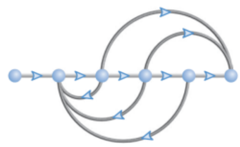
Time Response 4



Learning Outcome

After completing this chapter, the student will be able to

- use poles and zeros of transfer functions to determine the time response of a control system
- describe quantitatively the transient response of 1st-order systems
- write the general response of 2nd-order systems given the pole location
- find the damping ratio and natural frequency of a 2nd-order system
- find the settling time, peak time, percent overshoot, and rise time for an underdamped 2nd-order system
- approximate higher-order systems and systems with zeros as 1st/2nd-order systems
- describe the effects of nonlinearities on the system time response
- find the time response from the state-space representation

§1. Introduction

- This chapter
  - analyze of system transient response
  - demonstrate applications of the system representation by evaluating the transient response from the system model
  - evaluate the response of a subsystem prior to inserting it into the closed-loop system
  - describe a valuable analysis and design tool, poles and zeros
  - analyze the models to find the step response of 1st- and 2nd-order systems

§2. Poles, Zeros, and System Response

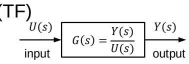
- The output response of a system is the sum of two responses
  - the forced response
  - the natural response
- Output response of a system depends on the positions of the poles and zeroes and their relationships

→ study

- the concept of poles and zeroes
- fundamental to the analysis and design of control systems
- simplifies the evaluation of a system's response

§2. Poles, Zeros, and System Response

- Transfer function (TF)



$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m}{a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

- The poles of a TF are roots of

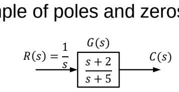
$$a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n=0$$

- The zeros of a TF are roots of

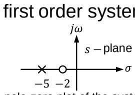
$$b_0s^m+b_1s^{m-1}+\cdots+b_{m-1}s+b_m=0$$

§2. Poles, Zeros, and System Response

- Example of poles and zeros of a first order system



system showing input and output



pole-zero plot of the system

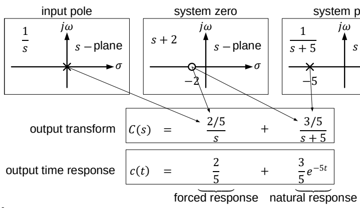
The unit step response of the system

$$C(s)=\frac{1}{s}\times\frac{s+2}{s+5}=\frac{s+2}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}=\frac{2/5}{s}+\frac{3/5}{s+5}$$

$$A=\left.\frac{s+2}{(s+5)}\right|_{s\to0}=\frac{2}{5},B=\left.\frac{s+2}{s}\right|_{s\to-5}=\frac{3}{5}$$

$$\to c(t)=\frac{2}{5}+\frac{3}{5}e^{-5t}$$

§2. Poles, Zeros, and System Response



Evolution of a system response

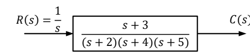
Remark

- A pole of the input function generates the form of the forced response
- A pole of the TF generates the form of the natural response
- A pole on the real axis generates an exponential response of e^{at}
- The zeros and poles generate the amplitudes for both the forced and natural responses

§2. Poles, Zeros, and System Response

- Ex.4.1 Evaluating Response Using Poles

Write the output, $c(t)$, in general terms, for the given system



Specify the forced and natural parts of the solution

Solution

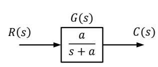
- Each system pole generates an exponential as part of the natural response
- The input's pole generates the forced response

Thus $C(s) = \underbrace{\frac{K_1}{s}}_{\text{forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{natural response}}$

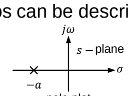
$\rightarrow c(t) = K_1 + K_2e^{-2t} + K_3e^{-4t} + K_4e^{-5t}$

§3. First Order Systems

- A first-order system without zeros can be described by the TF



first order system



pole plot

- If the input is step

$$C(s) = R(s)G(s) = \frac{1}{s} \times \frac{a}{s+a} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

- Taking the inverse transform, the step response is given by

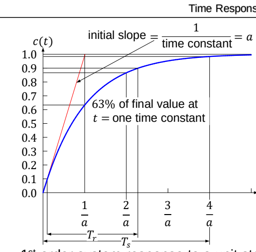
$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

- input pole, 0, generates the forced response $c_f(t) = 1$
- system pole, $-a$, generates the natural response $c_n(t) = -e^{-at}$

§3. First Order Systems

1. Time Constant T_c

- Time constant $T_c = \frac{1}{a}$



1st-order system response to a unit step

$$e^{-at}\Big|_{t=\frac{1}{a}} = e^{-1} = 0.37$$

$$c(t)\Big|_{t=\frac{1}{a}} = 1 - e^{-at}\Big|_{t=\frac{1}{a}} = 0.63$$

→The time constant is the time it takes for the step response to rise to 63% of its final value

$$\boxed{C(s) = R(s)G(s) = \frac{1}{s} \times \frac{a}{s+a} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a} \rightarrow c(t) = c_f(t) + c_n(t) = 1 - e^{-at}}$$

§3. First Order Systems

$$\frac{d}{dt}e^{-at}\Big|_{t=0} = -ae^{-at}\Big|_{t=0} = -a$$

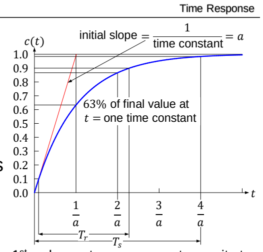
a: the initial rate of change of the exponential at $t = 0$

- Transient response specifications

T_c : time constant

T_r : rise time

T_s : settling time



1st-order system response to a unit step

- a relates to the speed at which the system responds to a step input: the farther the pole from the imaginary axis, the faster the transient response
- a has the units $1/s$, or frequency → the exponential frequency

§3. First Order Systems

2. Rise Time T_r

- The time for the waveform to go from 0.1 to 0.9 of its final value
- Rise time is found by solving

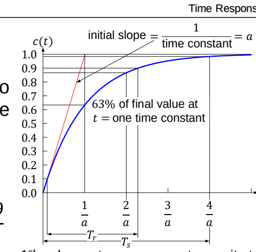
$c(t) = 1 - e^{-at}$

$c(t_1) = 0.1$

$$1 - e^{-at_1} = 0.1 \rightarrow t_1 = -\frac{\ln 0.9}{a}$$

$c(t_2) = 0.9$

$$1 - e^{-at_2} = 0.9 \rightarrow t_2 = -\frac{\ln 0.1}{a}$$



1st-order system response to a unit step

$$\rightarrow t_2 - t_1 = -\frac{\ln 0.1}{a} + \frac{\ln 0.9}{a} = -\frac{0.11}{a} + \frac{2.31}{a}$$

$$\rightarrow T_r = \frac{2.2}{a}$$

§3. First Order Systems

3. Settling Time T_s

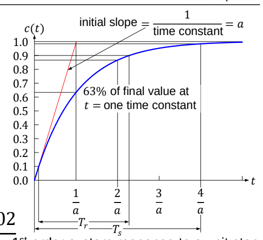
- The time for the response to reach, and stay within, 2% of its final value
- Settling time is found by solving

$c(t) = 1 - e^{-at}$

$c(T_s) = 0.98$

$$1 - e^{-aT_s} = 0.98 \rightarrow T_s = -\frac{\ln 0.02}{a}$$

$$\rightarrow T_s = \frac{4}{a}$$

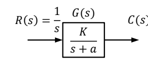


1^{st}-order system response to a unit step

§3. First Order Systems

4. First-Order Transfer Functions via Testing

- Consider a simple first-order system with the step input



$$C(s) = \frac{1}{s} \times \frac{K}{s+a} = \frac{K/a}{s} - \frac{K/a}{s+a}$$

If we can identify K and a from laboratory testing, we can obtain the TF of the system

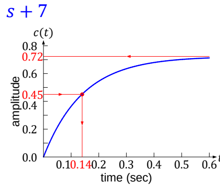
§3. First Order Systems

- Example using the plot of $G(s) = \dfrac{5}{s+7}$

From the plot

- Steady state value $c_{ss} = 0.72$
- Time constant

$0.63 \times c_{ss} = 0.45 \rightarrow T_c = 0.14s$



Therefore

$$T_c = \frac{1}{a} \rightarrow a = \frac{1}{0.14} = 7.14$$

$$\frac{K}{a} = c_{ss} \rightarrow K = ac_{ss} = 7.14 \times 0.72 = 5.14$$

$\rightarrow$ The TF of the plot $\quad G(s) = \dfrac{5.14}{s+7.14}$

Note: 1^{st} order system has no overshoot and nonzero initial slope

§3. First Order Systems

Skill-Assessment Ex.4.2

Problem Find the time constant, T_c, rising time, T_r, and settling time, T_s, of the system with the following TF

$$G(s) = \frac{50}{s+50}$$

Solution From the TF

$$a = 50$$

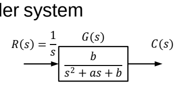
Time constant $\quad T_c = \dfrac{1}{a} = \dfrac{1}{50} = 0.02s$

Rising time $\quad T_r = \dfrac{2.2}{a} = \dfrac{2.2}{50} = 0.044s$

Settling time $\quad T_s = \dfrac{4}{a} = \dfrac{4}{50} = 0.08s$

§4. Second Order Systems: Introduction

- The general 2^{nd}-order system

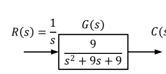


general

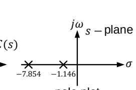
- Depending on component values, the second order system exhibits a wide range of responses such as
  - first-order system
  - damped oscillation
  - pure oscillation

§4. Second Order Systems: Introduction

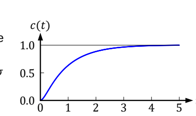
- Second-order systems, pole plots, and step responses
  - Overdamped



overdamped



pole plot



$$C(s) = \frac{9}{s(s+1.146)(s+7.854)}$$

$$\rightarrow c(t) = 1 + 0.171e^{-7.854t} - 0.171e^{-1.146t}$$

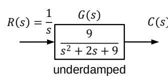
Poles: two real at $-\sigma_1$ and $-\sigma_2$

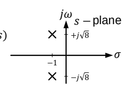
Natural response: $c_n(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$

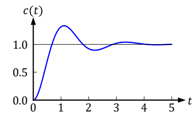
[r,p,k]=residue([9],[1,9,9,0]), $C(s) = 1/s \rightarrow c(t) = 1$, $C(s) = 1/(s+a) \rightarrow c(t) = e^{-at}$

§4. Second Order Systems: Introduction

- Underdamped







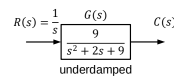
$$C(s)=\frac{9}{s(s^2+2s+9)}$$
$$=\frac{9}{s\left[s-\left(-1-j\sqrt{8}\right)\right]\left[s-\left(-1+j\sqrt{8}\right)\right]}$$

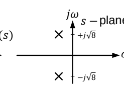
$\rightarrow c(t)=1-1.06e^{-t}\cos(\sqrt{8}t-19.47^0)$

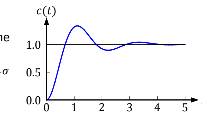
Poles: two complex at $-\sigma_d\pm j\omega_d$

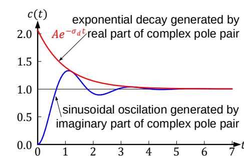
Natural response: $c_n(t)=Ae^{-\sigma_d t}\cos(\omega_d t-\phi)$

§4. Second Order Systems: Introduction



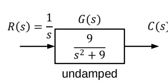


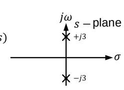


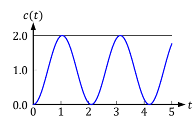


§4. Second Order Systems: Introduction

- Undamped







$$C(s)=\frac{9}{s(s^2+9)}=\frac{1}{s}-\frac{s}{s^2+3^2}$$

$\rightarrow c(t)=1-\cos 3t$

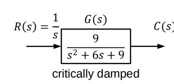
Poles: two imaginary at $\pm j\omega_1$

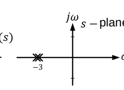
Natural response: $c_n(t)=A\cos(\omega_1 t-\phi)$

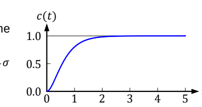
$C(s)=s/(s^2+a^2)\rightarrow c(t)=\cos at$

§4. Second Order Systems: Introduction

- Critically damped







$$C(s)=\frac{9}{s(s^2+6s+9)}$$
$$=\frac{1}{s}-\frac{1}{s+3}-\frac{3}{(s+3)^2}$$

$\rightarrow c(t)=1-3te^{-3t}-e^{-3t}$

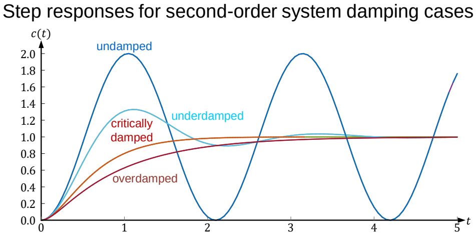
Poles: two real at $-\sigma_1$

Natural response: $c_n(t)=K_1e^{-\sigma_1 t}+K_2te^{-\sigma_1 t}$

$C(s)=a/(s+a)^2\rightarrow c(t)=ate^{-at}$

§4. Second Order Systems: Introduction

- Step responses for second-order system damping cases



§4. Second Order Systems: Introduction

Skill-Assessment Ex.4.3

Problem For each of the following transfer functions, write, by inspection, the general form of the step response

a. $G(s)=\dfrac{400}{s^2+12s+400}$

b. $G(s)=\dfrac{900}{s^2+90s+900}$

c. $G(s)=\dfrac{225}{s^2+30s+225}$

d. $G(s)=\dfrac{625}{s^2+625}$

§4. Second Order Systems: Introduction

Solution a. $G(s)=\dfrac{400}{s^2+12s+400}$

$$=\frac{400}{[s-(-6-j19.08)][s-(-6+j19.08)]}$$

The system has two complex poles at $-6\pm j19.08$

The general form of the step response

$c(t)=A+Be^{-6t}\cos(19.08t+\phi)$

b. $G(s)=\dfrac{900}{s^2+90s+900}=\dfrac{900}{(s+78.54)(s+11.46)}$

The system has two real poles at -78.54 and -11.46

The general form of the step response

$c(t)=A+Be^{-78.54t}+Ce^{-11.46t}$

$G(s)$ has poles at $-\sigma_d\pm j\omega_d$ → natural response $c_n(t)=Ae^{-\sigma_d t}\cos(\omega_d t-\phi)$
$G(s)$ has poles at $-\sigma_1,-\sigma_2$ → natural response $c_n(t)=K_1e^{-\sigma_1 t}+K_2e^{-\sigma_2 t}$

§4. Second Order Systems: Introduction

c. $G(s)=\dfrac{225}{s^2+30s+225}=\dfrac{400}{(s+15)^2}$

The system has two repeated real poles at -15

The general form of the step response

$c(t)=A+Be^{-15t}+Cte^{-15t}$

d. $G(s)=\dfrac{625}{s^2+625}=\dfrac{625}{s^2+25^2}$

The system has two imaginary poles at $\pm j25$

The general form of the step response

$c(t)=A+B\cos(25t+\phi)$

$G(s)$ has poles at $-\sigma_1,-\sigma_1$ → natural response $c_n(t)=K_1e^{-\sigma_1 t}+K_2te^{-\sigma_1 t}$
$G(s)$ has poles at $\pm j\omega_1$ → natural response $c_n(t)=A\cos(\omega_1 t-\phi)$

§5. The General Second Order Systems

1. Natural frequency ω_n

- The natural frequency of a second-order system is the frequency of oscillation of the system without damping

2. Damping ratio ζ

- The natural frequency of a second-order system is the frequency of oscillation of the system without damping

$$\zeta=\frac{\text{exponential decay frequency}}{\text{natural frequency}}=\frac{1}{2\pi}\frac{\text{natural period}}{\text{exponential time constant}}$$

- Consider the general second order system time constant

$$G(s)=\frac{b}{s^2+as+b}\equiv\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\rightarrow\omega_n=\sqrt{b}\tag{4.18}$$

$$\zeta=\frac{a}{2\sqrt{b}}\tag{4.20}$$

§5. The General Second Order Systems

- Ex.4.5 Finding ζ and ω_n For a Second-Order System

Given the transfer function of Eq. (4.23), find ζ and ω_n

$$G(s)=\frac{36}{s^2+4.2s+36}\tag{4.23}$$

Solution

From observation

- The natural frequency of the given system

$\omega_n=\sqrt{b}=\sqrt{36}=6$

- The damping ratio of the given system

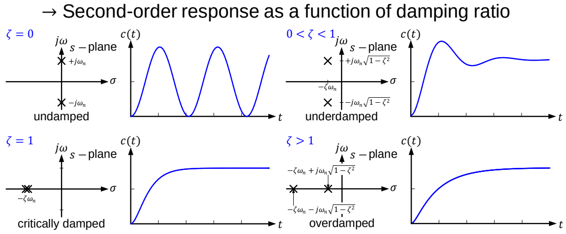
$\zeta=\dfrac{a}{2\sqrt{b}}=\dfrac{4.2}{2\times6}=0.35$

§5. The General Second Order Systems

- The poles of the transfer function

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\rightarrow p_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

→ Second-order response as a function of damping ratio

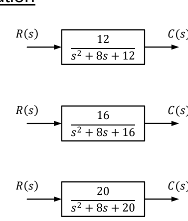


§5. The General Second Order Systems

- Ex.4.4 Characterizing Response from the Value of ζ

Find the value of ζ and report the kind of response expected

Solution



$\zeta=\dfrac{a}{2\sqrt{b}}=\dfrac{8}{2\sqrt{12}}=1.16$

→ overdamped

$\zeta=\dfrac{a}{2\sqrt{b}}=\dfrac{8}{2\sqrt{16}}=1$

→ critically damped

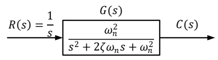
$\zeta=\dfrac{a}{2\sqrt{b}}=\dfrac{8}{2\sqrt{20}}=0.89$

→ underdamped

§6. Underdamped Second Order Systems

- The step response for the general second-order system

$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$



$$=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}+\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

$$\rightarrow c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$

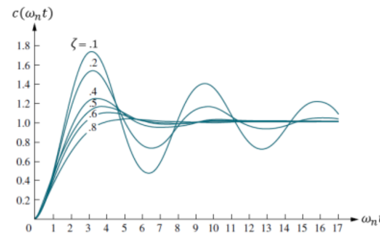
$$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right) \quad (4.28)$$

$$\phi=\tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$$

$\mathcal{L}\{e^{-at}\cos\omega t\}=\frac{s+a}{(s+a)^2+\omega^2}$, $\mathcal{L}\{e^{-at}\sin\omega t\}=\frac{\omega}{(s+a)^2+\omega^2}$ (2.34-35)

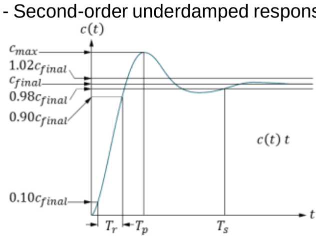
§6. Underdamped Second Order Systems

- Second-order underdamped responses for damping ratio values



§6. Underdamped Second Order Systems

- Second-order underdamped response specifications



- Rise time, T_r

$$T_r=t\big|_{0.9c_{final}}-t\big|_{0.1c_{final}}$$

- Peak time, T_p

$$T_p=t\big|_{c_{max}}$$

- Percent overshoot, $\%OS$

$$\%OS=\frac{c_{max}-c_{final}}{c_{final}}\times 100$$

- Settling time, T_s

The time required for the transient's damped oscillations to reach and stay within 2% of the steady-state value

§6. Underdamped Second Order Systems

1. Evaluation of T_p

$$T_p=t\big|_{c_{max}}=t\big|_{\dot{c}=0}$$

$$(4.28)\rightarrow \dot{c}(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t$$

$$\dot{c}(t)=0\rightarrow \omega_n\sqrt{1-\zeta^2}t=n\pi$$

$$\rightarrow t=\frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\rightarrow T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad (4.34)$$

$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$ (4.28)

§6. Underdamped Second Order Systems

2. Evaluation of $\%OS$

$$\%OS=\frac{c_{max}-c_{final}}{c_{final}}\times 100$$

$$(4.28)\rightarrow c_{max}=c(T_p)$$

$$=1-e^{-\zeta\pi/\sqrt{1-\zeta^2}}\left(\cos\pi+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$=1+e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

For the unit step $c_{final}=1$

$$\rightarrow \%OS=e^{-\zeta\pi/\sqrt{1-\zeta^2}}\times 100 \quad (4.38)$$

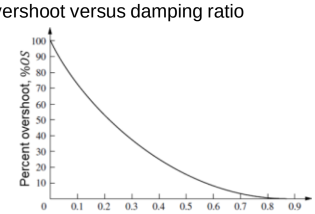
$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$ (4.28)

§6. Underdamped Second Order Systems

ζ can be found from given $\%OS$

$$(4.38)\rightarrow \zeta=-\frac{\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} \quad (4.39)$$

Percent overshoot versus damping ratio



$\%OS=e^{-\zeta\pi/\sqrt{1-\zeta^2}}\times 100$ (4.38)

§6. Underdamped Second Order Systems

3. Evaluation of T_s

- To find the settling time → to find the time it takes for the amplitude of the decaying sinusoid in (4.28) to reach 0.02

$$\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} = 0.02$$

$$\rightarrow T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

- An approximation for the settling time that will be used for all values of ζ

$$T_s = \frac{4}{\zeta\omega_n} \qquad (4.42)$$

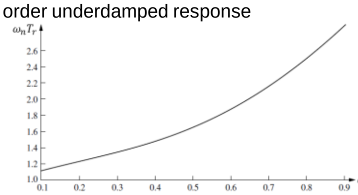
$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right) \qquad (4.28)$$

§6. Underdamped Second Order Systems

4. Evaluation of T_r

When a precise analytical relationship between rise time and damping ratio, ζ, cannot be found → using numerical method

- Ex. Normalized rise time versus damping ratio for a second-order underdamped response



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

$$\omega_n T_r \approx 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$$

with maximum error less than 0.5% for $0 < \zeta < 0.9$

§6. Underdamped Second Order Systems

- Ex.4.5 Finding T_p, $\%OS$, T_s, and T_r from a TF

Finding T_p, $\%OS$, T_s, and T_r from given transfer function

Solution $G(s) = \dfrac{100}{s^2 + 15s + 100}$

From observation

$$\omega_n = \sqrt{b} = \sqrt{100} = 10rad/s$$

$$\zeta = \frac{a}{2\sqrt{b}} = \frac{15}{2\sqrt{100}} = 0.75$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.75^2}} = 0.475s$$

$$OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = e^{-0.75\pi/\sqrt{1-0.75^2}} \times 100 = 2.84\%$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.75 \times 10} = 0.533s$$

§6. Underdamped Second Order Systems

- Ex.4.5 Finding T_p, $\%OS$, T_s, and T_r from a TF

Finding T_p, $\%OS$, T_s, and T_r from given transfer function

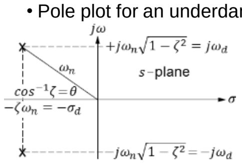
Solution $G(s) = \dfrac{100}{s^2 + 15s + 100}$

From observation

$$\omega_n = 10rad/s$$
$$\zeta = 0.75$$
$$T_p = 0.475s$$
$$OS = 2.84\%$$
$$T_s = 0.533s$$
$$\omega_n T_r \approx 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$$
$$= 1.76 \times 0.75^3 - 0.417 \times 0.75^2 + 1.039 \times 0.75 + 1$$
$$= 2.5370$$
$$\rightarrow T_r = 0.2537s$$

§6. Underdamped Second Order Systems

- The relation between peak time, percent overshoot, and settling time to the location of the poles
  - Pole plot for an underdamped second-order system



$$\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$$
$$\sigma_d \equiv \zeta\omega_n$$
$$\cos\theta = \zeta$$

ω_d : damped frequency of oscillation

σ_d : exponential damping frequency

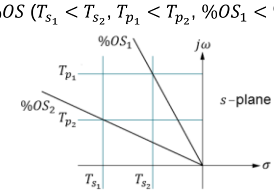
- The peak time and settling time in terms of the pole location

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \qquad (4.44)$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \qquad (4.45)$$

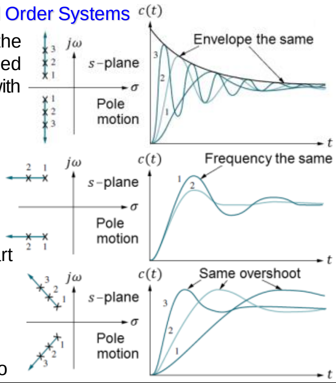
§6. Underdamped Second Order Systems

- Lines of constant peak time, T_p, settling time, T_s, and percent overshoot, $\%OS$ ($T_{s_1} < T_{s_2}$, $T_{p_1} < T_{p_2}$, $\%OS_1 < \%OS_2$)



§6. Underdamped Second Order Systems

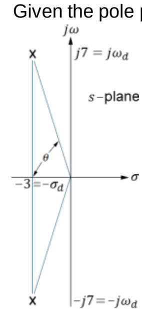
- Step responses of the 2nd-order underdamped systems as poles move with
  - constant real part
  - constant imaginary part
  - constant damping ratio



§6. Underdamped Second Order Systems

- <u>Ex.4.6</u> Finding T_p, $\%OS$, and T_s from Pole Location

Given the pole plot, find ζ; ω_n; T_p; $\%OS$, and T_s



<u>Solution</u>

The damping ratio

$$\zeta = \cos\theta = \cos\left[\tan^{-1}(7/3)\right] = 0.394$$

The natural frequency

$$\omega_n = \sqrt{7^2 + 3^2} = 7.616rad/s$$

The peak time

$$T_p = \pi/\omega_d = \pi/7 = 0.449s$$

The percent overshoot

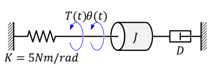
$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 26\%$$

The approximate settling time

$$T_s = 4/\sigma_d = 4/3 = 1.333s$$

§6. Underdamped Second Order Systems

- <u>Ex.4.7</u> Transient Response Through Component Design



Find J and D to yield 20% overshoot and a settling time of $2s$ for a step input of torque $T(t)$

<u>Solution</u>

The rotational mechanical system $G(s) = \dfrac{\frac{1}{J}}{s^2 + \frac{D}{J}s + \frac{K}{J}}$

Requirement $\%OS = 20 \rightarrow \zeta = \dfrac{-ln(\%OS/100)}{\sqrt{\pi^2 + ln^2(\%OS/100)}} = 0.456$

$$T_s = 4/\zeta\omega_n = 2 \rightarrow \omega_n = 2/0.456 = 4.386$$

Observation $\omega_n = \sqrt{K/J} = 4.386$

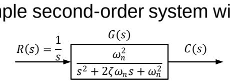
$$2\zeta\omega_n = D/J = 2 \times 0.456 \times 4.386 = 38.474$$

Therefore $D = 1.04Nms/rad$, $J = 0.26kgm^2$

§6. Underdamped Second Order Systems

5. Second-Order Transfer Functions via Testing

- Consider a simple second-order system with the step input



From testing data in laboratory

- The response curve for percent overshoot, $\%OS$, and settling time, T_s

  $\%OS, T_s \rightarrow \zeta, \omega_n$

- The expected steady-state values, c_{final}

  $c_{final} \rightarrow K$

- A problem at the end of the chapter illustrates the estimation of a second-order transfer function from the step response

§6. Underdamped Second Order Systems

Skill-Assessment Ex.4.5

<u>Problem</u> Find ζ, ω_n, T_s, T_p, T_r, $\%OS$ for $G(s) = \dfrac{361}{s^2 + 16s + 361}$

<u>Solution</u> $\omega_n = \sqrt{361} = 19rad/s$

$$2\zeta\omega_n = 16 \rightarrow \zeta = \frac{16}{2\omega_n} = \frac{16}{2 \times 19} = 0.421$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{8} = 0.5s$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{19\sqrt{1-0.421^2}} = 0.182s$$

$$\omega_n T_r = 1.4998 \rightarrow T_r = \frac{1.4998}{\omega_n} = \frac{1.4998}{19} = 0.079s$$

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = e^{\frac{-0.421\pi}{\sqrt{1-0.421^2}}} \times 100\% = 23.3\%$$

$\omega_n T_r \approx 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$

§6. Underdamped Second Order Systems

TryIt 4.1

Use the following MATLAB statements to calculate the answers to Skill-Assessment Exercise 4.5. Ellipses mean code continues on next line.

```
numg=361;
deng=[1 16 361];
omegan=sqrt(deng(3)...
/deng(1))
zeta=(deng(2)/deng(1))...
/(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt...
(1-zeta^2))
pos=100*exp(-zeta*...
pi/sqrt(1-zeta^2))
Tr=(1.768*zeta^3-...
0.417*zeta^2+1.039*...
zeta+1)/omegan
```

Matlab

```
numg=361;
deng=[1 16 361];
omegan=sqrt(deng(3)/deng(1))
zeta=(deng(2)/deng(1))/(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt(1-zeta^2))
pos=100*exp(-zeta*pi/sqrt(1-zeta^2))
Tr=(1.768*zeta^3 -0.417*zeta^2+1.039*zeta+1)/omegan
```

Result

$\zeta = 0.421$, $\omega_n = 19$, $T_s = 0.5s$, $T_p = 0.182s$, $T_r = 0.079s$, $\%OS = 23.3\%$

§7. Systems Response with Additional Poles

- The formulas describing percent overshoot, settling time, and peak time were derived only for a system with two complex poles and no zeros

  → for a system with more than two poles or with zeros?

- Before apply the derived formulas, a system with more than two poles or with zeros must be approximated as a 2nd-order system that has just two complex dominant poles

§7. Systems Response with Additional Poles

- Let us now look at the conditions that would have to exist in order to approximate the behavior of a three-pole system as that of a two-pole system

  Consider a three-pole system with complex poles, $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, and a third pole on the real axis, $-\alpha_r$

  The output transform

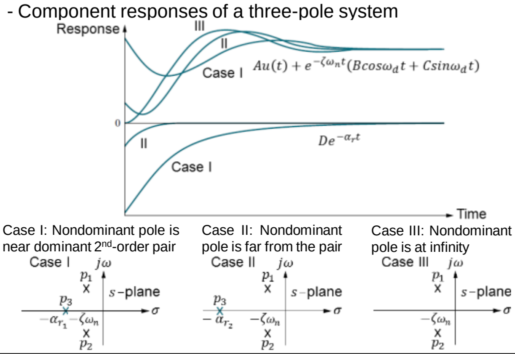
$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$$

  In time domain

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

§7. Systems Response with Additional Poles

- Component responses of a three-pole system



§7. Systems Response with Additional Poles

- Ex.4.8 Comparing Responses of Three-Pole Systems

  Find the step response of each of the TFs and compare them

$$T_1(s) = \frac{24.542}{s^2+4s+24.542},\ T_2(s) = \frac{10}{s+10}T_1(s),\ T_3(s) = \frac{3}{s+3}T_1(s)$$

  Solution

$$T_1(s) = \frac{24.542}{s^2+4s+24.542}$$

$$\rightarrow c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8^0)$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2+4s+24.542)}$$

$$\rightarrow c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34^0)$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2+4s+24.542)}$$

$$\rightarrow c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63^0)$$

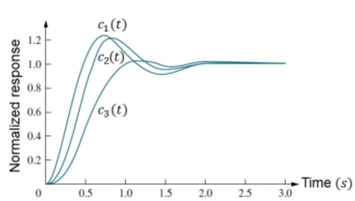
§7. Systems Response with Additional Poles

$$T_1(s) = \frac{24.542}{s^2+4s+24.542}$$

$$= \frac{24.542}{[s-(-2-j4.532)][s-(-2+j4.532)]}$$

$$\rightarrow c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8^0)$$

$c_1(t)$: the pure second-order system response



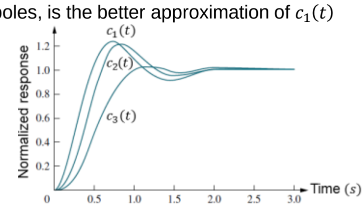
§7. Systems Response with Additional Poles

$$T_2(s) = \frac{245.42}{(s+10)(s^2+4s+24.542)}$$

$$= \frac{245.42}{(s+10)[s-(-2-j4.532)][s-(-2+j4.532)]}$$

$$\rightarrow c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34^0)$$

$c_2(t)$: with the third pole at -10 and farthest from the dominant poles, is the better approximation of $c_1(t)$



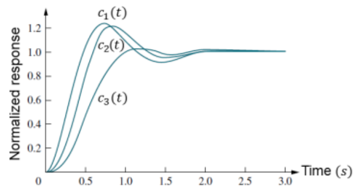
§7. Systems Response with Additional Poles

$$T_3(s) = \frac{73.626}{(s+3)(s^2+4s+24.542)}$$

$$= \frac{73.626}{(s+3)[s-(-2-j4.532)][s-(-2+j4.532)]}$$

$$\rightarrow c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63^0)$$

$c_3(t)$: with the third pole at -3, close to the dominant poles, yields the most error



§7. Systems Response with Additional Poles

MATLAB **ML**

Run ch4p2 in Appendix B

Learn how to use MATLAB to

- generate a step response for a transfer function
- plot the response directly or collect the points for future use
- solve Ex.4.8

§7. Systems Response with Additional Poles

Simulink **SL**

System responses can alternately be obtained using Simulink. Simulink is a software package that is integrated with MATLAB to provide a graphical user interface (GUI) for defining systems and generating responses

The reader is encouraged to study Appendix C, which contains a tutorial on Simulink as well as some examples. One of the illustrative examples, Ex.C.1, solves Ex.4.8 using Simulink

§7. Systems Response with Additional Poles

Gui Tool **GUIT**

Another method to obtain systems responses is through the use of MATLAB's LTI Viewer. An advantage of the LTI Viewer is that it displays the values of settling time, peak time, rise time, maximum response, and the final value on the step response plot

The reader is encouraged to study Appendix E, which contains a tutorial on the LTI Viewer as well as some examples. Ex.E.1 solves Ex.4.8 using the LTI Viewer

§7. Systems Response with Additional Poles

Skill-Assessment Ex.4.6

Problem Determine the validity of a second-order approximation for each of these two transfer functions

a. $G_1(s) = \frac{700}{(s+15)(s^2+4s+100)}$

b. $G_2(s) = \frac{360}{(s+4)(s^2+2s+90)}$

§7. Systems Response with Additional Poles

Solution

a. $G_1(s) = \frac{700}{[s-(-15)][s-(-2-j9.798)][s-(-2+j9.798)]}$

The dominant poles have a real part of -2 and the higher-order pole is at -15, i.e. more than five-times further

→ The 2nd-order approximation is valid

b. $G_2(s) = \frac{700}{[s-(-4)][s-(-1-j9.434)][s-(-1+j9.434)]}$

The dominant poles have a real part of -1 and the higher-order pole is at -4, i.e. not more than five-times further

→ The 2nd-order approximation is not valid

§7. Systems Response with Additional Poles

TryIt 4.2

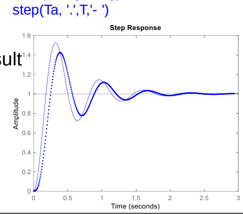
Use the following MATLAB and Control System Toolbox statements to investigate the effect of the additional pole in Skill-Assessment Exercise 4.6(a). Move the higher-order pole originally at −15 to other values by changing "a" in the code.

```
a=15
numga=100*a;
denga=conv([1 a],...
 [1 4 100]);
Ta=tf(numga,denga);
numg=100;
deng=[1 4 100];
T=tf(numg,deng);
step(Ta,'.',T,'-')
```

Matlab

```
a=15
numga=100*a;
denga=conv([1 a],[1 4 100]);
Ta=tf(numga,denga);
numg=100;
deng=[1 4 100];
T=tf (numg,deng);
step(Ta, '.',T,'- ')
```

Result

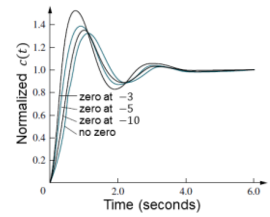


§8. Systems Response with Zeros

- Given the 2nd-order system

$$G(s)=\frac{1}{[s-(-1-j2.828)][s-(-1+j2.828)]}$$

- Consecutively add zeros at -3, -5, and -10. The results, normalized to the steady-state value



- the closer the zero is to the dominant poles, the greater its effect on the transient response
- as the zero moves away from the dominant poles, the response approaches that of the two-pole system

§8. Systems Response with Zeros

- This analysis can be reasoned via the partial-fraction expansion

$$T(s)=(s+a)\times\frac{1}{(s+b)(s+c)}$$
$$=\frac{s+a}{(s+b)(s+c)}$$
$$=\frac{A}{s+a}+\frac{B}{s+b}=\frac{\frac{-b+a}{-b+c}}{s+b}+\frac{\frac{-c+a}{-c+b}}{s+c}$$

If the zero is far from the poles, then a is large compared to b and c, and

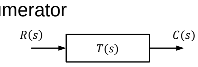
$$T(s)\approx=\frac{\frac{a}{-b+c}}{s+b}+\frac{\frac{a}{-c+b}}{s+c}=\frac{a}{(s+b)(s+c)}$$

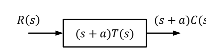
Hence, the zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response

§8. Systems Response with Zeros

- Another way to look at the effect of a zero, which is more general, is as follows (Franklin, 1991)

Let $C(s)$ be the response of a system, $T(s)$, with unity in the numerator





If adding a zero to the TF, yielding $(s+a)T(s)$, the Laplace transform of the response will be

$$(s+a)C(s)=sC(s)+aC(s) \qquad (4.70)$$

The response of a system with a zero consists of two parts

- the derivative of the original response, and
- a scaled version of the original response

§8. Systems Response with Zeros

- The response of a system with a zero

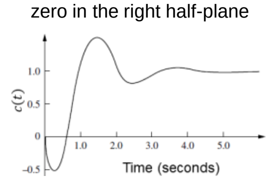
$$(s+a)C(s)=sC(s)+aC(s) \qquad (4.70)$$

- If a, the negative of the zero, is very large, the Laplace transform of the response is approximately $aC(s)$, or a scaled version of the original response
- If a is not very large, the response has an additional component consisting of the derivative of the original response, $sC(s)$
- As a becomes smaller, the derivative term contributes more to the response and has a greater effect

  For step responses, the derivative is typically positive at the start of a step response. Thus, for small values of a, we can expect more overshoot in second order systems because the derivative term will be additive around the first overshoot

§8. Systems Response with Zeros

- An interesting phenomenon occurs if a is negative, placing the zero in the right half-plane

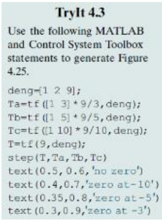


From Eq. (4.70): if the derivative term, $sC(s)$, is larger than the scaled response, $aC(s)$, the response will initially follow the derivative in the opposite direction from the scaled response

Notice that the response begins to turn toward the negative direction even though the final value is positive. A system that exhibits this phenomenon is known as a nonminimum-phase system

$(s+a)C(s)=sC(s)+aC(s)$ (4.70)

§8. Systems Response with Zeros



TryIt 4.3

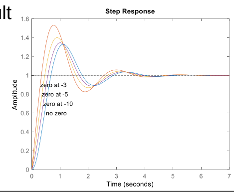
Use the following MATLAB and Control System Toolbox statements to generate Figure 4.25.

```
deng=[1 2 9];
Ta=tf([1 3]*9/3,deng);
Tb=tf([1 5]*9/5,deng);
Tc=tf([1 10]*9/10,deng);
T=tf(9,deng);
step(T,Ta,Tb,Tc)
text(0.5,0.6,'no zero')
text(0.4,0.7,'zero at-10')
text(0.35,0.8,'zero at-5')
text(0.3,0.9,'zero at -3')
```

Matlab

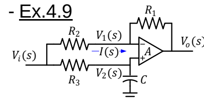
```
deng=[1 2 9];
Ta=tf([1 3]*9/3,deng); Tb=tf([1 5]*9/5,deng);
Tc=tf([1 10]*9/10,deng); T=tf(9,deng);
step(T,Ta,Tb,Tc)
text(0.5,0.6, 'no zero')
text(0.4,0.7, 'zero at -10')
text(0.35,0.8, 'zero at -5')
text(0.3,0.9, 'zero at -3')
```

Result



§8. Systems Response with Zeros

- Ex.4.9 TF of a Nonminimum-Phase System

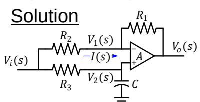


a. Find the TF, $V_o(s)/V_i(s)$, for the operational amplifier circuit

b. If $R_1=R_2$, this circuit is known as an all-pass filter, since it passes sine waves of a wide range of frequencies without attenuating or amplifying their magnitude (Dorf, 1993). We will learn more about frequency response in Chapter 10. For now, let $R_1=R_2$; $R_3C=0.1$, and find the step response of the filter. Show that component parts of the response can be identified with those in Eq. (4.70)

§8. Systems Response with Zeros

Solution



a. $V_1(s)=I(s)R_1+V_o(s)$

where, $I(s)=\dfrac{V_i(s)-V_o(s)}{R_1+R_2}$

$$\rightarrow V_1(s)=\frac{R_1V_i(s)+R_2V_o(s)}{R_1+R_2}$$

$$\frac{V_2(s)}{1/Cs}+\frac{V_2(s)-V_i(s)}{R_3}=0\rightarrow V_2(s)=V_i(s)\frac{1/Cs}{R_3+1/Cs}$$

The amplifier's gain A: $V_o(s)=A[V_2(s)-V_1(s)]$

$$\rightarrow\frac{V_o(s)}{V_i(s)}=\frac{A(R_2-R_1R_3Cs)}{(R_3Cs+1)[R_1+R_2(1+A)]}$$

$$\rightarrow\frac{V_o(s)}{V_i(s)}=\frac{R_2-R_1R_3Cs}{R_2R_3Cs+R_2}=-\frac{R_1\left(s-\frac{R_2}{R_1R_3C}\right)}{R_2\left(s+\frac{1}{R_3C}\right)}$$

§8. Systems Response with Zeros

b. Letting $R_1=R_2$ and $R_3C=1/10$

$$\frac{V_o(s)}{V_i(s)}=-\frac{R_1\left(s-\frac{R_2}{R_1R_3C}\right)}{R_2\left(s+\frac{1}{R_3C}\right)}=-\frac{s-\frac{1}{R_3C}}{s+\frac{1}{R_3C}}=-\frac{s-10}{s+10}$$

For a step input, evaluate the response as suggested by Eq.(4.70)

$$C(s)=-\frac{s-10}{s(s+10)}=-\frac{1}{s+10}+10\frac{1}{s(s+10)}$$
$$=sC_o(s)-10C_o(s)$$

where, $C_o(s)=-1/[s(s+10)]$ is the Lapalce transform of the response without a zero

$$C(s)=-\frac{1}{s+10}+\frac{1}{s}-\frac{1}{s+10}=\frac{1}{s}-\frac{2}{s+10}$$

or the response with a zero $c(t)=1-2e^{-10t}$

$(s+a)C(s)=sC(s)+aC(s)$ (4.70)

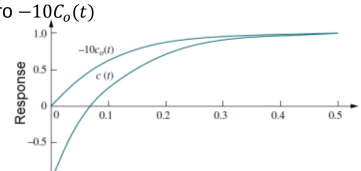
§8. Systems Response with Zeros

$$C_o(s)=-\frac{1}{s(s+10)}=-\frac{1}{10}\frac{1}{s}+\frac{1}{10}\frac{1}{s+10}$$

the response without a zero

$$c(t)=-\frac{1}{10}+\frac{1}{10}e^{-10t}$$

Step response of the nonminimum-phase network $c(t)$ and normalized step response of an equivalent network without the zero $-10C_o(t)$



§8. Systems Response with Zeros

- The pole-zero cancellation and its effect on our ability to make 2nd-order approximations to a system
  - Assume a three pole system with a zero as shown in Eq. (4.85)
  - If the pole term, $(s+p_3)$, and the zero term, $(s+z)$, cancel out, we are left with

$$T(s)=\frac{K(s+z)}{(s+p_3)(s^2+as+b)}=\frac{K\cancel{(s+z)}}{\cancel{(s+p_3)}(s^2+as+b)} \quad (4.85)$$

as a second-order TF

  - From another perspective, if the zero at $-z$ is very close to the pole at $-p_3$, then a partial-fraction expansion of Eq. (4.85) will show that the residue of the exponential decay is much smaller than the amplitude of the 2nd-order response

§8. Systems Response with Zeros

- Ex.4.10 Evaluating Pole-Zero Cancellation Using Residues

For each of the following response functions, determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

Solution

$$C_1(s) = \frac{1}{s} + (-3.5) \times \frac{1}{s+5} + 3.5 \times \frac{1}{s+6} + (-1) \times \frac{1}{s+3.5}$$

The residue of the pole at -3.5, which is closest to the zero at -4, is equal to 1 and is not negligible compared to the other residues. Thus, a second-order step response approximation cannot be made for $C_1(s)$

§8. Systems Response with Zeros

$$C_2(s) = 0.87\frac{1}{s} + (-5.3) \times \frac{1}{s+5} + 4.4 \times \frac{1}{s+6} + 0.033 \times \frac{1}{s+4.01}$$

The residue of the pole at -4.01, which is closest to the zero at -4, is equal to 0.033, about two orders of magnitude below any of the other residues

Hence, we make a 2nd-order approximation by neglecting the response generated by the pole at -4.01

$$C_2(s) \approx 0.87\frac{1}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6}$$

and the response $c_2(t)$ is approximately

$$c_2(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$$

§8. Systems Response with Zeros

TryIt 4.4

Use the following MATLAB and Symbolic Math Toolbox statements to evaluate the effect of higher-order poles by finding the component parts of the time response of $c_1(t)$ and $c_2(t)$ in Example 4.10.

```
syms s
C1=26.25*(s+4)/...
  (s*(s+3.5)*(s+5)*(s+6));
C2=26.25*(s+4)/...
  (s*(s+4.01)*(s+5)*(s+6));
c1=ilaplace(C1);
c1=vpa(c1,3);
'c1'
pretty(c1)
c2=ilaplace(C2);
c2=vpa(c2,3);
'c2'
pretty(c2);
```

Matlab

```
syms s
C1=26.25*(s+4)/(s*(s+3.5)*(s+5)*(s+6));
C2=26.25*(s+4)/(s*(s+4.01)*(s+5)*(s+6));
c1=ilaplace(C1);
c1=vpa(c1,3); 'c1', pretty (c1)
c2=ilaplace(C2);
c2=vpa(c2,3); 'c2', pretty(c2)
```

Result

```
c1
exp(-6.0 t) 3.5 - exp(-3.5 t) 1.0 - exp(-5.0 t) 3.5 + 1.0
c2
exp(-6.0 t) 4.4 - exp(-5.0 t) 5.3 + exp(-4.01 t) 0.0332 + 0.873
```


§8. Systems Response with Zeros

Skill-Assessment Ex.4.7

Problem Determine the validity of a 2nd-order approximation for each of these two TFs

a. $$G(s) = \frac{185.71(s+7)}{(s+6.5)(s+10)(s+20)}$$

b. $$G(s) = \frac{197.14(s+7)}{(s+6.9)(s+10)(s+20)}$$

§8. Systems Response with Zeros

Solution a. Expanding $G(s)$ by partial fractions

$$G(s) = \frac{1}{s} + \frac{0.8942}{s+20} + \frac{-1.5918}{s+10} + \frac{-0.3023}{s+6.5}$$

-0.3023 is not an order of magnitude less than residues of 2nd-order terms

→ a 2nd-order approximation is not valid

b. Expanding $G(s)$ by partial fractions

$$G(s) = \frac{1}{s} + \frac{0.9782}{s+20} + \frac{-1.9078}{s+10} + \frac{-0.0704}{s+6.9}$$

-0.0704 is an order of magnitude less than residues of 2nd-order terms

→ a 2nd-order approximation is valid

§9. Effects of Nonlinearities upon Time Response

- Qualitatively examine the effects of nonlinearities upon the time response of physical systems

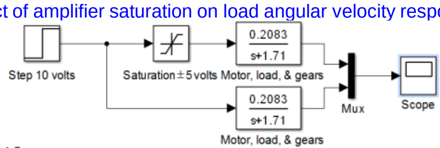
  In the following examples, we insert nonlinearities, such as saturation, dead zone, and backlash into a system to show the effects of these nonlinearities upon the linear responses. The responses were obtained using Simulink

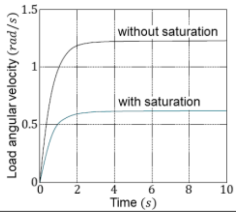
- Assume the motor and load from the Antenna Control and look at the load angular velocity

$$\omega_o(s) = 0.1s\theta_m(s) = E_a(s)\frac{0.2083}{s+1.71}$$

§9. Effects of Nonlinearities upon Time Response

Effect of amplifier saturation on load angular velocity response

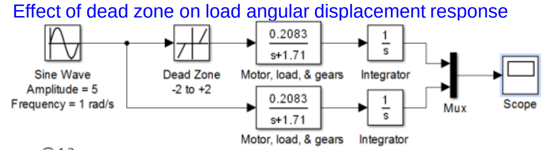


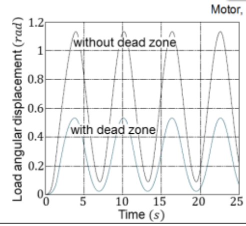


The effect of amplifier saturation is to limit the obtained velocity

§9. Effects of Nonlinearities upon Time Response

Effect of dead zone on load angular displacement response

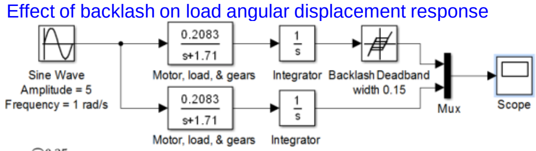


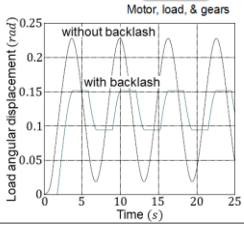


The effect of dead zone is lower the output amplitude

§9. Effects of Nonlinearities upon Time Response

Effect of backlash on load angular displacement response



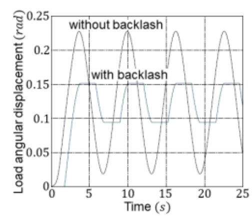


As the motor reverses direction, the output shaft remains stationary while the motor begins to reverse. When the gears finally connect, the output shaft itself begins to turn in the reverse direction

§9. Effects of Nonlinearities upon Time Response

Skill-Assessment Ex.4.8

Problem Use MATLAB's Simulink to reproduce the following figure



§10. Laplace Transform Solution of State Equations

- Consider the state and output equations

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \quad (4.92)$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \quad (4.93)$$

- Taking the Laplace transform of the state equation

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \quad (4.94)$$

$$\rightarrow (s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{x}(0) + \boldsymbol{B}\boldsymbol{U}(s)$$

$$\rightarrow \boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$

$$= \frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}[\boldsymbol{x}(0) + \boldsymbol{B}\boldsymbol{U}(s)] \quad (4.96)$$

- Taking the Laplace transform of the output equation

$$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s) \quad (4.97)$$

§10. Laplace Transform Solution of State Equations

Eigenvalues and Transfer Function Poles

- The poles of the TF determine the nature of the transient response of the system ⟹ Is there an equivalent quantity in the state-space representation that yields the same information?
- Consider the SISO system with the output, $Y(s)$, the input, $U(s)$ and assumed that $x(0) = 0$

$$\frac{Y(s)}{U(s)} = \boldsymbol{C}\frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{C}\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D}\det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} \quad (4.98)$$

The roots of the denominator of Eq. (4.98) are the poles of the system. Since the denominators of Eqs. (4.96) and (4.98) are identical, the system poles equal the eigenvalues. Hence, if a system is represented in state-space, the poles can be found from

$$det(s\boldsymbol{I} - \boldsymbol{A}) = 0$$

§10. Laplace Transform Solution of State Equations

- Ex.4.11 Laplace Transform Solution; Eigenvalues and Poles

Given the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\boldsymbol{x}$$

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

do the following

a. Solve the preceding state equation and obtain the output for the given exponential input

b. Find the eigenvalues and the system poles

§10. Laplace Transform Solution of State Equations

Solution

a. Find $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$

$$s\boldsymbol{I}-\boldsymbol{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix}$$

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1} = \frac{\operatorname{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}$$

$$= \frac{\begin{bmatrix} s^2+9s+26 & s+9 & 1 \\ -24 & s^2+9s & s \\ -24s & -(26s+24) & s^2 \end{bmatrix}}{s^3+9s^2+26s+24}$$

Find $\boldsymbol{U}(s)$

$$\boldsymbol{U}(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

§10. Laplace Transform Solution of State Equations

Find $\boldsymbol{X}(s)$

$$\boldsymbol{X}(s) = (s\boldsymbol{I}-\boldsymbol{A})^{-1}[\boldsymbol{x}(0)+\boldsymbol{B}\boldsymbol{U}(s)]$$

$$= \frac{\begin{bmatrix} s^2+9s+26 & s+9 & 1 \\ -24 & s^2+9s & s \\ -24s & -(26s+24) & s^2 \end{bmatrix}}{s^3+9s^2+26s+24}\left[\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{1}{s+1}\right]$$

$$= \frac{\begin{bmatrix} s^2+9s+26 & s+9 & 1 \\ -24 & s^2+9s & s \\ -24s & -(26s+24) & s^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \frac{2s+3}{s+1} \end{bmatrix}}{(s+2)(s+3)(s+4)}$$

$$= \frac{1}{(s+1)(s+2)(s+3)(s+4)}\begin{bmatrix} s^3+10s^2+37s+29 \\ 2s^2-21s-24 \\ s(2s^2-21s-24) \end{bmatrix}$$

§10. Laplace Transform Solution of State Equations

The output equation

$$Y(s) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix}$$

$$= X_1(s) + X_2(s)$$

$$= \frac{s^3+12s^2+16s+5}{(s+1)(s+2)(s+3)(s+4)}$$

$$= \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4}$$

$$\rightarrow y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}$$

b.

The denominator of the system's transfer function $\det(s\boldsymbol{I}-\boldsymbol{A}) = 0$ furnishes both the poles of the system and the eigenvalues -2; -3, and -4

§10. Laplace Transform Solution of State Equations

Symbolic Math SM

Run ch4sp1 in Appendix F

Learn how to use the Symbolic Math Toolbox to

- solve state equations for the output response using the Laplace transform
- solve Ex.4.11

§10. Laplace Transform Solution of State Equations

Skill-Assessment Ex.4.9

Problem Given the system represented in state space

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-t}, y = \begin{bmatrix} 1 & 3 \end{bmatrix}\boldsymbol{x}$$

$$\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

do the following

a. Solve for $y(t)$ using state-space equation and Laplace transform techniques

b. Find the eigenvalues and the system poles

Solution a. $s\boldsymbol{I}-\boldsymbol{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}$

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1} = \frac{\operatorname{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})} = \frac{\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}}{s^2+5s+6}$$

§10. Laplace Transform Solution of State Equations

$$\boldsymbol{U}(s)=\mathcal{L}\{e^{-t}\}=\frac{1}{s+1}$$

The state vector

$$\boldsymbol{X}(s)=(s\boldsymbol{I}-\boldsymbol{A})^{-1}[\boldsymbol{x}(0)+\boldsymbol{B}\boldsymbol{U}(s)]$$

$$=\frac{\begin{bmatrix}s+5 & 2\\ -3 & s\end{bmatrix}}{s^2+5s+6}\left[\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\frac{1}{s+1}\right]$$

$$=\frac{\begin{bmatrix}s+5 & 2\\ -3 & s\end{bmatrix}}{(s+2)(s+3)}\begin{bmatrix}0\\ \dfrac{s+2}{s+1}\end{bmatrix}$$

$$=\frac{1}{(s+1)(s+2)(s+3)}\begin{bmatrix}2(s^2+7s+7)\\ s^2-4s-6\end{bmatrix}$$

§10. Laplace Transform Solution of State Equations

The output

$$Y(s)=\begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}X_1(s)\\X_2(s)\end{bmatrix}$$

$$=\frac{5s^2+2s-4}{(s+1)(s+2)(s+3)}$$

$$=-\frac{0.5}{s+1}-\frac{12}{s+2}+\frac{17.5}{s+3}$$

$$\rightarrow y(t)=-0.5e^{-t}-12e^{-2t}+17.5e^{-3t}$$

b.

The eigenvalues are given by the roots of $\det(s\boldsymbol{I}-\boldsymbol{A})=s^2+5s+6=0$, or -2 and -3

§10. Laplace Transform Solution of State Equations

TryIt 4.5

Use the following MATLAB and Symbolic Math Toolbox statements to solve Skill-Assessment Exercise 4.9.

```
syms s
A=[0 2;-3 -5]; B=[0;1];
C=[1 3];X0=[2;1];
U=1/(s+1);
I=[1 0;0 1];
X=((s*I-A)^-1)*(X0+B*U);
Y=C*X;Y=simplify(Y);
y=ilaplace(Y);
pretty(y)
eig(A)
```

Matlab

```
syms s
A= [0 2;-3 -5];B=[0;1];C= [1 3];X0=[2;1];
U=1/(s+1);I= [1 0;0 1];
X=((s*I-A)^-1)*(X0+B*U);
Y=C*X;Y=simplify(Y);
y=ilaplace(Y);pretty(y)
eig(A)
```

Result

```
exp(-3 t) 35                    exp(-t)
--------------- - exp(-2 t) 12 - --------
      2                            2
ans =
  -2.0000
  -3.0000
```


§11. Time Domain Solution of State Equations

- The solution in the time domain is given directly by

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\boldsymbol{x}(0)+\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

$$=\underbrace{\boldsymbol{\Phi}(t)\boldsymbol{x}(0)}_{\text{the zero input response}}+\underbrace{\int_0^t\boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)d\tau}_{\text{convolution integral}}$$

$\boldsymbol{\Phi}(t)$: the state-transition matrix, $\boldsymbol{\Phi}(t)=e^{\boldsymbol{A}t}$

- For the unforced system

$$\mathcal{L}\{\boldsymbol{x}(t)\}=\mathcal{L}\{\boldsymbol{\Phi}(t)\boldsymbol{x}(0)\}=(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{x}(0)$$

$$\rightarrow\mathcal{L}\{\boldsymbol{\Phi}(t)\}=(s\boldsymbol{I}-\boldsymbol{A})^{-1}$$

$$\rightarrow\boldsymbol{\Phi}(t)=\mathcal{L}^{-1}\{(s\boldsymbol{I}-\boldsymbol{A})^{-1}\}$$

$$=\mathcal{L}^{-1}\left\{\frac{\text{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}\right\}$$

§11. Time Domain Solution of State Equations

- Ex.4.12 Time Domain Solution

Given the state equation and initial state vector

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\1\end{bmatrix}u(t)$$

$$\boldsymbol{x}(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

where $u(t)$ is a unit step, find the state-transition matrix and then solve for $x(t)$

Solution

The eigenvalues

$$\det(s\boldsymbol{I}-\boldsymbol{A})=0$$

$$\det\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix}\right)=\det\left(\begin{bmatrix}s & -1\\ 8 & s+6\end{bmatrix}\right)=s^2+6s+8=0$$

or $s_1=-2$ and $s_2=-4$

§11. Time Domain Solution of State Equations

Since each term of the state-transition matrix is the sum of responses generated by the poles (eigenvalues), we assume a state-transition matrix of the form

$$\boldsymbol{\Phi}(t)=\begin{bmatrix}K_1e^{-2t}+K_2e^{-4t} & K_3e^{-2t}+K_4e^{-4t}\\ K_5e^{-2t}+K_6e^{-4t} & K_7e^{-2t}+K_8e^{-4t}\end{bmatrix}$$

In order to find the values of the constants, we make use of the properties of the state-transition matrix

$$\boldsymbol{\Phi}(0)=\boldsymbol{I}\implies K_1+K_2=1\qquad K_3+K_4=0$$

$$K_5+K_6=0\qquad K_7+K_8=1$$

$$\dot{\boldsymbol{\Phi}}(0)=\boldsymbol{A}\implies -2K_1-4K_2=0\qquad -2K_3-4K_4=1$$

$$-2K_5-4K_6=-8\qquad -2K_7-4K_8=-6$$

$$\rightarrow K_1=2,\ K_2=-1,\ K_3=1/2,\ K_4=-1/2$$

$$K_5=-4,\ K_6=4,\ K_7=-1,\ K_8=2$$

§11. Time Domain Solution of State Equations

Therefore

$$\mathbf{\Phi}(t)=\begin{bmatrix}2e^{-2t}-4e^{-4t} & 0.5e^{-2t}-0.5e^{-4t}\\ -4e^{-2t}+4e^{-4t} & -e^{-2t}+2e^{-4t}\end{bmatrix}$$

$$\rightarrow \mathbf{\Phi}(t-\tau)\boldsymbol{B}=\begin{bmatrix}2e^{-2(t-\tau)}-4e^{-4(t-\tau)} & 0.5e^{-2(t-\tau)}-0.5e^{-4(t-\tau)}\\ -4e^{-2(t-\tau)}+4e^{-4(t-\tau)} & -e^{-2(t-\tau)}+2e^{-4(t-\tau)}\end{bmatrix}\times\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\begin{bmatrix}0.5e^{-2(t-\tau)}-0.5e^{-4(t-\tau)}\\ -e^{-2(t-\tau)}+2e^{-4(t-\tau)}\end{bmatrix}$$

$$\rightarrow \mathbf{\Phi}(t)\boldsymbol{x}(0)=\begin{bmatrix}2e^{-2t}-4e^{-4t} & 0.5e^{-2t}-0.5e^{-4t}\\ -4e^{-2t}+4e^{-4t} & -e^{-2t}+2e^{-4t}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2e^{-2t}-e^{-4t}\\ -4e^{-2t}+4e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

$$\rightarrow \int_0^t \mathbf{\Phi}(t-\tau)\boldsymbol{Bu}(\tau)d\tau=\int_0^t\begin{bmatrix}0.5e^{-2(t-\tau)}-0.5e^{-4(t-\tau)}\\ -e^{-2(t-\tau)}+2e^{-4(t-\tau)}\end{bmatrix}d\tau$$

$$=\begin{bmatrix}0.5e^{-2t}\int_0^t e^{2\tau}d\tau-0.5e^{-4t}\int_0^t e^{4\tau}d\tau\\ -e^{-2t}\int_0^t e^{2\tau}d\tau+2e^{-4t}\int_0^t e^{4\tau}d\tau\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{8}-\frac{1}{4}e^{-2t}+\frac{1}{8}e^{-4t}\\ \frac{1}{2}e^{-2t}-\frac{1}{2}e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

The final result

$$\boldsymbol{x}(t)=\mathbf{\Phi}(t)\boldsymbol{x}(0)+\int_0^t\mathbf{\Phi}(t-\tau)\boldsymbol{Bu}(\tau)d\tau$$

$$=\begin{bmatrix}2e^{-2t}-e^{-4t}\\ -4e^{-2t}+4e^{-4t}\end{bmatrix}+\begin{bmatrix}\frac{1}{8}-\frac{1}{4}e^{-2t}+\frac{1}{8}e^{-4t}\\ \frac{1}{2}e^{-2t}-\frac{1}{2}e^{-4t}\end{bmatrix}$$

$$\rightarrow \boldsymbol{x}(t)=\begin{bmatrix}\frac{1}{8}+\frac{7}{4}e^{-2t}-\frac{7}{8}e^{-4t}\\ -\frac{7}{2}e^{-2t}+\frac{7}{2}e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

- Ex.4.13 State-Transition Matrix via Laplace Transform

Find the state-transition matrix of Ex.4.12, using $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$

Solution

$$(s\boldsymbol{I}-\boldsymbol{A})=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix}=\begin{bmatrix}s & -1\\ 8 & s+6\end{bmatrix}$$

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{\mathrm{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{\begin{bmatrix}s+6 & 1\\ -8 & s\end{bmatrix}}{s^2+6s+8}=\begin{bmatrix}\frac{s+6}{s^2+6s+8} & \frac{1}{s^2+6s+8}\\ \frac{-8}{s^2+6s+8} & \frac{s}{s^2+6s+8}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

Expanding each term in the matrix on the right by partial fractions yields

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\begin{bmatrix}\frac{2}{s+2}-\frac{1}{s+4} & \frac{1/2}{s+2}-\frac{1/2}{s+4}\\ \frac{-4}{s+2}+\frac{4}{s+4} & \frac{-1}{s+2}+\frac{2}{s+4}\end{bmatrix}$$

Finally, taking the inverse Laplace transform of each term

$$\mathbf{\Phi}(t)=\begin{bmatrix}2e^{-2t}-4e^{-4t} & 0.5e^{-2t}-0.5e^{-4t}\\ -4e^{-2t}+4e^{-4t} & -e^{-2t}+2e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

Symbolic Math SM

Run ch4sp2 in Appendix F

Learn how to use the Symbolic Math Toolbox to

- solve state equations for the output response using the convolution integral
- solve Ex.4.12 and Ex.4.13

§11. Time Domain Solution of State Equations

Run ch4p3 in Appendix B

Learn how to use MATLAB to

- simulate the step response of systems represented in state space
- specify the range on the time axis for the plot

§11. Time Domain Solution of State Equations

Skill-Assessment Ex.4.10

Problem Given the system represented in state space

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 2\\ -2 & -5\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\ 1\end{bmatrix}e^{-2t}$$

$$y=\begin{bmatrix}2 & 1\end{bmatrix}\boldsymbol{x}$$

$$\boldsymbol{x}(0)=\begin{bmatrix}1\\ 2\end{bmatrix}$$

do the following

a. Solve for the state-transition matrix

b. Solve for the state vector using the convolution integral

c. Find the output, $y(t)$

§11. Time Domain Solution of State Equations

Solution a. $s\boldsymbol{I}-\boldsymbol{A}=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 2\\ -2 & -5\end{bmatrix}=\begin{bmatrix}s & -2\\ 2 & s+5\end{bmatrix}$

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{\operatorname{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{\begin{bmatrix}s+5 & 2\\ -2 & s\end{bmatrix}}{s^2+5s+4}=\frac{\begin{bmatrix}s+5 & 2\\ -2 & s\end{bmatrix}}{(s+1)(s+4)}$$

$$=\begin{bmatrix}\dfrac{4/3}{s+1}-\dfrac{1/3}{s+4} & \dfrac{2/3}{s+1}-\dfrac{2/3}{s+4}\\ \dfrac{-2/3}{s+1}+\dfrac{2/3}{s+4} & \dfrac{-1/3}{s+1}+\dfrac{4/3}{s+4}\end{bmatrix}$$

Taking the Laplace transform of each term, the state transition matrix is given by

$$\boldsymbol{\Phi}(t)=\begin{bmatrix}\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t}-\frac{2}{3}e^{-4t}\\ -\frac{2}{3}e^{-t}+\frac{2}{3}e^{-4t} & -\frac{1}{3}e^{-t}+\frac{4}{3}e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

b.

$$\boldsymbol{\Phi}(t-\tau)=\begin{bmatrix}\frac{4}{3}e^{-(t-\tau)}-\frac{1}{3}e^{-4(t-\tau)} & \frac{2}{3}e^{-(t-\tau)}-\frac{2}{3}e^{-4(t-\tau)}\\ -\frac{2}{3}e^{-(t-\tau)}+\frac{2}{3}e^{-4(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)}+\frac{4}{3}e^{-4(t-\tau)}\end{bmatrix}$$

$$\boldsymbol{Bu}(\tau)=\begin{bmatrix}0\\ e^{-2\tau}\end{bmatrix}$$

$$\boldsymbol{\Phi}(t-\tau)\boldsymbol{Bu}(\tau)=\begin{bmatrix}\frac{2}{3}e^{-\tau}e^{-t}-\frac{2}{3}e^{2\tau}e^{-4t}\\ -\frac{1}{3}e^{-\tau}e^{-t}+\frac{4}{3}e^{2\tau}e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

$$\int_0^t\boldsymbol{\Phi}(t-\tau)\boldsymbol{Bu}(\tau)d\tau$$

$$=\int_0^t\begin{bmatrix}\frac{2}{3}e^{-\tau}e^{-t}-\frac{2}{3}e^{2\tau}e^{-4t}\\ -\frac{1}{3}e^{-\tau}e^{-t}+\frac{4}{3}e^{2\tau}e^{-4t}\end{bmatrix}d\tau$$

$$=\begin{bmatrix}\frac{2}{3}e^{-t}\int_0^t e^{-\tau}d\tau-\frac{2}{3}e^{-4t}\int_0^t e^{2\tau}d\tau\\ -\frac{1}{3}e^{-t}\int_0^t e^{-\tau}d\tau+\frac{4}{3}e^{-4t}\int_0^t e^{2\tau}d\tau\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{3}e^{-t}-e^{-2t}+\frac{1}{3}e^{-4t}\\ -\frac{1}{3}e^{-t}+e^{-2t}-\frac{2}{3}e^{-4t}\end{bmatrix}$$

§11. Time Domain Solution of State Equations

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)+\int_0^t\boldsymbol{\Phi}(t-\tau)\boldsymbol{Bu}(\tau)d\tau$$

$$=\begin{bmatrix}\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t}-\frac{2}{3}e^{-4t}\\ -\frac{2}{3}e^{-t}+\frac{2}{3}e^{-4t} & -\frac{1}{3}e^{-t}+\frac{4}{3}e^{-4t}\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}$$

$$+\begin{bmatrix}\frac{2}{3}e^{-t}-e^{-2t}+\frac{1}{3}e^{-4t}\\ -\frac{1}{3}e^{-t}+e^{-2t}-\frac{2}{3}e^{-4t}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{10}{3}e^{-t}-e^{-2t}-\frac{4}{3}e^{-4t}\\ -\frac{5}{3}e^{-t}+e^{-2t}+\frac{8}{3}e^{-4t}\end{bmatrix}$$

c. $y(t)=\begin{bmatrix}2 & 1\end{bmatrix}\boldsymbol{x}=5e^{-t}-e^{-2t}$

§12. Performance Indices

- Performance index: A quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specifications
- A system is considered an optimum control system when the system parameters are adjusted so that the index reaches an extremum value, commonly a minimum value

§12. Performance Indices

1.The ISE Index

$$ISE = \int_0^T e^2(t)dt$$

ISE : Integral of the Square of the Error

T : the upper limit T is a finite time chosen somewhat arbitrarily so that the integral approaches a steady-state value. It is usually convenient to choose T as the settling time, T_s

$e(t)$: error

- The ISE performance index is easily adapted for practical measurements because a squaring circuit is readily obtained
- Furthermore, the squared error is mathematically convenient for analytical and computational purposes

§12. Performance Indices

2.The IAE Index

Integral of the Absolute magnitude of the Error

$$IAE = \int_0^T |e(t)|dt$$

3.The ITAE Index

Integral of Time multiplied by the Absolute magnitude of the Error

$$ITAE = \int_0^T t|e(t)|dt$$

4.The ITSE Index

Integral of Time multiplied by the Square of the Error

$$ITSE = \int_0^T te^2(t)dt$$

§12. Performance Indices

5.The ISTAE Index

Integral of the Squared Time multiplied by the Absolute magnitude of the Error

$$ISTAE = \int_0^T t^2|e(t)|dt$$

6.The ISTSE Index

Integral of the Squared Time multiplied by the Square of the Error

$$ISTSE = \int_0^T t^2e^2(t)dt$$

- The performance indices ITSE, ISTAE and ISTSE have not been applied to any great extent in practice because of the increased difficulty in handling them

§13. Case Studies