Capital Allocation to Risky Assets

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WU EXECUTIVE ACADEMY

- Suppose you want to build a financial portfolio.
- You need to choose:
  1. how to allocate resources within the portfolio (e.g., stocks vs. bonds),
  2. how much to invest in the risky portfolio, as opposed to in a risk-free alternative.
- In order to do the above, you need to know:
  1. expected returns,
  2. the degree of risk,
  3. your preference with respect to the risk-return trade-off.

- The task of building the (optimal) risky portfolio can be delegated to a professional.
- On the contrary, the choice of how much to invest in a risky portfolio depends on individual attitudes (i.e., preferences) toward risk and reward.
- Therefore, it is important to introduce an instrument to evaluate these preferences.
- This instrument is a utility function that we will use to evaluate and rank different portfolios.

Outline

Utility and risk aversion

Capital allocation

Risk tolerance and asset allocation

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How to choose among risky alternatives?

- Consider the following example:

Table 6.1

Available risky portfolios (Risk-free rate = 5%)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

- As the expected return increases, so does risk.
- How can investor choose among the three alternatives? The answer is not obvious.
- Let's take as measure of risk the return variance (if returns are normally distributed, this is a good measure of risk).

Utility and risk aversion

- We will assume investors can assign a numerical value to the utility of different portfolios and use a common utility function:

$$U = E(r) - \frac{1}{2}A\sigma^2.$$

- $E(r)$ represents expected return, while σ^2 is the variance of returns.
- A it's a parameter that captures risk aversion of an investor (i.e., the higher is A, the more the investor dislikes risk).

Certainty equivalence and risky returns

- We can interpret the value for the utility as a certainty equivalence:
$$r_{CE} \equiv U = E(r) - \frac{1}{2}A\sigma^2.$$
- We will conclude that a given portfolio is preferred to a risk-free alternative only if its certainty equivalent is larger than the risk-free rate.

Risk aversion

- An investor who is risk neutral, has a coefficient of risk aversion $A = 0$ and only looks at expected return.
- An investor who is risk lover (averse) has a coefficient of risk aversion $A < 0$ $(A > 0)$.

Risk-return trade-off

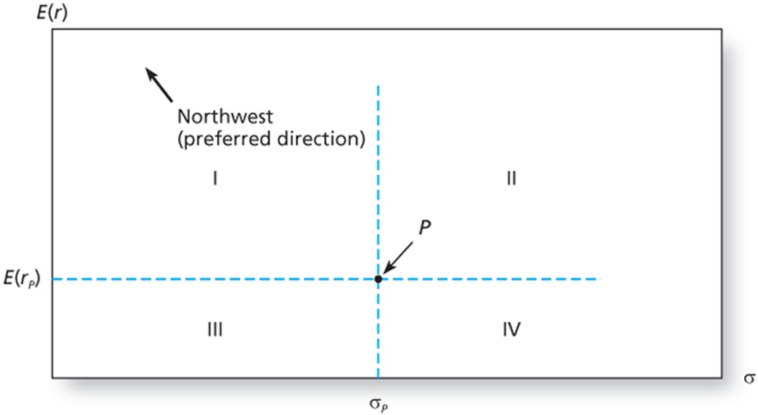

Figure: Trade-off between risk and return of a potential investment portfolio, P.

The mean-variance criterion is not complete!

- **Mean-Variance Criterion** $\Rightarrow$ Portfolio A is preferred to Portfolio B if and only:
  - $E(r_A) \geq E(r_B)$,
  - $\sigma_A \leq \sigma_B$,
  - and at least one of the inequality is strict.
- This criterion is not **complete**, because I cannot rank any alternatives on the basis this criterion.

Risk-return trade-off and risk-aversion I

- What can we say about portfolios that lie in the quadrants II and III?
- The answer depends on the degree of risk aversion.

Risk-return trade-off and risk-aversion II

- Recall the utility function we use is:

$$U = E(r) - \frac{1}{2}A\sigma^2.$$

- Therefore, the slope of the indifferences curves (or marginal rate of substitution) is:

$$MRS = \frac{\partial E(r)}{\partial \sigma} = A\sigma.$$

- The larger is risk aversion, the larger is the additional reward an investor demands for one additional unity of risk.

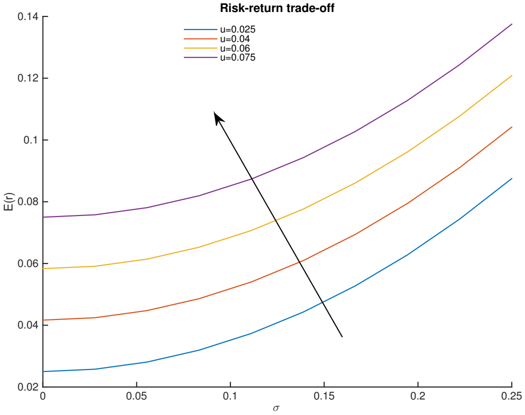

Figure: Trade-off between risk and return: indifference curves (arrow points in direction of higher utility)

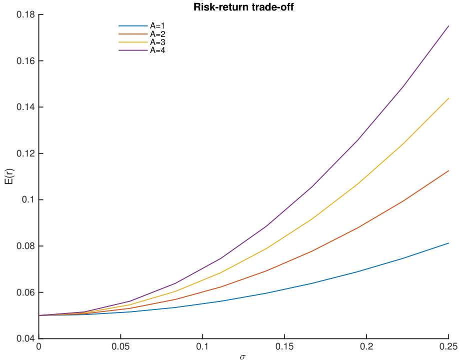

Figure: The effect of higher risk-aversion A.

- Each curve *represents the same level of utility*, for different levels of the risk aversion A.
- When $A \uparrow$, for a given level of risk σ, the investor requires a higher expected return to get to the same level of utility.

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Capital allocation: risky and risk-free portfolios

- One of the easiest and immediate investment strategies to control risk is to allocate a fraction of your wealth in risk-free (or low risk) assets (e.g., short-term government securities, money market instruments, etc.).

- By capital allocation, we mean an allocation among broad investment classes, rather than among the specific securities in each class (stock picking or stock selection).

"The most fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves? ... That decision has accounted for an astonishing 94% of the difference in total returns achieved by institutionally managed pension funds." John Bogle, former Chairman of the Vanguard Group.

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How to allocate capital between risky and risk-free portfolios

- Let's denote the portfolio of risky assets with P and the risk-free asset with F.
- Let's assume that the risky alternatives are two mutual funds: one invests in equity (E), and one in long-term fixed income bonds (B).
- For the time being, let's take as given the composition of the risky portfolios and **analyze how to allocate capital between the risky portfolios and the risk-free alternative**.

An example I

- The initial value of the portfolio is \$300,000.
- \$90,000 invested in a money mutual fund ($\simeq$ risk-free), and \$210,000 in the more risky funds (\$113,400 in equity, \$96,600 in bonds).
- The shares of E and B in the risky portfolio are:
  - $E : w_E = \frac{113{,}400}{210{,}000} = .54$
  - $B : w_B = \frac{96{,}600}{210{,}000} = .46$

An example II

- The weight of the risky portfolio with respect of the overall portfolio is:
  - $y = \frac{210{,}000}{300{,}000} = .7$
  - $1 - y = \frac{90{,}000}{300{,}000} = .3$
- And the weight of each asset class with respect of the overall portfolio is:
  - $E : \frac{113{,}400}{300{,}000} = .378$
  - $B : \frac{96{,}600}{300{,}000} = .322$
  - $P = E + B = 0.7.$

An example III

- We can think about the risky assets E and B as a single asset (i.e., a fund that invests both in equity and bonds).
- Let's assume that when we change the share of the risk-free assets in the overall portfolio, we do not change the relative proportions of the various risky assets within the risky portfolio.
- In this case, the probability distribution of the return of portfolio P does not change when we change y (i.e., when we change asset allocation).

Portfolio of one risky asset and a risk-free asset

- Suppose we need to determine y: the share of risky investment in the total portfolio.
- Denote with r_P the return on the risky portfolio (P); with $E(r_P)$ its expected return; with σ_P its standard deviation; and with r_f the risk-free return.
- The total portfolio return (r_C) is then equal to:

$$r_C = y \times r_P + (1 - y) \times r_f.$$

- And the expected return:

$$E(r_C) = y \times E(r_P) + (1 - y) \times r_f = r_f + y \times [E(r_P) - r_f].$$

Expected risky return

- How can we interpret the expected return:

$$E(r_C) = r_f + y \times [E(r_P) - r_f]?$$

  - **risk-free return**,
  - **risk premium**,
  - investors are risk averse!

The risk of a risky return

- The standard deviation of the portfolio return r_P is: σ_P.
- What is the standard deviation of r_C? $\sigma_C = y \times \sigma_P$
- We can *control* portfolio risk (measured by σ_C) simply by changing y.
  - For example, if $y = 0$ we can completely eliminate risk but we need to accept a return equal to r_f.
  - Generally, there exists a trade off between risk and return and it is expressed by the utility function so that $y \neq 0$.

Example: Portfolio of one risky asset and a risk-free asset

- Let's consider a simple numerical example:
  - $E(r_P) = 15\%$.
  - $\sigma_P = 22\%$.
  - $r_f = 7\%$.
- We can now represent the main characteristics of the total portfolio on a risk-return plane, for a given value of y. This graphical representation is called **capital allocation line**, and it describes the **investment opportunity set** with a risky asset and a risk-free asset.

Capital Allocation Line (CAL)

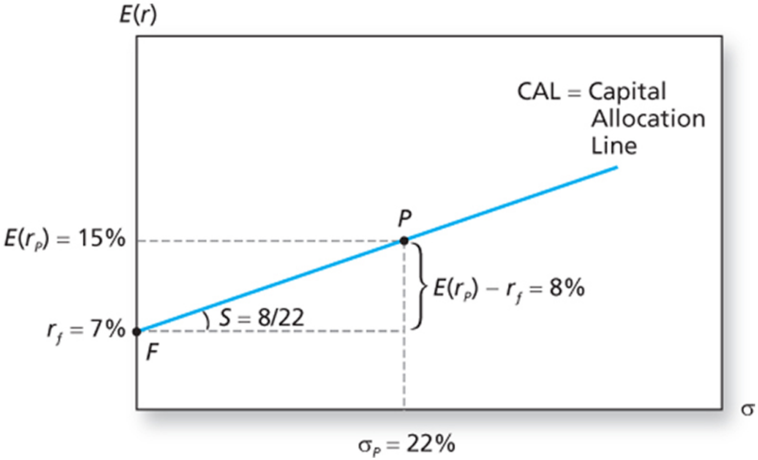

Figure: The investment opportunity set with a risky asset and a risk-free asset in the expected return-standard deviation plane.

CAL and critical combinations

- If $y = 0$ we are in F.
- If $y = 1$ we are in P.
- If $0 < y < 1$ we are on the straight line that connects F and P (i.e., linear combinations of the portfolios F and P).
- What is the slope? What is the economics intuition?

Risk-return specification of expected returns

- $\sigma_C = y\sigma_P \Rightarrow y = \frac{\sigma_C}{\sigma_P}$.
- $E(r_C) = r_f + y[E(r_P) - r_f] = r_f + \frac{\sigma_C}{\sigma_P}[E(r_P) - r_f]$.
- Slope:
$$S = \frac{\partial E(r_c)}{\partial \sigma_c} = \frac{[E(r_P) - r_f]}{\sigma_P}$$
.
- $E(r_C) = r_f + \underbrace{[\frac{E(r_P) - r_f}{\sigma_P}]}_{\text{price of risk}} \underbrace{\sigma_C}_{\text{amount of risk}}$.
- The market price of risk is the premium per unit of risk (σ_P) and depends only by the prices of available securities.
- This ratio is usually called *Sharpe Ratio* (SR).

Reward volatility ratio and SR

- To sum up:
  - The CAL, or capital allocation line, represents the investment opportunity set with a risky asset and a risk-free asset.
  - The slope of the CAL captures the trade-off between risk and reward, or the **reward-volatility ratio**.
  - The reward-volatility ratio is no more than the **Sharpe ratio**.

Leverage and the CAL

- Is it possible to choose portfolios to the right of P?
- Yes, if the investor can borrow (**leverage**), for example at the rate r_f.
- In this case, $y > 1$: how do we interpret the fact that $1 - y < 0$?

Different borrowing/lending rates and the CAL

- Usually, only (some) governments can borrow at the risk-free rate.
- Suppose investors must pay a spread with respect to the risk-free rate when they borrow so that: $r_f^B = 9\%$.
- In this case, the slope of the CAL changes in the subset of combinations that implies some degree of **leverage**.
- In the plot, the CAL will have a kink in P.

Portfolio of one risky asset and a risk-free asset

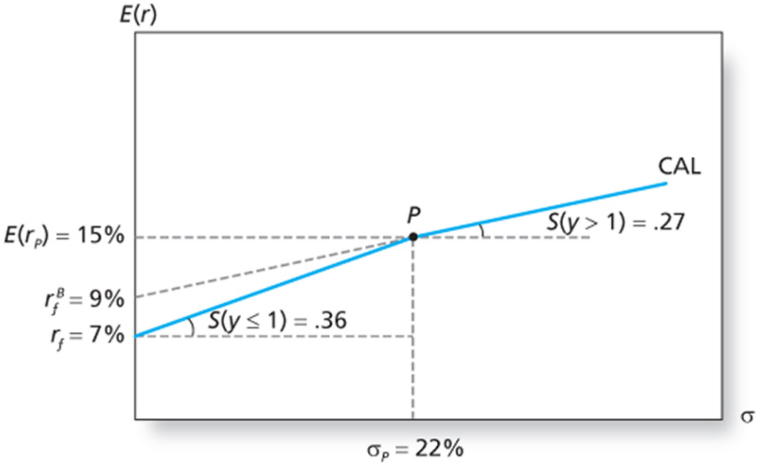

Figure: The opportunity set with differential borrowing and lending rates

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What about the optimal risky portfolio?

- We learnt that the CAL represents the set of investment opportunities.
- But what about the optimal portfolio? Or where on the CAL should we choose to be?
- In order to answer to these questions, we must consider the degree of risk tolerance, in other words investors' preference toward risk.

Risk tolerance and asset allocation

- Recall that the expected return on C is:

$$E(r_C) = r_f + y[E(r_P) - r_f].$$

- The variance of the return on C is:

$$\sigma_C^2 = y^2 \sigma_P^2.$$

- The investor will maximize the following utility:

$$\max_{\{y\}} U = E(r_C) - \frac{1}{2} A \sigma^2,$$

where both the expected return and risk increases with y.

Merton's formula

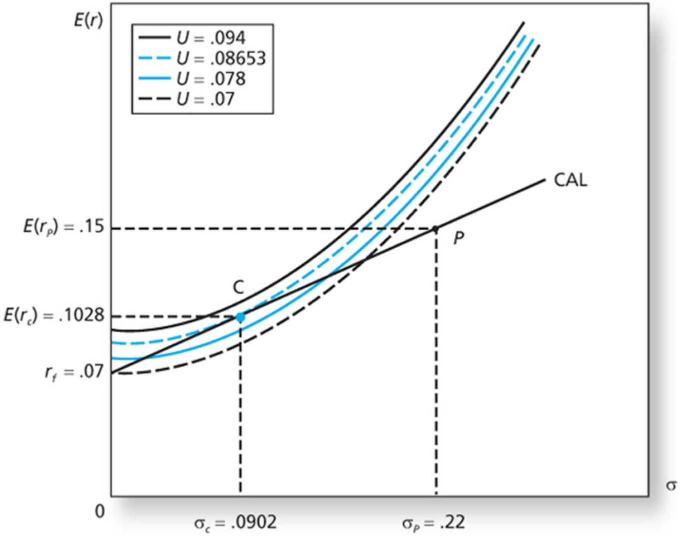
- The investors' optimization problem is as follows:

$$\max_{\{y\}} r_f + y[E(r_p) - r_f] - \frac{1}{2}Ay^2\sigma_p^2,$$

- Set FOC to zero to get the classic Merton's formula for the optimal portfolio allocation:

$$\frac{\partial U}{\partial y} = 0 \Rightarrow y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}.$$

Risk tolerance and asset allocation



Concept check

- What's the interpretation of values $y^* > 1$?
- Is it possible to find $y^* < 0$?
- How does y^* change with A?
- What is the effect on the optimal portfolio of changes in the Sharpe ratio?