05. Reduction of Multiple Subsystems

Learning Outcome

After completing this chapter, the student will be able to

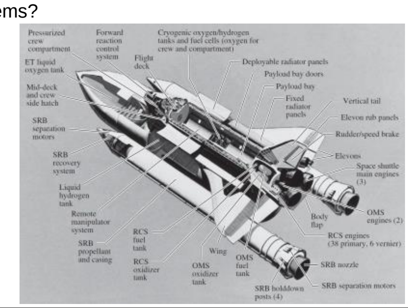
- Reduce a block diagram of multiple subsystems to a single block representing the transfer function from input to output
- Analyze and design transient response for a system consisting of multiple subsystems
- Convert block diagrams to signal-flow diagrams
- Find the transfer function of multiple subsystems using Mason's rule
- Represent state equations as signal-flow graphs
- Represent multiple subsystems in state space in cascade, parallel, controller canonical, and observer canonical forms
- Perform transformations between similar systems using transformation matrices and diagonalize a system matrix

§1. Introduction

- Represent multiple subsystems in two ways
  - block diagrams: for frequency-domain analysis and design
  - signal-flow graphs: for state-space analysis
- Develop techniques to reduce each representation to a single transfer function
  - Block diagram algebra will be used to reduce block diagrams
  - Mason's rule to reduce signal-flow graphs

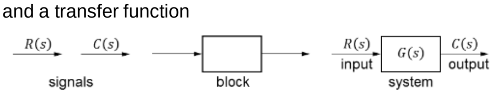
§2. Block Diagrams

- The space shuttle consists of multiple subsystems. Can you identify those that are control systems, or parts of control systems?

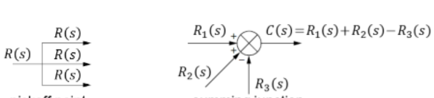


§2. Block Diagrams

- A subsystem is represented as a block with an input, an output, and a transfer function

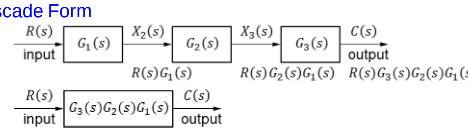


- Many systems are composed of multiple subsystems. When multiple subsystems are interconnected, a few more schematic elements must be added to the block diagram

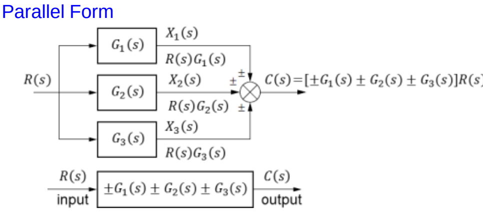


§2. Block Diagrams

Cascade Form

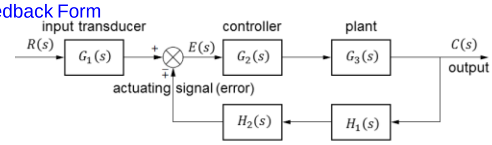


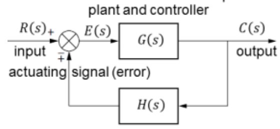
Parallel Form

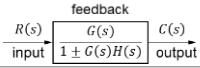


§2. Block Diagrams

Feedback Form

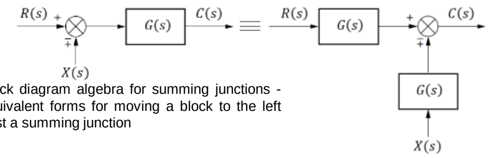




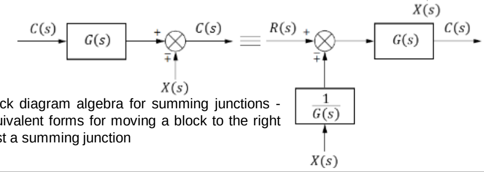


§2. Block Diagrams

Moving Blocks to Create Familiar Forms

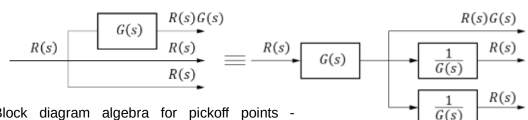


Block diagram algebra for summing junctions - equivalent forms for moving a block to the left past a summing junction

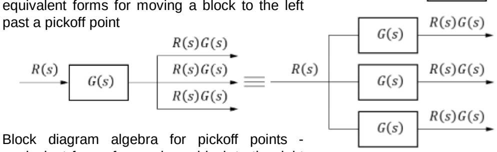


Block diagram algebra for summing junctions - equivalent forms for moving a block to the right past a summing junction

§2. Block Diagrams



Block diagram algebra for pickoff points - equivalent forms for moving a block to the left past a pickoff point

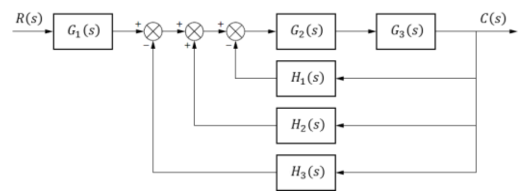


Block diagram algebra for pickoff points - equivalent forms for moving a block to the right past a pickoff point

§2. Block Diagrams

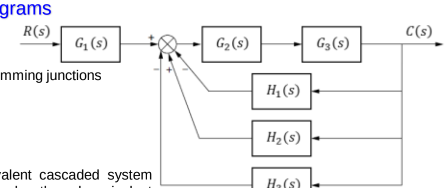
- Ex.5.1 Block Diagram Reduction via Familiar Forms

Reduce the block diagram to a single transfer function



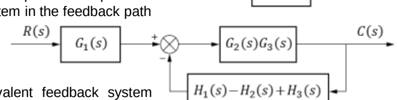
§2. Block Diagrams

Solution

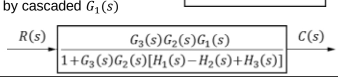


1. Collapse summing junctions

2. Form equivalent cascaded system in the forward path and equivalent parallel system in the feedback path



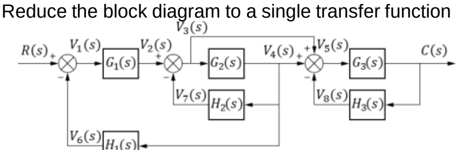
3. Form equivalent feedback system and multiply by cascaded $G_1(s)$



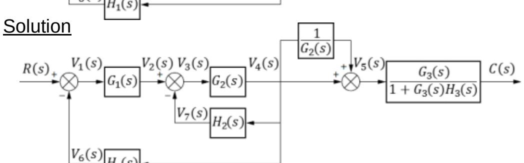
§2. Block Diagrams

- Ex.5.2 Block Diagram Reduction via Familiar Forms

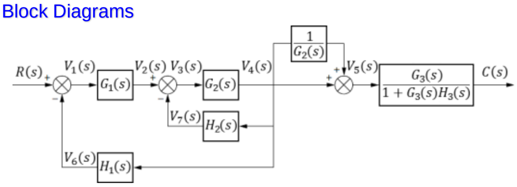
Reduce the block diagram to a single transfer function

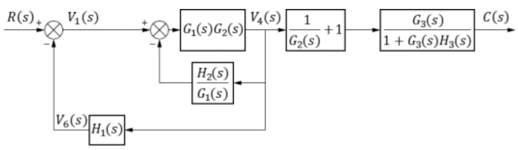


Solution

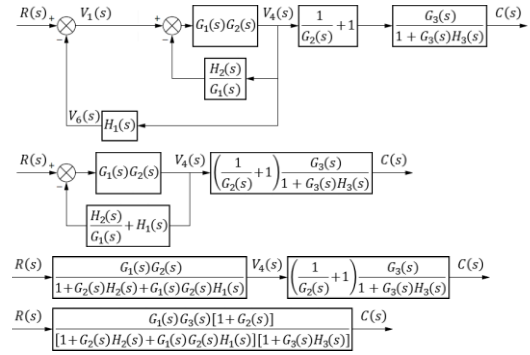


§2. Block Diagrams





§2. Block Diagrams



§2. Block Diagrams

MATLAB ML

Run ch5p1 in Appendix B

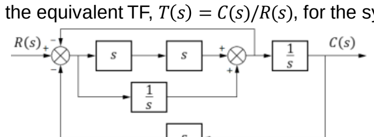
Learn how to use MATLAB to

- perform block diagram reduction

§2. Block Diagrams

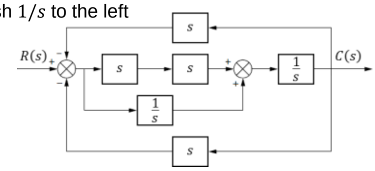
Skill-Assessment Ex.5.1

Problem Find the equivalent TF, $T(s)=C(s)/R(s)$, for the system

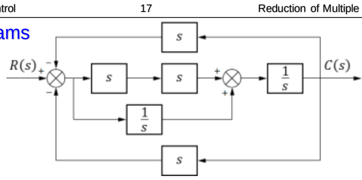


Solution

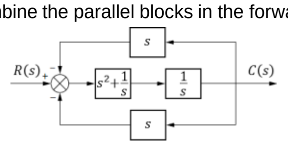
Push $1/s$ to the left



§2. Block Diagrams



Combine the parallel blocks in the forward path



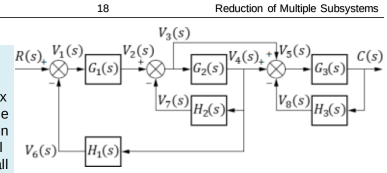
Apply the feedback formula, simplify, and get

$$T(s)=\frac{s^3+1}{2s^4+s^2+2s}$$

§2. Block Diagrams

TryIt 5.1

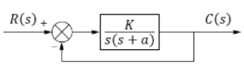
Use the following MATLAB and Control System Toolbox statements to find the closed loop transfer function of the system in Ex.5.2 if all $G_i(s)=1/(s+1)$ and all $H_i(s)=1/s$

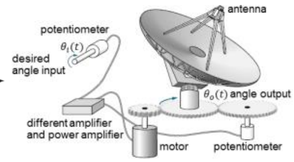



```
G1=tf(1,[1 1]); G2=G1; G3=G1;
H1=tf(1,[1 0]); H2=H1; H3=H1;
System=append(G1,G2,G3,H1,H2,H3);
input=1; output=3;
Q= [1 -4 0 0 0; 2 1 -5 0 0; 3 2 1 -5 -6;
4 2 0 0 0; 5 2 0 0 0; 6 3 0 0 0];
T=connect(System,Q,input,output);
T=tf(T); T=minreal(T)
```

§3. Analysis and Design of Feedback Systems

- Consider the system





which can model a control system such as the antenna azimuth position control system. For example, the transfer function, $K/s(s+a)$, can model the amplifiers, motor, load, and gears.

The closed-loop transfer function, $T(s)$, for this system

$$T(s) = \frac{K}{s^2 + as + K}$$

K : models the amplifier gain, that is, the ratio of the output voltage to the input voltage

§3. Analysis and Design of Feedback Systems

$$T(s) = \frac{K}{s^2 + as + K}$$

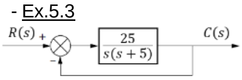
- As K varies, the poles move through the three ranges of operation of a second-order system
  - overdamped: $0 < K < a^2/4$ $\quad s_{1,2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2}$

    As K increases, the poles move along the real axis
  - critically damped: $K = a^2/4$ $\quad s_{1,2} = -\frac{a}{2}$
  - underdamped: $K > a^2/4$ $\quad s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2}$

    As K increases, the real part remains constant and the imaginary part increases. Thus, the peak time decreases and the percent overshoot increases, while the settling time remains constant

§3. Analysis and Design of Feedback Systems

- Ex.5.3 **Finding Transient Response**



Given the system, find the peak time, percent overshoot, settling time

Solution

The closed-loop transfer function

$$T(s) = \frac{25}{s^2 + 5s + 25} = \frac{5^2}{s^2 + 2 \times 0.5 \times 5s + 5^2}$$

and $\omega_n = \sqrt{25} = 5$, $\zeta = 0.5$. From these values of ζ and ω_n

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{5\sqrt{1-0.5^2}} = 0.726s$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 16.303$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5 \times 5} = 1.6s$$

§3. Analysis and Design of Feedback Systems

MATLAB ML

Run ch5p2 in Appendix B

Learn how to use MATLAB to

- perform block diagram reduction followed by an evaluation of the closed-loop system's transient response by finding, T_p, $\%OS$, and T_s
- generate a closed-loop step response
- solve Ex.5.3

§3. Analysis and Design of Feedback Systems

Simulink SL

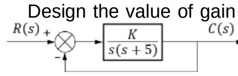
Learn how to use MATLAB's Simulink to

- explore the added capability of MATLAB's Simulink using Appendix C
- simulate feedback systems with nonlinearities through Ex.C.3 (p.842 Textbook)

§3. Analysis and Design of Feedback Systems

- Ex.5.4 **Gain Design for Transient Response**

Design the value of gain K for the feedback control system so that the system will respond with a 10% overshoot



Solution

The closed-loop transfer function

$$T(s) = \frac{\frac{K}{s(s+5)}}{1 + \frac{K}{s(s+5)}} = \frac{K}{s^2 + 5s + K}$$

$$= \frac{\left(\sqrt{K}\right)^2}{s^2 + 2 \times \frac{5}{2\sqrt{K}} \times \sqrt{K}s + \left(\sqrt{K}\right)^2}$$

and $\omega_n = \sqrt{K}$, $\zeta = 5/2\sqrt{K}$

§3. Analysis and Design of Feedback Systems

Percent overshoot is a function only of ζ

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 10\%$$

$$\Longrightarrow \zeta = 0.591$$

From this damping ratio

$$\zeta = \frac{5}{2\sqrt{K}}$$

$$\Longrightarrow K = \left(\frac{5}{2\zeta}\right)^2 = \left(\frac{5}{2 \times 0.591}\right)^2 = 17.9$$

Although we are able to design for percent overshoot in this problem, we could not have selected settling time as a design criterion because, regardless of the value of K, the real parts, -2.5, of the poles of $K/(s^2 + 5s + K)$ remain the same

§3. Analysis and Design of Feedback Systems

Skill-Assessment Ex.5.2

Problem For a unity feedback control system with a forward-path TF $G(s) = 16/s(s+a)$, design the value of a to yield a closed-loop step response that has 5% overshoot

WileyPLUS WPCS Control Solutions

Solution The closed-loop transfer function

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{16}{s^2 + as + 16} = \frac{4^2}{s^2 + 2 \times \frac{a}{8} \times 4s + 4^2}$$

and $\omega_n = 4$, $\zeta = a/8$

Percent overshoot

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

$$\Longrightarrow \zeta = \frac{-\ln(\%OS)}{\sqrt{\pi^2 + \ln^2(\%OS)}} = \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} = 0.69$$

$$\Longrightarrow a = 8\zeta = 8 \times 0.69 = 5.52$$

§3. Analysis and Design of Feedback Systems

TryIt 5.2

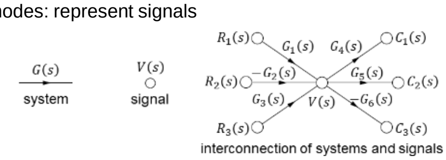
Use the following MATLAB and Control System Toolbox statements to find ζ, ω_n, $\%OS$, T_s, T_p, and T_r for the closed-loop unity feedback system described in Skill-Assessment Ex.5.2. Start with $a = 2$ and try some other values. A step response for the closed loop system will also be produced

$$G(s) = \frac{16}{s(s+a)}$$

```
a=2; numg=16; deng=poly([0 -a]);
G=tf(numg,deng);
T=feedback(G,1);
[numt,dent]=tfdata(T,'v');
wn=sqrt(dent(3))
z=dent(2)/(2*wn)
Ts=4/(z*wn)
Tp=pi/(wn*sqrt(1-z^2))
pos=exp(-z*pi/sqrt(1-z^2))*100
Tr=(1.76*z^3-0.417*z^2+1.039*z+1)/wn
step(T)
```


§4. Signal-Flow Graphs

- A signal-flow graph consists only of
  - branches: represent systems
  - nodes: represent signals



- A system is represented by a line with an arrow showing the direction of signal flow through the system. Adjacent to the line we write the transfer function. A signal is a node with the signal's name written adjacent to the node

§4. Signal-Flow Graphs

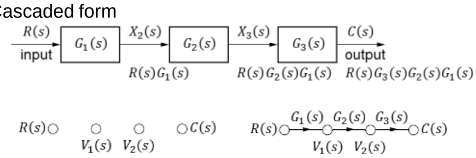
- Ex.5.5 Converting Common Block Diagrams to Signal-Flow Graphs

Convert the cascaded, parallel, and feedback forms of the following block diagrams into signal-flow graphs

Solution

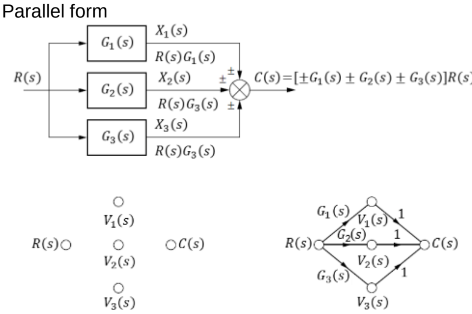
- Start by drawing the signal nodes for that system
- Next interconnect the signal nodes with system branches

a. Cascaded form



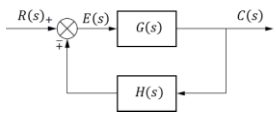
§4. Signal-Flow Graphs

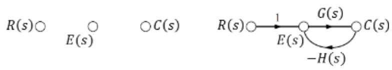
b. Parallel form



§4. Signal-Flow Graphs

c. Feedback form

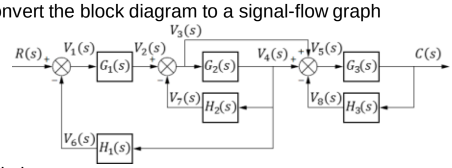




§4. Signal-Flow Graphs

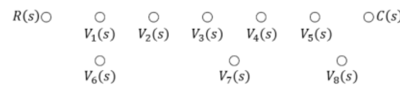
- Ex.5.6 Converting a Block Diagram to a Signal-Flow Graph

Convert the block diagram to a signal-flow graph

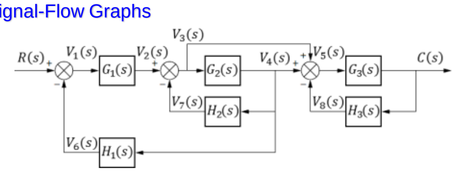


Solution

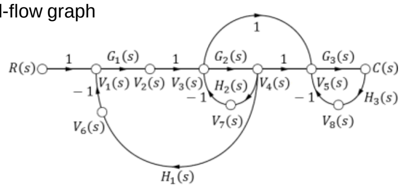
Signal nodes



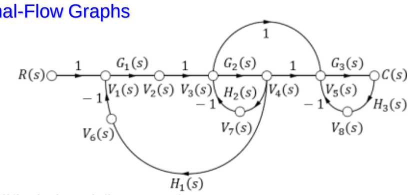
§4. Signal-Flow Graphs



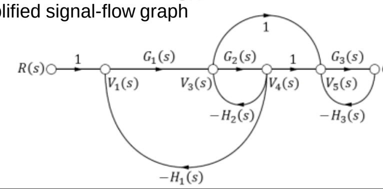
Signal-flow graph



§4. Signal-Flow Graphs



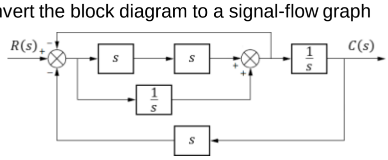
Simplified signal-flow graph



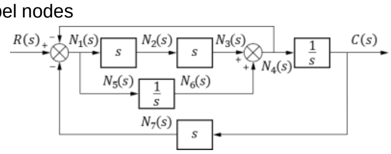
§4. Signal-Flow Graphs

Skill-Assessment Ex.5.3

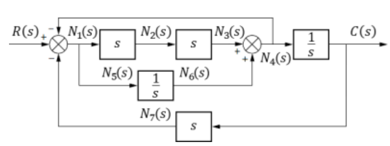
Problem Convert the block diagram to a signal-flow graph



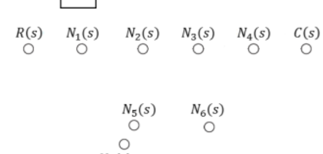
Solution Label nodes



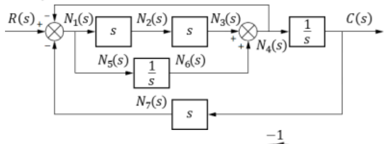
§4. Signal-Flow Graphs



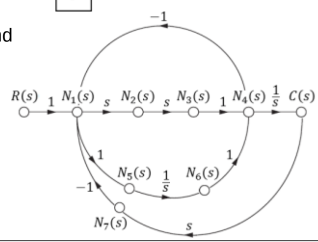
Draw nodes



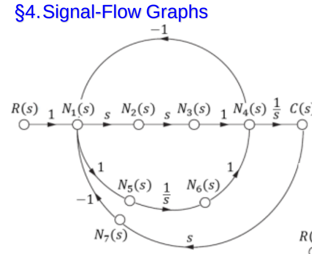
§4. Signal-Flow Graphs



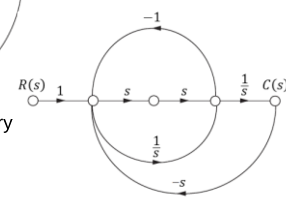
Connect nodes and label subsystems



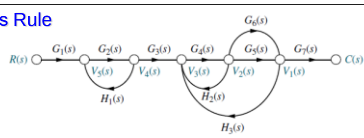
§4. Signal-Flow Graphs



Eliminate unnecessary



§5. Mason's Rule

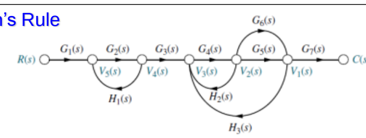


- Loop gain: the product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once

  Ex.

  $G_2(s)H_1(s)$
  $G_4(s)H_2(s)$
  $G_4(s)G_5(s)H_3(s)$
  $G_4(s)G_6(s)H_3(s)$

§5. Mason's Rule

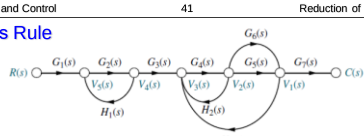


- Forward-path gain: the product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow

  Ex.

  $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
  $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

§5. Mason's Rule

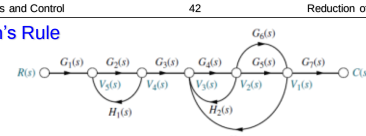


- Nontouching loops: loops that do not have any nodes in common

  Ex.

  Loop $G_2(s)H_1(s)$ does not touch loops $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$

§5. Mason's Rule



- Nontouching-loop gain: the product of loop gains from nontouching loops taken two, three, four, or more at a time

  Ex.

  The product of loop gain $G_2(s)H_1(s)$ and loop gain $G_4(s)H_2(s)$ is a nontouching-loop gain taken two at a time

  In summary, all three of the nontouching-loop gains taken two at a time $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
  $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
  $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

§5. Mason's Rule

- Mason's Rule

The transfer function, $C(s)/R(s)$, of a system represented by a signal-flow graph is

$$G(s)=\frac{C(s)}{R(s)}=\frac{\sum_k T_k\Delta_k}{\Delta}$$

k : number of forward paths

T_k : the k^{th} forward-path gain

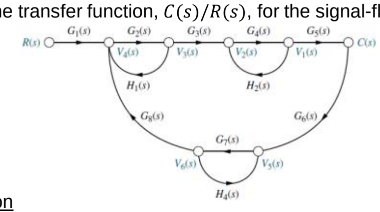
Δ : $1-\sum$ loop gains $+\sum$ nontouching-loop gains taken two at a time $-\sum$ nontouching-loop gains taken three at a time $+\sum$ nontouching-loop gains taken four at a time $-\cdots$

Δ_k: $\Delta-\sum$ loop gain terms in Δ that touch the k^{th} forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the k^{th} forward path

§5. Mason's Rule

- Ex.5.7 Transfer Function via Mason's Rule

Find the transfer function, $C(s)/R(s)$, for the signal-flow graph

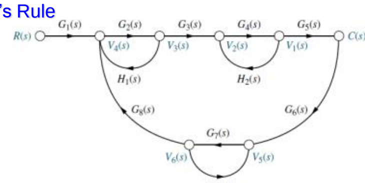


Solution

First, identify the forward-path gains

$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$

§5. Mason's Rule



Second, identify the loop gains

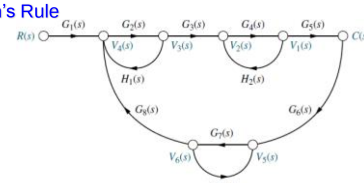
$G_2(s)H_1(s)$

$G_4(s)H_2(s)$

$G_7(s)H_4(s)$

$G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

§5. Mason's Rule



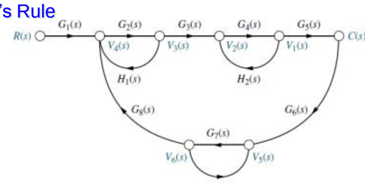
Third, identify the nontouching loops taken two at a time

- loop 1 does not touch loop 2: $G_2(s)H_1(s)G_4(s)H_2(s)$
- loop 1 does not touch loop 3: $G_2(s)H_1(s)G_7(s)H_4(s)$
- loop 2 does not touch loop 3: $G_4(s)H_2(s)G_7(s)H_4(s)$

Finally, the nontouching loops taken three at a time

- loops 1,2 and 3: $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

§5. Mason's Rule

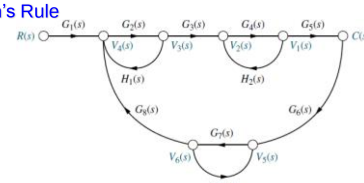


Form Δ

$$\begin{aligned}\Delta=1-[&G_2(s)H_1(s)+G_4(s)H_2(s)+G_7(s)H_4(s)\\&+G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)]\\+[&G_2(s)H_1(s)G_4(s)H_2(s)+G_2(s)H_1(s)G_7(s)H_4(s)\\&+G_4(s)H_2(s)G_7(s)H_4(s)]\\-[&G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned}$$

Δ : $1-\sum$ loop gains $+\sum$ nontouching-loop gains taken two at a time $-\sum$ nontouching-loop gains taken three at a time $+\sum$ nontouching-loop gains taken four at a time $-\cdots$

§5. Mason's Rule



Form Δ_k by eliminating from Δ the loop gains that touch the kth forward path

$\Delta_1=1-G_7(s)H_4(s)$

The transfer function

$$G(s)=\frac{T_1\Delta_1}{\Delta}=\frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1-G_7(s)H_4(s)]}{\Delta}$$

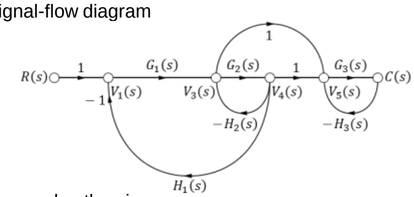
Δ_k: $\Delta-\sum$ loop gain terms in Δ that touch the k^{th} forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the k^{th} forward path

§5. Mason's Rule

Skill-Assessment Ex.5.4

Problem Use Mason's rule to find the transfer function of the signal-flow diagram

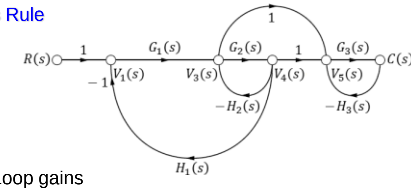
WileyPLUS WPCS Control Solutions



Solution Forward path gains

- $G_1G_2G_3$
- G_1G_3

§5. Mason's Rule



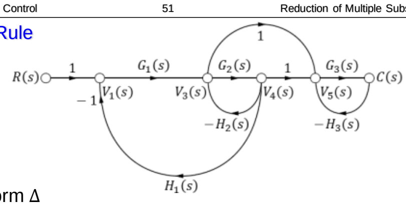
Loop gains

- $-G_1G_2H_1$
- $-G_2H_2$
- $-G_3H_3$

Nontouching loops

- $[-G_1G_2H_1][-G_3H_3] = G_1G_2G_3H_1H_3$
- $[-G_2H_2][-G_3H_3] = G_2G_3H_2H_3$

§5. Mason's Rule



Form Δ

$$\Delta = 1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3$$

Form Δ_k

$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

The transfer function

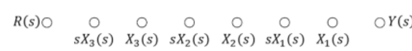
$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta} = \frac{G_1G_3[1+G_2]}{[1+G_2H_2+G_1G_2H_1][1+G_3H_3]}$$

§6. Signal-Flow Graphs of State Equations

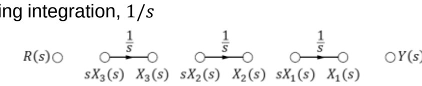
- Consider the following state and output equations

$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$

- First, identify state variables, x_1, x_2, and x_3; nodes, the input, r, and the output, y



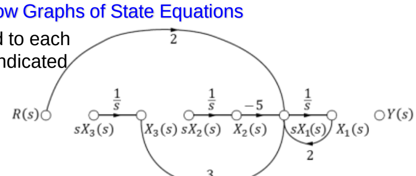
- Next interconnect the state variables and their derivatives with the defining integration, $1/s$



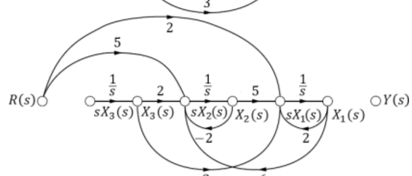
§6. Signal-Flow Graphs of State Equations

- Then, feed to each node the indicated signals

- $sX_1(s)$



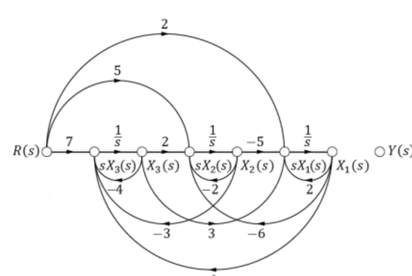
- $sX_2(s)$



$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$, $\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$

§6. Signal-Flow Graphs of State Equations

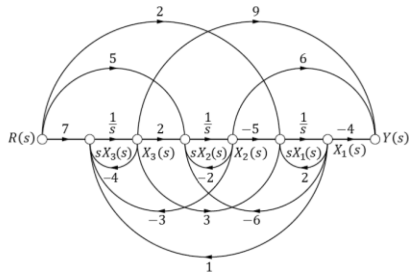
- $sX_3(s)$



$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$

§6. Signal-Flow Graphs of State Equations

- Finally, the output, y



$y = -4x_1 + 6x_2 + 9x_3$

§6. Signal-Flow Graphs of State Equations

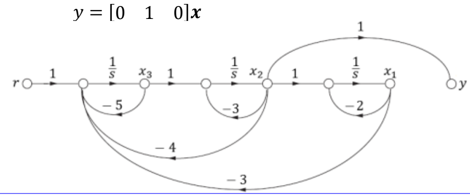
Skill-Assessment Ex.5.5

Problem Draw a signal-flow graph for the following state and output equations

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [0 \quad 1 \quad 0]\boldsymbol{x}$$

Solution

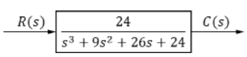


$\dot{x}_1 = -2x_1 + x_2$, $\dot{x}_2 = -3x_2 + x_3$, $\dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + r$, $y = x_2$

§7. Alternative Representations in State Space

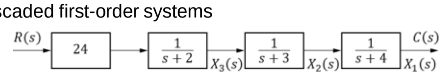
Cascade Form

- Consider the system



$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{24}{(s+2)(s+3)(s+4)} \quad (5.37)$$

- A block diagram representation of this system formed as cascaded first-order systems



Note: these state variables are not the phase variables

- Transforming each block into an equivalent differential equation and cross-multiplying

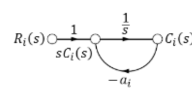
$$\frac{C_i(s)}{R_i(s)} = \frac{1}{s + a_i} \Longrightarrow (s + a_i)C_i(s) = R_i(s) \quad (5.39)$$

§7. Alternative Representations in State Space

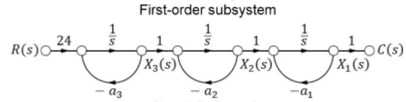
- Solving for $dc_i(t)/dt$ yields

$$(s + a_i)C_i(s) = R_i(s) \Longrightarrow \frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t)$$

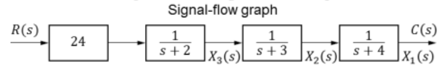
- Signal-flow graph



First-order subsystem

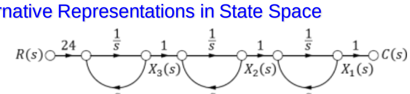


Signal-flow graph



Cascaded first-order systems

§7. Alternative Representations in State Space



- The state equations for the new representation of the system

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + 24r$$

with the system output

$$y = c(t) = x_1$$

- The state equations in vector-matrix form

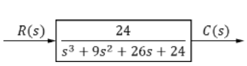
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0]\boldsymbol{x}$$

§7. Alternative Representations in State Space

Parallel Form

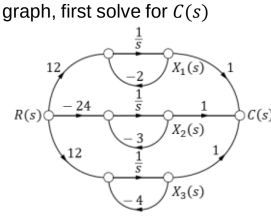
- Consider the system



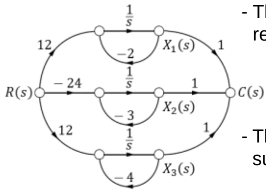
$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4} \quad (5.45)$$

- To arrive at a signal-flow graph, first solve for $C(s)$

$$C(s) = +R(s)\frac{12}{s+2} - R(s)\frac{24}{s+3} + R(s)\frac{12}{s+4}$$



§7. Alternative Representations in State Space



- The state equations for the new representation of the system

$$\dot{x}_1 = -2x_1 \qquad\qquad + 12r$$
$$\dot{x}_2 = \qquad -3x_2 \qquad - 24r$$
$$\dot{x}_3 = \qquad\qquad -4x_3 + 12r$$

- The output equation is found by summing the signals that give $c(t)$

$$y = c(t) = x_1 + x_2 + x_3$$

- The state equations in vector-matrix form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r \qquad (5.49)$$
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boldsymbol{x}$$

§7. Alternative Representations in State Space

MATLAB ML

Run ch5p3 in Appendix B

Learn how to use MATLAB to

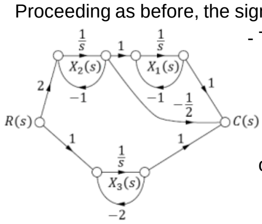
- use MATLAB to convert a transfer function to state space in a specified form
- solve the previous example by representing the transfer function in Eq.(5.45) by the state-space representation in parallel form of Eq.(5.49)

§7. Alternative Representations in State Space

- If the denominator of the TF has repeated real roots

$$\frac{C(s)}{R(s)} = \frac{s+3}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}$$

Proceeding as before, the signal-flow graph



- The state equations

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = \qquad -x_2 \qquad + 2r$$
$$\dot{x}_3 = \qquad\qquad -2x_3 + r$$
$$y = c(t) = x_1 - 0.5x_2 + x_3$$

or, in vector-matrix form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 1 & -0.5 & 1 \end{bmatrix} \boldsymbol{x}$$

- Note: the system matrix will not be diagonal

§7. Alternative Representations in State Space

Controller Canonical Form

- Consider the system

$$\frac{C(s)}{R(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24} \qquad (5.55)$$

- The phase-variable form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (5.56)$$

- Renumbering the phase variables in reverse order yields

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \quad (5.57)$$

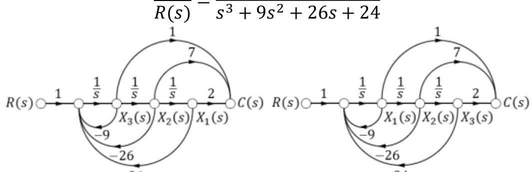
- Finally, rearranging in the controller canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -24 & -26 & -9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r, y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (5.58)$$

§7. Alternative Representations in State Space

- Signal-flow graphs for obtaining forms for

$$\frac{C(s)}{R(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$



phase-variable form controller canonical form

§7. Alternative Representations in State Space

TryIt 5.3

Use the following MATLAB and Control System Toolbox statements to convert the transfer function of Eq.(5.55) to the controller canonical state-space representation of Eqs. (5.58)

$$\frac{C(s)}{R(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24} \qquad (5.55)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -24 & -26 & -9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r \quad (5.58)$$

$$y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
numg=[1 7 2];
deng=[1 9 26 24];
[Acc,Bcc,Ccc,Dcc]=tf2ss(numg,deng)
```

§7.Alternative Representations in State Space

Observer Canonical Form

- Consider the system

$$\frac{C(s)}{R(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24}=\frac{\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}}{1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}} \qquad (5.59)$$

- Cross-multiplying yields

$$\left(\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right)R(s)=\left(1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right)C(s) \qquad (5.60)$$

- Combining terms of like powers of integration gives

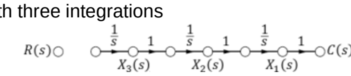
$$C=\frac{1}{s}\left\{(R-9C)+\frac{1}{s}\left[(7R-26C)+\frac{1}{s}(2R-24C)\right]\right\} (5.62)$$

This equation can be used to draw the signal-flow graph

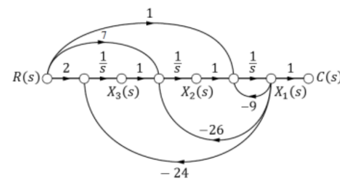
§7.Alternative Representations in State Space

$$C=\frac{1}{s}\left\{(R-9C)+\frac{1}{s}\left[(7R-26C)+\frac{1}{s}(2R-24C)\right]\right\} \qquad (5.62)$$

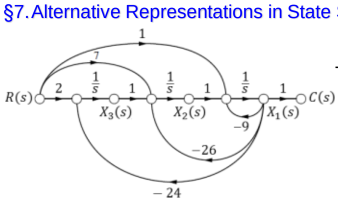
- Start with three integrations



- Signal-flow graph for observer canonical form variables



§7.Alternative Representations in State Space



- The state equation

$$\begin{aligned}\dot{x}_1&=-9x_1+x_2 &&+r\\ \dot{x}_2&=-26x_1 &&+x_3+7r\\ \dot{x}_3&=-24x_1 &&+2r\\ y&=c(t)=x_1\end{aligned}$$

- The state equations in vector-matrix form

$$\dot{\boldsymbol{x}}=\begin{bmatrix}-9&1&0\\-26&0&1\\-24&0&0\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}1\\7\\2\end{bmatrix}r \qquad (5.65)$$
$$y=\begin{bmatrix}1&0&0\end{bmatrix}\boldsymbol{x}$$

§7.Alternative Representations in State Space

TryIt 5.4
Use the following MATLAB and Control System Toolbox statements to convert the transfer function of Eq. (5.55) to the observer canonical state space representation of Eqs. (5.65)

$$\frac{C(s)}{R(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24} \qquad (5.55)$$

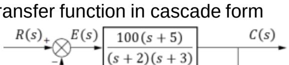
$$\dot{\boldsymbol{x}}=\begin{bmatrix}-9&1&0\\-26&0&1\\-24&0&0\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}1\\7\\2\end{bmatrix}r \qquad (5.65)$$
$$y=\begin{bmatrix}1&0&0\end{bmatrix}\boldsymbol{x}$$

```
numg=[1 7 2];
deng=[1 9 26 24];
[Acc,Bcc,Ccc,Dcc]=tf2ss(numg,deng);
Aoc=transpose(Acc)
Boc=transpose(Ccc)
Coc=transpose(Bcc)
```


§7.Alternative Representations in State Space

- Ex.5.8 State-Space Representation of Feedback Systems

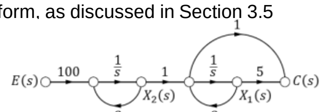
Represent the feedback control system in state space. Model the forward transfer function in cascade form



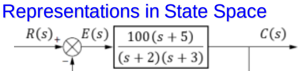
Solution

First, model the forward transfer function in cascade form

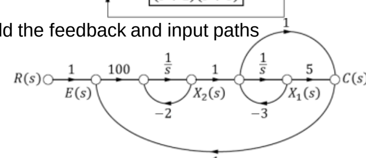
- The gain of 100, the pole at -2, -3 → in cascaded form
- The zero at -5 → obtained using the method for implementing zeros for a system represented in phase-variable form, as discussed in Section 3.5



§7.Alternative Representations in State Space



Next add the feedback and input paths



by inspection, write the state equations

$$\begin{aligned}\dot{x}_1&=-3x_1+x_2\\ \dot{x}_2&=\quad -2x_2+100(r-c)\end{aligned}$$

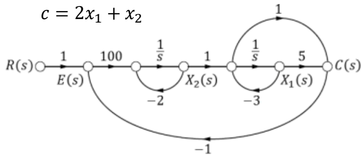
The output $c=5x_1+(x_2-3x_1)=2x_1+x_2$

Then

$$\begin{aligned}\dot{x}_1&=-3x_1+x_2\\ \dot{x}_2&=-200x_1-102x_2+100\end{aligned}$$

§7. Alternative Representations in State Space

Then
$$\dot{x}_1 = -3x_1 + x_2$$
$$\dot{x}_2 = -200x_1 - 102x_2 + 100$$
$$c = 2x_1 + x_2$$



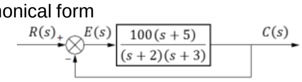
In vector-matrix form
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$
$$y = [2 \quad 1]\boldsymbol{x}$$

§7. Alternative Representations in State Space

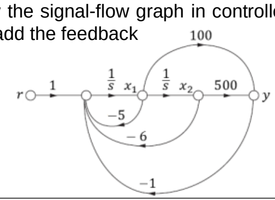
Skill-Assessment Ex.5.6

Problem Represent the feedback control system in state space. Model the forward transfer function in controller canonical form

WileyPLUS WPCS Control Solutions

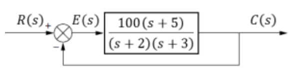


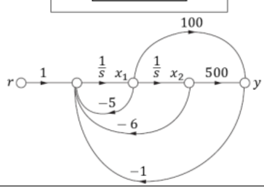
Solution Draw the signal-flow graph in controller canonical form and add the feedback



§7. Alternative Representations in State Space

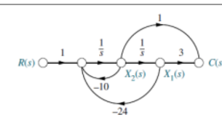
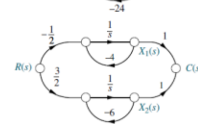
Writing the state equations from the signal-flow diagram
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -105 & -506 \\ 1 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$
$$y = [100 \quad 500]\boldsymbol{x}$$



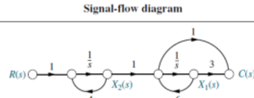
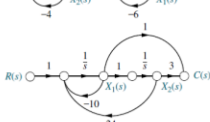
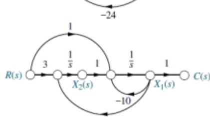


§7. Alternative Representations in State Space

- Writing the state equations from the signal-flow diagram
$$\frac{C(s)}{R(s)} = \frac{s+3}{(s+4)(s+6)}$$

Form	Transfer function	Signal-flow diagram	State equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$

§7. Alternative Representations in State Space

Form	Transfer function	Signal-flow diagram	State equations
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$

§8. Similarity Transformations

- A system represented in state space as
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

can be transformed to a similar system
$$\dot{\boldsymbol{z}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{P}\boldsymbol{z} + \boldsymbol{D}\boldsymbol{u}$$

For example, for 2-space
$$\boldsymbol{P} = [\boldsymbol{U}_{z_1} \quad \boldsymbol{U}_{z_1}] = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$
$$\boldsymbol{x} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \boldsymbol{P}\boldsymbol{z}$$

and
$$\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{x}$$

§8. Similarity Transformations

- Ex.5.9 Similarity Transformations on State Equations

Given the system represented in state space

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}$$

transform the system to a new set of state variables, z, where the new state variables are related to the original state variables, x, as follows

$$z_1 = 2x_1$$

$$z_2 = 3x_1 + 2x_2$$

$$z_3 = x_1 + 4x_2 + 5x_3$$

§8. Similarity Transformations

Solution

$$\begin{aligned} z_1 &= 2x_1 \\ z_2 &= 3x_1 + 2x_2 \\ z_3 &= x_1 + 4x_2 + 5x_3 \end{aligned} \quad \Rightarrow \boldsymbol{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\boldsymbol{x} = \boldsymbol{P}^{-1}\boldsymbol{x}$$

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\boldsymbol{CP} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}$$

§8. Similarity Transformations

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\boldsymbol{CP} = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}$$

The transformed system is

$$\dot{\boldsymbol{z}} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}u$$

$$y = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}\boldsymbol{z}$$

§8. Similarity Transformations

MATLAB ML

Run ch5p4 in Appendix B

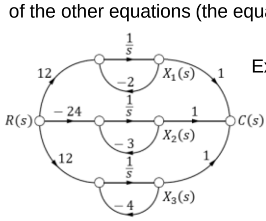
Learn how to use MATLAB to

- perform similarity transformations
- do Ex.5.9

§8. Similarity Transformations

Diagonalizing a System Matrix

- The parallel form of a signal-flow graph can yield a diagonal system matrix
- Advantage: each state equation is a function of only one state variable ⟹ each differential equation can be solved independently of the other equations (the equations are decoupled)



Example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix}r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\boldsymbol{x}$$

§8. Similarity Transformations

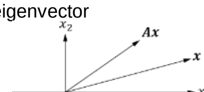
Diagonalizing a System Matrix

- Eigenvector

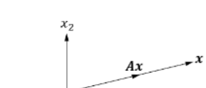
The eigenvectors of the matrix A are all vectors, $\boldsymbol{x}_i \neq \boldsymbol{0}$, which under the transformation A become multiples of themselves; that is

$$\boldsymbol{Ax}_i = \lambda_i\boldsymbol{x}_i, \quad \lambda_i \text{ : constant} \tag{5.80}$$

- If $\boldsymbol{Ax}$ is not collinear with $\boldsymbol{x}$ after the transformation, $\boldsymbol{x}$ is not an eigenvector
- If $\boldsymbol{Ax}$ is collinear with $\boldsymbol{x}$ after the transformation, $\boldsymbol{x}$ is an eigenvector



$\boldsymbol{x}$ is not an eigenvector



$\boldsymbol{x}$ is an eigenvector

§8. Similarity Transformations

- Eigenvalue

The eigenvalues of the matrix $\boldsymbol{A}$ are the values of λ_i that satisfy

$$\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i, \quad \lambda_i : \text{constant} \qquad (5.80)$$

for $\boldsymbol{x}_i \neq \boldsymbol{0}$

- To find the eigenvectors, rearrange Eq. (5.80). Eigenvectors, λ_i, satisfy

$$\boldsymbol{0} = (\lambda_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}_i \qquad (5.81)$$

$$\boldsymbol{x}_i = (\lambda_i \boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{0} = \frac{\mathrm{adj}(\lambda_i \boldsymbol{I} - \boldsymbol{A})}{\det(\lambda_i \boldsymbol{I} - \boldsymbol{A})}\boldsymbol{0}$$

Since $\boldsymbol{x}_i \neq \boldsymbol{0}$, a nonzero solution exists if

$$\det(\lambda_i \boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{0} \qquad (5.83)$$

From which λ_i, the eigenvalues, can be found

§8. Similarity Transformations

- Ex.5.10 Finding Eigenvectors

Find the eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

Solution

The eigenvectors, $\boldsymbol{x}_i$, satisfy Eq. (5.81). First, use $\det(\lambda_i \boldsymbol{I} - \boldsymbol{A}) = 0$ to find the eigenvalues, λ_i, for Eq. (5.81)

$$\begin{aligned} \det(\lambda_i \boldsymbol{I} - \boldsymbol{A}) &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 8 \\ &= (\lambda + 2)(\lambda + 4) \end{aligned}$$

from which the eigenvalues are $\lambda = -2$, and $\lambda = -4$

$\boldsymbol{0} = (\lambda_i \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}_i$ (5.81)

§8. Similarity Transformations

Using Eq. (5.80) successively with each eigenvalue, we have

$$\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$$

Using eigenvalue $\lambda = -2$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{aligned} -3x_1 + \ x_2 &= -2x_1 \\ x_1 - 3x_2 &= -2x_2 \end{aligned}$$

From which $x_1 = x_2$. Thus $\boldsymbol{x} = \begin{bmatrix} c \\ c \end{bmatrix}$

Using eigenvalue $\lambda = -4$, $\boldsymbol{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$

One choice of eigenvectors is $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$, λ_i: constant (5.80)

§8. Similarity Transformations

MATLAB ML

Run ch5p5 in Appendix B

Learn how to use MATLAB to diagonalize a system, is similar (but not identical) to Ex.5.11

§8. Similarity Transformations

Skill-Assessment Ex.5.7

Problem For the system represented in state space as follows

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}u, \ y = [1 \quad 4]\boldsymbol{x}$$

convert the system to one where the new state vector

$$\boldsymbol{z} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\boldsymbol{x}$$

Solution

$$\boldsymbol{P}^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix} \Rightarrow \boldsymbol{P} = \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$$

$$\boldsymbol{C}\boldsymbol{P} = [1 \quad 4]\begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = [0.8 \quad -1.4]$$

§8. Similarity Transformations

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$$

$$\boldsymbol{C}\boldsymbol{P} = [0.8 \quad -1.4]$$

The transformed system is

$$\dot{\boldsymbol{z}} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} -3 \\ -11 \end{bmatrix}u$$

$$y = [0.8 \quad -1.4]\boldsymbol{z}$$

§8. Similarity Transformations

Skill-Assessment Ex.5.8

Problem For the system represented in state space as follows

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}u,\ y = \begin{bmatrix} 1 & 4 \end{bmatrix}\boldsymbol{x}$$

find the diagonal system that is similar

Solution First find the eigenvalues

$$|\lambda_i \boldsymbol{I} - \boldsymbol{A}| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -3 \\ 4 & \lambda+6 \end{bmatrix}$$
$$= \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3)$$

From which the eigenvalues are -2 and -3

Now use $\boldsymbol{A}\boldsymbol{x}_i = \lambda \boldsymbol{x}_i$ for each eigenvalue, λ

Thus,

$$\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

§8. Similarity Transformations

$$\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For $\lambda = -2$

$$3x_1 + 3x_2 = 0$$
$$-4x_1 - 4x_2 = 0$$
$$\Longrightarrow x_1 = -x_2$$

For $\lambda = -3$

$$4x_1 + 3x_2 = 0$$
$$-4x_1 - 3x_2 = 0$$
$$\Longrightarrow x_1 = -0.75x_2$$

Let

$$\boldsymbol{P} = \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} \Longrightarrow \boldsymbol{P}^{-1} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix}$$

§8. Similarity Transformations

$$\boldsymbol{P} = \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} \Longrightarrow \boldsymbol{P}^{-1} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix}$$

Hence

$$\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$$
$$= \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$
$$\boldsymbol{P}^{-1}\boldsymbol{B} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}$$
$$\boldsymbol{C}\boldsymbol{P} = \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} = \begin{bmatrix} -2.121 & 2.6 \end{bmatrix}$$

The transformed system is

$$\dot{\boldsymbol{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}u$$
$$y = \begin{bmatrix} -2.121 & 2.6 \end{bmatrix}\boldsymbol{z}$$

§7. Alternative Representations in State Space

TryIt 5.5

Use the following MATLAB and Control System Toolbox statements to do Skill-Assessment Ex.5.8

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}u,\ y = \begin{bmatrix} 1 & 4 \end{bmatrix}\boldsymbol{x}$$

```
A=[1 3;-4 -6];
B=[1;3];
C=[1 4];
D=0;S=ss(A,B,C,D);
Sd=canon(S, 'modal')
```