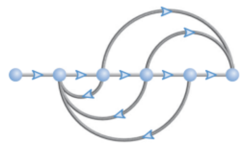
Modeling in the Time Domain 3



Chapter Objectives

After completing this chapter, the student will be able to

- find a mathematical model, called a state-space representation, for a linear, time invariant system
- model electrical and mechanical systems in state space
- convert a transfer function to state space
- convert a state-space representation to a transfer function
- linearize a state-space representation

§1. Introduction

Two approaches are available for the analysis and design of feedback control systems

- The classical, or frequency-domain, technique

Major disadvantage: can be applied only to linear, time-invariant systems or systems that can be approximated as such

Major advantage: rapidly provide stability and transient response information

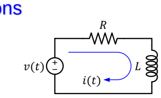
§1. Introduction

- The modern, or time domain, state-space technique
  - A unified method for modeling, analyzing, and designing a wide range of systems
  - Can be used to represent nonlinear systems that have backlash, saturation, and dead zone
  - Can handle, conveniently, systems with nonzero initial conditions
  - Can be used to represent time-varying systems, (for example, missiles with varying fuel levels or lift in an aircraft flying through a wide range of altitudes)
  - Can be compactly represented in state space for multiple-input, multiple-output systems
  - Can be used to represent systems with a digital computer in the loop or to model systems for digital simulation

§1. Introduction

- With a simulated system, system response can be obtained for changes in system parameters - an important design tool
- The state space approach is also attractive because of the availability of numerous state-space software packages for the personal computer

§2. Some Observations

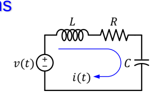


Select the current $i(t)$ as a variable, write the loop equation

$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

$$\rightarrow \frac{di(t)}{dt} = -\frac{R}{L}i(t) + \frac{1}{L}v(t)$$

§2. Some Observations



Select the current $i(t)$ as a variable, write the loop equation

$$i(t) = \frac{dq(t)}{dt}$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int idt = v(t)$$

$$\rightarrow \frac{dq(t)}{dt} = i(t)$$

$$\frac{di(t)}{dt} = -\frac{1}{LC}q(t) - \frac{R}{L}i(t) + \frac{1}{L}v(t)$$

§2. Some Observations

$$\frac{dq(t)}{dt} = i(t)$$

$$\frac{di(t)}{dt} = -\frac{1}{LC}q(t) - \frac{R}{L}i(t) + \frac{1}{L}v(t)$$

The state equation can be written in vector-matrix form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$

where, $x = \begin{bmatrix} q(t) \\ i(t) \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$, $u = v(t)$

The output equation can be written in vector-matrix form

$$y = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u$$

where, $y = v_L(t)$, $\boldsymbol{C} = \begin{bmatrix} -\frac{1}{C} & -R \end{bmatrix}$, $x = \begin{bmatrix} q(t) \\ i(t) \end{bmatrix}$, $D = 1$, $u = v(t)$

§3. The General State-Space Representation

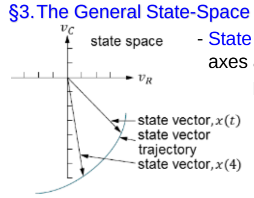
Review

- Linear combination: A linear combination of n variables, x_i, for $i = 1 \div n$, is given by the following sum, S

$$S = K_n x_n + K_{n-1}x_{n-1} + \cdots + K_1 x_1 \qquad K_i\text{: constant}$$

- Linear independence: A set of variables is said to be linearly independent if none of the variables can be written as a linear combination of the others
- System variable: Any variable that responds to an input or initial conditions in a system
- State variables: The smallest set of linearly independent system variables such that the values of the members of the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t \geq t_0$
- State vector: A vector whose elements are the state variables

§3. The General State-Space Representation



Graphic representation of state space and a state vector

- State space: n-dimensional space whose axes are the state variables

  In the figure
  - state variables: v_R and v_C
  - state trajectory can be thought of as being mapped out by the state vector, $x(t)$, for a range of t
- State equations: A set of n simultaneous, first-order differential equations with n variables, where the n variables to be solved are the state variables
- Output equation: The algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs

§3. The General State-Space Representation

- A system is represented in state space by the following equations

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

for $t \geq t_0$ and initial conditions, $\boldsymbol{x}(t_0)$, where

$\boldsymbol{x}$: state vector

$\dot{\boldsymbol{x}}$: derivative of the state vector with respect to time

$\boldsymbol{y}$: output vector — $\boldsymbol{u}$: input or control vector

$\boldsymbol{A}$: system matrix — $\boldsymbol{B}$: input matrix

$\boldsymbol{C}$: output matrix — $\boldsymbol{D}$: feedforward matrix

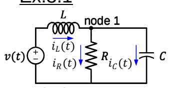
§4. Applying the State-Space Representation

In this section, the state-space formulation is applied to the representation of more complicated physical systems. The first step in representing a system is to select the state vector, which must be chosen according to the following considerations

1. A minimum number of state variables must be selected as components of the state vector. This minimum number of state variables is sufficient to describe completely the state of the system
2. The components of the state vector (that is, this minimum number of state variables) must be linearly independent

§4. Applying the State-Space Representation

- Ex.3.1 **Representing an Electrical Network**



Given the electrical network, find a state-space representation if the output is the current through the resistor

Solution

Step 1 Label all of the branch currents in the network. These include i_L, i_R, and i_C

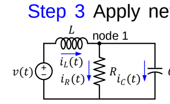
Step 2 Select the state variables by writing the derivative equation for all energy storage elements, L and C

$$C\frac{dv_C}{dt} = i_C \quad (3.22)$$

$$L\frac{di_L}{dt} = v_L \quad (3.23)$$

§4. Applying the State-Space Representation

Step 3 Apply network theory, such as Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L.



At Node 1,

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L \quad (3.24)$$

which yields i_C in terms of the state variables, v_C and i_L

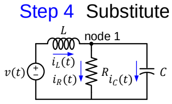
Around the outer loop,

$$v_L = -v_C + v(t) \quad (3.25)$$

which yields v_L in terms of the state variable, v_C, and the source, $v(t)$

§4. Applying the State-Space Representation

Step 4 Substitute (3.24), (3.25) into (3.22), (3.23) to obtain



$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L \quad (3.27.a)$$

$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v(t) \quad (3.27.b)$$

Step 5 Find the output equation. Since the output is $i_R(t)$

$$i_R = v_C/R \quad (3.28)$$

The state-space representation is found by representing Eqs. (3.27) and (3.28) in vector-matrix form

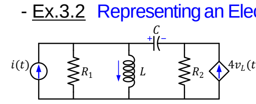
$$\begin{bmatrix}\dot{v}_C\\ \dot{i}_L\end{bmatrix} = \begin{bmatrix}-1/RC & 1/C\\ -1/L & 0\end{bmatrix}\begin{bmatrix}v_C\\ i_L\end{bmatrix} + \begin{bmatrix}0\\ 1/L\end{bmatrix}v(t)$$

$$i_R = \begin{bmatrix}1/R & 0\end{bmatrix}\begin{bmatrix}v_C\\ i_L\end{bmatrix}$$

$C\frac{dv_C}{dt} = i_C$ (3.22), $L\frac{di_L}{dt} = v_L$ (3.23), $i_C = -\frac{1}{R}v_C + i_L$ (3.24), $v_L = -v_C + v(t)$ (3.25)

§4. Applying the State-Space Representation

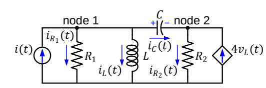
- Ex.3.2 **Representing an Electrical Network with a Dependent Source**



Find the state and output equations for the electrical network if the output vector is $y = [v_{R_2} \quad i_{R_2}]^T$

Solution

Step 1 Label all of the branch currents in the network



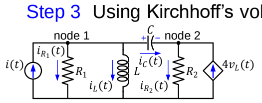
Step 2 Select the state variables by listing the voltage-current relationships for all of the energy-storage elements

$$L\frac{di_L}{dt} = v_L \qquad C\frac{dv_C}{dt} = i_C \quad (3.30)$$

Select the state variables: $x_1 = i_L$ and $x_2 = v_C$ (3.31)

§4. Applying the State-Space Representation

Step 3 Using Kirchhoff's voltage and current laws to find i_L, v_C in terms of the state variables



Around the mesh containing L and C

$$v_L = v_C + v_{R_2} = v_C + i_{R_2}R_2$$

At node 2, $i_{R_2} = i_C + 4v_L$

$$v_L = v_C + (i_C + 4v_L)R_2 = \frac{1}{1-4R_2}(v_C + i_C R_2) \quad (3.35)$$

At node 1

$$i_C = i - i_{R_1} - i_L = i - \frac{v_{R_1}}{R_1} - i_L = i - \frac{v_L}{R_1} - i_L \quad (3.36)$$

§4. Applying the State-Space Representation

Rewriting Eqs (3.35) and (3.36)

$$(1-4R_2)v_L - R_2 i_C = v_C$$

$$-(1/R_1)v_L \quad - i_C = i_L - i$$

Writing the result in vector-matrix form

$$v_L = \frac{1}{\Delta}[R_2 i_L - v_C - R_2 i] \quad (3.38)$$

$$i_C = \frac{1}{\Delta}\left[(1-4R_2)i_L + \frac{1}{R_1}v_C - (1-4R_2)i\right] \quad (3.39)$$

where,

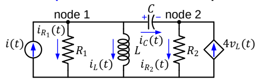
$$\Delta = -\left[(1-4R_2) + \frac{R_2}{R_1}\right]$$

$$\begin{bmatrix}\dot{i}_L\\ \dot{v}_C\end{bmatrix} = \begin{bmatrix}R_2/(L\Delta) & -1/(L\Delta)\\ (1-4R_2)/(C\Delta) & 1/(R_1 C\Delta)\end{bmatrix}\begin{bmatrix}i_L\\ v_C\end{bmatrix} + \begin{bmatrix}-R_2/(L\Delta)\\ (1-4R_2)/(C\Delta)\end{bmatrix}i$$

$v_L = \frac{1}{1-4R_2}(v_C + i_C R_2)$ (3.35), $i_C = i - \frac{v_L}{R_1} - i_L$ (3.36)

§4. Applying the State-Space Representation

Step 4 Derive the output equation



Since the specified output variables are v_{R_2} and i_{R_2}, note that around the mesh containing C, L, and R_2

$$v_{R_2} = -v_C + v_L \qquad (3.42.a)$$

$$i_{R_2} = i_C + 4v_L \qquad (3.42.b)$$

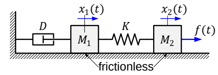
Substituting Eqs. (3.38) and (3.39) into Eq.(3.42)

$$\begin{bmatrix} v_{R_2} \\ i_{R_2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1+1/\Delta) \\ 1/\Delta & (1-4R_1)/(R_1\Delta) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i$$

$$\boxed{v_L = \frac{1}{\Delta}\left[R_2 i_L - v_C - R_2 i\right] \ (3.38), \quad i_C = \frac{1}{\Delta}\left[(1-4R_2)i_L + \frac{1}{R_1}v_C - (1-4R_2)i\right] \ (3.39)}$$

§4. Applying the State-Space Representation

- Ex.3.3 Representing a Translational Mechanical System



Find the state equations for the translational mechanical system

Solution

Find the Laplace-transformed equations of motion

$$+\begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_1\end{bmatrix} \times X_1(s) - \begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix} \times X_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_1\end{bmatrix}$$

$$+\left[M_1 s^2 + Ds + K\right]X_1 \qquad - KX_2 = 0$$

$$-\begin{bmatrix}\text{sum of impedances}\\ \text{between } x_1 \text{ and } x_2\end{bmatrix} \times X_1(s) + \begin{bmatrix}\text{sum of impedances}\\ \text{connected to the}\\ \text{motion at } x_2\end{bmatrix} \times X_2(s) = \begin{bmatrix}\text{sum of applied}\\ \text{forces at } x_2\end{bmatrix}$$

$$-KX_1 + \left[M_2 s^2 + K\right]X_2 = F$$

§4. Applying the State-Space Representation

Take the inverse Laplace transform assuming zero initial conditions

$$\left[M_1 s^2 + Ds + K\right]X_1 - KX_2 = 0 \rightarrow M_1\frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + K(x_1 - x_2) = 0 \quad (3.44)$$

$$-KX_1 + \left[M_2 s^2 + K\right]X_2 = F \rightarrow -Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t) \quad (3.45)$$

Let $\dfrac{dx_1}{dt} \equiv v_1$, $\dfrac{dx_2}{dt} \equiv v_2 \rightarrow \dfrac{d^2x_1}{dt^2} = \dfrac{dv_1}{dt}$, $\dfrac{d^2x_2}{dt^2} = \dfrac{dv_2}{dt}$

State equations

$$\frac{dx_1}{dt} = +v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt} = +v_2$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)$$

§4. Applying the State-Space Representation

$$\frac{dx_1}{dt} = \qquad + v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt} = \qquad + v_2$$

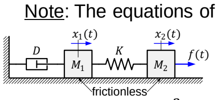
$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 \qquad - \frac{K}{M_2}x_2 \qquad + \frac{1}{M_2}f(t)$$

In vector matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K}{M_1} & -\dfrac{D}{M_1} & \dfrac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K}{M_2} & 0 & -\dfrac{K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{M_2} \end{bmatrix} f(t)$$

§4. Applying the State-Space Representation

Note: The equations of motion (3.44) and (3.45) can be derived directly from the figure using Newton's laws of motion



$$M_1\frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + K(x_1 - x_2) = 0$$

$$M_2\frac{d^2x_2}{dt^2} + K(x_2 - x_1) = f(t)$$

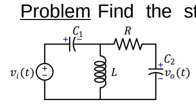
$$\rightarrow M_1\frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + K(x_1 - x_2) = 0 \qquad (3.44)$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t) \qquad (3.45)$$

§4. Applying the State-Space Representation

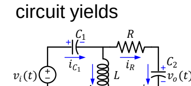
Skill-Assessment Ex.3.1

Problem Find the state-space representation of the electrical network with the output is $v_o(t)$



Solution

Identifying appropriate variables on the circuit yields

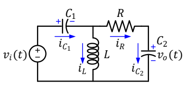


Writing the derivative relations

$$C_1\frac{dv_{C_1}}{dt} = i_{C_1} \qquad L\frac{di_L}{dt} = v_L \qquad C_2\frac{dv_{C_2}}{dt} = i_{C_2}$$

§4. Applying the State-Space Representation

Using Kirchhoff's current and voltage laws



$$i_{C_1} = i_L + i_R = i_L + \frac{1}{R}(v_L - v_{C_2})$$
$$v_L = -v_{C_1} + v_i$$
$$i_{C_2} = i_R = \frac{1}{R}(v_L - v_{C_2})$$

Substituting and rearranging

$$\frac{dv_{C_1}}{dt} = -\frac{1}{RC_1}v_{C_1} + \frac{1}{C_1}i_L - \frac{1}{RC_1}v_{C_2} + \frac{1}{RC_1}v_i$$
$$\frac{di_L}{dt} = -\frac{1}{L}v_{C_1} + \frac{1}{L}v_i$$
$$\frac{dv_{C_2}}{dt} = -\frac{1}{RC_2}v_{C_1} - \frac{1}{RC_2}v_{C_2} + \frac{1}{RC_2}v_i$$

The output $v_o = v_{C_2}$

§4. Applying the State-Space Representation

$$\frac{dv_{C_1}}{dt} = -\frac{1}{RC_1}v_{C_1} + \frac{1}{C_1}i_L - \frac{1}{RC_1}v_{C_2} + \frac{1}{RC_1}v_i$$
$$\frac{di_L}{dt} = -\frac{1}{L}v_{C_1} + \frac{1}{L}v_i$$
$$\frac{dv_{C_2}}{dt} = -\frac{1}{RC_2}v_{C_1} - \frac{1}{RC_2}v_{C_2} + \frac{1}{RC_2}v_i$$

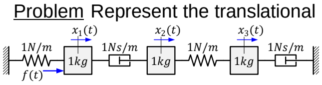
The output $v_o = v_{C_2}$, the equations in vector-matrix form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{v}_{C_1} \\ i_L \\ \dot{v}_{C_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} & -\frac{1}{RC_1} \\ -\frac{1}{L} & 0 & 0 \\ -\frac{1}{RC_2} & 0 & -\frac{1}{RC_2} \end{bmatrix} \begin{bmatrix} v_{C_1} \\ i_L \\ v_{C_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{L} \\ \frac{1}{RC_2} \end{bmatrix} v_i$$
$$y = [0 \quad 0 \quad 1]\boldsymbol{x}$$

§4. Applying the State-Space Representation

Skill-Assessment Ex.3.2

Problem Represent the translational mechanical system in state-space, where $x_3(t)$ is the output



Solution Writing the equations of motion

$$(s^2+s+1)X_1 - sX_2 = F$$
$$-sX_1 + (s^2+s+1)X_2 - X_3 = 0$$
$$-X_2 + (s^2+s+1)X_3 = 0$$

Taking the inverse Laplace transform and simplifying

$$\ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f$$
$$\ddot{x}_2 = +\dot{x}_1 - \dot{x}_2 - x_2 + x_3$$
$$\ddot{x}_3 = -\dot{x}_3 - x_3 + x_2$$

§4. Applying the State-Space Representation

Defining state variables

$$z_1 = x_1,\ z_2 = \dot{x}_1,\ z_3 = x_2,\ z_4 = \dot{x}_2,\ z_5 = x_3,\ z_6 = \dot{x}_3$$

Write the state equations

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f$$
$$= -z_2 - z_1 + z_4 + f$$
$$\dot{z}_3 = \dot{x}_2 = z_4$$
$$\dot{z}_4 = \ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3$$
$$= z_2 - z_4 - z_3 + z_5$$
$$\dot{z}_5 = \dot{x}_3$$
$$= z_6$$
$$\dot{z}_6 = \ddot{x}_3 = -\dot{x}_3 - x_3 + x_2$$
$$= -z_6 - z_5 + z_3$$

$\ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f,\quad \ddot{x}_2 = +\dot{x}_1 - \dot{x}_2 - x_2 + x_3,\quad \ddot{x}_3 = -\dot{x}_3 - x_3 + x_2$

§4. Applying the State-Space Representation

$$\dot{z}_1 = +z_2$$
$$\dot{z}_2 = -z_1 - z_2 + z_4 + f$$
$$\dot{z}_3 = +z_4$$
$$\dot{z}_4 = +z_2 - z_3 - z_4 + z_5$$
$$\dot{z}_5 = +z_6$$
$$\dot{z}_6 = +z_3 - z_5 - z_6$$

The output $y = z_5$, the equations in vector-matrix form

$$\dot{\boldsymbol{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t)$$
$$y = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]\boldsymbol{z}$$

§5. Converting a Transfer Function to State-Space

Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

y : output $\quad u$: input $\quad a_i$'s, b_0 : constant

Choose the output, y, and its derivatives as the state variables

$$x_1 = y,\ x_2 = \frac{dy}{dt},\ x_3 = \frac{d^2y}{dt^2},\ \cdots,\ x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

Differentiating both sides yields

$$\dot{x}_1 = \frac{dy}{dt},\ \dot{x}_2 = \frac{d^2y}{dt^2},\ \dot{x}_3 = \frac{d^3y}{dt^3},\ \cdots,\ \dot{x}_n = \frac{d^n y}{dt^n}$$

Define the sate variables

$$\dot{x}_1 \overset{\text{def}}{=} x_2,\ \dot{x}_2 \overset{\text{def}}{=} x_3,\ \dot{x}_3 \overset{\text{def}}{=} x_4,\ \cdots,\ \dot{x}_{n-1} \overset{\text{def}}{=} x_n$$
$$\dot{x}_n \overset{\text{def}}{=} -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + b_0u$$

§5. Converting a Transfer Function to State-Space

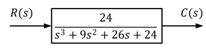
The phase-variable form of the state equations

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ \vdots\\ 0\\ b_0\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}$$

§5. Converting a Transfer Function to State-Space

- Ex.3.4 Converting a TF with Constant Term in Numerator



Find the state-space representation in phase-variable form for the TF

Solution

Step 1 Find the associated differential equation

$$\frac{C(s)}{R(s)}=\frac{24}{s^3+9s^2+26s+24}$$

$$\rightarrow (s^3+9s^2+26s+24)C(s)=24R(s)$$

Take the inverse Laplace transform, assuming zero initial conditions

$$\dddot{c}+9\ddot{c}+26\dot{c}+24c=24r$$

§5. Converting a Transfer Function to State-Space

Step 2 Select the state variables

Choosing the state variables $x_1=c$, $x_2=\dot{c}$, $x_3=\ddot{c}$

$$\dot{x}_1= \quad + x_2$$
$$\dot{x}_2= \quad + x_3$$
$$\dot{x}_3=-24x_1-26x_2-9x_3+24r$$
$$y=c=x_1$$

In vector-matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 24\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

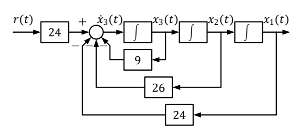
$\dddot{c}+9\ddot{c}+26\dot{c}+24c=24r$

§5. Converting a Transfer Function to State-Space

An equivalent block diagram of the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 24\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



§5. Converting a Transfer Function to State-Space

MATLAB ML

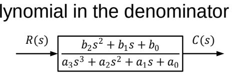
Run ch3p1 through ch3p4 in Appendix B

Learn how to use MATLAB to

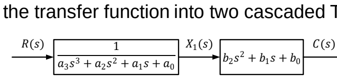
- represent the system matrix A, the input matrix B, and the output matrix C
- convert a transfer function to the state-space representation in phase-variable form
- solve Ex.3.4

§5. Converting a Transfer Function to State-Space

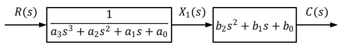
- If a TF has a polynomial in s in the numerator that is of order less than the polynomial in the denominator



the numerator and denominator can be handled separately: separate the transfer function into two cascaded TFs



§5.Converting a Transfer Function to State-Space



- The first TF with just the denominator is converted to the phase-variable representation in state space, phase variable x_1 is the output, and the rest of the phase variables are the internal variables of the first block
- The second TF with just the numerator yields

$$Y(s)=C(s)=(b_2s^2+b_1s+b_0)X_1(s)$$

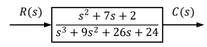
Taking the inverse Laplace transform with zeros initial conditions

$$y(t)=b_2\frac{d^2x_1}{dt^2}+b_1\frac{dx_1}{dt}+b_0x_1$$
$$=b_0x_1+b_1x_2+b_2x_3$$

Hence, the second block simply forms a specified linear combination of the state variables developed in the first block

§5.Converting a Transfer Function to State-Space

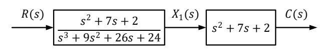
- Ex.3.5 Converting a TF with Polynomial in Numerator



Find the state-space representation of the transfer function

Solution

Step 1 Separate the system into two cascaded blocks



Step 2 Find the state equations for the block containing the denominator

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

§5.Converting a Transfer Function to State-Space

Step 3 Introduce the effect of the block with the numerator

The second block states that

$$C(s)=(b_2s^2+b_1s+b_0)X_1(s)=(s^2+7s+2)X_1(s)$$

Taking inverse Laplace transform with zero initial conditions

$$c=\ddot{x}_1+7\dot{x}_1+2x_1=x_3+7x_2+2x_1$$

The output equation

$$y=\begin{bmatrix}b_0&b_1&b_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2&7&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

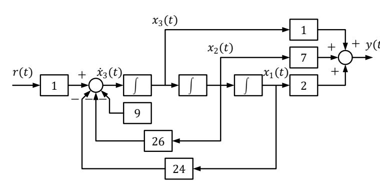
An equivalent block diagram of the system

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$
$$y=\begin{bmatrix}2&7&1\end{bmatrix}\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}^T$$

§5.Converting a Transfer Function to State-Space

An equivalent block diagram of the system

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r,$$
$$y=\begin{bmatrix}2&7&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$



§5.Converting a Transfer Function to State-Space

Skill-Assessment Ex.3.3

Problem Find the state equations and output equation for the phase-variable representation of the TF

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$$G(s)=\frac{2s+1}{s^2+7s+9}$$

Solution

First TF

$$\frac{X(s)}{R(s)}=\frac{1}{s^2+7s+9}\rightarrow(s^2+7s+9)X(s)=R(s)$$

Taking inverse Laplace transform with zero initial conditions

$$\ddot{x}+7\dot{x}+9x=r$$

Defining the state variables as $x_1=x$, $x_2=\dot{x}$

$$\dot{x}_1=x_2$$
$$\dot{x}_2=\ddot{x}=-7\dot{x}-x+r=-9x_1-7x_2+r$$

§5.Converting a Transfer Function to State-Space

Second TF

The output equation

$$c=2\dot{x}+x=x_1+2x_2$$

Putting all equation in vector-matrix form

$$\dot{\boldsymbol{x}}=\begin{bmatrix}0&1\\-9&-7\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\1\end{bmatrix}r$$
$$c=\begin{bmatrix}1&2\end{bmatrix}\boldsymbol{x}$$

$\dot{x}_2=-9x_1-7x_2+r,\quad c=x_1+2x_2$

§6. Converting from State Space to a Transfer Function

- Given the state and output equations

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

- Take the Laplace transform assuming zero initial conditions

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s)$$
$$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s)$$

- After some arrangement

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$
$$\boldsymbol{Y}(s) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right]\boldsymbol{U}(s)$$

The matrix $\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$: the transfer function matrix

- If $\boldsymbol{U}(s) = U(s)$ and $\boldsymbol{Y}(s) = Y(s)$ are scalars, the TF

$$T(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \qquad (3.73)$$

§6. Converting from State Space to a Transfer Function

Review of calculating the Inverse of a Matrix

Ex.: Find the inverse matrix of $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

• Step 1: Create a Matrix of Cofactors

For the first row

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & -2 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

For the second row

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ -2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ -2 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ -2 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ -2 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

For the third row

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 2 & -10 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 2 & -10 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 2 & -10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & -3 \\ 2 & -10 & 0 \end{bmatrix}$$

§6. Converting from State Space to a Transfer Function

• Step 2: Adjugate

Obtain the transpose of the matrix of cofactors, i.e. the adjoint of the matrix, by writing the rows as columns

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & -3 \\ 2 & -10 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & -10 \\ 2 & -3 & 0 \end{bmatrix}$$

• Step 3: Calculate the inverse matrix

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & -10 \\ 2 & -3 & 0 \end{bmatrix}}{det\left(\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}\right)} = \frac{1}{10}\begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & -10 \\ 2 & -3 & 0 \end{bmatrix}$$

§6. Converting from State Space to a Transfer Function

- Ex.3.6 State-Space Representation to Transfer Function

Find the TF, $T(s) = Y(s)/U(s)$, for given the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u, \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}$$

Solution

Find $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

§6. Converting from State Space to a Transfer Function

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

Then

$$T(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + [0]$$

$$= \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}$$

§6. Converting from State Space to a Transfer Function

MATLAB ML

Run ch3p5 in Appendix B

Learn how to use MATLAB to

- convert a state-space representation to a transfer function
- solve Ex.3.6

§6. Converting from State Space to a Transfer Function

Symbolic Math SM

Run ch3sp1 in Appendix F

Learn how to use the Symbolic Math Toolbox to

- write matrices and vectors
- solve Ex.3.6

§6. Converting from State Space to a Transfer Function

Skill-Assessment Ex.3.4

Problem Convert the state and output equations to a TF

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t), y = [1.5 \quad 0.625]\boldsymbol{x} \qquad (3.78)$$

Solution

$$s\boldsymbol{I} - \boldsymbol{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s^2 + 4s + 6}$$

$$G(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} = [1.5 \quad 0.625]\frac{\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}}{s^2 + 4s + 6}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{3s + 5}{s^2 + 4s + 6}$$

§6. Converting from State Space to a Transfer Function

TryIt 3.2

Use the following MATLAB and the Control System Toolbox statements to obtain the transfer function shown in Skill-Assessment Exercise 3.4 from the state-space representation of Eq. (3.78).

```
A=[-4 -1.5;4 0];
B=[2 0]';
C=[1.5 0.625];
D=0;
T=ss(A,B,C,D);
T=tf(T)
```

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t) \qquad (3.78)$$
$$y = [1.5 \quad 0.625]\boldsymbol{x}$$

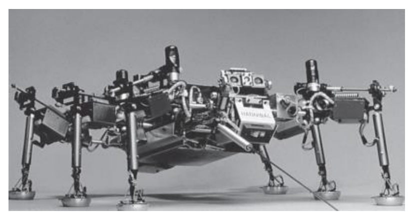
Matlab A=[-4 -1.5; 4 0]; B=[2 0]';
C=[1.5 0.625]; D=0;
T=ss(A,B,C,D); T=tf(T)

Result T =

```
      3 s + 5
  -----------------
  s^2 + 4 s + 6
```

Continuous-time transfer function

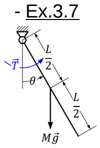
§7. Linearization

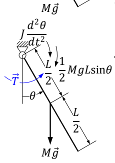


Walking robots, such as Hannibal shown here, can be used to explore hostile environments and rough terrain, such as that found on other planets or inside volcanoes

§7. Linearization

- Ex.3.7 Representing a Nonlinear System





First represent the simple pendulum in state space (Mg: weight, T: applied torque in the θ direction, and L: length). Assume the mass is evenly distributed, with the center of mass at $L/2$. Then linearize the state equations about the pendulum's equilibrium point - the vertical position with zero angular velocity

Solution

Drawing the free body diagram

Summing the torques

$$J\frac{d^2\theta}{dt^2} + \frac{MgL}{2}\sin\theta = T$$

§7. Linearization

$$J\frac{d^2\theta}{dt^2} + \frac{MgL}{2}\sin\theta = T$$

Letting $x_1 = \theta$, $x_2 = d\theta/dt$, the state equation

$$\dot{x}_1 = x_2 \qquad (3.80.a)$$

$$\dot{x}_2 = -\frac{MgL}{2J}\sin x_1 + \frac{T}{J} \qquad (3.80.b)$$

The nonlinear Eq. (3.80) represent a valid and complete model of the pendulum in state space even under nonzero initial conditions and even if parameters are time varying

To apply classical techniques and convert these state equations to a transfer function → The nonlinear must be linearized

§7. Linearization

Linearize the equation about the equilibrium point, $x_1 = \theta = 0$, $x_2 = d\theta/dt = 0$. Let x_1 and x_2 be perturbed about the equilibrium point, or

$$x_1 = 0 + \delta x_1$$
$$x_2 = 0 + \delta x_2$$

Using Eqs. (2.182)

$$\sin x_1 - \sin 0 = \left.\frac{d(\sin x_1)}{dx_1}\right|_{x_1=0} \delta x_1 = \delta x_1 \rightarrow \sin x_1 = \delta x_1$$

The state equations now become

$$\delta\dot{x}_1 = \delta x_2$$
$$\delta\dot{x}_2 = -\frac{MgL}{2J}\delta x_1 + \frac{T}{J}$$

$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$ (2.182), $\dot{x}_1 = x_2$ (3.80.a), $\dot{x}_2 = -\frac{MgL}{2J}\sin x_1 + \frac{T}{J}$ (3.80.b)

§7. Linearization

Skill-Assessment Ex.3.5

Problem Represent the translational mechanical system in state space about the equilibrium displacement. The spring is nonlinear $f_s(t) = 2x_s^2(t)$. The applied force is $f(t) = 10 + \delta f(t)$, where $\delta f(t)$ is a small force about the $10N$ constant value. Assume the output to be the displacement of the mass, $x(t)$

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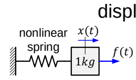

Solution

The equation of motion

$$\frac{d^2x}{dt^2} + 2x^2 = 10 + \delta f(t) \quad (1)$$

Letting $x = x_0 + \delta x$

$$\frac{d^2(x_0 + \delta x)}{dt^2} + 2(x_0 + \delta x)^2 = 10 + \delta f(t) \quad (2)$$

§7. Linearization

Linearize x^2 at x_0

$$(x_0 + \delta x)^2 - x_0^2 = \left.\frac{d(x^2)}{dx}\right|_{x_0} \delta x = 2x_0\delta x$$

$$\rightarrow (x_0 + \delta x)^2 = x_0^2 + 2x_0\delta x \quad (3)$$

Substituting Eq.(3) into Eq.(1)

$$\frac{d^2\delta x}{dt^2} + 4x_0\delta x = -2x_0^2 + 10 + \delta f(t) \quad (4)$$

The force of the spring at equilibrium $F = 10 = 2x_0^2 \rightarrow x_0 = \sqrt{5}$

Substituting this value of x_0 into Eq.(4)

$$\frac{d^2\delta x}{dt^2} + 4\sqrt{5}\delta x = \delta f(t)$$

$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$ (2.182), $\frac{d^2x}{dt^2} + 2x^2 = 10 + \delta f(t)$ (1)

§7. Linearization

$$\frac{d^2\delta x}{dt^2} + 4\sqrt{5}\delta x = \delta f(t)$$

Selecting the state variables $x_1 = \delta x$, $x_2 = \delta\dot{x}$

The state and output equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \delta\ddot{x} = -4\sqrt{5}x_1 + \delta f(t)$$
$$y = x_1$$

Converting to vector-matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4\sqrt{5} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta f(t)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

§8. Case Studies

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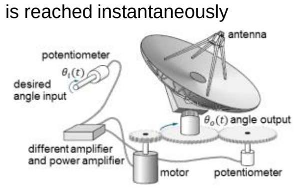
1. Antenna Control: State-Space Representation

Problem Find the state-space representation in phase-variable form for each dynamic subsystem in the antenna azimuth position control. By dynamic, we mean that the system does not reach the steady state instantaneously. A pure gain, on the other hand, is an example of a non dynamic system, since the steady state is reached instantaneously



§8. Case Studies

The transfer function of the power amplifier is given on the front endpapers as $G(s) = 100/(s + 100)$. We will convert this transfer function to its state-space representation. Letting $v_p(t)$ represent the power amplifier input and $e_a(t)$ represent the power amplifier output

$$G(s) = \frac{E_a(s)}{V_p(s)} = \frac{100}{s + 100} \quad (3.85)$$

Cross-multiplying, $(s + 100)E_a(s) = 100V_p(s)$, from which the differential equation can be written as

$$\frac{de_a(t)}{dt} + 100e_a(t) = 100v_p(t) \quad (3.86)$$

§8. Case Studies

2. Pharmaceutical Drug Absorption

Problem In the pharmaceutical industry we want to describe the distribution of a drug in the body. A simple model divides the process into compartments: the dosage, the absorption site, the blood, the peripheral compartment, and the urine. The rate of change of the amount of a drug in a compartment is equal to the input flow rate diminished by the output flow rate. Figure 3.16 summarizes the system. Here each xi is the amount of drug in that particular compartment (Lordi, 1972). Represent the system in state space, where the outputs are the amounts of drug in each compartment