Index Models

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EXECUTIVE
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- The Markowitz portfolio model has some weaknesses:
  - it requires the estimates of a large number of covariances.
  - the model does not provide guidelines to estimate the risk premia that are necessary to build the efficient frontier.

- We now want to look at index models, which simplify a lot the estimation of the variance-covariance matrix and improve the analysis of the risk premia offered by the different assets.
- These models allow to clearly and simply separate systematic from idiosyncratic risk.

- Suppose you want to use the Markowitz model.
- Suppose further you are considering 50 assets.
- In this case, your input list is as follows:
  1. 50 estimates of expected returns,
  2. 50 estimates of variances,
  3. 1,225 estimates of covariances (i.e., $(n^2 - n)/2$),
  4. for a total of 1,325 estimates!
- Suppose now you are interested in an even larger portfolio, for example including all the 3,000 stocks in the NYSE ...
- In this case you will need about 4.5 million estimates!

- In addition, estimation errors on the correlation coefficients can produce meaningless results.
- For example, consider:

		Correlation matrix		
Asset	SD(%)	A	B	C
A	20	1.00	0.90	0.90
B	20	0.90	1.00	0.00
C	20	0.90	0.00	1.00

- Try now building a portfolio with weights -1.00, 1.00 e 1.00 for assets A, B e C: you will find a **negative** variance!
- Therefore, in the example above the estimated correlation matrix must be mutually inconsistent (technically, a correlation matrix that cannot generate a negative portfolio variance is *positive definite*).

- Our goal now is to build a simpler model, that requires a smaller number of estimates.
- We can achieve this goal because often the covariances among assets are positive, since assets are influenced by the same factors (i.e., business cycle, term structure of interest rates, commodity prices, etc.).
- *Unexpected* variations in these factors cause *unexpected* variations in the returns of the different assets.
- Therefore, uncertainty on r_i depends on the uncertainty on e_i.

A Single-Factor Security Market

- Let's start by decomposing the return on asset i as follows:

$$r_i = E(r_i) + e_i$$

- $\Rightarrow E(r_i)$ is the **expected** component.
- $\Rightarrow e_i$ is the **unexpected** component, which we assume has zero mean and standard deviation σ_i.

A Single-Factor Security Market

- If asset returns can be approximated by normal distributions correlated one to the other, we will say that they are *jointly normally distributed*.
- When more than one variable drives normally distributed asset returns, these returns are said to have a *multivariate* normal distribution.
- We start with the simpler case in which only one variable drives asset returns: in this case we have a **single-factor security market**.
- Extensions to multiple factors are straightforward.

A Single-Factor Security Market

- Suppse that a **common factor** m influences the innovations in the asset returns (e_i).
- For example, we can think of m as a factor that affects all firms, maybe the business cycle.
- In this case, we can decompose the return on i as:

$$r_i = E(r_i) + \underbrace{m}_{\text{aggregate uncertainty}} + \underbrace{e_i}_{\text{idiosyncratic uncertainty on } i}$$

- Note that m does not have the i index since it is common to all firms.
- m captures macro **surprises** (i.e., unexpected macro shocks) and has mean zero and standard deviation σ_m.
- What is the correlation between m and e_i? $m \perp e_i$

A Single-Factor Security Market

- The correlation between m and e_i is equal to zero because e_i represent firm-specific risk, that is independent from aggregate risk.
- Therefore, the variance of r_i is equal to:

$$\sigma_i^2 = \sigma_m^2 + \sigma^2(e_i),$$

- The covariances between returns i and j are:

$$Cov(r_i, r_j) = Cov(m + e_i, m + e_j) = \sigma_m^2.$$

- $\Rightarrow$ Note how only the **common factor** generates correlation among the different assets.

A Single-Factor Security Market

- We can think that some assets are influenced by aggregate risk more than others.
- We call the sensitivity coefficient to aggregate risk β_i (beta).
- We are now ready to write the single-factor model:

$$r_i = E(r_i) + \beta_i m + e_i.$$

A Single-Factor Security Market

- **Systematic risk** of asset i is determined by the coefficient β_i: $\beta_i^2 \sigma_m^2$.
- Intuitively, the larger is β_i, the larger is the systematic risk component.
- **Total risk** is equal to:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2(e_i).$$

Concept check

- What do you expect the β of a *pro-cyclical* stock to look like? What about that of a *defensive* stock?

A Single-Factor Security Market

- The covariance between two assets also depend on their respective βs:

$$Cov(r_i, r_j) = Cov(\beta_i m + e_i, \beta_j m + e_j) = \beta_i \beta_j \sigma_m^2.$$

A Single-Factor Security Market

- The assumption of normality of asset returns guarantees that portfolio returns are also normal and that there is a linear relationship between security returns and the common factor.
- But what is the common factor? What characteristics should it have?
- We seek a variable that is observable, so that we can estimate its volatility as well as the sensitivity of individual securities to variation in its value.

A Single-Factor Security Market

- What variable can possibly be a good candidate to proxy the single factor?
- A good starting point is represented by the choice of the return on a broad market index, for example the S&P 500.
- If we use the return on the S&P 500 as proxy for m, we have a single-factor model, where the market index is the common factor.

A Single-Factor Security Market

- One of the advantages of using the S&P 500 it's that historical returns are observed on long time series.
- Define the excess returns, and the standard deviation of the market index (M) as follows:

$$R_M = r_m - r_f \text{ and } \sigma_M,$$

- Note how the index model is **linear**, in other words is given by the sum of different factors.
- Since the model is linear, we can estimate β as slope coefficient in a linear regression with a single regressor (and a constant).

A Single-Factor Security Market

- The regression equation is:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t),$$

  where $R_i = r_i - r_f$ represents the excess return.
- The intercept represents the expected return when the excess market return is zero.
- The slope is called the *beta* of an asset, and represents the sensitivity of the asset with respect to the single factor, or the market index.
- e_i represents a firm specific surprise (i.e., shock).

The expected return-beta relationship

- In order to determine the relationship between expected return and beta, we can take the expected value of the excess return on asset i:
$$E(R_i) = \alpha_i + \beta_i E(R_M).$$
- Part of the risk premium on asset i depends on the market risk premium (systematic, or aggregate, risk).
- We label α the risk premium that is uncorrelated with the market.

Risk and covariance in the single-index model

- Total risk = Systematic risk + Specific risk:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i).$$

- Covariance = product of the betas × market risk:

$$\beta_i \beta_j \sigma_M^2.$$

- Correlation = product of the correlations with the market index:

$$Corr(r_i, r_j) = \frac{\beta_i \beta_j \sigma_M^2}{\sigma_i \sigma_j} = \frac{\beta_i \sigma_M^2 \beta_j \sigma_M^2}{\sigma_i \sigma_M \sigma_j \sigma_M} = Corr(r_i, r_M) \times Corr(r_j, r_M).$$

Risk and covariance in the single-index model

- Therefore, the required parameters in the single index model are only $\alpha, \beta, \sigma(e)$ for the different portfolio assets, and the market risk premium and variance.
- Much simpler than the Markowitz model!

Risk and covariance in the single-index model

- We have now learnt what the advantages of the single index model are.
- But what are the costs?
  - strong restriction on the source of risk (macro vs. micro),
  - industry level events.
- Possible solution? Multi-factor models.

The index model and diversification

- Suppose you have chosen a portfolio with n assets with **equal weights**.
- The excess return for each asset i is:

$$R_i = \alpha_i + \beta_i R_M + e_i.$$

- The return on the portfolio p is:

$$R_p = \alpha_p + \beta_p R_M + e_p.$$

- It is possible to show that when $n \to \infty$ the fraction of risk that depends on non-market factor decreases, while the market risk fraction does not (*principle of diversification*).

The index model and diversification

- Excess portfolio return:

$$R_p = \sum_i^n w_i R_i = \frac{1}{n}\sum_i^n \alpha_i + (\frac{1}{n}\sum_i^n \beta_i)R_M + \frac{1}{n}\sum_i^n e_i$$

  where we assume uniform weights $w_i = 1/n$

- Therefore, the portfolio sensitivity to market risk is equal to the average beta:

$$\beta_p = \frac{1}{n}\sum_i^n \beta_i.$$

- In addition, there is the non-market risk component:

$$\alpha_p = \frac{1}{n}\sum_i^n \alpha_i.$$

- We define with $e_p = \frac{1}{n}\sum_i^n e_i$ the zero-mean variable.

The index model and diversification

- Portfolio variance is:

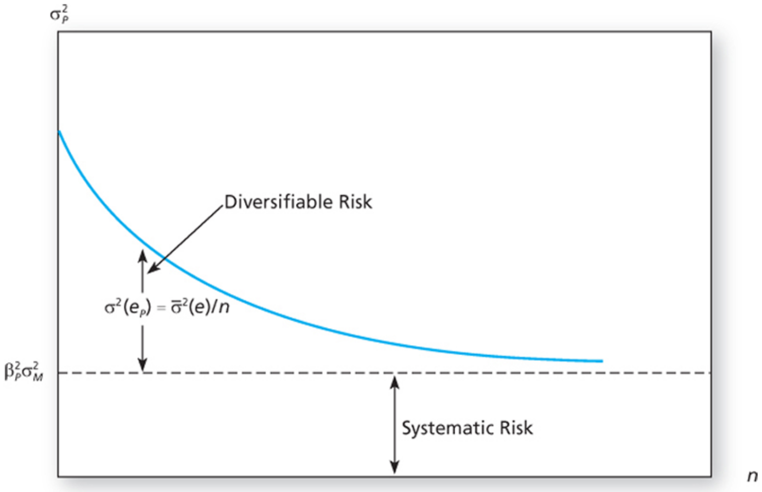
$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \sigma^2(e_p).$$

- Note how β_p does not decrease when n increases.
- Therefore, $\beta_p^2 \sigma_M^2$ does not disappear when $n \uparrow$ (if most assets have $\beta > 0$, and when $n \to \infty$ we should expect $\beta_p \to 1$).
- Since the e are independent, the non-systematic part of portfolio variance is:

$$\sigma^2(e_p) = \sum_i^n (\frac{1}{n})^2 \sigma^2(e_i) = \frac{1}{n} \overline{\sigma}^2(e).$$

- When $n \to \infty$, $\sigma^2(e_p) \to 0$.

The index model and diversification



Portfolio Construction and the Single-Index Model

- How do we prepare the input list for the single-index model?
  1. **Macro analysis** is used to estimate the risk premium and the voltility of the market index.
  2. **Statistical analysis** is used to estimate the betas for each stock and the variances of the residuals ($\sigma^2_{e_i}$).
  3. The **portfolio manager** uses the inputs from point 1 and 2 to estimate the expected return of each stock absent any contribution from security analysis. In this way, the manager obtains a measure of market expected return that is then used as **benchmark**.
  4. Firm-specific expected returns (**alphas**) are estimated using different models (not necessarily the linear model).
- Therefore, the values of the alphas describe incremental risk premium attributable to private information developed by the security analyst.

Portfolio Construction and the Single-Index Model

- In the linear model we have examined so far, the expected return of each stock not subject to security analysis is simply $\beta_i E(R_M)$
- Any additional firm-specific return depends on non-market factors analyzed by the security analyst.
- Note that what makes a particular stock more interesting to a portfolio manager is exactly the alpha (why?).
- Positive and negative alphas.

Portfolio Construction and the Single-Index Model

- Consider now only stocks included in the SP500.
- Suppose you can only invest in a subset of these stocks.
- In order to avoid an insufficient diversification, you can add to the portfolio the index itself.
- The index can be thought of as a passive portfolio, with a beta equal to 1 and alpha to zero.

Portfolio Construction and the Single-Index Model

- Suppose now you have n components of the SP500 in your portfolio, and the passive index that replicates the SP500.
- The input list that the portfolio manager requires is as follows:
  1. Risk premium on the SP500.
  2. Estimate of the standard deviation of the SP500.
  3. n estimate of the beta coefficients, n variances of the residuals, and n alphas.

Portfolio Construction and the Single-Index Model

- With the betas, alphas and market risk premium we can generate $n + 1$ expected returns.
- With the betas, variances of residuals, and variance of market index we can generate the variance-covariance matrix.
- Therefore, we can follow the same algorithm we have studies when we talked about optimal risky portfolios!

Portfolio Construction and the Single-Index Model

- We need to maximize the SR using a set of weights $w_1, ..., w_{n+1}$.
- SR:

$$SR = \frac{E(R_p)}{\sigma_p}.$$

- The expected portfolio return is:

$$E(r_p) = \alpha_p + E(R_M)\beta_P = \sum_{i=1}^{n+1} w_i\alpha_i + E(R_M)\sum_{i=1}^{n+1} w_i\beta_i.$$

- The standard deviation of the portfolio is:

$$\sigma_p = [\beta_p^2\sigma_M^2 + \sigma^2(e_p)]^{1/2} = [\sigma_M^2(\sum_{i=1}^{n} w_i\beta_i)^2 + \sum_{i=1}^{n+1} w_i^2\sigma^2(e_i)]^{1/2}.$$

Portfolio Construction and the Single-Index Model

- We can solve this constrained maximization problem using Matlab or similar software packages adding the feasibility constraints such that the sum of the weights equal 1.
- Or we could solve the problem with paper and pencil.

Portfolio Construction and the Single-Index Model

- The optimal portfolio depends on the combination of two components:
  1. an active portfolio (A) with n stocks,
  2. the market index, or passive portfolio (M).
- The task of security analysis is to uncover the individual stocks with non-zero alphas and take the right investment position on these assets.
- The cost of this strategy comes from the departure from an efficient diversification strategy.

Portfolio Construction and the Single-Index Model

- The positive contribution of an individual stock to the portfolio comes from the non-market risk premium (i.e., its alpha).
- The negative contribution comes instead from the increase in the portfolio variance, through firm specific risk.

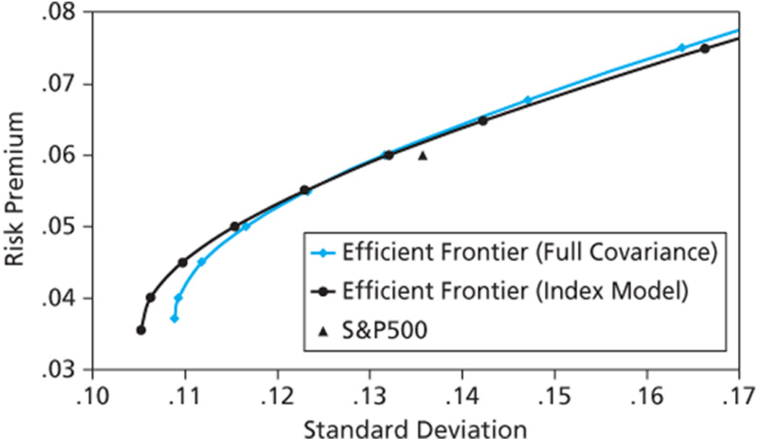
Portfolio Construction and the Single-Index Model

- Note that a stock with $\alpha < 0$ has a negative weight (short position).
- If it is not possible to do short-sales, stocks with $\alpha < 0$ are deleted from the portfolio.
- Note also that the passive portfolio is efficient only when alphas are all equal to zero.

Portfolio Construction and the Single-Index Model

- How can we compare the single-index model to the Markovitz model?
- Let's look at the efficient frontier.

Portfolio Construction and the Single-Index Model



Practical Aspects of Portfolio Management with the Index Model

- If the portfolio manager doesn't have special private info on the stock, she will base her estimates on:

$$E(r_i) = r_f + \beta_i[E(r_M - r_f)].$$

- The parameters' in the equation are estimated via OLS on the historical time series.
- Merrill Lynch: *Security Risk Evaluation* (beta book).
- Note how Merrill Lynch (as most investment banks) does not use excess returns:

$$r = a + br_M + e^*.$$

- We have used instead:

$$r - r_f = \alpha + \beta(r_M - r_f) + e.$$

Practical Aspects of Portfolio Management with the Index Model

- We can always re-write the equation line we have used so far as:

$$r = r_f + \alpha + \beta r_M - \beta r_f + e = \alpha + r_f(1 - \beta) + \beta r_M + e.$$

- As long as r_f is constant, the two regressions are identical and have the same slope.
- Also, note how Merrill Lunch does not include dividends in the computations of returns.

Practical Aspects of Portfolio Management with the Index Model

- Merrill Lynch also uses an *ad hoc* adjusted beta:

$$\text{Adjusted beta} = 2/3 \text{ sample beta} + 1/3\ (1).$$

Practical Aspects of Portfolio Management with the Index Model

- Betas estimated on historical data might non be the best estimates for future betas.
- Alternative approach:

$$\text{Current beta} = \text{a} + \text{b}(\text{Past beta}),$$

- and then:

$$\text{Forecast beta} = \text{a} + \text{b}(\text{Current beta}).$$

- Even more complicated estimation strategies are possible:

$$\text{Current beta} = \text{a} + b_1(\text{Past beta}) + b_2(\text{Firm Size}) + b_3(\text{Debt ratio}).$$

Practical Aspects of Portfolio Management with the Index Model

- Variables that can be used to forecast beta:
  - variance of revenues,
  - variance of cash-flows,
  - growth rate of revenue per share,
  - market capitalization,
  - dividend yield,
  - debt-to-assets ratio.
- Rosenberg e Guy: recommend the use of sectoral adjustment in the estimation of betas.

Practical Aspects of Portfolio Management with the Index Model

Table 8.4

Industry betas and adjustment factors

Industry	Beta	Adjustment Factor
Agriculture	0.99	−.140
Drugs and medicine	1.14	−.099
Telephone	0.75	−.288
Energy utilities	0.60	−.237
Gold	0.36	−.827
Construction	1.27	.062
Air transport	1.80	.348
Trucking	1.31	.098
Consumer durables	1.44	.132

Practical Aspects of Portfolio Management with the Index Model

- Suppose now you have estimated the following linear equation for a portfolio you believe to be underpriced:

$$R_p = 0.01 + 1.4 R_{SP500} + e_p,$$

  where R is an excess return.
- Hence, this portfolio has: $\alpha = 1\%$ and $\beta = 1.4$.
- Even if you believe this portfolio to be underpriced, you are concerned with **aggregate risk**.

Practical Aspects of Portfolio Management with the Index Model

- How can an investor take a position based on the estimate on α, that is independent on aggregate risk?
- A possibility is to build a **tracking portfolio** (T):
  - the tracking portfolio is designed to replicate the systematic component of returns,
  - the tracking portfolio must have the same beta on portfolio P, and the smallest idiosyncratic risk possible (**beta capture**),
  - the tracking portfolio will have a levered position on the SP500, in order to obtain a beta equal to 1.4,
  - in particular, portfolio T has a long position 1.4 on the SP500, and a short position equal to -0.4 on T-bills (i.e., the alpha of T is equal to zero).

Practical Aspects of Portfolio Management with the Index Model

- The combined portfolio (C) has the following excess return

  $$R_C = R_p - R_T = (0.01 + 1.4R_{SP500} + e_p) - 1.4R_{SP500} = 0.01 + e_p,$$

  where R is an excess return.
- The strategy of separating the research of α from the exposure to aggregate risk is called *alpha transport*.
- The new portfolio is also called *market neutral*.
- The one described is a typical **long-short** investment strategy, followed for example by many hedge funds.