# Equilibrium models: the CAPM

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So far we discussed portfolio analysis based on the mean-variance principle:

- We use the expression *mean-variance* since we assume that investors are interested only in the average returns and variances of portfolios.
  - This assumption is good as long as returns are normally distributed.

- Some of the main lessons we learnt from the mean-variance analysis:
  - You need to have the same risky portfolio regardless of the degree of risk aversion.
    - if you want less risk, you will combine the risky portfolio with a risk-free asset.
    - if you want more risk, you will borrow (leverage).
  - In large portfolios, it is important to look at the covariance, and not at the variance.

# What are the problems with the mean-variance analysis?

- Not too many!
  - Caveat: do not forget that according to this model, you should include in your analysis all assets, including human capital, real estate, etc.
- However, the Markowitz model does not say anything about the **origin of prices, returns, variances, covariances**, etc.
- In this sense, Markowitz is a partial equilibrium model.
- Now we want to consider an **equilibrium model** that let us understand how prices are determined in an efficient market. This is model is the capital asset pricing model, or simply **CAPM**.

# Markowitz vs. CAPM I

- Mean-variance analysis selects a portfolio, *given* expected returns and the covariances.
- In practice, we need a large number of estimates!
- Since estimating expected returns is not easy, it would be useful to have a model that tells us what *they should be*.

# Markowitz vs. CAPM II

- The CAPM is an equilibrium model that establishes a relationship between expected returns and covariances for all assets.
  - In equilibrium, all investors are happy of their portfolio holdings, and do not want to change them (this is the meaning of *equilibrium model*).
  - Note how the Markowtiz portfolio model is relevant for any investor, regardless of the fact that we are in an equilibrium, or that the CAPM is correct.

# Equilibrium pricing

- The approach we will follow asks the following question: *if all investors have an efficient portfolio, how should we price assets in order for them to be all held in equilibrium?*
  - For example, suppose that given the price & expected return combination produced by the model, there is no investor that wants to buy the IMB stock.
  - In this case, something in the model is not working.
  - The price of the IMB stock would be *too high*, and the expected return it offers *too low*.
  - The price of IBM should fall to the point in which, in the aggregate, all investors are happy to hold all the existing stocks.

What kind of prices (and risk-return relationship) are possible in equilibrium?

**The CAPM tries to answer to this question**.

# CAPM and the empirical evidence

- If the CAPM holds in the data, or not, is the subject of much academic debate (we will talk about it).
- Even in the case CAPM did not hold in the data, there are several reasons to study it:
  1. if it is *wrong*, then it means that is possible to *beat the market portfolio*, under the assumption that we are interested in expected returns and variances.
  2. furthermore, the mean-variance analysis and the CAPM provide us with a framework to study the relationship between risk & return.
- **The CAPM then is the backbone of more advanced and complicated asset pricing models**.

# The assumptions of the CAPM I

- We need a set of assumptions to formally derive the CAPM. Some of these can be *relaxed* without affecting results too much:
    1. No transaction costs.
    2. All assets are *tradable* and infinitely divisible.
    3. There are no taxes.
    4. All investors are *price takers*, and no one can directly influence the price of a stock.
    5. Investors consider only means and variances.
        - returns are normally distributed.
        - all investors have a quadratic utility.
    6. There are no constraints with respect to short-sales, or borrowing.
    7. Expectations are homogenous (i.e., all investors have same expectations).

# The assumptions of the CAPM II

- Note how assumptions $5 \rightarrow 7$ imply that **all investors solve the same optimization problem and face the same efficient frontier**.
- Therefore, you can think of a model with a representative investor.

# Tw-fund separation theorem

What did we learn from the solution of the optimal (passive) portfolio?

- All investors hold a linear combinations of two portfolios:
  - the risk-free asset,
  - the tangency portfolio.
- If all investors look at the same CAL, **then all investors have the same tangency portfolio**!

# CAPM and the tangency portfolio I

- What is the tangency portfolio?
  1. Markowitz: investors *should* hold the tangency portfolio.
  2. Equilibrium theory (*market clearing*):
     - the risk-free asset is in zero-supply: in other words, in the aggregate, lending and borrowing must cancel out.
     - the average investor *must* hold the market portfolio.
  3. CAPM: the tangency portfolio **must be the market portfolio**.

# CAPM and the tangency portfolio II

- What do we mean by "**market portfolio**"?
  - The market portfolio (or aggregate wealth portfolio) is a portfolio that includes all assets in proportions to their relative value (with respect to the total value).
  - This portfolio is the sum of stocks, bonds, real-estate, human capital, etc.

# What about individual assets?

- The CAL of the CAPM is also called **CML, or capital market line**: it describes optimal combinations of risk and return as follows:

$$E(r_e) = r_f + (\frac{E(r_m) - r_f}{\sigma_m})\sigma_e,$$

where with $r_e$ we denote the return of any *efficient* portfolio, and with $r_m$ the return on the market portfolio.

# The CML

- Note how the CML implies that **all investors should exclusively hold combinations of the market portfolio and the risk-free asset**.
- What can we say about inefficient portfolios (or about individual assets)? What can we say about their equilibrium expected return?

# Equilibrium portfolio model I

- Investors will want to keep an asset in their portfolio only if it provides an additional return in exchange of the contribution to portfolio risk.

# Equilibrium portfolio model II

- **Preview:** For each asset, the quantity of risk contributed to the portfolio will be exactly equal to the contribution in terms of expected return.
- Therefore, **the ratio between marginal return and marginal variance must be the same for each asset**.
  - The contribution in terms of expected return is equal to the expected excess return.
  - The contribution in terms of risk is proportional to the covariance with the portfolio return.
  - **This is the fundamental intuition for the standard CAPM formula, that relates the so called asset-$\beta$ and expected return**.

# Portfolio variance and covariances I

- How does variance change if we add a little bit of a stock to the market portfolio?

$$
\begin{aligned}
\sigma_m^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov(r_i, r_j) \\
&= \sum_{i=1}^{N} w_i [\sum_{j=1}^{N} w_j cov(r_i, r_j)] \\
&= \sum_{i=1}^{N} w_i cov(r_i, \underbrace{[\sum_{j=1}^{N} w_j r_j]}_{\text{vw mkt return}}).
\end{aligned}
$$

where the vw mkt return is the value-weighted market return (i.e., weights are the relative market cap).

# Portfolio variance and covariances II

- Therefore, what matters to determine the marginal increase in risk when we modify the quantity invested in a individual stock is the covariance with the portfolio return.

# Useful results: moments of combinations I

- Constants come out of expectations and expectations of sums are equal to sums of expectations.
- If $c$ and $d$ are numbers:

$$E(c \times R^a) = c \times E(R^a),$$

$$E(R^a + R^b) = E(R^a) + E(R^b).$$

- More in generally:

$$E[c \times R^a + d \times R^b] = c \times E(R^a) + d \times E(R^b).$$

# Useful results: moments of combinations II

- Variance of sums works like taking a square:

  $$var(cR^a + dR^b) = c^2 var(R^a) + d^2 var(R^b) + 2 \times cd \times cov(R^a, R^b).$$

- Covariances work linearly:

  $$cov(cR^a, dR^b) = cd \times cov(R^a, R^b).$$

# Formal derivation of the CAPM

1. Under our assumptions, all investors must hold the market portfolio.

2. Given our definition of equilibrium, all investors must be happy of their portfolios. If not, prices should adjust accordingly.

3. This means that in equilibrium no one can do anything to increase the Sharpe ratio of his/her portfolio.

# Semi-Formal derivation of the CAPM I

Suppose that you currently hold the market portfolio, and that you decide to invest a **small additional quantity** of funds equal to $\delta_{GM}$ in the GM stock, and that you finance your purchase at the risk-free rate.

# Semi-Formal derivation of the CAPM II

1. The return is then:

$$r_c = r_m - \delta_{GM} r_f + \delta_{GM} r_{GM}.$$

2. Expected return and variance are:

$$\begin{aligned} E(r_c) &= E(r_m) + \delta_{GM}(E(r_{GM}) - r_f) \\ \sigma_c^2 &= \sigma_m^2 + \delta_{GM}^2 + 2\delta_{GM} cov(r_{GM}, r_m) \end{aligned}$$

3. The variation ($\Delta$) in these two quantities is equal to:

$$\begin{aligned} \Delta E(r_c) &= \delta_{GM}(E(r_{GM}) - r_f) \\ \Delta \sigma_c^2 &= 2\delta_{GM} cov(r_{GM}, r_m), \end{aligned}$$

where we ignored $\delta_{GM}^2$, since it is a very small number for small values of $\delta_{GM}$.

# Semi-Formal derivation of the CAPM III

What happens if we invest $\delta$ additional funds in GM and a little less in IBM so that we leave the portfolio variance unchanged?

# Semi-Formal derivation of the CAPM IIII

1. The change in variance is equal to:

$$\Delta\sigma_c^2 = 2(\delta_{GM} cov(r_{GM}, r_m) + \delta_{IBM} cov(r_{IBM}, r_m)).$$

2. In order for this change to be equal to zero:

$$\delta_{IBM} = -\delta_{GM}(\frac{cov(r_{GM}, r_m)}{cov(r_{IBM}, r_m)}).$$

3. In this case, the variation in the portfolio expected return is:

$$\begin{aligned}\Delta E(r_c) &= \delta_{GM} E(r_{GM} - r_f) + \delta_{IBM} E(r_{IBM} - r_f) \\ &= \delta_{GM}[E(r_{GM} - r_f) - E(r_{IBM} - r_f)(\frac{cov(r_{GM}, r_m)}{cov(r_{IBM}, r_m)})]\end{aligned}$$

# Marginal benefit = marginal cost

- Recall that the investor holds the market portfolio, which is also the tangency portfolio.
- This portfolio has **the highest Sharpe ratio among all the possible portfolios**.
- Therefore, by definition, we cannot increase the expected return, by keeping variance constant. For this to be true:

$$\frac{E(r_{GM}) - r_f}{cov(r_{GM}, r_m)} = \frac{E(r_{IBM}) - r_f}{cov(r_{IBM}, r_m)} = \lambda.$$

- $\lambda$ denotes the ratio between marginal benefit and marginal cost.

# The security market line

- This result holds also for a portfolio of assets, not only for individual stocks.
- Let's use the market portfolio in place of the IMB stock:

$$\frac{E(r_{GM}) - r_f}{cov(r_{GM}, r_m)} = \frac{E(r_m) - r_f}{cov(r_m, r_m)} = \frac{E(r_m) - r_f}{\sigma_m^2} = \lambda,$$

  this means that:

$$\begin{aligned} E(r_{GM}) - r_f &= \frac{E(r_m) - r_f}{\sigma_m^2} cov(r_{GM}, r_m) \\ &= (E(r_m) - r_f) \underbrace{\frac{cov(r_{GM}, r_m)}{\sigma_m^2}}_{\beta_{GM}} \end{aligned}$$

- This is the formal representation of the SML, or security market line.

# SML vs. CML

- Every stock lies on the SML (in the plane expected return-beta).
- Only the market portfolio and the risk-free asset, and the linear combinations of the two, are on the CML (in the plane expected return-standard deviation).
- The SML represents return vs. systematic risk.
- The CML represents return vs. total risk (systematic + non-systematic).

# To sum-up I

- By the definition of tangency portfolio, investors are not able to get a higher Sharpe ratio by combining the tangency portfolio with any other assets.

- This constraint implies **a linear relationship** between the equilibrium return of an asset, and its $\beta$ with respect to the tangency portfolio

$$E(r_i) - r_f = [E(r_T) - r_f]\beta_i$$

# To sum-up II

- The CAPM states that the equilibrium tangency portfolio *is the market portfolio*.
- One possible interpretation of this result is that the excess return of asset $i$, $E(r_i) - r_f$, must be equal to the quantity of priced risk $\beta$ times **the market price of risk** $E(r_M) - r_f$.

# Statistical interpretation of an asset $\beta$

We can interpret $\beta$ as a linear regression coefficient (OLS):

- Consider the following regression equation:

$$r_i^e = \alpha_i + \beta_i r_m^e + \epsilon_i.$$

- The OLS slope coefficient is:

$$\beta_i = \frac{cov(r_i^e, r_m^e)}{\sigma_m^2},$$

which is equivalent to the relationship we have derived earlier.

# Systematic vs. idiosyncratic risk I

- The fraction of $r_i^e$ that *is explained* by the market return is $\beta_i r_m^e$: $\rightarrow$ this quantity represents the systematic (or also "market", or "aggregate") risk of an asset.

- The fraction that *is not explained* is instead $\epsilon_i$: $\rightarrow$ this part represents the "specific", or "idiosyncratic" risk, of an asset.

# Systematic vs. idiosyncratic risk II

- We have learnt that we can decompose total variance in:
    1. the systematic component $\beta_i^2 \sigma_m^2$.
    2. the idiosyncratic component $\sigma_\epsilon^2$.

# Food for thought

- The CAPM implies that *only* the systematic component is priced.
- Why don't investors look at non-systematic risk?
- What is the interpretation of the $R^2$ in the CAPM regression?

# Some definitions and clarifications

- In the CAPM framework, diversification and idiosyncratic risk can be confused. In fact, it is usually possible to find two definitions:

  1. A well diversified portfolio will have the minimum possible variance for a given level of expected return.
     - ALL minimum variance portfolios on the frontier are well diversified according to this definition.
     - These portfolios might also have idiosyncratic risk, defined as risk that is not correlated with the aggregate market.

  2. A well diversified portfolio will have zero idiosyncratic risk, or $\sigma_\epsilon^2 = 0$.
     - ONLY the market portfolio (and the combinations of this latter portfolio with the risk-free asset, i.e., all portfolios on the CAL) is well diversified according to this second definition.

# Deviations from CAPM

- What is the interpretation of $\alpha_i$ according to the CAPM?
- $\alpha_i$ represents deviations from the SML.
- For example, $\alpha_i > 0$ denotes the situation when a stock offers a return that is higher than the equilibrium return, for a given level of systematic risk.
- Therefore, $\alpha_i$ are pricing errors (or anomalies): if you believe the model is correct, than these are signals which drive your investment strategies.