of $\mathbf{S}$ is contained in $\mathcal{H}$. I don't see where $\mathbf{\Gamma S} = \mathbf{S \Gamma}$ is needed, in any case.

14. p.32, Lemma 4 and its proof, double check the indices in σ^*_{jk}. For instance, on p. 33, last sentence in paragraph 1, is $\langle \boldsymbol{J}_j \cdot \boldsymbol{E}_k \rangle = \langle \boldsymbol{J}_j \cdot \boldsymbol{e}_k \rangle$ equal to σ^*_{jk} or σ^*_{kj}? The same

question for the fourth line in eq.(71). If the latter (instead the current version), then in eq.(70), there should be no "transpose".

15. p.34, last 3 lines in paragraph 2, for $f \in \mathscr{H}_{\mathcal{V}}^1$, $\partial f/\partial x_j$ could be defined as stated here, except that the limit of $\{\partial f_n/\partial x_j\}$ should be sought for in $\|\cdot\|$ topology rather than $\|\cdot\|_1$ (which itself involves a derivative, according to the definition in eq.(45)). Also, it is not clear that $\{\partial^2 f_n/\partial x_j^2\}$ will have a limit, since $\mathscr{H}_{\mathcal{V}}^1$ is taken as the completion w.r.t. a norm that only concerns first order derivatives.

16. Examine the consequence of item 8 above on the norms appearing in Appendix D.

Here are list of typos I found:

17. p.7, last line $\phi(0, \boldsymbol{x})$ should be ϕ^δ.

18. p.23, line 4 in section C.1., in $\mathcal{V} = \times_{j=1}^d$, $\times$ should be $\otimes$?

19. p.31, line 1, $\mathcal{H}$ should be $\mathscr{F}$?

20. p.32, line 3 from bottom of paragraph 1, "$\mathbf{S} =$ ... is self-adjoint" should be "$-\imath\mathbf{S}$ ... is self-adjoint "?