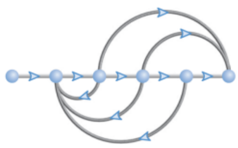
# Root Locus Techniques 8







## Learning Outcome

After completing this chapter, the student will be able to

- Define a root locus
- State the properties of a root locus
- Sketch a root locus
- Find the coordinates of points on the root locus and their associated gains
- Use the root locus to design a parameter value to meet a transient response specification for systems of order 2 and higher
- Sketch the root locus for positive-feedback systems
- Find the root sensitivity for points along the root locus

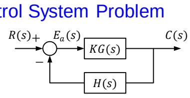




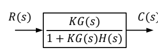
## §1. Introduction

Root Locus Technique was discovered by Evans in 1948

### The Control System Problem




Closed-loop system




Equivalent transfer function

Closed-loop TF: $T(s)=\dfrac{KG(s)}{1+KG(s)H(s)}$ Open-loop TF: $KG(s)H(s)$

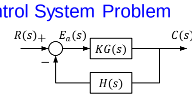
- From open-loop TF $KG(s)H(s)$
  - can determine the poles of $KG(s)H(s)$
  - variations in $K$ do not affect the location of any pole of this TF
- From the closed-loop TF $T(s)$
  - cannot determine the poles of $T(s)$ unless factor the denominator
  - the poles of $T(s)$ change with $K$



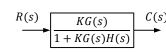


## §1. Introduction

### The Control System Problem




Closed-loop system




Equivalent transfer function

Closed-loop TF: $T(s)=\dfrac{KG(s)}{1+KG(s)H(s)}$ Open-loop TF: $KG(s)H(s)$

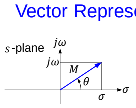
- The poles of $T(s)$ are a function of $K$ → find the poles for specific values of $K$ to estimate the system's transient response and stability
- The root-locus method displays the location of the poles of the closed-loop TF as a function of the gain factor $K$ of the open-loop TF



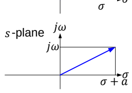


## §1. Introduction

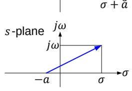
### Vector Representation of Complex Numbers




Any complex number, $s=\sigma+j\omega$, described in Cartesian coordinates can be graphically represented by a vector or in polar form $M\angle\theta$




If $F(s)=(s+a)$ then substituting the complex number $s=\sigma+j\omega$ yields $F(s)=(\sigma+a)+j\omega$, another complex number




$F(s)$ has a zero at $-a$. If translate the vector $a$ units to the left → alternate representation of the complex number that originates at the zero $-a$ and terminates on the point $s=\sigma+j\omega$

→ $s+a$ is a complex number and can be represented by a vector drawn from the zero of the function to the point $s$

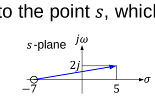




## §1. Introduction

→ $s+a$ is a complex number and can be represented by a vector drawn from the zero of the function to the point $s$

Ex. $(s+7)|_{s\to5+j2}$ is a complex number drawn from the zero of the function, $-7$, to the point $s$, which is $5+j2$



## §1. Introduction

Apply the concepts to a complicated function

$$F(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=\frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}} \quad (8.4)$$

The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point, $s$

Since each complex factor can be thought of as a vector

- the magnitude, $M$, of $F(s)$

$$M=\frac{\prod \text{zero lengths}}{\prod \text{pole lengths}}=\frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|} \quad (8.5)$$

- the angular, $\theta$, of $F(s)$

$$\theta=\sum \text{zero angles}-\sum \text{pole angles}$$

$$=\sum_{i=1}^{m}\angle(s+z_i)-\sum_{j=1}^{n}\angle(s+p_j) \quad (8.6)$$



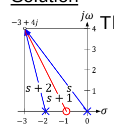


## §1. Introduction

- Ex.8.1 Evaluation of a complex function via vectors

Given $F(s)=\dfrac{s+1}{s(s+2)}$. Find $F(s)$ at the point $s=-3+j4$

Solution




The vector originating at

- the zero at $-1$ $\sqrt{20}\angle 116.6^0$
- the pole at $0$ $5\angle 126.9^0$
- the pole at $-2$ $\sqrt{17}\angle 104.0^0$

$$\rightarrow M\angle\theta=\frac{\sqrt{20}}{5\times\sqrt{17}}\angle(116.6^0-126.9^0-104.0^0)=0.217\angle-114.3^0$$

$M=\dfrac{\prod \text{zero lengths}}{\prod \text{pole lengths}}=\dfrac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|}$ (8.5)

$\theta=\sum \text{zero angles}-\sum \text{pole angles}=\sum_{i=1}^{m}\angle(s+z_i)-\sum_{j=1}^{n}\angle(s+p_j)$ (8.6)





## §1. Introduction

### Skill-Assessment Ex.8.1

Problem Given $F(s)=\dfrac{(s+2)(s+4)}{s(s+3)(s+6)}$

Find $F(s)$ at the point $s=-7+j9$ the following ways

a. Directly substituting the point into $F(s)$

b. Calculating the result using vectors

Solution a. Directly substituting the point into $F(s)$

$$F(-7+j9)=\frac{(-7+j9+2)(-7+j9+4)}{(-7+j9)(-7+j9+3)(-7+j9+6)}$$

$$=\frac{(-5+j9)(-3+j9)}{(-7+j9)(-7+j9)(-1+j9)}$$

$$=\frac{-66-j72}{944-j378}=-0.0339-j0.0899$$

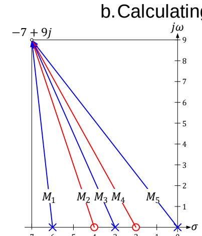
$$=0.096\angle-110.7^0$$





## §1. Introduction

b. Calculating the result using vectors




$$F(-7+j9)=\frac{\vec{M}_2\vec{M}_4}{\vec{M}_1\vec{M}_3\vec{M}_5}$$

$$=\frac{(-3+j9)(-5+j9)}{(-1+j9)(-4+j9)(-7+j9)}$$

$$=\frac{-66-j72}{944-j378}$$

$$=-0.0339-j0.0899$$

$$=0.096\angle-110.7^0$$





## §1. Introduction

### TryIt 8.1

Use the following MATLAB statements to solve the problem given in Skill-Assessment Ex.8.1.

```
s=-7+9j;
G=(s+2)*(s+4)/...
  (s*(s+3)*(s+6));
Theta=(180/pi)*angle(G)
M=abs(G)
```

$$G(s)=\frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

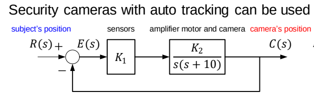
```
Theta =
  -110.6881
M =
   0.0961
```





## §2. Defining the Root Locus

Security cameras with auto tracking can be used to follow moving objects automatically




Block diagram

$R(s) \rightarrow \boxed{\dfrac{K}{s^2+10s+K}} \rightarrow C(s)$ $\quad K=K_1+K_2$

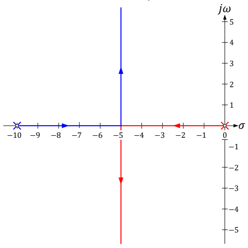
Closed-loop transfer function

- The variation of pole location for different values of gain, $K$

| $K$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| pole 1 | $0$ | $-9.47$ | $-8.87$ | $-8.16$ | $-7.24$ | $-5$ | $-5+j2.24$ | $-5+j3.16$ | $-5+j3.87$ | $-5+j4.47$ | $-5+j5$ |
| pole 2 | $-10$ | $-0.53$ | $-1.13$ | $-1.84$ | $-2.77$ | $-5$ | $-5-j2.24$ | $-5-j3.16$ | $-5-j3.87$ | $-5-j4.47$ | $-5-j5$ |

## §2. Defining the Root Locus

- root locus: the paths of the closed-loop poles as the gain $K$ is varied. The discussion will be limited to positive gain, or $K \geq 0$



- The root locus shows the changes in the transient response
  - $K < 25$: overdamped
    the poles are real
  - $K = 25$: critically damped
    the poles are real and multiple
  - $K > 25$: underdamped
    the poles are complex

| $K$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| pole 1 | $0$ | $-9.47$ | $-8.87$ | $-8.16$ | $-7.24$ | $-5$ | $-5+j2.24$ | $-5+j3.16$ | $-5+j3.87$ | $-5+j4.47$ | $-5+j5$ |
| pole 2 | $-10$ | $-0.53$ | $-1.13$ | $-1.84$ | $-2.77$ | $-5$ | $-5-j2.24$ | $-5-j3.16$ | $-5-j3.87$ | $-5-j4.47$ | $-5-j5$ |





## §3. Properties of the Root Locus

- The closed-loop TF for the system

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \tag{8.12}$$

- The pole, $s$, is the roots of the characteristics equation

$$KG(s)H(s) = -1$$
$$= 1\angle(2k+1)180^0 \quad k = 0, \pm 1, \dots \tag{8.13}$$

or value of $s$ is a closed-loop pole if

$$|KG(s)H(s)| = 1 \tag{8.14}$$

and

$$\angle KG(s)H(s) = (2k+1)180^0 \tag{8.15}$$

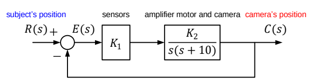
$$K = \frac{1}{|G(s)||H(s)|} \tag{8.16}$$





## §3. Properties of the Root Locus

- Ex.




The pole location varies with $K$

$$KG(s)H(s) = \frac{K}{s(s+10)}$$

$$K = 5,\ \ p = -9.47:\ KG(s)H(s) = \frac{5}{(-9.47)(-9.47+10)} = -1$$

$$K = 5,\ \ p = -0.53:\ KG(s)H(s) = \frac{5}{(-0.53)(-0.53+10)} = -1$$

$$K = 10, p = -8.87:\ KG(s)H(s) = \frac{10}{(-8.87)(-8.87+10)} = -1$$

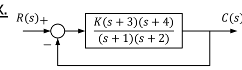
…

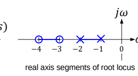


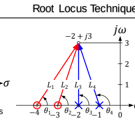


## §3. Properties of the Root Locus

- Ex.




The open-loop and closed-loop TF

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \tag{8.18}$$

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)} \tag{8.19}$$

Consider the point $-2 + j3$

If this point is a closed-loop pole, then *the angles of the zeros minus the angles of the poles must equal an odd multiple of* $180^0$

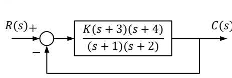
$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^0 + 71.57^0 - 90^0 - 108.43^0 = -70.55^0$$

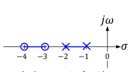
→ $-2 + j3$ is not a point on the root locus, or alternatively, $-2 + j3$ is not a closed-loop pole for any gain

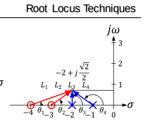




## §3. Properties of the Root Locus




Consider the point $-2 + j\frac{\sqrt{2}}{2}$

$$\sum_1^n \theta_{z_i} - \sum_1^m \theta_{p_i} = (2k+1)180^0 \tag{8.20}$$

$\rightarrow \theta_1 + \theta_2 - \theta_3 - \theta_4 = 180^0$

$\rightarrow -2 + j\frac{\sqrt{2}}{2}$ is a point on the root locus for some value of gain $K$

The gain $K$ for the point $-2 + j(\sqrt{2}/2)$

$$K = \frac{1}{|G(s)||H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}} \tag{8.21}$$

$$\rightarrow K = \frac{\vec{L}_3\vec{L}_4}{\vec{L}_1\vec{L}_2} = \frac{(\sqrt{2}/2) \times 1.22}{2.12 \times 1.22} = 0.33$$





## §3. Properties of the Root Locus

### Skill-Assessment Ex.8.2

Problem Given a unity feedback system that has the forward TF

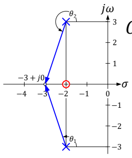
$$G(s) = \frac{K(s+2)}{(s^2 + 4s + 13)}$$

do the following

a. Calculate the angle of $G(s)$ at the point $(-3 + j0)$ by finding the algebraic sum of angles of the vectors drawn from the zeros and poles of $G(s)$ to the given point
b. Determine if the point specified in a is on the root locus
c. If the point specified in a is on the root locus, find the gain, $K$, using the lengths of the vectors

## §3. Properties of the Root Locus

Solution a. Directly substituting the point into $F(s)$



$$G(s)=\frac{K(s+2)}{(s^2+4s+13)}=\frac{K(s+2)}{[s-(-2+j3)][s-(-2-j3)]}$$

From the diagram

$$\sum \text{angles}=180^0-\theta_1-\theta_2$$

with $\theta_1=90^0+\tan^{-1}(1/3)=108.43^0$

$\theta_2=270^0-\tan^{-1}(1/3)=251.57^0$

$\rightarrow \sum \text{angles}=180^0$

b. Since the angle is $180^0$, the point $(-3+j0)$ is on the locus pole lengths

c. $K=\dfrac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$

$$=\frac{\sqrt{1^2+3^2}\times\sqrt{1^2+3^2}}{1}=10$$





## §3. Properties of the Root Locus

**TryIt 8.2**

Use MATLAB and the following statements to solve Skill-Assessment Ex.8.2

```
s=-3+0j;
G=(s+2)/(s^2+4*s+13);
Theta=(180/pi)*angle(G)
M=abs(G);
K=1/M
```

$$G(s)=\frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

```
s=-3+0j;
G=(s+2)*(s^2+4*s+13);
Theta=(180/pi)*angle(G)
M=abs(G)
```





## §4. Sketching the Root Locus

The following five rules allow us to sketch the root locus

1. Number of branches

   The number of branches of the root locus equals the number of closed-loop poles

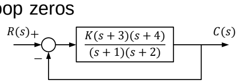
2. Symmetry

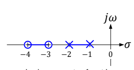
   The root locus is symmetrical about the real axis

3. Real-axis segments

   On the real axis, for $K>0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros

   Ex.








## §4. Sketching the Root Locus

4. Starting and ending points

   The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$

Proof

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}=\frac{K\frac{N_G(s)}{D_G(s)}}{1+K\frac{N_G(s)}{D_G(s)}\frac{N_H(s)}{D_H(s)}}=\frac{KN_G(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}$$

$$K\rightarrow 0:\ T(s)=\frac{KN_G(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}\approx\frac{KN_G(s)}{D_G(s)D_H(s)+\epsilon}$$

→ the closed-loop system poles at small gains approach the combined poles of $G(s)$ and $H(s)$: the root locus begins at the poles of $G(s)H(s)$, the open-loop TF

$$K\rightarrow\infty:T(s)=\frac{KN_G(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}\approx\frac{KN_G(s)}{\epsilon+KN_G(s)N_H(s)}$$

→ the closed-loop system poles at large gains approach the combined zeros of $G(s)$ and $H(s)$: the root locus ends at the zeros of $G(s)H(s)$, the open-loop TF





## §4. Sketching the Root Locus

5. Behavior at infinity

   The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, $\sigma_a$ and angle, $\theta_a$ as follows finite poles

$$\sigma_a=\frac{\sum\text{finite poles}-\sum\text{finite zeros}}{\#\text{ finite poles}-\#\text{ finite zeros}}=\frac{\sum_1^n p_i-\sum_1^m z_i}{n-m}\qquad(8.27)$$

$$\theta_a=\frac{(2k+1)\pi}{\#\text{ finite poles}-\#\text{ finite zeros}}=\frac{(2k+1)\pi}{n-m}\qquad(8.28)$$

where $k=0,\pm1,\pm2,\ldots$ and the angle is given in radians with respect to the positive extension of the real axis

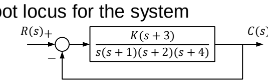




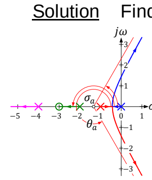
## §4. Sketching the Root Locus

- Ex.8.2 Sketching a root locus with asymptotes

Sketch the root locus for the system




Solution Find the asymptotes



The real-axis intercept

$$\sigma_a=\frac{\sum^m p_i-\sum_1^n z_i}{m-n}=\frac{(0-1-2-4)-(-3)}{4-1}$$

$$\rightarrow\sigma_a=-\frac{4}{3}$$

The angles of line that intersect at $-4/3$

$$\theta_a=\frac{(2k+1)\pi}{m-n}=\frac{(2k+1)\pi}{4-1}=\frac{(2k+1)\pi}{3}$$

$$\rightarrow\theta_a=(k,\theta_a)=[(0;\pi/3),(1;\pi),(2;5\pi/3)]$$

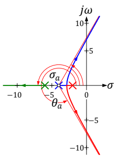
## §4. Sketching the Root Locus

### Skill-Assessment Ex.8.3

Problem Sketch the root locus and its asymptotes for a unity feedback system that has the forward TF

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

Solution Find the asymptotes



The real-axis intercept

$$\sigma_a = \frac{\sum_1^m p_i - \sum_1^n z_i}{m-n} = \frac{(-2-4-6)-(0)}{3-0} = -4$$

The angles of line that intersect at $-4$

$$\theta_a = \frac{(2k+1)\pi}{m-n} = \frac{(2k+1)\pi}{3-0} = \frac{(2k+1)\pi}{3}$$

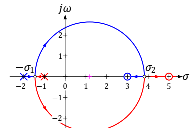
$$\rightarrow \theta_a = (k,\theta_a) = [(0;\pi/3),(1;\pi),(2;5\pi/3)]$$





## §5. Refining the Sketch

### Real-Axis Breakaway and Break-In Points



breakaway point: $-\sigma_1$

break-in point: $\sigma_2$

At the breakaway or break-in point, the branches of the root locus form an angle of $180^0/n$ with the real axis

$n$: the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis (Kuo, 1991)

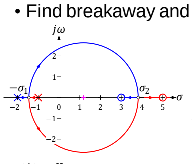
For the two poles shown, the branches at the breakaway point form $180^0/n = 180^0/2 = 90^0$ angles with the real axis





## §5. Refining the Sketch

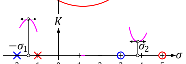
- Find breakaway and break-in point by maximize/minimize the gain



Eq. (8.13) $\rightarrow K = -\dfrac{1}{G(s)H(s)}$ (8.31)

Along the real axis Eq. (8.31) becomes

$$K = -\frac{1}{G(\sigma)H(\sigma)} \qquad (8.32)$$



To find breakaway and break-in point $\rightarrow$ differentiate (8.32) with respect to $\sigma$ and set the derivative equal to zero

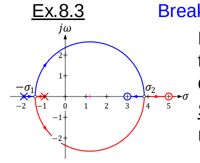
$KG(s)H(s) = -1 = 1\angle(2k+1)180^0 \quad k = 0,\pm1,\ldots$ (8.13)





## §5. Refining the Sketch

Ex.8.3 Breakaway and break-in points via differentiation



Find the breakaway and break-in points for the root locus of the given figure, using differential calculus

Solution

Using the open-loop poles and zeros

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{s^2+3s+2}$$

For all points along the real axis, $KG(\sigma)H(\sigma) = -1$

$$\frac{K(\sigma^2-8\sigma+15)}{\sigma^2+3\sigma+2} = -1 \rightarrow K = -\frac{\sigma^2+3\sigma+2}{\sigma^2-8\sigma+15} \rightarrow \frac{dK}{d\sigma} = \frac{11\sigma^2-26\sigma-61}{(\sigma^2-8\sigma+15)^2}$$

Setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = 0 \rightarrow \sigma = [-1.4530, 3.8166] \rightarrow \sigma_1 = -1.4530,\ \sigma_2 = 3.8166$$





## §5. Refining the Sketch

- Find breakaway and break-in point using transition method

Breakaway and break-in points satisfy the relationship

$$\sum_1^m \frac{1}{\sigma + z_i} = \sum_1^n \frac{1}{\sigma + p_i} \qquad (8.37)$$

$z_i, p_i$: the negative of the zero and pole values, respectively, of $G(\sigma)H(\sigma)$

Solving (8.37) for $\sigma$, the real-axis values that minimize or maximize $K$, yields the breakaway and break-in points without differentiating

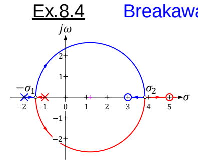
$KG(s)H(s) = -1 = 1\angle(2k+1)180^0 \quad k = 0,\pm1,\ldots$ (8.13)





## §5. Refining the Sketch

Ex.8.4 Breakaway and break-in points without differentiation



Find the breakaway and break-in points for the root locus of the given figure, without differentiating

Solution

Using (8.37)

$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2}$$

Simplifying

$$11\sigma^2 - 26\sigma - 61 = 0$$

$$\rightarrow (\sigma + 1.4530)(\sigma - 3.8166) = 0$$

$$\rightarrow \sigma_1 = -1.4530,\ \sigma_2 = 3.8166$$

$\sum_1^m \frac{1}{\sigma+z_i} = \sum_1^n \frac{1}{\sigma+p_i}$ (8.37)

## §5. Refining the Sketch

### The $j\omega$-Axis Crossings

Use the Routh-Hurwitz criterion to find the $j\omega$-axis crossing as follows

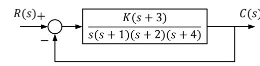
1. Forcing a row of zeros in the Routh table will yield the gain
2. Going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing





## §5. Refining the Sketch

- Ex.8.5 Frequency and gain at imaginary-axis crossing




For the given system, find the frequency and gain, $K$, for which the root locus crosses the imaginary axis. For what range of $K$ is the system stable?

Solution

The closed-loop transfer function for the system

$$T(s)=\frac{G(s)}{1+G(s)}$$

$$=\frac{K(s+3)}{s^4+7s^3+14s^2+(8+K)s+3K}$$





## §5. Refining the Sketch

$$T(s)=\frac{K(s+3)}{s^4+7s^3+14s^2+(8+K)s+3K}$$

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $14$ | $3K$ |
| $s^3$ | $7$ | $K+8$ | $0$ |
| $s^2$ | $-K+90$ | $21K$ | $0$ |
| $s^1$ | $\frac{-K^2-65K+720}{-K+90}$ | $0$ | $0$ |
| $s^0$ | $1$ | $0$ | $0$ |

Complete row of zeros → the possibility for imaginary axis roots

$$s^1{:}\,-K^2-65K+720=(K+74.65)(K-9.65)=0$$

$\rightarrow K=9.65$

Forming the even polynomial by using the $s^2$ row with $K=9.65$

$$s^2{:}\,(90-K)s^2+21K=80.35s^2+202.7=0$$

$\rightarrow s=\pm j1.59$

The system is stable for $0\le K\le 9.65$

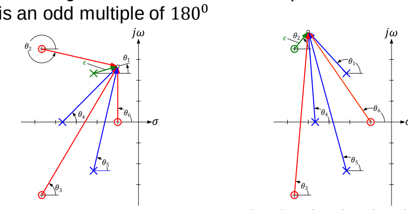




## §5. Refining the Sketch

### Angles of Departure and Arrival

Assume a point on the root locus $\epsilon$ close to a complex pole/zero, → sum of angles drawn from all finite poles and zeros to this point is an odd multiple of $180^0$



$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^0\rightarrow\theta_1$

a. angle of departure

$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^0\rightarrow\theta_2$

b. angle of arrival

Open-loop poles and zeros and calculation

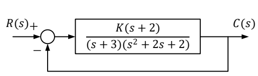




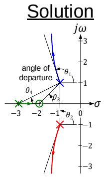
## §5. Refining the Sketch

- Ex.8.6 Angle of departure from a complex pole

Find the angle of departure from the complex poles and sketch the root locus




Solution




Calculate the sum of angles draw to a point $\epsilon$ close to the complex pole, $-1+j1$, in the second quadrant

$$-\theta_1-\theta_2+\theta_3-\theta_4=-\theta_1-90^0+\tan^{-1}\left(\frac{1}{1}\right)+\tan^{-1}\left(\frac{1}{2}\right)$$

$$=180^0$$

$\rightarrow\theta_1=-251.6^0=108.4^0$

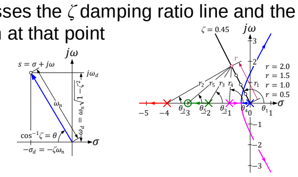




## §5. Refining the Sketch

### Plotting and Calibrating the Root Locus

Find the exact point at which the locus crosses the $\zeta$ damping ratio line and the gain at that point




$z=[z_1,z_2,\ldots]$
$p=[p_1,p_2,\ldots]$
$\zeta$
$r_{range}=r_{min}:dr:r_{max},n=\text{length}(r_{range})$
$[r,k]=\text{rlocus}(sys),m=\text{length}(r)$

```
for i = 1:m
    for j = 1:n
        θ(i) = Σθ_{z_i} − Σθ_{p_i}
        e(i) = 180 × |1 − |θ(i)/π||
    end
end
```

$[e_{min},i_{min}]=\min(e)$
$R=r_{range}(i_{min})$
$r_i=\tan^{-1}\frac{R\sin(\cos^{-1}\zeta)}{|p_i-R\zeta|}$
$K=\frac{\prod\text{pole lengths}}{\prod\text{zero lengths}}$

- using (8.20) to search a given line for the point
- using (8.21) to calculate the gain at that point

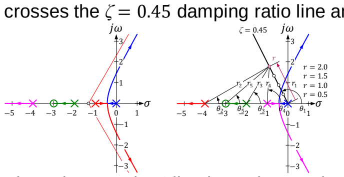
$$\sum_1^n\theta_{z_i}-\sum_1^m\theta_{p_i}=(2k+1)180^0 \qquad (8.20)$$

$$K=\frac{1}{|G(s)||H(s)|}=\frac{1}{M}=\frac{\prod\text{pole lengths}}{\prod\text{zero lengths}} \qquad (8.21)$$

## §5. Refining the Sketch

### Plotting and Calibrating the Root Locus

Ex.: Given the root locus → Find the exact point at which the locus crosses the $\zeta = 0.45$ damping ratio line and the gain at that point



| radius, $r$ | angle, $^0$ |
|---|---|
| 0.500 | −158.4 |
| 0.747 | −180.0 |
| 1.000 | −199.9 |
| 1.500 | −230.4 |
| 2.000 | −251.5 |

The point $r$ on the $\zeta$ line is on the root locus if

$$(2k+1)180^0 = \theta_5 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

$$= \tan^{-1}\frac{r\sin z}{|3 - r\zeta|} - (180^0 - \cos^{-1}\zeta) - \tan^{-1}\frac{r\sin z}{|4 - r\zeta|}$$

$$-\tan^{-1}\frac{r\sin z}{|2 - r\zeta|} - \tan^{-1}\frac{r\sin z}{|1 - r\zeta|}$$

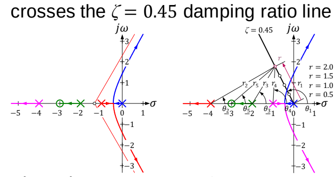




## §5. Refining the Sketch

### Plotting and Calibrating the Root Locus

Ex.: Given the root locus → Find the exact point at which the locus crosses the $\zeta = 0.45$ damping ratio line and the gain at that point



| radius, $r$ | angle, $^0$ |
|---|---|
| 0.500 | −158.4 |
| 0.747 | −180.0 |
| 1.000 | −199.9 |
| 1.500 | −230.4 |
| 2.000 | −251.5 |

The gain $K$ at $r = 0.7470$

$$K = \frac{|r_1||r_2||r_3||r_4|}{|r_5|} = \frac{0.7471 \times 3.7241 \times 1.7926 \times 0.9412}{2.7461}$$

$\rightarrow K = 1.7090$





## §5. Refining the Sketch

### Skill-Assessment Ex.8.4

Problem Given a unity feedback system that has the forward TF

$$G(s) = \frac{K(s+2)}{s^2 - 4s + 13}$$

do the following

a. Sketch the root locus

b. Find the imaginary-axis crossing

c. Find the gain, $K$, at the $j\omega$-axis crossing

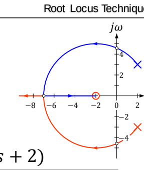
d. Find the break-in point

e. Find the angle of departure from the complex poles





## §5. Refining the Sketch



Solution a. The sketched root locus

$$G(s) = \frac{K(s+2)}{s^2 - 4s + 13}$$

b. The imaginary-axis crossing

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{K(s+2)}{s^2 - (K-4)s + (2K+13)}$$

The Routh table

| | | |
|---|---|---|
| $s^2$ | $1$ | $2K+13$ |
| $s^1$ | $K-4$ | $0$ |
| $s^0$ | $2K+13$ | $0$ |

from row of zero → $K = 4$

from $s^2$ row with $K = 4 \rightarrow s^2 + 21 = 0 \rightarrow s = \pm j\sqrt{21}$

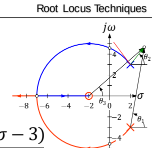
c. The gain, $K$, at the $j\omega$-axis crossing

From (b) → $K = 4$





## §5. Refining the Sketch



d. The break-in point

$$(8.32) \rightarrow K = -\frac{\sigma^2 - 4\sigma + 13}{\sigma + 2}$$

$$\rightarrow \frac{dK}{d\sigma} = -\frac{\sigma^2 + 4\sigma - 21}{(\sigma+2)^2} = -\frac{(\sigma+7)(\sigma-3)}{(\sigma+2)^2}$$

$$\frac{dK}{d\sigma} = 0 \rightarrow \sigma = -7$$

e. The angle of departure from the complex poles

Draw vectors to a point $\epsilon$ close to the complex pole

$$\theta_3 - \theta_2 - \theta_1 = 180^0$$

$$\rightarrow \tan^{-1}(3/4) - \theta_2 - 90^0 = 180^0$$

$$\rightarrow \theta_2 = \tan^{-1}(3/4) - 270^0 = -233.1^0 = 126.9^0$$

$$K = -\frac{1}{G(\sigma)H(\sigma)} \qquad (8.32)$$





## §6. An Example

### Basic Rules for Sketching the Root Locus

*1. Number of branches*

number of branches = number of closed-loop poles

*2. Symmetry*

The root locus is symmetrical about the real axis

*3. Real-axis segments*

The root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros

*4. Starting and ending points*

The root locus

- begins at the finite and infinite poles of $G(s)H(s)$, and
- ends at the finite and infinite zeros of $G(s)H(s)$

## §6. An Example

*5. Behavior at infinity*

The root locus → straight lines as asymptotes as the locus → ∞

The asymptotes are given by the real-axis intercept

$$\sigma_a = \frac{\sum_1^n p_i - \sum_1^m z_i}{n-m} \qquad \theta_a = \frac{(2k+1)\pi}{n-m} \qquad k = 0, \pm 1, \pm 2, \ldots$$

*6. Real-axis breakaway and break-in points*

$$K = -\frac{1}{G(\sigma)H(\sigma)} \rightarrow \frac{dK}{d\sigma} = 0 \rightarrow \sigma \rightarrow \begin{matrix} \sigma_{K_{max}} : \text{breaks away point} \\ \sigma_{K_{min}} : \text{breaks into point} \end{matrix}$$

*7. Calculation of $j\omega$-axis crossings*

Using the Routh-Hurwitz criterion

- Forcing a row of zeros in the Routh table will yield the gain
- Going back one row to the even polynomial equation and solving for the roots





## §6. An Example

Additional Rules for Refining the Sketch

*8. Angles of departure and arrival*

Assume a point $\epsilon$ close to the complex pole or zero. Add all angles drawn from all open-loop poles and zeros to this point

$$\sum_1^n \theta_{z_i} - \sum_1^m \theta_{p_i} = (2k+1)180^0$$

Solving for the unknown angle → the angle of departure/arrival

*9. Plotting and calibrating the root locus*

All points on the root locus satisfy $\angle G(s)H(s) = (2k+1)180^0$

The gain, $K$, at any point on the root locus is given by

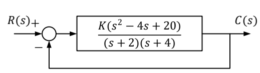
$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{finite pole lengths}}{\prod \text{finite zero lengths}}$$





## §6. An Example

- Ex.8.7 Sketching a root locus and finding critical points




Sketch the root locus for the system and find the following

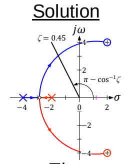
a. The exact point and gain where the locus crosses the 0.45 damping ratio line
b. The exact point and gain where the locus crosses the $j\omega$-axis
c. The breakaway point on the real axis
d. The range of $K$ within which the system is stable





## §6. An Example

Solution



First sketch the root locus

- the real-axis segment $\sigma = -2, -4$
- the root locus starts at the open-loop poles $(-2, -4)$ and ends at the open-loop zeros $(2 \pm j4)$

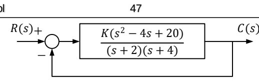
a. The exact point and gain where the locus crosses the 0.45 damping ratio line $\zeta$

Searching in polar coordinates, we find that the root locus crosses the $\zeta = 0.45$ line at $3.4\angle 116.7^0$ with a gain, $K = 0.417$





## §6. An Example



b. The exact point and gain where the locus crosses the $j\omega$-axis

The closed-loop TF for the system

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K(s^2-4s+20)}{(1+K)s^2 + (6-4K)s + (8+20K)}$$

The Routh table

| | | |
|---|---|---|
| $s^2$ | $1+K$ | $8+20K$ |
| $s^1$ | $6-4K$ | 0 |
| $s^0$ | $8+20K$ | 0 |

A complete row of zeros yields the possibility for imaginary axis roots

$$s^1 : 6 - 4K = 0 \rightarrow K = 1.5$$

Forming the even polynomial by using the $s^2$ row with $K = 1.5$

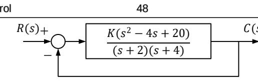
$$s^2 : (1+K)s^2 + (8+20K) = 2.5s^2 + 38 = 0 \rightarrow s = \pm j3.9$$

The root locus crosses the $j\omega$-axis at $\pm j3.9$ with a gain of $K = 1.5$





## §6. An Example



c. The breakaway point on the real axis

From Eq.(8.32)

$$K = -\frac{1}{G(\sigma)H(\sigma)} = -\frac{\sigma^2 + 6\sigma + 8}{\sigma^2 - 4\sigma + 20}$$

$$\rightarrow \frac{dK}{d\sigma} = \frac{10\sigma^2 - 24\sigma - 152}{(\sigma^2 - 4\sigma + 20)^2} = \frac{(\sigma - 5.279)(\sigma + 2.879)}{(\sigma^2 - 4\sigma + 20)^2}$$

$$\frac{dK}{d\sigma} = 0 \rightarrow \sigma = -2.879$$

d. From the answer to (b), the system is stable for $0 \le K \le 1.5$

$$K = -\frac{1}{G(\sigma)H(\sigma)} \qquad (8.32)$$

## §6. An Example

MATLAB ML

Run ch8p1 in Appendix B

Learn how to use MATLAB to

- plot and title a root locus
- overlay constant $\zeta$ and $\omega_n$ curves
- zoom into and zoom out from a root locus
- interact with the root locus to find critical points as well as gains at those points
- solve Ex.8.7





## §6. An Example

Skill-Assessment Ex.8.5

Problem Given a unity feedback system that has the forward TF

$$G(s)=\frac{K(s-2)(s-4)}{s^2+6s+25}$$

do the following

a. Sketch the root locus

b. Find the imaginary-axis crossing

c. Find the gain, $K$, at the $j\omega$-axis crossing

d. Find the break-in point

e. Find the point where the locus crosses $\zeta=0.5$ damping ratio line

f. Find the gain at the point where the locus crosses the $0.5$ damping ratio line

g. Find the range of gain, $K$, for which the system is stable

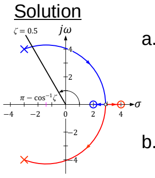




## §6. An Example

$$G(s)=\frac{K(s-2)(s-4)}{s^2+6s+25}$$

Solution




a. Sketch the root locus

- the real-axis segment $\sigma=2,4$
- the root locus starts at the open-loop poles $(-3\pm j4)$ and ends at the open-loop zeros $(2,4)$

b. Find the imaginary-axis crossing

$$T(s)=\frac{K(s-2)(s-4)}{(1+K)s^2+6(1-K)s+(25+8K)}$$

The Routh table

| | | |
|---|---|---|
| $s^2$ | $1+K$ | $25+8K$ |
| $s^1$ | $6-6K$ | $0$ |
| $s^0$ | $25+8K$ | $0$ |

from row of zero $\rightarrow K=1$

from $s^2$ row with $K=1\rightarrow 2s^2+33=0\rightarrow s=\pm j\sqrt{16.5}$

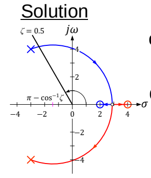




## §6. An Example

$$G(s)=\frac{K(s-2)(s-4)}{s^2+6s+25}$$

Solution




c. Find the gain, $K$, at the $j\omega$-axis crossing

From the answer to (b), the gain $K=1$

d. Find the break-in point

From (8.32)

$$K=-\frac{1}{G(\sigma)H(\sigma)}=-\frac{\sigma^2+6\sigma+25}{\sigma^2-6\sigma+8}$$

$$\rightarrow\frac{dK}{d\sigma}=\frac{12\sigma^2+34\sigma-198}{(\sigma^2-6\sigma+8)^2}=\frac{(\sigma-2.885)(\sigma+5.719)}{(\sigma^2-6\sigma+8)^2}$$

$$\frac{dK}{d\sigma}=0\rightarrow\sigma=2.885$$

$K=-\frac{1}{G(\sigma)H(\sigma)}$ (8.32)

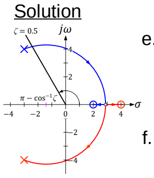




## §6. An Example

$$G(s)=\frac{K(s-2)(s-4)}{s^2+6s+25}$$

Solution




e. Find the point where the locus crosses the $0.5$ damping ratio line

Searching along $\zeta=0.5$ for the $180^0$ point, we find $s=-2.42+j4.18$

f. Find the gain at the point where the locus crosses the 0.5 damping ratio line

For the result in part (e), $K=0.108$

g. Find the range of gain, $K$, for which the system is stable

Using the result from part (c) and the root locus, $K<1$

$\sum_1^n\theta_{z_i}-\sum_1^m\theta_{p_i}=(2k+1)180^0$ (8.20)

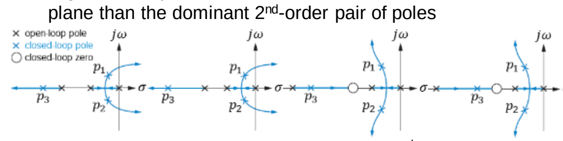
$K=\frac{1}{|G(s)||H(s)|}=\frac{1}{M}=\frac{\prod\text{pole lengths}}{\prod\text{zero lengths}}$ (8.21)





## §7. Transient Response Design via Gain Adjustment

- The formulas describing percent overshoot, settling time, and peak time were derived only for a system with two closed-loop complex poles and no closed-loop zeros
- Conditions for 2nd-order Approximations
  - Higher-order poles are much farther into the left half of the $s$-plane than the dominant 2nd-order pair of poles




  - Closed-loop zeros near closed-loop 2nd-order poles are canceled by proximity of higher-order closed-loop poles
  - Closed-loop zeros not canceled by proximity of higher-order closed-loop poles are far from closed-loop 2nd-order poles

## §7. Transient Response Design via Gain Adjustment

Design Procedure for Higher Order Systems

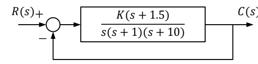
1. Sketch the root locus for the given system
2. Assume 2nd-order system without any zeros, find the gain, $K$, to meet transient response specification
3. Justify the 2nd-order assumption by
   - evaluating that all higher-order poles are much farther from the $j\omega$-axis than the dominant 2nd-order pair (5 times farther)
   - verifying that closed-loop zeros are approximately canceled by higher-order poles, or be sure that the zero is far removed from the dominant 2nd-order pole pair to yield approximately the same response obtained without the finite zero
4. If the assumptions cannot be justified, the solution will have to be simulated in order to be sure it meets the transient response specification





## §7. Transient Response Design via Gain Adjustment

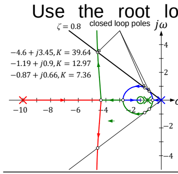
- Ex.8.8 Third-order system gain design




Given the system, design the value of gain, $K$, to yield $1.52\%$ overshoot. Also estimate the settling time, peak time, and steady-state error

Solution

Use the root locus program to search along the $\zeta = 0.8$ damping ratio line for the point where the root locus crosses $1.52\%$ overshoot line. Three points satisfy this criterion:




$$-0.87 \pm j0.66,\ K = 07.36$$
$$-1.19 \pm j0.90,\ K = 12.79$$
$$-4.60 \pm j3.45,\ K = 39.64$$





## §7. Transient Response Design via Gain Adjustment

To test our assumption of a 2nd-order system, calculate the location of the 3rd pole

| case | closed-loop poles | closed-loop zeros | gain | 3rd closed-loop pole | settling time | peak time | $K_v$ |
|---|---|---|---|---|---|---|---|
| 1 | $-0.87 + j0.66$ | $-1.5 + j0$ | 07.36 | $-9.25$ | 4.60 | 4.76 | 1.1 |
| 2 | $-1.19 + j0.90$ | $-1.5 + j0$ | 12.79 | $-8.61$ | 3.36 | 3.49 | 1.9 |
| 3 | $-4.6 + j3.45$ | $-1.5 + j0$ | 39.64 | $-1.80$ | 0.87 | 0.91 | 5.9 |

When the gain is set to meet the transient response, we have also designed the steady-state error. For the example, the steady-state error specification is given by $K_v$ and is calculated

$$K_v = \lim_{s\to0} sG(s) = \lim_{s\to0} s\frac{K(s+1.5)}{s(s+1)(s+10)} = \frac{K\times1.5}{1\times10} = 0.15K$$

$K = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}}$ (8.21) $T_s = \frac{4}{\zeta\omega_n}$, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$





## §7. Transient Response Design via Gain Adjustment

MATLAB
ML

Run ch8p2 in Appendix B

Learn how to use MATLAB to

- enter a value of percent overshoot from the keyboard
- draw the root locus and overlay the percent overshoot line requested
- interact with MATLAB and select the point of intersection of the root locus with the requested percent overshoot line
- interact with the root locus to find critical points as well as gains at those points
- solve Ex.8.8





## §7. Transient Response Design via Gain Adjustment

GUI Tool
GUIT

Explore the SISO Design Tool in Appendix E

The SISO Design Tool is a convenient and intuitive way to obtain, view, and interact with a system's root locus. Section D.7 describes the advantages of using the tool, while Section D.8 describes how to use it. For practice, you may want to apply the SISO Design Tool to some of the problems at the end of this chapter





## §7. Transient Response Design via Gain Adjustment

Skill-Assessment Ex.8.6

Problem Given a unity feedback system that has the forward-path TF

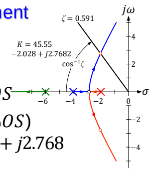
$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

do the following

a. Sketch the root locus
b. Using a second-order approximation, design the value of $K$ to yield $10\%$ overshoot for a unit-step input
c. Estimate the settling time, peak time, rise time, and steady-state error for the value of $K$ designed in (b)
d. Determine the validity of your 2nd-order approximation

## §7. Transient Response Design via Gain Adjustment

Solution




a. Sketch the root locus

b. Design the value of $K$ to yield $10\%OS$

Searching along the $\zeta = 0.591$ $(10\%OS)$ line for the $180^0$ point yields $-2.028 + j2.768$ with $K = 45.55$

c. Estimate $T_s$, $T_p$, $T_r$, $e_{step}(\infty)$

$$T_s = \frac{4}{|Re|} = \frac{4}{2.028} = 1.97s$$

$$T_p = \frac{\pi}{|Im|} = \frac{\pi}{2.768} = 1.13s$$

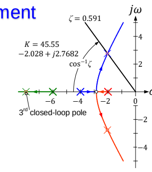
$$\omega_n T_r\Big|_{\zeta=0.591} \approx 1.8346 \rightarrow T_r = \frac{1.8346}{\sqrt{2.228^2 + 2.768^2}} = 0.53s$$

$\omega_n T_r \approx 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$





## §7. Transient Response Design via Gain Adjustment




$T_s = 1.97s$

$T_p = 1.13s$

$T_r = 0.53s$

System is Type 0

$$K_p = \frac{K}{2 \times 4 \times 6} = \frac{45.55}{48} = 0.949$$

$$\rightarrow e_{step}(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + 0.949} = 0.51$$

d. Determine the validity of your 2nd-order approximation

Searching the real axis to the left of $-6$ for the point whose gain is $45.55$ → the 3rd pole: $p_3 = -7.94$

The real part of the dominant pole: $p_{1,2} = -2.028$

→ $p_3$ is not five times further $p_{1,2}$

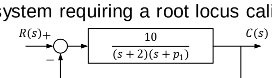
→ the 2nd-order approximation is not valid





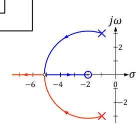
## §8. Generalized Root Locus

Consider the system requiring a root locus calibrated with $p_1$ as a parameter




The open-loop TF

$$KG(s)H(s) = \frac{10}{(s+2)(s+p_1)}$$




The closed-loop TF

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)} = \frac{10}{s^2+2s+10+p_1(s+2)} = \frac{\frac{10}{s^2+2s+10}}{1+\frac{p_1(s+2)}{s^2+2s+10}}$$

Implies that we have the system $KG(s)H(s) = \dfrac{p_1(s+2)}{s^2+2s+10}$ (8.62)

The root locus can now be sketched as a function of $p_1$, assuming the open-loop system (8.62)





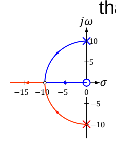
## §8. Generalized Root Locus

Skill-Assessment Ex.8.7

Problem Sketch the root locus for variations in the value of $p_1$, for a unity feedback system that has the following forward TF

$$G(s) = \frac{100}{s(s+p_1)}$$

Solution Find the closed-loop TF and put it the form that yields $p_1$ as the root locus variable




$$T(s) = \frac{G(s)}{1+G(s)} = \frac{100}{s^2+p_1 s+100} = \frac{100}{(s^2+100)+p_1 s}$$

$$= \frac{\frac{100}{s^2+100}}{1+\frac{p_1 s}{s^2+100}}$$

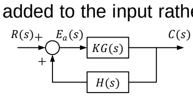
Hence, $KG(s)H(s) = \dfrac{p_1 s}{s^2+100}$ → the root locus





## §9. Root Locus for Positive-Feedback Systems

- The properties of the root locus change dramatically if the feedback signal is added to the input rather than subtracted




- The closed-loop TF $T(s) = \dfrac{KG(s)}{1 - KG(s)H(s)}$ (8.63)

- We now retrace the development of the root locus for the denominator of Eq. (8.63). Obviously, a pole, $s$, exists when

$$KG(s)H(s) = 1 = 1\angle k360^0 \quad k = 0, \pm 1, \pm 2, \ldots \quad (8.64)$$

the root locus for positive-feedback systems consists of all points on the $s$-plane where the angle of $KG(s)H(s) = k360^0$

→Refer the textbook





## §10. Pole Sensitivity

The root locus exhibits a nonlinear relationship between gain and pole location. Along some sections of the root locus

- very small changes in gain yield very large changes in pole location and hence performance → high sensitivity to changes in gain
- very large changes in gain yield very small changes in pole location → low sensitivity to changes in gain

We prefer systems with low sensitivity to changes in gain

## §10. Pole Sensitivity

- The sensitivity of a closed-loop pole, $s$, to gain, $K$

$$S_{s:K} = \frac{K}{s} \times \frac{\delta s}{\delta K} \qquad (8.69)$$

$s$ : the current pole location

$K$ : the current gain

- Converting the partials to finite increments, the actual change in the closed-loop poles can be approximated

$$\Delta s = s \times S_{s:K} \times \frac{\Delta K}{K} \qquad (8.70)$$

$\Delta s$ : the change in pole location

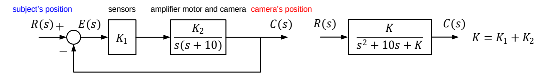
$\Delta K/K$ : the fractional change in the gain, $K$





## §10. Pole Sensitivity

- Ex.8.10 Root sensitivity of a closed-loop system to gain variations




Find the root sensitivity of the system at $s = -9.47$ and $-5 + j5$. Also calculate the change in the pole location for a 10% change in $K$

Solution

Differentiating the closed-loop TF denominator, w.r.t. $K$

$$2s\frac{\partial s}{\partial K} + 10\frac{\partial s}{\partial K} + 1 = 0 \rightarrow \frac{\partial s}{\partial K} = \frac{-1}{2s+10}$$

The sensitivity

$$S_{s:K} = \frac{K}{s}\frac{\delta s}{\delta K} = \frac{K}{s}\frac{-1}{2s+10}$$





## §10. Pole Sensitivity

$$S_{s:K}\Big|_{s=-9.47} = \frac{K}{s}\frac{-1}{2s+10}\Big|_{s=-9.47} \frac{5}{-9.47}\frac{-1}{2\times(-9.47)+10} = -0.059$$

The change in the pole location for a 10% change in $K$

$$\Delta s = s.S_{s:K}\frac{\Delta K}{K} = (-9.47)\times(-0.059)\times 10\% = 0.056$$

$$S_{s:K}\Big|_{s=-5+j5} = \frac{K}{s}\frac{-1}{2s+10}\Big|_{s=-9.47} \frac{50}{-5+j5}\frac{-1}{2\times(-5+j5)+10}$$

$$= 1/(1+j1) = (1/\sqrt{2})\angle -45^0$$

The change in the pole location for a 10% change in $K$

$$\Delta s = s.S_{s:K}\frac{\Delta K}{K} = (-5+j5)\times[1/(1+j1)]\times 10\% = -j5$$

| $K$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| pole 1 | $0$ | $-9.47$ | $-8.87$ | $-8.16$ | $-7.24$ | $-5$ | $-5+j2.24$ | $-5+j3.16$ | $-5+j3.87$ | $-5+j4.47$ | $-5+j5$ |
| pole 2 | $-10$ | $-0.53$ | $-1.13$ | $-1.84$ | $-2.77$ | $-5$ | $-5-j2.24$ | $-5-j3.16$ | $-5-j3.87$ | $-5-j4.47$ | $-5-j5$ |





## §10. Pole Sensitivity

Skill-Assessment Ex.8.9

Problem A negative unity feedback system has the forward TF

$$G(s) = \frac{K(s+1)}{s(s+2)}$$

If $K$ is set to 20, find the changes in closed-loop pole location for a 5% change in $K$

Solution The closed-loop TF

$$T(s) = \frac{K(s+1)}{s^2+(K+2)s+K}$$

Differentiating the denominator with respect to $K$

$$2s\frac{\partial s}{\partial K} + (K+2)\frac{\partial s}{\partial K} + s + 1 = (2s+K+2)\frac{\partial s}{\partial K} + s + 1 = 0$$

$$\rightarrow \frac{\partial s}{\partial K} = -\frac{s+1}{2s+K+2}$$





## §10. Pole Sensitivity

$$\frac{\partial s}{\partial K} = -\frac{s+1}{2s+K+2} \rightarrow S_{s:K} = \frac{K}{s}\frac{\partial s}{\partial K} = \frac{-K(s+1)}{s(2s+K+2)}$$

$$\rightarrow S_{s:K}\Big|_{K=20} = \frac{-10(s+1)}{s(s+11)}$$

The closed-loop poles when $K = 20$

$$D(s)\Big|_{K=20} = s^2 + 22s + 20 = (s+21.05)(s+0.95) = 0$$

For the pole at $-21.05$

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K} = -21.05\frac{-10\times(-21.05+1)}{-21.05\times(-21.05+11)}0.05 = -0.9975$$

For the pole at $-0.95$

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K} = -0.95\frac{-10\times(-0.95+1)}{-0.95\times(-0.95+11)}0.05 = -0.0025$$





## §11. Case Study