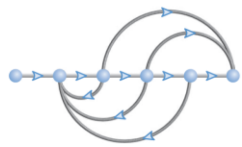
Stability

6



Learning Outcome

After completing this chapter, the student will be able to

- Make and interpret a basic Routh table to determine the stability of a system
- Make and interpret a Routh table where either the first element of a row is zero or an entire row is zero
- Use a Routh table to determine the stability of a system represented in state space

§1. Introduction

- Three requirements enter into the design of a control system
  - transient response
  - stability, and
  - steady-state errors
- Stability is the most important system specification. If a system is unstable, transient response and steady-state errors are moot points. An unstable system cannot be designed for a specific transient response or steady-state error requirement
- What, then, is stability? There are many definitions for stability, depending upon the kind of system or the point of view. In this section, discussion is limited to linear, time-invariant (LTI) systems

§1. Introduction

- The output of a system can be controlled if the steady-state response consists of only the forced response. But the total response of a system is the sum of the forced and natural responses, or

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

- There are two definitions for stability, using
  - the natural response, and
  - the total response (BIBO)

§1. Introduction

- Consider the general TF

$$G(s) = \frac{R(s)}{C(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

The response

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

$c_{forced}(t)$: the forced response

$c_{natural}(t)$: the natural response $c_{natural}(t) = \sum_{i=1}^{n} \lambda_i e^{p_i t}$

p_i : the poles of the system, or the roots of $D(s) = 0$

The system is

- stable if $\lim_{t\to\infty} c_{natural}(t) \to 0$
- unstable if $\lim_{t\to\infty} c_{natural}(t) \to \infty$

§1. Introduction

- Using the natural response, a linear, time-invariant system is
  - stable if the natural response approaches zero as time approaches infinity
  - unstable if the natural response approaches infinity as time approaches infinity
  - marginally stable if the natural response neither decays nor grows but remains constant or oscillates
- Using the total response (BIBO), a linear, time-invariant system is
  - stable if every bounded input yields a bounded output
  - unstable if any bounded input yields an unbounded output

§1. Introduction

- How do we determine if a system is stable?

Poles placed in the left half-plane (LHP)

Poles placed in LHP yield either pure exponential decay or damped sinusoidal natural responses. These natural responses decay to zero as $t \to \infty$

→ if the closed-loop system poles are in the LHP and hence have a negative real part, the system is stable

Poles in the right half-plane (RHP)

Poles in RHP yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses. These natural responses approach infinity as $t \to \infty$

→ if the closed-loop system poles are in the RHP and hence have a positive real part, the system is unstable

§2. Routh-Hurwitz Criterion

- Routh-Hurwitz criterion for stability (Routh, 1905)
  - Generate a data table called a Routh table, and
  - Interpret the Routh table to tell how many closed-loop system poles are in LHP, RHP, and on the $j\omega$-axis
- Why study the Routh-Hurwitz criterion when modern calculators and computers can tell us the exact location of system poles?
  - The power of the method lies in design rather than analysis.

  For example, it is easy to determine the range of the unknown parameter in the denominator of a TF to yield stability

§2. Routh-Hurwitz Criterion

Generating a Basic Routh Table

- Look at the equivalent closed-loop TF

$$R(s) \rightarrow \boxed{\frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \rightarrow C(s)$$

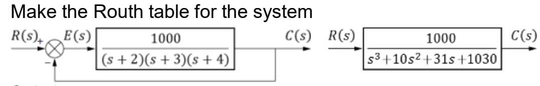
- Create the Routh table

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	$b_1 = \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_2 = \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	$b_3 = \cdots$	
s^{n-3}	$c_1 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1}$	$c_2 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1}$	$c_3 = \cdots$	
$\cdots$	$\cdots$	$\cdots$		
s^0		0		

§2. Routh-Hurwitz Criterion

- Ex.6.1 Creating a Routh Table

Make the Routh table for the system



Solution

Find the equivalent closed-loop system

Create the Routh table

s^3	1	31	0
s^2	10	1030	0
s^1	$\frac{-\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{10} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0$	
s^0	$\frac{-\begin{vmatrix} 10 & 1030 \\ -72 & 0 \end{vmatrix}}{-72} = 1030$		

§2. Routh-Hurwitz Criterion

Interpreting the Basic Routh Table

- Routh-Hurwitz criterion

number of roots in the RHP = number of sign changes in the first column

- Ex. two sign changes in the first column

  two poles exist in the RHP

  ⟹ the system is unstable

s^3	1	31	0
s^2	10	1030	0
s^1	$\frac{-\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{10} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0$	
s^0	$\frac{-\begin{vmatrix} 10 & 1030 \\ -72 & 0 \end{vmatrix}}{-72} = 1030$		

§2. Routh-Hurwitz Criterion

Note

For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below

s^3	1	31	0
s^2	1 $\cancel{10}$	103 $\cancel{1030}$	0
s^1	$\frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{1} = 0$

§2. Routh-Hurwitz Criterion

Skill-Assessment Ex.6.1

Problem Make a Routh table and tell how many roots of the following polynomial are in the RHP and in the LHP

WileyPLUS WPCS Control Solutions

$$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

Solution Create the Routh table

s^7	3	6	7	2	0
s^6	9	4	8	6	
s^5	$\frac{14}{3}$	$\frac{39}{9}$	0	0	0
s^4	$-\frac{61}{14}$	8	6	0	0
s^3	$\frac{787}{61}$	$\frac{392}{61}$	0	0	

§2. Routh-Hurwitz Criterion

s^7	3	6	7	2	0
s^6	9	4	8	6	
s^5	$\frac{14}{3}$	$\frac{39}{9}$	0	0	0
s^4	$-\frac{61}{14}$	8	6	0	0
s^3	$\frac{787}{61}$	$\frac{392}{61}$	0	0	
s^2	$\frac{8004}{787}$	6	0		
s^1	$-\frac{1581}{1334}$	0			
s^0	6				

→ Four poles in the RHP and three poles in the LHP

$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$

§3. Routh-Hurwitz Criterion: Special Cases

Zero Only in the First Column

- An epsilon, ε, is assigned to replace the zero in the first column. The value ε is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined
- Ex.6.2 Stability via Epsilon Method

Determine the stability of the closed-loop TF

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Solution

s^5	1	3	5
s^4	2	6	3
s^3	$\frac{-\begin{vmatrix}1 & 3\\ 2 & 6\end{vmatrix}}{2} = 0$	$\frac{-\begin{vmatrix}1 & 5\\ 2 & 3\end{vmatrix}}{2} = -3.5$	$\frac{-\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix}}{2} = 0$

§3. Routh-Hurwitz Criterion: Special Cases

s^5	1	3	5
s^4	2	6	3
s^3	ε	$\frac{-\begin{vmatrix}1 & 5\\ 2 & 3\end{vmatrix}}{2} = -3.5$	$\frac{-\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix}}{2} = 0$
s^2	$\frac{-\begin{vmatrix}2 & 6\\ \varepsilon & -3.5\end{vmatrix}}{\varepsilon} = \frac{6\varepsilon - 7}{\varepsilon}$	$\frac{-\begin{vmatrix}2 & 3\\ \varepsilon & 0\end{vmatrix}}{\varepsilon} = 3$	$\frac{-\begin{vmatrix}2 & 0\\ \varepsilon & 0\end{vmatrix}}{\varepsilon} = 0$
s^1	$\frac{-\begin{vmatrix}\varepsilon & -3.5\\ \frac{6\varepsilon-7}{\varepsilon} & 3\end{vmatrix}}{\frac{6\varepsilon-7}{\varepsilon}} = \frac{42\varepsilon - 49 - 6\varepsilon^2}{12\varepsilon - 14}$	$\frac{-\begin{vmatrix}\varepsilon & 0\\ \frac{6\varepsilon-7}{\varepsilon} & 0\end{vmatrix}}{\frac{6\varepsilon-7}{\varepsilon}} = 0$	
s^0	$\frac{-\begin{vmatrix}\frac{6\varepsilon-7}{\varepsilon} & 3\\ \frac{42\varepsilon-49-6\varepsilon^2}{12\varepsilon-14} & 0\end{vmatrix}}{\frac{42\varepsilon-49-6\varepsilon^2}{12\varepsilon-14}} = 3$		

§3. Routh-Hurwitz Criterion: Special Cases

Label	First column	$\varepsilon = +$	$\varepsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	ε	$+$	$-$
s^2	$\frac{6\varepsilon - 7}{\varepsilon}$	$-$	$+$
s^1	$\frac{42\varepsilon - 49 - 6\varepsilon^2}{12\varepsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

The system is unstable and has two poles in the RHP

§3. Routh-Hurwitz Criterion: Special Cases

TryIt 6.1

Use the following MATLAB statement to find the poles of the closed-loop transfer function in Eq. (6.2)

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \quad (6.2)$$

```
roots([1 2 3 6 5 3])
```

§3. Routh-Hurwitz Criterion: Special Cases

MATLAB ML

Run ch6sp1 in Appendix F

Learn how to use MATLAB to

- use MATLAB to calculate the values of cells in a Routh table even if the table contains symbolic objects, such as ε
- see that the Symbolic Math Toolbox and MATLAB yield an alternate way to generate the Routh table for Ex.6.2

§3. Routh-Hurwitz Criterion: Special Cases

Zero Only in the First Column - Alternative method (Phillips, 1991)

- Ex.6.3 Stability via Reverse Coefficients

Determine the stability of the closed-loop TF

$$T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3} \qquad (6.6)$$

Solution

Write a polynomial that has the reciprocal roots of the denominator of Eq. (6.6)

$$s^5+2s^4+3s^3+6s^2+5s+3$$

$$D(s)=3s^5+5s^4+6s^3+3s^2+2s+1 \qquad (6.7)$$

§3. Routh-Hurwitz Criterion: Special Cases

$$s^5+2s^4+3s^3+6s^2+5s+3$$

$$D(s)=3s^5+5s^4+6s^3+3s^2+2s+1 \qquad (6.7)$$

Form the Routh table using Eq. (6.7)

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

Since there are two sign changes, the system is unstable and has two right-half-plane poles

This is the same as the result obtained in Ex.6.2. Notice that the above table does not have a zero in the first column

§3. Routh-Hurwitz Criterion: Special Cases

Entire Row is Zero

- Ex.6.4 Stability via Routh Table with Row of Zeros

Determine the number of RHP poles in the closed-loop TF

$$T(s)=\frac{10}{s^5+7s^4+6s^3+42s^2+8s+56}$$

Solution

s^5	1	6	8
s^4	1 $\cancel{7}$	6 $\cancel{42}$	8 $\cancel{56}$
s^3	1 4 $\cancel{0}$	3 $\cancel{12}$ $\cancel{0}$	0 $\cancel{0}$ $\cancel{0}$
s^2	3	8	0
s^1	1/3	0	0
s^0	8	0	0

$\Rightarrow$ no RHP poles

Replace all row of zeros by $P(s)=s^4+6s^2+8$

Differentiate $P(s)$ with respect to s $dP(s)/ds=4s^3+12s+0$

§3. Routh-Hurwitz Criterion: Special Cases

- Ex.6.5 Pole Distribution via Routh Table with Row of Zeros

Determine the position of the TF

$$T(s)=\frac{20}{s^8+s^7+12s^6+22s^5+39s^4+59s^3+48s^2+38s+20}$$

Solution

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	−1 $\cancel{-10}$	−2 $\cancel{-20}$	1 $\cancel{10}$	2 $\cancel{20}$	0
s^5	1 $\cancel{20}$	3 $\cancel{60}$	2 $\cancel{40}$	0	0
s^4	1	3	2	0	0
s^3	2 4 $\cancel{0}$	3 6 $\cancel{0}$	0 $\cancel{0}$	0	0
s^2	3 $\cancel{3/2}$	4 $\cancel{2}$	0	0	0
s^1	1/3	0	0	0	0
s^0	4	0	0	0	0

§3. Routh-Hurwitz Criterion: Special Cases

Interpreting the Routh table

- The sub-polynomial $P(s)=s^4+3s^2+2=(s^2+1)(s^2+2)=0$ has 4 imaginary roots → 4 poles on the $j\omega$-axis
- Two signs changed in the first column → 2 poles on RHP
- 2 poles on LHP

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	−1 $\cancel{-10}$	−2 $\cancel{-20}$	1 $\cancel{10}$	2 $\cancel{20}$	0
s^5	1 $\cancel{20}$	3 $\cancel{60}$	2 $\cancel{40}$	0	0
s^4	1	3	2	0	0
s^3	2 4 $\cancel{0}$	3 6 $\cancel{0}$	0 $\cancel{0}$	0	0
s^2	3 $\cancel{3/2}$	4 $\cancel{2}$	0	0	0
s^1	1/3	0	0	0	0
s^0	4	0	0	0	0

§3. Routh-Hurwitz Criterion: Special Cases

Skill-Assessment Ex.6.2

Problem Find how many poles of the following closed-loop system, $T(s)$, are in the RHP, LHP, and on the $j\omega$-axis

$$T(s)=\frac{s^3+7s^2-21s+10}{s^6+s^5-6s^4-s^2-s+6}$$

Solution Create the Routh table

s^6	1	-6	-1	6
s^5	1	0	-1	0
s^4	-6	0	6	0
s^3	-24 $\cancel{0}$	0	0	0
s^2	ε $\cancel{0}$	6	0	0
s^1	$\frac{144}{\varepsilon}$	0	0	0
s^0	6	0	0	0

§3. Routh-Hurwitz Criterion: Special Cases

Interpreting the Routh table

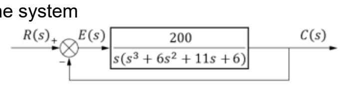
- $P(s)=-6s^4+6=-6(s^2+1)(s^2-1)=0$ has 2 imaginary roots → 2 poles on the $j\omega$-axis
- There is two sign change in the first column → the polynomial has two RHP pole
- Two left poles in the LHP

s^6	1	-6	-1	6
s^5	1	0	-1	0
s^4	-6	0	6	0
s^3	-24 $\cancel{0}$	0	0	0
s^2	ε $\cancel{0}$	6	0	0
s^1	$\frac{144}{\varepsilon}$	0	0	0
s^0	6	0	0	0

§4. Routh-Hurwitz Criterion: Additional Examples

- Ex.6.6 Standard Routh-Hurwitz

Find the number of poles in the LHP, the RHP, and on the $j\omega$-axis for the system



Solution

The closed-loop TF

$$T(s)=\frac{200}{s^4+6s^3+11s^2+6s+200)}$$

The Routh table for the denominator

s^4	1	11	200
s^3	1 $\cancel{6}$	1 $\cancel{6}$	
s^2	1 $\cancel{10}$	20 $\cancel{200}$	
s^1	-19		
s^0	20		

§4. Routh-Hurwitz Criterion: Additional Examples

Interpreting the Routh table

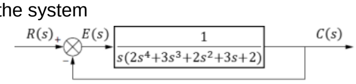
- No zero row
  → there is no pole on the $j\omega$-axis
- There is two sign change in the first column
  → the polynomial has two RHP pole
- The system has 4 poles
  → two left poles in the LHP

s^4	1	11	200
s^3	1 $\cancel{6}$	1 $\cancel{6}$	
s^2	1 $\cancel{10}$	20 $\cancel{200}$	
s^1	-19		
s^0	20		

§4. Routh-Hurwitz Criterion: Additional Examples

- Ex.6.7 Routh-Hurwitz with Zero in First Column

Find the number of poles in the LHP, the RHP, and on the $j\omega$-axis for the system



Solution

The closed-loop TF

$$T(s)=\frac{\frac{1}{s(2s^4+3s^3+2s^2+3s+2)}}{1+\frac{1}{s(2s^4+3s^3+2s^2+3s+2)}}$$

$$=\frac{1}{2s^5+3s^4+2s^3+3s^2+2s+1}$$

§4. Routh-Hurwitz Criterion: Additional Examples

The Routh table for the denominator

s^5	2	2	2
s^4	3	3	1
s^3	ε $\cancel{0}$	$4/3$	
s^2	$\frac{3\varepsilon-4}{\varepsilon}$	1	
s^1	$\frac{12\varepsilon-16-3\varepsilon^2}{9\varepsilon-12}$		
s^0	1		

$$0<\varepsilon\ll 1\rightarrow\frac{3\varepsilon-4}{\varepsilon}<0,\frac{12\varepsilon-16-3\varepsilon^2}{9\varepsilon-12}>0$$

There are two sign changes, and the system is unstable, with two poles in the RHP. The remaining poles are in the LHP

$$T(s)=\frac{1}{2s^5+3s^4+2s^3+3s^2+2s+1}$$

§4. Routh-Hurwitz Criterion: Additional Examples

Alternative solution using the reciprocal roots

$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$$

s^5		1	3	3
s^4		2	2	2
s^3		2	2	
s^2	ε	$\cancel{0}$	2	
s^1		$\frac{2\varepsilon-4}{\varepsilon}$		
s^0		2		

$$0 < \varepsilon \ll 1 \rightarrow \frac{2\varepsilon - 4}{\varepsilon} < 0$$

There are two sign changes, and the system is unstable, with two poles in the RHP. The remaining poles are in the LHP

$T(s) = \frac{1}{2s^5+3s^4+2s^3+3s^2+2s+1}$

§4. Routh-Hurwitz Criterion: Additional Examples

MATLAB ML

Run ch6p1 in Appendix B

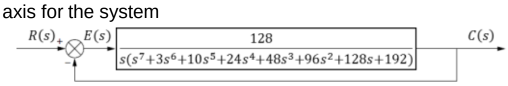
Learn how to use MATLAB to

- perform block diagram reduction to find $T(s)$, followed by an evaluation of the closed-loop system's poles to determine stability
- do Ex.6.7

§4. Routh-Hurwitz Criterion: Additional Examples

- Ex.6.8 Routh-Hurwitz with Row of Zeros

Find the number of poles in the LHP, the RHP, and on the $j\omega$-axis for the system



Solution

The closed-loop TF

$$T(s) = \frac{G(s)}{1+G(s)}$$

$$= \frac{128}{s^8+3s^7+10s^6+24s^5+48s^4+96s^3+128s^2+192s+128)}$$

§4. Routh-Hurwitz Criterion: Additional Examples

The Routh table for the denominator

s^8			1			10			48			128	128
s^7		1	~~3~~		8	~~24~~		32	~~96~~		64	~~192~~	
s^6		1	~~2~~		8	~~16~~		32	~~64~~		64	~~128~~	
s^5	3	~~6~~	~~0~~	16	~~32~~	~~0~~	32	~~64~~	~~0~~	0	~~0~~	~~0~~	
s^4		1	~~8/3~~		8	~~64/3~~		24	~~64~~				
s^3		−1	~~−8~~		−5	~~−40~~							
s^2		1	~~3~~		8	~~24~~							
s^1			3										
s^0			8										

$$P(s) = s^6 + 8s^4 + 32s^2 + 64$$
$$dP(s)/ds = 6s^5 + 32s^3 + 64s + 0$$

$T(s) = \frac{128}{s^8+3s^7+10s^6+24s^5+48s^4+96s^3+128s^2+192s+128)}$

§4. Routh-Hurwitz Criterion: Additional Examples

Interpreting the Routh table

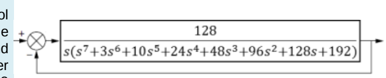
s^8			1			10			48			128	128
s^7		1	~~3~~		8	~~24~~		32	~~96~~		64	~~192~~	
s^6		1	~~2~~		8	~~16~~		32	~~64~~		64	~~128~~	
s^5	3	~~6~~	~~0~~	16	~~32~~	~~0~~	32	~~64~~	~~0~~	0	~~0~~	~~0~~	
s^4		1	~~8/3~~		8	~~64/3~~		24	~~64~~				
s^3		−1	~~−8~~		−5	~~−40~~							
s^2		1	~~3~~		8	~~24~~							
s^1			3										
s^0			8										

- $P(s) = (s^4 + 4s^2 + 16)(s^2 + 4) \rightarrow$ 2 poles on the $j\omega$-axis
- Two sign change in the first column → 2 RHP pole
- The system has 8 poles → ~~6~~ 4 left poles in the LHP

§4. Routh-Hurwitz Criterion: Additional Examples

TryIt 6.2

Use MATLAB, The Control System Toolbox, and the following statements to find the closed-loop transfer function, $T(s)$, for Ex.6.8 and the closed-loop poles

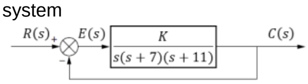



```
numg=128;
deng=[1 3 10 24 48 96 128 192 0];
G=tf(numg,deng);
T=feedback(G,1)
poles=pole(T)
```

§4. Routh-Hurwitz Criterion: Additional Examples

- Ex.6.9 Stability Design via Routh-Hurwitz

Find the number of poles in the LHP, the RHP, and on the $j\omega$-axis for the system



Solution

The closed-loop TF

$$T(s)=\frac{\frac{K}{s(s+7)(s+11)}}{1+\frac{K}{s(s+7)(s+11)}}=\frac{K}{s^3+18s^2+77s+K}$$

§4. Routh-Hurwitz Criterion: Additional Examples

The Routh table for the denominator

s^3	1	77
s^2	18	K
s^1	$\frac{1386-K}{18}$	
s^0	K	

- $K>1386$ the system is unstable
- $K<1386$ the system is stable
- $K=1386$

s^3	1	77
s^2	18	1386
s^1	36 $\cancel{0}$	
s^0	1386	

$P(s)=18s^2+1386$: 2 poles in the $j\omega$-axis, 1 pole in the LHP

$T(s)=\frac{K}{s^3+18s^2+77s+K}$

§4. Routh-Hurwitz Criterion: Additional Examples

Run ch6p2 in Appendix B

Learn how to use MATLAB to

- set up a loop to search for the range of gain to yield stability
- do Ex.6.9

§4. Routh-Hurwitz Criterion: Additional Examples

Learn how to use the Symbolic Math Toolbox to

- run ch6sp2 in Appendix F
- calculate the values of cells in a Routh table even if the table contains symbolic objects, such as a variable gain, K
- solve Ex.6.9

§4. Routh-Hurwitz Criterion: Additional Examples

The Routh-Hurwitz criterion is often used in limited applications to factor polynomials containing even factors

- Ex.6.10 Factoring via Routh-Hurwitz

Factor the polynomial $P_0(s)=s^4+3s^3+30s^2+30s+200$

Solution

The Routh table

s^4		1		30	200
s^3	1	$\cancel{3}$	$\cancel{10}$	$\cancel{30}$	
s^2	$\cancel{1}$	$\cancel{20}$	10	$\cancel{200}$	0
s^1	2	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	0
s^0		10			0

From the zero row $P(s)=s^2+10$, then

$$\begin{aligned}P_0(s)&=s^4+3s^3+30s^2+30s+200\\&=(s^2+10)(s^2+3s+20)\\&=(s+j3.1623)(s-j3.1623)\\&\quad\times(s+15+j4.213)(s+15-j4.213)\end{aligned}$$

§4. Routh-Hurwitz Criterion: Additional Examples

Skill-Assessment Ex.6.3

Problem For a unity feedback system with the forward TF

WileyPLUS WPCS Control Solutions

$$G(s)=\frac{K(s+20)}{s(s+2)(s+3)}$$

find the range of K to make the system stable

Solution The closed-loop TF

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{K(s+20)}{s^3+5s^2+(6+K)s+20K}$$

The Routh table

s^3	1	$6+K$
s^2	5	$20K$
s^1	$\frac{30-15K}{5}$	
s^0	$20K$	

The system is stable

$$\begin{cases}30-15K>0\\20K>0\end{cases}$$

$\rightarrow 0<K<2$

§5. Stability in State Space

- The system transfer function

$$T(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}$$
$$=\boldsymbol{C}\frac{\mathrm{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}\boldsymbol{B}+\boldsymbol{D}$$
$$=\frac{N(s)}{\det(s\boldsymbol{I}-\boldsymbol{A})}$$

The characteristic equation $\det(s\boldsymbol{I}-\boldsymbol{A})=0$

- The eigenvalues of the matrix A are identical to the system's poles before cancellation of common poles and zeroes in the TF

→ the stability of a system represented in state space can be determined by finding the eigenvalues of the system matrix, A, and determining their locations on the s-plane

§5. Stability in State Space

- Ex.6.11 Stability in State Space

Given the system

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0&3&1\\2&8&1\\-10&-5&-2\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}10\\0\\0\end{bmatrix}u,\ y=\begin{bmatrix}1&0&0\end{bmatrix}\boldsymbol{x}$$

find out how many poles are in the LHP, in the RHP, and on the $j\omega$-axis

Solution

Find $(s\boldsymbol{I}-\boldsymbol{A})$

$$s\boldsymbol{I}-\boldsymbol{A}=\begin{bmatrix}s&0&0\\0&s&0\\0&0&s\end{bmatrix}-\begin{bmatrix}0&3&1\\2&8&1\\-10&-5&-2\end{bmatrix}$$
$$=\begin{bmatrix}s&-3&-1\\-2&s-8&-1\\10&5&s+2\end{bmatrix}$$

§5. Stability in State Space

$$\det(s\boldsymbol{I}-\boldsymbol{A})=\det\begin{bmatrix}s&-3&-1\\-2&s-8&-1\\10&5&s+2\end{bmatrix}$$
$$=s\begin{vmatrix}s-8&-1\\5&s+2\end{vmatrix}-(-3)\begin{vmatrix}-2&-1\\10&s+2\end{vmatrix}+(-1)s\begin{vmatrix}-2&s-8\\10&5\end{vmatrix}$$
$$=s^3-6s^2-7s-52$$

Using this polynomial, form the Routh table

s^3		1		−7
s^2	−3	~~−6~~	−26	~~−52~~
s^1	−1	~~−47/3~~	0	~~0~~
s^0		−26		

One sign change in the first column, the system has one RHP pole and two LHP poles. It is therefore unstable

§5. Stability in State Space

MATLAB ML

Learn how to use MATLAB to

- run ch6sp3 in Appendix B
- determine the stability of a system represented in state space by finding the eigenvalues of the system matrix
- do Ex.6.11

§5. Stability in State Space

Skill-Assessment Ex.6.4

Problem For the following system represented in state space, find out how many poles are in the LHP, in the RHP, and on the $j\omega$-axis

WileyPLUS WPCS Control Solutions

$$\dot{\boldsymbol{x}}=\begin{bmatrix}2&1&1\\1&7&1\\-3&4&-5\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}r,\ y=\begin{bmatrix}0&1&0\end{bmatrix}\boldsymbol{x}$$

Solution Find $|s\boldsymbol{I}-\boldsymbol{A}|$

$$s\boldsymbol{I}-\boldsymbol{A}=\begin{bmatrix}s&0&0\\0&s&0\\0&0&s\end{bmatrix}-\begin{bmatrix}2&1&1\\1&7&1\\-3&4&-5\end{bmatrix}$$
$$=\begin{bmatrix}s-2&-1&-1\\-1&s-7&-1\\3&-4&s+5\end{bmatrix}$$
$$\rightarrow\det(s\boldsymbol{I}-\boldsymbol{A})=s^3-4s^2-33s+51$$

§5. Stability in State Space

$$\det(s\boldsymbol{I}-\boldsymbol{A})=s^3-4s^2-33s+51$$

Form the Routh table

s^3	1	−33
s^2	−4	51
s^1	−20.25	
s^0	51	

There are two sign changes. Thus, there are two RHP poles and one LHP pole

§5. Stability in State Space

TryIt 6.3
Use the following MATLAB statements to find the eigenvalues of the system described in Skill-Assessment Ex.6.4

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\boldsymbol{x}$$

```
A=[2 1 1; 1 7 1; -3 4 -5];
Eig=eig (A)
```


§6. Case Studies