Efficient Diversification

Nicola Borri

LUISS University

Portfolio Investment & Management—Professional MBA
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EXECUTIVE
ACADEMY

> *One should always divide his wealth into three parts: a third in land, a third in merchandise, and a third ready to hand.*

Rabbi Issac bar Aha, 4th century AD

> *Diversification is a protection against ignorance.*

Warren Buffet

Outline

Outline

Diversification and Portfolio Risk

Portfolio with two Risky Assets

Asset Allocation with Stocks, Bonds and Bills

The Markowitz Portfolio Optimization Model

- Capital allocation:
  - it determines investors' exposure to risk,
  - it depends on the degree of risk aversion, and on the expectations about the risk-reward characteristics of the risky portfolios.
- We will show that there **exists** an optimal portfolio.
- We will show the benefits of a two-stage investment strategy (*two-fund separation theorem*):
  1. capital allocation,
  2. asset selection.
- We will show that the two steps can be solved independently.

- We will consider portfolios with a short-term horizon.
- This assumption is not too problematic: even when the horizon is long-term, we can always rebalance the portfolio more frequently.
- For short horizons, the skewness of returns is absent (while it is a characteristics of returns compounded for long period of time).
- For this reason, the assumption about normality of holding period returns (HPR) is sufficient (note: as a result, we would focus on the mean and variance of portfolio returns).

Diversification and Portfolio Risk

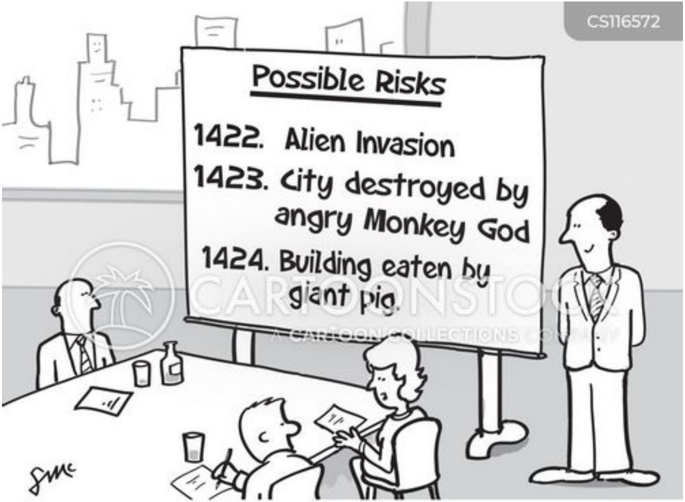
- Suppose you have only one stock in your portfolio: DELL (tech sector).
- What are the sources of risk?
  1. risk that originates in the general economic conditions (business cycle, inflation, etc.): these are referred to as macro risk factors.
  2. risk that is **specific** to DELL (e.g., success in R&D, competitors, etc.).
- Suppose you decide to diversify your portfolio simply by investing half of your wealth in DELL, and the other half in ExxonMobil (oil sector).
- What could happen in this case?

Portfolio diversification I

- Suppose that the stock's specific risk influences the two companies differently: in this case, **portfolio diversification** reduces portfolio risk.
- But, if this is the case, why then limiting diversification to only two stocks?
- By increasing the number of stocks in the portfolio, it is possible to further diversify firm-specific risk.

Portfolio diversification II

- However, even if we increase the number of stocks in our portfolio, we cannot diversify out macro risk, since the latter influences *all* the stocks in the same way.
- Therefore, if all stocks are influenced similarly by the business cycle we cannot *easily* hedge the exposure from business cycle risk.
- Think about the recent COVID shock: stock indices around the world collapsed all at the same time ... it would have been very hard to diversify that kind of risk.



"Well he certainly does a very thorough risk analysis."

Systematic vs idiosyncratic risk

- If all the risks are firm specific, then diversification can reduce risk almost completely: this is called **insurance principle**.
- On the contrary, when common sources of risk influence all the firms, even a good diversification cannot reduce risk to zero.
- Risk that cannot be diversified is referred to as market risk, or **systematic risk**.
- Risk that can be diversified is referred to as non-systematic risk, **or idiosyncratic risk**.

Risk and number of stocks in a portfolio I

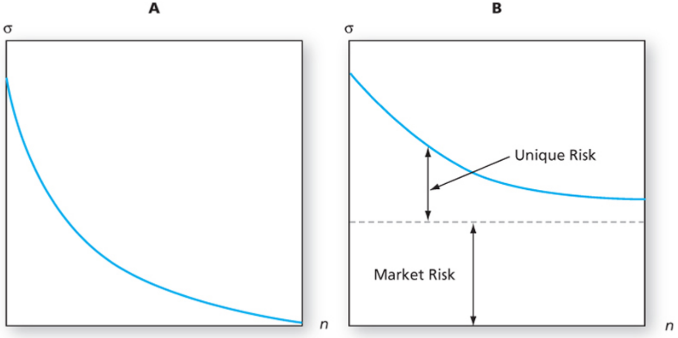


Figure: Portfolio risk as a function of the number of stocks in the portfolio

Risk and number of stocks in a portfolio II

- In the figure in the next slide you can see the effects of diversification using NYSE data.
- Diversification is obtained with a *naive* strategy that uses the same number of shares for each stock (**equally weighted portfolio**).

Portfolio diversification (NYSE, equally weighted)

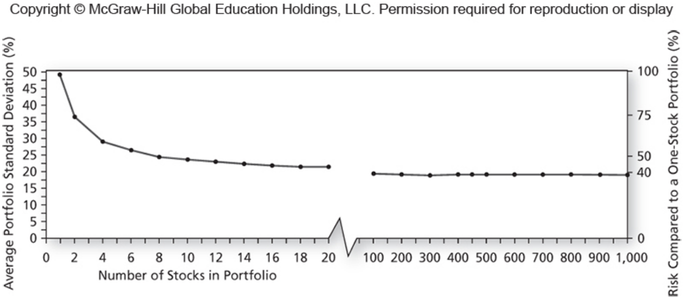


Figure: Portfolio diversification (NYSE, equally weighted)

Empirical evidence

- The mean annual SD for an average stock is 49%.
- The mean annual covariance between stocks is 0.037, and the correlation is equal to 39%.
- Since the covariance is **positive**, even a very large equity portfolio is *risky*.

Outline

Towards an efficient diversification

- Is it possible to improve on a equally weighted diversification?
- We would like to determine the most efficient diversification: **the one that minimizes risk for each given level of expected return**.
- For the time being, let's simplify and let's think at a diversification strategy with only two assets: a fixed-income fund that invests in debt securities (D) and an equity fund that invests in stocks (E).

Two risky assets

Table 7.1

Descriptive statistics for two mutual funds

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%
Covariance, $Cov(r_D, r_E)$	72	
Correlation coefficient, ρ_{DE}	.30	

Recall the definition of correlation coefficient: $\rho_{DE} = \frac{Cov(r_D, r_E)}{\sigma_D \sigma_E}$, and $\rho_{DE} \in [-1, 1]$.

Portfolio with two risky assets

- A share w_D is invested in the fixed-income fund.
- A share $1 - w_D$ is instead invested in the equity fund.
- The return of the portfolio is simply:

$$r_P = w_D r_D + w_E r_E.$$

Expected portfolio return and variance

- The expected return of the portfolio is:

$$E(r_P) = w_D E(r_D) + w_E E(r_E),$$

- And the variance of the portfolio is:

$$\sigma_P^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E),$$

and depends on the covariance!

The portfolio variance

- Note how portfolio variance is not simply the weighted sum of individual variances.
- To understand the formula for portfolio variance, recall that: $Cov(r_D, r_D) = \sigma_D^2$.
- Therefore, we can re-write the formula for the portfolio variance as:

$$\sigma_P^2 = w_D w_D Cov(r_D, r_D) + w_E w_E Cov(r_E, r_E) + 2 w_D w_E Cov(r_D, r_E).$$

Variance-covariance matrix

- To understand the computation of the portfolio variance, a useful visual tool is the **bordered covariance matrix**: the covariance matrix with the portfolio weights for each fund placed on the borders.

- To find portfolio variance, first multiply each element in the covariance matrix by the pair of portfolio weights in its row and column, then add up all the terms.

Bordered covariance matrix

Table 7.2 Computation of portfolio variance from the covariance matrix

A. Bordered Covariance Matrix		
Portfolio Weights	w_D	w_E
w_D	$\mathrm{Cov}(r_D, r_D)$	$\mathrm{Cov}(r_D, r_E)$
w_E	$\mathrm{Cov}(r_E, r_D)$	$\mathrm{Cov}(r_E, r_E)$
B. Border-Multiplied Covariance Matrix		
Portfolio Weights	w_D	w_E
w_D	$w_D w_D \mathrm{Cov}(r_D, r_D)$	$w_D w_E \mathrm{Cov}(r_D, r_E)$
w_E	$w_E w_D \mathrm{Cov}(r_E, r_D)$	$w_E w_E \mathrm{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \mathrm{Cov}(r_D, r_D) + w_E w_D \mathrm{Cov}(r_E, r_D)$	$w_D w_E \mathrm{Cov}(r_D, r_E) + w_E w_E \mathrm{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \mathrm{Cov}(r_D, r_D) + w_E w_D \mathrm{Cov}(r_E, r_D) + w_D w_E \mathrm{Cov}(r_D, r_E) + w_E w_E \mathrm{Cov}(r_E, r_E)$	

Figure: Bordered covariance matrix

Note that most modern statistical software will immediately *estimate* the covariance matrix for you.

The benefits of diversification

- Note how the variance is not a simple weighted sum of the variances of the individual assets.
- Portfolio variance is smaller when the covariance term is negative (i.e., that's how diversification reduces the idiosyncratic risk component).
- More in general, the standard deviation of a portfolio is always smaller than the weighted average of the standard deviations of the individual components **unless these assets are perfectly positively correlated**.

Diversification with two assets

- Recall the formula for the Pearson correlation coefficient:

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E.$$

- Therefore:

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \rho_{DE}\sigma_D\sigma_E.$$

- If $\rho_{DE} = 1$, the expression above is a perfect square and:

$$\sigma_p = w_D\sigma_D + w_E\sigma_E.$$

Hedges

- We will say that an asset is a "hedge" when it has a negative correlation with respect to the other assets.
- Note how a "hedge" reduces total risk, leaving the expected return unchanged:

$$E(r_P) = w_D E(r_D) + w_E E(r_E),$$

and

$$\sigma_P^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \rho_{DE} \sigma_D \sigma_E.$$

- Other things equal, we will always prefer to add to our portfolios assets with low or, even better, negative correlation with our existing position.
- Portfolios of less than perfectly correlated assets always offer some degree of diversification benefit.

Perfect hedge

- Recall that the minimum value of the correlation coefficient is $\rho = -1$. In this case:

$$\sigma_P^2 = (w_D\sigma_D - w_E\sigma_E)^2.$$

- Therefore, now we are in the position to easily construct a **perfect hedge**: we simply need to find the weights that minimize the variance by solving:

$$w_D\sigma_D - w_E\sigma_E = 0.$$

Perfect hedge and the risk-free rate

- The perfect hedge portfolio has $\sigma = 0$, i.e., no risk.
- Therefore, the perfect hedge is a risk-free asset.
- **By no arbitrage**, all risk-free assets must offer the same expected return (abstracting from differences in liquidity, or other asset-specific characteristics, such as pledgeability as collateral).
- Then, the expected return of the perfect hedge IS the risk-free rate.

Solving for the perfect hedge

- We can thus solve for the optimal weights:

$$w_D = \frac{\sigma_E}{\sigma_D + \sigma_E},$$

- and $w_E = (1 - w_D)$, or :

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E}.$$

What happens when we change w_D and w_E?

Changing the weights w_D and w_E

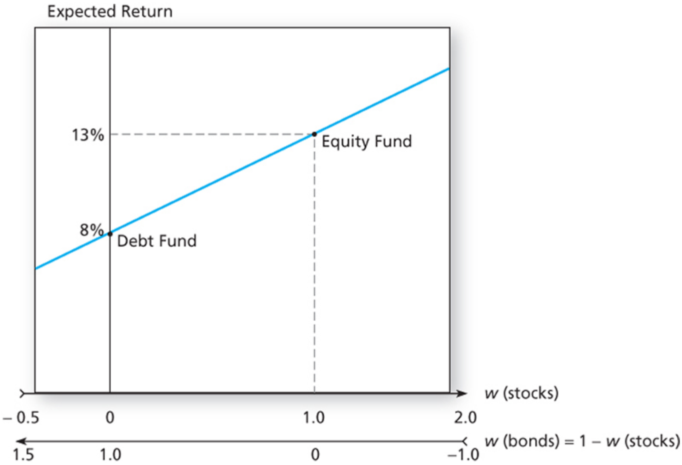


Figure: Portfolio expected return as a function of investment proportions.

Changing the weights w_D and w_E

- When w_D changes from 0 to 1, the expected return changes from 13% to 8%.
- What happens if $w_D > 1$ and $w_E < 0$, maintaining the feasibility constraint $w_D + w_E = 1$?

Portfolio expected return

- If $w_D > 1$ and $w_E < 0$ the strategy implies **selling short** the equity fund, and buying (going long) the debt fund.

- Note how the relationship between expected portfolio return and shares of the equity and debt fund is linear:

$$E(r_P) = w_D E(r_D) + (1 - w_D) E(r_E).$$

Portfolio variance

- Variations of w_D e w_E also determine changes in the portfolio variance:

$$\sigma_P^2 = w_D^2\sigma_D^2 + (1 - w_D)^2\sigma_E^2 + 2w_D(1 - w_D)Cov(r_D, r_E).$$

- In the figure in the next slide we can observe the relationship between portfolio weights and standard deviation.

Portfolio SD

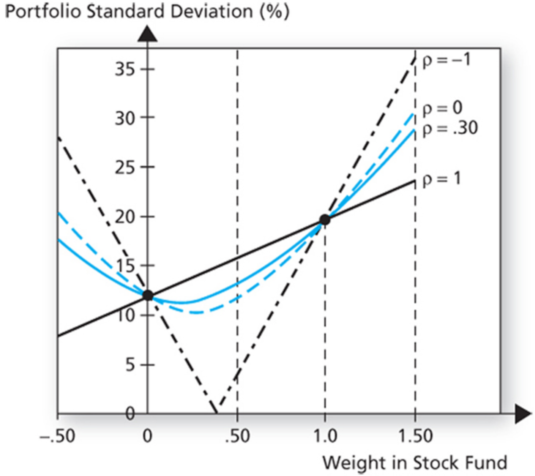


Figure: Portfolio standard deviation as a function of investment proportions.

Note how for $-1 \leq \rho < 1$ the relationship is convex. Why?

Minimum variance portfolio

- What is the minimum value for the standard deviation that is it possible to reach?
- Let's first start from:

$$\sigma_P^2 = w_D^2\sigma_D^2 + (1 - w_D)^2\sigma_E^2 + 2w_D(1 - w_D)Cov(r_D, r_E).$$

- Compute then the derivative of the variance with respect to w_D and set it equal to zero:

$$w_{min}(D) = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_E^2 + \sigma_D^2 - 2Cov(r_D, r_E)}.$$

- In our example: $w_{min}(D) = .82$ and $w_{min}(E) = .18$, while the variance of the minimum variance portfolio is 11.45%.

Minimum variance portfolio

- Note how the minimum variance portfolio has a standard deviation that is smaller than that of the individual assets.
- This result depends on the benefits of diversification.
- Only for $\rho = 1$ there are no benefits from diversification.
- In the previous figure, you can also see the relationship between portfolio standard deviation and weights for the case of two assets that are a perfect hedge (i.e., the case $\rho = 1$).

Portfolio risk and expected return

- Now our goal is to plot the relationship between **portfolio risk** and **expected return**.
- In the next slides, you can see a figure that describes the portfolio opportunity set for different values of ρ.
- The portfolio opportunity set shows the possible combinations of risk and expected return that it is possible to achieve by combining the two assets in a portfolio (i.e., for different values of w_D, w_E.).

Portfolio opportunity set

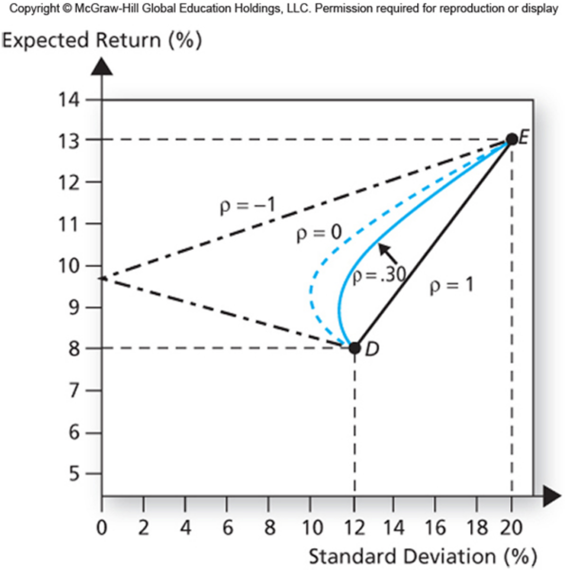


Figure: Portfolio expected return as a function of standard deviation.

Portfolio opportunity set

- When $\rho = 1$ there are no benefits from diversification.
- There are benefits from diversification when the set of investment opportunities is pushed in the direction N-W (for $\rho < 1$).
- The optimal choice within the portfolio opportunity set depends on investors's risk aversion.
- In order to find the portfolio that gives the highest utility given the degree of risk aversion A, we need to maximize the investor's utility:

$$U = E(r) - 0.5A\sigma_p^2,$$

using as constraints the formulas for the expected return and variance of the portfolio.

Outline

Introducing the risk-free asset

- We now introduce an additional asset: a risk-free asset (e.g., a T-bill).
- We consider T-bill as risk-free assets.
- Suppose that T-bills have a return of 5%.
- The return and risk of the equity and debt fund are still those from table 7.1.
- Let's start by plotting the portfolio opportunity set.

Opportunity set

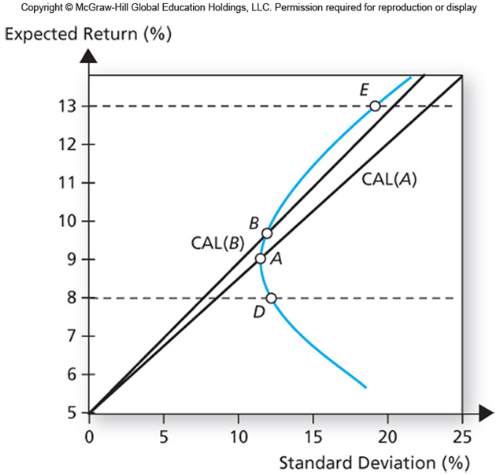


Figure: The opportunity set of the debt and equity funds and two feasible CALs.

Finding the optimal risky portfolio

- Portfolio A correspond to the minimum variance portfolio.
- Portfolio A ($w_D = .82$) has an expected return of 8.9% and a SD of 11.45% (cf. Table 3 in textbook).
- In the previous figure, the first CAL starts off from the risk-free return and goes through A.
- If we buy A, then the SR is equal to:

$$SR = \frac{8.9 - 5}{11.45} = .34.$$

- Recall that the SR is also the slope of the CAL.

Finding the optimal risky portfolio

- Now consider the second CAL, that goes through portfolio B ($w_D = .7$).
- Portfolio B has an expected return of 9.5% and a SD of 11.70% (cf. Table 3).
- In this case, the SR is equal to .38.
- Therefore, portfolio B **dominates** portfolio A ($B \succ A$).

The tangency portfolio

- Should we stop in B, or is it worth continuing?
- Note that if we take a CAL with a slightly steeper slope, we can further increase the SR.
- This argument holds up to the point where we reach the tangency with the investment opportunity set.
- Tangency occurs for the highest possible SR.
- We denote the corresponding portfolio with P.

The tangency portfolio and opportunity set

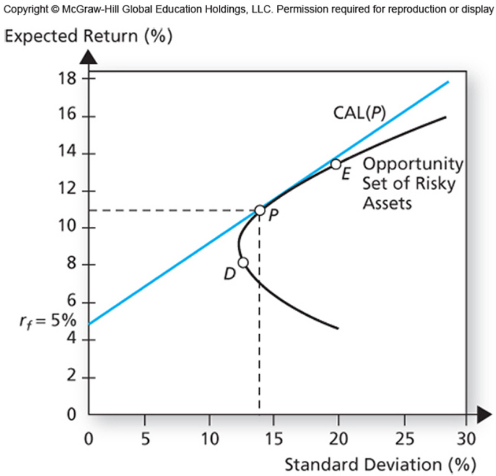


Figure: The opportunity set of the debt and equity funds with the optimal CAL and the optimal risky portfolio.

Mean-variance efficient portfolio

- The risk-return characteristics of P are:
  - $E(r_P) = 11\%$,
  - $\sigma_P = 14.2\%$.
- This portfolio is usually called the tangency portfolio and in combination with the risk-free asset will let us build the steepest possible CAL.
- Sometimes, this portfolio is also called mean-variance efficient, or MVE.
- Why P is the optimal risky portfolio?

Max SR portfolio

- Goal: find weights w_D and w_E that define the CAL with the steepest slope (i.e., the weights that define the risky portfolio with the largest SR).
- We want to maximize the slope of the CAL for each possible portfolio p:
- What is the slope of the CAL?

$$\max_{w} S_p = \frac{E(r_p) - r_f}{\sigma_p}.$$

Max SR portfolio

- In order to solve this optimization problem, first we can replace in S_p the formulas for the expected return and SD:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}.$$

- Expected return is:

$$E(r_p) = w_D E(r_D) + w_E E(r_E) = 8w_D + 13w_E.$$

- SD:

$$\sigma_p = [w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_e \rho_{DE} \sigma_D \sigma_E]^{1/2}.$$

- Note that we need to also add the constraint that the weights sum up to 1 ($\sum w = 1$).

Finding the max SR portfolio

- To solve this optimization problem, we follow the standard procedure:

  1. Substitute in S_p the expressions for $E(r_p)$ and σ_p,
  2. Substitute in S_p $(1 - w_D)$ in place of w_E,
  3. Take derivative of S_p with respect to w_D,
  4. Set the derivative to zero and solve for w_D.

Analytical solution

- Solution:

$$w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]Cov(R_D, R_E)},$$

- and:

$$w_E = 1 - w_D.$$

- Note how R is an excess return.

Optimal allocation (complete portfolio)

- Now we can combine what we learnt so far to find the optimal allocation.
- In order to do this, we need an additional piece of information: the degree of risk aversion of the invstor.
- The optimal allocation in a risky asset is equal to:

$$y^* = \frac{E(r_p) - r_f}{A\sigma_p^2}.$$

- In our example, $E(r_p) = .11$, $r_f = .05$ and $\sigma_p = .142$. Se A = 4 so that:

$$y^* = .7439$$

.

Optimal allocation (complete portfolio)

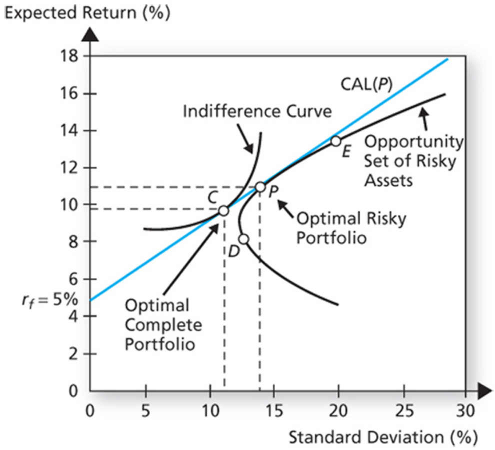


Figure: Determination of the optimal complete portfolio.

Optimal allocation

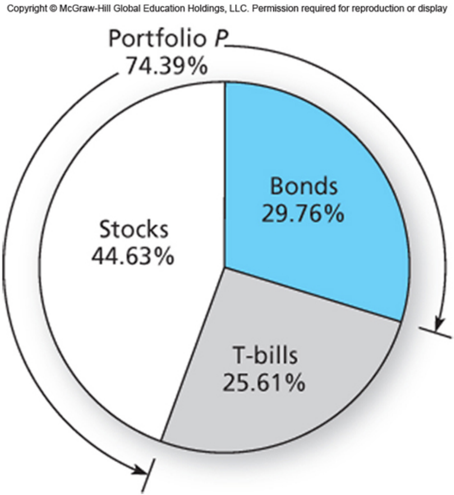


Figure: The proportions of the optimal complete portfolio.

To sum up

- How did we get to the overall portfolio?
  1. First, we defined the return characteristics of each asset (expected return, variance, etc.).
  2. Second, we computed the risky portfolio P.
  3. Third, we computed the characteristics of P using the weights computed in the previous step.
  4. Fourth, we allocated wealth between the risky portfolio and the risk-free asset.

- Note that what we showed has a precise implication in terms of investment strategies: the optimal risky portfolio is the same for every investor, regardless of his/her risk aversion.

- Investors can control risk not by choosing a different composition of the risky portfolio, but rather by choosing an allocation between the risky portfolio and the risk-free asset.

- The risky portfolio must be diversified.

- These results are very different from advice like:
  - *you are young, you can take risks. Buy a couple of high-tech stocks with good growth potential.*
  - *you are close to retirement, it is better to move all of your savings in cash or risk-less securities, and leave nothing invested in risky assets.*

- Our results depend on:
  - One between:
    1. all returns are normally distributed,
    2. investors look only at average returns and variance.
  - All assets are *tradable*.
  - There are no transaction costs.

The Markowitz Portfolio Optimization Model

- It is possible to generalize the method we learnt to build the optimal portfolio to the general case of multiple risky assets and one risk-free asset.
- The first step requires defining the risk-return opportunities available to the investor.
- This set is described by the **minimum variance frontier** of risky assets.
- The frontier describes the minimum variance that can be achieved for a given level of risk (required inputs: expected returns, variances and covariances).

The Markowitz Portfolio Optimization Model

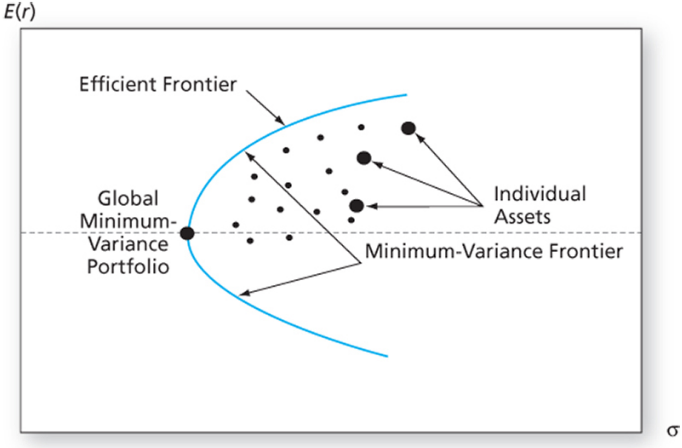


Figure: The minimum-variance frontier of risky assets.

The Markowitz Portfolio Optimization Model

- Note how all the individual assets are to the right and within the minimum variance frontier.
- This result is always true when short-selling is possible.
- All the portfolios on the minimum variance frontier and above the (global) minimum variance portfolio are candidates for the role of optimal risky portfolio.
- The subset of the minimum variance frontier above the (global) minimum variance portfolio is called **efficient frontier of risky assets**.

The Markowitz Portfolio Optimization Model

- We can now introduce the risk-free asset.
- We need to find the CAL with the highest risk-return trade off (i.e., the highest SR, or highest slope).
- In the next slide, you can see how this CAL is the one that is tangent to the efficient frontier and the tangency portfolio P is the optimal risky portfolio.

The Markowitz Portfolio Optimization Model

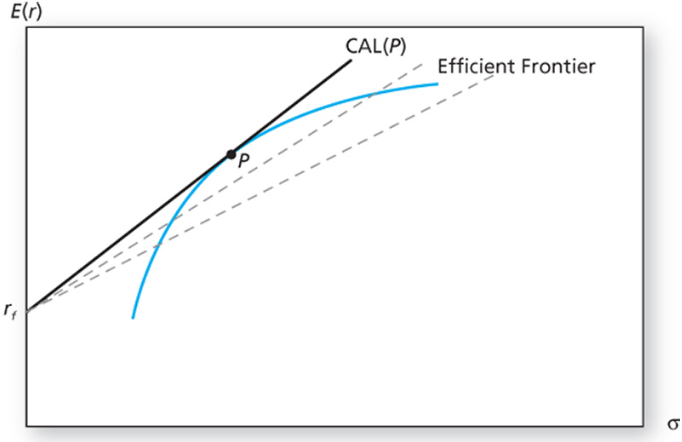


Figure: The efficient frontier of risky assets with the optimal CAL.

The Markowitz Portfolio Optimization Model

- When the investor has found the optimal risky portfolio, she has to choose the optimal combination of P and the risk-free asset.
- As usual, this choice depends on the risk-preferences of the investor.

The Markowitz Portfolio Optimization Model

- Suppose now that the portfolio manager has information on the expected returns, variances and covariances of n risky assets.
- In practice, we can think of a $(n, 1)$ vector that contains all the expected returns, and a (n, n) symmetric matrix whose n main diagonal elements are the estimates of the variances, while the $n^2 - n = n(n-1)$ off-diagonal elements are the estimates for the covariances for all the assets' pairs.
- Note that since the matrix is symmetric, the number of covariances is $n(n-1)/2$.

The Markowitz Portfolio Optimization Model

- Note that if n is large, the task of the portfolio manager is not an easy one.
- For example, if $n = 50$, he needs 50 estimates of expected returns, 50 estimates of variances, and 1,225 estimates of covariances.

The Markowitz Portfolio Optimization Model

- The expected portfolio return is:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i).$$

- Portfolio variance is:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j).$$

- Note that σ_p^2 can also be computed using the bordered covariance matrix.

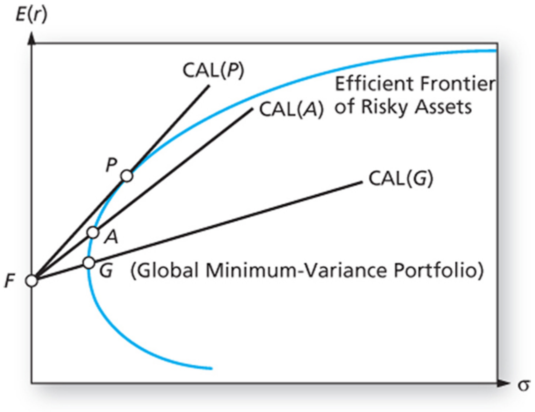
The Markowitz Portfolio Optimization Model

- Harry Markowitz in 1952 developed a model to identify the efficient frontier of risky assets.
- The basic idea is simple: for each level of risk, find the portfolio with the highest expected return.
- In practice, a portfolio manager has a software tool that receives as inputs the vector of expected returns and the variance-covariance matrix and produces as output the efficient frontier.
- Usually, these tools allow the portfolio manager to choose options like: "short-selling is not allowed", or "dividend-yield must be greater than ...", etc.
- Mathematically, each "option" acts as an additional constraint in the optimization.

The Markowitz Portfolio Optimization Model

- Once we find the efficient frontier, we can introduce the risk-free asset.
- Exactly as in the two-assets example, we need to find the CAL with the highest slope, and tangent to the efficient frontier.

The Markowitz Portfolio Optimization Model



- Portfolio P maximizes the trade-off risk-return.

The Markowitz Portfolio Optimization Model

- How is the Markowits model practically used?

  1. Fix different levels for the expected returns, then look for portfolios with the lowest standard deviation for each of these levels. Discard the bottom half of the frontier.
  2. Fix different levels for the standard deviations, then look for portfolios with the highest possible expected returns for each of these levels.
  3. Consider from the beginning the risk-free rate, and maximize the SR of portfolio P (only constraint is the feasibility constraint that weights sum-up to 1). If you follow this strategy you will not obtain the full frontier.

The Markowitz Portfolio Optimization Model

- If you directly maximize the SR to obtain P, you will not obtain the efficient frontier.
- However, an important property of frontier portfolios is that any portfolio formed by combining two portfolios on the minimum-variance frontier will also be on the frontier.
- For example, we can easily look for portfolio G, the global minimum-variance portfolio by minimizing variance with no constraint on the expected return.
- Or, we can look for portfolio Z on the inefficient branch of the frontier with zero covariance with P, the optimal risky portfolio, by minimizing standard deviation subject to zero covariance with P.

The Markowitz Portfolio Optimization Model

- According to this model, a portfolio manager will choose P regardless of the preferences of the investor that do not enter the problem up to this point.
- Investors' preferences matter for the choice of the optimal combination of the risky portfolio and the risk-free asset.
- This result is known as **separation property**.
- Clearly, an immediate implication of this result is that financial advisory firms enjoy large economies of scale (i.e., the cost of finding P is spread among all the investors).