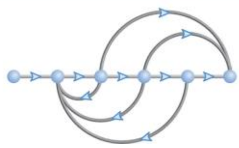
Modeling in the Frequency Domain 2



Chapter Objectives

After completing this chapter, the student will be able to

- find the Laplace transform of time functions and the inverse Laplace transform
- find the transfer function (TF) from a differential equation and solve the differential equation using the transfer function
- find the transfer function for linear, time-invariant electrical networks
- find the TF for linear, time-invariant translational mechanical systems
- find the TF for linear, time-invariant rotational mechanical systems
- find the TF for gear systems with no loss and for gear systems with loss
- find the TF for linear, time-invariant electromechanical systems
- produce analogous electrical and mechanical circuits
- linearize a nonlinear system in order to find the TF

§1. Introduction

Transform Requirements into a Physical System → Draw a Functional Block Diagram → Create a Schematic → Develop a Mathematical Model (Block Diagram) → Reduce Block Diagram → Analyze and Design

- Mathematical models from schematics of physical systems
  - transfer functions in the frequency domain
  - state equations in the time domain
- Block diagram representation of
  - a system

input $r(t)$ → system → output $c(t)$

  - an interconnection of subsystems

input $r(t)$ → subsystem → subsystem → subsystem → output $c(t)$

§2. Laplace Transform Review

- Transforms: a mathematical conversion from one way of thinking to another to make a problem easier to solve

problem in original way of thinking → transform → solution in transform way of thinking → inverse transform → solution in original way of thinking

- The Laplace transform the problem in time-domain to problem in s-domain, then applying the solution in s-domain, and finally using inverse transform to converse the solution back to the time-domain

problem in time domain → Laplace transform → solution in s domain → Inverse Laplace transform → solution in time domain

- The Laplace transform is named in honor of mathematician and astronomer Pierre-Simon Laplace (1749-1827)
- Others: Fourier transform, z-transform, wavelet transform, …

§2. Laplace Transform Review

- The Laplace transform of the function $f(t)$ for $t>0$ is defined by the following relationship

$$F(s)=\mathcal{L}\{f(t)\}=\int_{0-}^{+\infty} f(t)e^{-st}dt \qquad (2.1)$$

s : complex frequency variable, $s=\sigma+j\omega$ with s, ω are real numbers, $s\in C$ for which makes $F(s)$ convergent

$\mathcal{L}$: Laplace transform

F : a complex-valued function of complex numbers

- The inverse Laplace transform of the function $F(s)$ for $t>0$ is defined by the following relationship

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds=f(t)u(t) \qquad (2.2)$$

u : the unit step function, $u(t)=\begin{cases}1 & if\ t>0\\ 0 & if\ t<0\end{cases}$

§2. Laplace Transform Review

- The Laplace transform table

Table 2.1 Laplace transform table

No.	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{-at}u(t)$	$\frac{1}{s+a}$
6	$\sin\omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos\omega t u(t)$	$\frac{s}{s^2+\omega^2}$

§2. Laplace Transform Review

- Ex.2.1 Laplace Transform of a Time Function

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$

Solution

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$
$$= \int_0^\infty Ae^{-at}e^{-st}dt$$
$$= A\int_0^\infty e^{-(a+s)t}dt$$
$$= -\frac{A}{s+a}e^{-(a+s)t}\Big|_0^\infty$$
$$\rightarrow F(s) = \frac{A}{s+a} \quad (2.3)$$

Table 2.1 Laplace transform table

No.	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{-at}u(t)$	$\frac{1}{s+a}$
6	$\sin\omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos\omega t u(t)$	$\frac{s}{s^2+\omega^2}$

§2. Laplace Transform Review

Table 2.2 Laplace transform theorem

No.	Theorem	Name
1	$\mathcal{L}\{f(t)\} = F(s) = \int_{0_-}^\infty f(t)e^{-st}dt$	Definition
2	$\mathcal{L}\{kf(t)\} = kF(s)$	Linearity theorem
3	$\mathcal{L}\{f_1(t)+f_2(t)\} = F_1(s)+F_2(s)$	Linearity theorem
4	$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$	Frequency shift theorem
5	$\mathcal{L}\{f(t-T)\} = e^{-sT}F(s)$	Time shift theorem
6	$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7	$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0_-)$	Differentiation theorem
8	$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0_-) - f'(0_-)$	Differentiation theorem
9	$\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - \sum_{k=1}^n s^{n-k}f^{k-1}(0_-)$	Differentiation theorem
10	$\mathcal{L}\left\{\int_{0-}^t f(\tau)d\tau\right\} = F(s)/s$	Integration theorem
11	$\mathcal{L}\{f(\infty)\} = \lim_{s\to0} sF(s)$	Final value theorem
12	$\mathcal{L}\{f(0_+)\} = \lim_{s\to\infty} sF(s)$	Initial value theorem

§2. Laplace Transform Review

- Ex.2.2 Inverse Laplace Transform

Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$

Solution

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = tu(t)$$
$$f_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$
$$= e^{-3t}f(t)$$
$$\rightarrow f_1(t) = e^{-3t}tu(t)$$

$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$ (Table 2.1 – 3)
$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$ (Table 2.2 – 4)

§2. Laplace Transform Review

Partial-Fraction Expansion

$$F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5}$$
$$= (s+1) + \frac{2}{s^2+s+5}$$
$$\rightarrow f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathcal{L}^{-1}\underbrace{\left\{\frac{2}{s^2+s+5}\right\}}_{F(s)}$$

Using partial-fraction expansion to expand function like $F(s)$ into a sum of terms and then find the inverse Laplace transform for each term

$\mathcal{L}\{\delta(t)\} = 1$ (Table 2.1 – 1)
$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) \rightarrow \mathcal{L}\left\{\frac{d\delta(t)}{dt}\right\} = s$ (Table 2.2 – 7)

§2. Laplace Transform Review

Case 1. Roots of the Denominator of $F(s)$ are Real and Distinct

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} \quad (2.8)$$

$\lim_{s\to-1}[(2.8)\times(s+1)]$

$$\rightarrow \lim_{s\to-1}\left\{\frac{2}{(s+2)}\right\} = \lim_{s\to-1}\left\{K_1 + \frac{(s+1)K_2}{s+2}\right\} \quad (2.9)$$
$$\rightarrow K_1 = 2$$

$\lim_{s\to-2}[(2.8)\times(s+2)]$

$$\rightarrow \lim_{s\to-2}\left\{\frac{2}{(s+1)}\right\} = \lim_{s\to-2}\left\{\frac{(s+2)K_1}{s+1} + K_2\right\}$$
$$\rightarrow K_2 = -2$$
$$\rightarrow F(s) = \frac{2}{s+1} - \frac{2}{s+2} \Longrightarrow f(t) = \left(2e^{-t} - 2e^{-2t}\right)u(t) \quad (2.10)$$

§2. Laplace Transform Review

In general, given an $F(s)$ whose denominator has real and distinct roots, a partial-fraction expansion

$$F(s) = \frac{N(s)}{D(s)}$$
$$= \frac{N(s)}{(s+p_1)(s+p_2)\dots(s+p_i)\dots(s+p_n)}$$
$$= \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \dots + \frac{K_i}{s+p_i} + \dots + \frac{K_n}{s+p_n} \quad (2.11)$$

To find K_i

- multiply (2.11) by $(s+p_i)$
- let s approach $-p_i$

§2. Laplace Transform Review

- <u>Ex.2.3</u> Laplace Transform Solution of a Differential Equation

Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

<u>Solution</u>

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$\rightarrow Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$

$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0_-)$ (Table 2.2 – 7)

$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0_-) - f'(0_-)$ (Table 2.2 – 8)

§2. Laplace Transform Review

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$

Evaluate the residue K_i

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_2 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

$$\rightarrow Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$$

Hence $y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right)u(t)$

§2. Laplace Transform Review

$$Y(s) = \frac{32}{s(s^2+12s+32)}$$

$$= \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$$

$$y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right)u(t) \quad (2.20)$$

The $u(t)$ in (2.20) shows that the response is zero until $t = 0$

Unless otherwise specified, all inputs to systems in the text will not start until $t = 0$. Thus, output responses will also be zero until $t = 0$

For convenience, the $u(t)$ notation will be eliminated from now on. Accordingly, the output response

$$y(t) = 1 - 2e^{-4t} + e^{-8t} \quad (2.21)$$

§2. Laplace Transform Review

MATLAB ML

Run ch2p1 through ch2p8 in Appendix B

Learn how to use MATLAB to

- represent polynomials
- find roots of polynomials
- multiply polynomials, and
- find partial-fraction expansions

§2. Laplace Transform Review

MATLAB

$$Y(s) = \frac{32}{s^3+12s^2+32s} = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$$

Matlab [r,p,k] = residue([32],[1,12,32,0])

Result r = [1, −2, 1], p = [−8, −4, 0], k = [ ]

$$Y(s) = \underbrace{0}_{k} + \underbrace{1}_{r_1}\frac{1}{s-\underbrace{(-8)}_{p_1}} + \underbrace{(-2)}_{r_2}\frac{1}{s-\underbrace{(-4)}_{p_2}} + \underbrace{1}_{r_3}\frac{1}{s-\underbrace{(0)}_{p_3}}$$

$$= \frac{1}{s+8} - \frac{2}{s+4} + \frac{1}{s}$$

§2. Laplace Transform Review

Case 2. Roots of the Denominator of $F(s)$ are Real and Repeated

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad (2.22)$$

$$= \frac{K_1}{s+1} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2} \quad (2.23)$$

$\lim\limits_{s\to -1}[(2.23) \times (s+1)]$

$$\rightarrow \lim_{s\to -1}\left\{\frac{2}{(s+2)}\right\} = \lim_{s\to -1}\left\{K_1 + \frac{(s+1)K_2}{s+2}\right\}$$

$$\rightarrow K_1 = 2$$

$\lim\limits_{s\to -2}[(2.23) \times (s+2)^2]$

$$\rightarrow \frac{2}{(s+1)} = \frac{(s+2)^2K_1}{s+1} + K_2 + (s+2)K_3 \quad (2.24)$$

$$\rightarrow K_2 = -2$$

§2. Laplace Transform Review

$$F(s)=\frac{K_1}{s+1}+\frac{K_2}{(s+2)^2}+\frac{K_3}{s+2} \tag{2.23}$$

$$K_1=2,\ K_2=-2$$

$$\frac{2}{(s+1)}=\frac{(s+2)^2K_1}{s+1}+K_2+(s+2)K_3 \tag{2.24}$$

Differentiate (2.24) with respect to s

$$\frac{-2}{(s+1)^2}=\frac{(s+2)sK_1}{(s+1)^2}+K_3 \tag{2.25}$$

$K_1=2,\ s\to -2\to K_3=-2$

$$\to Y(s)=\frac{2}{s+1}-\frac{2}{(s+2)^2}-\frac{2}{s+2}$$

Hence

$$y(t)=2e^{-t}-2te^{-2t}-2e^{-2t} \tag{2.26}$$

§2. Laplace Transform Review

TryIt 2.1

Use the following MATLAB and Control System Toolbox statement to form the linear, time-invariant (LTI) transfer function of Eq. (2.22).

```
F=zpk([],[-1 -2 -2],2)
```

$$F(s)=\frac{2}{(s+1)(s+2)^2} \tag{2.22}$$

Matlab `F=zpk([], [-1 -2 -2],2)`

Result

```
F =
        2
  -----------------
  (s+1) (s+2)^2
Continuous-time zero/pole/gain model
```


§2. Laplace Transform Review

$$F(s)=\frac{2}{s^3+5s^2+8s+4}=\frac{2}{s+1}-\frac{2}{(s+2)^2}-\frac{2}{s+2}$$

Matlab `[r,p,k] = residue([2],[1,5,8,4])`

Result r = [−2, −2, 2], p = [−2, −2, −1], k = [ ]

$$F(s)=\underbrace{0}_{k}+\underbrace{(-2)}_{r_1}\frac{1}{s-(\underbrace{-2}_{p_1})}+(\underbrace{-2}_{r_2})\frac{1}{[s-(\underbrace{-2}_{p_2})]^2}+\underbrace{2}_{r_3}\frac{1}{s-(\underbrace{-1}_{p_3})}$$

$$=-\frac{2}{(s+2)^2}-\frac{2}{s+2}+\frac{2}{s+1}$$

§2. Laplace Transform Review

In general, given an $F(s)$ whose denominator has real and distinct roots, a partial-fraction expansion

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{(s+p_1)^r(s+p_2)\dots(s+p_i)\dots(s+p_n)}$$

$$=\frac{K_1}{(s+p_1)^r}+\frac{K_2}{(s+p_1)^{r-1}}+\dots+\frac{K_r}{s+p_1}$$

$$+\frac{K_{r+1}}{s+p_2}+\dots+\frac{K_i}{s+p_i}+\dots+\frac{K_n}{s+p_n} \tag{2.27}$$

To find K_i

- multiply (2.27) by $(s+p_1)^r$ to get $F_1(s)=(s+p_1)^rF(s)$
- let s approach $-p_i$

$$K_i=\frac{1}{(i-1)!}\frac{d^{i-1}F_1(s)}{ds^{i-1}}\bigg|_{s\to p_1}\qquad i=1,2,\dots,r;\ 0!=1$$

§2. Laplace Transform Review

Case 3. Roots of the Denominator of $F(s)$ are Complex or Imaginary

$$F(s)=\frac{3}{s(s^2+2s+5)} \tag{2.30}$$

$$=\frac{K_1}{s}+\frac{K_2s+K_3}{s^2+2s+5} \tag{2.31}$$

$\lim_{s\to 0}[(2.31)\times s]\to K_1=3/5$

First multiplying (2.31) by the lowest common denominator, $s(s^2+2s+5)$, and clearing the fraction

$$3=K_1(s^2+2s+5)+(K_2s+K_3)s \tag{2.32}$$

$$\to 3=\left(K_2+\frac{3}{5}\right)s^2+\left(K_3+\frac{6}{5}\right)s+3 \tag{2.33}$$

Balancing the coefficients: $K_2=-3/5$, $K_3=-6/5$

$$F(s)=\frac{3}{s(s^2+2s+5)}=\frac{3}{5}\frac{1}{s}-\frac{3}{5}\frac{s+2}{s^2+2s+5}$$

§2. Laplace Transform Review

$$F(s)=\frac{3}{5}\frac{1}{s}-\frac{3}{5}\frac{s+2}{s^2+2s+5}$$

$$=\frac{3}{5}\frac{1}{s}-\frac{3}{5}\frac{(s+1)+(1/2)2}{(s+1)^2+2^2}$$

$$\to f(t)=\frac{3}{5}-\frac{3}{5}e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right) \tag{2.38}$$

or

$$f(t)=0.6-0.671e^{-t}\cos(2t-\phi) \tag{2.41}$$

$\mathcal{L}\{Ae^{-at}\cos\omega t+Be^{-at}\sin\omega t\}=\frac{B(s+a)+B\omega}{(s+a)^2+\omega^2}$ (Table 2.1 – 6&7)

§2. Laplace Transform Review

TryIt 2.2

Use the following MATLAB statements to help you get Eq. (2.26).

```
numf=2;
denf=poly([-1 -2 -2]);
[r,p,k]=residue...
 (numf,denf)
```

$$F(s) = \frac{2}{(s+1)(s+2)^2} \qquad (2.22)$$

$$y(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \qquad (2.26)$$

Matlab
```
numf=2;
denf=poly([-1 -2 -2]);
[r,p,k]=residue(numf,denf)
```

Result r = [-2 -2 2], p = [-2 -2 -1], k = []

$$F(s) = \underbrace{0}_{k} + \underbrace{-2}_{r_1}\frac{1}{[s-(\underbrace{-2}_{p_1})]^2} + (\underbrace{-2}_{r_2})\frac{1}{s-(\underbrace{-2}_{p_2})} + \underbrace{2}_{r_3}\frac{1}{s-(\underbrace{-1}_{p_3})}$$

$$= -2\frac{1}{(s+2)^2} - 2\frac{1}{s+2} + 2\frac{1}{s+1}$$

§2. Laplace Transform Review

TryIt 2.3

Use the following MATLAB and Control System Toolbox statement to form the LTI transfer function of Eq. (2.30).

```
F=tf([3],[1 2 5 0])
```

$$F(s) = \frac{3}{s(s^2+2s+5)} \qquad (2.30)$$

Matlab
```
F=tf([3],[1 2 5 0])
```

Result
```
F =
          3
  ----------------------
  s^3 + 2 s^2 + 5 s
  Continuous-time transfer function
```


§2. Laplace Transform Review

In general, given an $F(s)$ whose denominator has complex or purely imaginary roots, a partial-fraction expansion

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s+p_1)(s^2+as+b)\ldots}$$

$$= \frac{K_1}{(s+p_1)} + \frac{K_2 s + K_3}{(s^2+as+b)} + \cdots \qquad (2.42)$$

To find K_i

- the K_i 's in (2.42) are found through balancing the coefficients of the equation after clearing fractions
- put $(K_2 s + K_3)/(s^2+as+b)$ in to the form
$$\frac{B(s+a)+B\omega}{(s+a)^2+\omega^2}$$

§2. Laplace Transform Review

TryIt 2.4

Use the following MATLAB and Symbolic Math Toolbox statements to get Eq. (2.38) from Eq. (2.30).

```
syms s
f=ilaplace...
 (3/(s*(s^2+2*s+5)));
pretty(f)
```

$$F(s) = \frac{3}{s(s^2+2s+5)} \qquad (2.30)$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) \qquad (2.38)$$

Matlab
```
syms s; f=ilaplace(3/(s*(s^2+2*s+5))); pretty(f)
```

Result
```
f =
3/5 - (3*exp(-t)*(cos(2*t) + sin(2*t)/2))/5
              /            sin(2 t) \
   exp(-t) | cos(2 t) + ---------- | 3
3             \                2   /
- - --------------------------------------
5             5
```


§2. Laplace Transform Review

TryIt 2.5

Use the following MATLAB statements to help you get Eq. (2.47).

```
numf=3
denf=[1 2 5 0]
[r,p,k]=residue(numf,denf)
```

$$F(s) = \frac{3}{s(s^2+2s+5)} \qquad (2.30)$$

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{s+1+j2} + \frac{2-j1}{s+1+j2}\right) \qquad (2.47)$$

Matlab
```
numf=3; denf=[1 2 5 0]; [r,p,k]=residue(numf,denf)
```

Result r=[-0.3+0.15i; -0.3-0.15i; 0.6]; p=[-1+2i; -1-2i; 0]; k=[]

$$F(s) = \underbrace{0}_{k} + (\underbrace{-0.3+j0.15}_{r_1})\frac{1}{s-(\underbrace{-1+j2}_{p_1})}$$

$$+(\underbrace{-0.3-j0.15}_{r_2})\frac{1}{s-(\underbrace{-1-2j}_{p_2})} + (\underbrace{0.6}_{r_3})\frac{1}{s-(\underbrace{0}_{p_3})}$$

$$= -\frac{0.3-j0.15}{s+1-2j} - \frac{0.3+j0.15}{s+1+2j} + 0.6\frac{1}{s}$$

§2. Laplace Transform Review

Symbolic Math SM

Run ch2sp1 and ch2sp2 in Appendix F

Learn how to use the Symbolic Math Toolbox to

- construct symbolic objects
- find the inverse Laplace transforms of frequency functions
- find the Laplace of time functions

§2. Laplace Transform Review

Skill-Assessment Ex.2.1

Problem Find the Laplace transform of

$$f(t) = te^{-5t}$$

Solution

$$F(s) = \mathcal{L}\{te^{-5t}\} = \frac{1}{(s+5)^2}$$

Matlab `syms t s F; f = t*exp(-5*t); F=laplace(f, s); pretty(F)`

Result

```
    1
---------
       2
(s + 5)
```


§2. Laplace Transform Review

Skill-Assessment Ex.2.2

Problem Find the inverse Laplace transform of

$$F(s) = \frac{10}{s(s+2)(s+3)^2}$$

Solution Expanding $F(s)$ by partial fractions

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)^2} + \frac{D}{s+3}$$

where,

$$A = \left.\frac{10}{(s+2)(s+3)^2}\right|_{s\to 0} = \frac{5}{9}, B = \left.\frac{10}{s(s+3)^2}\right|_{s\to -2} = -5$$

$$C = \left.\frac{10}{s(s+2)}\right|_{s\to -3} = \frac{10}{3}, D = (s+3)^2\left.\frac{dF(s)}{ds}\right|_{s\to -3} = \frac{40}{9}$$

$$\to F(s) = \frac{5}{9}\frac{1}{s} - 5\frac{1}{s+2} + \frac{10}{3}\frac{1}{(s+3)^2} + \frac{40}{9}\frac{1}{s+3}$$

§2. Laplace Transform Review

$$F(s) = \frac{5}{9}\frac{1}{s} - 5\frac{1}{s+2} + \frac{10}{3}\frac{1}{(s+3)^2} + \frac{40}{9}\frac{1}{s+3}$$

Taking the inverse Laplace transform

$$f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$$

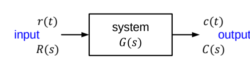
Matlab `syms s; f=ilaplace(10/(s*(s+2)*(s+3)^2)); pretty(f)`

Result

```
exp(-3 t) 40                  t exp(-3 t) 10   5
--------------- - exp(-2 t) 5 + --------------- + -
      9                              3          9
```


§3. The Transfer Function

- The transfer function (TF) of a component is the quotient of the Laplace transform of the output divided by the Laplace transform of the input, with all initial conditions assumed to be zero



- TFs are defined only for linear time invariant systems
- The input-output relationship of a control system $G(s)$ $C(s)$

$$a_n\frac{d^nc(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0c(t)$$
$$= b_m\frac{d^mr(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0r(t)$$

$c(t)$: output $r(t)$: input a_i's, b_i's: constant

§3. The Transfer Function

$$a_n\frac{d^nc(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0c(t)$$
$$= b_m\frac{d^mr(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0r(t)$$

- Taking the Laplace transform of both sides with zero initial conditions

$$a_ns^nC(s) + a_{n-1}s^{n-1}C(s) + \cdots + a_0C(s)$$
$$= b_ms^mR(s) + b_{m-1}s^{m-1}R(s) + \cdots + b_0R(s)$$

- The TF

$$G(s) \equiv \frac{C(s)}{R(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0} \qquad (2.53)$$

- The output of the system can be written in the form

$$C(s) = G(s)R(s) \qquad (2.54)$$

§3. The Transfer Function

- Ex.2.4 TF for a Differential Equation

Find the TF represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Solution

Taking the Laplace transform with zero initial conditions

$$sC(s) + 2C(s) = R(s)$$

The TF

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

§3.The Transfer Function

MATLAB ML

Run ch2p9 through ch2p12 in Appendix B

Learn how to use MATLAB to

- create TFs with numerators and denominators in polynomial or factored form
- convert between polynomial and factored forms
- plot time functions

§3.The Transfer Function

Symbolic Math SM

Run ch2sp3 in Appendix F

Learn how to use the Symbolic Math Toolbox to

- simplify the input of complicated TFs as well as improve readability
- enter a symbolic TF and convert it to a linear time-invariant (LTI) object as presented in Appendix B, ch2p9

§3.The Transfer Function

- Ex.2.5 System Response from the TF

Given $G(s) = 1/(s+2)$, find the response, $c(t)$ to an input, $r(t) = u(t)$, a unit step, assuming zero initial conditions

Solution

For a unit step

$$r(t) = u(t) \rightarrow R(s) = 1/s$$

The output

$$C(s) = R(s)G(s) = \frac{1}{s}\frac{1}{s+2}$$

Expanding by partial fractions

$$C(t) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$$

Taking the inverse Laplace transform

$$c(t) = 0.5 - 0.5e^{-2t} \quad (2.60)$$

§3.The Transfer Function

TryIt 2.6

Use the following MATLAB and Symbolic Math Toolbox statements to help you get Eq. (2.60).

```
syms s
C=1/(s*(s+2))
C=ilaplace(C)
```

$$G(s) = \frac{1}{s+2}$$

$$R(s) = \frac{1}{s}$$

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \quad (2.60)$$

Matlab

```
syms s
C=1/(s*(s+2))
C=ilaplace(C)
```

Result

```
C =
1/2 - exp(-2*t)/2
```

$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$

§3.The Transfer Function

TryIt 2.7

Use the following MATLAB statements to plot Eq. (2.60) for t from 0 to 1 sat intervals of 0.01 s.

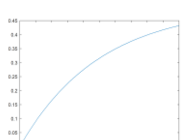
```
t=0:0.01:1;
plot...
(t,(1/2-1/2*exp(-2*t)))
```

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \quad (2.60)$$

Matlab

```
t=0:0.01:1;
plot(t,(1/2-1/2*exp(-2*t)))
```

Result



§3.The Transfer Function

Skill-Assessment Ex.2.3

Problem Find the TF, $G(s) = C(s)/R(s)$, corresponding to the differential equation

$$\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$$

Solution Taking the Laplace transform with zero initial conditions

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

Collecting terms

$$(s^3 + 3s^2 + 7s + 5)C(s) = (s^2 + 4s + 3)R(s)$$

The TF

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

§3. The Transfer Function

Skill-Assessment Ex.2.4

Problem Find the differential equation corresponding to the TF

$$G(s)=\frac{2s+1}{s^2+6s+2}$$

Solution The TF

$$G(s)=\frac{C(s)}{R(s)}=\frac{2s+1}{s^2+6s+2}$$

Cross multiplying

$$\frac{d^2c}{dt^2}+6\frac{dc}{dt}+2c=2\frac{dr}{dt}+r$$

§3. The Transfer Function

Skill-Assessment Ex.2.5

Problem Find the ramp response for a system whose TF

$$G(s)=\frac{s}{(s+4)(s+8)}$$

Solution For a ramp response

$$r(t)=tu(t)\rightarrow R(s)=\frac{1}{s^2}$$

The output

$$\begin{aligned}C(s)&=R(s)G(s)\\&=\frac{1}{s^2}\frac{s}{(s+4)(s+8)}\\&=\frac{A}{s}+\frac{B}{s+4}+\frac{C}{s+8}\end{aligned}$$

§3. The Transfer Function

$$C(s)=\frac{1}{s(s+4)(s+8)}=\frac{A}{s}+\frac{B}{s+4}+\frac{C}{s+8}$$

where,

$$A=\left.\frac{1}{(s+4)(s+8)}\right|_{s\rightarrow 0}=\frac{1}{32}$$

$$B=\left.\frac{1}{s(s+8)}\right|_{s\rightarrow -4}=-\frac{1}{16}$$

$$C=\left.\frac{1}{s(s+4)}\right|_{s\rightarrow -8}=\frac{1}{32}$$

$$\rightarrow C(s)=\frac{1}{32}\frac{1}{s}-\frac{1}{16}\frac{1}{s+4}+\frac{1}{32}\frac{1}{s+8}$$

The ramp response

$$c(t)=\frac{1}{32}-\frac{1}{16}e^{-4t}+\frac{1}{32}e^{-8t}$$

§4. Electrical Network Transfer Functions

Components and the relationships between voltage and current and between voltage and charge under zero initial conditions

Table 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-Current $v(t)-i(t)$	Current-Voltage $i(t)-v(t)$	Voltage-Charge $v(t)-q(t)$	Impedance $Z(s)=V(s)/I(s)$	Admittance $Z(s)=I(s)/V(s)$
Capacitor	$v(t)=\frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t)=C\frac{dv(t)}{dt}$	$v(t)=\frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t)=Ri(t)$	$i(t)=\frac{1}{R}v(t)$	$v(t)=R\frac{dq(t)}{dt}$	R	$\frac{1}{R}=G$
Inductor	$v(t)=L\frac{di(t)}{dt}$	$i(t)=\frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t)=L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

$v(t)$: voltage, V $i(t)$: current, A $q(t)$: charge, Q

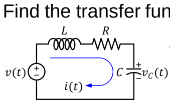
C : capacitor, F R : resistor, Ω L : inductor, H

§4. Electrical Network Transfer Functions

Simple Circuits via Mesh Analysis

- Ex.2.6 Transfer Function - Single Loop via the Differential Equation

Find the transfer function $V_C(s)/V(s)$



Solution

The voltage loop

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int_0^1 i(\tau)d\tau=v(t)$$

Using the relationships $i(t)=dq(t)/dt$ and $q=Cv_C$

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=v(t)$$

$$\rightarrow LC\frac{d^2v_C}{dt^2}+RC\frac{dv_C}{dt}+v_C=v(t)$$

§4. Electrical Network Transfer Functions

$$LC\frac{d^2v_C}{dt^2}+RC\frac{dv_C}{dt}+v_C=v(t)$$

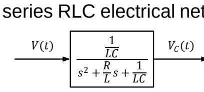
Taking Laplace transform assuming zero initial conditions

$$\left(LCs^2+RCs+1\right)V_C(s)=V(s)$$

Solving for the transfer function

$$\frac{V_C(s)}{V(s)}=\frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Block diagram of series RLC electrical network



§4. Electrical Network Transfer Functions

Impedance

- A resistance resists or "impedes" the flow of current. The corresponding relation is $v/i = R$. Capacitance and inductance elements also impede the flow of current
- An impedance is a generalization of the resistance concept and is defined as the ratio of a voltage transform $V(s)$ to a current transform $I(s)$ and thus implies a current source
- Standard symbol for impedance

$$Z(s) \equiv \frac{V(s)}{I(s)} \quad (2.70)$$

- Kirchhoff's voltage law to the transformed circuit

$$\boxed{[\text{ Sum of Impedances }] \times I(s) = [\text{ Sum of Applied Voltages }]} \quad (2.72)$$

§4. Electrical Network Transfer Functions

- The impedance of a resistor is its resistance

$$Z(s) = R$$

- For a capacitor

$$v(t) = \frac{1}{C}\int_0^t idt \to V(s) = \frac{I(s)}{C(s)s}$$

The impedance of a capacitor

$$Z(s) = \frac{1}{Cs}$$

- For an inductor

$$v(t) = L\frac{di}{dt} \to V(s) = LI(s)s$$

The impedance of a inductor

$$Z(s) = Ls$$

§4. Electrical Network Transfer Functions

Series and Parallel Impedances

- The concept of impedance is useful because the impedances of individual elements can be combined with series and parallel laws to find the impedance at any point in the system
- The laws for combining series or parallel impedances are extensions to the dynamic case of the laws governing series and parallel resistance elements

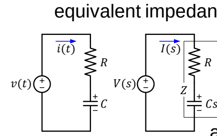
§4. Electrical Network Transfer Functions

- Series Impedances

- Two impedances are in *series* if they have the *same current*. If so, the total impedance is the sum of the individual impedances

$$Z(s) = Z_1(s) + Z_2(s)$$

- Example: A resistor R and capacitor C in series have the equivalent impedance



$$Z(s) = R + \frac{1}{Cs} = \frac{RCs+1}{Cs}$$

$$\to \frac{V(s)}{I(s)} \equiv Z(s) = \frac{RCs+1}{Cs}$$

and the differential equation model is

$$C\frac{dv}{dt} = RC\frac{di}{dt} + i(t)$$

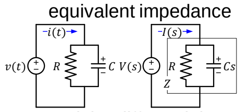
§4. Electrical Network Transfer Functions

- Parallel Impedances

- Two impedances are in *parallel* if they have the *same voltage difference* across them. Their impedances combine by the reciprocal rule

$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}$$

- Example: A resistor R and capacitor C in parallel have the equivalent impedance



$$\frac{1}{Z(s)} = \frac{1}{1/Cs} + \frac{1}{R}$$

$$\to \frac{V(s)}{I(s)} \equiv Z(s) = \frac{R}{RCs+1}$$

and the differential equation model

$$RC\frac{dv}{dt} + v = Ri(t)$$

§4. Electrical Network Transfer Functions

Admittance

$$Y(s) \equiv \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

In general, admittance is complex

- The real part of admittance is called conductance

$$G = \frac{1}{R}$$

- The imaginary part of admittance is called susceptance

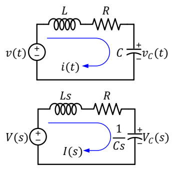
To apply Kirchhoff's voltage law to the transformed circuit

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively
2. Replace the component values with their impedance values

§4. Electrical Network Transfer Functions

- Ex.2.7 Transfer Function - Single Loop via Transform Method

Find the TF $V_C(s)/V(s)$



Solution

The mesh equation using impedances

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

$$\rightarrow \frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

The voltage across the capacitor

$$V_C(s) = I(s)\frac{1}{Cs}$$

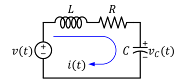
$$\rightarrow V_C(s)/V(s)$$

§4. Electrical Network Transfer Functions

Simple Circuits via Nodal Analysis

- Ex.2.8 Transfer Function - Single Node via Transform Method

Find the transfer function $V_C(s)/V(s)$



Solution

The TF can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \ \rightarrow \frac{V_C(s)}{V(s)}$$

$\frac{V_C(s)}{I/Cs}$: the current flowing out of the node through the capacitor

$\frac{V_C(s) - V(s)}{R + Ls}$: the current flowing out of the node through the series resistor and inductor

§4. Electrical Network Transfer Functions

Complex Circuits via Mesh Analysis

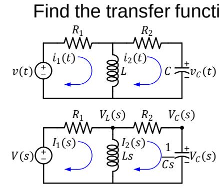
To solve complex electrical networks - those with multiple loops and nodes – using mesh analysis

1. Replace passive element values with their impedances
2. Replace all sources and time variables with their Laplace transform
3. Assume a transform current and a current direction in each mesh
4. Write Kirchhoff's voltage law around each mesh
5. Solve the simultaneous equations for the output
6. Form the TF

§4. Electrical Network Transfer Functions

- Ex.2.10 Transfer Function – Multiple Loops

Find the transfer function $I_2(s)/V(s)$



Solution

Convert the network into Laplace transforms

Summing voltages around each mesh through which the assumed currents flow

$$R_1 I_1 + Ls I_1 - Ls I_2 = V$$

$$Ls I_2 + R_2 I_2 + \frac{1}{Cs} I_2 - Ls I_1 = 0$$

or

$$\begin{aligned} (R_1 + Ls) I_1 \qquad\qquad - Ls I_2 &= V \\ -Ls I_1 + \left(Ls + R_2 + \frac{1}{Cs}\right) I_2 &= 0 \end{aligned} \qquad (2.80)$$

§4. Electrical Network Transfer Functions

Review of Cramer's Rule

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows

$$Ax = b$$

A : $(n \times n)$ matrix has a nonzero determinant

x : the column vector of the variables $x = (x_1, \dots, x_n)^T$

b : the column vector of known parameters

The system has a unique solution, whose individual values for the unknowns are given by

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1, \dots, n$$

A_i : the matrix formed by replacing the i^{th} column of A by the column vector b

§4. Electrical Network Transfer Functions

$$\begin{aligned} (R_1 + Ls) I_1 \qquad\qquad - Ls I_2 &= V \\ -Ls I_1 + \left(Ls + R_2 + \frac{1}{Cs}\right) I_2 &= 0 \end{aligned} \qquad (2.80)$$

Using Cramer's rule

$$I_2 = \frac{\begin{vmatrix} R_1 + Ls & V \\ -Ls & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & Ls + R_2 + \frac{1}{Cs} \end{vmatrix}}$$

$$= \frac{0 - (-Ls)V}{(R_1 + Ls)\left(Ls + R_2 + \frac{1}{Cs}\right) - L^2 s^2}$$

$$\rightarrow I_2 = \frac{LCs^2 V}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1} \qquad (2.81)$$

§4. Electrical Network Transfer Functions

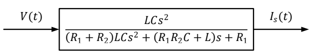
$$I_2 = \frac{LCs^2V}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1} \qquad (2.81)$$

Forming the transfer function

$$G(s) = \frac{I_2(s)}{V(s)}$$

$$= \frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1} \qquad (2.82)$$

The network is now modeled as the transfer function of figure



§4. Electrical Network Transfer Functions

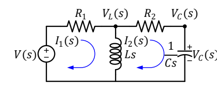
Note

$$(R_1+Ls)I_1 \qquad -LsI_2 = V$$

$$+\left[\begin{array}{l}\text{sum of impedances}\\ \text{around mesh 1}\end{array}\right]\times I_1(s) - \left[\begin{array}{l}\text{sum of impedances}\\ \text{common to the two}\\ \text{meshes}\end{array}\right]\times I_2(s) = \left[\begin{array}{l}\text{sum of applied}\\ \text{voltages}\\ \text{around mesh 1}\end{array}\right]$$

$$-LsI_1 + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2 = 0$$

$$-\left[\begin{array}{l}\text{sum of impedances}\\ \text{common to the two}\\ \text{meshes}\end{array}\right]\times I_1(s) + \left[\begin{array}{l}\text{sum of impedances}\\ \text{around mesh 2}\end{array}\right]\times I_2(s) = \left[\begin{array}{l}\text{sum of applied}\\ \text{voltages}\\ \text{around mesh 2}\end{array}\right]$$



§4. Electrical Network Transfer Functions

Symbolic Math SM

Run ch2sp4 in Appendix F

Learn how to use the Symbolic Math Toolbox to

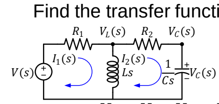
- solve simultaneous equations using Cramer's rule
- solve for the transfer function in (2.82) using (2.80)

§4. Electrical Network Transfer Functions

Complex Circuits via Nodal Analysis

- Ex.2.11 Transfer Function – Multiple Nodes

Find the transfer function $V_C(s)/V(s)$



Solution

Sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$

$$\frac{V_L-V}{R_1}+\frac{V_L}{Ls}+\frac{V_L-V_C}{R_2}=0$$

$$\frac{V_C}{1/Cs}+\frac{V_C-V_L}{R_2}=0 \qquad (2.85)$$

or

$$\left(G_1+G_2+\frac{1}{Ls}\right)V_L \qquad -G_2V_C = VG_1$$

$$-G_2V_L + (G_2+Cs)V_C = 0 \qquad (2.86)$$

§4. Electrical Network Transfer Functions

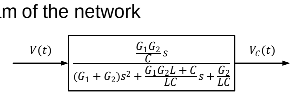
$$\left(G_1+G_2+\frac{1}{Ls}\right)V_L \qquad -G_2V_C = VG_1$$

$$-G_2V_L + (G_2+Cs)V_C = 0 \qquad (2.86)$$

Solving for the transfer function

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1G_2}{C}s}{(G_1+G_2)s^2+\frac{G_1G_2L+C}{LC}s+\frac{G_2}{LC}} \qquad (2.87)$$

Block diagram of the network



§4. Electrical Network Transfer Functions

- Another way to write node equations is to replace voltage sources by current sources

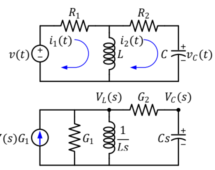
In order to handle multiple-node electrical networks → do perform the following steps

1. Replace passive element values with their admittances
2. Replace all sources and time variables with their Laplace transform
3. Replace transformed voltage sources with transformed current sources
4. Write Kirchhoff's current law at each node
5. Solve the simultaneous equations for the output
6. Form the transfer function

§4. Electrical Network Transfer Functions

- Ex.2.12 Transfer Function – Multiple Nodes with Current Sources

Find the transfer function $V_C(s)/V(s)$



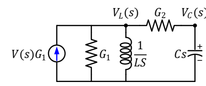
Solution

Convert all impedances to admittances and all voltage sources in series with an impedance to current sources in parallel with an admittance using Norton's theorem

Norton's Theorem

Any collection of batteries and resistances with two terminals is electrically equivalent to an ideal current source i in parallel with a single resistor r. The value of r is the same as that in the Thevenin equivalent and the current i can be found by dividing the open circuit voltage by r

§4. Electrical Network Transfer Functions



Using the general relationship $I(s) = Y(s)V(s)$ and summing currents at the node $V_L(s)$

$$G_1V_L(s) + \frac{1}{Ls}V_L(s) + G_2[V_L(s) - V_C(s)] = G_1V(s) \quad (2.88)$$

Summing the currents at the node $V_C(s)$

$$CV_C(s) + G_2[V_C(s) - V_L(s)] = 0 \quad (2.89)$$

Solving (2.88) and (2.89), forming the transfer function

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1G_2}{C}s}{(G_1+G_2)s^2 + \frac{G_1G_2L + C}{LC}s + \frac{G_2}{LC}}$$

§4. Electrical Network Transfer Functions

Note

$$G_1V_L(s) + \frac{1}{Ls}V_L(s) + G_2[V_L(s) - V_C(s)] = G_1V(s) \quad (2.88)$$

$$+\begin{bmatrix}\text{sum of admittances}\\ \text{connected to node 1}\end{bmatrix} \times V_L(s) - \begin{bmatrix}\text{sum of admittances}\\ \text{common to the two}\\ \text{nodes}\end{bmatrix} \times V_C(s) = \begin{bmatrix}\text{sum of applied}\\ \text{currents at}\\ \text{node 1}\end{bmatrix}$$

$$CV_C(s) + G_2[V_C(s) - V_L(s)] = 0 \quad (2.89)$$

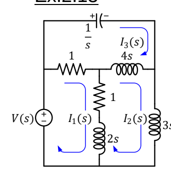
$$-\begin{bmatrix}\text{sum of admittances}\\ \text{common to the two}\\ \text{nodes}\end{bmatrix} \times V_L(s) + \begin{bmatrix}\text{sum of admittances}\\ \text{connected to node 2}\end{bmatrix} \times V_C(s) = \begin{bmatrix}\text{sum of applied}\\ \text{currents at}\\ \text{node 2}\end{bmatrix}$$

§4. Electrical Network Transfer Functions

A Problem-Solving Technique

Sum impedances around a mesh in the case of mesh equations

- Ex.2.13 Mesh Equations via Inspection



Write the mesh equations for the network

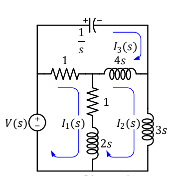
Solution

The mesh equations for loop 1

$$+(2s+2)I_1 - (2s+1)I_2 - I_3 = V \quad (2.94.a)$$

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{around mesh 1}\end{bmatrix} \times I_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{common to mesh 1}\\ \text{and mesh 2}\end{bmatrix} \times I_2(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{common to mesh 1}\\ \text{and mesh 3}\end{bmatrix} \times I_3(s) = \begin{bmatrix}\text{sum of applied}\\ \text{voltages}\\ \text{around mesh 1}\end{bmatrix}$$

§4. Electrical Network Transfer Functions

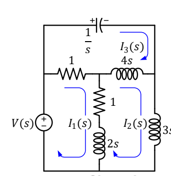


The mesh equations for loop 2

$$-(2s+1)I_1 + (9s+1)I_2 - 4sI_3 = 0 \quad (2.94.b)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{common to mesh 1}\\ \text{and mesh 2}\end{bmatrix} \times I_1(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{around mesh 2}\end{bmatrix} \times I_2(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{common to mesh 2}\\ \text{and mesh 3}\end{bmatrix} \times I_3(s) = \begin{bmatrix}\text{sum of applied}\\ \text{voltages}\\ \text{around mesh 2}\end{bmatrix}$$

§4. Electrical Network Transfer Functions



The mesh equations for loop 3

$$-I_1 - 4sI_2 + \left(4s + 1 + \frac{1}{s}\right)I_3 = 0 \quad (2.94.c)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{common to mesh 1}\\ \text{and mesh 3}\end{bmatrix} \times I_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{common to mesh 2}\\ \text{and mesh 3}\end{bmatrix} \times I_2(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{around mesh 3}\end{bmatrix} \times I_3(s) = \begin{bmatrix}\text{sum of applied}\\ \text{voltages}\\ \text{around mesh 3}\end{bmatrix}$$

§4. Electrical Network Transfer Functions

TryIt 2.8

Use the following MATLAB and Symbolic Math Toolbox statements to help you solve for the electrical currents in Eq. (2.94).

```
syms s I1 I2 I3 V
A=[(2*s+2) -(2*s+1) -1
 -(2*s+1) (9*s+1) -4*s
 -1 -4*s.. (4*s+1+1/s)];
B=[I1;I2;I3];
C=[V;0;0];
B=inv(A)*C;
pretty(B)
```

$$+(2s+2)I_1-(2s+1)I_2-I_3=V \qquad (2.94.a)$$

$$-(2s+1)I_1+(9s+1)I_2-4sI_3=0 \qquad (2.94.b)$$

$$-I_1-4sI_2+\left(4s+1+\frac{1}{s}\right)I_3=0 \qquad (2.94.c)$$

Matlab

```
syms s I1 I2 I3 V;
A=[(2*s+2) -(2*s+1) -1; -(2*s+1) (9*s+1) -4*s; -1 -4*s (4*s+1+1/s)];
B=[I1;I2;I3]; C=[V;0;0];
B=inv(A)*C;
pretty(B)
```


§4. Electrical Network Transfer Functions

Result

```
/           3      2               \
| V (20 s  + 13 s  + 10 s + 1) |
| ----------------------------------- |
|               #1                  |
|          3      2                 |
| V (8 s  + 10 s  + 3 s + 1)  |
| -------------------------------- |
|               #1                  |
|              2                    |
|     V s (8 s  + 13 s + 1)      |
|      -------------------------    |
\               #1                 /
where
              4       3       2
  #1 == 24 s  + 30 s  + 17 s  + 16 s + 1
```

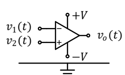
$$\frac{(20s^3+13s^2+10s+1)V}{24s^4+30s^3+17s^2+16s+1}$$

$$\frac{(8s^3+10s^2+3s+1)V}{24s^4+30s^3+17s^2+16s+1}$$

$$\frac{s(8s^2+13s+1)V}{24s^4+30s^3+17s^2+16s+1}$$

§4. Electrical Network Transfer Functions

Operational Amplifiers

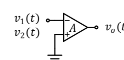


- An operational amplifier (op-amp) is an electronic amplifier used as a basic building block to implement transfer functions
- Op-amp has the following characteristics
  1. Differential input, $v_2(t)-v_1(t)$
  2. High input impedance, $Z_i=\infty$ (ideal)
  3. Low output impedance, $Z_o=0$ (ideal)
  4. High constant gain amplification, $A=\infty$ (ideal)
- The output, $v_o(t)$, is given by

$$v_o(t)=A[v_2(t)-v_1(t)] \qquad (2.95)$$

§4. Electrical Network Transfer Functions

Inverting Operational Amplifiers

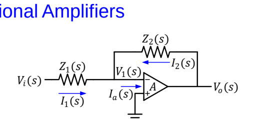


- If $v_2(t)$ is grounded, the amplifier is called an inverting operational amplifier
- The output, $v_o(t)$, is given by

$$v_o(t)=-Av_1(t) \qquad (2.96)$$

§4. Electrical Network Transfer Functions

Inverting Operational Amplifiers



$Z_i(s)=\infty \rightarrow I_a(s)=0$

$I_1(s)=-I_2(s)$

$A=\infty \rightarrow v_1(t)\approx 0$

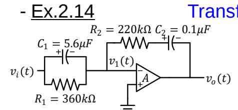
$$I_1(s)=\frac{V_i(s)}{Z_1(s)}=-I_2(s)=-\frac{V_o(s)}{Z_2(s)}$$

The transfer function of the inverting operational amplifier

$$\frac{V_o(s)}{V_i(s)}=-\frac{Z_2(s)}{Z_1(s)} \qquad (2.97)$$

§4. Electrical Network Transfer Functions

- Ex.2.14 Transfer Function – Inverting Op-Amp Circuit



Find the transfer function $V_o(s)/V_i(s)$

Solution

The impedances

$$\frac{1}{Z_1}=\frac{1}{1/C_1s}+\frac{1}{R_1}\rightarrow Z_1=\frac{1}{C_1s+\dfrac{1}{R_1}}=\frac{360\times10^3}{2.016s+1}$$

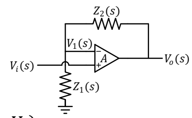
$$Z_2=R_2+\frac{1}{C_2s}=220\times10^3+\frac{10^7}{s}$$

The transfer function

$$\frac{V_o(s)}{V_i(s)}=-\frac{Z_2(s)}{Z_1(s)}=-\frac{220\times10^3+\dfrac{10^7}{s}}{\dfrac{360\times10^3}{2.016s+1}}=-1.232\frac{s^2+45.95s+22.55}{s}$$

§4. Electrical Network Transfer Functions

Noninverting Operational Amplifiers



$$V_o = A(V_i - V_1)$$

$$V_1 = \frac{Z_1}{Z_1 + Z_2} V_o$$

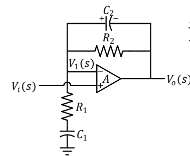
The transfer function of the noninverting operational amplifier

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} \qquad (2.104)$$

§4. Electrical Network Transfer Functions

- Ex.2.15 Transfer Function – Noninverting Op-Amp Circuit

Find the transfer function $V_o(s)/V_i(s)$



Solution

The impedances

$$Z_1 = R_1 + \frac{1}{C_1 s}$$

$$Z_2 = \frac{R_2 \frac{1}{C_2 s}}{R_2 + \frac{1}{C_2 s}}$$

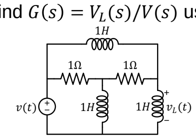
The transfer function

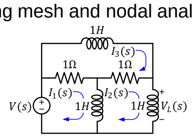
$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1}$$

§4. Electrical Network Transfer Functions

Skill Assessment Ex.2.6

Problem Find $G(s) = V_L(s)/V(s)$ using mesh and nodal analysis





Solution

Mesh analysis

Writing the mesh equations

$$\begin{aligned} (s+1)I_1 \quad & -sI_2 \quad & -I_3 = V \\ -sI_1 + (2s+1)I_2 \quad & & -I_3 = 0 \\ -I_1 \quad & -I_2 + (s+2)I_3 = 0 & \end{aligned}$$

§4. Electrical Network Transfer Functions

$$\begin{aligned} (s+1)I_1 \quad & -sI_2 \quad & -I_3 = V \\ -sI_1 + (2s+1)I_2 \quad & & -I_3 = 0 \\ -I_1 \quad & -I_2 + (s+2)I_3 = 0 & \end{aligned}$$

Solving the mesh equation for I_2

$$I_2 = \frac{\begin{vmatrix} s+1 & V & -1 \\ -s & 0 & -1 \\ -1 & 0 & s+2 \end{vmatrix}}{\begin{vmatrix} s+1 & -s & -1 \\ -s & 2s+1 & -1 \\ -1 & -1 & s+2 \end{vmatrix}} = \frac{(s^2+2s+1)V}{s(s^2+5s+2)}$$

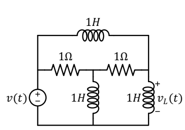
The voltage across L

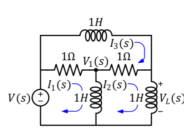
$$V_L = sI_2 = \frac{(s^2+2s+1)V}{s^2+5s+2}$$

$$\rightarrow G(s) = \frac{V_L}{V} = \frac{s^2+2s+1}{s^2+5s+2}$$

§4. Electrical Network Transfer Functions

Nodal analysis





Writing the nodal equations

$$\left(\frac{1}{s} + 2\right)V_1 \qquad - V_L = \quad V$$

$$-V_1 + \left(\frac{2}{s} + 1\right)V_L = \frac{1}{s}V$$

§4. Electrical Network Transfer Functions

$$\left(\frac{1}{s} + 2\right)V_1 \qquad - V_L = \quad V$$

$$-V_1 + \left(\frac{2}{s} + 1\right)V_L = \frac{1}{s}V$$

Solving the nodal equation for V_L

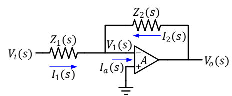
$$V_L = \frac{\begin{vmatrix} \frac{1}{s}+2 & V \\ -1 & \frac{1}{s} \end{vmatrix}}{\begin{vmatrix} \frac{1}{s}+2 & -1 \\ -1 & \frac{2}{s}+1 \end{vmatrix}} = \frac{(s^2+2s+1)V}{s^2+5s+2}$$

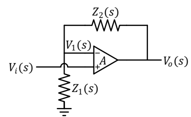
$$\rightarrow G(s) = \frac{V_L}{V} = \frac{s^2+2s+1}{s^2+5s+2}$$

§4. Electrical Network Transfer Functions

Skill Assessment Ex.2.7

Problem If $Z_1(s)$is the impedance of a $10\mu F$ capacitor and $Z_2(s)$ is the impedance of a $100k\Omega$ resistor, find the transfer function, $G(s) = V_o(s)/V_i(s)$ if these components are used with (a) an inverting op-amp and (b) a noninverting op-amp





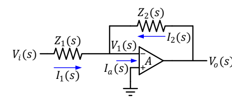
Solution

$$Z_1 = Z_C = \frac{1}{Cs} = \frac{1}{10^{-5}s} = \frac{10^5}{s}$$

$$Z_2 = Z_R = R = 10^5$$

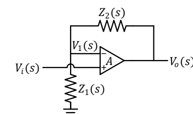
§4. Electrical Network Transfer Functions

(a) Inverting Op-Amp



$$G(s) = -\frac{Z_2}{Z_1} = -\frac{10^5}{\frac{10^5}{s}} = -s$$

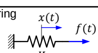
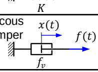
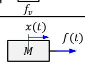
(b) Noninverting Op-Amp



$$G(s) = \frac{Z_1 + Z_2}{Z_1} = \frac{\frac{10^5}{s} + 10^5}{\frac{10^5}{s}} = s + 1$$

§5. Translational Mechanical System Transfer Functions

Table 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

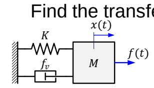
Component	Force - Velocity	Force - Displacement	Impedance $Z_M(s) = \frac{F(s)}{X(s)}$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v\frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

$f(t)$: force, N $v(t)$: velocity, m/s f_v : damping coefficient, Ns/m
$x(t)$: displacement, m M : mass, $kg(= Ns^2/m)$ K : stiffness coefficient, N/m

§5. Translational Mechanical System Transfer Functions

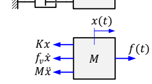
- Ex.2.16 Transfer Function - One Equation of Motion

Find the transfer function $X(s)/F(s)$



Solution

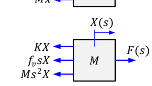
Free body diagram



Using Newton's law to sum all of the forces

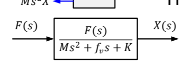
$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking Laplace transform



$$Ms^2X(s) + f_vsX(s) + KX(s) = F(s)$$

The transfer function



$$G(s) = \frac{X(s)}{F(s)} = \frac{F(s)}{Ms^2 + f_vs + K}$$

§5. Translational Mechanical System Transfer Functions

Impedance

- Define impedance for mechanical components

$$Z_M(s) \equiv \frac{F(s)}{X(s)}$$

$$\rightarrow F(s) = Z_M(s)X(s)$$

$\boxed{\text{Sum of Impedances} \times X(s) = \text{Sum of Applied Forces}}$

- The impedance of a spring is its stiffness coefficient

$$F(s) = KX(s) \quad \rightarrow Z_M(s) = K \qquad (2.112)$$

- For the viscous damper

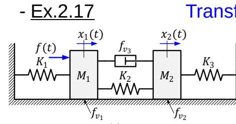
$$F(s) = f_vsX(s) \quad \rightarrow Z_M(s) = f_vs \qquad (2.113)$$

- For the mass

$$F(s) = Ms^2X(s) \quad \rightarrow Z_M(s) = Ms^2 \qquad (2.114)$$

§5. Translational Mechanical System Transfer Functions

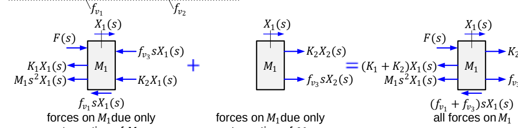
- Ex.2.17 Transfer Function - Two Degrees of Freedom



Find the transfer function $X_2(s)/F(s)$

Solution

Free body diagram of M_1

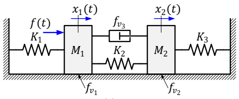


The Laplace transform of the equation of motion of M_1

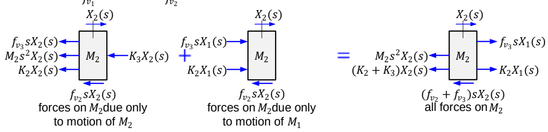
$$+[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1 - (f_{v_3}s + K_2)X_2 = F$$

$\boxed{\text{Sum of Impedances} \times X(s) = \text{Sum of Applied Forces}}$

§5. Translational Mechanical System Transfer Functions



Free body diagram of M_2



The Laplace transform of the equation of motion of M_2

$$-(f_{v_3}s+K_2)X_1+\left[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)\right]X_2=0$$

$\boxed{\text{Sum of Impedances} \times X(s) = \text{Sum of Applied Forces}}$

§5. Translational Mechanical System Transfer Functions

The Laplace transform of the equations of motion

$$+\left[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)\right]X_1-(f_{v_3}s+K_2)X_2=F$$
$$-(f_{v_3}s+K_2)X_1+\left[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)\right]X_2=0$$

The Laplace transform of the equations of motion

$$X_2=\frac{\begin{vmatrix}\left[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)\right] & F\\ -(f_{v_3}s+K_2) & 0\end{vmatrix}}{\Delta}=\frac{(f_{v_3}s+K_2)F}{\Delta}$$

where

$$\Delta=\begin{vmatrix}\left[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)\right] & -(f_{v_3}s+K_2)\\ -(f_{v_3}s+K_2) & \left[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)\right]\end{vmatrix}$$

The transfer function

$$G(s)=\frac{X_2(s)}{F(s)}=\frac{f_{v_3}s+K_2}{\Delta}$$

§5. Translational Mechanical System Transfer Functions

Note

The Laplace transform of the equations of motion of M_1

$$+\left[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)\right]X_1-(f_{v_3}s+K_2)X_2=F$$

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_1\end{bmatrix}\times X_1(s)-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix}\times X_2(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_1\end{bmatrix}$$

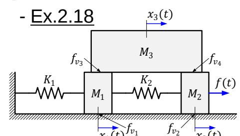
The Laplace transform of the equations of motion of M_2

$$-(f_{v_3}s+K_2)X_1+\left[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)\right]X_2=0$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix}\times X_1(s)+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_2\end{bmatrix}\times X_2(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_2\end{bmatrix}$$

§5. Translational Mechanical System Transfer Functions

- Ex.2.18



Equations of Motion by Inspection

Write the equations of motion for the mechanical network

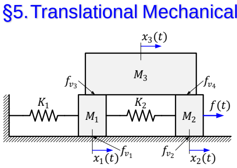
Solution

The Laplace transform of the equations of motion of M_1

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_1\end{bmatrix}\times X_1(s)-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix}\times X_2(s)$$
$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_3\end{bmatrix}\times X_3(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_1\end{bmatrix}$$

$$+\left[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)\right]X_1-K_2X_2-f_{v_3}sX_3=0$$

§5. Translational Mechanical System Transfer Functions

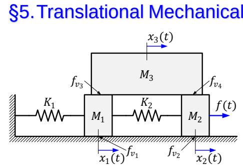


The Laplace transform of the equations of motion of M_2

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix}\times X_1(s)+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_2\end{bmatrix}\times X_2(s)$$
$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_2 \text{ and } x_3\end{bmatrix}\times X_3(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_2\end{bmatrix}$$

$$-K_2X_1+\left[M_2s^2+(f_{v_2}+f_{v_4})s+K_2\right]X_2-f_{v_4}sX_3=F$$

§5. Translational Mechanical System Transfer Functions



The Laplace transform of the equations of motion of M_3

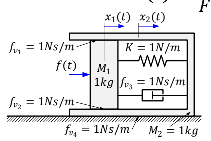
$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_3\end{bmatrix}\times X_1(s)-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_2 \text{ and } x_3\end{bmatrix}\times X_2(s)$$
$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_3\end{bmatrix}\times X_3(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_3\end{bmatrix}$$

$$-f_{v_3}sX_1-f_{v_4}sX_2+\left[M_3s^2+(f_{v_3}+f_{v_4})s\right]X_3=0$$

§5. Translational Mechanical System Transfer Functions

Skill-Assessment Ex.2.8

Problem Find the transfer function $G(s)=\dfrac{X_2(s)}{F(s)}$



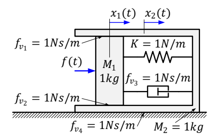
Solution

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_1\end{bmatrix}\times X_1(s)-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix}\times X_2(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_1\end{bmatrix}$$

$$+\left[M_1s^2+\left(f_{v_1}+f_{v_2}+f_{v_3}\right)s+K\right]X_1$$
$$-\left[\left(f_{v_1}+f_{v_2}+f_{v_3}\right)s+K\right]X_2=F$$

$$\rightarrow +(s^2+3s+1)X_1-(3s+1)X_2=F$$

§5. Translational Mechanical System Transfer Functions



$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix}\times X_1(s)+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_2\end{bmatrix}\times X_2(s)=\begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_2\end{bmatrix}$$

$$-\left[\left(f_{v_1}+f_{v_2}+f_{v_3}\right)s+K\right]X_1$$
$$+\left[M_2s^2+\left(f_{v_1}+f_{v_2}+f_{v_3}+f_{v_4}\right)s+K\right]X_2=0$$

$$\rightarrow -(3s+1)X_1+(s^2+4s+1)X_2=0$$

§5. Translational Mechanical System Transfer Functions

$$+(s^2+3s+1)X_1 \quad -(3s+1)X_2=F$$
$$-(3s+1)X_1+(s^2+4s+1)X_2=0$$

The solution for X_2

$$X_2=\frac{\begin{vmatrix}s^2+3s+1 & F\\ -(3s+1) & 0\end{vmatrix}}{\Delta}=\frac{(3s+1)F}{\Delta}$$

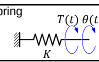
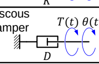
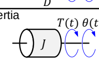
where

$$\Delta=\begin{vmatrix}s^2+3s+1 & -(3s+1)\\ -(3s+1) & s^2+4s+1\end{vmatrix}$$
$$=s(s^3+7s^2+5s+1)$$

$$\rightarrow G(s)=\frac{X_2(s)}{F(s)}=\frac{3s+1}{s(s^3+7s^2+5s+1)}$$

§6. Rotational Mechanical System Transfer Functions

Table 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-Angular Velocity	Torque-Angular Displacement	Impedance $Z_M(s)=\dfrac{T(s)}{\theta(s)}$
Spring 	$T(t)=K\int_0^t \omega(\tau)d\tau$	$T(t)=K\theta(t)$	K
Viscous damper 	$T(t)=D\omega(t)$	$T(t)=D\dfrac{d\theta(t)}{dt}$	Ds
Inertia 	$T(t)=J\dfrac{d\omega(t)}{dt}$	$T(t)=J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

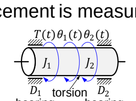
$T(t)$: torque, Nm $\quad$ $\theta(t)$: angular, rad $\quad$ K : spring coefficient, Nm/rad

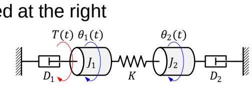
D : coefficient of viscous friction, Nms/rad $\quad$ J : moment of inertia, kgm^2

§6. Rotational Mechanical System Transfer Functions

- Ex.2.19 Transfer Function – Two Equations of Motion

Find the TF, $\theta_2(s)/T(s)$, for the rotational system shown in figure. The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right



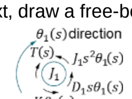


Solution

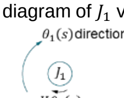
First, obtain the schematic from the physical system

§6. Rotational Mechanical System Transfer Functions

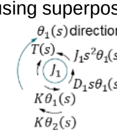
Next, draw a free-body diagram of J_1 và J_2, using superposition



Torques on J_1 due only to the motion of J_1

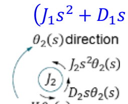


Torques on J_1 due only to the motion of J_2

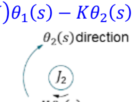


Final free-body diagram for J_1

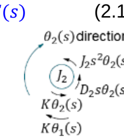
$$(J_1s^2+D_1s+K)\theta_1(s)-K\theta_2(s)=T(s) \qquad (2.127.a)$$



Torques on J_2 due only to the motion of J_2



Torques on J_2 due only to the motion of J_1



Final free-body diagram for J_2

$$-K_1\theta_1(s)+(J_2s^2+D_2s+K)\theta_2(s)=0 \qquad (2.127.b)$$

§6. Rotational Mechanical System Transfer Functions

$$(J_1s^2+D_1s+K)\theta_1(s) \qquad - K\theta_2(s) = T(s) \quad (2.127.a)$$

$$-K_1\theta_1(s)+(J_2s^2+D_2s+K)\theta_2(s)=0 \quad (2.127.b)$$

The solution for θ_2

$$\theta_2 = \frac{\begin{vmatrix} J_1s^2+D_1s+K & T \\ -K & 0 \end{vmatrix}}{\Delta} = \frac{KT}{\Delta}$$

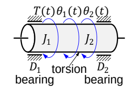
where

$$\Delta = \begin{vmatrix} J_1s^2+D_1s+K & -K \\ -K & J_2s^2+D_2s+K \end{vmatrix}$$

$$\rightarrow G(s) = \frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$

§6. Rotational Mechanical System Transfer Functions

Note



$$(J_1s^2+D_1s+K)\theta_1(s) - K\theta_2(s) = T(s) \quad (2.127.a)$$

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_1\end{bmatrix}\times\theta_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix}\times\theta_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_1\end{bmatrix}$$

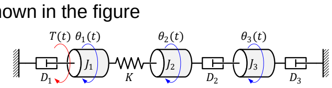
$$-K_1\theta_1(s)+(J_2s^2+D_2s+K)\theta_2(s)=0 \quad (2.127.b)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix}\times\theta_1(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_2\end{bmatrix}\times\theta_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_2\end{bmatrix}$$

§6. Rotational Mechanical System Transfer Functions

- Ex.2.20 Equations of Motion by Inspection

Write the Laplace transform of the equations of motion for the system shown in the figure



Solution

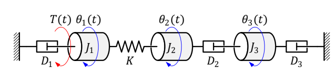
The Laplace transform of the equations of motion of J_1

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_1\end{bmatrix}\times\theta_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix}\times\theta_2(s)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_3\end{bmatrix}\times\theta_3(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_1\end{bmatrix}$$

$$+[J_1s^2+D_1s+K]\theta_1 - K\theta_2 - 0\theta_3 = T(s) \quad (2.131.a)$$

§6. Rotational Mechanical System Transfer Functions



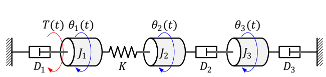
The Laplace transform of the equations of motion of J_2

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix}\times\theta_1(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_2\end{bmatrix}\times\theta_2(s)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_3 \text{ and } \theta_2\end{bmatrix}\times\theta_3(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_2\end{bmatrix}$$

$$-K\theta_1 + [J_2s^2+D_2s+K]\theta_2 - D_2s\theta_3 = 0 \quad (2.131.b)$$

§6. Rotational Mechanical System Transfer Functions



The Laplace transform of the equations of motion of J_3

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_3\end{bmatrix}\times\theta_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_2 \text{ and } \theta_3\end{bmatrix}\times\theta_2(s)$$

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_3\end{bmatrix}\times\theta_3(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_3\end{bmatrix}$$

$$-0\theta_1 - D_2s\theta_2 + [J_3s^2+D_3s+D_2s]\theta_3 = 0 \quad (2.131.c)$$

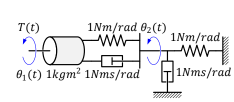
§6. Rotational Mechanical System Transfer Functions

Skill-Assessment Ex.2.9

Problem

Find the transfer function

$$G(s) = \frac{\theta_2(s)}{T(s)}$$



Solution

The equations of motion

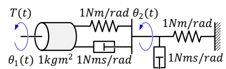
$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_1\end{bmatrix}\times\theta_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix}\times\theta_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_1\end{bmatrix}$$

$$+(s^2+s+1)\theta_1(s) \ - (s+1)\theta_2(s) = T(s)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix}\times\theta_1(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_2\end{bmatrix}\times\theta_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_2\end{bmatrix}$$

$$-(s+1)\theta_1(s) + (2s+2)\theta_2(s) = 0$$

§6. Rotational Mechanical System Transfer Functions



The equations of motion

$$(s^2+s+1)\theta_1(s) \quad -(s+1)\theta_2(s) = T(s)$$

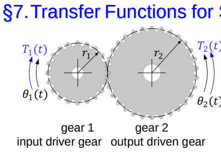
$$-(s+1)\theta_1(s) + (2s+2)\theta_2(s) = 0$$

Solving for $\theta_2(s)$

$$\theta_2 = \frac{\begin{vmatrix} s^2+s+1 & T \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} s^2+s+1 & -(s+1) \\ -(s+1) & 2s+2 \end{vmatrix}} = \frac{(s+1)T}{2s^3+3s^2+2s+1}$$

$$\rightarrow G(s) = \frac{\theta_2(s)}{T(s)} = \frac{s+1}{2s^3+3s^2+2s+1}$$

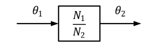
§7. Transfer Functions for Systems with Gears



Kinematic relationship

$$\theta_2 r_2 = \theta_1 r_1 \rightarrow \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

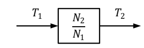
Power on gears



$$T_1\dot{\theta}_1 = T_2\dot{\theta}_2$$

The ratio of torques on two gears

$$\frac{T_2}{T_1} = \frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{N_2}{N_1}$$



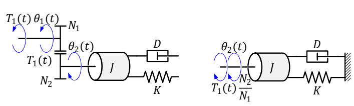
θ_1, θ_2 : rotation angles of gear 1 and 2, rad

r_1, r_2 : radius of gear 1 and 2, m

N_1, N_2 : number of teeth of gear 1 and 2

T_1, T_2 : torques on gear 1 and 2, Nm

§7. Transfer Functions for Systems with Gears



a. rotational system driven by gears b. equivalent system at the output after reflection of input torque

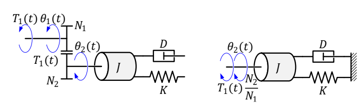
What happens to mechanical impedances that are driven by gears?

(a) : gears driving a rotational inertia, spring, and viscous damper

(b) : an equivalent system at θ_1 without the gears

Can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?

§7. Transfer Functions for Systems with Gears



a. rotational system driven by gears b. equivalent system at the output after reflection of input torque

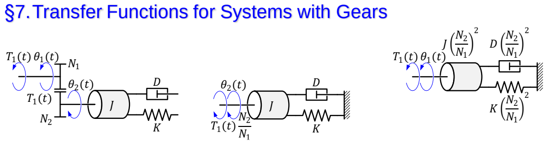
T_1 can be reflected to the output by multiplying by N_2/N_1

$$(Js^2+Ds+K)\theta_2(s) = T_1(s)\frac{N_2}{N_1} \quad (2.131)$$

Convert $\theta_2(s)$ into an equivalent $\theta_1(s)$, so that

$$(Js^2+Ds+K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1} \quad (2.132)$$

§7. Transfer Functions for Systems with Gears



a. rotational system driven by gears b. equivalent system at the output after reflection of input torque c. equivalent system at the input after reflection of impedances

$$(Js^2+Ds+K)\theta_2(s) = T_1(s)(N_2/N_1) \quad (2.131)$$

$$(Js^2+Ds+K)(N_1/N_2)\theta_1(s) = T_1(s)(N_2/N_1) \quad (2.132)$$

$$\rightarrow \left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s) \quad (2.133)$$

Thus, the load can be thought of as having been reflected from the output to the input

§7. Transfer Functions for Systems with Gears

Generalizing the results

Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

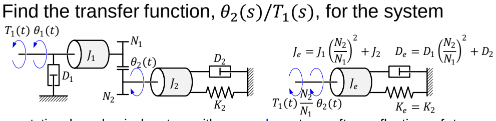
$$\left(\frac{\text{number of teeth of gear on destination shaft}}{\text{number of teeth of gear on source shaft}}\right)^2$$

where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft

§7. Transfer Functions for Systems with Gears

- Ex.2.21 Transfer Function - System with Lossless Gears

Find the transfer function, $\theta_2(s)/T_1(s)$, for the system



a. rotational mechanical system with gears b. system after reflection of torques and impedances to the output shaft

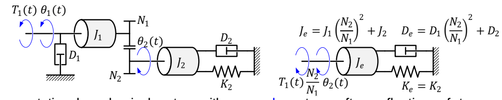
Solution

Reflect the impedances (J_1 and D_1) and torque (T_1) on the input shaft to the output, where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by (N_2/N_1)

The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1} \qquad (2.139)$$

§7. Transfer Functions for Systems with Gears



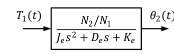
a. rotational mechanical system with gears b. system after reflection of torques and impedances to the output shaft

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)(N_2/N_1) \qquad (2.139)$$

where, $J_e = J_1(N_2/N_1)^2 + J_2$, $D_e = D_1(N_2/N_1)^2 + D_2$, $K_e = K_2$

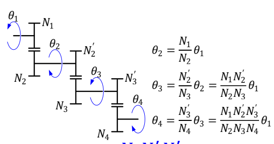
Solving for $G(s)$

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}$$



§7. Transfer Functions for Systems with Gears

- In order to eliminate gears with large radii, a gear train is used to implement large gear ratios by cascading smaller gear ratios.



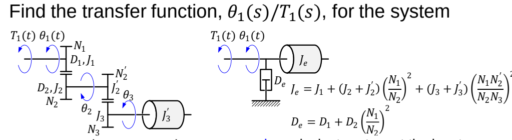
$$\theta_4 = \frac{N_1 N_2' N_3'}{N_2 N_3 N_4}\theta_1$$

- For gear trains, the equivalent gear ratio is the product of the individual gear ratios

§7. Transfer Functions for Systems with Gears

- Ex.2.22 Transfer Function – Gears with Loss

Find the transfer function, $\theta_1(s)/T_1(s)$, for the system



a. system using a gear train b. equivalent system at the input

Solution

Reflect all of the impedances to the input shaft, θ_1

The equation of motion can now be written as

$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s)$$

The transfer function

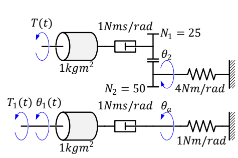
$$G(s) = \theta_1(s)/T_1(s) = 1/(J_e s^2 + D_e s)$$

§7. Transfer Functions for Systems with Gears

Skill-Assessment Ex.2.10

Problem Find the TF

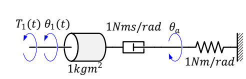
$$G(s) = \frac{\theta_2(s)}{T(s)}$$



Solution Transforming the network to one without gears by reflecting the $4Nm/rad$ spring to the left and multiplying by $(25/50)^2$

$$4[Nm/rad] \times \left(\frac{25}{50}\right)^2 = 1[Nm/rad]$$

§7. Transfer Functions for Systems with Gears



The equation of motion

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_1\end{bmatrix} \times \theta_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix} \times \theta_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_1\end{bmatrix}$$

$$+(s^2 + s)\theta_1(s) \qquad - s\theta_a(s) = T(s)$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } \theta_1 \text{ and } \theta_2\end{bmatrix} \times \theta_1(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } \theta_2\end{bmatrix} \times \theta_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{torques at } \theta_2\end{bmatrix}$$

$$-s\theta_1(s) + (s+1)\theta_a(s) = 0$$

§7. Transfer Functions for Systems with Gears

The equation of motion

$$(s^2+s)\theta_1(s) \qquad -s\theta_a(s) = T(s)$$
$$-s\theta_1(s) + (s+1)\theta_a(s) = 0$$

Solving for $\theta_a(s)$

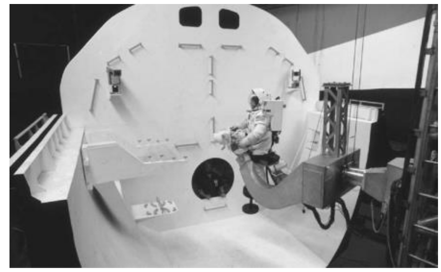
$$\theta_a(s) = \frac{\begin{vmatrix} s^2+s & T \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} s^2+s & -s \\ -s & s+1 \end{vmatrix}} = \frac{sT(s)}{s^3+s^2+s}$$

$$\rightarrow \frac{\theta_a(s)}{T(s)} = \frac{1}{s^2+s+1}$$

The transfer function

$$\frac{\theta_2(s)}{T(s)} = \frac{\frac{1}{2}\theta_a(s)}{T(s)} = \frac{1/2}{s^2+s+1}$$

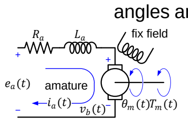
§8. Electromechanical System Transfer Functions



NASA flight simulator robot arm with electromechanical control system components

§8. Electromechanical System Transfer Functions

- A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input
- Derive the transfer function for the armature-controlled dc servomotor (Mablekos, 1980)
  - Fixed field: a magnetic field is developed by stationary permanent magnets or a stationary electromagnet
  - Armature: a rotating circuit, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force



$$F = Bli_a(t)$$

B : the magnetic field strength

l : the length of the conductor

§8. Electromechanical System Transfer Functions

- A conductor moving at right angles to a magnetic field generates a voltage at the terminals of the conductor equal to

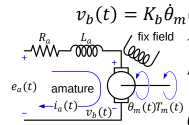
$$e = Blv$$

e : the voltage

v : the velocity of the conductor normal to the magnetic field

- The current-carrying armature is rotating in a magnetic field, its voltage is proportional to speed

$$v_b(t) = K_b\dot{\theta}_m(t) \qquad (2.144)$$



$v_b(t)$: back electromotive force (back emf)

K_b : a constant of proportionality called the back emf constant

$\dot{\theta}_m(t)$: the angular velocity of the motor

§8. Electromechanical System Transfer Functions

- Taking the Laplace transform

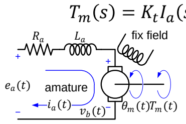
$$V_b(s) = K_b s\theta_m(s) \qquad (2.145)$$

- The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$

$$R_aI_a(s) + L_asI_a(s) + V_b(s) = E_a(s) \qquad (2.146)$$

- The torque developed by the motor is proportional to the armature current

$$T_m(s) = K_tI_a(s) \qquad (2.147)$$



T_m : the torque developed by the motor

K_t : the motor torque constant, which depends on the motor and magnetic field characteristics

§8. Electromechanical System Transfer Functions

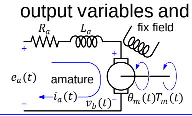
- Rearranging Eq.(2.147)

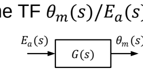
$$I_a(s) = \frac{1}{K_t}T_m(s) \qquad (2.148)$$

- To find the TF of the motor, first substitute Eqs. (2.145) and (2.148) into (2.146), yielding

$$\frac{(R_a+L_as)T_m(s)}{K_t} + K_bs\theta_m(s) = E_a(s) \qquad (2.149)$$

- Then, find $T_m(s)$ in terms of $\theta_m(s)$, separate the input and output variables and obtain the TF $\theta_m(s)/E_a(s)$

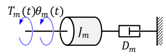




$V_b(s) = K_bs\theta_m(s)$ (2.145), $\quad R_aI_a(s) + L_asI_a(s) + V_b(s) = E_a(s)$ (2.146)

§8. Electromechanical System Transfer Functions

- A typical equivalent mechanical loading on a motor



J_m : the equivalent inertia at the armature and includes both the armature inertia and, the load inertia reflected to the armature

D_m : the equivalent viscous damping at the armature and includes both the armature viscous damping and, the load viscous damping reflected to the armature

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s) \quad (2.150)$$

- Substituting Eq.(2.150) into Eq.(2.149)

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \quad (2.151)$$

$\frac{(R_a+L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$ (2.149)

§8. Electromechanical System Transfer Functions

- Assume that the armature inductance, L_a, is small compared to the armature resistance, R_a, which is usual for a dc motor, Eq. (2.151) becomes

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s) \quad (2.152)$$

- After simplification

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a}\frac{1}{J_m}}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t}{R_a}K_b\right)\right]} \quad (2.153)$$

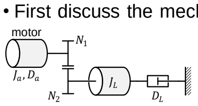
- The form of Eq.(2.153)

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)} \quad (2.154)$$

$\frac{(R_a+L_a s)(J_m s^2+D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$ (2.151)

§8. Electromechanical System Transfer Functions

- First discuss the mechanical constants, J_m and D_m. Consider the figure: a motor with inertia J_a and damping D_a at the armature driving a load consisting of inertia J_L and damping D_L



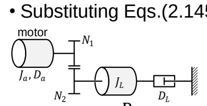
Assuming that all inertia and damping values shown are known, J_L and D_L can be reflected back to the armature as some equivalent inertia and damping to be added to J_a and D_a, respectively → The equivalent inertia, J_m, and equivalent damping, D_m, at the armature

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 \quad (2.155.a)$$

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 \quad (2.155.b)$$

§8. Electromechanical System Transfer Functions

- Substituting Eqs.(2.145), (2.148) into Eq. (2.146), with $L_a = 0$



$$\frac{R_a}{K_t}T_m(s) + K_b s\theta_m(s) = E_a(s) \quad (2.156)$$

Taking the inverse Laplace transform

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t) \quad (2.157)$$

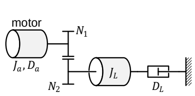
- When the motor is operating at steady state with a dc voltage input

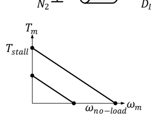
$$\frac{R_a}{K_t}T_m + K_b\omega_m = e_a \quad (2.158)$$

$$\rightarrow T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a \quad (2.159)$$

$V_b(s) = K_b s\theta_m(s)$ (2.145), $R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$ (2.146), $I_a(s) = \frac{1}{K_t}T_m(s)$ (2.148)

§8. Electromechanical System Transfer Functions





Torque-speed curves with an armature voltage, e_a, as a parameter

The stall torque

$$T_{stall} = \frac{K_t}{R_a}e_a \quad (2.160)$$

The no-load speed

$$\omega_{no-load} = \frac{e_a}{K_b} \quad (2.161)$$

The electrical constants of the motor

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} \quad (2.162)$$

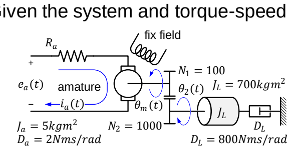
$$K_b = \frac{e_a}{\omega_{no-load}} \quad (2.163)$$

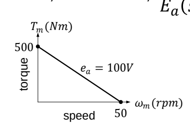
The electrical constants, K_t/R_a and K_b, can be found from a dynamometer test of the motor, which would yield T_{stall} and $\omega_{no-load}$ for a given e_a

§8. Electromechanical System Transfer Functions

- Ex.2.23 Transfer Function-DC Motor and Load

Given the system and torque-speed curve, find the TF, $\frac{\theta_L(s)}{E_a(s)}$

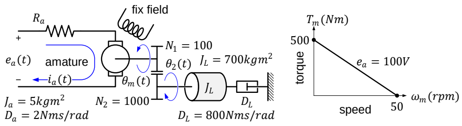




§8. Electromechanical System Transfer Functions

Solution

Find the mechanical constants



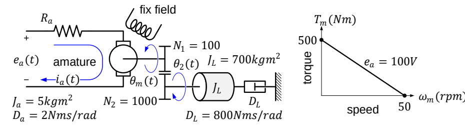
$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700 \times \left(\frac{100}{1000}\right)^2 = 12$$

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800 \times \left(\frac{100}{1000}\right)^2 = 10$$

§8. Electromechanical System Transfer Functions

Find the electrical constants from the torque-speed curve

$T_{stall} = 500$, $\omega_{no-load} = 50$, $e_a = 100$



$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{500}{100} = 5$$

$$K_b = \frac{e_a}{\omega_{no-load}} = \frac{100}{50} = 2$$

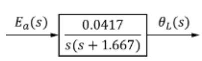
§8. Electromechanical System Transfer Functions

The transfer function $\theta_m(s)/E_a(s)$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a}\frac{1}{J_m}}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t}{R_a}K_b\right)\right]}$$

$$= \frac{5 \times \frac{1}{12}}{s\left[s + \frac{1}{12} \times (10 + 5 \times 2)\right]}$$

$$= \frac{0.417}{s(s + 1.667)}$$



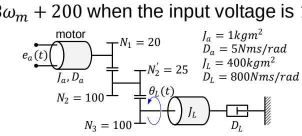
The transfer function $\theta_L(s)/E_a(s)$

$$\frac{\theta_L(s)}{E_a(s)} = \frac{\theta_m(s)\frac{N_1}{N_2}}{E_a(s)} = \frac{0.417 \times \frac{100}{1000}}{s(s + 1.667)} = \frac{0.0417}{s(s + 1.667)}$$

§8. Electromechanical System Transfer Functions

Skill-Assessment Ex.2.11

Problem Find the TF, $G(s) = \theta_L(s)/E_s(s)$, for the motor and load system. The torque-speed curve is given by $T_m = -8\omega_m + 200$ when the input voltage is $100volts$



Solution

Find the mechanical constants

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\frac{N_2'}{N_3}\right)^2 = 1 + 400 \times \left(\frac{20}{100} \times \frac{25}{100}\right)^2 = 2$$

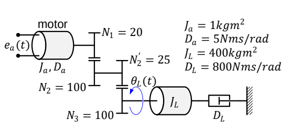
$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\frac{N_2'}{N_3}\right)^2 = 5 + 800 \times \left(\frac{20}{100} \times \frac{25}{100}\right)^2 = 7$$

§8. Electromechanical System Transfer Functions

Find the electrical constants from the torque-speed eq.

$\omega_m = 0 \quad \rightarrow T_m = 200$

$T_m = 0 \quad \rightarrow \omega_{no-load} = 200/8 = 25$



$$\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{200}{100} = 2$$

$$K_b = \frac{E_a}{\omega_{no-load}} = \frac{100}{25} = 4$$

$T_m = -8\omega_m + 200$

§8. Electromechanical System Transfer Functions

Substituting all values into the motor transfer function

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a}\frac{1}{J_m}}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t}{R_a}K_b\right)\right]}$$

$$= \frac{2 \times \frac{1}{2}}{s\left[s + \frac{1}{2}(7 + 2 \times 4)\right]}$$

$$= \frac{1}{s(s + 7.5)}$$

The transfer function $\theta_L(s)/E_a(s)$

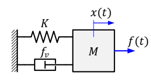
$$\frac{\theta_L(s)}{E_a(s)} = \frac{\theta_m(s)\frac{N_1}{N_2}\frac{N_2'}{N_3}}{E_a(s)} = \frac{\frac{20}{100} \times \frac{25}{100}}{s(s + 7.5)} = \frac{0.05}{s(s + 7.5)}$$

§9. Electric Circuit Analogs

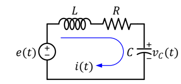
- Electric circuit analog: an electric circuit that is analogous to a system from another discipline
- The mechanical systems can be represented by the equivalent electric circuits
- Analogs can be obtained by comparing the describing equations, such as the equations of motion of a mechanical system, with either electrical mesh or nodal equations
  - when compared with mesh equations, the resulting electrical circuit is called a series analog
  - when compared with nodal equations, the resulting electrical circuit is called a parallel analog

§9. Electric Circuit Analogs

Series Analog



mechanical system



desired electrical representation

Consider the translational mechanical system, the equation of motion

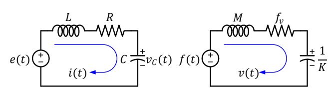
$$(Ms^2 + f_v s + K)X(s) = F(s) = \frac{Ms^2 + f_v s + K}{s} sX(s)$$

$$\rightarrow \left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s)$$

Kirchhoff's mesh equation for the simple series RLC network

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$

§9. Electric Circuit Analogs



desired electrical representation series analog

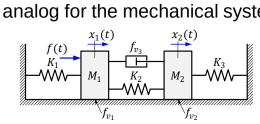
Parameters for series analog

mass	$= M$	$\rightarrow$ inductor	$L = M$ henries
viscous damper	$= f_v$	$\rightarrow$ resistor	$R = f_v$ ohms
spring	$= K$	$\rightarrow$ capacitor	$C = 1/K$ farads
applied force	$= f(t)$	$\rightarrow$ voltage source	$e(t) = f(t)$
velocity	$= v(t)$	$\rightarrow$ mesh current	$i(t) = v(t)$

§9. Electric Circuit Analogs

- Ex.2.24 Converting a Mechanical System to a Series Analog

Draw a series analog for the mechanical system



Solution

The equations of motion with $X(s) \rightarrow V(s)$

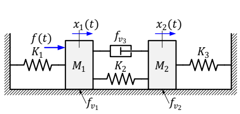
$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{K_1 + K_2}{s}\right]V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right)V_2(s) = F(s)$$

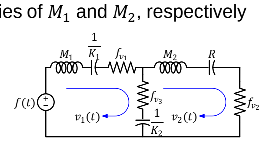
$$-\left(f_{v_3} + \frac{K_2}{s}\right)V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{K_2 + K_3}{s}\right]V_2(s) = 0$$

§9. Electric Circuit Analogs

Coefficients represent sums of electrical impedance. Mechanical impedances associated with M_1 form the first mesh, where impedances between the two masses are common to the two loops. Impedances associated with M_2 form the second mesh

$v_1(t)$ and $v_2(t)$are the velocities of M_1 and M_2, respectively



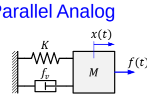


$$+\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{K_1+K_2}{s}\right]V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right)V_2(s) = F(s)$$

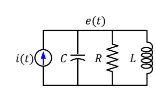
$$-\left(f_{v_3} + \frac{K_2}{s}\right)V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{K_2+K_3}{s}\right]V_2(s) = 0$$

§9. Electric Circuit Analogs

Parallel Analog



mechanical system



desired electrical representation

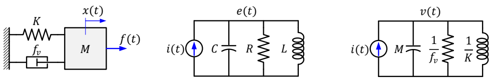
Consider the translational mechanical system, the equation of motion

$$\left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s)$$

Kirchhoff's nodal equation for the simple parallel RLC network

$$\left(Cs + \frac{1}{Rs} + \frac{1}{Ls}\right)E(s) = I(s)$$

§9. Electric Circuit Analogs



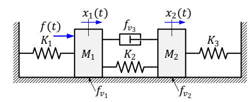
Parameters for parallel analog

mass	$= M$	$\rightarrow$	capacitor	$C = M$ farads
viscous damper	$= f_v$	$\rightarrow$	resistor	$R = 1/f_v$ ohms
spring	$= K$	$\rightarrow$	inductor	$L = 1/K$ henries
applied force	$= f(t)$	$\rightarrow$	current source	$i(t) = f(t)$
velocity	$= v(t)$	$\rightarrow$	node voltage	$e(t) = v(t)$

§9. Electric Circuit Analogs

- Ex.2.25 Converting a Mechanical System to a Parallel Analog

Draw a parallel analog for the mechanical system



Solution

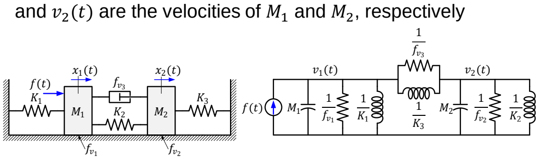
The equations of motion with $X(s) \rightarrow V(s)$

$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{K_1 + K_2}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{K_2 + K_3}{s}\right] V_2(s) = 0$$

§9. Electric Circuit Analogs

Coefficients represent sums of electrical admittances. Admittances associated with M_1 form the elements connected to the first node, where mechanical admittances between the two masses are common to the two nodes. Mechanical admittances associated with M_2 form the elements connected to the second node $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2, respectively



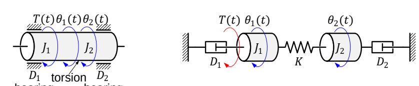
$$+\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{K_1 + K_2}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{K_2 + K_3}{s}\right] V_2(s) = 0$$

§9. Electric Circuit Analogs

Skill-Assessment Ex.2.12

Problem Draw a series and parallel analog for the rotational mechanical system



Solution

The equations of motion

$$+(J_1 s^2 + D_1 s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

§9. Electric Circuit Analogs

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

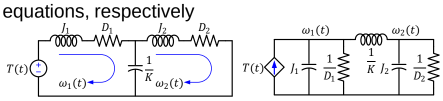
$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

Letting $\theta_1(s) = \omega_1(s)/s$, $\theta_2(s) = \omega_2(s)/s$

$$\left(J_1 s + D_1 + \frac{K}{s}\right)\omega_1(s) - \frac{K}{s}\omega_2(s) = T(s)$$

$$-\frac{K}{s}\omega_1(s) + \left(J_2 s + D_2 + \frac{K}{s}\right)\omega_2(s) = 0$$

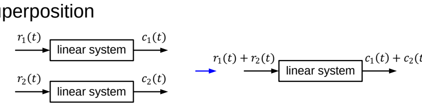
From these equations, draw both series and parallel analogs by considering these to be mesh or nodal equations, respectively



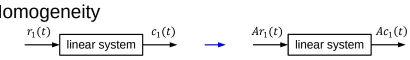
series analog parallel analog

§10. Nonlinearities

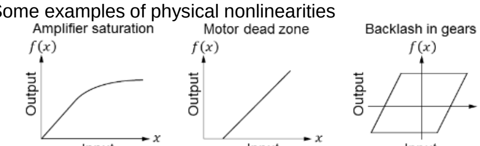
- A linear system possesses two properties
  - Superposition



  - Homogeneity



- Some examples of physical nonlinearities

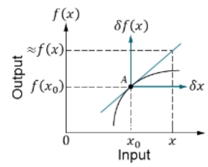


§11.Linearization

- To obtain transfer function from a nonlinear system
  - recognize the nonlinear component and write the nonlinear differential equation
  - find the steady-state solution is called equilibrium
  - linearize the nonlinear differential equation
  - take the Laplace transform of the linearized differential equation, assuming zero initial conditions
  - separate input and output variables and form the transfer function

§11.Linearization

- Assume a nonlinear system operating at point A, $[x_0, f(x_0)]$, small changes in the input can be related to changes in the output about the point by way of the slope of the curve at the point A



$$f(x) - f(x_0) \approx m_a(x - x_0)$$
$$\rightarrow \delta f(x) \approx m_a \delta x$$
$$\rightarrow f(x) \approx f(x_0) + m_a(x - x_0)$$
$$\approx f(x_0) + m_a \delta x$$

m_a : the slope of the curve at point A

δx : small excursions of the input about point A

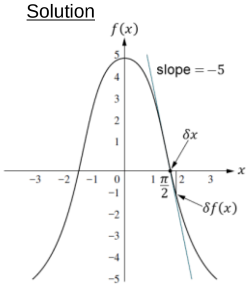
$\delta f(x)$: small changes in the output related by the slope at point A

§11.Linearization

- Ex.2.26 Linearizing a Function

Linearize $f(x) = 5\cos x$ about $x = \pi/2$

Solution



Using the linearized equation

$$f(x) \approx f(x_0) + m_a \delta x$$

where

$$f\left(\frac{\pi}{2}\right) = 5\cos\left(\frac{\pi}{2}\right) = 0$$

$$m_a = \left.\frac{df}{dx}\right|_{x=\frac{\pi}{2}} = \left.(-5\sin x)\right|_{x=\frac{\pi}{2}} = -5$$

The system can be presented as

$$f(x) \approx -5\delta x$$

for small excursions of x about $\pi/2$

§11.Linearization

Taylor series expansion

Taylor series expansion expresses the value of a function in terms of the value of that function at a particular point, the excursion away from that point, and derivatives evaluated at that point

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{(x - x_o)}{1!} + \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} \frac{(x - x_o)^2}{2!} + \cdots$$

For small excursions of x from x_0, the higher-order terms can be neglected

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_o)$$

§11.Linearization

- Ex.2.27 Linearizing a Differential Equation

Linearize the following equation for small excursion about $x = \pi/4$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0$$

Solution

The presence of the term $\cos x$ makes this equation nonlinear

Since we want to linearize the equation about $x = \pi/4$, we let $x = \pi/4 + \delta x$, where δx is the small excursion about $\pi/4$

$$\frac{d^2(\delta x + \pi/4)}{dt^2} + 2\frac{d(\delta x + \pi/4)}{dt} + \cos(\delta x + \pi/4) = 0$$

$$\frac{d^2(\delta x + \pi/4)}{dt^2} = \frac{d^2 \delta x}{dt^2}$$

$$\frac{d(\delta x + \pi/4)}{dt} = \frac{d\delta x}{dt}$$

§11.Linearization

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

$$f(x) = \cos x = \cos(\delta x + \pi/4)$$

$$f(x_0) = f(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$$

$$x - x_0 = \delta x$$

$$\left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{d\cos x}{dx}\right|_{x=\pi/4} = -\sin(\pi/4) = -\sqrt{2}/2$$

$$\rightarrow \cos(\delta x + \pi/4) = \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\delta x$$

The linearized differential equation

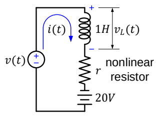
$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} - \frac{\sqrt{2}}{2}\delta x = -\frac{\sqrt{2}}{2}$$

Solve this equation for δx, and obtain $x = \delta x + \pi/4$

§11.Linearization

- Ex.2.28 Transfer Function-Nonlinear Electrical Network

Find the transfer function, $V_L(s)/V(s)$, for the electrical network, which contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where i_r and v_r are the resistor current and voltage, respectively. Also, $v(t)$ is a small-signal source



Solution

From the voltage-current relationship

$$i_r = 2e^{0.1v_r}$$

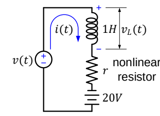
$$\rightarrow v_r = 10\ln(0.5i_r) = 10\ln(0.5i)$$

Applying Kirchhoff's voltage law around the loop

$$L\frac{di}{dt} + 10\ln(0.5i) - 20 = v(t)$$

§11.Linearization

- Evaluate the equilibrium solution
  - Set the small-signal source, $v(t)$, equal to zero



- Evaluate the steady-state current

In the steady state $v_L(t) = Ldi/dt$ and di/dt, given a constant battery source. Hence, the resistor voltage, v_r, is $20V$

$$i_r = 2e^{0.1v_r} = 2e^{0.1\times 20} = 14.78A$$

$$\rightarrow i_0 = i_r = 14.78A$$

i_0 is the equilibrium value of the network current $\rightarrow i = i_0 + \delta i$

$$L\frac{di}{dt} + 10\ln(0.5i) - 20 = v(t)$$

$$\rightarrow L\frac{d(i_0+\delta i)}{dt} + 10\ln[0.5(i_0+\delta i)] - 20 = v(t)$$

§11.Linearization

$$f(i) = f(i_0) + \left.\frac{df}{di}\right|_{i=i_0}(i - i_0)$$

$$f(i) = \ln(0.5i) = \ln[0.5(i_0+\delta i)]$$

$$f(i_0) = \ln(0.5i_0)$$

$$i - i_0 = \delta i$$

$$\left.\frac{df}{di}\right|_{i=i_0} = \left.\frac{d\ln(0.5i)}{di}\right|_{i=i_0} = \left.\frac{1}{i}\right|_{i=i_0} = \frac{1}{i_0}$$

$$\rightarrow \ln[0.5(i_0+\delta i)] = \ln(0.5i_0) + \frac{1}{i_0}\delta i$$

The linearized equation

$$L\frac{d\delta i}{dt} + 10\left(\ln(0.5i_0) + \frac{1}{i_0}\delta i\right) - 20 = v(t)$$

$L\frac{d(i_0+\delta i)}{dt} + 10\ln[0.5(i_0+\delta i)] - 20 = v(t)$

§11.Linearization

The linearized equation with $L = 1H$, $i_0 = 14.78A$

$$\frac{d\delta i}{dt} + 0.677\delta i = v(t) \rightarrow \delta i(s) = \frac{V(s)}{s+0.677}$$

The voltage across the inductor about the equilibrium point

$$v_L(t) = L\frac{d(i_0+\delta i)}{dt} = L\frac{d\delta i}{dt} \rightarrow V_L(s) = Ls\delta i(s) = s\delta i(s)$$

The voltage across the inductor about the equilibrium point

$$V_L(s) = s\frac{V(s)}{s+0.677}$$

The final transfer function

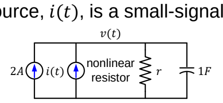
$$\frac{V_L(s)}{V(s)} = \frac{s}{s+0.677}$$

for small excursions about $i = 14.78A$ or, equivalently, about $v(t) = 0$

§11.Linearization

Skill-Assessment Ex.2.13

Problem Find the linearized TF $G(s) = V(s)/I(s)$, for the electrical network. The network contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = e^{v_r}$. The current source, $i(t)$, is a small-signal generator



Solution

The nodal equation

$$C\frac{dv}{dt} + i_r - 2 = i(t)$$

But $C = 1$, $v = v_0 + \delta v$, $i_r = e^{v_r} = e^v = e^{v_0+\delta v}$

$$\frac{d(v_0+\delta v)}{dt} + e^{v_0+\delta v} - 2 = i(t)$$

§11.Linearization

Linearize e^v

$$f(v) = f(v_0) + \left.\frac{df}{dv}\right|_{v=v_0}(v - v_0)$$

$$f(v) = e^v = e^{v_0+\delta v}$$

$$f(v_0) = e^{v_0}$$

$$v - v_0 = \delta v$$

$$\left.\frac{df}{dv}\right|_{v=v_0} = \left.\frac{de^v}{dv}\right|_{v=v_0} = \left.e^v\right|_{v=v_0} = e^{v_0}$$

$$\rightarrow e^{v_0+\delta v} = e^{v_0} + e^{v_0}\delta v$$

The linearized equation

$$\frac{d\delta v}{dt} + e^{v_0} + e^{v_0}\delta v - 2 = i(t)$$

$\frac{d(v_0+\delta v)}{dt} + e^{v_0+\delta v} - 2 = i(t)$

§11.Linearization

Setting $i(t) = 0$ and letting the circuit reach steady state, the capacitor acts like an open circuit. Thus, $v_0 = v_r$ with $i_r = 2$. But, $i_r = e^{v_r}$ or $v_r = \ln i_r$. Hence, $v_0 = \ln 2 = 0.693$

$$\frac{d\delta v}{dt} + e^{v_0} + e^{v_0}\delta v - 2 = i(t)$$

$$\rightarrow \frac{d\delta v}{dt} + 2\delta v = i(t)$$

Taking the Laplace transform

$$(s+2)\delta v(s) = I(s)$$

The transfer function

$$\frac{\delta v(s)}{I(s)} = \frac{V(s)}{I(s)} = \frac{1}{s+2}$$

about equilibrium

§12.Case Studies