Notes on Factor Models

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1 Mean-variance frontier and beta representation (Cochrane, chapter 5.1)

The expected return-beta representation of a factor model is:

$$E(R^i) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots, \qquad i = 1, \dots, N \tag{1}$$

where the beta terms are from multiple time-series (contemporaneous) regressions of returns on factors f for each asset i:

$$R^i_t = a_i + \beta_{i,a} f^a_t + \beta_{i,b} f^b_t + \dots + \epsilon^i_t \qquad t = 1, \dots, T \tag{2}$$

Equation (1) is the beta model. Note that both γ and the λ are common across all assets. The expected return-beta model is a restriction that the intercept is the same for all assets in the time-series regressions. With one single factor, all assets expected returns should line up on a straight line as function of the corresponding asset beta.

The point of the beta model (1) is to explain the variation in average returns across assets (i.e., the cross-section of asset returns). The betas in (1) are the explanatory variables, which vary assets by assets. The γ and λ are common across all assets, and are the intercept and slopes in the cross-sectional relationship.

One way to estimate the model free parameters γ, λ and to test the model (1), is to run a cross-sectional regression of average returns on betas:

$$E(R^i) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots + \alpha_i, \qquad i = 1, \dots, N$$

where the β_i are right-hand variables (i.e., the "data"), the γ and λ are the intercept and slopes coefficients that we estimate, and the α_i are pricing errors. The model predicts $\alpha_i = 0$ and they should be statistically insignificant and economically small in a test.

Special cases

- if there is a constant risk-free rate R^f,then:
$$R^f = \gamma$$
as its betas are all zero in equation (1).[1] For this reason, γ is also called the zero-beta rate.
- if we use excess returns directly, then γ cancels out in equation (1):
$$E(R^{ei}) = \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots, \qquad i = 1, \dots, N$$
where $R^{ei} = R^i - R^j$. In this case, $\beta_{i,a}$ represents the regression coefficient of the excess return R^{ei} on the factors.
- if the factors are also returns or excess returns (i.e., factors are tradable), then we can estimate the λ coefficients directly rather than using the cross-sectional regression. Each factor has a beta of one on itself and zero on all other factors. Then:
$$E(f^a_t) = \lambda_a$$
and we can write the beta model as:
$$E(R^{ei}) = \beta_{i,a} E(f^a_t) + \beta_{i,b} E(f^b_t) + \dots + \alpha_i \qquad i = 1, \dots, N$$

Important difference between the time-series and cross-sectional beta-pricing model. The time-series regression model (2) will in general have a different intercept a_i for each return i, while the intercept γ is the same for all assets in the beta pricing equation (1). The beta pricing equation is a restriction on expected returns, and thus imposes a restriction on intercepts in the time-series regressions. In the special case that the factors are themselves excess returns, the restriction is particularly simple: the time-series regression intercepts should all be zero. In this case, we can avoid the cross-sectional regression entirely, since there are no free parameters left.

[1] The betas are all zero because the risk-free rate is known ahead of time.

2 From discount factors to beta representation (Cochrane, chapter 6.1)

Start from Euler equation $1 = E(MR^i)$ applied to returns. Use definition of covariance to get $E(R^i) = 1/E(M) - Cov(M, R^i)/E(M)$. Recall that $R^f = 1/E(M)$, which is also equal to the zero-beta rate γ. Then we can write a single-beta representation:

$$E(R^i) = \gamma + \left(\frac{Cov(M, R^i)}{Var(M)}\right)\left(-\frac{Var(M)}{E(M)}\right) = \gamma + \beta_{i,m}\lambda_m$$

3 Factor models and discount factors (Cochrane, chapter 6.3)

Beta pricing models are equivalent to linear models for the discount factor M:

$$E(R^i) = \gamma + \lambda'\beta_i \quad \Leftrightarrow \quad M = a + b'f \tag{3}$$

Consider a linear factor model with de-meaned factors (e.g., excess returns) and where we normalize the risk-free rate $E(M) = 1$:

$$M = 1 + b'\left[f - E(f)\right]$$

Theorem: given the model

$$M = 1 + b'\left[f - E(f)\right] \qquad , \qquad 0 = E(MR^e)$$

one can find λ such that:

$$E(R^e) = \beta'\lambda \tag{4}$$

where β are the multipe regression coefficients of excess returns R^e on the factors. Conversly, given λ in (4), we can find b such that the model (3) holds.

Proof: Recall that $E(M) = 1$ and $M = 1 + \left[f - E(f)\right]' b$. Then

$$0 = E(MR^e) = E(R^e)E(M) + Cov(M, R^e)$$

and

$$Cov(M, R^e) = Cov(1 + b'\left[f - E(f)\right], R^e) = Cov(f'b, R^e) = Cov(R^e, f')b$$

Therefore:

$$0 = E(R^e) + Cov(R^e, f')b \quad \Rightarrow E(R^e) = -Cov(R^e, f')b$$

From covariance to beta is quick:

$$E(R^e) = -Cov(R^e, f')Var(f)^{-1}Var(f)b = \beta'\lambda$$

Therefore, λ and b are related by

$$\lambda = -Var(f)b \tag{5}$$

4 Cross-sectional regressions (Cochrane, chapter 12.2)

Start from factor model $E(R^{ei}) = \beta_i'\lambda, 1 = 1, ..., N$, which says that average returns should be proportional to betas. Even if the model is true, the cross-sectional relationship will not work out perfectly in each sample. This is the motivation for running a cross-sectional regression. First, find estimates of the betas from time-series regressions:

$$R_t^{ei} = a_i + \beta_i' f_t + \epsilon_t^i, \qquad t = 1, ..., T \quad \text{for each } i. \tag{6}$$

Then, estimate the factor premia λ from a regression across assets of average returns on the betas:

$$E_T(R^{ei}) = \beta_i'\lambda + \alpha_i, \qquad i = 1, ..., N. \tag{7}$$

Note that the cross-sectional regression residuals α_i are the pricing errors. This is know as two-pass regression estimate. Note that you can run the cross-sectional regression (7) with or without a constant. The theory says the constant or zero-beta excess return should be zero. You can impose this restriction, or estimate a constant and see if it turns out to be small.

Time Series vs. Cross Section

You can run the cross-sectional regression when the factor is not a return. The time-series test, instead, requires factors that are returns, so that you can estimate factor risk premia by $\hat{\lambda} = E_T(f)$. Note that the asset pricing model does predict a restriction on the intercepts in the time-series regression. To see this, first impose the restriction $E(R^{ei}) = \beta_i'\lambda$. Then, you can write the time-series regression model (6) as

$$R_t^{ei} = \beta_i'\lambda + \beta_i'(f_t - E(f)) + \epsilon_t^i, \qquad t = 1, ..., T \qquad \text{for each } i.$$

Thus, the intercept restriction is

$$a_i = \beta_i'(\lambda - E(f)) \tag{8}$$

Note that setting $\lambda = E(f)$ results in a zero intercept. However, you can also see that without an estimate of λ, you cannot check this intercept restriction. In a sense, setting $\lambda = E(f)$ implicitly assumes that the model is correct. When the factor is not a return, we can only use the cross-sectional method. When the factor is a return, we can compare the time-series and cross-sectional method. The time-series method estimates the factor risk premium as the sample mean of the factor. Hence, the factor receives a zero pricing error in each sample. The predicted zero-beta excess return is also zero. Think of of a one-factor model and a scatter plot of average asset returns vs. betas, where one of the asset is the factor. Then, the time-series regression method draws a line through the origin and through the factor, *ignoring all the other points*. The cross-sectional regression method, instead, picks the slope and intercept to best fit all the points. In fact, if the factor is a return, the GLS cross-sectional regression, including the factor as test asset, is identical to the time-series regression.

5 Testing for Priced Factors: Lambdas or b's? (Cochrane, chapter 13.4)

Recall that bs are not the same as βs. The former are the regression coefficients of M on f, while the latter are the regression coefficients of R^i on f. Assume mean-zero factors, excess returns, and normalize the risk-free rate to $E(M) = 1$. Recalling equation (5), the parameters b and λ are related by

$$\lambda = E(ff')b.$$

When the factors are orthogonal, the variance-covariance matrix $E(ff')$ is diagonal, and each $\lambda_j = 0$ if and only if the corresponding $b_j = 0$. The distinction between b and λ only matters when the factors are correlated, and $E(ff')$ is not diagonal. λ_j captures whether factor f_j is priced. b captures whether factor f_j is marginally useful in pricing assets, given the presence of other factors. λ_j is proportional to the single regression coefficient of M on f (i.e., $\lambda_j = cov(M, f_j)$). b_j is the multiple coefficient of M on f_j given all the other factors. Therefore, testing $\lambda_j = 0$ asks the question: "is factor j correlated with the true discount factor?". When you ask the question "should I include factor j given the other factors?", you want to ask the multiple regression question.