07. Steady-State Error

Learning Outcome

After completing this chapter, the student will be able to

- Find the steady-state error for a unity feedback system
- Specify a system's steady-state error performance
- Design the gain of a closed-loop system to meet a steady-state error specification
- Find the steady-state error for disturbance inputs
- Find the steady-state error for nonunity feedback systems
- Find the steady-state error sensitivity to parameter changes
- Find the steady-state error for systems represented in state space

§1. Introduction

- Control systems analysis and design focus on three specifications
  - transient response
  - stability
  - steady-state errors

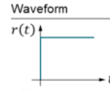
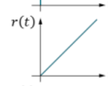
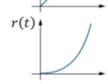
  taking into account the robustness of the design along with economic and social considerations
- Control system design entails trade-offs between desired
  - transient response,
  - steady-state error, and
  - the requirement that the system be stable

§1. Introduction

Definition and Test Inputs

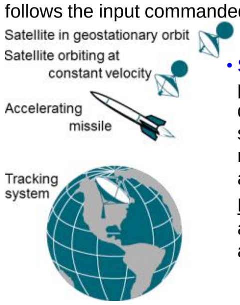
- Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$. Test inputs used for steady-state error analysis and design are summarized

Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	step	constant position	1	$\frac{1}{s}$
	ramp	constant velocity	t	$\frac{1}{s^2}$
	parabola	constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

§1. Introduction

- In order to explain how these test signals are used, let us assume a position control system, where the output position follows the input commanded position

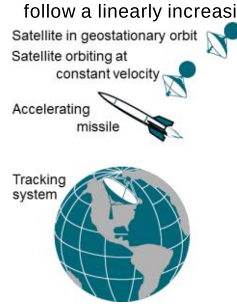


- Step inputs represent constant position and thus are useful in determining the ability of the control system to position itself with respect to a stationary target, such as a satellite in geostationary orbit

  Ex.: An antenna position control is a system that can be tested for accuracy using step inputs

§1. Introduction

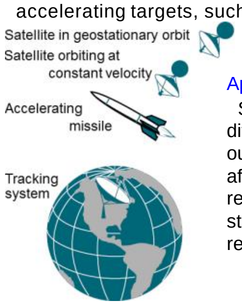
- Ramp inputs represent constant-velocity inputs to a position control system, and can be used to test a system's ability to follow a linearly increasing input or, equivalently, to track a constant velocity target



Ex.: A position control system that tracks a satellite that moves across the sky at a constant angular velocity, would be tested with a ramp input to evaluate the steady-state error between the satellite's angular position and that of the control system

§1. Introduction

- Parabola inputs represent constant acceleration inputs to position control systems and can be used to represent accelerating targets, such as the missile, to determine the steady-state error performance

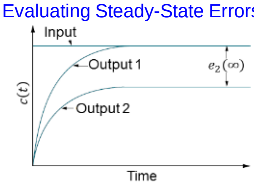


Application to Stable Systems

Since we are concerned with the difference between the input and the output of a feedback control system after the steady state has been reached, our discussion is limited to stable systems, where the natural response approaches zero as $t \to \infty$

§1. Introduction

Evaluating Steady-State Errors

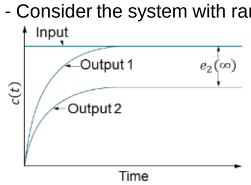


Steady-state error with step input

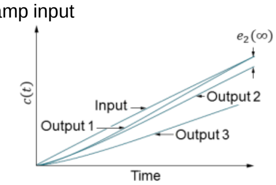
- Consider the system with step input
  - output 1 has zero steady-state error
  - output 2 has a finite steady-state error, $e_2(\infty)$

§1. Introduction

- Consider the system with ramp input



Steady-state error with step input

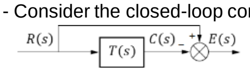


Steady-state error with step ramp input

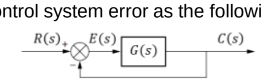
- output 1 has zero steady-state error
- output 2 has a finite steady-state error, $e_2(\infty)$, as measured vertically after the transients have died down
- output 3 has infinite steady-state error, as measured vertically after the transients have died down, if the output's slope is different from that of the input

§1. Introduction

- Consider the closed-loop control system error as the following



Closed-loop control system error
a. general representation



Closed-loop control system error
b. representation for unity feedback systems

$G(s)$: system transfer function

$T(s)$: closed-loop transfer function

$E(s)$: error, the difference between the input and the output

$\Longrightarrow$ study the steady-state, or final, value of $e(t)$

- First, study and derive expressions for the steady-state error for unity feedback systems
- Then, expand to nonunity feedback systems

§1. Introduction

Sources of Steady-State Error

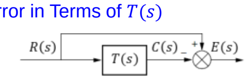
- Many steady-state errors in control systems arise from nonlinear sources, such as backlash in gears or a motor that will not move unless the input voltage exceeds a threshold

$\Longrightarrow$ study the steady state errors that arise from the configuration of the system itself and the type of applied input

- steady-state error for unity feedback systems
- static error constants and system type

§2. Steady-State Error for Unity Feedback Systems

Steady-State Error in Terms of $T(s)$



Closed-loop control system error - general representation

The error between the input, $R(s)$, and the output, $C(s)$

$$E(s) = R(s) - C(s) \quad (7.2)$$

$$C(s) = R(s)T(s) \quad (7.3)$$

$$\Longrightarrow E(s) = R(s)[1 - T(s)] \quad (7.4)$$

Applying the final value theorem

$$e(\infty) = \lim_{t\to\infty} e(t)$$

$$= \lim_{s\to 0} sE(s) \quad (7.5)$$

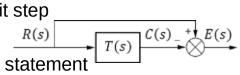
$$= \lim_{s\to 0} sR(s)[1 - T(s)] \quad (7.6)$$

§2. Steady-State Error for Unity Feedback Systems

- Ex.7.1 **Steady-State Error in Terms of $T(s)$**

Find the steady-state error for the system $T(s) = 5/(s^2+7s+10)$ if the input is a unit step

Solution



From the problem statement

$$T(s) = \frac{5}{s^2+7s+10},\qquad R(s) = \frac{1}{s}$$

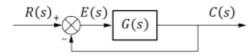
The error $E(s) = R(s)[1-T(s)] = \dfrac{s^2+7s+5}{s(s^2+7s+10)}$

Since $T(s)$ is stable and, subsequently, $E(s)$ does not have RHP poles or $j\omega$ poles other than at the origin, apply the final value theorem

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s^2+7s+5}{s^2+7s+10} = \frac{1}{2}$$

§2. Steady-State Error for Unity Feedback Systems

Steady-State Error in Terms of $G(s)$



Closed-loop control system error - representation for unity feedback systems

Consider the unity feedback control system

$$E(s) = R(s) - C(s) \qquad (7.8)$$

$$C(s) = E(s)G(s) \qquad (7.9)$$

$$\Longrightarrow E(s) = \frac{R(s)}{1+G(s)} \qquad (7.10)$$

Apply the final value theorem

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)} \qquad (7.11)$$

§2. Steady-State Error for Unity Feedback Systems

Step Input

$$e(\infty) = e_{\text{step}}(\infty) = \lim_{s\to 0}\frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} \qquad (7.12)$$

In order to have zero steady-state error, the dc gain, $\lim_{s\to 0} G(s)$, of the forward transfer function

$$\lim_{s\to 0} G(s) = \infty \qquad (7.13)$$

To satisfy Eq. (7.13), $G(s)$ must take on the following form

$$G(s) = \frac{(s+z_1)\cdots}{s^n(s+p_1)\cdots} \Longrightarrow s^n(s+p_1)(s+p_2)\cdots \to 0 \quad (7.14)$$

$n \ge 1$: at least one pure integration in the forward path

$n = 0$: $\lim_{s\to 0} G(s) = \dfrac{z_1 z_2\cdots}{p_1 p_2\cdots} \ne \infty \quad \Longrightarrow$ finite steady-state error

§2. Steady-State Error for Unity Feedback Systems

Ramp Input

$$e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s\to 0}\frac{s(1/s^2)}{1+G(s)} = \frac{1}{\lim_{s\to 0} sG(s)} \qquad (7.16)$$

To have zero steady-state error for a ramp input

$$\lim_{s\to 0} sG(s) = \infty \qquad (7.17)$$

To satisfy Eq. (7.17), $G(s)$ must take on the following form

$$G(s) = \frac{(s+z_1)\cdots}{s^n(s+p_1)\cdots} \Longrightarrow s^n(s+p_1)(s+p_2)\cdots \to 0 \quad (7.14)$$

$n \ge 2$: at least two integrations in the forward path

$n = 1$: $\lim_{s\to 0} sG(s) = \dfrac{z_1 z_2\cdots}{p_1 p_2\cdots} \ne \infty \Longrightarrow$ finite steady-state error

$n = 0$: $\lim_{s\to 0} sG(s) = 0 \quad \Longrightarrow$ diverging ramps

§2. Steady-State Error for Unity Feedback Systems

Parabolic Input

$$e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s\to 0}\frac{s(1/s^3)}{1+G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)} \qquad (7.20)$$

To have zero steady-state error for a parabolic input

$$\lim_{s\to 0} s^2G(s) = \infty \qquad (7.21)$$

To satisfy Eq. (7.21), $G(s)$ must take on the following form

$$G(s) = \frac{(s+z_1)\cdots}{s^n(s+p_1)\cdots} \Longrightarrow s^n(s+p_1)(s+p_2)\cdots \to 0 \quad (7.14)$$

$n \ge 3$: at least three integrations in the forward path

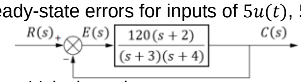
$n = 2$: $\lim_{s\to 0} s^2G(s) = \dfrac{z_1 z_2\cdots}{p_1 p_2\cdots} \ne \infty \Longrightarrow$ finite steady-state error

$n \le 1$: $\lim_{s\to 0} s^2G(s) = 0 \quad \Longrightarrow$ infinite steady-state error

§2. Steady-State Error for Unity Feedback Systems

- Ex.7.2 **Steady-State Errors for Systems with No Integrations**

Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, $5t^2(t)$ to the system



The function $u(t)$ is the unit step

Solution

The closed-loop system is stable

$5u(t)$ or $5/s$: $\quad e_{\text{step}}(\infty) = 5\times\dfrac{1}{1+\lim_{s\to 0} G(s)} = \dfrac{5}{1+20} = \dfrac{5}{21}$

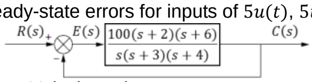
$5tu(t)$ or $5/s^2$: $\quad e_{\text{ramp}}(\infty) = 5\times\dfrac{1}{\lim_{s\to 0} sG(s)} = \dfrac{5}{0} = \infty$

$5t^2u(t)$ or $10/s^3$: $\quad e_{\text{parabola}}(\infty) = 10\times\dfrac{1}{\lim_{s\to 0} s^2G(s)} = \dfrac{10}{0} = \infty$

§2. Steady-State Error for Unity Feedback Systems

- Ex.7.3 Steady-State Errors for Systems with One Integration

Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, $5t^2(t)$ to the system



The function $u(t)$ is the unit step

Solution

The closed-loop system is stable

$5u(t)$ or $5/s$: $e_{\text{step}}(\infty) = 5 \times \dfrac{1}{1+\lim\limits_{s\to 0} G(s)} = \dfrac{5}{\infty} = 0$

$5tu(t)$ or $5/s^2$: $e_{\text{ramp}}(\infty) = 5 \times \dfrac{1}{\lim\limits_{s\to 0} sG(s)} = \dfrac{5}{100} = \dfrac{1}{20}$

$5t^2u(t)$ or $10/s^3$: $e_{\text{parabola}}(\infty) = 10 \times \dfrac{1}{\lim\limits_{s\to 0} s^2G(s)} = \dfrac{10}{0} = \infty$

§2. Steady-State Error for Unity Feedback Systems

Skill-Assessment Ex.7.1

Problem A unity feedback system has the following forward TF

WileyPLUS WPCS Control Solutions

$$G(s) = \frac{10(s+20)(s+30)}{s(s+25)(s+35)}$$

a. Find the steady-state error for the following inputs $15u(t)$, $15tu(t)$, and $15t^2(t)$

b. Repeat for

$$G(s) = \frac{10(s+20)(s+30)}{s^2(s+25)(s+35)(s+50)}$$

§2. Steady-State Error for Unity Feedback Systems

Solution

a. First check stability

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{10s^2+500s+6000}{s^3+70s^2+1375s+6000}$$
$$= \frac{10(s+30)(s+20)}{(s+26.03)(s+37.89)(s+6.085)}$$

$\Longrightarrow$ the closed-loop system is stable

$5u(t)$: $e_{\text{step}}(\infty) = \dfrac{15}{1+\lim\limits_{s\to 0} G(s)} = \dfrac{5}{1+\infty} = 0$

$5tu(t)$: $e_{\text{ramp}}(\infty) = \dfrac{15}{\lim\limits_{s\to 0} sG(s)} = \dfrac{15}{\dfrac{10\times 20\times 30}{25\times 35}} = 2.1875$

$5t^2u(t)$: $e_{\text{parabola}}(\infty) = \dfrac{30}{\lim\limits_{s\to 0} s^2G(s)} = \dfrac{30}{0} = \infty$

§2. Steady-State Error for Unity Feedback Systems

b. First check stability

$$T(s) = \frac{G(s)}{1+G(s)}$$
$$= \frac{10s^2+500s+6000}{s^5+110s^4+3875s^3+4.37\times 10^4 s^2+500s+6000}$$
$$= \frac{10(s+30)(s+20)}{(s+50.01)(s+35)(s+25)(s^2-7.189\times 10^{-4}s+0.1372)}$$

$\Longrightarrow$ From the second-order term in the denominator, the system is unstable. Instability could also be determined using the Routh-Hurwitz criteria on the denominator of $T(s)$

Since the system is unstable, calculations about steady-state error cannot be made

§3. Static Error Constants and System Type

Static Error Constants

- The relationships for steady-state error

• For a step input, $u(t)$ $\quad e_{\text{step}}(\infty) = \dfrac{1}{1+\lim\limits_{s\to 0} G(s)}$

• For a ramp input, $tu(t)$ $\quad e_{\text{ramp}}(\infty) = \dfrac{1}{\lim\limits_{s\to 0} sG(s)}$

• For a parabolic input, $\frac{1}{2}t^2u(t)$ $\quad e_{\text{parabola}}(\infty) = \dfrac{1}{\lim\limits_{s\to 0} s^2G(s)}$

- The limits static error constants

• Position constant, K_p $\quad K_p = \lim\limits_{s\to 0} G(s)$

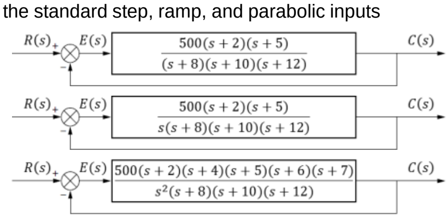
• Velocity constant, K_v $\quad K_v = \lim\limits_{s\to 0} sG(s)$

• Acceleration constant, K_a $\quad K_a = \lim\limits_{s\to 0} s^2G(s)$

§3. Static Error Constants and System Type

- Ex.7.4 Steady-State Error via Static Error Constants

Evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs

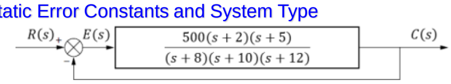


Solution

All closed-loop systems are indeed stable

§3. Static Error Constants and System Type

a.



- The limits static error constants

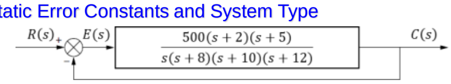
$$K_p = \lim_{s\to 0} G(s) = \frac{500\times 2\times 5}{8\times 10\times 12} = 5.208$$
$$K_v = \lim_{s\to 0} sG(s) = 0$$
$$K_a = \lim_{s\to 0} s^2G(s) = 0$$

- The steady-state error

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+5.208} = 0.161$$
$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = \frac{1}{0} = \infty$$
$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

§3. Static Error Constants and System Type

b.



- The limits static error constants

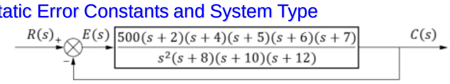
$$K_p = \lim_{s\to 0} G(s) = \infty$$
$$K_v = \lim_{s\to 0} sG(s) = \frac{500\times 2\times 5\times 6}{8\times 10\times 12} = 31.25$$
$$K_a = \lim_{s\to 0} s^2G(s) = 0$$

- The steady-state error

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$
$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = \frac{1}{31.25} = 0.032$$
$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

§3. Static Error Constants and System Type

c.



- The limits static error constants

$$K_p = \lim_{s\to 0} G(s) = \infty$$
$$K_v = \lim_{s\to 0} sG(s) = \infty$$
$$K_a = \lim_{s\to 0} s^2G(s) = \frac{500\times 2\times 4\times 5\times 6\times 7}{8\times 10\times 12} = 875$$

- The steady-state error

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$
$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = \frac{1}{\infty} = 0$$
$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \frac{1}{875} = 1.14\times 10^{-3}$$

§3. Static Error Constants and System Type

MATLAB ML

Run ch7p1 in Appendix B

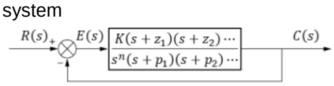
Learn how to use MATLAB to

- test the system for stability
- evaluate static error constants
- calculate steady-state error
- solve Ex.7.4 with System (b)

§3. Static Error Constants and System Type

System Type

- The values of the static error constants, again, depend upon the form of $G(s)$, especially the number of pure integrations in the forward path
- Given the system



Feedback control system for defining system type

define system type to be the value of n in the denominator or, equivalently, the number of pure integrations in the forward path

- $n=0$: type 0 system
- $n=1$: type 1 system
- $n=2$: type 2 system

§3. Static Error Constants and System Type

- Relationships between input, system type, static error constants, and steady-state errors

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p=$ Constant	$\frac{1}{1+K_p}$	$K_p=\infty$	0	$K_p=\infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v=0$	∞	$K_v=$ Constant	$\frac{1}{K_v}$	$K_v=\infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a=0$	∞	$K_a=0$	∞	$K_a=$ Constant	$\frac{1}{K_a}$

§3. Static Error Constants and System Type

Skill-Assessment Ex.7.2

Problem A unity feedback system has the following forward TF

WileyPLUS WPCS Control Solutions

$$G(s)=\frac{1000(s+8)}{(s+7)(s+9)}$$

a. Evaluate system type, K_p, K_v, and K_a

b. Use your answers to (a.) to find the steady-state errors for the standard step, ramp, and parabolic inputs

Solution The system is stable

§3. Static Error Constants and System Type

a.

The closed-loop transfer function

$$T(s)=\frac{G(s)}{1+G(s)}$$

$$=\frac{1000(s+8)}{(s+9)(s+7)+1000(s+8)}$$

$$=\frac{1000(s+8)}{s^2+1016s+8063}$$

The system is Type 0. Therefore

$$K_p=\lim_{s\to 0}G(s)=\frac{1000\times 8}{7\times 9}=127$$

$$K_v=\lim_{s\to 0}sG(s)=0$$

$$K_a=\lim_{s\to 0}s^2G(s)=0$$

§3. Static Error Constants and System Type

b.

The steady-state error

$$e_{\text{step}}(\infty)=\frac{1}{1+K_p}=\frac{1}{1+127}=7.8\times 10^{-3}$$

$$e_{\text{ramp}}(\infty)=\frac{1}{K_v}=\frac{1}{0}=\infty$$

$$e_{\text{parabola}}(\infty)=\frac{1}{K_a}=\frac{1}{0}=\infty$$

§3. Static Error Constants and System Type

TryIt 7.1

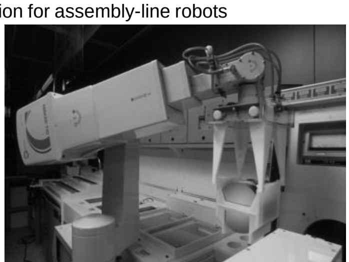
Use MATLAB, the Control System Toolbox, and the following statements to find K_p,$e_{\text{step}}(\infty)$, and the closed-loop poles to check for stability for the system of Skill-Assessment Ex.7.2

$$G(s)=\frac{1000(s+8)}{(s+7)(s+9)}$$

```
numg=1000*[1 8];
deng=poly([-7 -9]);
G=tf(numg,deng);
Kp=dcgain(G)
estep=1/(1+Kp)
T=feedback(G,1);
poles=pole(T)
```


§4. Steady-State Error Specifications

A robot used in the manufacturing of semiconductor random-access memories (RAMs) similar to those in personal computers. Steady-state error is an important design consideration for assembly-line robots



§4. Steady-State Error Specifications

- The specifications for a control system's transient response
  - damping ratio, ζ
  - settling time, T_s
  - peak time, T_p
  - percent overshoot, $\%OS$
- The specification of a static error constant
  - the position constant, K_p
  - velocity constant, K_v
  - acceleration constant, K_a

§4. Steady-State Error Specifications

- For example, if a control system has the specification $K_v = 1000$, we can draw several conclusions
  - The system is stable
  - The system is of Type 1
  - A ramp input is the test signal
  - The steady-state error between the input ramp and the output ramp is $1/K_v$ per unit of input slope

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

§4. Steady-State Error Specifications

- Ex.7.5 Interpreting the Steady-State Error Specification

What information is contained in the specification $K_p = 1000$?

Solution

The system is stable

The system is Type 0

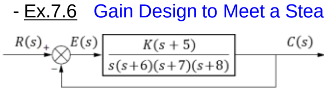
The input test signal is a step

The error per unit step is $e_{\text{step}}(\infty) = \dfrac{1}{1+K_p} = \dfrac{1}{1+1000} = \dfrac{1}{1001}$

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

§4. Steady-State Error Specifications

- Ex.7.6 Gain Design to Meet a Steady-State Error Specification



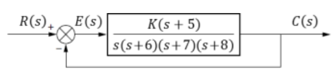
Find the value of K so that there is 10% error in the steady state

Solution

Since the system is Type 1, the error stated in the problem must apply to a ramp input; only a ramp yields a finite error in a Type 1 system

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

§4. Steady-State Error Specifications



$$e(\infty) = \frac{1}{K_v} = 0.1$$

$$\Rightarrow K_v = 10 = \lim_{s\to 0} sG(s) = \frac{K\times 5}{6\times 7\times 8}$$

$$\Rightarrow K = 672$$

Applying the Routh-Hurwitz criterion, we see that the system is stable at this gain

Although this gain meets the criteria for steady-state error and stability, it may not yield a desirable transient response

§4. Steady-State Error Specifications

MATLAB ML

Run ch7p2 in Appendix B

Learn how to use MATLAB to

- find the gain to meet a steady-state error specification
- solves Ex.7.6

§4. Steady-State Error Specifications

Skill-Assessment Ex.7.3

Problem A unity feedback system has the following forward TF

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$$G(s) = \frac{K(s+2)}{(s+14)(s+18)}$$

Find the value of K to yield a 10% error in the steady state

Solution The system is stable for positive K

For a step input

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = 0.1$$

$$\Rightarrow K_p = 9 = \lim_{s\to 0} G(s) = \frac{12\times K}{14\times 18}$$

$$\Rightarrow K = 189$$

§4. Steady-State Error Specifications

TryIt 7.2

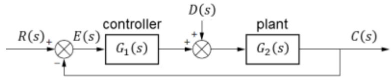
Use MATLAB, the Control System Toolbox, and the following statements to solve Skill-Assessment Ex.7.3 and check the resulting system for stability

$$G(s)=\frac{K(s+2)}{(s+14)(s+18)}$$

```
numg=[1 12];
deng=poly([-14 -18]);
G=tf(numg,deng);
Kpdk=dcgain(G);
estep=0.1;
K=(1/estep-1)/Kpdk
T=feedback(G,1);
poles=pole(T)
```


§5. Steady-State Error for Disturbances

- Feedback control systems are used to compensate for disturbances or unwanted inputs that enter a system, result that regardless of these disturbances, the system can be designed to follow the input with small or zero error



Feedback control system showing disturbance

- Consider a feedback control system with a disturbance, $D(s)$, injected between the controller and the plant

$$C(s)=E(s)G_1(s)G_2(s)+D(s)\,G_2(s) \quad (7.58)$$

$$C(s)=R(s)-E(s) \quad (7.59)$$

$$E(s)=\frac{1}{1+G_1(s)G_2(s)}R(s)-\frac{G_2(s)}{1+G_1(s)G_2(s)}D(s) \quad (7.60)$$

§5. Steady-State Error for Disturbances

- To find the steady-state value of the error, apply the final value theorem to Eq. (7.60) and obtain

$$\begin{aligned} e(\infty)&=\lim_{s\to0}sE(s)\\ &=\lim_{s\to0}\frac{s}{1+G_1(s)G_2(s)}R(s)-\lim_{s\to0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)\\ &=e_R(\infty)+e_D(\infty)\end{aligned}$$

where,

- $e_R(\infty)$: the steady-state error due to $R(s)$

$$e_R(\infty)=\lim_{s\to0}\frac{s}{1+G_1(s)G_2(s)}R(s)$$

- $e_D(\infty)$: the steady-state error due to the disturbance $D(s)$

$$e_D(\infty)=-\lim_{s\to0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)$$

$$E(s)=\frac{1}{1+G_1(s)G_2(s)}R(s)-\frac{G_2(s)}{1+G_1(s)G_2(s)}D(s) \quad (7.60)$$

§5. Steady-State Error for Disturbances

- Assume a step disturbance, $D(s)=1/s$

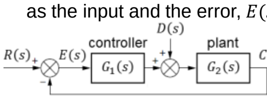
Substituting this value into the second term of Eq. (7.61), $e_D(\infty)$, the steady-state error component due to a step disturbance is found to be

$$e_D(\infty)=-\lim_{s\to0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}\frac{1}{s}=-\frac{1}{\lim\limits_{s\to0}\dfrac{1}{G_2(s)}+\lim\limits_{s\to0}G_1(s)} \quad (7.62)$$

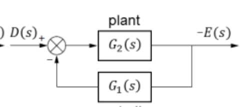
This equation shows that the steady-state error produced by a step disturbance can be reduced by increasing the dc gain of $G_1(s)$ or decreasing the dc gain of $G_2(s)$

§5. Steady-State Error for Disturbances

- Rearrange the system so that the disturbance, $D(s)$, is depicted as the input and the error, $E(s)$, as the output, with $R(s)=0$



Feedback control system



Rearrange feedback system to show disturbance as input and error as output, with $R(s)=0$

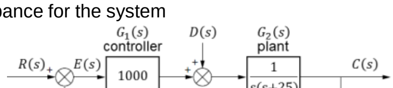
To minimize the steady-state value of $E(s)$, we must either

- increase the dc gain of $G_1(s)$ so that a lower value of $E(s)$ will be fed back to match the steady-state value of $D(s)$, or
- decrease the dc value of $G_2(s)$, which then yields a smaller value of $e(\infty)$ as predicted by the feedback formula

§5. Steady-State Error for Disturbances

- Ex.7.7 Steady-State Error Due to Step Disturbance

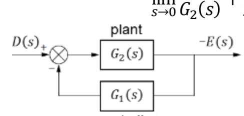
Find the steady-state error component due to a step disturbance for the system



Solution

The system is stable

$$e_D(\infty)=-\frac{1}{\lim\limits_{s\to0}\dfrac{1}{G_2(s)}+\lim\limits_{s\to0}G_1(s)}=-\frac{1}{0+1000}=-\frac{1}{1000}$$

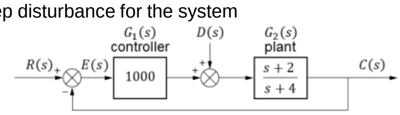


The result shows that the steady-state error produced by the step disturbance is inversely proportional to the dc gain of $G_1(s)$. The dc gain of $G_2(s)$ is infinite in this example

§5. Steady-State Error for Disturbances

Skill-Assessment Ex.7.4

Problem Evaluate the steady-state error component due to a step disturbance for the system



Solution The system is stable

For a step input

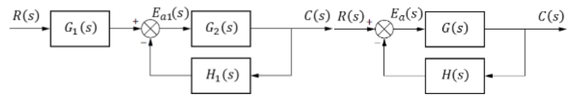
$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\dfrac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)}$$

$$= \frac{1}{2 + 1000}$$

$$= -9.98 \times 10^{-4}$$

§6. Steady-State Error for Nonunity Feedback Systems

- A general feedback system, showing the input transducer, $G_1(s)$, controller and plant, $G_2(s)$, and feedback, $H_1(s)$



- Pushing the input transducer to the right past the summing junction yields the general nonunity feedback system, where

  $G(s) = G_1(s)G_2(s)$

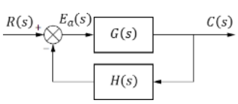
  $H(s) = H_1(s)/G_1(s)$

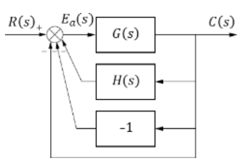
  $E_a(s)$: actuating signal, $E_a(s) \neq E(s) = C(s) - R(s)$

  If $r(t)$ and $c(t)$ have the same units, the steady-state error can be found, $e(\infty) = r(\infty) - c(\infty)$

§6. Steady-State Error for Nonunity Feedback Systems

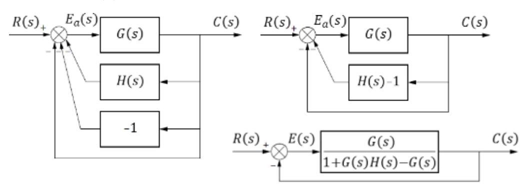
- Form a unity feedback system by adding and subtracting unity feedback paths. This step requires that input and output units be the same





§6. Steady-State Error for Nonunity Feedback Systems

- Combine $H(s)$ with the negative unity feedback



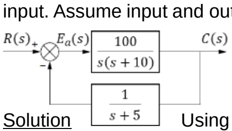
- Combine the feedback system consisting of $G(s)$ and $[H(s) - 1]$, leaving an equivalent forward path and a unity feedback

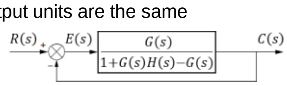
  Notice that the final figure shows $E(s) = R(s) - C(s)$ explicitly

§6. Steady-State Error for Nonunity Feedback Systems

- Ex.7.8 Steady-State Error for Nonunity Feedback Systems

  Find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same





Solution Using the equivalent forward TF

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{\dfrac{100}{s(s+10)}}{1 + \dfrac{100}{s(s+10)}\dfrac{1}{s+5} - \dfrac{100}{s(s+10)}}$$

$$\Rightarrow G_e(s) = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

§6. Steady-State Error for Nonunity Feedback Systems

The equivalent forward TF

$$G_e(s) = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

Thus, the system is Type 0, since there are no pure integrations

The appropriate static error constant is then K_p

$$K_p = \lim_{s\to 0} G_e(s) = \frac{100 \times 5}{-400} = -\frac{5}{4}$$

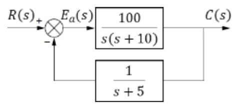
The steady-state error

$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - 5/4} = -4$$

The negative value for steady-state error implies that the output step is larger than the input step

§6. Steady-State Error for Nonunity Feedback Systems

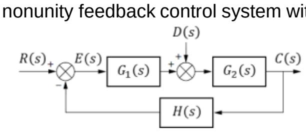
TryIt 7.3
Use MATLAB, the Control System Toolbox, and the following statements to find $G_e(s)$ in Ex.7.8




```
G=zpk([],[0 -10],100);
H=zpk([],- 5,1);
Ge=feedback(G,(H-1));
'Ge(s)'
Ge=tf(Ge)
T=feedback(Ge,1);
'Poles of T(s)'
pole(T)
```


§6. Steady-State Error for Nonunity Feedback Systems

- Consider a nonunity feedback control system with disturbance



- The steady-state error for this system, $e(\infty)=c(\infty)-r(\infty)$

$$e(\infty)=\lim_{s\to 0} sE(s)$$

$$=\lim_{s\to 0} s\left\{\left[1-\frac{G_1G_2}{1+G_1G_2H}\right]R-\frac{G_2}{1+G_1G_2H}D\right\} \quad (7.69)$$

with step inputs and step disturbances, $R(s)=D(s)=1/s$

$$e(\infty)=\left[1-\frac{\lim_{s\to 0}(G_1G_2)}{\lim_{s\to 0}(1+G_1G_2H)}\right]-\frac{\lim_{s\to 0}G_2}{\lim_{s\to 0}(1+G_1G_2H)} \quad (7.70)$$

§6. Steady-State Error for Nonunity Feedback Systems

$$e(\infty)=\left[1-\frac{\lim_{s\to 0}(G_1G_2)}{\lim_{s\to 0}(1+G_1G_2H)}\right]-\frac{\lim_{s\to 0}G_2}{\lim_{s\to 0}(1+G_1G_2H)} \quad (7.70)$$

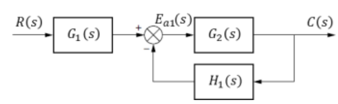
For zero error

$$\frac{\lim_{s\to 0}(G_1G_2)}{\lim_{s\to 0}(1+G_1G_2H)}=1,\qquad \frac{\lim_{s\to 0}G_2}{\lim_{s\to 0}(1+G_1G_2H)}=0 \quad (7.71)$$

The above two equations can always be satisfied if

- the system is stable
- $G_1(s)$ is a Type 1 system
- $G_2(s)$ is a Type 0 system
- $H(s)$ is a Type 0 system with a dc gain of unity

§6. Steady-State Error for Nonunity Feedback Systems



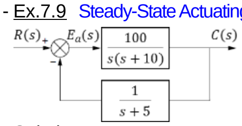
- The steady-state value of the actuating signal, $E_{a1}(s)$, for a general feedback system

$$e_{a1}(\infty)=\lim_{s\to 0}\frac{sR(s)G_1(s)}{1+G_2(s)H_1(s)} \quad (7.72)$$

Note: There is no restriction that the input and output units be the same, since we are finding the steady-state difference between signals at the summing junction, which do have the same units

§6. Steady-State Error for Nonunity Feedback Systems

- Ex.7.9 Steady-State Actuating Signal for Nonunity Feedback Systems



Find the steady-state actuating signal for the system with a unit step input. Repeat for a unit ramp input

Solution

$$e_a(\infty)=\lim_{s\to 0}\frac{sR(s)G_1(s)}{1+G_2(s)H_1(s)}=\lim_{s\to 0}\frac{sR(s)\times 1}{1+\frac{100}{s(s+10)}\times\frac{1}{s+5}}$$

For step input $e_a(\infty)=\lim_{s\to 0}\frac{s(1/s)\times 1}{1+\frac{100}{s(s+10)}\times\frac{1}{s+5}}=0$

For ramp input $e_a(\infty)=\lim_{s\to 0}\frac{s(1/s^2)\times 1}{1+\frac{100}{s(s+10)}\times\frac{1}{s+5}}=\frac{1}{2}$

§6. Steady-State Error for Nonunity Feedback Systems

Skill-Assessment Ex.7.5

Problem

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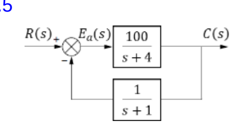


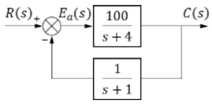
a. Find the steady-state error, $e(\infty)=c(\infty)-r(\infty)$, for a unit step input given the nonunity feedback system. Repeat for a unit ramp input. Assume input and output units are the same

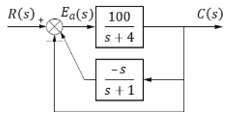
b. Find the steady-state actuating signal, $e_a(\infty)$, for a unit step input given the nonunity feedback system. Repeat for a unit ramp input

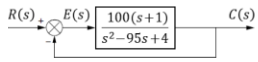
§6. Steady-State Error for Nonunity Feedback Systems

Solution The system is stable

Create a unity-feedback system



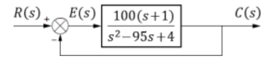




$$H_e(s) = \frac{1}{s+1} - 1 = \frac{-s}{s+1}$$

$$G_e(s) = \frac{G(s)}{1+G(s)H_e(s)} = \frac{100(s+1)}{s^2-95s+4}$$

§6. Steady-State Error for Nonunity Feedback Systems



a. The steady-state error, $e(\infty) = c(\infty) - r(\infty)$

The system is Type 0, $K_p = \lim_{s\to 0} G(s) = 100/4 = 25$

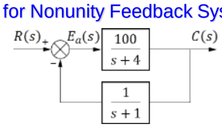
The steady-state error

$$e_{\text{step}}(\infty) = 1/(1+K_p) = 0.0385$$

$$e_{\text{ramp}}(\infty) = \infty$$

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

§6. Steady-State Error for Nonunity Feedback Systems



b. The steady-state actuating signal, $e_a(\infty)$

$$e_a(\infty) = \lim_{s\to 0} \frac{sR(s)G_1(s)}{1+G_2(s)H_1(s)} = \lim_{s\to 0} \frac{sR(s)\times 1}{1+\frac{100}{s+4}\times\frac{1}{s+1}}$$

step input $e_a(\infty) = \lim_{s\to 0} \frac{s(1/s)\times 1}{1+\frac{100}{s+4}\times\frac{1}{s+1}} = \frac{1}{104} = 0.0385$

ramp input $e_a(\infty) = \lim_{s\to 0} \frac{s(1/s^2)\times 1}{1+\frac{100}{s+4}\times\frac{1}{s+1}} = \frac{1}{0} = \infty$

§7. Sensitivity

Sensitivity: the ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero

$$S_{F:P} = \lim_{\Delta P\to 0} \frac{\text{Fractional change in the function}, F}{\text{Fractional change in the parameter}, P}$$

$$= \lim_{\Delta P\to 0} \frac{\Delta F/F}{\Delta P/P}$$

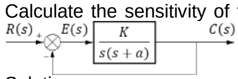
$$= \lim_{\Delta P\to 0} \frac{P\Delta F}{F\Delta P}$$

$$\Rightarrow S_{F:P} = \frac{P\,\delta F}{F\,\delta P} \tag{7.75}$$

§7. Sensitivity

- Ex.7.10 Sensitivity of a Closed-Loop Transfer Function

Calculate the sensitivity of the closed-loop transfer function to changes in the parameter a. How would you reduce the sensitivity?



Solution

The closed-loop transfer function

$$T(s) = \frac{K}{s^2+as+K}$$

The sensitivity

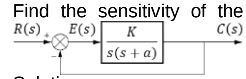
$$S_{T:a} = \frac{a}{T}\frac{\delta T}{\delta a} = \frac{a}{\frac{K}{s^2+as+K}}\frac{-Ks}{\left(\frac{K}{s^2+as+K}\right)^2} = \frac{-as}{s^2+as+K}$$

An increase in K reduces the sensitivity of the closed-loop transfer function to changes in the parameter a

§7. Sensitivity

- Ex.7.11 Sensitivity of Steady-State Error with Ramp Input

Find the sensitivity of the steady-state error to changes in parameter K and parameter a with ramp inputs



Solution

The steady-state error for the system

$$e(\infty) = 1/K_v = a/K$$

The sensitivity of $e(\infty)$ to changes in parameter a

$$S_{e:a} = \frac{a}{e}\frac{\delta e}{\delta a} = \frac{a}{a/K}\frac{1}{K} = 1$$

The sensitivity of $e(\infty)$ to changes in parameter K

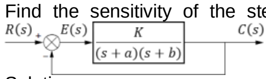
$$S_{e:K} = \frac{K}{e}\frac{\delta e}{\delta K} = \frac{K}{a/K}\frac{-a}{K^2} = -1$$

There is no reduction or increase in sensitivity

§7. Sensitivity

- Ex.7.12 Sensitivity of Steady-State Error with Step Input

Find the sensitivity of the steady-state error to changes in parameter K and parameter a for the system with a step input



Solution

The steady-state error for this Type 0 system

$$e(\infty)=\frac{1}{1+K_p}=\frac{1}{1+\dfrac{K}{ab}}=\frac{ab}{ab+K}$$

The sensitivity of $e(\infty)$ to changes in parameter a

$$S_{e:a}=\frac{a}{e}\frac{\delta e}{\delta a}=\frac{a}{\dfrac{ab}{ab+K}}\frac{(ab+K)b-ab^2}{(ab+K)^2}=\frac{K}{ab+K}$$

§7. Sensitivity

The steady-state error for this Type 0 system

$$e(\infty)=\frac{1}{1+K_p}=\frac{1}{1+\dfrac{K}{ab}}=\frac{ab}{ab+K}$$

The sensitivity of $e(\infty)$ to changes in parameter a

$$S_{e:a}=\frac{a}{e}\frac{\delta e}{\delta a}=\frac{a}{\dfrac{ab}{ab+K}}\frac{(ab+K)b-ab^2}{(ab+K)^2}=\frac{K}{ab+K}$$

The sensitivity of $e(\infty)$ to changes in parameter K

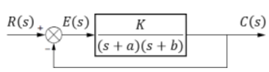
$$S_{e:K}=\frac{K}{e}\frac{\delta e}{\delta K}=\frac{K}{\dfrac{ab}{ab+K}}\frac{-ab}{(ab+K)^2}=\frac{-K}{ab+K}$$

The sensitivity to changes in parameter K and parameter a is less than unity for positive a and b. Thus, feedback in this case yields reduced sensitivity to variations in both parameters

§7. Sensitivity

TryIt 7.4

Use MATLAB, the Symbolic Math Toolbox, and the following statements to find $S_{e:a}$ in Ex.7.12

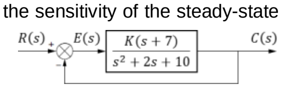



```
syms K a b s
G=K/((s+a)*(s+b));
Kp=subs(G,s,0);
e=1/(1+Kp);
Sea=(a/e)*diff(e,a);
Sea=simple(Sea);
'Sea'
pretty(Sea)
```


§7. Sensitivity

Skill-Assessment Ex.7.6

Problem Find the sensitivity of the steady-state error to changes in K



Solution

The system is Type 0

$$K_p=\lim_{s\to 0}G(s)=7K/10$$

The steady-state error for this Type 0 system

$$e(\infty)=\frac{1}{1+K_p}=\frac{1}{1+7K/10}=\frac{10}{10+7K}$$

The sensitivity of $e(\infty)$ to changes in parameter K

$$S_{e:K}=\frac{K}{e}\frac{\delta e}{\delta K}=\frac{K}{\dfrac{10}{10+7K}}\frac{(-10)\times 7}{(10+7K)^2}=-\frac{7K}{10+7K}$$

§8. Steady-State Error for Systems in State Space

Analysis via Final Value Theorem

- Consider the closed-loop system represented in state space

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}r,\; y=\boldsymbol{C}\boldsymbol{x} \qquad (7.84)$$

- The Laplace transform of the error

$$E(s)=R(s)-Y(s) \qquad (7.85)$$

$$Y(s)=R(s)T(s) \qquad (7.86)$$

$T(s)$: the closed-loop transfer function

$$\Longrightarrow E(s)=R(s)[1-T(s)] \qquad (7.87)$$

- Using Eq.(3.73) for $T(s)$

$$E(s)=R(s)\left[1-\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\right] \qquad (7.88)$$

- Applying the final value theorem

$$\lim_{s\to 0}sE(s)=\lim_{s\to 0}\{sR(s)\left[1-\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\right]\} \qquad (7.89)$$

$$T(s)=\frac{Y(s)}{R(s)}=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D} \qquad (3.73)$$

§8. Steady-State Error for Systems in State Space

- Ex.7.13 Steady-State Error Using the Final Value Theorem

Evaluate the steady-state error for the system with unit step and unit ramp inputs. Use the final value theorem

$$\boldsymbol{A}=\begin{bmatrix}-5 & 1 & 0\\ 0 & -2 & 1\\ 20 & -10 & 1\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0\\0\\1\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}-1 & 1 & 0\end{bmatrix}$$

Solution

The steady-state error

$$e(\infty)=\lim_{s\to 0}\left[sR(s)\left(1-\frac{s+4}{s^3+6s^2+13s+20}\right)\right]$$

$$=\lim_{s\to 0}\left[sR(s)\frac{s^3+6s^2+12s+16}{s^3+6s^2+13s+20}\right]$$

For a unit step, $R(s)=1/s$, and $e(\infty)=4/5$. For a unit ramp, $R(s)=1/s^2$, and $e(\infty)=\infty$. Notice that the system behaves like a Type 0 system

§8. Steady-State Error for Systems in State Space

TryIt 7.5

Use MATLAB, the Symbolic Math Toolbox, and the following statements to find the steady-state error for a step input to the system of Ex.7.13

$$\boldsymbol{A}=\begin{bmatrix}-5 & 1 & 0\\ 0 & -2 & 1\\ 20 & -10 & 1\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0\\0\\1\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}-1 & 1 & 0\end{bmatrix}$$

```
syms s
A= [-5 1 0; 0 -2 1; 20 -10 1];
B=[0;0;1];
C=[-1 1 0];
I=[1 0 0; 0 1 0; 0 0 1];
E=(1/s)*[1-C*[(s*I-A)^-1]*B];
error=subs(s*E,s,0)
```


§8. Steady-State Error for Systems in State Space

Analysis via Input Substitution

Consider the closed-loop system represented in state space

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}r,\ y=\boldsymbol{C}\boldsymbol{x} \qquad (7.84)$$

Step Inputs

If the input is a unit step, $r=1$, a steady-state solution, $\boldsymbol{x}_{ss}$

$$\boldsymbol{x}_{ss}=[V_1\ \ V_2\ \ \cdots\ \ V_n]^T=\boldsymbol{V},\ V_i \text{ is constant} \qquad (7.92)$$

$$\dot{\boldsymbol{x}}_{ss}=\boldsymbol{0} \qquad (7.93)$$

Eq.(7.84)$\Longrightarrow \boldsymbol{0}=\boldsymbol{A}\boldsymbol{V}+\boldsymbol{B},\ y_{ss}=\boldsymbol{C}\boldsymbol{V}$ (7.94)

$$\boldsymbol{V}=-\boldsymbol{A}^{-1}\boldsymbol{B} \qquad (7.95)$$

The steady-state error

$$e(\infty)=1-y_{ss}=1-\boldsymbol{C}\boldsymbol{V}=1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \qquad (7.96)$$

§8. Steady-State Error for Systems in State Space

Ramp Inputs

If the input is an unit ramp, $r=t$, a steady-state solution, $\boldsymbol{x}_{ss}$

$$\boldsymbol{x}_{ss}=[V_1t+W_1\ \ V_2t+W_2\ \ \cdots\ \ V_nt+W_n]^T$$

$$=\boldsymbol{V}t+\boldsymbol{W},\ W_i,\ V_i \text{ are constants} \qquad (7.97)$$

$$\dot{\boldsymbol{x}}_{ss}=[V_1\ \ V_2\ \ \cdots\ \ V_n]^T=\boldsymbol{V} \qquad (7.98)$$

$$\Longrightarrow \boldsymbol{V}=\boldsymbol{A}(\boldsymbol{V}t+\boldsymbol{W})+\boldsymbol{B}t,\ y_{ss}=\boldsymbol{C}(\boldsymbol{V}t+\boldsymbol{W}) \qquad (7.99)$$

In order to balance Eq.(7.99)

equate the matrix coefficients of t $\quad \boldsymbol{A}\boldsymbol{V}=-\boldsymbol{B}$ or $\boldsymbol{V}=-\boldsymbol{A}^{-1}\boldsymbol{B}$

equating constant terms $\quad \boldsymbol{A}\boldsymbol{W}=\boldsymbol{V}$ or $\boldsymbol{W}=\boldsymbol{A}^{-1}\boldsymbol{V}$

Substituting into (7.99) yields

$$y_{ss}=\boldsymbol{C}\left[-\boldsymbol{A}^{-1}\boldsymbol{B}t+\boldsymbol{A}^{-1}(-\boldsymbol{A}^{-1}\boldsymbol{B})\right]=-\boldsymbol{C}\left[\boldsymbol{A}^{-1}\boldsymbol{B}t+(\boldsymbol{A}^{-1})^2\boldsymbol{B}\right]$$

The steady-state error

$$e(\infty)=\lim_{t\to\infty}(t-y_{ss})=\lim_{t\to\infty}\left[\left(1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}\right)t+\boldsymbol{C}(\boldsymbol{A}^{-1})^2\boldsymbol{B}\right] \quad (7.103)$$

§8. Steady-State Error for Systems in State Space

- Ex.7.14 — **Steady-State Error Using Input Substitution**

Evaluate the steady-state error for the system with unit step and unit ramp inputs. Use input substitution

$$\boldsymbol{A}=\begin{bmatrix}-5 & 1 & 0\\ 0 & -2 & 1\\ 20 & -10 & 1\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0\\0\\1\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}-1 & 1 & 0\end{bmatrix}$$

Solution

For a unit step input, the steady-state error

$$e(\infty)=1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}$$
$$=1-0.2$$
$$=0.8$$

For a ramp input, the steady-state error

$$e(\infty)=\lim_{t\to\infty}\left[\left(1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}\right)t+\boldsymbol{C}(\boldsymbol{A}^{-1})^2\boldsymbol{B}\right]$$
$$=\lim_{t\to\infty}(0.8t+0.08)$$
$$=\infty$$

§8. Steady-State Error for Systems in State Space

Skill-Assessment Ex.7.7

Problem: Find the steady-state error for a step input using both the final value theorem and input substitution methods

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$$\boldsymbol{A}=\begin{bmatrix}0 & 1\\ -3 & -6\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0\\1\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}1 & 1\end{bmatrix}$$

Solution: Using the final value theorem

$$e_{\text{step}}(\infty)=\lim_{s\to0}sR(s)\left[1-\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\right]$$
$$=\lim_{s\to0}\left(1-\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s & -1\\ 3 & s+6\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$=\lim_{s\to0}\left(1-\begin{bmatrix}1 & 1\end{bmatrix}\frac{\begin{bmatrix}s+6 & 1\\ -3s & s\end{bmatrix}}{s^2+6s+3}\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$=\lim_{s\to0}\frac{s^2+5s+2}{s^2+6s+3}$$
$$=2/3$$

§8. Steady-State Error for Systems in State Space

Using input substituition

$$e_{\text{step}}(\infty)=1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}$$
$$=1+\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -3 & -6\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$
$$=1+\begin{bmatrix}1 & 1\end{bmatrix}\frac{\begin{bmatrix}-6 & -1\\ 3 & 0\end{bmatrix}}{3}\begin{bmatrix}0\\1\end{bmatrix}$$
$$=1+\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{3}\\ 0\end{bmatrix}$$
$$=\frac{2}{3}$$