Bond Prices, Returns, and Yields

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Outline

Outline

Time notation

- Use subscripts to denote when things happen: e.g., P_{2014} is the price at the end of year 2014.
- Price at time t is P_t, interest rate at time t is R_t, etc.
- Sometimes, when timing is really important, more precise notation: e.g., $R_{t,t+1}$ will denote the return from time t to $t+1$.
- Today is usually time 0.

Returns

- R denote gross returns: e.g.,

$$R = \frac{\$\text{ payoff}}{\$\text{ investment}}$$

- For a stock that pays dividend D_t:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{\$\text{ payoff at } t+1}{\$\text{ investment at } t}$$

- A gross return is a number like 1.1, or 10% return.
- The *net* return is:

$$r_{t+1} = R_{t+1} - 1$$

- The *percent* return is:

$$100 \times r_{t+1}$$

Concept Review

Suppose you buy 1 share in a stock for \$10 at time t, and you sell it a $t+1$ for the same price of \$10. At $t+1$ you also receive a dividend of \$1. What is your gross return between t and $t+1$?

$$R = \frac{10+1}{10} = 1.1$$

while the net return is equal to 0.1 (for a percent return of 10%).

Returns

- The *log* or *continuously compounded* return is:

$$r_t = \ln R_t$$

  for example, $\ln(1.10) = 0.09531$ or 9.531%.
- An investment remunerated at the continuously compounded return r_t yields every period:

$$e^{r_t} = e^{\ln R_t} = R_t$$

- The *real return* corrects for inflation:

$$R_{t+1}^{\text{real}} = \frac{\text{Goods back at } t+1}{\text{Goods paid at } t}$$

Concept Review

Note that net and log returns are not the same (even if we use same notation r). However, for small values of the net return they are very close. To see this, take a first order approximation (i.e., Taylor's expansion) of $\ln R = \ln[1 + r]$ around $r \approx 0$ to find that

$$\ln R \approx r = R - 1$$

However, whenever you use and/or compute log returns keep in mind the correct interpretation in terms of *continuously compounded* return.

Returns

- The price index is defined as:

$$PI_t \equiv \frac{\$_t}{Goods_t}; \quad \Pi_{t+1} \equiv \frac{PI_{t+1}}{PI_t}.$$

- We can use CPI data to find real returns:

$$R^{\text{real}}_{t+1} = \frac{\$_{t+1}\frac{Goods_{t+1}}{\$_{t+1}}}{\$_t\frac{Goods_t}{\$_t}} = \frac{\$_{t+1}\frac{1}{PI_{t+1}}}{\$_t\frac{1}{PI_t}} = R^{\text{nominal}}_{t+1}\frac{PI_t}{PI_{t+1}} = \frac{R^{\text{nominal}}_{t+1}}{\Pi_{t+1}}.$$

- Therefore, we simply divide the gross return by the gross inflation rate to get gross real return.

Returns

- What about the Fisher equation? This is true for *log* returns:

$$\ln R_{t+1}^{\text{real}} = \ln R_{t+1}^{\text{nominal}} - \ln \Pi_{t+1}.$$

- It is *approximately* true that we can do the same for *net* returns:

$$\frac{R^{\text{nominal}}}{\Pi} = \frac{1 + r^{\text{nominal}}}{1 + \pi} \approx 1 + r^{\text{nominal}} - \pi,$$

  the approximation is okay when the inflation rate is low.
- Example: $r^{\text{nominal}} = 5\%$, $\pi = 2\%$:

$$1 + 5\% - 2\% = 1.03 \qquad \frac{1 + 5\%}{1 + 2\%} = 1.029.$$

Taxes and the Real Rate of Interest

- Real rates for investors depend also on taxes which usually are levied on *nominal* rates.
- For example, consider a tax t on the nominal interest payments:
- The after-tax real interest rate is *approximately* equal to:

$$i(1-t)-\pi=(r+\pi)(1-t)-\pi=r(1-t)-\pi t,$$

and falls as inflation rises.

example

TIPS

- We have seen that in order to compute real rates you can subtract inflation from nominal rates.
- In the markets there are also securities that *pay directly real rates*.
- An example are TIPS (Treasury Inflation Protected Securities).

TIPS

- TIPS respond to the demand by investors of safe assets that can protect against inflation.
- TIPS are issued by the US Treasury and are inflation-indexed bonds.
- The principal is adjusted to the CPI: when CPI rises (falls), the principal adjusts upward (downward).
- The coupon rate is constant but generates a different amount of interest when multiplied by the inflation adjusted principal.
- TIPS are offered in 5Y, 10Y and 30Y maturities.

Returns

- Following what we said about real returns, you can find *dollar* returns of international securities denominated in non-dollar currency.
- Suppose you have an Italian security that pays a gross Euro (€) return:

$$R_{t+1}^{IT,€} = \frac{€ \ \text{ payoff at } t+1}{€ \ \text{ investment at } t}.$$

- Define the exchange rate as:

$$E_t = \frac{\$_t}{€_t},$$

then the return *in dollar* of the Italian security is:

$$R_{t+1}^{IT,\$} = \frac{\$_{t+1}}{\$_t} = \frac{€ \text{ payoff at } t+1}{€ \text{ investment at } t} \frac{\$_{t+1}/€_{t+1}}{\$_t/€_t} = R_{t+1}^{IT,€} \frac{E_{t+1}}{E_t}.$$

Returns on UK stock market since Brexit

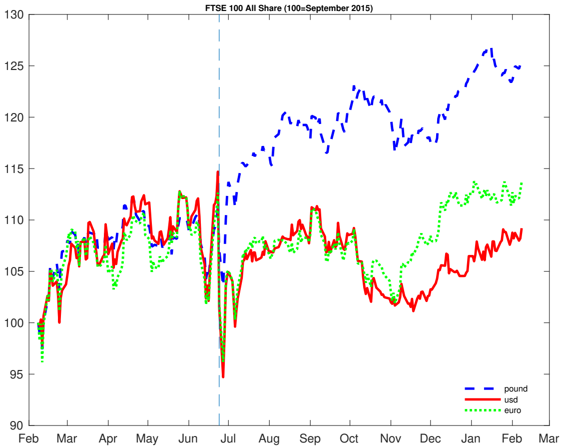


Figure: Cumulated returns on FTSE 100 All Share 09/2015 – 01/2017. Vertical dashed line is the "Brexit" referendum result.

Compound returns

- Suppose you hold an instrument that pays 10% per year for 10 years.
- What do you get for a \$1 investment?
- The correct answer is not \$2, or $\$1 + 10 \times 0.1$, since you earn interest on interest.
- The right answer is the *compound return*.

Compound returns

- Denote with V_t the value at time t.
- Suppose the period return is constant and equal to R.
- Then:

$$\begin{aligned} V_1 &= RV_0 = (1+r)V_0 \\ V_2 &= R \times (RV_0) = R^2 V_0 \\ V_T &= R^T V_0. \end{aligned}$$

- Therefore, R^T is the *compound return*.

Compound returns

- Recall that:

$$\ln(ab) = \ln a + \ln b; \quad \ln(a^2) = 2\ln a.$$

- Therefore:

$$\begin{aligned} \ln V_1 &= \ln R + \ln V_0 \\ \ln V_T &= T \ln R + \ln V_0. \end{aligned}$$

- In other words, the compound *log* return is T *times* the one-period log return
- Log returns are often very handy Details logs in economics

Compound returns

- Log returns are often very handy ‣ Details logs in economics
- Consider a multi-period problem. The T period return is:

$$R_1 \times R_2 \times \ldots \times R_T,$$

while the T period log return is:

$$\ln R_1 + \ln R_2 + \ldots + \ln R_T.$$

- With log returns, we can also subtract rather than divide to get the exact real returns or exchange conversions:

$$\begin{aligned} R^{\text{real}} &= \frac{R^{\text{nominal}}}{\Pi} \rightarrow \ln(R^{\text{real}}) = \ln R^{\text{nominal}} - \ln \Pi \\ R^{\$} &= R^{€}\frac{E_{t+1}}{E_t} \rightarrow \ln R^{\$} = \ln R^{€} + \Delta e_{t+1} \end{aligned}$$

where $\Delta e = \ln E_{t+1} - \ln E_t$.

Within period compounding

- Suppose a bond that pays 10% is compounded semi-annually: i.e., two payments of 5% each are made at 6 months intervals. In this case, the total annual gross return is:
$$(1.05)(1.05) = 1.1025 = 10.25\% > 10\%.$$
- What if it is compounded quarterly?
$$(1.025)^4 = 1.1038 = 10.38\% > 10.25\%.$$
- Compounded N times:
$$(1 + \frac{r}{N})^N.$$
- If we compound *continuously*:
$$\lim_{N\to\infty}(1 + \frac{r}{N})^N = 1 + r + \frac{1}{2}r^2 + \ldots = e^r.$$

Concept Review

- A stated rate of 10% continuously compounded is really a gross return of $e^{10\%} = 1.1057 = 10.517\%$.
- What is the three year return of a security that pays a stated rate of 10%, compounded semiannually?
- It must be:
$$(1 + \frac{0.1}{2})^{2\times 3}.$$
- Similarly, the continuously compounded T year return is: e^{rT}.

Risk with Non-Normal Distributions

- Two methods are common to measure risk and account for the potential non-normality of returns:
  1. Value-at-Risk (VaR)
  2. Expected Shortfall
- Additional measures: LPSD Sortino ratio

Value-at-Risk

- The **value-at-risk** (VaR) highlights the potential loss from extreme negative returns, i.e., the loss corresponding to a very low percentile of the return distribution.
- The VaR is another name for the quantile of a distribution (q): the value below which lies $q\%$ of the values:

$$Prob\{x \leq VaR_{5\%}\} = 5\%.$$

- For example, the median of the distribution is the 50% quantile.

Value-at-Risk

- Practitioners commonly use the **5% quantile** as the VaR of the distribution $\Rightarrow$ this tells us that with a probability of 5%, we can expect a loss equal to or greater than the VaR.
- For a normal distribution, the 5% VaR lies 1.65 standard deviations below the mean.
- If the distribution is not normal, the VaR gives useful information about the magnitude of loss we can expect in a bad scenario.

Value-at-Risk

- To obtain a sample estimate of VaR, first sort observations from high to low.
- The VaR is the return at the 5th percentile of the sample distribution.
- Almost always, 5% of the number of observations will not be an integer, so you will need to interpolate.

Value-at-Risk

- For example, suppose you have 84 returns ordered from largest to lowest.
- In this case, 5% of the number of observations is 4.2 (not an integer), and we must interpolate between the bottom fifth and fourth observation.
- Suppose the bottom five returns are: r_5, r_4, r_3, r_2, r_1, then:

  $$VaR = r_4 + 0.2 \times (r_5 - r_4),$$

  where the 0.2 is the coefficient from the linear interpolation. Details
- In practice, most software packages return directly the interpolated value when we compute a given quantile.

Value-at-Risk

- Consider a portfolio of stocks.
- Assume that this portfolio has a 1-day 5% VaR of $1 million.
- This means that there is a 5% probability that the portfolio will fall in value by more than $1 million over a one day period, assuming that markets are normal and there is no trading (e.g., a loss of $1 million or more is expected on 1 day in 20).
- Note that common parameters for VaR are 1% and 5% probabilities and one day and two week horizon.

Expected Shortfall

- The **expected shortfall (ES)** provides the answer to the question: "Assuming the terminal value of the portfolio falls in the bottom 5% of possible outcomes, what is its expected value?"
- The 5% VaR is the outcome at the upper boundary of these worst-case outcomes. By construction the VaR is always higher than the ES.
- ES improves on VaR because it accounts for the entire tail of the distribution.
- Note that ES is sometimes also called conditional tail expectations (CTE).

Expected Shortfall

- Go back to the example used to describe VaR.
- Assume equal probabilities for all values.
- We need to average across the bottom 5% of observations.
- In practice we would compute:

$$ES = \frac{r_1 + r_2 + r_3 + r_4 + 0.2r_5}{4.2} < VaR$$

ES with normal returns

Outline

Notation

- We need to distinguish bonds with different maturities.
- I denote maturity with a superscript in parentheses: for example, $P^{(4)}$ is the price of a four year zero-coupon bond.
- Recall: a zero coupon is a bond that pays no coupons, but only the face value at maturity.

Present value

- Start by ignoring uncertainty: all future cash flows are known (no default risk) and all future interest rates are known.
- Any bond is a claim to a sequence of cash flows: $\{CF_1, CF_2, \ldots, CF_N\}$.
- We can write its value as:

$$P = \sum_{j=1}^{N} \frac{CF_j}{R_0 R_1 \ldots R_{j-1}}, \tag{1}$$

where R_0 is the interest rate from 0 to 1, etc.

Present value

- Problem with the formula: where do we get the interest rate R_j?
- Suppose you know what interest rates banks will charge, and the borrowing and lending rates are equal, then these are the rates to use (by arbitrage). However, this only happens in textbooks!
- More commonly, you can use the market price of zero-coupon bonds. These should be equal to:

$$P^{(N)} = \frac{1}{R_0 R_1 \dots R_{N-1}}.$$

- Then, we can apply these prices in the formula (1) for the generic bond with cash flows CF_j:

$$P = \sum_{j=1}^{N} P^{(j)} CF_j.$$

Present value

- Note that the formula

$$P = \sum_{j=1}^{N} P^{(j)} CF_j,$$

says that any bond can be repackaged as a combinations of zero-coupon bonds (with weights, or quantities, equal to CF_j).

- Note also that we can infer zero prices from the prices of coupon bonds, and then use those zero prices to price other coupon bonds.

Yield

Definition

The yield (to maturity) is defined as that fictional, constant, known, annual, interest rate that justifies the quoted price of a bond, assuming that the bond does not default.

Yield

- Note that the definition of yield in the previous slide is a bit different - but in fact the same - from the one you might remember (for example, *the yield to maturity is the constant rate that equates a bond price to the present value of its cash flows*).
- More formally, the yield of a zero coupon bond is the number $Y^{(N)}$ that satisfies:

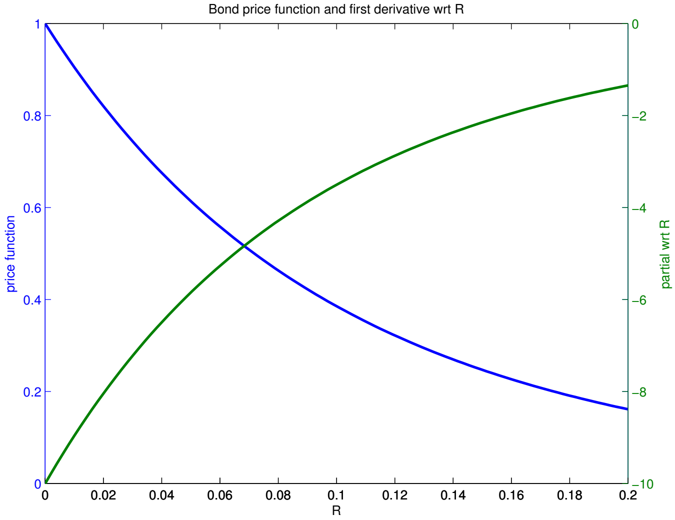
$$P^{(N)} = \frac{1}{[Y^{(N)}]^N}.$$

- Hence:

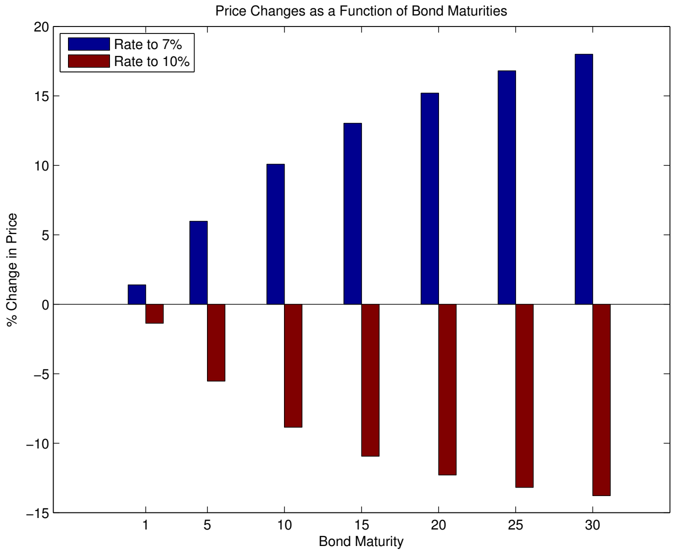
$$Y^{(N)} = \frac{1}{[P^{(N)}]^{1/N}}; \quad \ln Y^{(N)} = -\frac{1}{N} \ln P^{(N)}.$$

Yield

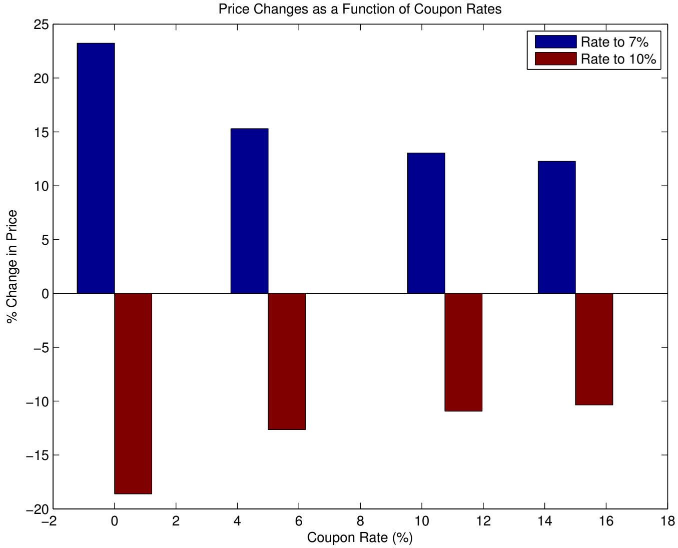
- In general, for coupon bonds, the yield of any stream of cash flows is the constant number Y that satisfies:
$$P = \sum_{j=1}^{N} \frac{CF_j}{[Y]^j}.$$
- In practice, you need to numerically search for the value Y that solves this equation given the cash flows and the price.
- So long as all cash flows are positive, this is fairly straightforward using a computer (postive & negative cash flows).
- To sum up, the yield is just a convenient way to quote the price of a bond.



Notes: This figure plots the price (blue line, left axis) of a zero coupon bond as function of R and its partial derivative (green line, right axis) with respect to R. The bond is a zero coupon bond with face value \$1 and years to maturity $T = 10$.



Notes: The longer the maturity of the bond, the more sensitive it is to changes in interest rates.



Notes: The smaller the coupon rate on the bond, the more sensitive it is to changes in interest rates. The bond coupon rate is the ratio between annual coupon payment and face value at maturity.

Annuities

- An annuity is a **constant** cash flow that occurs at regular intervals for a fixed period of time.
- The present value of an annuity can be calculated by taking each cash flow and discounting it back to the present, and adding up the present values (using formulas from previous slides).
- Alternatively, there is a short cut that can be used:

$$PV_t = \frac{CF}{R-1}[1-(\frac{1}{R})^N], \tag{2}$$

where CF is the annuity, R the constant gross discount rate and N the number of years.

The Math Behind the Short Cut ...

- The present value of an annuity can be written as:

$$PV_t = \sum_{i=1}^{N} \frac{CF}{R^i} = \frac{CF}{R}(1 + \frac{1}{R} + ... + \frac{1}{R^{N-1}}).$$

- Note how the sum in the last term is a geometric series and we can apply the standard formula ▸ Details to get:

$$PV_t = \frac{CF}{R} R \frac{1 - (\frac{1}{R})^N}{R - 1}.$$

- For further details check the entry geometric progression on wikipedia.

Application: Valuing a Plain Vanilla Bond

- A plain vanilla (or straight) bond is a bond that pays interest at regular intervals, and at maturity pays back the principal.
- Suppose you are trying to value this bond with a 15-year maturity and a 10.75% coupon rate. The current interest rate on bonds of this risk level is 8.5%.
- The PV of the cash flows on this bond is equal to:

$$\underbrace{PV(A = \$107.5, 8.5\%, 15)}_{\text{PV interest payments}} + \underbrace{\$1000/1.085^{15}}_{\text{PV principal}} = \$1,186.85,$$

where A denotes the specified annuity whose PV is computed by equation 2.

Application: Valuing a Plain Vanilla Bond

- If interest rates rise to 10%, the PV of the cash flows is:

$$PV(A = \$107.5, 10\%, 15) + \$1,000/1.10^{15} = \$1,057.05.$$

- As a result, the $\%\Delta$ in price in equal to -10.94%.
- On the contrary, if the interest rate falls to 7%, the PV is equal to $\$1,341.55$ and the $\%\Delta$ in price is equal to 13.03%.
- This asymmetric response to interest rate changes is called **convexity**.
- Convexity is related to the sensitivity of the **duration** of a bond to a change in the interest rate. We will refresh the concept of duration soon.

Present Value of a Growing Annuity

- A growing annuity is a cash flow growing at a constant rate (let's called it g) for a specified period of time.
- The present value of a growing annuity is equal to Details:

$$PV = \frac{A}{R-1-g}[1-(\frac{1+g}{R})^N]. \tag{3}$$

- Note that for $R-1=g$ the formula is not defined. In this case, the present value is simply equal to the sum of the annuities over the period, without the growth effect.

Application: The Value of an Oil Field

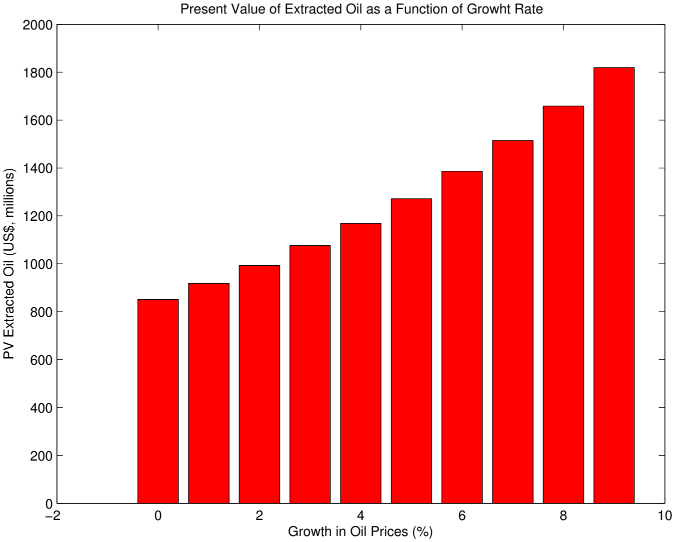
- Consider the case of an oil field. You have the right to extract oil for the next 20 years, over which you plan to extract 1 million barrels per year. The price per barrel is \$100 currently, but it is expected to increase 2% per year. The appropriate discount rate is 10%.

- The present value of the oil that *will be* extracted from this oil field can be estimated as follows:

$$PV = \$100 \times 1,000,000 \times 1.02 \times [\frac{1-(\frac{1.02}{1.1})^{20}}{0.1-0.02}] = \$993,382,320.$$

- Note that, since I am computing the PV of the oil that will be extracted starting one year from now, I use $\$100 \times 1.02$ as the starting oil price.

PV of Oil Field as a Function of Expected Growth Rate



Practice Questions

- If both the growth rate and the discount rate go up by 1%, will the PV of oil to be extracted from the oil field increase, decrease or stay the same?

Practice Questions

- The PV of oil to be extracted from the oil field increases when both the growth rate and the discount rate go up by 1%.
- You can check that by simply plugging the new values for R and g and compare the new and old results.
- In particular, the PV increases from \$993.38 million to \$999.08 million.
- Note that this is true as long as $R - 1 > g$. The opposite holds when $R - 1 < g$.

Perpetuity

- A perpetuity is a constant flow (A) at regular intervals **forever**.
- A perpetuity corresponds to the case of an annuity when $N \to \infty$.
- The present value of a perpetuity is:

$$PV = \frac{A}{R-1},$$

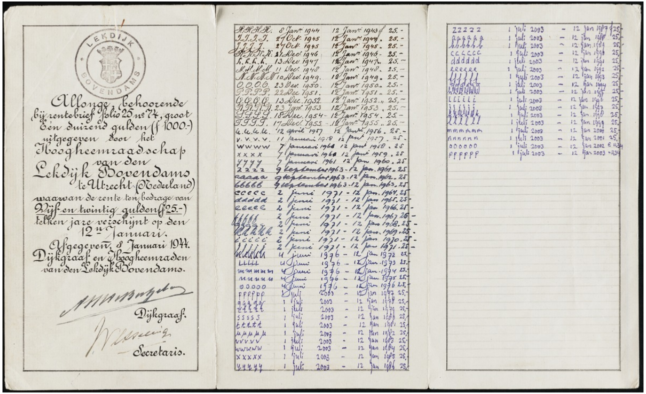
where the denominator is the net rate.

Concept Check: Valuing a Consol Bond

- A consol bond is a bond that has no maturity and pays a fixed coupon.
- Assume that you have a 6% coupon consol bond. The value of this bond, if the interest rate is 9%, is as follows:

$$\text{Value of Consol Bond} = \$60/.09 = \$667.$$

Yale's perpetuity: issued in 1648!



Yale still receives 136.20 euro per year from the holding of a perpetuity issued by the Dutch Water Authority in 1648 and written on goat skin. For more details check this article from Bloomberg.

Growing Perpetuities

- A growing perpetuity is a cash flow that is expected to grow at a **constant** rate **forever**. The present value of a growing perpetuity is:

$$\text{PV of Growing Perpetuity} = \frac{CF_1}{R - 1 - g},$$

where CF_1 is the expected cash flow *next year*, g is the constant growth rate and R is the discount rate.

Concept Check: Valuing a Stock with Growing Dividends

- Consider a stock that has paid dividends per share of \$2.00 in 2009.
- Its earnings and dividends have grown at 6% per year between 2000 and 2009 and are expected to growth at the same rate in the long term.
- The rate of return required by investors on stocks of equivalent risk is currently 10%.
- What is the value of this stock?
  - Current dividends per share $= \$2.00$,
  - Expected growth in earnings and dividends $= 6\%$,
  - Discount rate $= 10\%$,
  - Value of stock $= (\$2.00 \times 1.06)/(0.10 - 0.06) = \$53$.
  - This is the so called **Gordon model** of stock valuation.

Price/earnings (P/E) ratio

- You might think that perpetuities are rare or unrealistic.
- In fact, many types of assets have characteristics that are similar to perpetuities.
- Examples are income-oriented real estate, preferred shares, etc.
- Stocks are commonly noted as trading at a certain P/E ratio: this is recognized as a variation on the perpetuity or growing perpetuity formula.

Repo

Repos (1/2)

- Suppose that bank A has a government bond worth \$100 and needs cash.
- Bank A can sell the bond to bank B today and at the same time commit to repurchase the same bond tomorrow at a slightly higher price.
- This kind of transaction is called Repo, or repurchase agreement.
- The standard maturity of a Repo contract is *overnight* (even though it is possible to find Repos with alternative longer maturities).

Repos (2/2)

	Bank A	Bank B
Today	$+99.9$	-99.9
Tomorrow	-100	$+100$

- A Repo contract is like a collateralized loan: if tomorrow Bank A does not have the money to repurchase the bond, Bank B can keep the bond that covers the value of the defaulted loan.
- Note that typically Repos are over-collateralized, i.e., the value of the bond is larger that the cash lent.
- The percentage difference between the value of the collateral posted and the cash lent is called "margin", or "haircut".
- The interest rate implicit in the repo contract is the repo rate.

Repos, haircuts, and fire sales

Table 5.1

Initial 5 percent haircut		Subsequent 10 percent haircut	
Assets	Liabilities	Assets	Liabilities
ABS 100	Repo 95	ABS 50	Repo 45
	Equity 5		Equity 5
100	100	50	50

Table 5.2

Initial price: 100 Haircut 10 percent		Subsequent price: 95 Haircut 10 percent	
Assets	Liabilities	Assets	Liabilities
ABS 100	Repo 90	ABS 50	Repo 45
	Equity 10		Equity 5
100	100	50	50

Source: Freixas et al. (2015), chapter 5 and Brunnermeier and Pedersen (2009)

Forward rates

Forward rate

Definition

The forward rate of interest is the rate at which you can contract *today* to borrow or lend money starting at period N, to be paid back at period $N+1$.

Forward rate

- From the prices of zero-coupon bonds we can find implied future rates, or forward rates.
- Recall that:

$$P^{(N)} = \frac{1}{R_0 R_1 \dots R_{N-1}}.$$

- Then:

$$R_N = \frac{P^{(N)}}{P^{(N+1)}}.$$

- Why is this a forward rate? The easy way to understand this is by synthesizing a forward contract using a set of zero coupon bonds (see next slide).

Forward rate

- Suppose you bought one N period zero and simultaneously sold x $N+1$ period zero coupon bonds.
- What are the cash flows at every date?

Forward rate

	Buy N-period zero	Sell x N+1-period zeros	Net cash flow
Today 0	$-P^{(N)}$	$+xP^{(N+1)}$	$xP^{(N+1)} - P^{(N)}$
Time N	1		1
Time N+1		$-x$	$-x$

Forward rate

- Now choose x so that today net cash flow is zero:

$$x = \frac{P^{(N)}}{P^{(N+1)}}.$$

- Therefore, you pay or get nothing today, you get 1 at time N and you pay $P^{(N)}/P^{(N+1)}$ at time N+1. You've just synthesised a contract signed today for a loan from N to N+1, or a forward rate!
- To sum up:

$$F_N = \text{Forward rate } N \to N+1 = \frac{P^{(N)}}{P^{(N+1)}},$$

and:

$$\ln F_N = \ln P^{(N)} - \ln P^{(N+1)}.$$

Forward rate

- When are forward rates useful?
- Examples:
  1. You are planning an investment that requires you to borrow money sometime in the future and you want to lock in a given rate.
  2. You think you know where interest rates are going and you want to speculate.

Holding period returns

- Suppose you buy at time t a N-period bond and then sell it after one period, when it has become a N-1 period bond.
- Your return is:

$$HPR_{t+1}^{(N)} = \frac{P_{t+1}^{(N-1)}}{P_t^{(N)}},$$

and:

$$\ln HPR_{t+1}^{(N)} = \ln P_{t+1}^{(N-1)} - \ln P_t^{(N)}.$$

- Note the notation: we date this return (from t to $t+1$) as $t+1$ because you realize the value at $t+1$.

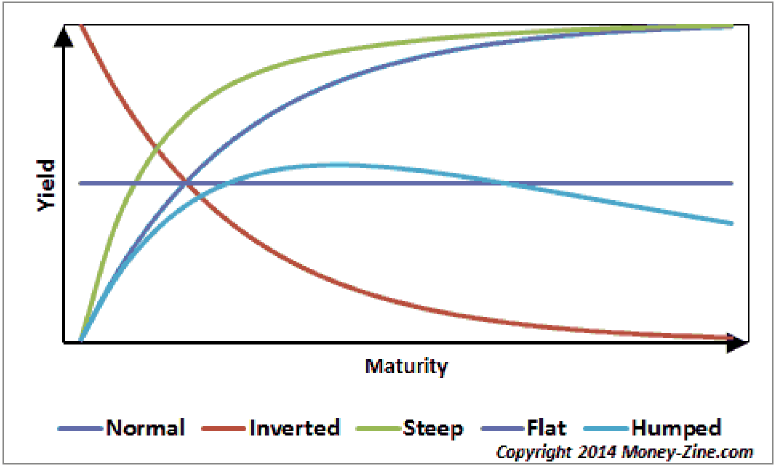
Yield curve

Yield curve

Definition

The yield curve is a plot of yields of zero coupon bonds as a function of their maturity.

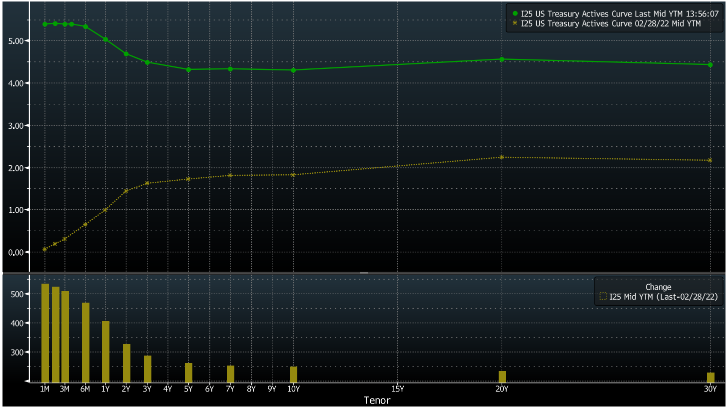
Yield curves



US Govt Yield curves

Bloomberg
I25 Mid YTM (Last-02/28/22)

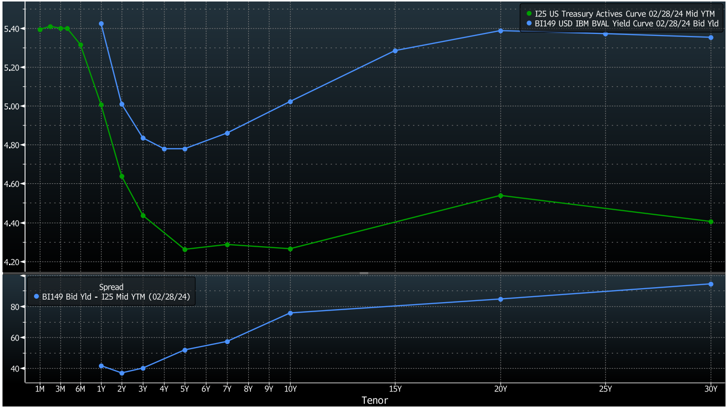
I25 US Treasury Actives Curve Last Mid YTM ~:%H:%M:%S~
I25 US Treasury Actives Curve 02/28/22 Mid YTM



US Govt & Corp Yield curves

Bloomberg

I25 US Treasury Actives Curve 02/28/24 Mid YTM
BI149 USD IBM BVAL Yield Curve 02/28/24 Bid Yld
BI149 Bid Yld - I25 Mid YTM (02/28/24)



Yield curve with no uncertainty

- Suppose we know where interest rates are going (or future yields on one period bonds). Then the present value formula is:

$$P_0^{(N)} = (\frac{1}{R_0}\frac{1}{R_1}\dots\frac{1}{R_{N-1}}) = (\frac{1}{Y_0^{(1)}}\frac{1}{Y_1^{(1)}}\dots\frac{1}{Y_{N-1}^{(1)}}).$$

- Recall the definition of yield:

$$P^{(N)} = \frac{1}{[Y^{(N)}]^N}.$$

- Therefore:

$$Y_0^{(N)} = \left[\frac{1}{P_0^{(N)}}\right]^{1/N} = (Y_0^{(1)}Y_1^{(1)}\dots Y_{N-1}^{(1)})^{1/N}, \tag{4}$$

  which is the same as saying that the yield on an N-period zero is the geometric average of future (one period) interest rates.

Yield curve with no uncertainty

- If you don't like geometric averages, or want something more compact, take logs of equation (4):

$$\ln Y_0^{(N)} = \frac{1}{N}(\ln Y_0^{(1)} + \ln Y_1^{(1)} + \ldots + \ln Y_{N-1}^{(1)}),$$

  or, *the log yield on an N-period zero is the arithmetic average of future interest rates.*
- Interpreation:
  - The right hand side of the yield curve formula represents one way of getting a dollar from now to N periods from now, by rolling over one-period bonds.
  - The left hand side expresses another way of getting a dollar from now to N periods from now, but buying a long (N) period zero coupon bond.
  - With no uncertainty, the two strategies must give the same return, by arbitrage.

Yield curve with uncertainty

- What if we do not know future interest rates?
- If investors are risk-neutral enough, then they will buy N-period zeros or roll-over one period bonds depending on which of the two strategies promises to perform better *on average*.
- Note that the previous is not a pure arbitrage argument, since there is no sure profits to be made.
- We can add a risk-premium factor to catch any error.

Yield curve with uncertainty

Definition

The N-period yield is the average of **expected** future one-period yields, perhaps plus a risk-premium.

Yield curve with uncertainty

- More formally:

$$Y_0^{(N)} = E_0[(Y_0^{(1)} Y_1^{(1)} \dots Y_{N-1}^{(1)})^{\frac{1}{N}}] + \text{ risk premium},$$

  or

$$\ln Y_0^{(N)} = \frac{1}{N} E_0(\ln Y_0^{(1)} + \ln Y_1^{(1)} + \dots + \ln Y_{N-1}^{(1)}) + \text{ risk premium}.$$

- Note that the expressions above are a tautology unless we characterize the risk premium.
- Recall the *expectation hypothesis* from undergrad textbook: there is no risk premium $\rightarrow$ it is a good approximation when the risk premium is small and does not vary much over time.
- More advanced term-structure models try to quantify the size and movement of the risk-premium term.

Forward yield curve

- Suppose *we were right* and we knew the correct future interest rate.
- Then by arbitrage it must be true that:

$$F^{(N)} = R_{N,N+1},$$

or that the forward rate is equal to the future spot rate.

Forward yield curve

- Note that: forward rate = to the future spot rate *implies* the yield curve.
- Start by considering the forward rate one step ahead: $F^{(1)} = R_{1,2}$.
- Replace the forward rate in forward rate formula:

$$\frac{P^{(1)}}{P^{(2)}} = R_{1,2} \equiv F^{(1)}.$$

- Use $R_{0,1} = 1/P^{(1)}$ and $P^{(2)} = 1/[Y^{(2)}]^2 \rightarrow Y^{(2)} = 1/\sqrt{P^{(2)}}$:

$$[Y^{(2)}]^2 = R_{0,1} R_{1,2} \rightarrow Y^{(2)} = [R_{0,1} R_{1,2}]^{1/2}.$$

- With uncertainty, add a risk premium: forward rate = expected future spot rate + risk premium.

Holding period return yield curve

- Consider two strategies to get money from today to tomorrow:
  1. hold a N-period zero coupon bond for a period, sell it as a N-1 period zero coupon bond.
  2. hold a 1-period zero coupon bond for a period.
- If investors (or bond traders) are risk-neutral, we expect holding period returns from the two strategies to be the same, except for a small risk premium.
- Formally:

$$E_0\left(HPR_{t+1}^{(N)}\right) = E_0\left(HPR_{t+1}^{(M)}\right) + \text{risk premium}.$$

Holding period return yield curve

- The result that expected holding period returns on bonds with different maturities are the same implies the yield curve.
- To see this, consider the no uncertainty case:

$$
\begin{aligned}
HPR_1^{(2)} &= HPR_1^{(1)} \\
\frac{P_1^{(1)}}{P_0^{(2)}} &= \frac{1}{P_0^{(1)}} \\
\frac{[Y_0^{(2)}]^2}{Y_1^{(1)}} &= Y_0^{(1)} \\
Y_0^{(2)} &= [Y_0^{(1)} Y_1^{(1)}]^{1/2}.
\end{aligned}
$$

Duration

Duration

- We know how to find the value of a bond.
- How does this value change when interest rates change?
- *Duration* is the answer to this question: it's the main measure of sensitivity of a bond's price to interest rate changes.

Duration

- Duration is the sensitivity of prices (P) to yields (Y). Formally:

$$D = -\frac{\text{\% Change in P}}{\text{\% Change in Y}} = -\frac{Y}{P}\frac{dP}{dY} = -\frac{d\ln P}{d\ln Y}.$$

- From the definition above, it's very easy to find the duration of a zero-coupon bond:

$$\begin{aligned} P^{(N)} &= \frac{1}{Y^N} \\ D &= -\frac{Y}{P}\frac{dP}{dY} = \frac{Y}{P}N\frac{1}{Y^{N+1}} = N. \end{aligned}$$

- Therefore, for zeros: duration = maturity.

Duration

- For coupon bonds, recall that $P = \sum_{j=1}^{N} \frac{CF_j}{Y^j}$.
- Apply duration formula:

$$-\frac{Y}{P}\frac{dP}{dY} = \frac{Y}{P}\sum_{j=1}^{N} j \frac{CF_j}{Y^{j+1}} = \frac{1}{P}\sum_{j=1}^{N} j \frac{CF_j}{Y^j} = \sum_{j=1}^{N} j \frac{CF_j / Y^j}{\sum_{j=1}^{N} CF_j / Y^j}.$$

- This is equivalent to say:

$$\text{Duration} = \sum_{\text{cash flows}} \text{duration of cash flows} \times \frac{\text{value of cash flow}}{\text{total value}}.$$

Duration

Definition

The duration of any bond = value-weighted average of durations of its individual cash flows

Duration

- Note that the duration of coupon bonds is less than their maturity.
- Modified duration is the percentage change in price for a one *percentage point* change in yield, rather than a one *percent* change in yield:

$$\text{M-duration} \equiv -\frac{\%\ \Delta \text{ in P}}{\Delta \text{ in Y}} = -\frac{1}{P}\frac{dP}{dY} = \frac{1}{Y}\left(-\frac{Y}{P}\frac{dP}{dY}\right) = \frac{1}{Y} \times \text{duration.}$$

Immunization

- How can we structure a portfolio so that it is not sensitive to interest rate changes?
  1. *Dedicated portfolio*: for each cash flow of assets or liabilities buy or sell a corresponding zero-coupon bond. Note that this is usually expensive since it requires lots of buying and selling.
  2. *Duration matching*: change two assets or liabilities so that 1) the present value of assets = the present value of liabilities and 2) the duration of assets = the duration of liabilities. Your total position will be insensitive to interest rate changes.

Mortgage-based securities

MBS

- Consider a bank that makes a \$100K loan to a subprime borrower that wants to buy a house[1].
- Assume that the borrower will default with a probability $p = 10\%$.
- In case of default, the bank can recover 50% of the face value of the loan (e.g., \$50K) by foreclosing the house.
- This is a risky loan!

[1] This example is from A. Mian and A. Sufi, *House of Debt*, Chicago Press 2014.

MBS

- By tranching the loan the bank can offload some of the risk.
- The bank creates two tranches of equal face value (e.g., \$50K):
  1. senior: paid first in case of default,
  2. junior: paid only after senior creditors are paid in full.
- Clearly, the senior tranche has zero risk: even in the event of default, creditors are paid in full.
- The junior tranche is risky: with 10% probability creditors lose everything.
- Since the senior tranche has zero risk, it will typically receive a very good credit rating (e.g. AAA) and can be sold to investors looking for safe investments (cf. Ben Bernanke and the global savings glut).

MBS

- Can the bank do better?
- Yes, by tranching + pooling.
- Suppose the bank pools together two subprime loans with same default probability and face value \$100K.
- The bank can create two tranches:
  1. senior: \$100K.
  2. junior: \$100K.
- Note that the senior tranche is risk-free: even if both loans are in default, senior creditors are paid in full.
- What about the junior tranche? If both loans default at the same time, we go back to the example without pooling.
- But what if defaults are uncorrelated ...

MBS

- Suppose the loans' probabilities of default are independent of each other.
- Recall Bayes rule in the case of two independent events: $P(A \cap B) = P(A)P(B)$.
- In our example there is 1% chance (i.e., $10\% \times 10\%$) that both loans are in default and 18% chance that only 1 loan is in default (e.g. $2 \times 90\% \times 10\%$).
- The bank can further tranche the junior tranche into:
  1. senior mezzanine tranche,
  2. subordinate equity tranche.
- Now the mezzanine tranche loses money only 1% of the time, when both loans default.

MBS

- As you probably have guessed, this process goes on if the bank can do more pooling and tranching.
- *In the limit*, the bank could claim that 90% (i.e., 1 minus probability of default on individual loan) of the MBS are super-safe (no default risk) and receive a AAA credit rating.
- Typically, the super-safe tranche is sold to investors looking for safe investments (e.g. MMFs).
- The bank then needs only 10% of the loan value to stay in business.
- Note that if defaults are perfectly correlated, there could never be more than 50% of the mortgage pool that is safe.

APPENDIX AND ADDITIONAL MATERIAL

Formula for a Geometric Series

- The general form of a geometric series is:

$$\sum_{k=0}^{N} ax^k = a + ax + ax^2 + \dots$$

  where $x \neq 0$ and a is a scalar.

- We can write a convenient formula for the sum of a geometric series:

$$\sum_{k=0}^{N} ax^k = \frac{a(1 - x^{N+1})}{1 - x}.$$

- In the case of an infinite geometric series, the formula is more convenient:

$$\sum_{k=0}^{\infty} ax^k = \frac{a}{1 - x}.$$

Back

Present Value of a Growing Annuity

- The formula for the present value of a finite growing annuity comes from the formula for a geometric series:

$$\frac{A}{1+R} + \frac{A(1+g)}{(1+R)^2} + \ldots + \frac{A(1+g)^{N-1}}{(1+R)^N} = \frac{A}{R-g}[1 - (\frac{1+g}{1+R})^N].$$

Back

The use of logs in economics[2]

- Suppose you invest P_1 for 1 year at the rate r. At the end of the year, your investment will be worth $P_2 = (1+r)P_1$.
- Sometimes, the interest might be "compounded": for example, you get your money back plus 2% at the end of 6 months, and then can earn 2% on both your original principal, as well as on the first 6 months interest:

$$P_2 = (1+r/2)(1+r/2)P_1 = (1+r/2)^2 P_1.$$

- The interest might be compounded even more frequently, and the formula $(1+r/n)^n$ converges to:

$$\lim_{n\to\infty}(1+r/n)^n = e^r.$$

- Therefore, if you earn an interest r, that is compounded continuously, at the end of the year your investment will have grown by:

$$P_2/P_1 = e^r.$$

[2] The following slides draw from Jim Hamilton's blog note.

The use of logs in economics

- In economics, we often take natural logs, for example: $\ln(P_2/P_1) = r$, and using log properties:
$$\ln P_2 - \ln P_1 = r.$$
- Therefore, if P is a stock price, the formula above says that taking the difference between the log of the stock price in year 2 and 1 gives you the holding period return quoted in terms of a continuously compounded rate.
- For low values of r, the continuously compounded return is almost the same as the non-compounded:
$$\ln(1+r) \simeq r.$$
- For example, assume $r = 4\%$. If the rate is continuously compounded and the initial investment is $P_1 = 1$, then $P_2 = e^{.04} = 1.0408$.
- In the example above, the percentage change is $(P_2 - P_1)/P_1 = 0.0408$ and the log change $\ln P_2 - \ln P_1 = 0.04$

The use of logs in economics

- Anytime you see a graph measured on a log scale, an increase of 0.01 on that scale corresponds very close to a 1% increase.

- Or, a straight line over time plotted in logs corresponds to growth at a constant percentage rate each year.

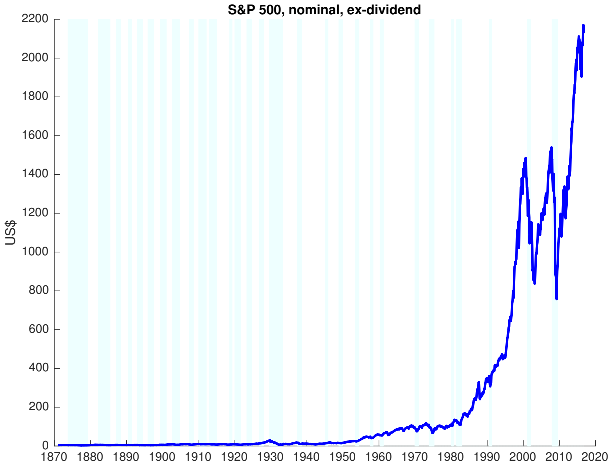
The use of logs in economics

- Consider the following example: if your portfolio goes up by 50% (e.g., from \$100 to \$150), and then declines by 50% (e.g., from \$150 to \$75), your average return is 0%, but you are not even back where you started!
- In fact, you end up 25% below where you started!
- By contrast, if your portfolio goes up by 0.5 in log terms, and then falls by 0.5 in log terms, you end up exactly where you started.
- For example:

$$\begin{aligned} \ln P_2 - \ln P_1 &= 0.5 \\ \ln P_3 - \ln P_2 &= -0.5 \\ (\ln P_3 - \ln P_2) &+ (\ln P_2 - \ln P_1) = \ln P_3 - \ln P_1 = 0 \end{aligned}$$

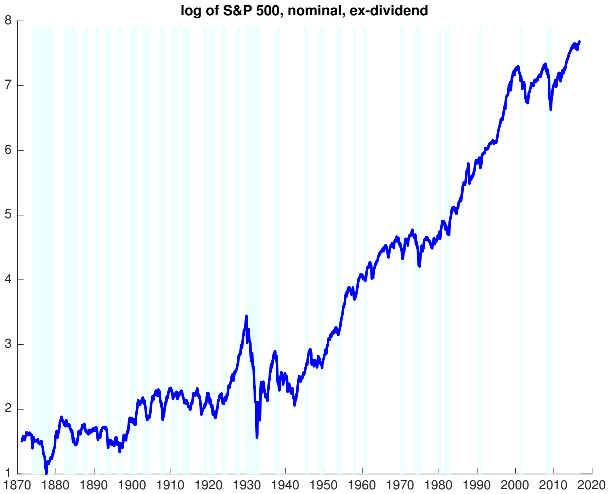
and therefore: $\ln P_3 - \ln P_1 = 0 \rightarrow P_3/P_1 = 1 \rightarrow P_3 = P_1$.

The use of logs in economics



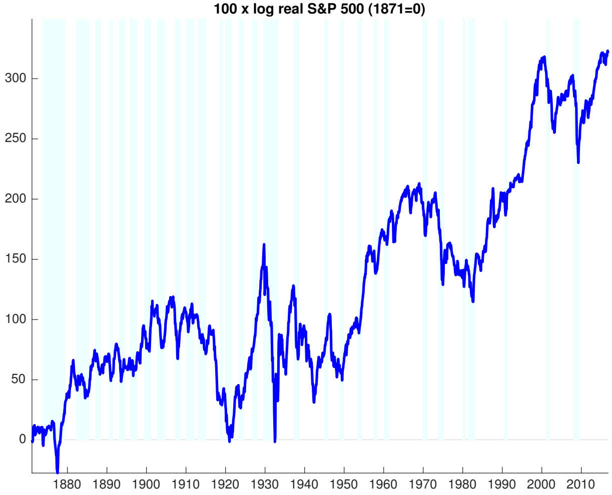
Notes: S&P Composite nominal index ex-dividends. Data are from Robert Shiller, at monthly frequency, for the period 1871.01 - 2016.01. Shaded areas are NBER official recessions.

The use of logs in economics



Notes: S&P Composite nominal index ex-dividends: log scale. Data are from Robert Shiller, at monthly frequency, for the period 1871.01 - 2016.01. Shaded areas are NBER official recessions.

The use of logs in economics



Notes: S&P Composite real index ex-dividends: log scale and normalized to 0 at start of sample. Data are from Robert Shiller, at monthly frequency, for the period 1871.01 - 2016.01. Note how the numbers on vertical axis denote cumulated log returns. Shaded areas are NBER official recessions.

The use of logs in economics

- Plotted on the standard scale, the graph of the overall US stock price index would show nothing for the first century, whereas the most recent decade appears extremely volatile.
- Plotted on a log scale, a vertical move of 0.01 corresponds to a 1% change at any point in the figure: on this scale, recent volatility is modest with respect to that around the Great Depression.
- The last graph deflate nominal returns from the effect of inflation.

Back

Yield with positive and negative cash flows

- The yield to maturity is the internal rate of return (IRR) of an investment with initial value P (i.e., the bond price) and cash-flows equal to the coupons and principal payments.
- For the case of a bond, cash flows are typically all positive.
- However, for other type of investments this is not always the case: for example, at some point before maturity you might have to put in more money in the investment.
- When cash-flows change sign, the IRR (and so the yield-to-maturity) has generally multiple, and possibly complex, solutions.
- Note that all solutions are *legitimate*, in the sense that they set the present value of cash flows equal to the current price.
- However, we can disregard some of the solutions on the ground of economic reasoning: for example, given limited liability, solutions that are < -1.

Back

Yield with positive and negative cash flows

- In Matlab you can find the IRR using the function `irr` that uses as inputs a vector of cash flows, where negative cash flows represent money invested and positive cash flows payouts.
- Matlab deals with the multiple solutions problem as follows:
  - in case of more than one strictly positive rates, Matlab picks the minimum;
  - in case of no strictly positive rates, but multiple non-positive rates, Matlab picks the maximum;
  - in case no real-valued exists, Matlab will return `NaN` (i.e., not a number).
- Note that `irr` assumes that the cash-flows are periodic (i.e., equally spaced).

Back

Yield with positive and negative cash flows

- Matlab has also alternative functions that will perform the same task, but with different and more sophisticated options.
- For example, `cfyield` computes the yield to maturity for cash flows given price and let you input the cash-flow specific dates.
- Refer to Matlab Help for more details and alternative options.

Back

Yield with positive and negative cash flows

- A useful application is in the context of mutual funds.
- Investors pour money in and out of mutual funds.
- How to measure the average return of the mutual fund?
- Suppose you have data on net quarterly flows (NF) to a mutual fund, and the initial and final total net asset value (NAV) of the fund.
- In this case, the average quarterly (gross) return R solves:

$$NAV_t = \sum_{j=1}^{T} \frac{NF_{t+j}}{R^j} + \frac{NAV_{t+T}}{R^T},$$

and we can use `irr` to back it out.

Back

Yield with positive and negative cash flows

- To avoid the complications arising from the possible multiplicity of solutions, sometimes an *ad hoc* approximation called the Dietz method is used.

- With reference to the example from the previous slide, the quarterly gross return using Dietz formula is:

$$R = \left(\frac{NAV_{t+T} - NAV_t + \sum_{j=1}^{T} NF_{t+j}}{NAV_t + 0.5 \sum_{j=1}^{T} NF_{t+j}} + 1\right)^{1/T}.$$

Back

Taxes and the Real Rate of Interest: An Example

- Suppose that inflation runs at the rate of 8% and your investment yields 12%.
- Suppose further that your marginal tax rate is constant at 26%.
- In this case, your before-tax real rate is 4%.
- In an inflation protected tax system, only your real rate should be taxed: $4\%(1 - 0.26) = 2.96\%$.
- However, if the tax code does not recognize that the first 8% of your return is just compensation for inflation, then your after-tax return is additionally reduced by $8\% \times 0.26 = 2.08\%$.
- Your after-tax real rate is just 0.88%!
- Currently inflation is very low: not a big deal. But what if inflation goes up again?

back

Expected Shortfall

- For the special case of normally distributed returns, a formula for the ES is:

$$ES = \frac{1}{.05}\exp(\mu)N[-\sigma - F(.95)] - 1,$$

  where:

  - .05 is the chosen significance level,
  - μ is the mean of the continuously compounded returns,
  - σ is the standard deviation,
  - N is the cumulative standard normal,
  - F is the inverse of the cumulative standard normal.

back

Lower Partial Standard Deviation

- The **lower partial standard deviation** (LPSD) is the standard deviation computed solely from values below the expected *excess return*.
- Excess returns are returns in excess of a risk-free rate.
- It improves on standard deviation of returns because:
  1. it evaluates separately negative outcomes,
  2. it takes the risk-free as benchmark of alternative portfolio.
- It is a measure of **downside risk**.
- Notice that LPSD does not take into account the frequency of negative excess returns (since it's an average).

back

Sortino ratio

- The Sortino ratio is an additional measure of downside risk.
- It penalizes only returns below a given threshold rate of return, while the Sharpe ratio penalizes *both* up- and down-side volatility equally.
- The Sortino ratio is calculated as:

$$S = \frac{R - T}{DR},$$

  where R denotes the average realized return on a portfolio, T is the required or target rate of return, and DR is the target semi-deviation (i.e, the squared root of the target semi-variance).
- In general, we can compute DR as:

$$DR = \sqrt{\int_{-\infty}^{T} (T - r)^2 f(r) dr},$$

  where r is the random variable representing returns from the distribution $f(r)$.

back