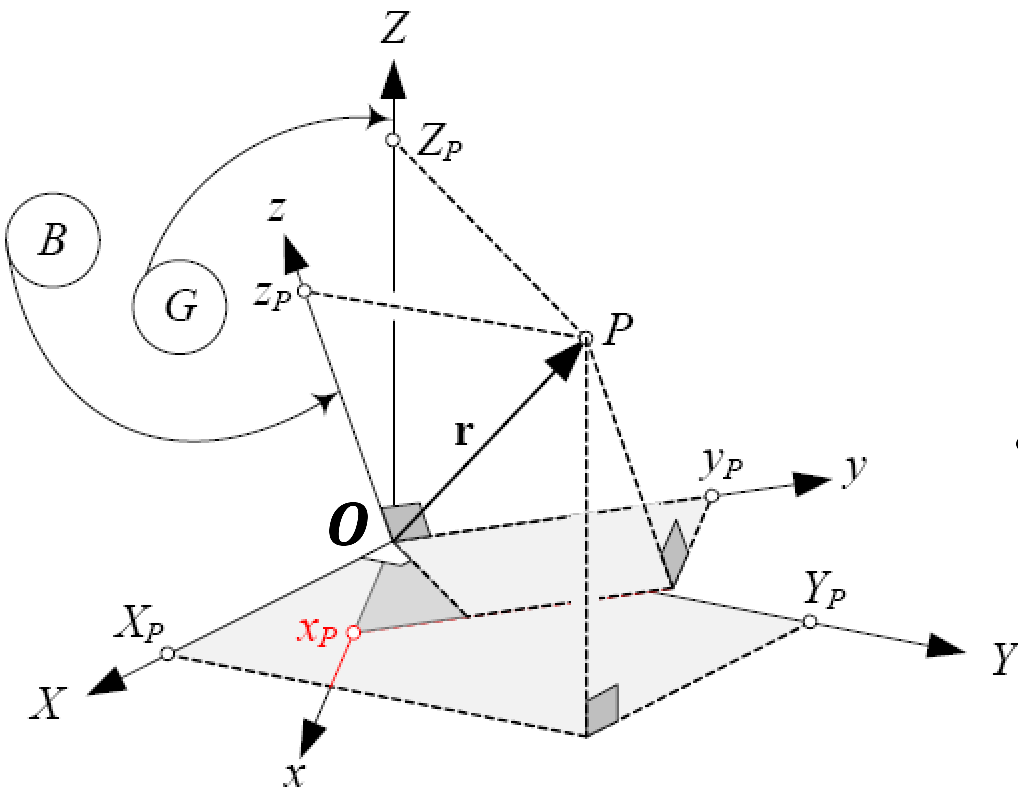
DE MO

03. Spatial Descriptions and Transformations

CONTENTS

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- Consider a rigid body ***B*** with a fixed point ***O*** ⟹ Rotation about the fixed point ***O*** is the only possible motion of the body ***B***



- Consider
  - local coordinate frame $Oxyz$ attaches to the rigid body ***B***
  - global coordinate frame $OZYZ$
- Determine
  - **Transformation matrices** for point ***P*** between two coordinates

- Point ***O*** of the body ***B*** is fixed to the ground ***G*** and is the origin of both coordinate frames

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

Trong sach co chung minh

- Rigid body $\boldsymbol{B}$ rotates $\boldsymbol{\alpha}$ **degrees** about the $\boldsymbol{Z}$-axis of the global coordinate
- $\boldsymbol{P}$ has local coordinate: ${}^{B}\boldsymbol{r}_P = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}$ and global coordinate: ${}^{G}\boldsymbol{r}_P = \begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix}$

$\Longrightarrow$ Relation between the two coordinates:

$$\begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} = \begin{bmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}$$

or

$${}^{G}\boldsymbol{r}_P = \boldsymbol{Q}_{Z,\alpha}\, {}^{B}\boldsymbol{r}_P \qquad (2.1)$$

where $\boldsymbol{Q}_{Z,\alpha}$ is the **Z-rotation matrix**

$$\boldsymbol{Q}_{Z,\alpha} = \begin{bmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.3)$$

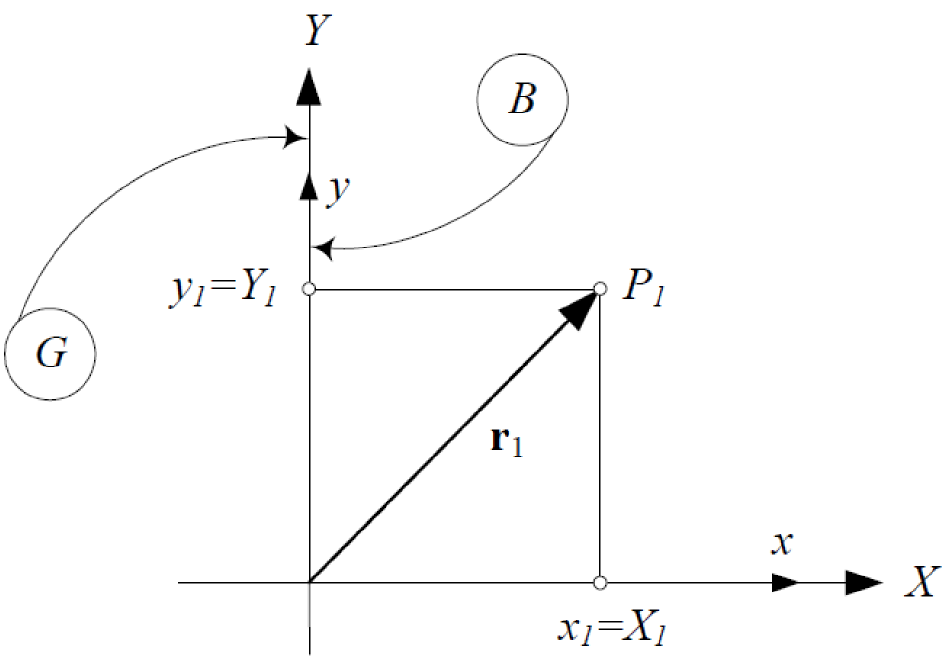
1. ROTATION ABOUT GLOBAL CARTESIAN AXES

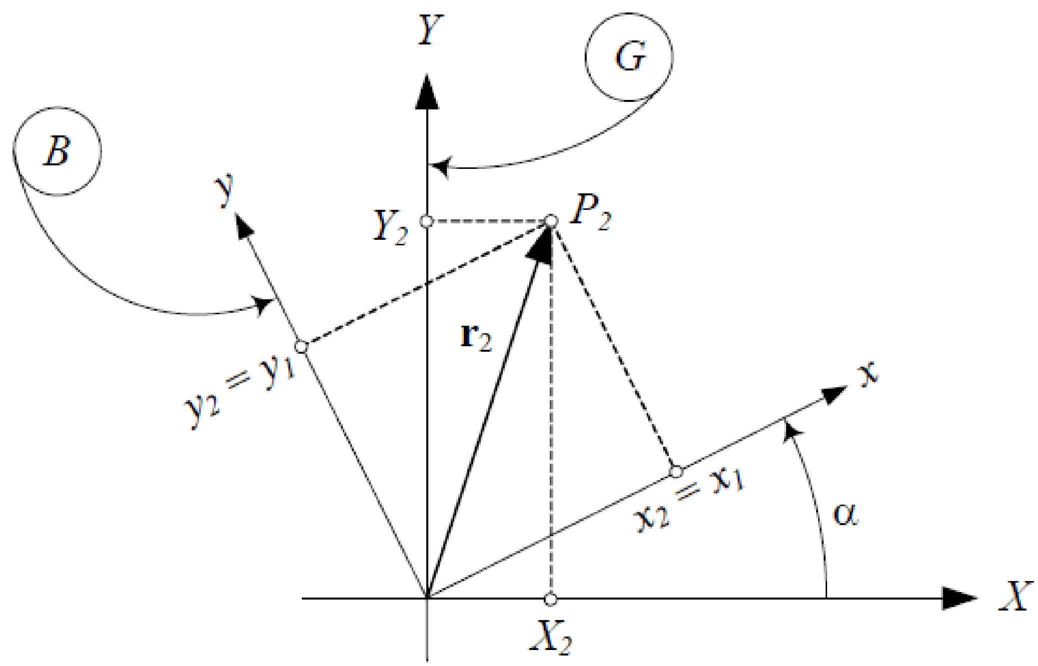
Before rotation:

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

After rotation:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$





1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- Similarly, rotation β degrees about the Y-axis, and γ degrees about the X-axis of the global frame relate the local and global coordinates of point $\boldsymbol{P}$ by the following equations

$$ {}^{G}\boldsymbol{r} = \boldsymbol{Q}_{Y,\beta}\, {}^{B}\boldsymbol{r} \tag{2.4}$$

$$ {}^{G}\boldsymbol{r} = \boldsymbol{Q}_{X,\gamma}\, {}^{B}\boldsymbol{r} \tag{2.5}$$

Where $\boldsymbol{Q}_{Y,\beta}$ is the ***Y*-rotation matrix**

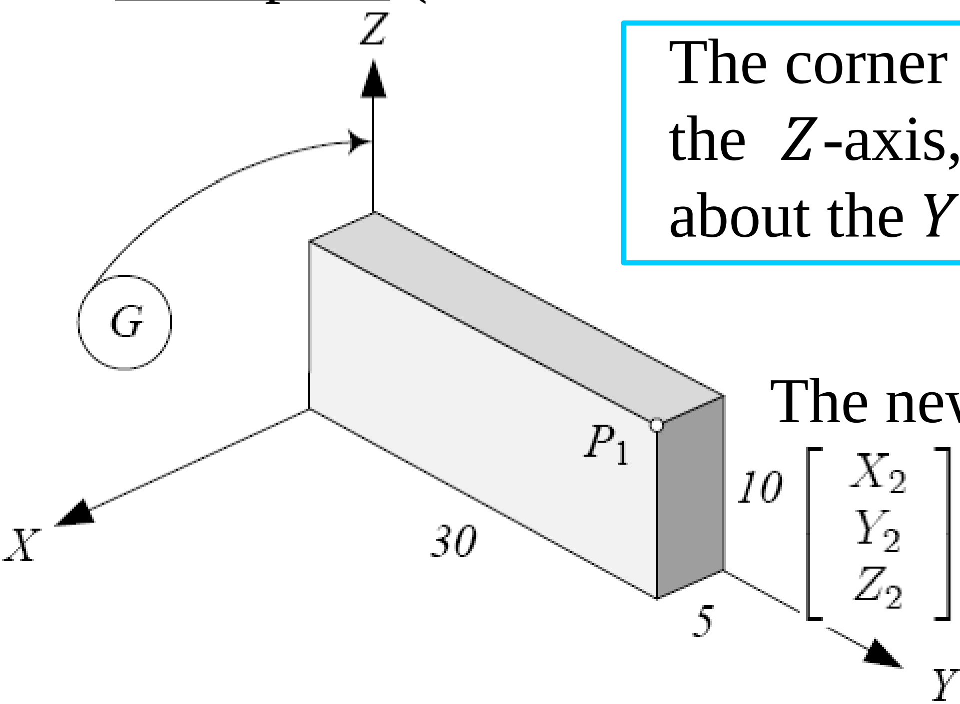
$$\boldsymbol{Q}_{Y,\beta} = \begin{bmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{bmatrix} \tag{2.6}$$

and $\boldsymbol{Q}_{X,\gamma}$ is the ***X*-rotation matrix**

$$\boldsymbol{Q}_{X,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & -sin\gamma \\ 0 & sin\gamma & cos\gamma \end{bmatrix} \tag{2.7}$$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- <u>Example 3</u> (*Successive rotation about global axes*)



The corner $\boldsymbol{P}(5,30,10)$ of the slab rotates 30^0 about the Z-axis, then 30^0 about the X-axis, and 90^0 about the Y-axis $\Longrightarrow$ Find the **final position** of $\boldsymbol{P}$?

Q=QyQxQz

The new global position of $\boldsymbol{P}$ after first rotation

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix}$$

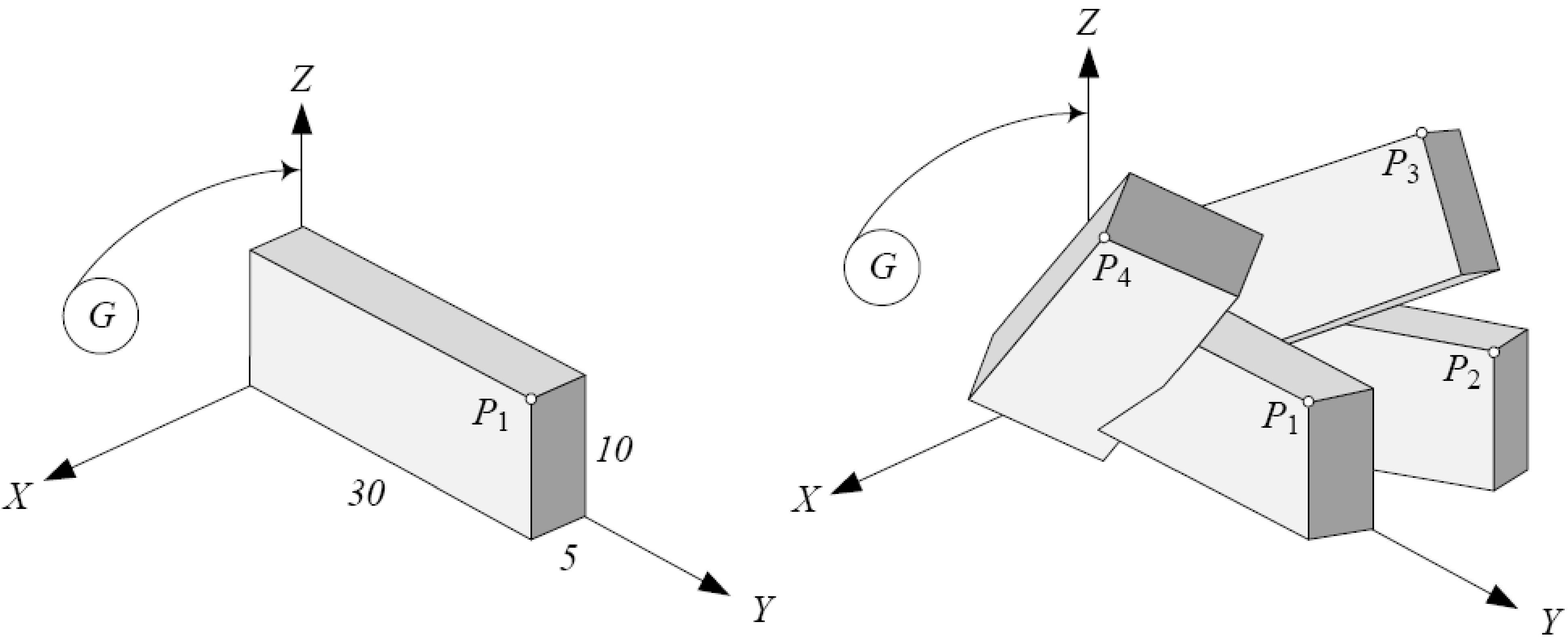
After the second rotation

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix}$$

And after the last rotation

$$\begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \cos 90 & 0 & \sin 90 \\ 0 & 1 & 0 \\ -\sin 90 & 0 & \cos 90 \end{bmatrix} \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix} = \begin{bmatrix} 22.90 \\ 19.66 \\ 10.68 \end{bmatrix}$$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES



The slab and the point ***P*** in first, second, third, and fourth positions

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- Example 4 (*Time dependent global rotation*)

A rigid body $\boldsymbol{B}$ continuously turns about Y-axis of $\boldsymbol{G}$ at a rate of $0.3rad/s$
$\Longrightarrow$ Find the **global position** and **global velocity** of the body point $\boldsymbol{P}$?

The rotation transformation matrix of the body is

$$^{G}\boldsymbol{Q}_B = \begin{bmatrix} cos0.3t & 0 & sin0.3t \\ 0 & 1 & 0 \\ -sin0.3t & 0 & cos0.3t \end{bmatrix}$$

Any point of $\boldsymbol{B}$ will move on a circle with radius $R = \sqrt{X^2 + Z^2}$ parallel to (X, Z)-plane

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} cos0.3t & 0 & sin0.3t \\ 0 & 1 & 0 \\ -sin0.3t & 0 & cos0.3t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xcos0.3t + zsin0.3t \\ y \\ zcos0.3t - xsin0.3t \end{bmatrix}$$

$$X^2 + Z^2 = (xcos0.3t + zsin0.3t)^2 + (zcos0.3t - xsin0.3t)^2$$
$$= x^2 + z^2 = R^2$$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

After $t = 1s$, the point ${}^{B}\boldsymbol{r} = [1 \quad 0 \quad 0]^T$ will be seen at

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos 0.3 & 0 & \sin 0.3 \\ 0 & 1 & 0 \\ -\sin 0.3 & 0 & \cos 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.955 \\ 0 \\ -0.295 \end{bmatrix}$$

After $t = 2s$, the point ${}^{B}\boldsymbol{r} = [1 \quad 0 \quad 0]^T$ will be seen at

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} cos 0.6 & 0 & \sin 0.6 \\ 0 & 1 & 0 \\ -\sin 0.6 & 0 & \cos 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.825 \\ 0 \\ -0.564 \end{bmatrix}$$

Taking a time derivative of ${}^{G}\boldsymbol{r}_P = \boldsymbol{Q}_{Y,\beta}\,{}^{B}\boldsymbol{r}_P$ to get the global velocity vector of any point $\boldsymbol{P}$

$$\begin{aligned} {}^{G}\boldsymbol{v}_P &= \dot{\boldsymbol{Q}}_{Y,\beta}\,{}^{B}\boldsymbol{r}_P \\ &= 0.3 \begin{bmatrix} z cos 0.3t - x sin 0.3t \\ 0 \\ -x cos 0.3t - z sin 0.3t \end{bmatrix} \end{aligned}$$

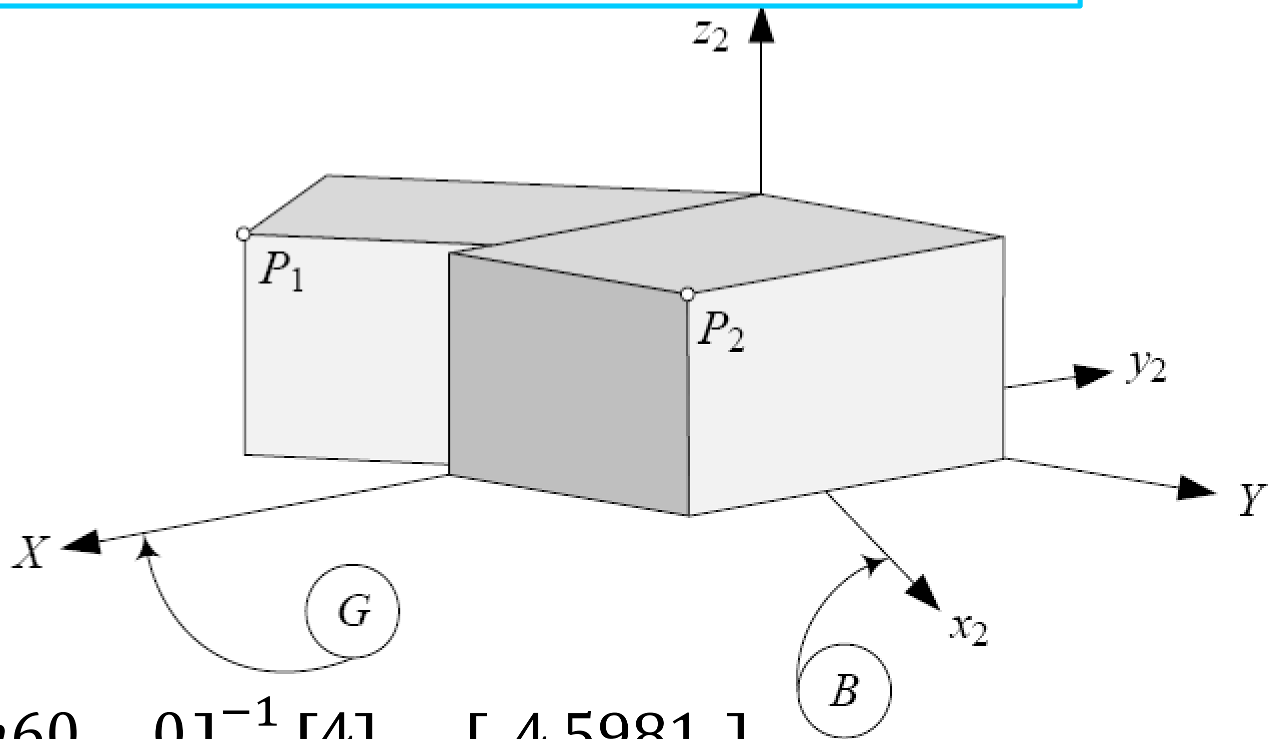
1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- <u>Example 5</u> (*Global rotation, local position*)

A point $\boldsymbol{P}$ moved to ${}^{G}\boldsymbol{r}_2 = [4,3,2]^T$ after rotating 60^0 about Z-axis

$\Longrightarrow$ Find **local position** of $\boldsymbol{P}$ in the local coordinate?

Nhan ma tran nghich dao



$$ {}^{B}\boldsymbol{r}_2 = \boldsymbol{Q}_{Z,60}^{-1}\, {}^{G}\boldsymbol{r}_2 $$

$$ \Longrightarrow \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} cos60 & -sin60 & 0 \\ sin60 & cos60 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.5981 \\ -1.9641 \\ 2.0 \end{bmatrix} $$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- The final global position of a point $\boldsymbol{P}$ (${}^{G}\boldsymbol{r}$) in a rigid body $\boldsymbol{B}$ with position vector ${}^{B}\boldsymbol{r}$, after a sequence of rotations $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3, \ldots, \boldsymbol{Q}_n$ about the global axes can be found by

$$ {}^{G}\boldsymbol{r} = {}^{G}\boldsymbol{Q}_B \, {}^{B}\boldsymbol{r} \tag{2.35} $$

Where the **global rotation matrix**

$$ {}^{G}\boldsymbol{Q}_B = \boldsymbol{Q}_n \ldots \boldsymbol{Q}_3 \boldsymbol{Q}_2 \boldsymbol{Q}_1 \tag{2.36} $$

A rotation matrix is **orthogonal**

$$ \boldsymbol{Q}^T = \boldsymbol{Q}^{-1} \tag{2.37} $$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- <u>Example 6</u> (*Successive global rotation matrix*)

  The **global rotation matrix** after a rotation $\boldsymbol{Q}_{Z,\alpha}$ followed by $\boldsymbol{Q}_{Y,\beta}$ and then $\boldsymbol{Q}_{X,\gamma}$ is

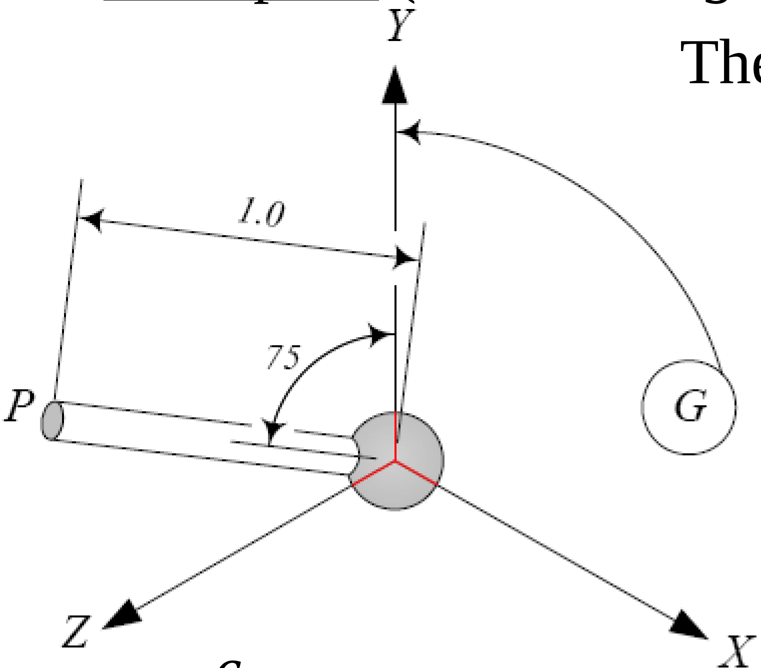
$$
{}^{G}\boldsymbol{Q}_{B} = \boldsymbol{Q}_{X,\gamma}\boldsymbol{Q}_{Y,\beta}\boldsymbol{Q}_{Z,\alpha} \tag{2.38}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & -sin\gamma \\ 0 & sin\gamma & cos\gamma \end{bmatrix} \begin{bmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{bmatrix} \begin{bmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & -c\beta s\alpha & s\beta \\ c\gamma s\alpha + c\alpha s\beta s\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -c\beta s\gamma \\ s\alpha s\gamma - c\alpha c\gamma s\beta & c\alpha s\gamma + c\gamma s\alpha s\beta & c\beta c\gamma \end{bmatrix}
$$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- Example 7 (*Successive global rotations, global position*)



The end point $\boldsymbol{P}$ of the arm is located at

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ l cos\theta \\ l sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 cos75^0 \\ 1 sin75^0 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.26 \\ 0.97 \end{bmatrix} \quad (2.39)$$

The rotation matrix to find the new position of the end point $\boldsymbol{P}$: rotate -29^0 about X-axis, then 30^0 about Z-axis, and 132^0 about X-axis is

$${}^{G}\boldsymbol{Q}_B = \boldsymbol{Q}_{X,132}\boldsymbol{Q}_{Z,30}\boldsymbol{Q}_{X,-29} = \begin{bmatrix} 0.87 & -0.44 & -0.24 \\ -0.33 & -0.15 & -0.93 \\ 0.37 & 0.89 & -0.27 \end{bmatrix} \quad (2.40)$$

and $$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 0.87 & -0.44 & -0.24 \\ -0.33 & -0.15 & -0.93 \\ 0.37 & 0.89 & -0.27 \end{bmatrix} \begin{bmatrix} 0.00 \\ 0.26 \\ 0.97 \end{bmatrix} = \begin{bmatrix} -0.3472 \\ -0.9411 \\ -0.0305 \end{bmatrix} \quad (2.41)$$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- Example 8 (*Twelve independent triple global rotations*)

There are 12 different independent combinations of triple rotations about the global axes which transform a rigid body coordinate frame ***B*** from the coincident position with a global frame ***G*** to any final orientation by only **three rotations about the global axes provided that no two consequence rotations are about the same axis**:

$$\begin{array}{llll}
\boldsymbol{Q}_{X,\gamma}\boldsymbol{Q}_{Y,\beta}\boldsymbol{Q}_{Z,\alpha} & \boldsymbol{Q}_{Z,\gamma}\boldsymbol{Q}_{Y,\beta}\boldsymbol{Q}_{X,\alpha} & \boldsymbol{Q}_{X,\gamma}\boldsymbol{Q}_{Y,\beta}\boldsymbol{Q}_{X,\alpha} & \boldsymbol{Q}_{X,\gamma}\boldsymbol{Q}_{Z,\beta}\boldsymbol{Q}_{X,\alpha} \\
\boldsymbol{Q}_{Y,\gamma}\boldsymbol{Q}_{Z,\beta}\boldsymbol{Q}_{X,\alpha} & \boldsymbol{Q}_{Y,\gamma}\boldsymbol{Q}_{X,\beta}\boldsymbol{Q}_{Z,\alpha} & \boldsymbol{Q}_{Y,\gamma}\boldsymbol{Q}_{Z,\beta}\boldsymbol{Q}_{Y,\alpha} & \boldsymbol{Q}_{Y,\gamma}\boldsymbol{Q}_{X,\beta}\boldsymbol{Q}_{Y,\alpha} \\
\boldsymbol{Q}_{Z,\gamma}\boldsymbol{Q}_{X,\beta}\boldsymbol{Q}_{Y,\alpha} & \boldsymbol{Q}_{X,\gamma}\boldsymbol{Q}_{Z,\beta}\boldsymbol{Q}_{Y,\alpha} & \boldsymbol{Q}_{Z,\gamma}\boldsymbol{Q}_{X,\beta}\boldsymbol{Q}_{Z,\alpha} & \boldsymbol{Q}_{Z,\gamma}\boldsymbol{Q}_{Y,\beta}\boldsymbol{Q}_{Z,\alpha}
\end{array}$$

1. ROTATION ABOUT GLOBAL CARTESIAN AXES

- Example 9 (*Order of rotation, and order of matrix multiplication*)

Changing the order of global rotation matrices is equivalent to changing the order of rotations. The position of a point ***P*** of a rigid body ***B*** is located at ${}^{B}\boldsymbol{r}_P = [1 \quad 2 \quad 3]^T$

Its global position after rotation 30^0 about X-axis and then 45^0 about Y-axis is at

$$\left({}^{G}\boldsymbol{r}_P\right)_1 = \boldsymbol{Q}_{Y,45}\boldsymbol{Q}_{X,30}\,{}^{B}\boldsymbol{r}_P = \begin{bmatrix} 0.53 & -0.84 & 0.13 \\ 0.0 & 0.15 & 0.99 \\ -0.85 & -0.52 & 0.081 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.76 \\ 3.27 \\ -1.64 \end{bmatrix}$$

And if we change the order of rotation then its position would be at:

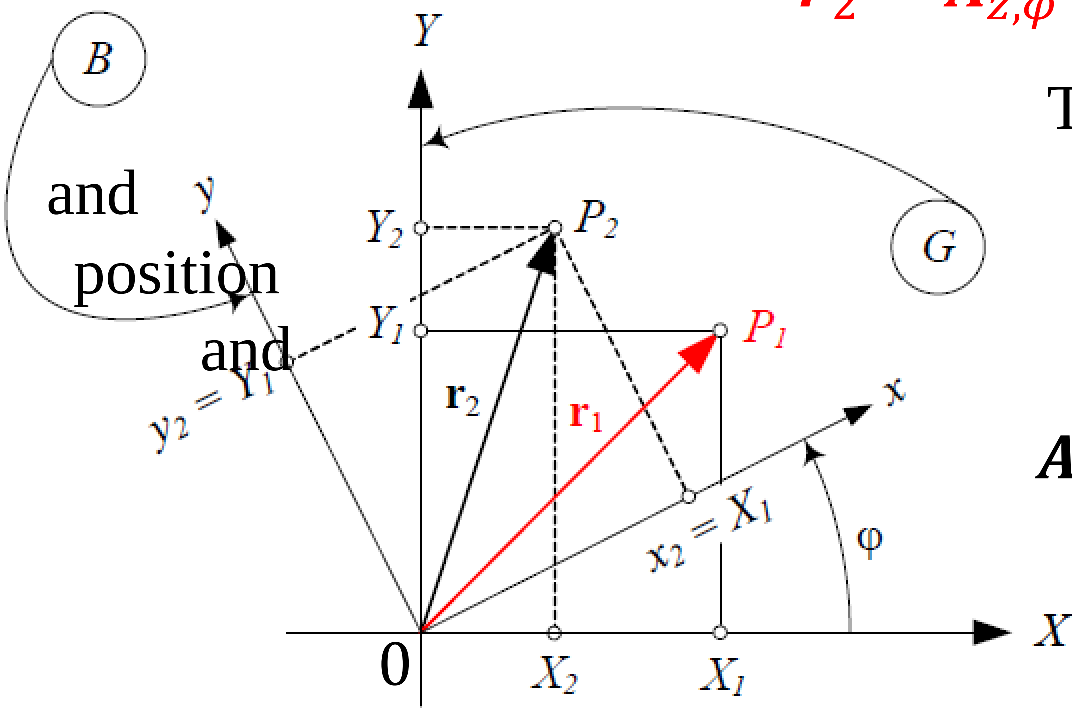
$$\left({}^{G}\boldsymbol{r}_P\right)_2 = \boldsymbol{Q}_{X,30}\boldsymbol{Q}_{Y,45}\,{}^{B}\boldsymbol{r}_P = \begin{bmatrix} 0.53 & 0.0 & 0.85 \\ -0.84 & 0.15 & 0.52 \\ -0.13 & -0.99 & 0.081 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3.08 \\ 1.02 \\ -1.86 \end{bmatrix}$$

These two final positions of ***P*** are $d = \left|\left({}^{G}\boldsymbol{r}_P\right)_1 - \left({}^{G}\boldsymbol{r}_P\right)_2\right| = 4.456$ apart

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- The rigid body ***B*** undergoes a rotation φ about the *z*-axis of its local coordinate frame
- Coordinates of any point ***P*** of the rigid body in local and global coordinates frame are related by the following equation

$$^{B}\boldsymbol{r}_2 = \boldsymbol{A}_{z,\varphi}\, {}^{G}\boldsymbol{r}_2$$



The vector $^{G}\boldsymbol{r}_2 = [X_2 \quad Y_2 \quad Z_2]^T$ and $^{B}\boldsymbol{r}_2 = [x_2 \quad y_2 \quad z_2]^T$ are the position vectors of the point in local and global frames respectively

$\boldsymbol{A}_{z,\varphi}$ is ***z*-rotation matrix**

$$\boldsymbol{A}_{z,\varphi} = \begin{bmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- Similarly, rotation θ about the y-axis and rotation ψ about the x-axis are described by the **y-rotation matrix $\boldsymbol{A}_{y,\theta}$** and the **$x$-rotation matrix $\boldsymbol{A}_{x,\psi}$** respectively

$$\boldsymbol{A}_{y,\theta} = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$$

$$\boldsymbol{A}_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi & sin\psi \\ 0 & -sin\psi & cos\psi \end{bmatrix}$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- Example 12 (*Local rotation*, *local position*)

  If a local coordinate frame $Oxyz$ has been rotated 60^0 about the z-axis and a point $\boldsymbol{P}$ in the global coordinate frame $OXYZ$ is at $[4 \quad 3 \quad 2]$

  Its **coordinates in the local coordinate frame** $Oxyz$ are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cos60 & sin60 & 0 \\ -sin60 & cos60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.60 \\ -1.97 \\ 2.0 \end{bmatrix}$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- Example 13 (*Local rotation*, *global position*)

  If a local coordinate frame $Oxyz$ has been rotated 60^0 about the z-axis and a point $\boldsymbol{P}$ in the local coordinate frame $Oxyz$ is at $[4 \quad 3 \quad 2]$

  Its **position in the global coordinate frame** $OXYZ$ is at

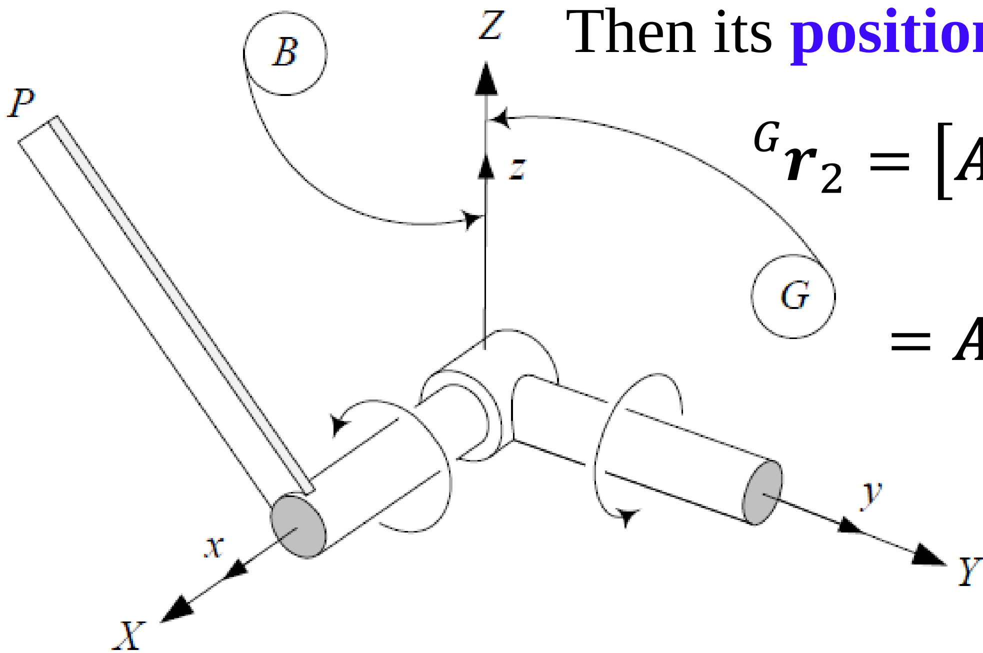
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} cos60 & sin60 & 0 \\ -sin60 & cos60 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.60 \\ 4.96 \\ 2.0 \end{bmatrix}$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- Example 14 (*Successive local rotation, global position*)

The arm shown in Fig 2.11 has two actuators. The first actuator rotates the arm -90^0 about y-axis and then the second actuator rotates the arm 90^0 about x-axis. If the end point $\boldsymbol{P}$ is at

$$^B\boldsymbol{r}_P = [9.5 \quad -10.1 \quad 10.1]^T$$



Then its **position in the global coordinate frame** is at

$$^G\boldsymbol{r}_2 = \left[\boldsymbol{A}_{x,90}\boldsymbol{A}_{y,-90}\right]^{-1}\,{}^B\boldsymbol{r}_P = \boldsymbol{A}_{y,-90}^{-1}\boldsymbol{A}_{x,90}^{-1}\,{}^B\boldsymbol{r}_P$$

$$= \boldsymbol{A}_{y,-90}^{T}\boldsymbol{A}_{x,90}^{T}\,{}^B\boldsymbol{r}_P = \begin{bmatrix} 10.1 \\ -10.1 \\ 9.5 \end{bmatrix}$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- The final global position of a point $\boldsymbol{P}$ in a rigid body $\boldsymbol{B}$ with position vector $\boldsymbol{r}$, after some rotation $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \cdots, \boldsymbol{A}_n$ about the local axes, can be found by

$$ {}^{B}\boldsymbol{r} = {}^{B}\boldsymbol{A}_{G}\, {}^{G}\boldsymbol{r} $$

  where

$$ {}^{B}\boldsymbol{A}_{G} = \boldsymbol{A}_n \cdots \boldsymbol{A}_3 \boldsymbol{A}_2 \boldsymbol{A}_1 $$

- ${}^{B}\boldsymbol{A}_{G}$ is called the **local rotation matrix** and it maps the **global coordinates** to their corresponding **local coordinates**

- Rotation about the local coordinate axis is conceptually interesting because in a sequence of rotations, each rotation is about one of the axes of the local coordinate frame, which has been moved to its new global position during the last rotation

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- <u>Example 15</u> (*Successive local rotation, local position*)

A local coordinate frame $\boldsymbol{B}(Oxyz)$ that initially is coincident with a global coordinate frame $\boldsymbol{G}(OXYZ)$ undergoes a rotation $\varphi = 30^0$ about the z-axis, then $\theta = 30^0$ about the x-axis, and then $\psi = 30^0$ about the y-axis.

The local rotation matrix is

$$^{B}\boldsymbol{A}_G = \boldsymbol{A}_{y,30}\boldsymbol{A}_{x,30}\boldsymbol{A}_{z,30} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix}$$

The global coordinate of $\boldsymbol{P}$ is $X = 5, Y = 30, Z = 10$, so the **coordinate of $\boldsymbol{P}$ in the local frame** is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 18.35 \\ 25.35 \\ 7.0 \end{bmatrix}$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- <u>Example 16</u> (*Successive local rotation*)

  The **rotation matrix** for a body point $\boldsymbol{P}(x, y, z)$ after rotation $\boldsymbol{A}_{z,\varphi}$ followed by $\boldsymbol{A}_{x,\theta}$ and $\boldsymbol{A}_{y,\psi}$ is

$$^{B}\boldsymbol{A}_{G} = \boldsymbol{A}_{y,\psi}\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{z,\varphi}$$

$$= \begin{bmatrix} c\varphi c\psi - s\theta s\varphi s\psi & c\psi s\varphi + c\varphi s\theta s\psi & -c\theta s\psi \\ -c\theta s\varphi & c\theta c\varphi & s\theta \\ c\varphi s\psi + s\theta c\psi s\varphi & s\varphi s\psi - c\varphi s\theta c\psi & c\theta c\psi \end{bmatrix} \qquad (2.97)$$

2. ROTATION ABOUT LOCAL CARTESIAN AXES

- <u>Example 17</u> (*Twelve independent triple local rotations*)

  Any 2 independent orthogonal coordinate frames with a common origin can be related by a sequence of three rotations about the local coordinate axes, where no two successive rotations may be about the same axis. In general, there are **12 different independent combinations** of triple rotation about local axes

$1-\boldsymbol{A}_{x,\psi}\boldsymbol{A}_{y,\theta}\boldsymbol{A}_{z,\varphi}$

$2-\boldsymbol{A}_{y,\psi}\boldsymbol{A}_{z,\theta}\boldsymbol{A}_{x,\varphi}$

$3-\boldsymbol{A}_{z,\psi}\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{y,\varphi}$

$4-\boldsymbol{A}_{z,\psi}\boldsymbol{A}_{y,\theta}\boldsymbol{A}_{x,\varphi}$

$5-\boldsymbol{A}_{y,\psi}\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{z,\varphi}$

$6-\boldsymbol{A}_{x,\psi}\boldsymbol{A}_{z,\theta}\boldsymbol{A}_{y,\varphi}$

$7-\boldsymbol{A}_{x,\psi}\boldsymbol{A}_{y,\theta}\boldsymbol{A}_{x,\varphi}$

$8-\boldsymbol{A}_{y,\psi}\boldsymbol{A}_{z,\theta}\boldsymbol{A}_{y,\varphi}$

$9-\boldsymbol{A}_{z,\psi}\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{z,\varphi}$

$10-\boldsymbol{A}_{x,\psi}\boldsymbol{A}_{z,\theta}\boldsymbol{A}_{x,\varphi}$

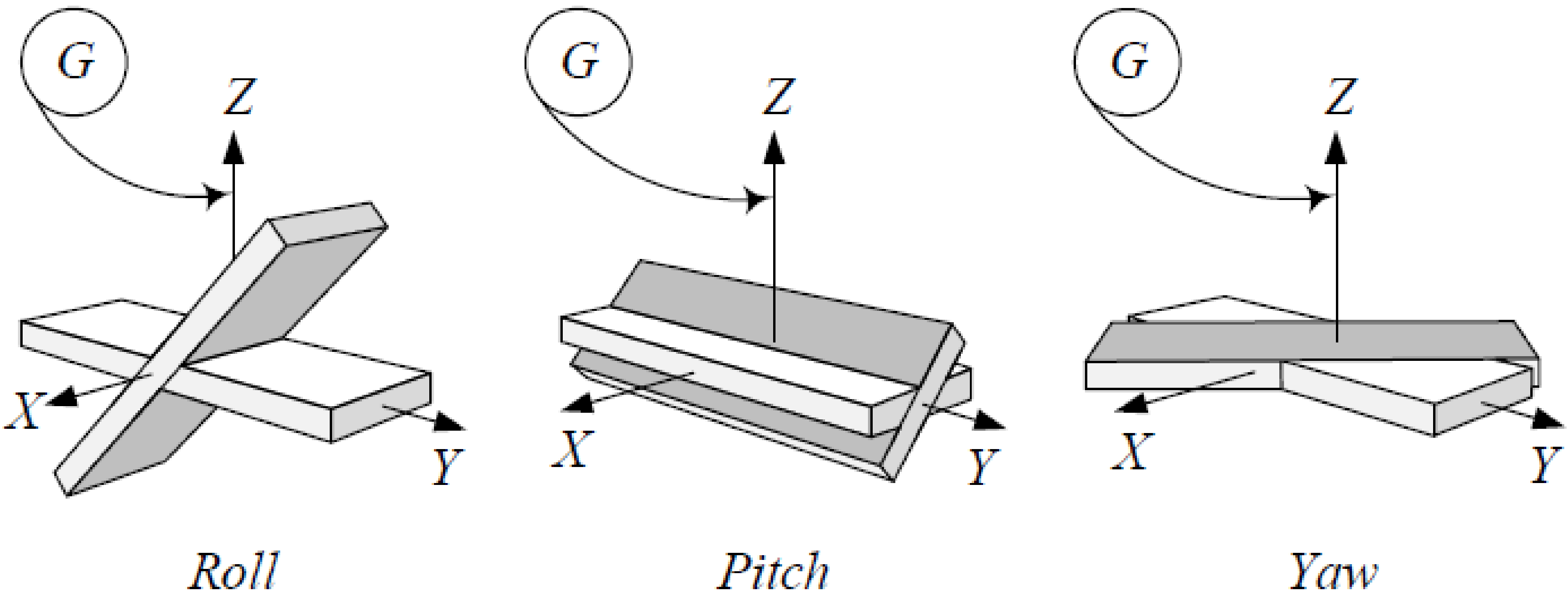
$11-\boldsymbol{A}_{y,\psi}\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{y,\varphi}$

$12-\boldsymbol{A}_{z,\psi}\boldsymbol{A}_{y,\theta}\boldsymbol{A}_{z,\varphi}$

3. ROLL-PITCH-YAW ANGLES

- The rotation about the X-axis of the global coordinate frame is called a ***roll***, the rotation about the Y-axis of the global coordinate frame is called a ***pitch***, and the rotation about the Z-axis of the global coordinate frame is called a ***yaw***

- This figure illustrates 45^0 roll, pitch, and yaw rotation about the axes of a global coordinate frame



3. ROLL-PITCH-YAW ANGLES

- Given the roll, pitch, and yaw angles, we can compute the **overall rotation matrix**

- The global ***roll-pitch-yaw rotation matrix*** is

$$
{}^{G}\boldsymbol{Q}_{B} = \boldsymbol{Q}_{Z,\gamma}\boldsymbol{Q}_{Y,\beta}\boldsymbol{Q}_{X,\alpha}
$$

$$
= \begin{bmatrix} c\beta c\gamma & -c\alpha s\gamma + c\gamma s\alpha s\beta & s\alpha s\gamma + c\alpha c\gamma s\beta \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -c\gamma s\alpha + c\alpha s\beta s\gamma \\ -s\beta & c\beta s\alpha & c\alpha c\beta \end{bmatrix} \tag{2.55}
$$

3. ROLL-PITCH-YAW ANGLES

- Also we are able to compute the equivalent roll, pitch, and yaw angles when a rotation matrix is given. Suppose that r_{ij} indicates the element of row i and column j of the roll-pitch-yaw rotation matrix (2.55)
- Then the **roll angle** is

neu ra 2 nghiem thi tuy tinh hinh thuc te se loai cac phuong an

$$\alpha = tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

and the **pitch angle** is

$$\beta = -sin^{-1}(r_{31})$$

and the **yaw angle** is

$$\gamma = tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

3. ROLL-PITCH-YAW ANGLES

- Example 11 (*Determination of roll-pitch-yaw angles*)

Determine the required **roll-pitch-yaw angles** to make the x-axis of the body coordinate $\boldsymbol{B}$ parallel to $\boldsymbol{u} = \hat{\boldsymbol{I}} + 2\hat{\boldsymbol{J}} + 3\hat{\boldsymbol{K}}$, while y-axis remains in (X, Y)-plane?

We have

cau nay chua hieu

dau tien tim duoc vector don vi
sau do ap dung cong thuc

$${}^{G}\hat{\boldsymbol{\imath}} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{1}{\sqrt{14}}\hat{\boldsymbol{I}} + \frac{2}{\sqrt{14}}\hat{\boldsymbol{J}} + \frac{3}{\sqrt{14}}\hat{\boldsymbol{K}}$$

$${}^{G}\hat{\boldsymbol{\jmath}} = (\hat{\boldsymbol{I}} \cdot \hat{\boldsymbol{\jmath}})\hat{\boldsymbol{I}} + (\hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\jmath}})\hat{\boldsymbol{J}} = cos\theta\hat{\boldsymbol{I}} + sin\theta\hat{\boldsymbol{J}}$$

The axes ${}^{G}\hat{\boldsymbol{\imath}}$ and ${}^{G}\hat{\boldsymbol{\jmath}}$ must be orthogonal, therefore

$$\begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \cdot \begin{bmatrix} cos\theta \\ sin\theta \\ 0 \end{bmatrix} = 0 \Rightarrow \theta = -26.56^{0}$$

3. ROLL-PITCH-YAW ANGLES

Find ${}^G\hat{\boldsymbol{k}}$ by a cross product: ${}^G\hat{\boldsymbol{k}} = {}^G\hat{\boldsymbol{\imath}} \times {}^G\hat{\boldsymbol{\jmath}} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \times \begin{bmatrix} 0.894 \\ -0.447 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.358 \\ 0.717 \\ -0.597 \end{bmatrix}$

Hence, the transformation matrix ${}^G\boldsymbol{Q}_B$ is

$$
{}^G\boldsymbol{Q}_B = \begin{bmatrix} \hat{\boldsymbol{I}} \cdot \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{I}} \cdot \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{I}} \cdot \hat{\boldsymbol{k}} \\ \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{k}} \\ \hat{\boldsymbol{K}} \cdot \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{K}} \cdot \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{K}} \cdot \hat{\boldsymbol{k}} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} & 0.894 & 0.358 \\ 2/\sqrt{14} & -0.447 & 0.717 \\ 3/\sqrt{14} & 0 & -0.597 \end{bmatrix}
$$

Now it is possible to determine the required roll-pitch-yaw angles

$$
\alpha = tan^{-1}\left(\frac{r_{32}}{r_{33}}\right) = tan^{-1}\left(\frac{0}{-0.597}\right) = 0
$$

$$
\beta = -sin^{-1}(r_{31}) = -sin^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx -0.93 \text{ rad}
$$

$$
\gamma = tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) = tan^{-1}\left(\frac{2/\sqrt{14}}{1/\sqrt{14}}\right) \approx 1.1071 \text{ rad}
$$

4. EULER ANGLES

- The rotation about the Z-axis of the global coordinate is called ***precession***, the rotation about the x-axis of the local coordinate is called ***nutation***, and the rotation about the z-axis of the local coordinate is called ***spin***
- The precession-nutation-spin rotation angles are also called ***Euler angles***
- ***Euler angles rotation matrix*** has many application in rigid body kinematics
- To find Euler angles rotation matrix to go from the global frame $G(OXYZ)$ to the final body frame $B(Oxyz)$, we employ a body frame $B'(Ox'y'z')$ that shown in Fig 2.12 that before the first rotation coincides with the global frame

4. EULER ANGLES

- Let there be at first a rotation φ about the z'-axis. Because Z-*axis* and z'-*axis* are coincident, we have

$$^{B'}\boldsymbol{r} = {}^{B'}\boldsymbol{A}_G\,{}^{G}\boldsymbol{r}$$

$$^{B'}\boldsymbol{A}_G = \boldsymbol{A}_{z,\varphi} = \begin{bmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

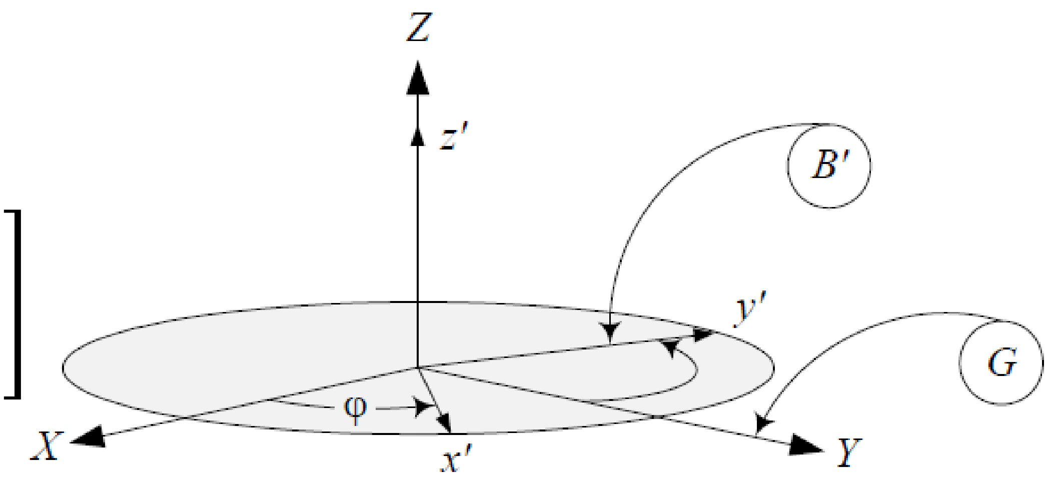


FIGURE 2.12. First Euler angle.

- Next we consider the $\boldsymbol{B}'(Ox'y'z')$ frame as a new fixed global frame and introduce a new body frame $\boldsymbol{B}''(Ox''y''z'')$. Before the second rotation, the two frame coincide

4. EULER ANGLES

- Then, we execute a θ rotation about x''-axis as shown in Fig 2.13. The transformation between $\boldsymbol{B}'(Ox'y'z')$ and $\boldsymbol{B}''(Ox''y''z'')$ is

$$^{B''}\boldsymbol{r} = {}^{B''}\boldsymbol{A}_{B'}\,{}^{B'}\boldsymbol{r}$$

$$^{B''}\boldsymbol{A}_{B'} = \boldsymbol{A}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix}$$

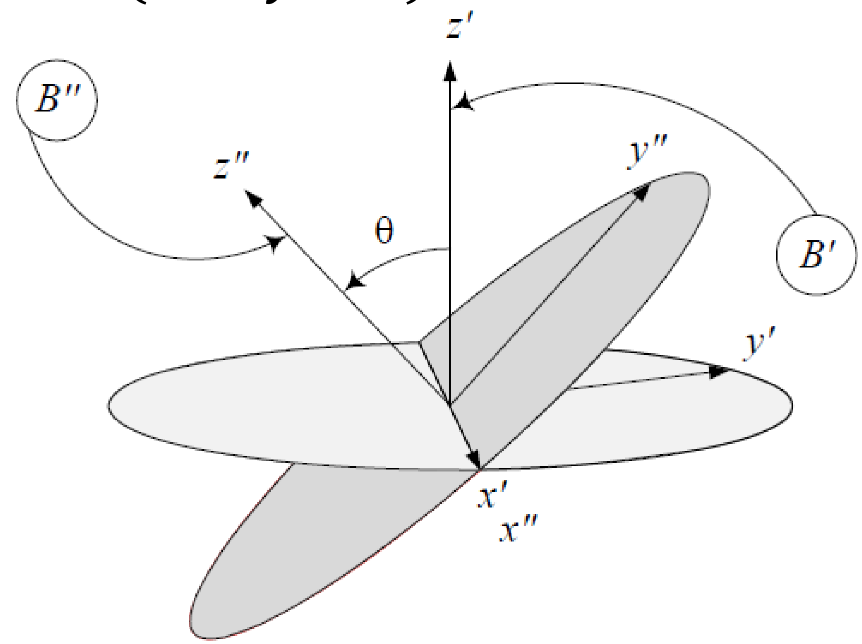


FIGURE 2.13. Second Euler angle.

- Finally, we consider the $\boldsymbol{B}''(Ox''y''z'')$ frame as a new fixed global frame and consider the final body frame $\boldsymbol{B}(Oxyz)$ to coincide with $\boldsymbol{B}''$ before the third rotation

4. EULER ANGLES

- We now execute a ψ rotation about the z''-*axis* as shown in Fig 2.14. The transformation between $\boldsymbol{B}''(Ox''y''z'')$ and $\boldsymbol{B}(Oxyz)$ is

$$^{B}\boldsymbol{r} = {}^{B}\boldsymbol{A}_{B''}\,{}^{B''}\boldsymbol{r}$$

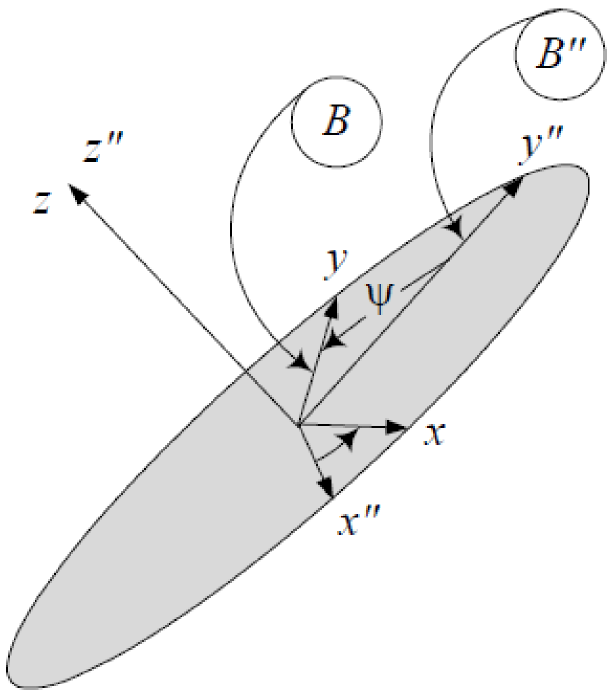


FIGURE 2.14. Third Euler angle.

$$^{B}\boldsymbol{A}_{B''} = \boldsymbol{A}_{z,\psi} = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By the rule of composition of rotation, the transformation from $\boldsymbol{G}(OXYZ)$ to $\boldsymbol{B}(Oxyz)$ is

$$^{B}\boldsymbol{r} = {}^{B}\boldsymbol{A}_{G}\,{}^{G}\boldsymbol{r}$$

$$^{B}\boldsymbol{A}_{G} = \boldsymbol{A}_{z,\psi}\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{z,\varphi}$$

$$= \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & c\psi s\varphi + c\theta c\varphi s\psi & s\theta s\psi \\ -c\varphi s\psi - c\theta c\psi s\varphi & -s\varphi s\psi + c\theta c\varphi c\psi & s\theta c\psi \\ s\theta s\varphi & -c\varphi s\theta & c\theta \end{bmatrix}$$

4. EULER ANGLES

- Example 18 (*Euler angle rotation matrix*)

  The **Euler** or **precession-nutation-spin rotation matrix** for $\varphi = 79.15^0$, $\theta = 41.41^0$, and $\psi = -40.7^0$ would be found by substituting φ, θ and ψ in equation (2.106)

$$^{B}\boldsymbol{A}_{G} = \boldsymbol{A}_{z,-40.7}\boldsymbol{A}_{x,41.41}\boldsymbol{A}_{z,79.15}$$

$$= \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix}$$

4. EULER ANGLES

- Example 19 (*Euler angles of a local rotation matrix*)

  The local rotation matrix after rotation 30^0 about the z-axis, then rotation 30^0 about the x-axis, and then 30^0 about the y-axis is

$$^{B}\boldsymbol{A}_G = \boldsymbol{A}_{y,30}\boldsymbol{A}_{x,30}\boldsymbol{A}_{z,30}$$

$$= \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix}$$

  The **Euler angles** of the corresponding rotation matrix are

$$\theta = cos^{-1}(r_{33}) = cos^{-1}(0.75) = 41.41^0$$

$$\varphi = -tan^{-1}\left(\frac{r_{31}}{r_{32}}\right) = -tan^{-1}\left(\frac{0.65}{-0.125}\right) = 79.15^0$$

$$\psi = tan^{-1}\left(\frac{r_{13}}{r_{23}}\right) = tan^{-1}\left(\frac{-0.43}{0.50}\right) = -40.7^0$$

4. EULER ANGLES

- Example 20 (*Relative rotation matrix of two bodies*)

Consider a rigid body $\boldsymbol{B}_1$ with an orientation matrix ${}^{B_1}\boldsymbol{A}_G$ made by Euler angles $\varphi = 30^0$, $\theta = -45^0$, $\psi = 60^0$, and another rigid body $\boldsymbol{B}_2$ having $\varphi = 10^0$, $\theta = 25^0$, $\psi = -15^0$, the individual rotation matrices are

$${}^{B_1}\boldsymbol{A}_G = \boldsymbol{A}_{z,60}\boldsymbol{A}_{x,-45}\boldsymbol{A}_{z,30} \qquad {}^{B_2}\boldsymbol{A}_G = \boldsymbol{A}_{z,-15}\boldsymbol{A}_{x,25}\boldsymbol{A}_{z,10}$$

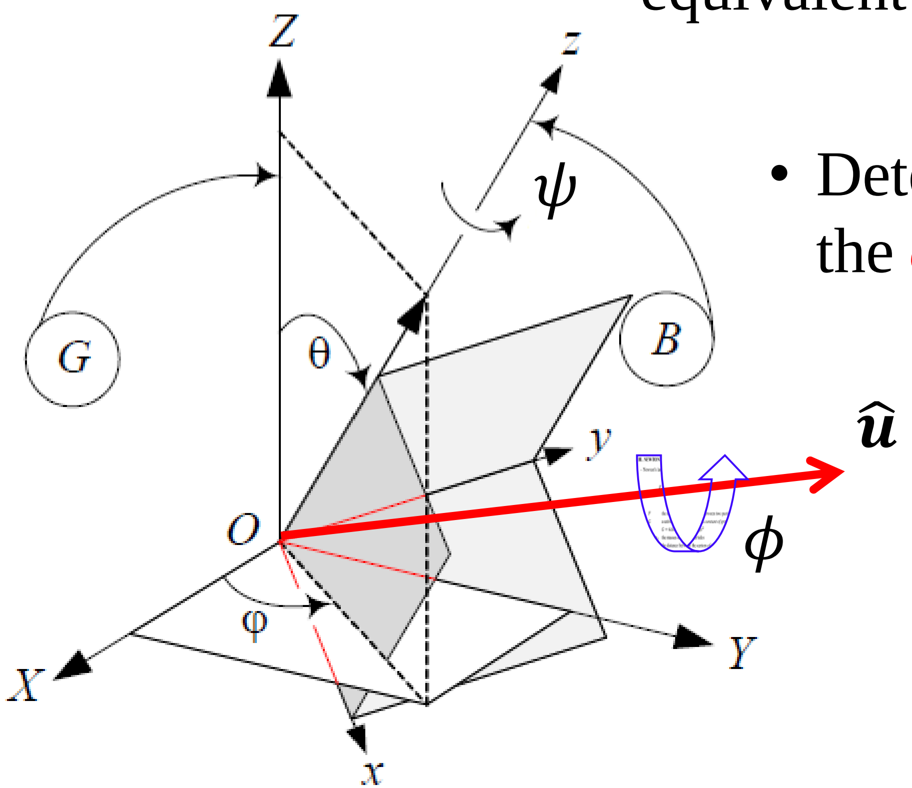
$$= \begin{bmatrix} 0.127 & 0.78 & -0.612 \\ -0.927 & -0.127 & -0.354 \\ -0.354 & 0.612 & 0.707 \end{bmatrix} \qquad = \begin{bmatrix} 0.992 & -0.0633 & -0.109 \\ 0.103 & 0.907 & 0.408 \\ 0.0734 & -0.416 & 0.906 \end{bmatrix}$$

The **relative rotation matrix** to map $\boldsymbol{B}_2$ to $\boldsymbol{B}_1$ may be found by

$${}^{B_1}\boldsymbol{A}_{B_2} = {}^{B_1}\boldsymbol{A}_G\,{}^{G}\boldsymbol{A}_{B_2} = \begin{bmatrix} 0.992 & 0.103 & 0.0734 \\ -0.0633 & 0.907 & -0.416 \\ -0.109 & 0.408 & 0.906 \end{bmatrix}$$

5. AXIS-ANGLE ROTATION

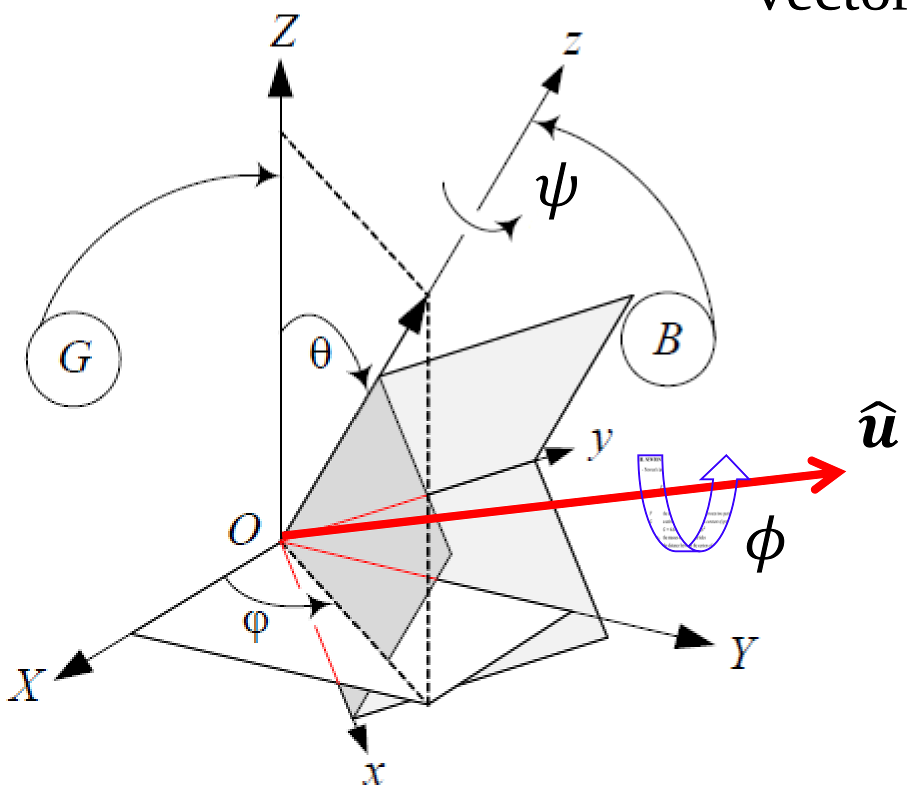
- We can **decompose** any rotation ϕ of a rigid body with a fixed point $\boldsymbol{O}$, about a globally fixed axis $\hat{\boldsymbol{u}}$ into **three rotations** about three given non coplanar axes
- The **final orientation** of a rigid body after a finite number of rotations is equivalent to a **unique rotation** about a **unique axis**



- Determination of the **angle** and **axis** is called the ***orientation kinematics*** of rigid bodies

5. AXIS-ANGLE ROTATION

- **Two parameters** are necessary to define the **direction** of a line through $\boldsymbol{O}$ and **one** is necessary to define the **amount of rotation** of a rigid body about this line
- Let the body frame $\boldsymbol{B}(Oxyz)$ rotate ϕ about a line indicated by a unit vector $\hat{\boldsymbol{u}}$ with direction cosines u_1, u_2, u_3



$$\hat{\boldsymbol{u}} = u_1\hat{\boldsymbol{I}} + u_2\hat{\boldsymbol{J}} + u_3\hat{\boldsymbol{K}}$$

$$\sqrt{u_1^2 + u_2^2 + u_3^2} = 1$$

This is called **axis-angle representation** of a rotation

5. AXIS-ANGLE ROTATION

- A transformation matrix ${}^{G}\boldsymbol{R}_{B}$ that maps the coordinates in the local frame $\boldsymbol{B}(Oxyz)$ to the corresponding coordinates in the global frame $\boldsymbol{G}(OXYZ)$

$$ {}^{G}\boldsymbol{r} = {}^{G}\boldsymbol{R}_{B}\,{}^{B}\boldsymbol{r} $$

$$ {}^{G}\boldsymbol{R}_{B} = \boldsymbol{R}_{\hat{u},\phi} = \mathbf{I}\cos\phi + \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^{T}\,vers\,\phi + \check{\boldsymbol{u}}\sin\phi $$

$$ {}^{G}\boldsymbol{R}_{B} = \begin{bmatrix} u_1^2 vers\phi + c\phi & u_1u_2 vers\phi - u_3 s\phi & u_1u_3 vers\phi + u_2 s\phi \\ u_1u_2 vers\phi + u_3 s\phi & u_2^2 vers\phi + c\phi & u_2u_3 vers\phi - u_1 s\phi \\ u_1u_3 vers\phi - u_2 s\phi & u_2u_3 vers\phi + u_1 s\phi & u_3^2 vers\phi + c\phi \end{bmatrix} \tag{3.5} $$

where

$$ vers\,\phi = versine\,\phi = 1 - \cos\phi = 2sin^2\frac{\phi}{2} $$

and $\check{\boldsymbol{u}}$ is the **skew-symmetric matrix** corresponding to the vector $\hat{\boldsymbol{u}}$

$$ \check{\boldsymbol{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} $$

5. AXIS-ANGLE ROTATION

- A matrix $\breve{\boldsymbol{u}}$ is skew-symmetric if

$$\breve{\boldsymbol{u}}^T = -\breve{\boldsymbol{u}}$$

- Given a transformation matrix ${}^G\boldsymbol{R}_B$ we can obtain the axis $\hat{\boldsymbol{u}}$ and angle ϕ of the rotation by

$$\breve{\boldsymbol{u}} = \frac{1}{2\sin\phi}\left({}^G\boldsymbol{R}_B - {}^G\boldsymbol{R}_B^T\right)$$

$$\cos\phi = \frac{1}{2}\left(tr\left({}^G\boldsymbol{R}_B\right) - 1\right)$$

Where

$$tr\left({}^G\boldsymbol{R}_B\right) = r_{11} + r_{22} + r_{33}$$

5. AXIS-ANGLE ROTATION

- <u>Example 40</u> (*Axis-angle rotation when* $\hat{\boldsymbol{u}} = \widehat{\boldsymbol{K}}$)

  If the local frame $\boldsymbol{B}(Oxyz)$ rotates about the Z-axis, then

$$\hat{\boldsymbol{u}} = \widehat{\boldsymbol{K}}$$

  And the transformation matrix (3.5) reduces to

$$^{G}\boldsymbol{R}_{B} = \begin{bmatrix} 0vers\phi + c\phi & 0vers\phi - 1s\phi & 0vers\phi + 0s\phi \\ 0vers\phi + 1s\phi & 0vers\phi + c\phi & 0vers\phi - 0s\phi \\ 0vers\phi - 0s\phi & 0vers\phi + 0s\phi & 1vers\phi + c\phi \end{bmatrix}$$

$$= \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  Which is equivalent to the rotation matrix about the Z-axis of global frame in (2.20)

5. AXIS-ANGLE ROTATION

- Example 41 (*Rotation about a rotated local axis*)

  If the body coordinate frame $Oxyz$ rotate φ about the global Z-axis, then the x-axis would be along $\hat{\boldsymbol{u}}_x$

$$\hat{\boldsymbol{u}}_x = {}^G\boldsymbol{R}_{Z,\varphi}\hat{\boldsymbol{\imath}} = \begin{bmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} cos\varphi \\ sin\varphi \\ 0 \end{bmatrix}$$

  Rotation θ about $\hat{\boldsymbol{u}}_x = (cos\varphi)\hat{\boldsymbol{I}} + (sin\varphi)\hat{\boldsymbol{J}}$ is defined by Rodriguez's formula (3.5)

$${}^G\boldsymbol{R}_{\hat{u}_x,\theta} = \begin{bmatrix} cos^2\varphi\ vers\theta + c\theta & c\varphi\ s\varphi\ vers\theta & s\varphi s\theta \\ c\varphi\ s\varphi\ vers\theta & sin^2\varphi\ vers\theta + c\theta & -c\varphi s\theta \\ -s\varphi s\theta & c\varphi s\theta & c\theta \end{bmatrix}$$

5. AXIS-ANGLE ROTATION

Now, rotation φ about the global Z-axis followed by rotation θ about the local x-axis is transformed by

$$
{}^{G}\boldsymbol{R}_{B} = {}^{G}\boldsymbol{R}_{\hat{u}_x,\theta}\, {}^{G}\boldsymbol{R}_{Z,\varphi}
$$

$$
= \begin{bmatrix} cos\varphi & -cos\theta sin\varphi & sin\theta sin\varphi \\ sin\varphi & cos\theta cos\varphi & -cos\varphi sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}
$$

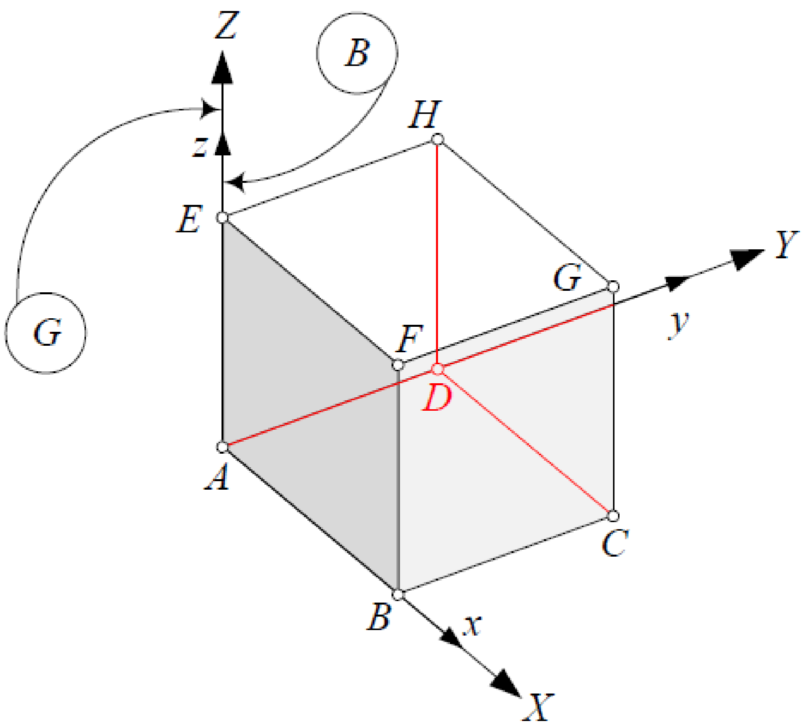
That must be equal to

$$
\left[\boldsymbol{A}_{x,\theta}\boldsymbol{A}_{Z,\varphi}\right]^{-1} = \boldsymbol{A}_{Z,\varphi}^{T}\boldsymbol{A}_{x,\theta}^{T}
$$

5. AXIS-ANGLE ROTATION

- Example 42 (*Axis and angle of rotation*)

  Consider a cubic rigid body with a fixed point at $\boldsymbol{A}$ and a unit length of edges as is shown in Fig 3.2. If we turn the cube 45^0 about $\boldsymbol{u} = [1 \quad 1 \quad 1]^T$ then we can find the global coordinates of its corner using Rodriguez transformation matrix as follows



$$\phi = \frac{\pi}{4} \qquad \hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\sqrt{3}} = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}$$

$$\boldsymbol{R}_{\hat{u},\phi} = \boldsymbol{I} cos\phi + \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^T vers\,\phi + \breve{\boldsymbol{u}} \sin\phi$$

$$= \begin{bmatrix} 0.80474 & -0.31062 & 0.50588 \\ 0.50588 & 0.80474 & -0.31062 \\ -0.31062 & 0.50588 & 0.80474 \end{bmatrix}$$

5. AXIS-ANGLE ROTATION

The local coordinates of the corners are

	${}^{B}\boldsymbol{r}_{B}$	${}^{B}\boldsymbol{r}_{C}$	${}^{B}\boldsymbol{r}_{D}$	${}^{B}\boldsymbol{r}_{E}$	${}^{B}\boldsymbol{r}_{F}$	${}^{B}\boldsymbol{r}_{G}$	${}^{B}\boldsymbol{r}_{H}$
x	1	1	0	0	1	1	0
y	0	1	1	0	0	1	1
z	0	0	0	1	1	1	1

The global coordinates of the corners after the rotation are

	${}^{G}\boldsymbol{r}_{B}$	${}^{G}\boldsymbol{r}_{C}$	${}^{G}\boldsymbol{r}_{D}$	${}^{G}\boldsymbol{r}_{E}$	${}^{G}\boldsymbol{r}_{F}$	${}^{G}\boldsymbol{r}_{G}$	${}^{G}\boldsymbol{r}_{H}$
X	0.804	0.495	-0.31	0.505	1.310	1	0.196
Y	0.505	1.31	0.804	-0.31	0.196	1	0.495
Z	-0.31	0.196	0.505	0.804	0.495	1	1.31

The mid point of the cube by $\boldsymbol{P}$

$${}^{G}\boldsymbol{r}_{P} = \frac{1}{2}\left({}^{G}\boldsymbol{r}_{B} + {}^{G}\boldsymbol{r}_{H}\right) = \frac{1}{2}\left({}^{G}\boldsymbol{r}_{F} + {}^{G}\boldsymbol{r}_{D}\right) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

5. AXIS-ANGLE ROTATION

- <u>Example 43</u> (*Axis and angle of a rotation matrix*)

  A body coordinate frame, $\boldsymbol{B}$, undergoes three Euler rotation $(\varphi, \theta, \psi) = (30^0, 45^0, 60^0)$ with respect to a global frame $\boldsymbol{G}$. The rotation matrix to transform coordinates of $\boldsymbol{B}$ to $\boldsymbol{G}$ is

$$ {}^{G}\boldsymbol{R}_{B} = {}^{B}\boldsymbol{R}_{G}^{T} = \left[\boldsymbol{R}_{z,\psi}\boldsymbol{R}_{x,\theta}\boldsymbol{R}_{z,\varphi}\right]^{T} = \boldsymbol{R}_{z,\varphi}^{T}\boldsymbol{R}_{x,\theta}^{T}\boldsymbol{R}_{z,\psi}^{T} $$

$$ = \begin{bmatrix} 0.12683 & -0.92678 & 0.35355 \\ 0.78033 & -0.12683 & -0.61237 \\ 0.61237 & 0.35355 & 0.70711 \end{bmatrix} $$

  The unique **angle-axis of rotation** for this rotation matrix are

$$ \phi = cos^{-1}\left(\frac{1}{2}\left(tr\left({}^{G}\boldsymbol{R}_{B}\right) - 1\right)\right) $$

$$ = cos^{-1}\,(-0.14645) = 98^{0} $$

5. AXIS-ANGLE ROTATION

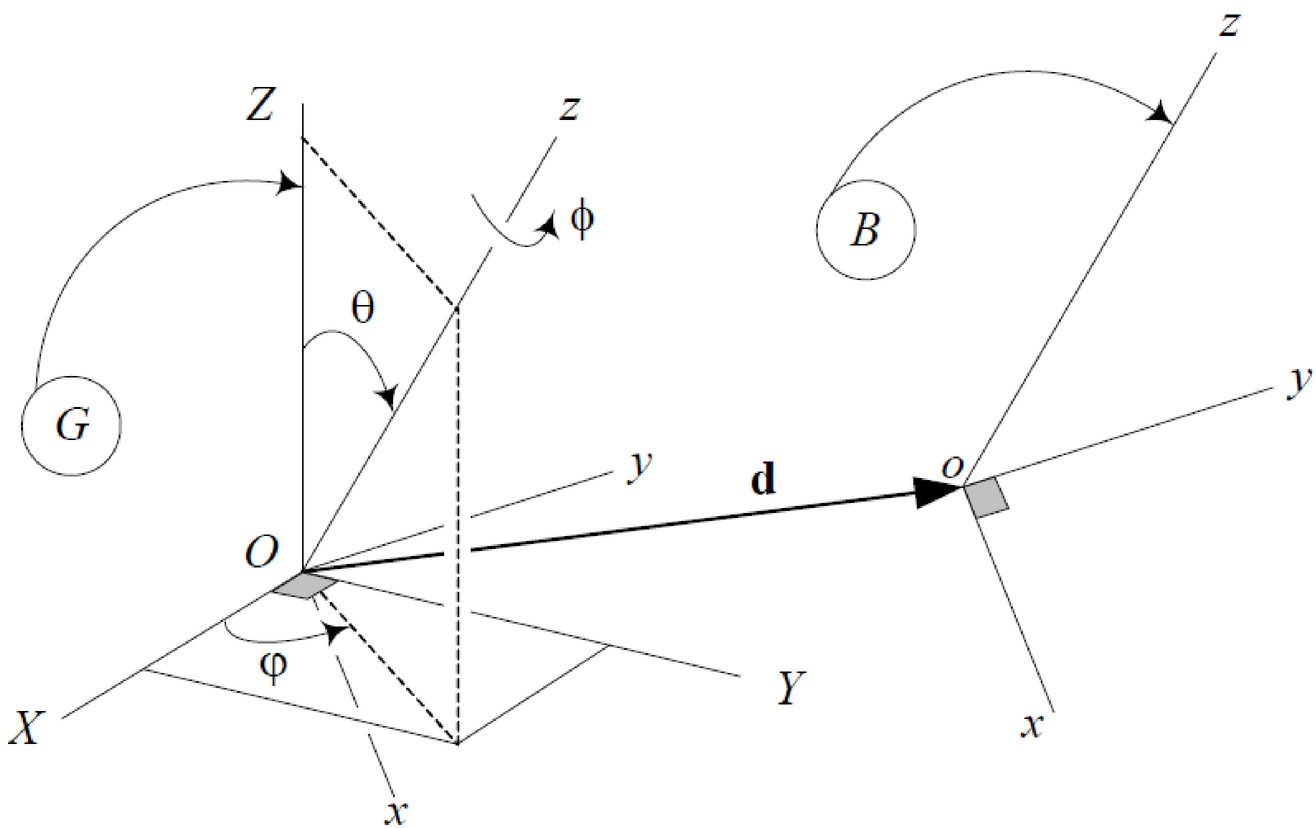
$$\breve{\boldsymbol{u}} = \frac{1}{2sin\phi}\left({}^{G}\boldsymbol{R}_{B} - {}^{G}\boldsymbol{R}_{B}^{T}\right) = \begin{bmatrix} 0.0 & -0.86285 & -0.13082 \\ 0.86285 & 0.0 & -0.48822 \\ 0.13082 & 0.48822 & 0.0 \end{bmatrix}$$

$$\hat{\boldsymbol{u}} = \begin{bmatrix} 0.48822 \\ -0.13082 \\ 0.86285 \end{bmatrix}$$

As a double check, we may verify the angle-axis rotation formula and derive the same rotation matrix.

6. RIGID BODY MOTION

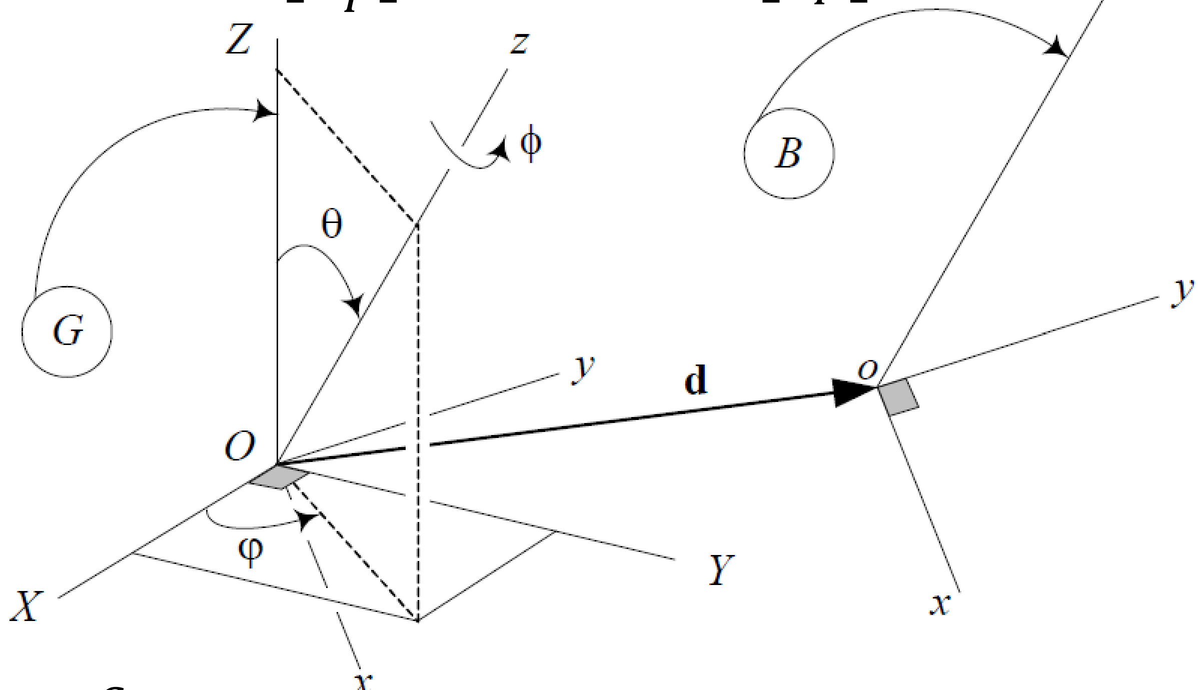
- A **rotation** ϕ about an axis $\hat{\boldsymbol{u}}$ and a **displacement** $\boldsymbol{d}$ is the general motion of a rigid body $\boldsymbol{B}$ in a global frame $\boldsymbol{G}$
- The **rigid body motion** can be defined by a **4 × 4 matrix**



6. RIGID BODY MOTION

$$^{G}\boldsymbol{r}_P = {}^{G}\boldsymbol{R}_B\,{}^{B}\boldsymbol{r}_P + {}^{G}\boldsymbol{d} \tag{4.1}$$

$$^{G}\boldsymbol{r}_P = \begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} \qquad {}^{B}\boldsymbol{r}_P = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} \qquad {}^{G}\boldsymbol{d} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix}$$



$^{G}\boldsymbol{d}$: **displacement** / **translation** of $\boldsymbol{B}$ with respect to $\boldsymbol{G}$

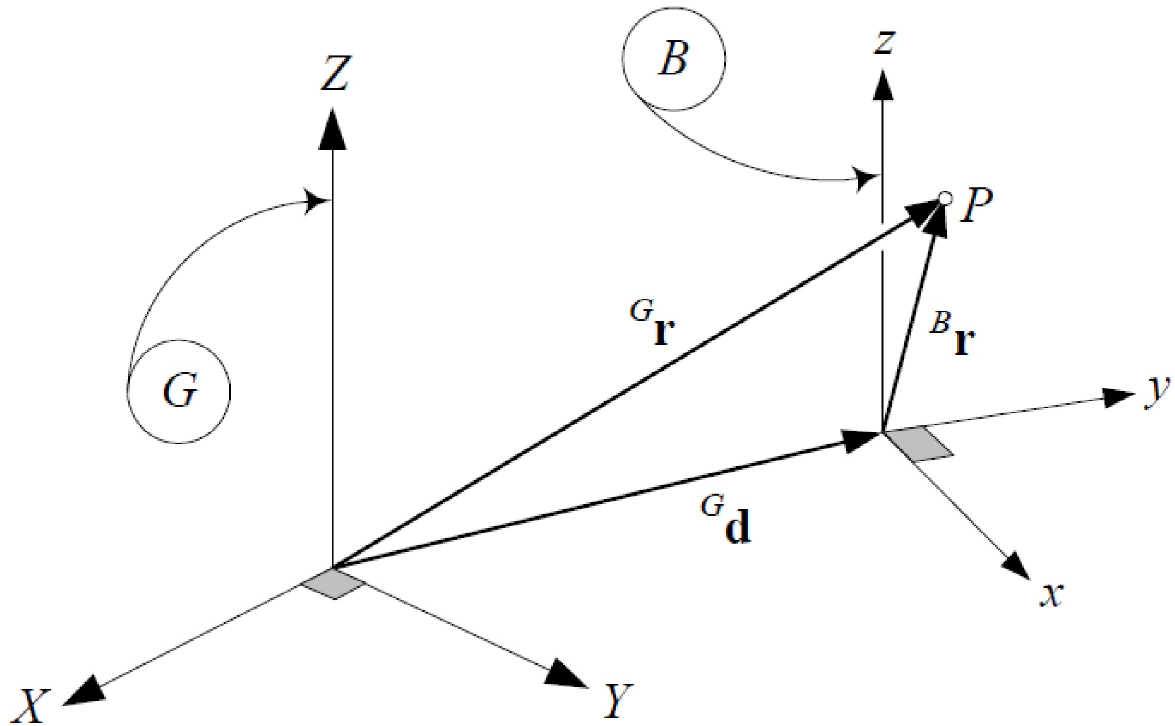
$^{G}\boldsymbol{R}_B$: **rotation matrix** to map $^{B}\boldsymbol{r}$ to $^{G}\boldsymbol{r}$ when $^{G}\boldsymbol{d} = 0$

6. RIGID BODY MOTION

- Example 71 (*Translation and rotation of a body coordinate frame*)

A body coordinate frame $\boldsymbol{B}(oxyz)$, that is originally coincident with global coordinate frame $\boldsymbol{G}(OXYZ)$, rotates 45^0 about the X-axis and translates to $[3\ 5\ 7]^T$

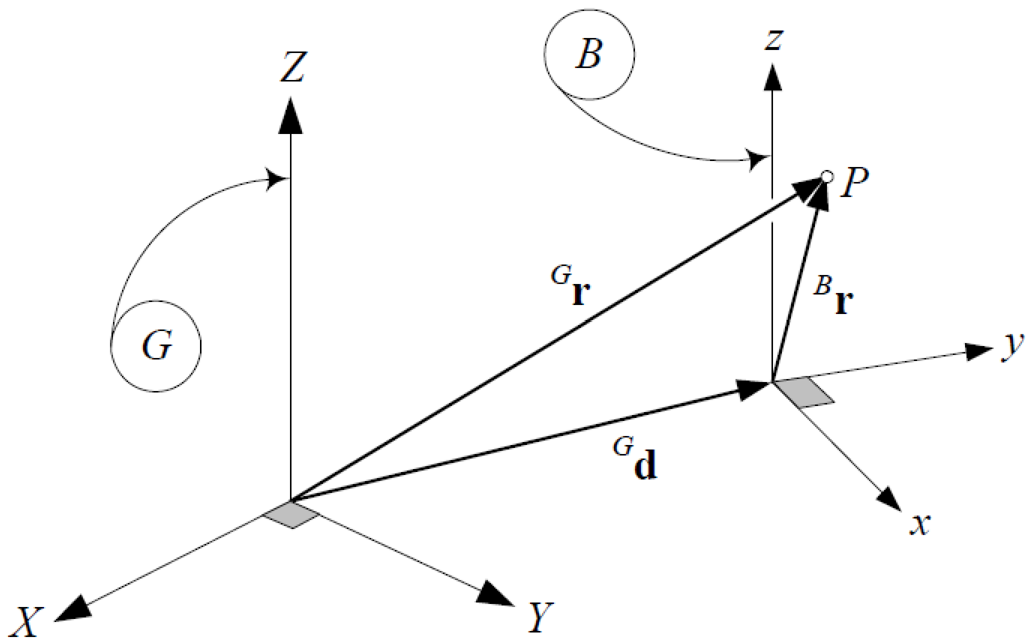
$\Longrightarrow$ Find the **global position** of a point at ${}^{B}\boldsymbol{r} = [x\ y\ z]^T$?



6. RIGID BODY MOTION

The **global position** of a point at ${}^{B}\boldsymbol{r} = [x\ y\ z]^T$

$$
\begin{aligned}
{}^{G}\boldsymbol{r}_P &= {}^{G}\boldsymbol{R}_B\,{}^{B}\boldsymbol{r}_P + {}^{G}\boldsymbol{d} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos45 & -sin45 \\ 0 & sin45 & cos45 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \\
&= (x+3)\hat{\boldsymbol{I}} + (0.707y - 0.707z + 5)\hat{\boldsymbol{J}} + (0.707y + 0.707z + 7)\hat{\boldsymbol{K}}
\end{aligned}
$$

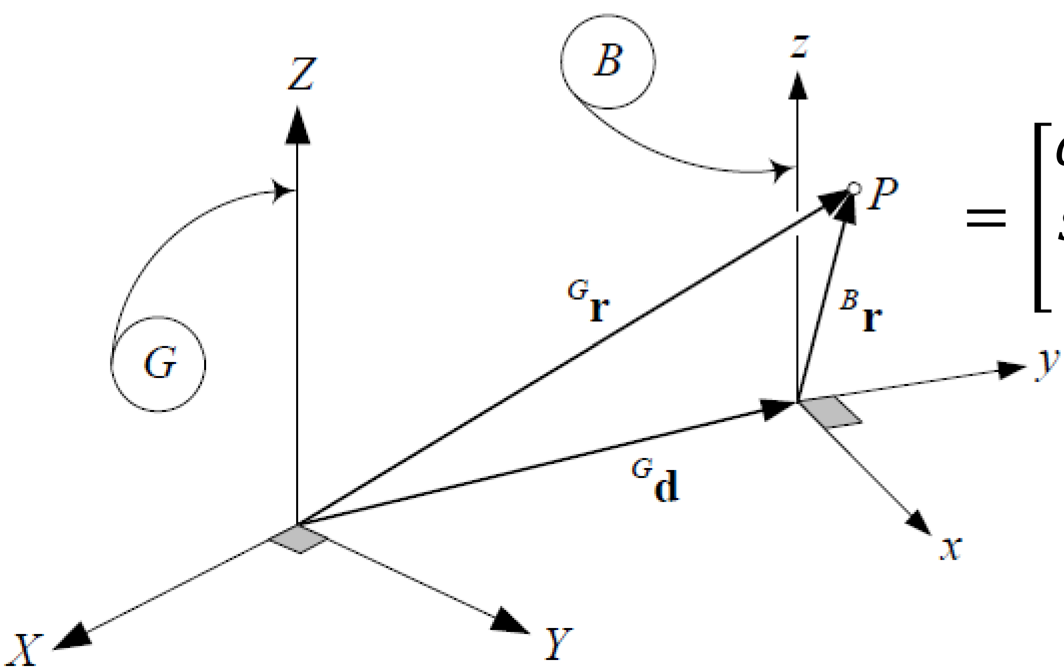


6. RIGID BODY MOTION

- Example 72 (*Moving body coordinate frame*)

Fig 4.2 show a point $\boldsymbol{P}$ at ${}^{B}\boldsymbol{r}_{P} = 0.1\hat{\boldsymbol{\imath}} + 0.3\hat{\boldsymbol{\jmath}} + 0.3\hat{\boldsymbol{k}}$ in a body frame $\boldsymbol{B}$, which is rotated 50^0 about the Z-axis, and translated -1 along X, 0.5 along Y, and 0.2 along the Z axes

The position of $\boldsymbol{P}$ in **global coordinate frame** is



$${}^{G}\boldsymbol{r}_{P} = {}^{G}\boldsymbol{R}_{B}\,{}^{B}\boldsymbol{r}_{P} + {}^{G}\boldsymbol{d}$$

$$= \begin{bmatrix} cos50 & -sin50 & 0 \\ sin50 & cos50 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0.5 \\ 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -1.166 \\ 0.769 \\ 0.5 \end{bmatrix}$$

6. RIGID BODY MOTION

- <u>Example 73</u> (*Rotation of a translated rigid body*)

  Point $\boldsymbol{P}$ of a rigid body $\boldsymbol{B}$ has an initial position vector

$$^{B}\boldsymbol{r}_P = [1 \quad 2 \quad 3]^T$$

  If the body rotates 45^0 about the x-axis, and then translates to $^{G}\boldsymbol{d} = [4 \quad 5 \quad 6]^T$, the **final position** of $\boldsymbol{P}$ would be

$$^{G}\boldsymbol{r} = {}^{B}\boldsymbol{R}_{x,45}^{T}\, {}^{B}\boldsymbol{r}_P + {}^{G}\boldsymbol{d}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos45 & -sin45 \\ 0 & sin45 & cos45 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

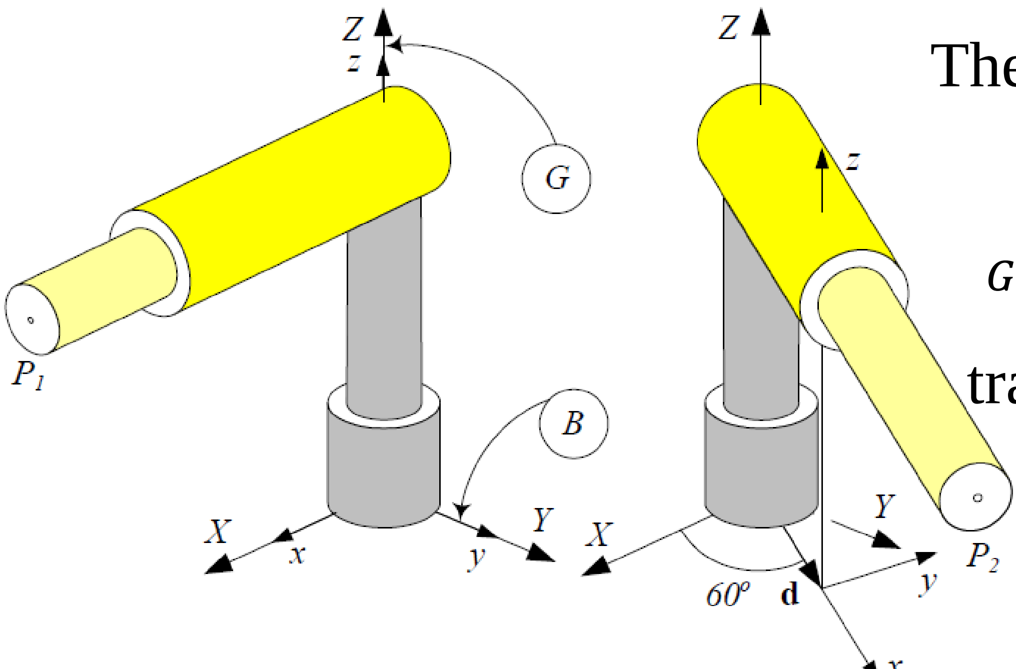
$$= \begin{bmatrix} 5.0 \\ 4.23 \\ 9.53 \end{bmatrix}$$

Note that rotation occurs with the assumption that $^{G}\boldsymbol{d} = 0$

6. RIGID BODY MOTION

- <u>Example 74</u> (*Arm rotation plus elongation*)

Position vector of point $\boldsymbol{P}_1$ at the tip of an arm shown in Fig 4.3(a) is at ${}^G\boldsymbol{r}_{P_1} = {}^B\boldsymbol{r}_{P_1} = [1350 \quad 0 \quad 900]^T$ mm. The arm rotates 60^0 about the global Z-axis, and elongates by $d = 720.2\hat{\imath}$ mm. The final configuration of the arm is shown in Fig 4.3(b)



(a) (b)

The new position vector of $\boldsymbol{P}$ is

$$ {}^G\boldsymbol{r}_{P_2} = {}^G\boldsymbol{R}_B\, {}^B\boldsymbol{r}_{P_1} + {}^G\boldsymbol{d} $$

${}^G\boldsymbol{R}_B = \boldsymbol{R}_{Z,60}$ is the rotation matrix to transform $\boldsymbol{r}_{P_2}$ to $\boldsymbol{r}_{P_1}$ when ${}^G\boldsymbol{d} = 0$

$$ {}^G\boldsymbol{R}_B = \begin{bmatrix} cos60 & -sin60 & 0 \\ sin60 & cos60 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

6. RIGID BODY MOTION

The translation vector in the body coordinate frame is ${}^{B}\boldsymbol{d} = [720.2 \quad 0 \quad 0]^T$, so ${}^{G}\boldsymbol{d}$ would be found by a transformation

$$ {}^{G}\boldsymbol{d} = {}^{G}\boldsymbol{R}_B\, {}^{B}\boldsymbol{d} $$

$$ = \begin{bmatrix} cos60 & -sin60 & 0 \\ sin60 & cos60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 720.2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 360.10 \\ 623.71 \\ 0.0 \end{bmatrix} $$

Therefore, the **final global position** of the tip of the arm is at

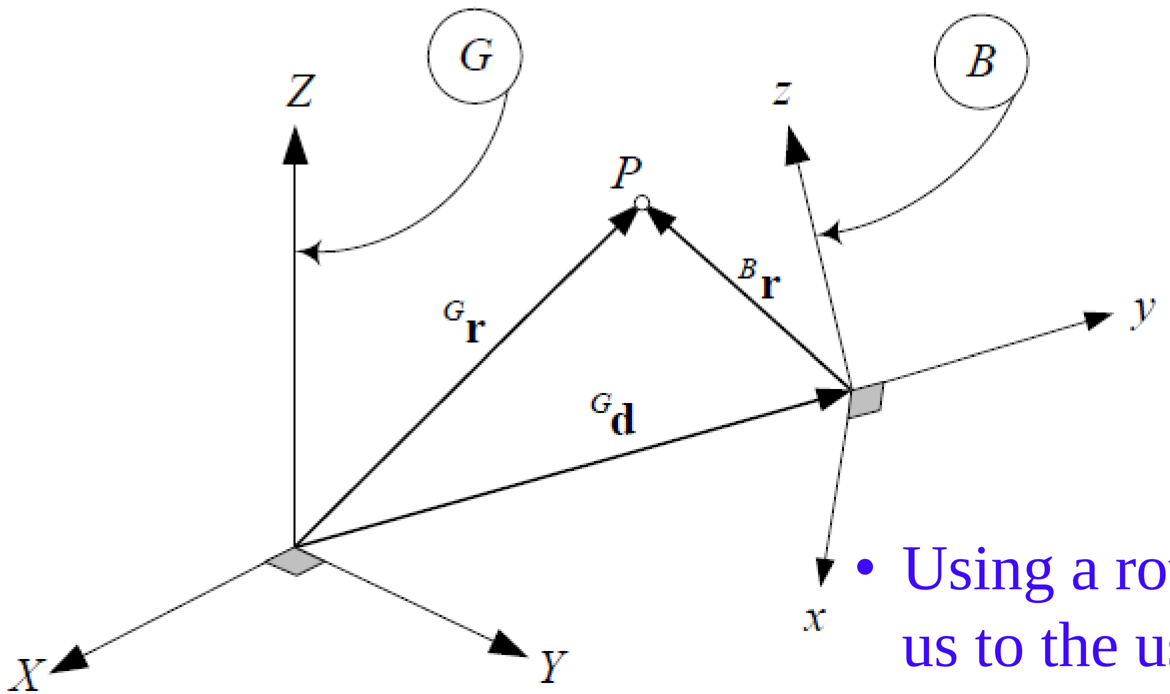
$$ {}^{G}\boldsymbol{r}_{P_2} = {}^{G}\boldsymbol{R}_B\, {}^{B}\boldsymbol{r}_{P_1} + {}^{G}\boldsymbol{d} $$

$$ = \begin{bmatrix} cos60 & -sin60 & 0 \\ sin60 & cos60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1350 \\ 0 \\ 900 \end{bmatrix} + \begin{bmatrix} 360.1 \\ 623.7 \\ 0.0 \end{bmatrix} $$

$$ = \begin{bmatrix} 1035.1 \\ 1792.8 \\ 900.0 \end{bmatrix} $$

7. HOMOGENEOUS TRANSFORMATION

- An arbitrary point $\boldsymbol{P}$ of a rigid body attached to the local frame $\boldsymbol{B}$ is denoted by ${}^{B}\boldsymbol{r}_{P}$ and ${}^{G}\boldsymbol{r}_{P}$ in different frames. The vector ${}^{G}\boldsymbol{d}$ indicates the position of origin $\boldsymbol{o}$ of the body frame in the global frame. Therefore, a general motion of a rigid body $\boldsymbol{B}(oxyz)$ in the global frame $\boldsymbol{G}(OXYZ)$ is a combination of **rotation** ${}^{G}\boldsymbol{R}_{B}$ and **translation** ${}^{G}\boldsymbol{d}$



$${}^{G}\boldsymbol{r} = {}^{G}\boldsymbol{R}_{B}\,{}^{B}\boldsymbol{r} + {}^{G}\boldsymbol{d}$$

- Using a rotation matrix plus a vector leads us to the use of homogeneous coordinates

7. HOMOGENEOUS TRANSFORMATION

- Introducing a new 4x4 ***homogeneous transformation matrix*** ${}^{G}\boldsymbol{T}_{B}$, helps us show a rigid motion by a single matrix transformation

$${}^{G}\boldsymbol{r} = {}^{G}\boldsymbol{T}_{B}\,{}^{B}\boldsymbol{r}$$

Where

$${}^{G}\boldsymbol{T}_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & X_0 \\ r_{21} & r_{22} & r_{23} & Y_0 \\ r_{31} & r_{32} & r_{33} & Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\equiv \begin{bmatrix} & {}^{G}\boldsymbol{R}_{B} & & {}^{G}\boldsymbol{d} \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} {}^{G}\boldsymbol{R}_{B} & {}^{G}\boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

And $\quad {}^{G}\boldsymbol{r} = \begin{bmatrix} X_P \\ Y_P \\ Z_P \\ 1 \end{bmatrix} \qquad {}^{B}\boldsymbol{r} = \begin{bmatrix} x_P \\ y_P \\ z_P \\ 1 \end{bmatrix} \qquad {}^{G}\boldsymbol{d} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$

7. HOMOGENEOUS TRANSFORMATION

- The **homogeneous transformation matrix** ${}^{G}\boldsymbol{T}_{B}$ is a 4x4 matrix that maps a homogeneous position vector from one frame to another
- Representation of an n-component position vector by an $(n+1)$-component vector is called homogeneous coordinate representation
- The appended element is a scale factor, ω; hence, in general, homogeneous representation of a vector $\boldsymbol{r} = [x \quad y \quad z]^T$ is

$$\boldsymbol{r} = \begin{bmatrix} \omega x \\ \omega y \\ \omega z \\ \omega \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \omega \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

- Example 76 (*Rotation and translation of a body coordinate frame*)

A body coordinate frame $\boldsymbol{B}(oxyz)$, that is originally coincident with global coordinate frame $\boldsymbol{G}(OXYZ)$, rotates 45^0 about the X-axis and translates to $[3 \quad 5 \quad 7 \quad 1]^T$

The matrix representation of the global position of a point is

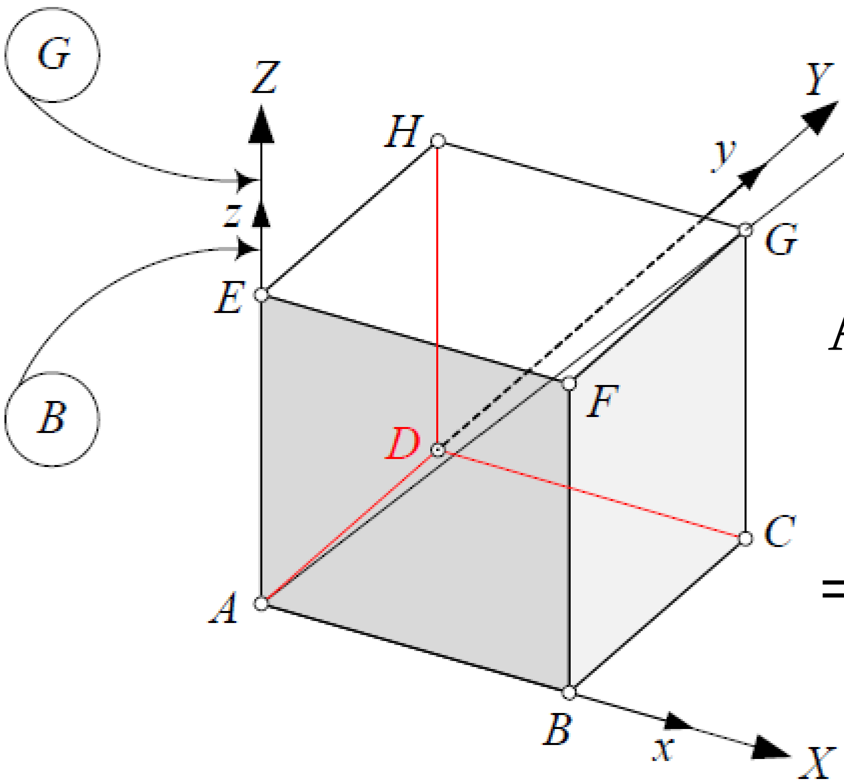
$$ {}^G\boldsymbol{r} = {}^G\boldsymbol{T}_B \, {}^B\boldsymbol{r} $$

$$ \Rightarrow \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & cos45 & -sin45 & 5 \\ 0 & sin45 & cos45 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

7. HOMOGENEOUS TRANSFORMATION

- Example 77 (*An axis-angle rotation and a translation*)

  Consider a cubic rigid body with a unit length of edges at the corner of the first quadrant. If we turn the cube 45^0 about $\mathbf{u} = [1 \quad 1 \quad 1]^T$ then



$$\phi = \frac{\pi}{4}$$

$$\hat{\boldsymbol{u}} = \frac{\mathbf{u}}{\sqrt{3}} = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}$$

And its Rodriguez transformation matrix is

$$\boldsymbol{R}_{\hat{u},\phi} = \boldsymbol{I}\, cos\phi + \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^T\, vers\phi + \breve{\boldsymbol{u}}\, sin\phi$$

$$= \begin{bmatrix} 0.80474 & -0.31062 & 0.50588 \\ 0.50588 & 0.80474 & -0.31062 \\ -0.31062 & 0.50588 & 0.80474 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

Translating the cube by ${}^{G}\boldsymbol{d} = [1 \quad 1 \quad 1]^T$ generates the following homogeneous transformation matrix

$$ {}^{G}\boldsymbol{T}_B = \begin{bmatrix} 0.80474 & -0.31062 & 0.50588 & 1 \\ 0.50588 & 0.80474 & -0.31062 & 1 \\ -0.31062 & 0.50588 & 0.80474 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

From the local coordinates of the corners of the upper face, the global coordinates of the corners after the motion are

	${}^{G}\boldsymbol{r}_E$	${}^{G}\boldsymbol{r}_F$	${}^{G}\boldsymbol{r}_G$	${}^{G}\boldsymbol{r}_H$
X	1.505	2.310	2	1.196
Y	0.689	1.196	2	1.495
Z	1.804	1.495	2	2.31

7. HOMOGENEOUS TRANSFORMATION

- <u>Example 78</u> (*Decomposition of* ${}^G\boldsymbol{T}_B$ *into translation and rotation*)

Homogeneous transformation matrix ${}^G\boldsymbol{T}_B$ can be decomposed to a matrix multiplication of a **pure rotation matrix** ${}^G\boldsymbol{R}_B$, and a **pure translation matrix** ${}^G\boldsymbol{D}_B$

$${}^G\boldsymbol{T}_B = {}^G\boldsymbol{D}_B\,{}^G\boldsymbol{R}_B$$

$$= \begin{bmatrix} 1 & 0 & 0 & X_0 \\ 0 & 1 & 0 & Y_0 \\ 0 & 0 & 1 & Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & X_0 \\ r_{21} & r_{22} & r_{23} & Y_0 \\ r_{31} & r_{32} & r_{33} & Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that decomposition of a transformation to translation and rotation is **not interchangeable**

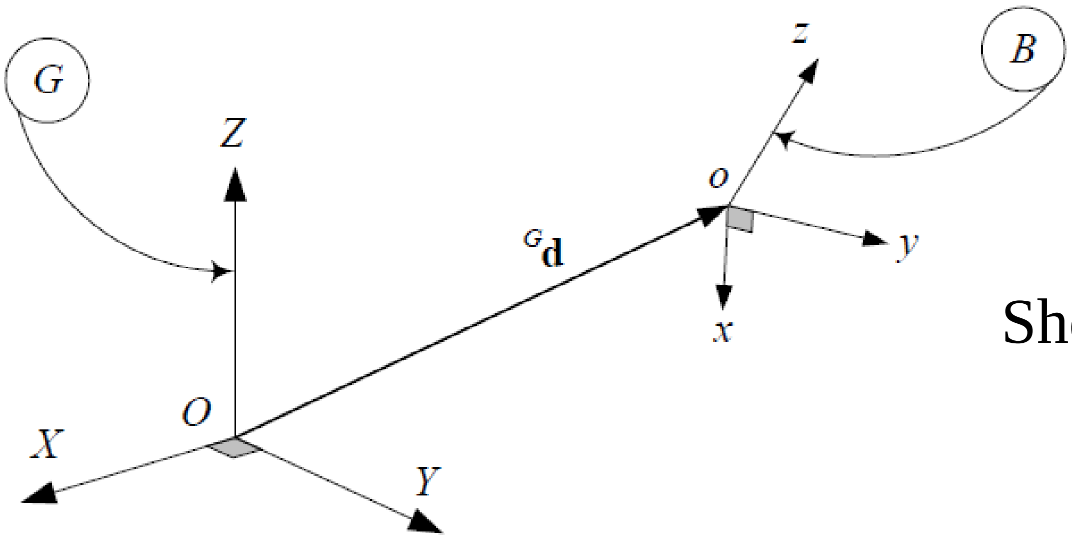
$${}^G\boldsymbol{T}_B = {}^G\boldsymbol{D}_B\,{}^G\boldsymbol{R}_B \neq {}^G\boldsymbol{R}_B\,{}^G\boldsymbol{D}_B$$

7. HOMOGENEOUS TRANSFORMATION

- The advantage of simplicity to work with homogeneous transformation matrices come with the penalty of losing the orthogonality property. If we show ${}^{G}\boldsymbol{T}_{B}$ by

$${}^{G}\boldsymbol{T}_{B}=\begin{bmatrix}\boldsymbol{I} & {}^{G}\boldsymbol{d}\\ 0 & 1\end{bmatrix}\begin{bmatrix}{}^{G}\boldsymbol{R}_{B} & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}{}^{G}\boldsymbol{R}_{B} & {}^{G}\boldsymbol{d}\\ 0 & 1\end{bmatrix}$$

then



$${}^{B}\boldsymbol{T}_{G}={}^{G}\boldsymbol{T}_{B}^{-1}=\begin{bmatrix}{}^{G}\boldsymbol{R}_{B} & {}^{G}\boldsymbol{d}\\ 0 & 1\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}{}^{G}\boldsymbol{R}_{B}^{T} & -{}^{G}\boldsymbol{R}_{B}^{T}\,{}^{G}\boldsymbol{d}\\ 0 & 1\end{bmatrix}$$

Showing that

$${}^{G}\boldsymbol{T}_{B}^{-1}\,{}^{G}\boldsymbol{T}_{B}=\boldsymbol{I}_{4}$$

7. HOMOGENEOUS TRANSFORMATION

- Example 88 (*Inverse of a homogeneous transformation matrix*)

Assume that

$$ {}^{G}\boldsymbol{T}_{B} = \begin{bmatrix} 0.643 & -0.766 & 0 & -1 \\ 0.766 & 0.643 & 0 & 0.5 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{G}\boldsymbol{R}_{B} & {}^{G}\boldsymbol{d} \\ 0 & 1 \end{bmatrix} $$

Then

$$ {}^{G}\boldsymbol{R}_{B} = \begin{bmatrix} 0.643 & -0.766 & 0 \\ 0.766 & 0.643 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{G}\boldsymbol{d} = \begin{bmatrix} -1 \\ 0.5 \\ 0.2 \end{bmatrix} $$

Therefore $$ {}^{B}\boldsymbol{T}_{G} = \begin{bmatrix} {}^{G}\boldsymbol{R}_{B}^{T} & -\,{}^{G}\boldsymbol{R}_{B}^{T}\,{}^{G}\boldsymbol{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.643 & 0.766 & 0 & 0.26 \\ -0.766 & 0.643 & 0 & -1.087 \\ 0 & 0 & 1 & -0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

7. HOMOGENEOUS TRANSFORMATION

- Example 89 (*Transformation matrix and coordinate of points*)

It is possible and sometimes convenient to describe a rigid body motion in terms of know displacement of specified points fixed in the body

Assume $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are 4 points at two different positions

$$\boldsymbol{A}_1(2,4,1) \qquad \boldsymbol{B}_1(2,6,1) \qquad \boldsymbol{C}_1(1,5,2) \qquad \boldsymbol{D}_1(3,5,2)$$

$$\boldsymbol{A}_2(5,1,1) \qquad \boldsymbol{B}_2(7,1,1) \qquad \boldsymbol{C}_2(6,2,1) \qquad \boldsymbol{D}_2(6,2,3)$$

There must be a transformation matrix $\boldsymbol{T}$ to map the initial positions to the final

$$[\boldsymbol{T}]\begin{bmatrix} 2 & 2 & 1 & 3 \\ 4 & 6 & 5 & 5 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 6 & 6 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

And hence

$$[\boldsymbol{T}] = \begin{bmatrix} 5 & 7 & 6 & 6 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 3 \\ 4 & 6 & 5 & 5 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5 & 7 & 6 & 6 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & -1/2 & 7/2 \\ 0 & 1/2 & -1/2 & -3/2 \\ -1/2 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

- <u>Example 90</u> (*Quick inverse transformation*)

For numerical calculation, it is more practical to decompose a transformation matrix into translation times rotation, and take advantage of the inverse of matrix multiplication

Consider a transformation matrix

$$[\boldsymbol{T}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = [\boldsymbol{D}][\boldsymbol{R}]$$

$$= \begin{bmatrix} 1 & 0 & 0 & r_{14} \\ 0 & 1 & 0 & r_{24} \\ 0 & 0 & 1 & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

Therefore

$$\boldsymbol{T}^{-1} = [\boldsymbol{D}\boldsymbol{R}]^{-1} = \boldsymbol{R}^{-1}\boldsymbol{D}^{-1} = \boldsymbol{R}^{T}\boldsymbol{D}^{-1}$$

$$= \begin{bmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -r_{14} \\ 0 & 1 & 0 & -r_{24} \\ 0 & 0 & 1 & -r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

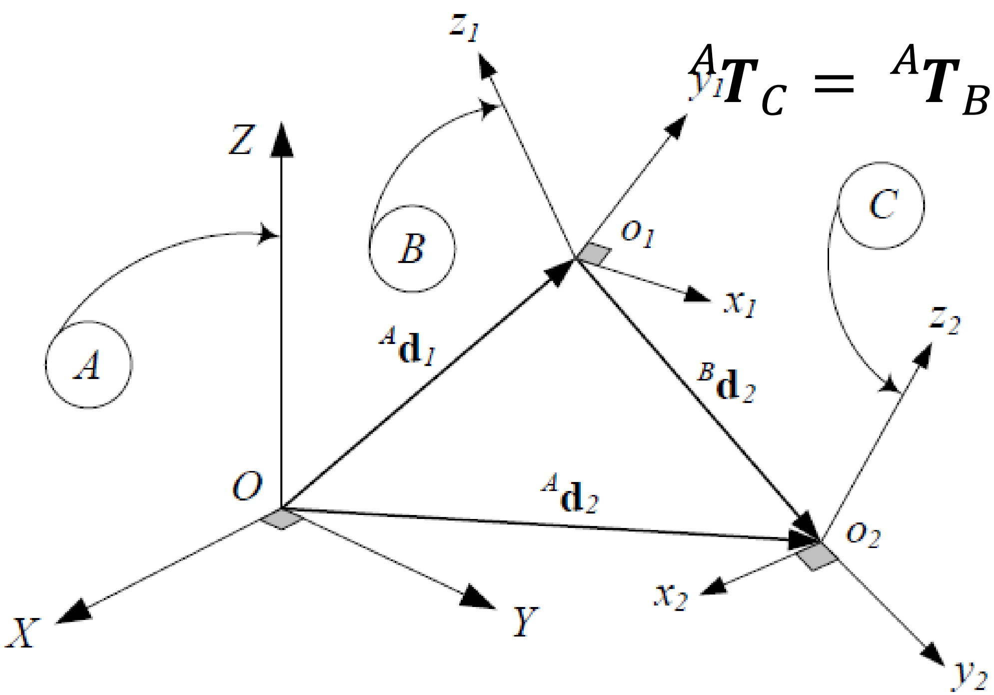
$$= \begin{bmatrix} r_{11} & r_{21} & r_{31} & -r_{11}r_{14} - r_{21}r_{24} - r_{31}r_{34} \\ r_{12} & r_{22} & r_{32} & -r_{12}r_{14} - r_{22}r_{24} - r_{32}r_{34} \\ r_{13} & r_{23} & r_{33} & -r_{13}r_{14} - r_{23}r_{24} - r_{33}r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

The transformation matrices to transform coordinates from frame $\boldsymbol{B}$ to $\boldsymbol{A}$, and from frame $\boldsymbol{C}$ to $\boldsymbol{B}$ are

$$^{A}\boldsymbol{T}_{B} = \begin{bmatrix} ^{A}\boldsymbol{R}_{B} & ^{A}\boldsymbol{d}_{1} \\ 0 & 1 \end{bmatrix} \qquad ^{B}\boldsymbol{T}_{C} = \begin{bmatrix} ^{B}\boldsymbol{R}_{C} & ^{B}\boldsymbol{d}_{2} \\ 0 & 1 \end{bmatrix}$$

Hence, the transformation matrix from $\boldsymbol{C}$ to $\boldsymbol{A}$ is



$$^{A}\boldsymbol{T}_{C} = {}^{A}\boldsymbol{T}_{B}\,{}^{B}\boldsymbol{T}_{C} = \begin{bmatrix} ^{A}\boldsymbol{R}_{B} & ^{A}\boldsymbol{d}_{1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} ^{B}\boldsymbol{R}_{C} & ^{B}\boldsymbol{d}_{2} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ^{A}\boldsymbol{R}_{B}\,{}^{B}\boldsymbol{R}_{C} & ^{A}\boldsymbol{R}_{B}\,{}^{B}\boldsymbol{d}_{2} + {}^{A}\boldsymbol{d}_{1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ^{A}\boldsymbol{R}_{C} & ^{A}\boldsymbol{d}_{2} \\ 0 & 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

And therefore, the inverse transformation is

$$ {}^{C}\boldsymbol{T}_{A} = \begin{bmatrix} {}^{A}\boldsymbol{R}_{C}^{T} & -{}^{A}\boldsymbol{R}_{C}^{T}\,{}^{A}\boldsymbol{d}_{2} \\ 0 & 1 \end{bmatrix} $$

The value of homogeneous coordinates are better appreciated when several displacements occur in succession which, for instance, can be written as

$$ {}^{G}\boldsymbol{T}_{4} = {}^{G}\boldsymbol{T}_{1}\,{}^{1}\boldsymbol{T}_{2}\,{}^{2}\boldsymbol{T}_{3}\,{}^{3}\boldsymbol{T}_{4} $$

Rather than

$$ {}^{G}\boldsymbol{R}_{4}\,{}^{4}\boldsymbol{r}_{P} + {}^{G}\boldsymbol{d}_{4} $$

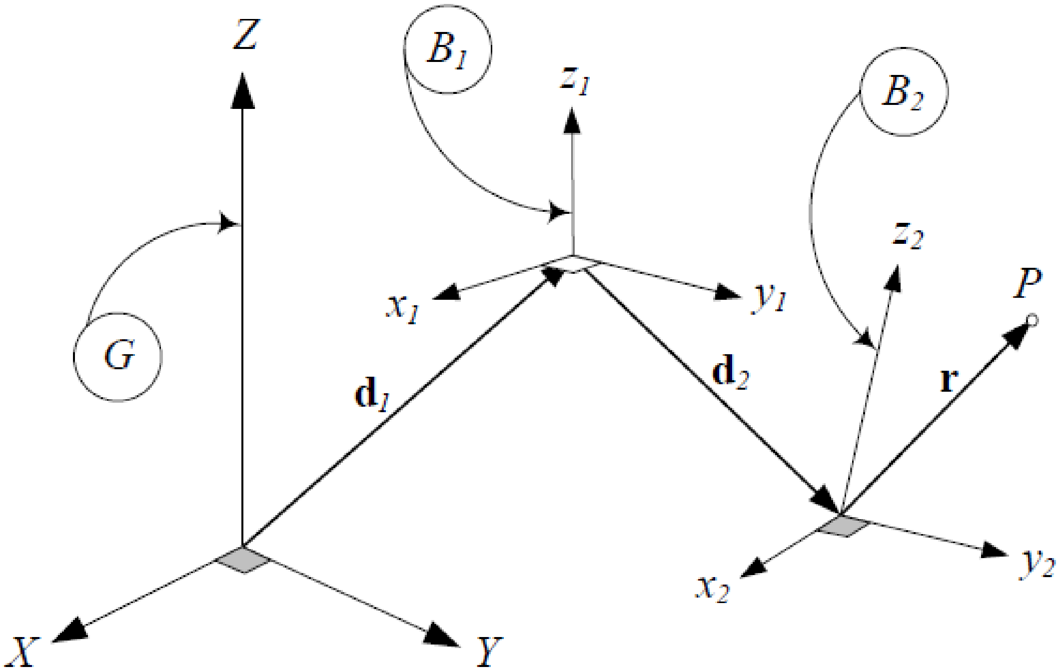
$$ = {}^{G}\boldsymbol{R}_{1}\left({}^{1}\boldsymbol{R}_{2}\left({}^{2}\boldsymbol{R}_{3}\left({}^{3}\boldsymbol{R}_{4}\,{}^{4}\boldsymbol{r}_{P} + {}^{3}\boldsymbol{d}_{4}\right) + {}^{2}\boldsymbol{d}_{3}\right) + {}^{1}\boldsymbol{d}_{2}\right) + {}^{G}\boldsymbol{d}_{1} $$

7. HOMOGENEOUS TRANSFORMATION

- Example 92 (*Homogeneous transformation for multiple frames*)

The coordinates of $\boldsymbol{P}$ in the global frame $\boldsymbol{G}(OXYZ)$ can be found by using the homogeneous transformation matrices

The position of $\boldsymbol{P}$ in frame $\boldsymbol{B}_2(x_2y_2z_2)$ is indicated by ${}^2\boldsymbol{r}_P$. Therefore, its position in frame $\boldsymbol{B}_1(x_1y_1z_1)$ is



$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^1\boldsymbol{R}_2 & {}^1\boldsymbol{d}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

And therefore, its position in the global frame $\boldsymbol{G}(OXYZ)$ would be

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{G}\boldsymbol{R}_1 & {}^{G}\boldsymbol{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^{G}\boldsymbol{R}_1 & {}^{G}\boldsymbol{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{1}\boldsymbol{R}_2 & {}^{1}\boldsymbol{d}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}$$

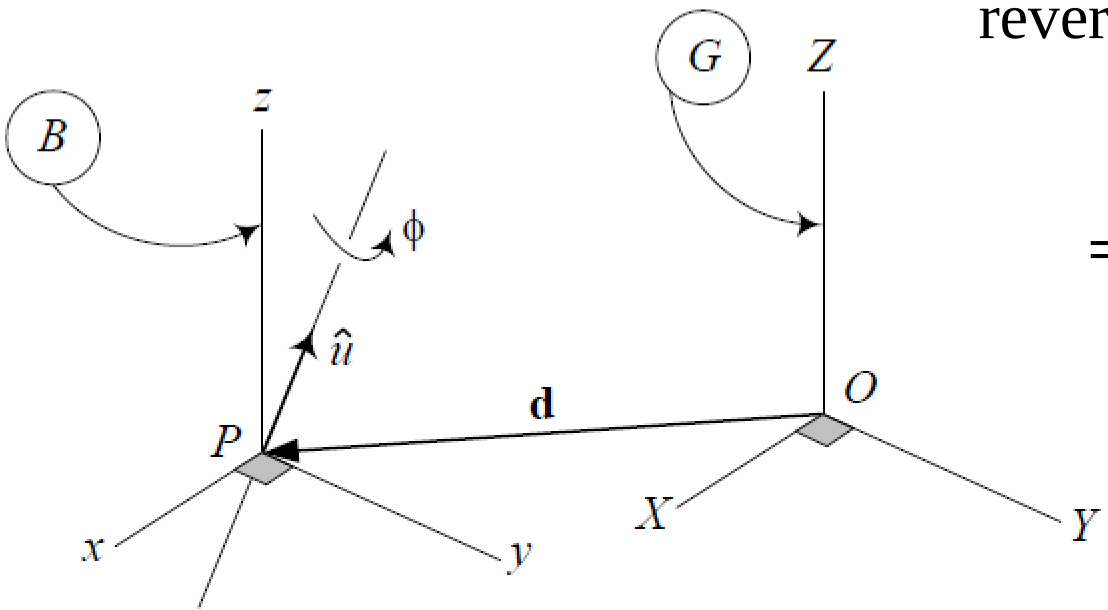
$$= \begin{bmatrix} {}^{G}\boldsymbol{R}_1\, {}^{1}\boldsymbol{R}_2 & {}^{G}\boldsymbol{R}_1\, {}^{1}\boldsymbol{d}_2 + {}^{G}\boldsymbol{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

- Example 93 (*Rotation about an axis not going through origin*)

  The homogeneous transformation matrix can represent rotations about an axis going through a point different from the origin

  We set a local frame $\boldsymbol{B}$ at point $\boldsymbol{P}$ parallel to the global frame $\boldsymbol{G}$. Then, a rotation around $\hat{\boldsymbol{u}}$ can be expressed as a translation along $-\boldsymbol{d}$, to bring the body frame $\boldsymbol{B}$ to the global frame $\boldsymbol{G}$, followed by a rotation $\hat{\boldsymbol{u}}$ and a reverse translation along $\boldsymbol{d}$



$$ {}^{G}\boldsymbol{T}_B = \boldsymbol{D}_{\hat{d},d}\boldsymbol{R}_{\hat{u},\phi}\boldsymbol{D}_{\hat{d},-d} $$

$$ = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{\hat{u},\phi} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{d} \\ 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} \boldsymbol{R}_{\hat{u},\phi} & \boldsymbol{d} - \boldsymbol{R}_{\hat{u},\phi}\boldsymbol{d} \\ 0 & 1 \end{bmatrix} $$

7. HOMOGENEOUS TRANSFORMATION

Where

$$\boldsymbol{R}_{\hat{u},\phi} = \begin{bmatrix} u_1^2 vers\phi + c\phi & u_1u_2 vers\phi - u_3 s\phi & u_1u_3 vers\phi + u_2 s\phi \\ u_1u_2 vers\phi + u_3 s\phi & u_2^2 vers\phi + c\phi & u_2u_3 vers\phi - u_1 s\phi \\ u_1u_3 vers\phi - u_2 s\phi & u_2u_3 vers\phi + u_1 s\phi & u_3^2 vers\phi + c\phi \end{bmatrix}$$

And

$$\boldsymbol{d} - \boldsymbol{R}_{\hat{u},\phi}\boldsymbol{d} =$$

$$\begin{bmatrix} d_1(1-u_1^2)vers\phi - u_1 vers\phi(d_2u_2 + d_3u_3) + s\phi(d_2u_3 - d_3u_2) \\ d_2(1-u_2^2)vers\phi - u_2 vers\phi(d_3u_3 + d_1u_1) + s\phi(d_3u_1 - d_1u_3) \\ d_3(1-u_3^2)vers\phi - u_3 vers\phi(d_1u_1 + d_2u_2) + s\phi(d_1u_2 - d_2u_1) \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

- <u>Example 94</u> (A *rotating cylinder*)

Imagine a cylinder with radius $R = 2$ that its axis $\hat{\boldsymbol{u}}$ is at $\boldsymbol{d}$

$$\hat{\boldsymbol{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{d} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

If the cylinder turns 90^0 about its axis then every point on the periphery of the cylinder will move 90^0 on a circle parallel to (x, y)-plane. The transformation of this motion is

$${}^{G}\boldsymbol{T}_{B} = \boldsymbol{D}_{\hat{d},d}\boldsymbol{R}_{\hat{u},\phi}\boldsymbol{D}_{\hat{d},-d} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{\hat{K},\frac{\pi}{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. HOMOGENEOUS TRANSFORMATION

Consider a point on the cylinder that was on the origin. After the rotation, the point would be seen at

$$^{G}\boldsymbol{r} = {}^{G}\boldsymbol{T}_{B}\, {}^{B}\boldsymbol{r}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$