A Utility-Based Framework to Price Assets

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Asset pricing theory tries to understand the prices or values of claims to uncertain payments

E.g., What is the value of a stock? *It is the value of a claim on its future uncertain dividends.*

- Prices and returns
- How to value an asset?
    - delay
    - risk
- Corrections for risk are much more important determinants of many asset values
    - E.g., over the last 50 years the real return on US stocks was $\approx 9\%$ on average; of this only about 1% is due to the riskless interest rate (i.e., delay)

- Positive vs. normative financial economics: description of the way the world *does* work or the way the world *should* work?
- Price equals expected discounted payoff
- Absolute versus relative pricing
- After we learn how to compute asset values, learn how to take derivatives: e.g., $\frac{\partial V}{\partial R}$, or $\frac{\partial V_1}{\partial V_m}$, etc.: risk-management, portfolio analysis, scenario analysis, etc.

Modern approaches to asset pricing

- Broadly speaking there are four classes of models

  1. Macro-finance models (e.g., Campbell and Cochrane (1999), Bansal and Yaron (2004), Barro (2006)).
  2. Empirical models with traded factors (e.g., Fama and French (1992), Fama and French (1993), Asness et al. (2013)).
  3. Empirical models with non-traded factors (e.g., Chen et al. (1986)).
  4. Euler equation models of a class of investors (e.g., Vissing-Jørgensen (2002)).

General approach used in this class

- Discount factor (or GMM) approach to asset pricing:
  - One framework for different classes of assets (stocks, bonds, currencies, options, etc):

    $$P_{it} = E_t(M_{t+1}X_{it+1}), \qquad i = 1, \ldots, N \qquad T = 1, \ldots, T$$

    where P_{it} denotes the i-asset price, M_{t+1} the stochastic discount factor (SDF) and X_{it+1} the i-asset payoff.
  - $M_{t+1} = f(\text{data, parameters})$; $f(.)$ is the model.
  - Empirical procedures: Hansen (1982)'s General Method of Moments (GMM) and Fama and MacBeth (1973).
  - In short, pick free parameters of the model to make it fits best, which usually means to minimize pricing errors and evaluate the model by examining how big those pricing errors are.

Focus of this class: macro-finance models

- Macro-Finance (see, e.g., the review paper by Cochrane (2017)):
    - Expected returns vary across time and across assets in ways that are linked to macroeconomic variables, or variables that also forecast macroeconomic events.
    - Sizable excess returns predicted by macro variables: do we have a good model to understand this?
        - Campbell and Cochrane (1999) ("habit models")), Bansal and Yaron (2004) ("long-run risk models"), Barro (2006) ("disaster risk models")?
    - Investment and interest rates.
    - Risk aversion and the business cycle.
    - Cost of business cycles? Lucas (2003) or Croce (2013)?

Outline

Outline

- The problem:
  - An investor must decide how much to save and how much to consume, and what portfolio of assets to hold.
  - In order to answer this question, we need to introduce a simplified model of human behaviour.
  - We do this through a utility function that captures basic features of human psychology: the evidence that people prefer money now, and money that is not very risky (standard technical assumption: $u(.)$ increasing + concave).

Solving the problem

- Find the value at time t of a payoff X_{t+1} at time $t+1$.
- Example: if you buy a stock today at price P_t, the payoff next period is the stock price P_{t+1} plus dividend D_{t+1}:

$$X_{t+1} = P_{t+1} + D_{t+1}.$$

- The payoff X_{t+1} is a random variable.
- At date t, an investor does not know exactly how much she will get from her investment at date $t+1$.

Solving the problem

- Let Q be the original consumption level (if the investor bought none of the asset; e.g., labor income).
- Let ξ be the amount of the asset the investor chooses to buy.
- Assume the investor can freely buy or sell as much of the payoff X_{t+1} at the price P_t as they wish.
- Let β be the subjective discount factor.

Solving the problem

- We can write the investor's optimization problem as follows:

$$\max_{\xi} \; u(C_t) + E_t[\beta u(C_{t+1})]$$

subject to[1]:

$$\begin{aligned} C_t &= Q_t - P_t\xi, \\ C_{t+1} &= Q_{t+1} + X_{t+1}\xi \end{aligned}$$

[1] In addition, in equilibrium $C_t, C_{t+1} > 0$. As the objective function is well-behaved (i.e., utility function is increasing and concave), this problem has an interior solution.

Considerations

- We assume that the investor can buy or sell *any quantity of the assets*, i.e., no short sales or borrowing constraints, and therefore an interior solution to the maximization problem.

- Discounting the future by β captures investors' impatience (subjective discount factor, typically ≈ 0.95 on an annual basis).

Considerations

- Utility function captures desire for more consumption, rather than intermediate objectives like mean and variance of portfolio returns.
- The period utility function is increasing and concave.
- The curvature of the utility function generates aversion to risk and to intertemporal substitution (i.e., the investor prefers a consumption stream that is steady over time and across states of nature).

CRRA Preferences

- We will often use a convenient power utility form (or CRRA):

$$u(C) = \frac{1}{1-\gamma} C^{1-\gamma}.$$

▸ case $\gamma = 1$

- The marginal utility of consumption is equal to:

$$u'(C) = C^{-\gamma}.$$

- Note that for $\gamma > 1$ the utility ($u(C) < 0$) is negative: *is this a problem*?

Equilibrium condition

- Substitute the constraints into the objective.
- Set the derivative with respect to ξ to zero:

$$P_t u'(C_t) = E_t[\beta u'(C_{t+1}) X_{t+1}],$$

  where $P_t u'(C_t)$ is the loss in utility if the investor buys another unit of the asset, and $E_t[\beta u'(C_{t+1}) X_{t+1}]$ is the expected discounted increase in utility he/she obtains from the extra payoff X_{t+1}.
- **The investor continues to buy or sell the asset until the marginal loss equals the marginal gain**.

Asset prices are function of consumption and payoffs

- The basic pricing formula is thus:

$$P_t = E_t[\beta \frac{u'(C_{t+1})}{u'(C_t)} X_{t+1}]. \tag{1}$$

- Note that this is a joint restriction on asset returns and consumption.

Stochastic discount factor

- Let M_{t+1} be the stochastic discount factor (SDF) defined as:

$$M_{t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)}.$$

- Then, the basic pricing formula can be expressed as:

$$P_t = E_t[M_{t+1} X_{t+1}]. \tag{2}$$

- Equation (2) is the cornerstone of modern asset pricing theory and can be used to price any asset. Much of asset pricing evolves around models of the SDF (i.e., the $f(.)$ which defines the SDF).

Comments

- Note that the equilibrium condition $P = E(MX)$ holds after everybody has done his/her investment choices.
- An individual will *observe payoffs and prices* and line-up consumption accordingly
  - $\hookrightarrow$ For example, if the price of an asset is *too low*, the individual will buy a lot of it, reduce current consumption and thereby increase future consumption.
- Also note that the equilibrium condition describes a marginal investment: it is not appropriate to describe a large asset purchase from a venture capitalist, a hedge-fund, etc.

Stochastic discount factor with no uncertainty

- If there is no uncertainty, we can express prices via the standard present value formula:

$$P_t = \frac{1}{R_t^f} X_{t+1},$$

  where R_t^f is the gross risk-free rate: at t lend \$1 and get a sure payoff of R_t^f at $t+1$
- In this case $1/R_t^f$ is the discount factor and we price assets by an arbitrage argument (which one?).

Prices, payoffs and returns

- The formula

$$P_t = E_t[M_{t+1}X_{t+1}]$$

  applies to many cases:

  - For stocks, the payoff X_{t+1} is the price next period P_{t+1} and the dividend D_{t+1}.
  - We divide the payoff by the price to obtain the gross return ($R_{t+1} = \frac{X_{t+1}}{P_t}$).

Prices, payoffs and returns

- The formula

$$P_t = E_t[M_{t+1}X_{t+1}]$$

applies to many cases:

  - For a one-period bond, the payoff is 1.
  - You buy it at price P_t and you get 1 dollar next period.
  - Alternatively, we can think of a return on a bond in the following way: you pay 1 dollar today, and you receive R_{t+1} dollars tomorrow. The Euler equation is thus:

$$E_t[M_{t+1}R_{t+1}] = 1.$$

A return is a payoff with price of one.

Risk-free rate

- A risk-free interest rate corresponds to the following case: you pay 1 dollar today, and you receive R_t^f dollars tomorrow, where R_t^f is known at date t (which is why it is risk-free).

- What does it really mean "risk-free"?

Scaling prices: returns and P/D

- Asset prices are generally non-stationary.
- Think in terms of returns or price-dividend ratios: they are typically stationary.
- The return on a stock is X_{t+1}/P_t.
- The price-dividend ratio is P_t/D_t and it is the price of the following payoff:

$$X_{t+1} = (\frac{P_{t+1}}{D_{t+1}} + 1)\frac{D_{t+1}}{D_t}. \tag{3}$$

Excess returns

- Excess returns:
    - if you borrow a dollar at the interest rate R^f and invest it in an asset with return R, you pay no money out-of-pocket today and get the payoff $R - R^f$.
    - This is a payoff with a zero price.
    - Zero price does not mean zero payoff: *it is a bet in which the value of the chance of losing exactly balances the value of the chance of winning.*
    - We denote any such difference between returns as an excess returns, R^e.

Prices, payoffs and returns

- Real (denominated in goods) or nominal stochastic discount factors and returns:
  - Assume that prices P_t^n and payoffs X_{t+1}^n are nominal.
  - Let Π_t be the price index (e.g., CPI) at date t.
  - Then the Euler equation is:

$$P_t^n = E_t[\underbrace{M_{t+1}\frac{\Pi_t}{\Pi_{t+1}}}_{M_{t+1}^n} X_{t+1}^n].$$

Risk-free rate and power utility

- Start from $E_t(M_{t+1}R_{t+1}) = 1$.
- The risk-free rate R_t^f is:

$$R_t^f = 1/E_t[M_{t+1}]. \tag{4}$$

- Assume power utility: $u(C) = C^{1-\gamma}/(1-\gamma)$.
- Marginal utility is $u'(C) = C^{-\gamma}$.
- The risk-free rate is then:

$$R_t^f = 1/E_t[\beta(\frac{C_{t+1}}{C_t})^{-\gamma}].$$

Certainty case

- Let us assume for now that there is no uncertainty in the economy. The risk-free rate is equal to:

$$R_t^f = \frac{1}{\beta}[\frac{C_{t+1}}{C_t}]^{\gamma}. \tag{5}$$

- Three effects link R^f to macro factors. Real interest rates are:
  1. higher when people are more impatient, i.e β small.
  2. higher when consumption growth is higher.
  3. more sensitive to consumption growth if γ large.

Log-normality assumption

- If z follows a Normal distribution, then:

$$E(e^z) = e^{E(z)+\frac{1}{2}\sigma^2(z)}.$$

- We will use this result in many derivations (▸ Details).
- Note that in continuos time you can get most of the following results without relying on the assumption of normality.

Uncertainty case

- We will use lower case letters for log variables.
- Let Δc_{t+1} denote the log consumption growth:

$$\Delta c_{t+1} = \ln \frac{C_{t+1}}{C_t} = \ln(C_{t+1}) - \ln(C_t).$$

- Assume that Δc_{t+1} is normally distributed (i.e., gross consumption growth is lognormally distributed).
- For example:

$$\Delta c_{t+1} = \mu + \sigma \epsilon_{t+1}.$$

Risk free under uncertainty

- The risk-free rate is known in advance so we can pull it out of the expectation:

$$
\begin{aligned}
R_t^f &= \{E_t[\beta(\frac{C_{t+1}}{C_t})^{-\gamma}]\}^{-1}, \\
&= [\beta E_t(e^{-\gamma \Delta c_{t+1}})]^{-1}, \\
&= [\beta e^{-\gamma E_t(\Delta c_{t+1}) + \frac{\gamma^2}{2}\sigma_t^2(\Delta c_{t+1})}]^{-1}
\end{aligned}
$$

- If $\forall t: \; \mu = E_t(\Delta c_{t+1})$ and $\sigma^2 = \sigma_t^2(\Delta c_{t+1})$, then the risk-free rate is constant

$$
R^f = [\beta e^{-\gamma\mu + \frac{\gamma^2}{2}\sigma^2}]^{-1}
$$

Risk free under uncertainty

- Let us define δ as $\beta = e^{-\delta}$ and $r_t^f = \ln(R_t^f)$.
- Then the log of the gross risk-free rate is:

$$
\begin{aligned}
r_t^f &= \delta + \gamma E_t(\Delta c_{t+1}) - \frac{\gamma^2}{2}\sigma_t^2(\Delta c_{t+1}), \\
r^f &= \delta + \gamma\mu - \frac{\gamma^2\sigma^2}{2}.
\end{aligned}
$$

Risk free: Uncertainty vs. Certainty

- Compare:

$$r_t^f = \delta + \gamma E_t(\Delta c_{t+1}) - \frac{\gamma^2}{2}\sigma_t^2(\Delta c_{t+1}).$$

  with:

$$R_t^f = \frac{1}{\beta}[\frac{C_{t+1}}{C_t}]^\gamma \longrightarrow r_t^f = \delta + \gamma \Delta c_{t+1}.$$

- The new term σ^2 captures precautionary savings (cf. concern for downside risk): $\Rightarrow$ When consumption is more volatile, people want to save more, driving down interest rates.

- Concern for low consumption states ▸ details.

Magnitudes

- Note that with low risk-aversion (γ), the precautionary savings effect is small.

- For example, for reasonable values of $\sigma = 2\%$ at annual frequency, when $\gamma = 1$ (log utility) the precautionary savings effect is equal to 2bp (i.e., 0.02%).

γ and power utility

- For power utility, the curvature parameter γ simultaneously controls ▸ IES ▸ risk aversion ▸ St Petersburg paradox:

  1. intertemporal substitution - aversion to consumption stream that varies over time,
  2. risk aversion - aversion to a consumption stream that varies across states of nature,
  3. and precautionary savings.

Term structure

- If the risk-free rate is constant, then the term structure of interest rates (i.e., interest rates function of maturity) is flat.
- Consider a 1-period zero-coupon asset that pays 1 unit at $t+1$:

$$P_{t,t+1} = E_t[M_{t,t+1} \times 1_{t+1}] = (R^f)^{-1}.$$

- Consider a 2-period zero-coupon asset that pays 1 unit at $t+2$:

$$\begin{aligned} P_{t,t+2} &= E_t[M_{t,t+2} \times 1_{t+2}] \\ &= E_t[M_{t,t+1} P_{t+1,t+2}] \\ &= E_t[M_{t,t+1} E_{t+1}[M_{t+1,t+2}]] \\ &= E_t[M_{t,t+1} M_{t+1,t+2}] \\ &= (R^f)^{-1} \times (R^f)^{-1} \\ &= (R^f)^{-2} \end{aligned}$$

where we used the law of iterated expectations (‣ details) and $M_{t,t+2}$ is the SDF from period t to $t+2$ (i.e., for CRRA utility $M_{t,t+2} = \beta^2 (\frac{C_{t+2}}{C_t})^{-\gamma}$).

Stock returns

- A stock is an asset (a tree) that pays a dividend (a fruit) $D_t = C_t$ in all the states of the world:

$$P_t = E_t[\beta(\frac{C_{t+1}}{C_t})^{-\gamma}(P_{t+1} + D_{t+1})].$$

- What can be consumed is given by the aggregate dividend paid as in Lucas (1978), because of the assumption that dividends (fruits) cannot be stored and, thus, must be consumed (recall: $u(.)$ is increasing and concave)

The PD ratio

- The price/dividend ratio is then equal to:

$$\frac{P_t}{D_t} \equiv \frac{P_t}{C_t} = E_t[\beta(\frac{C_{t+1}}{C_t})^{1-\gamma}(1+\frac{P_{t+1}}{D_{t+1}})], \tag{6}$$

where (6) is obtained by multiplying and dividing the term in expectation by $D_{t+1} = C_{t+1}$.

The PD ratio

- Iterate forward, impose transversality condition and assume that log consumption growth is i.i.d. normal to get ▸ Details:

$$\frac{P_t}{C_t} \equiv \frac{P}{C} = \frac{\beta e^{(1-\gamma)\mu + (1-\gamma)^2 \frac{\sigma^2}{2}}}{1 - \beta e^{(1-\gamma)\mu + (1-\gamma)^2 \frac{\sigma^2}{2}}}.$$

- **The price/dividend ratio is constant**. Intuition: the expectations of future consumption growth are independent on current consumption.
- Note how this result resembles the formula for a growing perpetuity with no uncertainty.

Stock returns

Price/dividend ratio

When expectations of future consumption and dividend growth are independent of the state today, and utility is CRRA, the price/dividend ratio is constant.

Risk premium

- The expected return on equity is constant:

$$E_t\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right) = E_t\left(\frac{\frac{P_{t+1}}{D_{t+1}}+1}{\frac{P_t}{D_t}}\frac{D_{t+1}}{D_t}\right) = \frac{1}{\beta}e^{\gamma\mu+(2\gamma-\gamma^2)\frac{\sigma^2}{2}}.$$

- The geometric risk premium is then:

$$\frac{E(R)}{R^f} = \frac{\frac{1}{\beta}e^{\gamma\mu+(2\gamma-\gamma^2)\frac{\sigma^2}{2}}}{\frac{1}{\beta}e^{\gamma\mu-\frac{\gamma^2\sigma^2}{2}}} = e^{\gamma\sigma^2}.$$

- For the US, $\sigma = 2\%$. Hence, the geometric risk premium is a small number: $\gamma \times 0.0004$, or 0.4 bp for log utility.

Realized returns

- Realized returns are equal to:

$$R_{t+1} = \frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}} \frac{D_{t+1}}{D_t} = \frac{1}{\beta} e^{(\gamma-1)\mu + (2\gamma - \gamma^2 - 1)\frac{\sigma^2}{2}} e^{\mu + \sigma\epsilon_{t+1}}.$$

- Therefore, realized returns are i.i.d. lognormal.

Volatility puzzle

- We have seen how the price/dividend ratio implied by the standard model is constant. The data counterpart is instead very volatile.
- In the model, since the P/D ratio is constant, the volatility of returns R_{t+1} is approximately equal to the volatility of dividend growth:

$$\sigma(R_{t+1}) \simeq \sigma(\frac{D_{t+1}}{D_t}).$$

- This is counterfactual. This result is equivalent to Shiller et al. (1981).

Shiller et al. (1981)

- Define with $P^*_{i,t}$ the realization of the fundamental value of stock i at time t (i.e., the discounted sum of realized future dividends).
- Rational investors set $P_{i,t} = E_t(P^*_{i,t})$.
- Forecast errors $P_{i,t} - P^*_{i,t}$ must be uncorrelated with any information available at t (including $P_{i,t}$).
- Since $P^*_{i,t} \equiv P_{i,t} + (P^*_{i,t} - P_{i,t})$, then:

$$Var(P^*_{i,t}) = Var(P_{i,t}) + Var(P^*_{i,t} - P_{i,t}).$$

- As a result, in a no-arbitrage market:

$$Var(P^*_{i,t}) > Var(P_{i,t}).$$

Shiller et al. (1981)

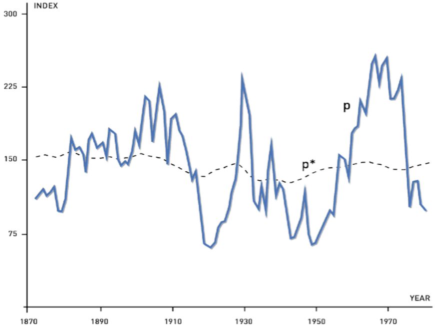


Figure 2: Real Standard and Poor's Composite Stock Price Index (solid line p) and *ex post* rational price (dotted line p^*), 1871–1979, both detrended by dividing a long-run exponential growth factor. The variable p^* is the present value of actual subsequent real detrended dividends, subject to an assumption about the present value in 1979 of dividends thereafter.

Figure: Figure is from Shiller (AER, 1981). For more details see Cochrane's blog post "Bob Shiller's Nobel".

Volatility puzzle

- P^* is the ex-post rational price.
- The actual price varies much more than P^*.
- Bob Shiller interpreted it as psychological and social dynamics, optimism and pessimism.
- Alternative explanation is that prices are not forecasting dividend growth, but long-run returns, i.e., the denominators in the "present value" formula (Cochrane's view).
- Read recent discussion by Nagel (2024).

Outline

Risk correction

- By definition, the covariance between M and X is:

$$cov(M, X) = E(MX) - E(M)E(X).$$

- Thus, we can rewrite our basic pricing formula as:

$$P_t = E_t(M_{t+1}X_{t+1}) = E_t(M_{t+1})E_t(X_{t+1}) + cov_t(M_{t+1}, X_{t+1}).$$

Risk correction

- Substitute in the pricing formula the risk-free rate ($R_t^f = 1/E_t[M_{t+1}]$), we obtain:

$$P_t = \frac{E_t(X_{t+1})}{R_t^f} + cov_t(M_{t+1}, X_{t+1}).$$

- The first term is the standard discounted present-value formula. The second term is a risk-adjustment.

Risk correction

- Let us substitute back for M in terms of consumption in previous equation:

$$P_t = \frac{E_t(X_{t+1})}{R_t^f} + \frac{cov_t[\beta u'(C_{t+1}), X_{t+1}]}{u'(C_t)}.$$

- Marginal utility $U'(C)$ declines as C rises.
- **An asset's price is lowered if its payoff covaries positively with consumption (i.e., if it pays high (low) when consumption is high (low) and the marginal utility low (high)). Insurance is an extreme example.**

Risk correction

An asset's price is lowered if its payoff covaries positively with consumption.

- Intuition:
  - Investors do not like uncertainty about consumption.
  - If you buy an asset whose payoff covaries positively with consumption, it means that it pays off well when you are already feeling wealthy and it pays off badly when you are already feeling poor.
  - Thus, **that asset will make your consumption stream more volatile**.

Risk correction: example

- Consider what happens to the volatility of consumption if the individual investor buys a little more (ξ) of payoff X:

$$\sigma^2(C + \xi X) = \sigma^2(C) + 2\xi cov(C, X) + \xi^2\sigma^2(X),$$

- For small portfolio changes: $\xi^2\sigma^2(X) \simeq 0$.
- It's the **covariance** between consumption and payoff that determines the effect of adding a bit more of each payoff on the volatility of consumption (**not the volatility of the payoff!**).

Risk correction

- For returns the basic pricing equation is:

$$E_t[M_{t+1}R^i_{t+1}] = 1.$$

- Although expected returns can vary across time and assets, *expected discounted returns should always be the same, 1.*

Risk correction

- Apply the covariance decomposition to the basic pricing equation for returns and use the definition of risk-free rate to obtain:

$$\begin{aligned} E_t(R^i_{t+1}) - R^f_t &= -R^f_t \, cov_t[M_{t+1}, R^i_{t+1}], \\ &= -\frac{cov_t[u'(C_{t+1}), R^i_{t+1}]}{E_t[u'(C_{t+1})]}. \end{aligned}$$

- All assets have an expected return equal to the risk-free rate, plus a risk adjustment.
- Assets whose returns covary positively with consumption make consumption more volatile, and so must promise higher expected returns to induce investors to hold them.
- Note that assets that covary negatively with consumption, such as insurance, can offer expected rates of return that are lower than the risk free rate, or even negative (net) expected returns.

Idiosyncratic risk

- You might think that an asset with a volatile payoff is "risky", and thus should have a large risk correction.
- However, if the payoff is uncorrelated with M, the asset receives *no* risk correction to its price and pays an expected return equal to the risk-free rate!
- This prediction holds even if X is very volatile and investors highly risk averse.
- Why? No first order effect on the variance of consumption stream:

$$\sigma^2(C + \xi X) = \sigma^2(C) + 2\xi cov(C, X) + \xi^2 \sigma^2(X).$$

Idiosyncratic risk

Only the component of payoff correlated with the SDF generates an extra return.

- *Idiosyncratic* risk, uncorrelated with the SDF, generates no premium.
- Only *systematic* risk generates risk correction.

Idiosyncratic risk

- Decompose the payoff as follows:

$$X = proj(X|M) + \varepsilon.$$

- Projecting X on M is like regressing X on M without a constant (i.e., X and M are centered around zero):

$$proj(X|M) = \frac{E(MX)}{E(M^2)} M.$$

Idiosyncratic risk

- The residuals ε are orthogonal to the right-hand side variable M: $E(M\varepsilon) = 0$, which means that the price of ε is zero.
- The price of the projection of X on M is the price of X:

$$P(proj(X|M)) = E(M\frac{E(MX)}{E(M^2)}M) = E(MX) = P(X).$$

- The projection of X on M is that part of X which is perfectly correlated with M.
- The idiosyncratic component of any payoff is that part that is uncorrelated with M.
- Only the systematic part of payoff accounts for its price.

β-pricing model

- Start from: $E_t(M_{t+1}R^i_{t+1}) = 1$.
- Rewrite it using the definition of covariance as:

$$E_t(R^i_{t+1}) - R^f_t = -R^f_t \, cov_t[M_{t+1}, R^i_{t+1}].$$

- Multiply and divide by $var_t[M_{t+1}]$ and rearrange terms:

$$E_t(R^i_{t+1}) = R^f_t + \underbrace{\left(-\frac{cov_t[M_{t+1}, R^i_{t+1}]}{var_t[M_{t+1}]}\right)}_{\beta_{i,M}} \underbrace{\left(\frac{var_t[M_{t+1}]}{E_t[M_{t+1}]}\right)}_{\lambda_M},$$

where $\beta_{i,M}$ is the regression coefficient of the return R^i on M.

β-pricing model

$E(R^i_{t+1}) = R^f_t + \beta_{i,M}\lambda_M$ is a beta-representation or beta-pricing model. Note that λ_M is independent of the asset i and depends on the volatility of the SDF. It is called the market price of risk. $\beta_{i,M}$ is the quantity of risk. **The excess return of asset *i* is equal to the quantity of risk of this asset times the price of risk**.

Consumption CAPM (CCAPM)

- Start from, assuming power utility:

$$E_t[\beta(\frac{C_{t+1}}{C_t})^{-\gamma} R^i_{t+1}] = 1.$$

- Assume that R^i_{t+1} and $\frac{C_{t+1}}{C_t}$ are jointly lognormal.
- Then:

$$E_t[e^{\log \beta - \gamma \Delta c_{t+1} + r^i_{t+1}}] = 1,$$

  where lower case letters denote logs.

Consumption CAPM (CCAPM)

- Apply log-normal trick to the return on asset i and to the risk-free rate to get the CCAPM ▸ Details:

$$E_t[\log(\frac{R^i_{t+1}}{R^f_{t+1}})] + \frac{1}{2} Var_t \log R^i_{t+1} = \log E_t(\frac{R^i_{t+1}}{R^f_{t+1}}) = \gamma Cov_t(\Delta c_{t+1}, \log R^i_{t+1}).$$

- Intuition:
  - Differences in average excess returns (corrected for Jensen's inequality term) depend only on covariances with consumption growth;
  - Assumption of joint log normality means that we cannot apply same reasoning to assets with highly nonlinear payoffs (for example, options).

β-pricing model: general setup (I/II)

- In a more general set up, rewrite Euler equation in terms of the log SDF m_{t+1} and the log return r^i_{t+1}:

$$E_t(m_{t+1}) + E_t(r^i_{t+1}) + \frac{1}{2}Var_t(m_{t+1}) +$$
$$+\frac{1}{2}Var_t(r^i_{t+1}) + Cov_t(m_{t+1}, r^i_{t+1}) = 0.$$

- Similar equation holds for the risk-free rate r^f_t.
- Subtract risk-free rate from expected log return:

$$E_t(r^i_{t+1}) - r^f_t + \frac{1}{2}Var_t(r^i_{t+1}) = -Cov_t(m_{t+1}, r^i_{t+1}).$$

β-pricing model: general setup (II/II)

- Let $\tilde{r}^{e,i}_{t+1}$ be the excess return corrected for the Jensen term: $\tilde{r}^{e,i}_{t+1} = r^i_{t+1} - r^f_t + \frac{1}{2} Var_t(r^i_{t+1})$. Then:

$$E_t(\tilde{r}^{e,i}_{t+1}) = -Cov_t(m_{t+1}, \tilde{r}^{e,i}_{t+1}).$$

CCAPM

- For power utility the log SDF depends only on consumption growth and is equal to:

$$m_{t+1} = \log \beta - \gamma g - \gamma(\Delta c_{t+1} - g),$$

  where g is the average consumption growth.
- In this case, the expected excess return is equal to:

$$E_t(\tilde{r}^{e,i}_{t+1}) = \gamma cov_t(\Delta c_{t+1} - g, \tilde{r}^{e,i}_{t+1}).$$

- Assets whose returns covary positively with consumption must promise positive expected returns to induce investors to hold them.

Fama-MacBeth procedure (I/II)

- Assume that we have many excess returns.
- A first stage regression of each excess return $\tilde{r}^{e,i}_{t+1}$ (corrected for the Jensen term) on a constant and consumption growth gives a corresponding slope coefficient of:

$$\beta_{i,\Delta c} = cov(\Delta c_{t+1}, r^{e,i}_{t+1}) / \sigma^2_{\Delta c_{t+1}}.$$

- A second stage regression of all average (i.e., expected) excess returns $E_T(\tilde{r}^{e,i})$ on a constant and the corresponding $\beta_{i,\Delta c}$ leads to the estimation of the market price of risk λ.
- This two-stage procedure is advocated by Fama and MacBeth (1973).

Fama-MacBeth procedure (II/II)

- The expected excess return can be rewritten in terms of the market price of risk $\lambda_{\Delta c}$ and the amount of risk β:

$$E_t(\tilde{r}^{e,i}_{t+1}) = \underbrace{cov_t[\Delta c_{t+1}, r^{e,i}_{t+1}]/\sigma^2_{\Delta c_{t+1}}}_{\textit{Amount of risk } \beta_{i,\Delta c}} \quad \underbrace{\gamma \sigma^2_{\Delta c_{t+1}}}_{\textit{Price of risk } \lambda_{\Delta c}} \quad .$$

- The market price of risk λ measures the expected excess return per unit of risk.
- Expected returns should increase linearly with their betas on consumption growth.

Intuition

- The factor risk premium $\lambda_{\Delta c}$ is determined by risk aversion and the volatility of consumption.

- Intuition: The more risk averse people are, or the riskier their environment, the larger an expected return premium one must pay to get investors to hold risky (high beta) assets.

Mean-variance frontier

- Start again from equation $E_t(M_{t+1}R_{t+1}) = 1$:

$$1 = E_t(M_{t+1}R^i_{t+1}) = E_t(M_{t+1})E_t(R^i_{t+1}) + \rho_t(M, R^i)\sigma_t(R^i)\sigma_t(M),$$

  where ρ_t is the correlation coefficient $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$.

- Then:

$$E_t(R^i_{t+1}) - R^f_t = -\rho_t(M, R^i)\frac{\sigma_t(M)}{E_t(M)}\sigma_t(R^i).$$

- This is an equilibrium relation (i.e., remember: it comes from the optimality condition): if someone is holding these assets it must be that their returns compensate for their risk.

Mean-variance frontier

The set of means and variances of asset returns is limited.

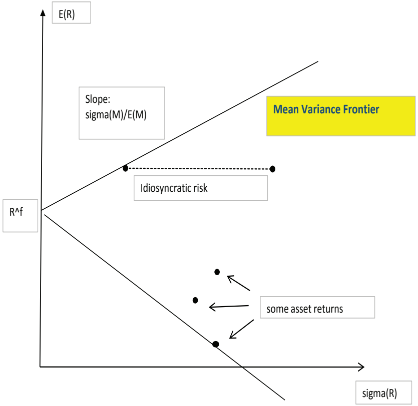
Mean-variance frontier

- Correlation coefficients cannot be $> 1 \Rightarrow$ all assets priced by the SDF M must obey:

$$|E(R^i) - R^f| \leq \frac{\sigma(M)}{E(M)}\sigma(R^i).$$

- The Hansen and Jagannathan (1991)'s mean-variance frontier is determined by $|\rho_{M,R^i}| = 1$.
- In a graph of expected returns $E(R)$ as a function of their volatility $\sigma(R)$, all assets must lie inside the wedge-shaped mean-variance region.

Mean-variance frontier



Mean-variance frontier

- The mean-variance frontier answers a naturally interesting question: how much mean return can you get for a given level of variance?
- The Hansen and Jagannathan (1991)'s bounds are popular because any model that does not satisfy them will be at odds with the data.

Implications

- All frontier returns R^{mv} are perfectly correlated with the SDF and with each other (the frontier is generated for $|\rho_t(M, R^i)| = 1$).
- Returns on the upper part of the frontier are perfectly negatively correlated with the discount factor and hence positively correlated with consumption. They are maximally risky and get the highest expected returns.
- Returns on the lower part of the frontier are perfectly positively correlated with the discount factor and hence negatively correlated with consumption. They provide the best insurance against consumption fluctuations.

How to construct a mean-variance efficient return?

- Consider a payoff $M/E(M^2)$.
- Its price is $E(M^2)/E(M^2) = 1$, so it is a return.
- It is on the mean-variance frontier.
- Thus, if we know M, we can construct a mean-variance efficient return.
- Notice that M is not on the mean-variance frontier. In fact, $P(M) = E(M^2) > 0$, so it is not a return.

Spanning frontier returns

- All frontier returns are perfectly correlated with each other.
- We can span any frontier return from two such returns (this is also called the two-fund theorem).
- Example: pick R^f and any single frontier return R^m:

$$R^{mv} = R^f + a(R^m - R^f),$$

for some number a.

Mean-variance efficient return and pricing information

- In addition, there exist constants a, b, c, d such that:

$$M = a + bR^{mv} \quad \text{and} \quad R^{mv} = d + eM.$$

- Thus, any mean-variance efficient return carries all pricing information.
- Given a mean-variance efficient return and the risk-free rate, we can find a discount factor that prices all assets and vice versa.

Roll (1977)'s theorem

- Given a discount factor, expected returns can be described in a single-beta representation using any mean-variance efficient return (except the risk-free rate):

$$E(R^i) = R^f + \beta_{i,mv}[E(R^{mv}) - R^f].$$

- This is Roll (1977)'s theorem (see also Roll and Ross (1994)).

Roll (1977)'s theorem

- Proof:
  - Start from: $E(R^i) = R^f + \beta_{i,M}\lambda_M$.
  - Recall that $|\rho(M, R^{mv})| = 1$.
  - Thus: $E(R^i) = R^f + \beta_{i,mv}\lambda_M$.
  - Note that: $E(R^{mv}) = R^f + \beta_{i,mv}\lambda_M = R^f + \lambda_M$.
  - Therefore: $E(R^i) = R^f + \beta_{i,mv}[E(R^{mv}) - R^f]$.
  - Last step depends on the fact that the beta models applies to every return including R^{mv} itself, and R^{mv} has a β of 1 on itself.

Intuition

- *Even though the means and standard deviations of returns fill out the space inside the mean-variance frontier, a graph of mean returns versus betas should yield a straight line.*

Idiosyncratic and Systematic Risk

- In the plot of the mean variance frontier we have seen that the return is decomposed into a priced or systematic component and a residual or idiosyncratic component.
- The priced part is perfectly correlated with the discount factor, and hence perfectly correlated with any frontier return.
- The residual part generates no expected return, and it is uncorrelated with the discount factor or any frontier return.

Idiosyncratic and Systematic Risk

- Asset inside the frontier or even on the lower portion of the frontier are not “worse” than assets on the frontier. The frontier and its internal region characterize equilibrium asset returns, with rational investors happy to hold all assets.

Conclusions

- A utility-based framework to price any assets.

- Basic pricing equation (Euler equation): $P = E(MX)$.

- Risk correction and asset prices.

What is next?

A look at the data and a discussion of the stylzed facts any asset pricing model should account for.

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Appendix

Proof: Log-normal trick (1/2)

- Recall: if r is distributed as $N(\mu, \sigma)$, then its density is:

$$f(r) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(r-\mu)^2/2\sigma^2} dr.$$

- If $\mu = 0$ and R is $LN(\mu, \sigma)$ then:

$$\begin{aligned}
E(R) = E(e^r) &= \int_{-\infty}^{\infty} e^r \frac{1}{\sqrt{2\pi}\sigma} e^{-r^2/2\sigma^2} dr \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{(2r\sigma^2 - r^2)/2\sigma^2} dr \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(r-\sigma^2)^2+\sigma^4}{2\sigma^2}} dr \\
&= e^{\frac{1}{2}\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(r-\sigma^2)^2}{2\sigma^2}} dr}_{=1 \text{ bc } N(\sigma^2, \sigma^2)} \\
&= e^{\frac{1}{2}\sigma^2}.
\end{aligned}$$

Proof: Log-normal trick (2/2)

- In the general case where $\mu \neq 0$, consider the following change of variable: $y = r - \mu$ e $dy = dr$.
- Then:

$$\begin{aligned} E(R) = E(e^r) &= \int_{-\infty}^{\infty} e^r \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(r-\mu)^2}{2\sigma^2}} \, dr \\ &= \int_{-\infty}^{\infty} e^{\mu+y} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{y^2}{2\sigma^2}} \, dy \\ &= e^{\mu} \int_{-\infty}^{\infty} e^{y} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{y^2}{2\sigma^2}} \, dy \\ &= e^{\mu + \frac{1}{2}\sigma^2}. \end{aligned}$$

▸ Back

Proof: Constant price/dividend ratio (I/II)

- Start from Euler equation for the price/dividend ratio 6 and recall that $D = C$:

$$\begin{aligned}
\frac{P_t}{C_t} &= E_t[\beta(\frac{C_{t+1}}{C_t})^{-\gamma}(1+\frac{P_{t+1}}{C_{t+1}})\frac{C_{t+1}}{C_t}] \\
\frac{P_t}{C_t} &= E_t[\beta(\frac{C_{t+1}}{C_t})^{1-\gamma}(1+\frac{P_{t+1}}{C_{t+1}})]
\end{aligned}$$

- Solve the recursion by iterating forward for T periods and using the law of iterated expectations:

$$\frac{P_t}{C_t} = E_t\sum_{j=1}^{T}[\beta^j(\frac{C_{t+j}}{C_t})^{1-\gamma}] + \underbrace{\beta^T E_t[(\frac{C_{t+T}}{C_t})^{1-\gamma}\frac{P_{t+T}}{C_t}]}_{\to 0}.$$

- Second term goes to 0 for $T \to \infty$: transversality or no bubble condition.

Back

Proof: Constant price/dividend ratio (II/II)

- We can write:

$$
\begin{aligned}
\frac{P_t}{C_t} &= \sum_{j\geq 1} \beta^j E_t[(\frac{C_{t+j}}{C_t})^{1-\gamma}] \\
&= \sum_{j\geq 1} \beta^j E_t[(\frac{C_{t+j}}{C_{t+j-1}})^{1-\gamma} \times \ldots (\frac{C_{t+1}}{C_t})^{1-\gamma}] \\
&= \sum_{j\geq 1} \beta^j E_t[\prod_{k=0}^{j-1} (\frac{C_{t+j-k}}{C_{t+j-1-k}})^{1-\gamma}] \\
&= \sum_{j\geq 1} \beta^j [e^{(1-\gamma)\mu+(1-\gamma)^2\frac{\sigma^2}{2}}]^j \\
&= \frac{\beta e^{(1-\gamma)\mu+(1-\gamma)^2\frac{\sigma^2}{2}}}{1-\beta e^{(1-\gamma)\mu+(1-\gamma)^2\frac{\sigma^2}{2}}},
\end{aligned}
$$

using the fact that log consumption growth is i.i.d. normal, the lognormal formula and the limit of a geometric series.

CCAPM

- Start from Euler equation for returns under power utility:

$$E_t[\beta(\frac{C_{t+1}}{C_t})^{-\gamma} R^i_{t+1}] = 1.$$

- Assume the log of the return and the log consumption growth are jointly normal. Then:

$$\begin{aligned} 1 &= E_t[e^{log\beta - \gamma\Delta c_{t+1} + r^i_{t+1}}], \\ 1 &= e^{log\beta - \gamma E_t \Delta c_{t+1} + E_t r^i_{t+1} + \frac{\gamma^2}{2}\sigma^2_{\sigma\Delta c} + \frac{1}{2}Var_t(r^i_{t+1}) - \gamma Cov_t(\Delta c_{t+1}, r^i_{t+1})}, \\ 0 &= log\beta - \gamma E_t \Delta c_{t+1} + E_t r^i_{t+1} + \frac{\gamma^2}{2}\sigma^2_{\sigma\Delta c} + \frac{1}{2}Var_t(r^i_{t+1}) - \gamma Cov_t(\Delta c_{t+1}, r^i_{t+1}). \end{aligned}$$

- Do same for risk-free rate:

$$r^f_{t+1} = -log\beta + \gamma E_t \Delta c_{t+1} - \frac{\gamma^2}{2}\sigma^2_{\Delta c}.$$

- Subtract the risk-free rate from return on asset i:

$$E_t[\log(\frac{R^i_{t+1}}{R^f_{t+1}})] + \frac{1}{2}Var_t \log R^i_{t+1} = \log E_t(\frac{R^i_{t+1}}{R^f_{t+1}}) = \gamma Cov_t(\Delta c_{t+1}, \log R^i_{t+1}).$$

Back

Martingale

A martingale is a sequence of random variables for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values at current time.

Back

Law of iterated expectations

- Simple version of the law of iterated expectations (LIE) says that (see Woolridge (2010)):

$$E(Y) = E_X[E(Y|X)].$$

- Think of X as a discrete vector taking possible values $C_1, C_2, \ldots, C_N$ with probabilities $p_1, p_2, \ldots, p_N$ then:

$$E(Y) = p_1 E(Y|X = c_1) + p_2 E(Y|X = c_2) + \ldots + p_N E(Y|X = c_N).$$

- $E(Y)$ is a weighted average of the averages.
- In words, the unconditional expectation of the conditional expectation of Y conditional on X is equal to the unconditional expectation of Y.

Law of iterated expectations

- With time-series, the LIE implies that your current best guess of your best guess next period of the realization of X two periods from now is equal to your current best guess of X two periods from now:

$$E_t[E_{t+1}(X_{t+2})] = E_t(X_{t+2}).$$

Back

Elasticity of inter-temporal substitution

- Consider a utility function over consumption bundles of this kind:

$$U = U(c_t, c_{t+1}, \ldots, c_{t+j}, \ldots).$$

- The elasticity of inter-temporal substitution (IES) between period $t+j$ and $t+i$ is equal to:

$$\text{IES} = -\frac{\frac{\partial(c_{t+j}/c_{t+i})}{c_{t+j}/c_{t+i}}}{\frac{\partial(U_{c_{t+j}}/U_{c_{t+i}})}{U_{c_{t+j}}/U_{c_{t+i}}}} = -\frac{\partial \ln(c_{t+j}/c_{t+i})}{\partial \ln(U_{c_{t+j}}/U_{c_{t+i}})},$$

  where U_c denote the partial derivative of utility with respect to consumption.

- The IES measures the inverse of the percentage change in marginal utility at j and i that one percent change in the ratio of consumption at the same dates leads to.

▸ Back

IES for additively separable CRRA utility functions

$$\begin{aligned}
\text{Utility function:} \quad U(c_i) &= \frac{c_i^{1-\gamma}}{1-\gamma} \\
\text{Marginal utility:} \quad U_c &= c^{-\gamma} \\
\text{Marginal utility ratio:} \quad \frac{U_{c_{t+j}}}{U_{c_{t+i}}} &= \left(\frac{c_{t+j}}{c_{t+i}}\right)^{-\gamma} \\
\text{Logarithmic form:} \quad \ln\left(\frac{U_{c_{t+j}}}{U_{c_{t+i}}}\right) &= -\gamma \ln\left(\frac{c_{t+j}}{c_{t+i}}\right) \\
\text{Rearranging:} \quad \ln\left(\frac{c_{t+j}}{c_{t+i}}\right) &= -\frac{1}{\gamma} \ln\left(\frac{U_{c_{t+j}}}{U_{c_{t+i}}}\right) \\
\text{IES:} \quad -\frac{\partial \ln\left(\frac{c_{t+j}}{c_{t+i}}\right)}{\partial \ln\left(\frac{U_{c_{t+j}}}{U_{c_{t+i}}}\right)} &= \frac{1}{\gamma}
\end{aligned}$$

Back

Elasticity of inter-temporal substitution

- Note that the marginal utility of consumption is decreasing in consumption.
- Therefore, we know that when $(c_{t+j}/c_{t+i}) \uparrow$, then $(U_{c_{t+j}}/U_{c_{t+i}}) \downarrow$
- The (inverse of the) IES says by how much.
- For CRRA utility, IES $= \frac{1}{\gamma}$.
- The larger is γ, the smaller the IES, and the less substitutable consumption between periods j and i is.
- In an intertemporal setup, the larger is γ, the more an individual will want to keep a smooth intertemporal consumption profile.

Back

Coefficient of relative risk-aversion

- The Arrow-Pratt coefficient of relative risk-aversion (RRA) is defined as:

$$\mathrm{RRA} = \frac{-cU_{cc}}{U_c}.$$

- For the CRRA utility functions:

$$\mathrm{RRA} = \frac{-c(-\gamma)c^{-\gamma-1}}{c^{-\gamma}} = \gamma.$$

- Therefore, the RRA is the inverse of the IES.
- The parameter γ controls at the same time IES and RRA.

▸ Back

St. Petersburg paradox (I/II)

- Consider a bet involving the toss of a fair coin.
- Start with \$2.
- The game then is as follows:
    1. toss a coin
    2. if "heads" double the money on the plate
    3. if "tail" game ends and win what is on the plate
- The expected payoff is

$$\frac{1}{2}(2) + \frac{1}{4}(4) + \frac{1}{8}(8) + \ldots = 1 + 1 + \ldots = \infty.$$

- Therefore, the bet is a random variable with a infinite expected value.

▸ Back

St. Petersburg paradox (II/II)

- What is the value of this bet?
- If investor are risk-neutral (i.e., linear utility), the price should be ∞.
- Experimental investigations show investors are willing to pay little for this bet. How come?
- Assume investors are averse to risk and have log utility and initial wealth W. The change in utility after paying c for the bet is:

$$\Delta E(u) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left[\ln(W + 2^k - c) - \ln(W) \right] < \infty$$

  where $k > 0$ is the number of times a coin has been tossed.
- If $W = \$1m$, then $c = \$20.88$; if $W = \$1K$, then $c = \$10.95$.

Back

CRRA utility

- To prove the CRRA utility function converges to log utility as $\gamma \longrightarrow 1$, we use l'Hôpital's rule.[2]
- Differentiate both the numerator and the denominator with respect to γ, and then take the limit of the ratio as $\gamma \longrightarrow 1$.

$$
\begin{aligned}
\text{Original expression:} \quad & \frac{C^{1-\gamma}-1}{1-\gamma} \\
\text{Numerator:} \quad & f(\gamma) = C^{1-\gamma} - 1 = e^{(1-\gamma)\ln C} - 1 \\
\text{Denominator:} \quad & g(\gamma) = 1-\gamma \\
\text{Derivative of the numerator:} \quad & f'(\gamma) = -C^{1-\gamma}\ln C \\
\text{Derivative of the denominator:} \quad & g'(\gamma) = -1 \\
\lim_{\gamma\to 1}\frac{f(\gamma)}{g(\gamma)} \quad = \quad & \lim_{\gamma\to 1}\frac{f'(\gamma)}{g'(\gamma)} = \lim_{\gamma\to 1} C^{1-\gamma}\ln C = \ln C
\end{aligned}
$$

back

[2] We add a -1 to the numerator to get the exact result. Note that adding or subtracting a constant does not change the utility function.

Precautionary savings (I/II)

- Consumption variance has an effect on the *level of consumption*: higher variance lowers consumption *today*, and therefore increases saving and decreases the interest rate.

- This effect depends on the characteristics of the utility function: it is the positive third derivative of the utility function that leads to precautionary savings.

▸ back

Precautionary savings (II/II)

- You can see this through a second-order Taylor approximation.
- Let $\bar{C} = E(C)$.
- Take a Taylor approximation around $C = \bar{C}$ of $u'(C)$:

$$u'(C) \approx u'(\bar{C}) + u''(\bar{C})(C - \bar{C}) + \frac{1}{2}u'''(\bar{c})(C - \bar{C})^2$$

- Take the expected value:

$$E[u'(C)] \approx u'(\bar{C}) + \frac{1}{2}u'''(\bar{c})Var(C).$$

- If and only if $u''' > 0$, a rise in the variance of consumption, holding $\bar{C}$ constant, raises $E[u'(C)]$.

back