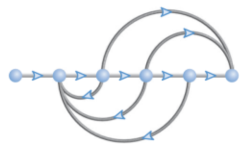
Design via State Space 12



Learning Outcome

After completing this chapter, the student will be able to

- Design a state-feedback controller using pole placement for systems represented in phase-variable form to meet transient response specifications
- Determine if a system is controllable
- Design a state-feedback controller using pole placement for systems not represented in phase-variable form to meet transient response specifications
- Design a state-feedback observer using pole placement for systems represented in observer canonical form
- Determine if a system is observable
- Design a state-feedback observer using pole placement for systems not represented in observer canonical form
- Design steady-state error characteristics for systems represented in state space

§1. Introduction

- Frequency domain methods of design do not allow to specify all poles in systems of order higher than 2 because they do not allow for a sufficient number of unknown parameters to place all of the closed-loop poles uniquely

  → State-space methods solve this problem by introducing

  (1) other adjustable parameters, and

  (2) the technique for finding these parameter values, so that we can properly place all poles of the closed-loop system

§1. Introduction

- Frequency domain methods do allow the specification of closed-loop zero locations, which time domain methods do not allow through placement of the lead compensator zero

  This is a disadvantage of state-space methods, since the location of the zero does affect the transient response

  Also, a state-space design may prove to be very sensitive to parameter changes

§1. Introduction

- There is a wide range of computational support for state-space methods; many software packages support the matrix algebra required by the design process

  However, as mentioned before, the advantages of computer support are balanced by the loss of graphic insight into a design problem that the frequency domain methods yield

- This chapter considers only an introduction to state-space design only to linear systems

§2. Controller Design

- An n^{th}-order feedback control system has an n^{th}-order closed-loop characteristic equation of the form

$$1s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad (12.1)$$

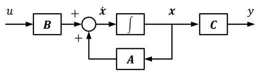
  Since the coefficient of the highest power of s is unity, there are n coefficients whose values determine the system's closed-loop pole locations. Thus, if we can introduce n adjustable parameters into the system and relate them to the coefficients in Eq. (12.1), all of the poles of the closed-loop system can be set to any desired location

§2.Controller Design

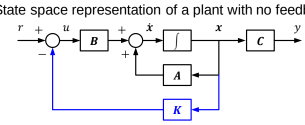
Topology for Pole Placement

- Consider the closed-loop system represented in state space

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u \quad (12.2.a)$$
$$y = \boldsymbol{Cx} \quad (12.2.b)$$



State space representation of a plant with no feedback



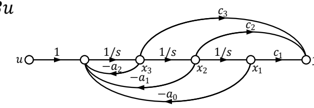
State space representation of a plant with state variable feedback

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u = \boldsymbol{Ax} + \boldsymbol{B}(-\boldsymbol{Kx} + r) = (\boldsymbol{A} - \boldsymbol{BK})x + \boldsymbol{B}r \quad (12.3.a)$$
$$y = \boldsymbol{Cx} \quad (12.3.b)$$

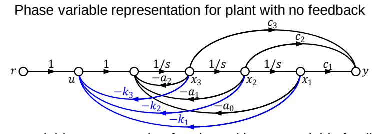
§2.Controller Design

- A plant signal-flow graph in phase-variable (controller canonical) form

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u \quad (12.2.a)$$
$$y = \boldsymbol{Cx} \quad (12.2.b)$$



Phase variable representation for plant with no feedback



Phase variable representation for plant with state variable feedback

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u = \boldsymbol{Ax} + \boldsymbol{B}(-\boldsymbol{Kx} + r) = (\boldsymbol{A} - \boldsymbol{BK})x + \boldsymbol{B}r \quad (12.3.a)$$
$$y = \boldsymbol{Cx} \quad (12.3.b)$$

§2.Controller Design

Pole Placement for Plants in Phase-Variable Form

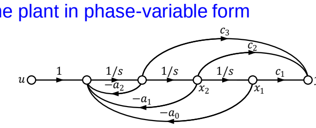
To apply pole-placement methodology to plants represented in phase-variable form

1. Represent the plant in phase-variable form
2. Feed back each phase variable to the input through a gain k_i
3. Find the characteristic equation for the closed-loop system represented in Step 2
4. Decide upon all closed-loop pole locations and determine an equivalent characteristic equation
5. Equate like coefficients of the characteristic equations from Steps 3 and 4 and solve for k_i

§2.Controller Design

Pole Placement for Plants in Phase-Variable Form

1. Represent the plant in phase-variable form



The phase-variable representation of the plant is given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \boldsymbol{C} = [c_1 \quad c_2 \quad \cdots \quad c_n] \quad (12.4)$$

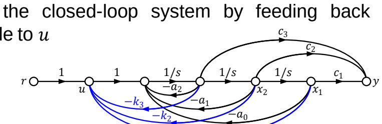
The characteristic equation of the plant

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (12.5)$$

§2.Controller Design

2. Feed back each phase variable to the input through a gain k_i

Form the closed-loop system by feeding back each state variable to u



$$u = -\boldsymbol{Kx} \quad (12.6)$$

where k_i: the phase variables' feedback gains

The system matrix, $\boldsymbol{A} - \boldsymbol{BK}$, for the closed-loop system

$$\boldsymbol{A} - \boldsymbol{BK} = \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \cdots & -(a_{n-1} + k_n) \end{bmatrix} \quad (12.8)$$

§2.Controller Design

3. Find the characteristic equation for the closed-loop system represented in Step 2

$$\det\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\right) = 0$$
$$\rightarrow s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0 \quad (12.9)$$

(12.9) can be derived from (12.5) by adding the appropriate k_i to each coefficient

4. Decide upon all closed-loop pole locations and determine an equivalent characteristic equation

$$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_2 s^2 + d_1 s + d_0 = 0 \quad (12.10)$$

$\boxed{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (12.5)}$

§2.Controller Design

5.Equate like coefficients of the characteristic equations from Steps 3 and 4 and solve for k_i

Equating Eqs. (12.9) and (12.10) to obtain

$$d_i = a_i + k_{i+1} \qquad i = 0,1,2,\dots,n-1$$
$$k_{i+1} = d_i - a_i$$

Note: • For systems represented in phase-variable form, the numerator polynomial is formed from the coefficients of the output coupling matrix, $\boldsymbol{C}$

• The plan and closed-loop system are both in phase-variable form and have the same output coupling matrix → the numerators of their transfer functions are the same

$s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0$ (12.9)
$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_2 s^2 + d_1 s + d_0 = 0$ (12.10)

§2.Controller Design

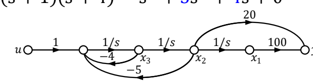
- Ex.12.1 Controller Design for Phase-Variable Form

Design the phase-variable feedback gains to yield $\%OS = 9.5\%$ and $T_s = 0.74s$ $\quad G(s) = \dfrac{20(s+5)}{s(s+1)(s+4)}$

Solution

Represent the plant in phase-variable form

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)} = \frac{1}{s^3 + 5s^2 + 4s + 0} \times (20s + 100)$$



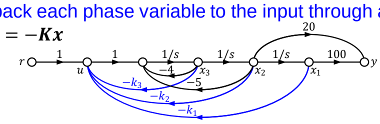
$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, \quad y = [100 \quad 20 \quad 0]\boldsymbol{x}$$

$$\rightarrow s^3 + 5s^2 + 4s + 0 = 0$$

§2.Controller Design

Feed back each phase variable to the input through a gain k_i

$$u = -\boldsymbol{Kx}$$



Find the characteristic equation for the closed-loop system

The closed-loop system's state equations

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r, \; y = [100 \quad 20 \quad 0]\boldsymbol{x}$$

The closed-loop system's characteristic equation

$$\det\big(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\big) = 0$$
$$\rightarrow s^3 + (5 + k_3)s^2 + (4 + k_2)s + k_1 = 0 \qquad (12.16)$$

§2.Controller Design

Decide the closed-loop pole locations and determine an equivalent characteristic equation

The second-order system with the desired performances

$$\zeta = -\frac{ln(\%OS/100)}{\sqrt{\pi^2 + ln^2(\%OS/100)}} = -\frac{ln(9.5/100)}{\sqrt{\pi^2 + ln^2(9.5/100)}} = 0.5996$$

$$\omega_n = \frac{4}{\zeta T_s} = \frac{4}{0.5996 \times 0.74} = 9.0147$$

$$\rightarrow G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{81.2648}{[s + (5.4 - j7.2)][(s + (5.4 + j7.2)]}$$

3rd-order system → select another closed-loop pole: $p_3 = -5.1$

The desired characteristic equation

$$(s + 5.4 - j7.2)(s + 5.4 + j7.2)(s + 5.1) = 0$$
$$\rightarrow s^3 + 15.9s^2 + 136.08s + 413.1 = 0 \qquad (12.17)$$

§2.Controller Design

Equate like coefficients of the characteristic equations from Steps 3 and 4 and solve for k_i

Equating Eqs. (12.16) and (12.17) to obtain

$$k_1 = 413.1, \; k_2 = 132.08, \; k_3 = 10.9$$

The state-space representation of the closed-loop system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -413.1 & -136.08 & -15.9 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (12.19.a)$$

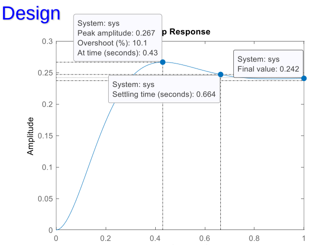
$$y = [100 \quad 20 \quad 0]\boldsymbol{x} \qquad (12.19.b)$$

The closed-loop transfer function

$$T(s) = \frac{20(s+5)}{s^3 + 15.9s^2 + 136.08s + 413.1}$$

$s^3 + (5 + k_3)s^2 + (4 + k_2)s + k_1 = 0$ (12.16)
$s^3 + 15.9s^2 + 136.08s + 413.1 = 0$ (12.17)

§2.Controller Design



From the simulation results: $\%OS = 10.1\%$, $T_s = 0.664s$

The steady-state response approaches 0.242 instead of unity, there is a large steady-state error → Design techniques to reduce this error are discussed in Section 12.8

§2. Controller Design

Run ch12p1 in Appendix B

Learn how to use MATLAB to

- design a controller for phase variables using pole placement
- solve Ex.12.1

§2. Controller Design

Skill-Assessment Ex.12.1

Problem For the plant

$$G(s)=\frac{100(s+10)}{s(s+3)(s+12)}$$

represented in the state space in phase-variable form by

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -36 & -15\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

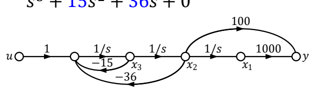
$$y=[1000 \quad 100 \quad 0]\boldsymbol{x}$$

design the phase-variable feedback gains to yield $\%OS=5\%$ and $T_p=0.3s$

§2. Controller Design

Solution Represent the plant in phase-variable form

$$G(s)=\frac{1}{s^3+15s^2+36s+0}\times(100s+1000)$$



$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -36 & -15\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$y=[1000 \quad 100 \quad 0]\boldsymbol{x}$$

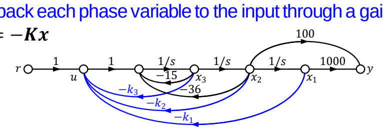
$$\rightarrow s^3+5s^2+4s+0=0$$

$G(s)=\frac{100(s+10)}{s(s+3)(s+12)}$

§2. Controller Design

Feed back each phase variable to the input through a gain k_i

$$u=-\boldsymbol{K}\boldsymbol{x}$$



Find the characteristic equation for the closed-loop system

The closed-loop system's state equations

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -k_1 & -(k_2+36) & -(k_3+15)\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$y=[1000 \quad 100 \quad 0]\boldsymbol{x}$$

The closed-loop system's characteristic equation

$$s^3+(k_3+15)s^2+(k_2+36)s+k_1=0 \qquad (1)$$

§2. Controller Design

Decide the closed-loop pole locations and determine an equivalent characteristic equation

The second-order system with the desired performances

$$\zeta=\frac{-ln(\%OS/100)}{\sqrt{\pi^2+ln^2(\%OS/100)}}=\frac{-ln(0.05)}{\sqrt{\pi^2+ln^2(0.05)}}=0.6901$$

$$\omega_n=\frac{\pi}{T_p\sqrt{1-\zeta^2}}=\frac{\pi}{0.3\sqrt{1-0.6901^2}}=14.4699$$

$$\rightarrow s^2+2\zeta\omega_n s+\omega_n^2=s^2+19.97s+209.4=0$$

3rd-order system → select the third pole -10 to cancel the zero at -10 → the desired characteristic equation

$$(s^2+19.97s+209.4)(s+10)=0$$

$$\rightarrow s^3+29.97s^2+409.1s+2094=0 \qquad (2)$$

§2. Controller Design

Equate like coefficients of the characteristic equations from Steps 3 and 4 and solve for k_i

Equating Eqs. (1) and (2) to obtain

$$k_1=2094,\ k_2=373.1,\ k_3=14.97$$

The state-space representation of the closed-loop system

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2094 & -409.1 & -29.97\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

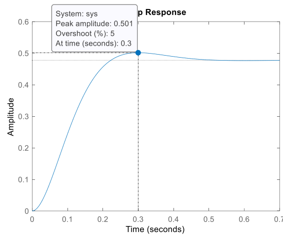
$$y=[1000 \quad 100 \quad 0]\boldsymbol{x}$$

The closed-loop transfer function

$$T(s)=\frac{100(s+10)}{s^3+29.97s^2+409.1s+2094}$$

$s^3+(k_3+15)s^2+(k_2+36)s+k_1=0$ (1)
$s^3+29.97s^2+409.1s+2094=0$ (2)

§2. Controller Design



$$T(s) = \frac{100(s+10)}{s^3 + 29.97s^2 + 409.1s + 2094}$$

From the simulation results: $\%OS = 50.1\%$, $T_p = 0.3s$

§2. Controller Design

TryIt 12.1

Use MATLAB, the Control System Toolbox, and the following statements to solve for the phase-variable feedback gains to place the poles of the system in Skill-Assessment Ex.12.1 at $-3-j5$; $-3+j5$, and -10

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -36 & -15 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1000 & 100 & 0 \end{bmatrix}\boldsymbol{x}$$

```
A=[0 1 0; 0 0 1; 0 -36 -15];
B=[0;0;1];
poles=[-3+5j,-3-5j,-10];
K=acker(A,B,poles)
```


§3. Controllability

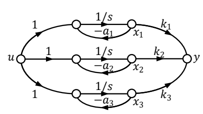
The system is controllable if an input to a system can take every state variable from a desired initial state to a desired final state

Controllability by Inspection

A system with distinct eigenvalues and a diagonal system matrix is controllable if the input coupling matrix B does not have any rows that are zero

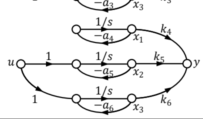
- Controllable system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$



- Uncontrollable system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -a_4 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & -a_6 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$



§3. Controllability

The Controllability Matrix

An n^{th}-order plant whose state equation is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$

is completely controllable if the matrix

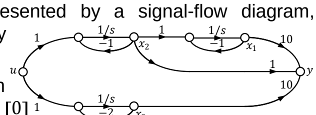
$$\boldsymbol{C_M} = [\boldsymbol{B} \;\; \boldsymbol{AB} \;\; \boldsymbol{A}^2\boldsymbol{B} \;\; \cdots \;\; \boldsymbol{A}^{n-1}\boldsymbol{B}]$$

is of rank n, where $\boldsymbol{C_M}$ is called the controllability matrix

§3. Controllability

- Ex.12.2 Controllability via the Controllability Matrix

Given the system, represented by a signal-flow diagram, determine its controllability



Solution

The system state equation

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

There is the zero in the B matrix, this configuration leads to uncontrollability only if the poles are real and distinct. In this case, the system has multiple poles at -1

The controllability matrix $\boldsymbol{C_M} = \left[\boldsymbol{B} \;\; \boldsymbol{AB} \;\; \boldsymbol{A}^2\boldsymbol{B}\right] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$

$|\boldsymbol{C_M}| = -1 \neq 0 \Longrightarrow \text{rank}\{\boldsymbol{C_M}\} = 3$: the system is controllable

§3. Controllability

MATLAB ML

Run ch12p2 in Appendix B

Learn how to use MATLAB to

- test a system for controllability
- solve Ex.12.2

§3. Controllability

Skill-Assessment Ex.12.2

Problem Determine whether the system

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$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

is controllable

Solution The controllability matrix

$$\boldsymbol{C_M} = [\boldsymbol{B}\ \ \boldsymbol{AB}\ \ \boldsymbol{A}^2\boldsymbol{B}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & -9 \\ 1 & -1 & 16 \end{bmatrix}$$

$\rightarrow |\boldsymbol{C_M}| = 80 \neq 0$

$\text{rank}\{\boldsymbol{C_M}\} = 3$: the system is controllable

§3. Controllability

TryIt 12.2

Use MATLAB, the Control System Toolbox, and the following statements to solve Skill-Assessment Ex.12.2

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

```
A=[-1 1 2; 0 -1 5; 0 3 -4];
B=[2;1;1];
Cm=ctrb(A,B)
Rank=rank(Cm)
```


§4. Alternative Approaches to Controller Design

1st method: Matching the coefficients of $\det(sI - (A - BK))$ with the coefficients of the desired characteristic equation

- Ex.12.3 Controller Design by Matching Coefficients

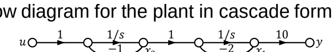
Design state feedback for the plant represented in cascade form to yield $OS\% = 15\%$, $T_s = 0.5s$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)}$$

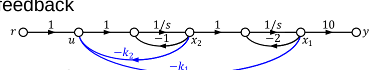
Solution

The signal-flow diagram for the plant in cascade form

- open loop



- with feedback



The state equations

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 1 \\ -k_1 & -(k_2+1) \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r,\ y = [10\ \ 0]\boldsymbol{x}$$

§4. Alternative Approaches to Controller Design

The state equations

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 1 \\ -k_1 & -(k_2+1) \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r,\ y = [10\ \ 0]\boldsymbol{x}$$

The characteristics equation

$$s^2 + (k_2+3)s + (2k_2 + k_1 + 2) = 0 \tag{12.32}$$

The desired characteristic equation

$$\zeta = -\frac{\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = -\frac{\ln(15/100)}{\sqrt{\pi^2 + \ln^2(15/100)}} = 0.5169$$

$$\omega_n = \frac{4}{T_s\zeta} = \frac{4}{0.5 \times 0.5169} = 15.4769 rad/s$$

$$\rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 16s + 239.5 = 0 \tag{12.33}$$

Equating the coefficients of Eqs. (12.32) and (12.33)

$$\left.\begin{aligned} k_2 + 3 &= 16 \\ 2k_2 + k_1 + 2 &= 239.5 \end{aligned}\right\} \rightarrow \begin{aligned} k_1 &= 211.5 \\ k_2 &= 13 \end{aligned}$$

§4. Alternative Approaches to Controller Design

2nd method: Transforming the system to phase variables, designing the feedback gains, and transforming the designed system back to its original state-variable representation

- Assume a plant not represented in phase-variable form

$$\dot{\boldsymbol{z}} = \boldsymbol{Az} + \boldsymbol{B}u,\ y = \boldsymbol{Cz} \tag{12.34}$$

Controllability matrix

$$\boldsymbol{C_{Mz}} = [\boldsymbol{B}\ \ \boldsymbol{AB}\ \ \boldsymbol{A}^2\boldsymbol{B} \cdots \boldsymbol{A}^{n-1}\boldsymbol{B}] \tag{12.35}$$

- Assume that the system can be transformed into the phase-variable ($\boldsymbol{x}$) representation with the transformation

$$\boldsymbol{z} = \boldsymbol{Px} \tag{12.36}$$

Substituting this transformation into Eqs. (12.34)

$$\dot{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{APx} + \boldsymbol{P}^{-1}\boldsymbol{B}u,\ y = \boldsymbol{CPx} \tag{12.37}$$

§4. Alternative Approaches to Controller Design

$$\dot{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{APx} + \boldsymbol{P}^{-1}\boldsymbol{B}u,\ y = \boldsymbol{CPx} \tag{12.37}$$

Controllability matrix

$$\begin{aligned} \boldsymbol{C_{Mx}} &= \Big[\boldsymbol{P}^{-1}\boldsymbol{B}\ \ (\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{B})\ \ (\boldsymbol{P}^{-1}\boldsymbol{AP})^2(\boldsymbol{P}^{-1}\boldsymbol{B})\ \cdots \\ &\qquad (\boldsymbol{P}^{-1}\boldsymbol{AP})^{n-1}(\boldsymbol{P}^{-1}\boldsymbol{B})\Big] \\ &= [\boldsymbol{P}^{-1}\boldsymbol{B}\ \ (\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{B})\ \ (\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{B}) \\ &\qquad \cdots (\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{AP}) \cdots (\boldsymbol{P}^{-1}\boldsymbol{B})] \\ &= \boldsymbol{P}^{-1}[\boldsymbol{B}\ \ \boldsymbol{AB}\ \ \boldsymbol{A}^2\boldsymbol{B} \cdots \boldsymbol{A}^{n-1}\boldsymbol{B}] \end{aligned} \tag{12.38}$$

Substituting Eq. (12.35) into (12.38) and solving for $\boldsymbol{P}$

$$\boldsymbol{P} = \boldsymbol{C_{Mz}}\boldsymbol{C_{Mx}^{-1}} \tag{12.39}$$

$\rightarrow$ the transformation matrix $\boldsymbol{P}$ can be found from the two controllability matrices

$$\boldsymbol{C_{Mz}} = [\boldsymbol{B}\ \ \boldsymbol{AB}\ \ \boldsymbol{A}^2\boldsymbol{B} \cdots \boldsymbol{A}^{n-1}\boldsymbol{B}] \tag{12.35}$$

§4. Alternative Approaches to Controller Design

- Design the feedback gains, $u = -\boldsymbol{K}_x \boldsymbol{x} + r$

$$\dot{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{x} + \boldsymbol{P}^{-1}\boldsymbol{B}u$$
$$= \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{x} - \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{K}_x\boldsymbol{x} + \boldsymbol{P}^{-1}\boldsymbol{B}r$$
$$= \left(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} - \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{K}_x\right)\boldsymbol{x} + \boldsymbol{P}^{-1}\boldsymbol{B}r \quad (12.40.a)$$
$$y = \boldsymbol{C}\boldsymbol{P}\boldsymbol{x} \quad (12.40.b)$$

Since this equation is in phase-variable form, the zeros of this closed-loop system are determined from the polynomial formed from the elements of $\boldsymbol{CP}$

§4. Alternative Approaches to Controller Design

$$\dot{\boldsymbol{x}} = \left(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} - \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{K}_x\right)\boldsymbol{x} + \boldsymbol{P}^{-1}\boldsymbol{B}r,\ y = \boldsymbol{C}\boldsymbol{P}\boldsymbol{x} \quad (12.40)$$

- Transform Eqs. (12.40) from phase variables back to the original representation using $\boldsymbol{x} = \boldsymbol{P}^{-1}\boldsymbol{z}$

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} - \boldsymbol{B}\boldsymbol{K}_x\boldsymbol{P}^{-1}\boldsymbol{z} + \boldsymbol{B}r = \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_x\boldsymbol{P}^{-1}\right)\boldsymbol{z} + \boldsymbol{B}r \quad (12.41.a)$$
$$y = \boldsymbol{C}\boldsymbol{z} \quad (12.41.b)$$

- Comparing Eqs. (12.41) with (12.3), to find the state variable feedback gain, $\boldsymbol{K}_z$, for the original system

$$\boldsymbol{K}_z = \boldsymbol{K}_x\boldsymbol{P}^{-1} \quad (12.42)$$

The TF of this closed-loop system is the same as the TF for Eqs. (12.40), since Eqs. (12.40) and (12.41) represent the same system. Thus, the zeros of the closed-loop TF are the same as the zeros of the uncompensated plant, based upon the development in Section 12.2

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(-\boldsymbol{K}\boldsymbol{x} + r) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}r,\ y = \boldsymbol{C}\boldsymbol{x} \quad (12.3)$$

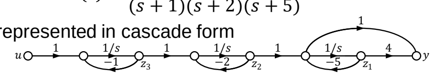
§4. Alternative Approaches to Controller Design

- Ex.12.4 Controller Design by Transformation

Design a state-variable feedback controller to yield a $\%OS = 20.8\%$ and $T_s = 4s$ for a plant

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+5)}$$

that is represented in cascade form



Solution

Original system

The state equations

$$\dot{\boldsymbol{z}} = \boldsymbol{A}_z\boldsymbol{z} + \boldsymbol{B}_z u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$
$$y = \boldsymbol{C}_z\boldsymbol{z} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\boldsymbol{z}$$

§4. Alternative Approaches to Controller Design

The controllability matrix

$$\boldsymbol{C}_{Mz} = \left[\boldsymbol{B}_z\ \ \boldsymbol{A}_z\boldsymbol{B}_z\ \ \boldsymbol{A}_z^2\boldsymbol{B}_z\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

Since $\det(\boldsymbol{C}_{Mz}) = -1 \neq 0$, the system is controllable

The characteristic equation

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = s^3 + 8s^2 + 17s + 10 = 0$$

Phase-variable representation of the system

Using the coefficients of the above equation to write

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_x\boldsymbol{x} + \boldsymbol{B}_x u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$
$$y = \boldsymbol{C}_x\boldsymbol{x} = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix}\boldsymbol{x}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{A}_z\boldsymbol{z} + \boldsymbol{B}_z u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u,\ y = \boldsymbol{C}_z\boldsymbol{z} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\boldsymbol{z}$$

§4. Alternative Approaches to Controller Design

The output equation was written using the coefficients of the numerator of $G(s)$, since the transfer function must be the same for the two representations

The controllability matrix, $\boldsymbol{C}_{Mx}$, for the phase-variable system

$$\boldsymbol{C}_{Mx} = \left[\boldsymbol{B}_x\ \ \boldsymbol{A}_x\boldsymbol{B}_x\ \ \boldsymbol{A}_x^2\boldsymbol{B}_x\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -8 \\ 1 & -8 & 47 \end{bmatrix} \quad (12.48)$$

Calculate the transformation matrix

The transformation matrix between the two systems

$$\boldsymbol{P} = \boldsymbol{C}_{Mz}\boldsymbol{C}_{Mx}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 7 & 1 \end{bmatrix} \quad (12.49)$$

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+5)},\ \boldsymbol{C}_{Mz} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

§4. Alternative Approaches to Controller Design

Design the controller using the phase-variable representation

The desired closed-loop system

- $\left.\begin{array}{l} OS = 20.8\% \\ T_s = 4s \end{array}\right\} \rightarrow$ the designed closed-loop system: $s^2 + 2s + 5$
- The closed-loop zero will be at $s = -4$ → choose the third closed-loop pole to cancel the closed-loop zero
- The total characteristic equation of the desired closed-loop system

$$D(s) = (s+4)\left(s^2 + 2s + 5\right)$$
$$= s^3 + 6s^2 + 13s + 20$$
$$= 0 \quad (12.50)$$

§4. Alternative Approaches to Controller Design

The designed closed-loop system

- The state equations for the phase-variable form with state-variable feedback

$$\dot{\boldsymbol{x}} = (\boldsymbol{A}_x - \boldsymbol{B}_x\boldsymbol{K}_x)\boldsymbol{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(10+k_{1_x}) & -(17+k_{2_x}) & -(8+k_{3_x}) \end{bmatrix}\boldsymbol{x}$$

$$y = \boldsymbol{C}_x\boldsymbol{x} = [4 \quad 1 \quad 0]\boldsymbol{x}$$

- The characteristic equation

$$\det\left(s\boldsymbol{I} - (\boldsymbol{A}_x - \boldsymbol{B}_x\boldsymbol{K}_x)\right)$$
$$= s^3 + \left(8 + k_{3_x}\right)s^2 + \left(17 + k_{2_x}\right)s + \left(10 + k_{1_x}\right)$$
$$= 0 \qquad (12.52)$$

§4. Alternative Approaches to Controller Design

Find $\boldsymbol{K}_x$

$$s^3 + 6s^2 + 13s + 20 = 0 \qquad (12.50)$$

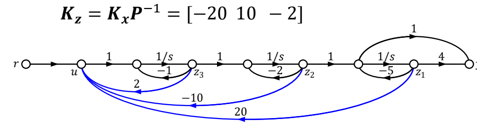
$$s^3 + \left(8 + k_{3_x}\right)s^2 + \left(17 + k_{2_x}\right)s + \left(10 + k_{1_x}\right) = 0 \quad (12.52)$$

Comparing Eq.(12.50) with (12.52)

$$\boldsymbol{K}_x = \left[k_{1_x} \ k_{2_x} \ k_{3_x}\right] = [10 \ \ -4 \ \ -2]$$

Transform the controller back to the original system

$$\boldsymbol{K}_z = \boldsymbol{K}_x\boldsymbol{P}^{-1} = [-20 \ \ 10 \ \ -2]$$



Designed system with state-variable feedback

§4. Alternative Approaches to Controller Design

Verify the design

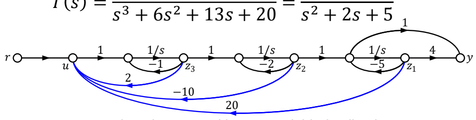
The state equations for the designed system

$$\dot{\boldsymbol{z}} = (\boldsymbol{A}_z - \boldsymbol{B}_z\boldsymbol{K}_z)\boldsymbol{z} + \boldsymbol{B}_z r = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 20 & -10 & 1 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \boldsymbol{C}_z\boldsymbol{z} = [-1 \quad 1 \quad 0]\boldsymbol{z}$$

The closed-loop TF

$$T(s) = \frac{s+4}{s^3 + 6s^2 + 13s + 20} = \frac{1}{s^2 + 2s + 5}$$



Designed system with state-variable feedback

§4. Alternative Approaches to Controller Design

MATLAB ML

Run ch12p3 in Appendix B

Learn how to use MATLAB to

- design a controller for a plant not represented in phase-variable form
- see that MATLAB does not require transformation to phase-variable form
- solve Ex.12.4

§4. Alternative Approaches to Controller Design

Skill-Assessment Ex.12.3

<u>Problem</u> Design a linear state-feedback controller to yield 20% overshoot and a settling time of $2s$ for a plant

WileyPLUS WPCS Control Solutions

$$G(s) = \frac{s+6}{(s+9)(s+8)(s+7)}$$

<u>Solution</u> First check controllability

$$\boldsymbol{C}_{Mz} = \left[\boldsymbol{B} \ \ \boldsymbol{AB} \ \ \boldsymbol{A}^2\boldsymbol{B}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix}$$

$|\boldsymbol{C}_{Mz}| = -1 \neq 0 \Longrightarrow \mathrm{rank}\{\boldsymbol{C}_M\} = 3$: the system is controllable

Now find the desired characteristic equation

$$\left.\begin{matrix} OS = 20\% \\ T_s = 2s \end{matrix}\right\} \rightarrow \begin{matrix} \xi = 0.456 \\ \omega_n = 4.386 \end{matrix}$$

$$\Longrightarrow s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + 4s + 19.24 = 0$$

§4. Alternative Approaches to Controller Design

To cancel the zero at -6, adding a pole at -6 yields the resulting desired characteristic equation

$$\left(s^2 + 4s + 19.24\right)(s+6)$$
$$= s^3 + 10s^2 + 43.24s + 115.45 = 0$$

Since $G(s) = \dfrac{s+6}{(s+9)(s+8)(s+7)}$

$$= \frac{s+6}{s^3 + 24s^2 + 191s + 504}$$

We can write the phase-variable representation

$$\boldsymbol{A}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504 & -191 & -24 \end{bmatrix}, \boldsymbol{B}_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{C}_p = [6 \ \ 1 \ \ 0]$$

$s^2 + 4s + 19.24 = 0$

§4. Alternative Approaches to Controller Design

The compensated system matrix in phase-variable form

$$\boldsymbol{A}_p - \boldsymbol{B}_p\boldsymbol{K}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(504+k_1) & -(191+k_2) & -(24+k_3) \end{bmatrix}$$

The characteristic equation for this system

$$s\boldsymbol{I} - \left(\boldsymbol{A}_p - \boldsymbol{B}_p\boldsymbol{K}_p\right)$$
$$= s^3 + (24+k_3)s^2 + (191+k_2)s + (504+k_1)$$

Equating coefficients of this equation with the coefficients of the desired characteristic equation yields the gains

$$\boldsymbol{K}_p = [k_1 \;\; k_2 \;\; k_3]$$
$$= [-388.55 \;\; -147.76 \;\; -14]$$

$s^3 + 10s^2 + 43.24s + 115.45 = 0$

§4. Alternative Approaches to Controller Design

Now develop the transformation matrix to transform back to the z-system

$$\boldsymbol{C}_{\boldsymbol{Mz}} = [\boldsymbol{B}_z \;\; \boldsymbol{A}_z\boldsymbol{B}_z \;\; \boldsymbol{A}_z^2\boldsymbol{B}_z] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix}$$

$$\boldsymbol{C}_{\boldsymbol{Mp}} = [\boldsymbol{B}_p \;\; \boldsymbol{A}_p\boldsymbol{B}_p \;\; \boldsymbol{A}_p^2\boldsymbol{B}_p] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -24 \\ 1 & -24 & 385 \end{bmatrix}$$

Therefore

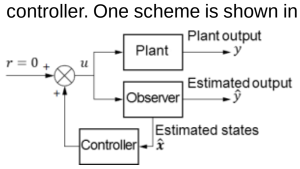
$$\boldsymbol{P} = \boldsymbol{C}_{\boldsymbol{Mz}}\boldsymbol{C}_{\boldsymbol{Mp}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 56 & 15 & 1 \end{bmatrix}$$

Hence,

$$\boldsymbol{K}_z = \boldsymbol{K}_p\boldsymbol{C}_{\boldsymbol{Mp}}^{-1}$$
$$= [-388.55 \;\; -147.76 \;\; -14]\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 56 & 15 & 1 \end{bmatrix}$$
$$= [-40.23 \;\; 62.24 \;\; -14]$$

§5. Observer Design

- Controller design relies upon access to the state variables for feedback through adjustable gains
- Some of the state variables may not be available at all, or it is too costly to measure them or send them to the controller
- If the state variables are not available because of system configuration or cost, it is possible to estimate the states. Estimated states, rather than actual states, are then fed to the controller. One scheme is shown in the figure



An observer, sometimes called an estimator, is used to calculate state variables that are not accessible from the plant. Here the observer is a model of the plant

§5. Observer Design

- Let's look at the disadvantages of such configuration. Assume the plant

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u,\; y = \boldsymbol{C}\boldsymbol{x} \qquad (12.57)$$

and an observer

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}u,\; \hat{y} = \boldsymbol{C}\hat{\boldsymbol{x}} \qquad (12.58)$$

Subtracting Eqs. (12.58) from (12.57) to obtain

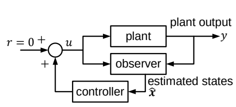
$$\dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}(\boldsymbol{x} - \hat{\boldsymbol{x}}),\; y - \hat{y} = \boldsymbol{C}(\boldsymbol{x} - \hat{\boldsymbol{x}}) \qquad (12.59)$$

Thus, the dynamics of the difference between the actual and estimated states is unforced, and if the plant is stable, this difference, due to differences in initial state vectors, approaches zero

The speed of convergence $\boldsymbol{x} \to \hat{\boldsymbol{x}}$ is too slow → using feedback to increase the speed of convergence between the actual and estimated states

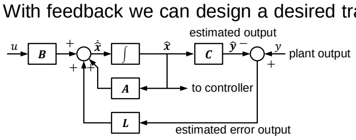
§5. Observer Design

- Use feedback to increase the speed of convergence between the actual and estimated states



The error between the outputs of the plant and the observer is fed back to the derivatives of the observer's states. The system corrects to drive this error to zero

With feedback we can design a desired transient response into



the observer that is much quicker than that of the plant or controlled closed-loop system

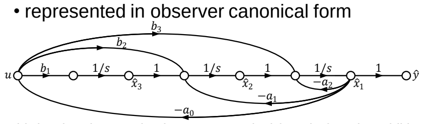
§5. Observer Design

- In designing a controller, the controller canonical (phase-variable) form yields an easy solution for the controller gains

In designing an observer, the observer canonical form yields the easy solution for the observer gains

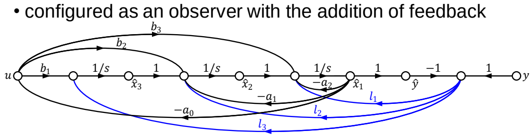
§5. Observer Design

- Example a third-order plant
  - represented in observer canonical form



Third-order observer in observer canonical form before the addition of the feedback

  - configured as an observer with the addition of feedback



Third-order observer in observer canonical form after the addition of the feedback

§5. Observer Design

- The design of the observer is separate from the design of the controller
- Similar to the design of the controller vector, $\boldsymbol{K}$, the design of the observer consists of evaluating the constant vector, $\boldsymbol{L}$, so that the transient response of the observer is faster than the response of the controlled loop in order to yield a rapidly updated estimate of the state vector
  - Find the state equations for the error between the actual state vector and the estimated state vector, $\boldsymbol{x}-\hat{\boldsymbol{x}}$
  - Find the characteristic equation for the error system and evaluate the required $\boldsymbol{L}$ to meet a rapid transient response for the observer

§5. Observer Design

- The state equations of the observer

$$\dot{\hat{\boldsymbol{x}}}=\boldsymbol{A}\hat{\boldsymbol{x}}+\boldsymbol{B}u+\boldsymbol{L}(y-\hat{y}),\ \hat{y}=\boldsymbol{C}\hat{\boldsymbol{x}} \qquad (12.60)$$

- The state equations for the plant

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}u,\ y=\boldsymbol{C}\boldsymbol{x} \qquad (12.61)$$

- Subtracting Eqs. (12.60) from (12.61) to obtain

$$\dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}=\boldsymbol{A}(\boldsymbol{x}-\hat{\boldsymbol{x}})-\boldsymbol{L}(y-\hat{y}),\ y-\hat{y}=\boldsymbol{C}(\boldsymbol{x}-\hat{\boldsymbol{x}}) \qquad (12.62)$$

$\boldsymbol{x}-\hat{\boldsymbol{x}}$: the error between the actual state vector and the estimated state vector

$y-\hat{y}$: the error between the actual output and the estimated output

- Subtracting the output equation into the state equation to obtain

$$\dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}=(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})(\boldsymbol{x}-\hat{\boldsymbol{x}}),\ y-\hat{y}=\boldsymbol{C}(\boldsymbol{x}-\hat{\boldsymbol{x}}) \qquad (12.63)$$

§5. Observer Design

$$\dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}=(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})(\boldsymbol{x}-\hat{\boldsymbol{x}}),\ y-\hat{y}=\boldsymbol{C}(\boldsymbol{x}-\hat{\boldsymbol{x}}) \qquad (12.63)$$

or $$\dot{\boldsymbol{e}}_x=(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\boldsymbol{e}_x,\ y-\hat{y}=\boldsymbol{C}(\boldsymbol{x}-\hat{\boldsymbol{x}}) \qquad (12.64)$$

$\boldsymbol{e}_x$: the estimated state error, $\boldsymbol{e}_x=\boldsymbol{x}-\hat{\boldsymbol{x}}$

- Eq. (12.64a) is unforced. If the eigenvalues are all negative, the estimated state vector error, $\boldsymbol{e}_x$, will decay to zero. The design then consists of solving for the values of $\boldsymbol{L}$ to yield a desired characteristic equation or response for Eq. (12.64). The characteristic equation is found from Eq. (12.64) to be

$$\det\big(\lambda\boldsymbol{I}-(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\big)=0 \qquad (12.65)$$

Now we select the eigenvalues of the observer to yield stability and a desired transient response that is faster than the controlled closed-loop response. These eigenvalues determine a characteristic equation that we set equal to Eq. (12.65) to solve for $\boldsymbol{L}$

§5. Observer Design

- Ex.12.5 Observer Design for Observer Canonical Form

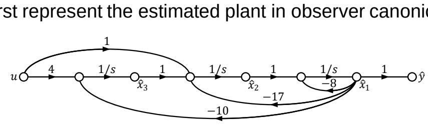
Design an observer for the plant

$$G(s)=\frac{s+4}{(s+1)(s+2)(s+5)}=\frac{s+4}{s^3+8s^2+17s+10}$$

which is represented in observer canonical form. The observer will respond 10 times faster than the controlled loop designed in Ex.12.4

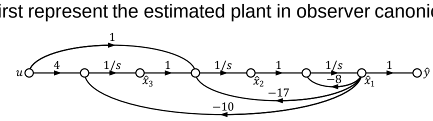
Solution

1. First represent the estimated plant in observer canonical form

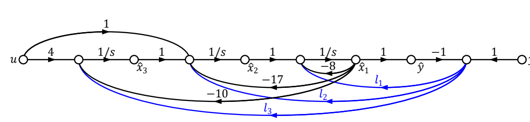


§5. Observer Design

1. First represent the estimated plant in observer canonical form

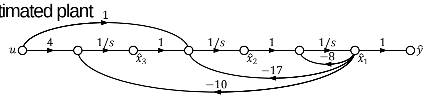


2. Now form the difference between the plant's actual output, y, and the observer's estimated output, $\hat{y}$, and add the feedback paths from this difference to the derivative of each state variable



§5. Observer Design

3. Next find the characteristic polynomial. The state equations for the estimated plant



$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}u = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix}\hat{\boldsymbol{x}} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}u, \hat{y} = \boldsymbol{C}\hat{\boldsymbol{x}} = [1 \quad 0 \quad 0]\hat{\boldsymbol{x}}$$

The observer error

$$\dot{\boldsymbol{e}}_x = (\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e}_x = \begin{bmatrix} -(8 + l_1) & 1 & 0 \\ -(17 + l_2) & 0 & 1 \\ -(10 + l_3) & 0 & 0 \end{bmatrix}\boldsymbol{e}_x$$

The characteristic polynomial

$$s^3 + (8 + l_1)s^2 + (17 + l_2)s + (10 + l_3) = 0 \qquad (12.74)$$

§5. Observer Design

4. Now evaluate the desired polynomial, set the coefficients equal to those of Eq. (12.74), and solve for the gains, l_i. From Eq. (12.50), the closed-loop controlled system has dominant second-order poles at $-1 \pm j2$. To make our observer 10 times faster, we design the observer poles to be at $-10 \pm j20$. We select the third pole to be 10 times the real part of the dominant second-order poles, or -100. Hence, the desired characteristic polynomial

$$(s + 100)(s^2 + 20s + 500) =$$
$$s^3 + 120s^2 + 2500s + 50{,}000 = 0 \qquad (12.75)$$

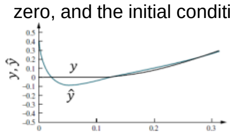
Equating Eqs. (12.74) and (12.75) to obtain

$$l_1 = 112,\ l_2 = 2483,\ l_3 = 49{,}990$$

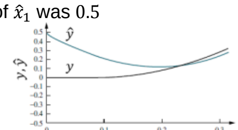
$$(s + 4)(s^2 + 2s + 5) = s^3 + 6s^2 + 13s + 20 = 0 \qquad (12.50)$$
$$s^3 + (8 + l_1)s^2 + (17 + l_2)s + (10 + l_3) = 0 \qquad (12.74)$$

§5. Observer Design

A simulation of the observer with an input of $r(t) = 100t$ is shown in the figure. The initial conditions of the plant were all zero, and the initial condition of $\hat{x}_1$ was 0.5



Simulation showing response of observer
a. closed-loop



Simulation showing response of observer
b. open-loop with observer gains disconnected

Since the dominant pole of the observer is $-10 \pm j20$, the expected settling time should be about $0.4s$. It is interesting to note the slower response in the figure, where the observer gains are disconnected, and the observer is simply a copy of the plant with a different initial condition

§5. Observer Design

MATLAB ML

Run ch12p4 in Appendix B

Learn how to use MATLAB to

- design an observer using pole placement
- solve Ex.12.5

§5. Observer Design

Skill-Assessment Ex.12.4

<u>Problem</u> Design an observer for the plant

WileyPLUS WPCS Control Solutions

$$G(s) = \frac{s + 6}{(s + 9)(s + 8)(s + 7)}$$

whose estimated plant is represented in state space in observer canonical form as

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}u = \begin{bmatrix} -24 & 1 & 0 \\ -191 & 0 & 1 \\ -504 & 0 & 0 \end{bmatrix}\hat{\boldsymbol{x}} + \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}u$$

$$\hat{y} = \boldsymbol{C}\hat{\boldsymbol{x}} = [1 \quad 0 \quad 0]\hat{\boldsymbol{x}}$$

The observer will respond 10 times faster than the controlled loop designed in Skill-Assessment Ex.12.3

§5. Observer Design

<u>Solution</u> The plant is given by

$$G(s) = \frac{s + 6}{(s + 9)(s + 8)(s + 7)} = \frac{20}{s^3 + 14s^2 + 56s + 64}$$

The characteristic polynomial for the plant with phase-variable state feedback

$$s^3 + (k_3 + 14)s^2 + (k_2 + 56)s + (k_3 + 64) = 0$$

The desired characteristic equation

$$(s + 53.33)(s^2 + 10.67s + 106.45) =$$
$$s^3 + 64s^2 + 675.48s + 5676.98 = 0$$

based upon 15% overshoot, $T_s = 0.75s$, and a third pole ten times further from the imaginary axis than the dominant poles

Comparing the two characteristic equations

$$k_1 = 5612.98,\ k_2 = 619.48, \text{ and } k_3 = 50$$

§5. Observer Design

TryIt 12.3
Use MATLAB, the Control System Toolbox, and the following statements to solve Skill-Assessment Ex.12.4

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}u = \begin{bmatrix} -24 & 1 & 0 \\ -191 & 0 & 1 \\ -504 & 0 & 0 \end{bmatrix}\hat{\boldsymbol{x}} + \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}u$$

$$\hat{y} = \boldsymbol{C}\hat{\boldsymbol{x}} = [1 \quad 0 \quad 0]\hat{\boldsymbol{x}}$$

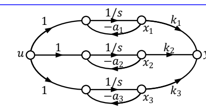
```
A=[-24 1 0; -191 0 1; -504 0 0];
C=[1 0 0]
pos=20
Ts=2
z=(-log(pos/100))/(sqrt(pi^2 +log(pos/100)^2));
wn=4/(z*Ts);
r=roots([1,2*z*wn,wn^2]);
poles=10*[r' 10*real(r(1))]
l=acker(A',C',poles)'
```


§6. Observability

Recall that the ability to control all of the state variables is a requirement for the design of a controller. State-variable feedback gains cannot be designed if any state variable is uncontrollable. Uncontrollability can be viewed best with diagonalized systems. The signal-flow graph showed clearly that the uncontrollable state variable was not connected to the control signal of the system

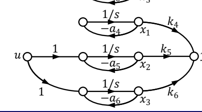
- Controllable system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u$$

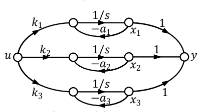


- Uncontrollable system

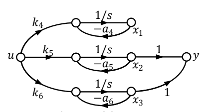
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -a_4 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & -a_6 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}u$$



§6. Observability



observable system

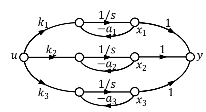


unobservable system

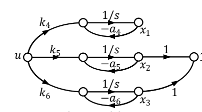
The ability to observe a state variable from the output is best seen from the diagonalized system. Here x_1 is not connected to the output and could not be estimated from a measurement of the output

If the initial-state vector, $x(t_0)$, can be found from $u(t)$ and $y(t)$ measured over a finite interval of time from t_0, the system is said to be observable; otherwise the system is said to be unobservable

§6. Observability



observable system

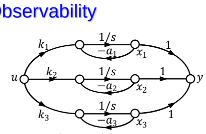


unobservable system

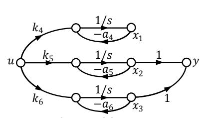
Simply stated, observability is the ability to deduce the state variables from a knowledge of the input, $u(t)$, and the output, $y(t)$

Pole placement for an observer is a viable design technique only for systems that are observable

§6. Observability



observable system



unobservable system

Observability by Inspection

The system can be explored from the output equation of a diagonalized system

- for the observable system

$$y = \boldsymbol{C}\boldsymbol{x} = [1 \ \ 1 \ \ 1]\boldsymbol{x}$$

- for the unobservable system

$$y = \boldsymbol{C}\boldsymbol{x} = [0 \ \ 1 \ \ 1]\boldsymbol{x}$$

§6. Observability

The Observability Matrix

An n^{th}-order plant whose state equation is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$

is completely observable if the matrix

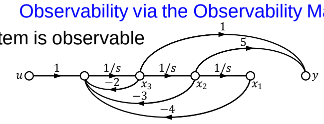
$$\boldsymbol{O_M} = [\boldsymbol{C} \ \ \boldsymbol{CA} \ \ \boldsymbol{CA^2} \ \ \cdots \ \ \boldsymbol{CA^{n-1}}]^T$$

is of rank n, where $\boldsymbol{O_M}$ is called the observability matrix

§6. Observability

- Ex.12.6 Observability via the Observability Matrix

Determine if the system is observable



Solution

The state and output equations for the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}\boldsymbol{x}$$

The observability matrix

$$\boldsymbol{O_M} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA^2} \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ -4 & -3 & 3 \\ -12 & -13 & -9 \end{bmatrix}$$

$\det(\boldsymbol{O_M}) = -344$, $\text{rank}(\boldsymbol{O_M}) = 3 \Longrightarrow$ system is observable

§6. Observability

MATLAB ML

Run ch12p5 in Appendix B

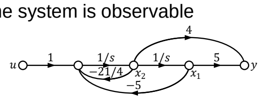
Learn how to use MATLAB to

- test a system for observability
- solve Ex.12.6

§6. Observability

- Ex.12.7 Unobservability via the Observability Matrix

Determine if the system is observable



Solution

The state and output equations for the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \begin{bmatrix} 0 & 1 \\ -5 & -21/4 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 5 & 4 \end{bmatrix}\boldsymbol{x}$$

The observability matrix

$$\boldsymbol{O_M} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -20 & -16 \end{bmatrix}$$

$\det(\boldsymbol{O_M}) = 0$, $\text{rank}(\boldsymbol{O_M}) < 3 \Longrightarrow$ system is unobservable

§6. Observability

Skill-Assessment Ex.12.5

WileyPLUS WPCS Control Solutions

Problem Determine whether the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \begin{bmatrix} -2 & 1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}u$$

$$y = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 4 & 6 & 8 \end{bmatrix}\boldsymbol{x}$$

is observable

Solution The observability matrix

$$\boldsymbol{O_M} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA^2} \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ -64 & -80 & -78 \\ 674 & 848 & 814 \end{bmatrix}$$

$\det(\boldsymbol{O_M}) = -1576$, $\text{rank}(\boldsymbol{O_M}) = 3$

$\Longrightarrow$ system is observable

§6. Observability

TryIt 12.4

Use MATLAB, the Control System Toolbox, and the following statements to solve Skill-Assessment Ex.12.5

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \begin{bmatrix} -2 & 1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}u$$

$$y = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 4 & 6 & 8 \end{bmatrix}\boldsymbol{x}$$

```
A = [-2 -1 -3; 0 -2 1; -7 -8 -9]
C=[4 6 8]
Om=obsv(A,C)
Rank=rank(Om)
```


§7. Alternative Approaches to Observer Design

- Assume a plant not represented in observer canonical form

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}u,\ y = \boldsymbol{C}\boldsymbol{z} \qquad (12.84)$$

- The observability matrix

$$\boldsymbol{O_{M_z}} = [\boldsymbol{C}\ \ \boldsymbol{CA}\ \ \boldsymbol{CA^2}\ \ \cdots\ \ \boldsymbol{CA^{n-1}}]^T \qquad (12.85)$$

- Now assume that the system can be transformed to the observer canonical form, $\boldsymbol{x}$, with the transformation

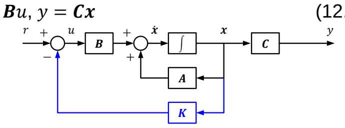
$$\boldsymbol{z} = \boldsymbol{P}\boldsymbol{x} \qquad (12.86)$$

- Substituting Eq. (12.86) into Eqs. (12.84) and pre-multiplying the state equation by $\boldsymbol{P}^{-1}$, we find that the state equations in observer canonical form are

§8. Steady-State Error Design via Integral Control

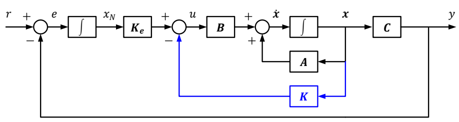
- Consider the controlled system

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u,\; y = \boldsymbol{Cx} \qquad (12.112)$$



- An additional state variable x_N has been added at the output of the leftmost integrator. The error is the derivative of this variable

$$\dot{x}_N = r - \boldsymbol{Cx} \qquad (12.111)$$



§8. Steady-State Error Design via Integral Control

- Rewritten as augmented vectors and matrices

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{x}_N\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ -\boldsymbol{C} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix} + \begin{bmatrix}\boldsymbol{B}\\ 0_N\end{bmatrix}u + \begin{bmatrix}\boldsymbol{0}\\ 1\end{bmatrix}r,\; y = [\boldsymbol{C} \quad 0]\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix} \qquad (12.113)$$

but

$$u = -\boldsymbol{Kx} + K_e x_N = -[\boldsymbol{K} \quad -K_e]\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix} \qquad (12.114)$$

- Substituting Eq. (12.114) into (12.113) and simplifying

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{x}_N\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}-\boldsymbol{BK} & \boldsymbol{B}K_e\\ -\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix} + \begin{bmatrix}\boldsymbol{0}\\ 1\end{bmatrix}r,\; y = [\boldsymbol{C} \quad 0]\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix} \qquad (12.115)$$

The system type has been increased, and we can use the characteristic equation associated with Eq. (12.115) to design $\boldsymbol{K}$ and K_e to yield the desired transient response

$\dot{x}_N = r - \boldsymbol{Cx} = -\boldsymbol{Cx} + \boldsymbol{r}$ (12.111), $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u,\; y = \boldsymbol{Cx}$ (12.112)

§8. Steady-State Error Design via Integral Control

- Ex.12.10 — Design of Integral Control

Consider the plant

$$\dot{\boldsymbol{x}} = \begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}0\\1\end{bmatrix}u,\; y = [1 \quad 0]\boldsymbol{x} \qquad (12.16)$$

a. Design a controller without integral control to yield a 10% overshoot and a settling time of $0.5s$. Evaluate the steady-state error for a unit step input

b. Repeat the design of (a) using integral control. Evaluate the steady-state error for a unit step input

Solution

a. From T_s and $\%OS$ the desired characteristic polynomial

$$s^2 + 16s + 183.1 \qquad (12.117)$$

The characteristic polynomial for the controlled plant

$$s^2 + (5 + k_2)s + (3 + k_1) \qquad (12.118)$$

§8. Steady-State Error Design via Integral Control

$$s^2 + 16s + 183.1 \qquad (12.117)$$

$$s^2 + (5 + k_2)s + (3 + k_1) \qquad (12.118)$$

Equating the coefficients of the above two Eqs. to get

$$K = [k_1 \quad k_2] = [180.1 \quad 11] \qquad (12.19)$$

From Eqs. (12.3), the controlled plant with state-variable feedback represented in phase variable form

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} + \boldsymbol{B}r = \begin{bmatrix}0 & 1\\ -183.1 & -16\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}0\\1\end{bmatrix}r \qquad (12.120)$$

$$y = \boldsymbol{Cx} = [1 \quad 0]\boldsymbol{x}$$

Using Eq.(7.96), the steady-state error for a step input

$$e(\infty) = 1 + [1 \quad 0]\begin{bmatrix}0 & 1\\ -183.1 & -16\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix} = 0.995 \qquad (12.121)$$

$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u = \boldsymbol{Ax} + \boldsymbol{B}(-\boldsymbol{Kx} + r) = (\boldsymbol{A} - \boldsymbol{BK})x + \boldsymbol{B}r,\; y = \boldsymbol{Cx}$ (12.3)

$e(\infty) = 1 - y_{ss} = 1 - \boldsymbol{CV} = 1 + \boldsymbol{CA}^{-1}\boldsymbol{B}$ (7.96)

§8. Steady-State Error Design via Integral Control

b. Use Eqs. (12.115) to represent the integral-controlled plant

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix} = \begin{bmatrix}\left(\begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[k_1 \quad k_2]\right) & \begin{bmatrix}0\\1\end{bmatrix}K_e\\ -[1 \quad 0] & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$= \begin{bmatrix}0 & 1 & 0\\ -(3+k_1) & -(5+k_2) & K_e\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} \qquad (12.122)$$

Using (12.16), (3.73), the TF of the plant, $G(s) = \dfrac{1}{s^2 + 5s + 3}$

$\dot{\boldsymbol{x}} = \begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}0\\1\end{bmatrix}u,\; y = [1 \quad 0]\boldsymbol{x}$ (12.16)

$T(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$ (3.73)

§8. Steady-State Error Design via Integral Control

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -(3+k_1) & -(5+k_2) & K_e\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}$$

Augment (12.117) with a third pole, $s + 100$, which has a real part greater than five times that of the desired dominant 2nd-order poles. The desired 3rd-order closed-loop system characteristic polynomial

$$(s + 100)(s^2 + 16s + 183.1)$$

$$= s^3 + 116s^2 + 1783.1s + 18{,}310 \qquad (12.123)$$

The characteristic polynomial for the system of Eqs. (12.112)

$$s^3 + (5 + k_2)s^2 + (3 + k_1)s + K_e \qquad (12.124)$$

$G(s) = \frac{1}{s^2+5s+3}$, $s^2 + 16s + 183.1 = [s - (8 - 10.9133i)][s - (8 + 10.9133i)]$ (12.117)

§8. Steady-State Error Design via Integral Control

$$(s+100)(s^2+16s+183.1)$$
$$= s^3+116s^2+1783.1s+18,310 \quad (12.123)$$
$$s^3+(5+k_2)s^2+(3+k_1)s+K_e \quad (12.124)$$

Matching coefficients from Eqs. (12.123) and (12.124)

$$k_1=1780.1,\ k_2=111,\ K_e=18,310 \quad (12.125)$$

Substituting these values into Eqs. (1.122) yields this closed-loop integral control system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -1783.1 & -116 & 18,310\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$

$$y=[1\quad 0\quad 0][x_1\quad x_2\quad x_3]^T \quad (12.122)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -(3+k_1) & -(5+k_2) & K_e\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r,\ y=[1\quad 0\quad 0]\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} \quad (12.122)$$

§8. Steady-State Error Design via Integral Control

In order to check our assumption for the zero, now find the closed-loop TF

$$T(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}=\frac{18,310}{s^3+116s^2+1783.1s+18,310} \quad (12.127)$$

The TF matches our design $\Longrightarrow$ The desired transient response

The steady-state error for a unit step input

$$e(\infty)=1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \quad (7.96)$$
$$=1+[1\quad 0\quad 0]\begin{bmatrix}0 & 1 & 0\\ -1783.1 & -116 & 18,310\\ -1 & 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}=0$$

The system behaves like Type 1 system (12.128)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -(3+k_1) & -(5+k_2) & K_e\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r,\ y=[1\quad 0\quad 0]\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} \quad (12.122)$$

§8. Steady-State Error Design via Integral Control

Skill-Assessment Ex.12.7

Problem Design an integral controller for the plant

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}u=\begin{bmatrix}0 & 1\\ -7 & -9\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$y=\boldsymbol{C}\boldsymbol{x}=[4\quad 1]\boldsymbol{x}$$

to yield a step response with 10% overshoot, a peak time of 2s and zero steady-state error

Solution The desired characteristic equation

$$\xi=-\frac{\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}=-\frac{\ln(10/100)}{\sqrt{\pi^2+\ln^2(10/100)}}=0.591$$
$$\omega_n=\frac{\pi}{T_p\sqrt{1-\xi^2}}=\frac{4}{2\sqrt{1-0.591^2}}=1.948rad/s$$
$$\Rightarrow s^2+2\xi\omega_n s+\omega_n^2=s^2+2.3s+3.79=0 \quad (*)$$

§8. Steady-State Error Design via Integral Control

$$s^2+2.3s+3.79=0 \quad (*)$$

The characteristic polynomial for the controlled plant

$$s^2+(9+k_2)s+(7+k_1)=0 \quad (**)$$

Equating the coefficients of the above two Eqs. to get

$$K=[k_1\quad k_2]=[-3.21\quad -6.70]$$

The controlled plant with state-variable feedback

$$\dot{\boldsymbol{x}}=(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}+\boldsymbol{B}r=\begin{bmatrix}0 & 1\\ -3.79 & -2.30\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\ 1\end{bmatrix}r$$
$$y=\boldsymbol{C}\boldsymbol{x}=[4\quad 1]\boldsymbol{x}$$

The steady-state error for a step input

$$e(\infty)=1+[4\quad 1]\begin{bmatrix}0 & 1\\ -3.79 & -2.30\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}=-0.0554$$

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}u=\begin{bmatrix}0 & 1\\ -7 & -9\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\ 1\end{bmatrix}u,\ y=\boldsymbol{C}\boldsymbol{x}=[4\quad 1]\boldsymbol{x},\ e(\infty)=1-y_{ss}=1+\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}$$

§8. Steady-State Error Design via Integral Control

The integral-controlled plant

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}0 & 1\\ -3.79 & -2.30\end{bmatrix} & \begin{bmatrix}0\\ 1\end{bmatrix}K_e\\ -[4\quad 1] & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$
$$=\begin{bmatrix}0 & 1 & 0\\ -3.79 & -2.30 & K_e\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$
$$y=[1\quad 0\quad 0]\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix}$$

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{x}_N\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}K_e\\ -\boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix}+\begin{bmatrix}\boldsymbol{0}\\ 1\end{bmatrix}r,\ y=[\boldsymbol{C}\quad 0]\begin{bmatrix}\boldsymbol{x}\\ x_N\end{bmatrix} \quad (12.115)$$

§8. Steady-State Error Design via Integral Control

The TF of the original system

$$G(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}$$
$$=[4\quad 1]\begin{bmatrix}s & -1\\ 7 & s+9\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}$$
$$=[4\quad 1]\frac{\begin{bmatrix}s+9 & 1\\ -7 & s\end{bmatrix}}{s(s+9)+7}\begin{bmatrix}0\\ 1\end{bmatrix}$$
$$=\frac{s+4}{s^2+9s+7}$$

Adding a pole at -4 which corresponds to the original system's zero location, yield the characteristic equation

$$(s^2+2.3s+3.97=0)(s+4)=s^3+6.3s^2+13s+15.16$$

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}u=\begin{bmatrix}0 & 1\\ -7 & -9\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\ 1\end{bmatrix}u,\ y=\boldsymbol{C}\boldsymbol{x}=[4\quad 1]\boldsymbol{x}$$