Solving $aE\overline{E} + b\overline{E} + c = 0$

We want to determine, for which triples $(a, b, c) \in \mathbb{C}^3$, the equation

$$a\,E\overline{E} \;+\; b\,\overline{E} \;+\; c \;=\; 0$$

admits a solution $E \in \mathbb{C}$.

The case $a = 0$

When $a = 0$, the equation reduces to

$$b\,\overline{E} + c = 0.$$

$$\text{If } b \neq 0, \qquad \overline{E} = -\frac{c}{b} \quad \Longrightarrow \quad E = -\frac{\overline{c}}{\overline{b}}.$$

So a solution always exists, for every $c \in \mathbb{C}$.

If $b = 0$, then the equation is simply $c = 0$.

- If $b = 0$ and $c = 0$: every E is a solution.
- If $b = 0$ and $c \neq 0$: no solution exists.

Thus for $a = 0$, a solution exists unless

$$(b, c) = (0, c \neq 0).$$

From now on, assume $a \neq 0$

We divide the original equation by a and define

$$b' = \frac{b}{a}, \qquad c' = \frac{c}{a},$$

so the equation becomes

$$E\overline{E} + b'\,\overline{E} + c' = 0.$$

Rewrite as

$$c' = -\,E\overline{E} + b'\,\overline{E}.$$

Let us decompose E into magnitude and direction:

$$E = E_{\text{size}}\,E_{\text{dir}}, \qquad |E_{\text{dir}}| = 1.$$

Then

$$c' = -\,E_{\text{size}}^2 \;+\; b' E_{\text{size}}\,\overline{E_{\text{dir}}}.$$

For fixed E_{size}, varying E_{dir} traces out a circle in the complex plane. Write

$$b' = b_\Sigma\,e^{i\theta}, \qquad b_\Sigma = |b'|,$$

and rotate coordinates so that the circle lies on the real axis. Then $c' = c_x + ic_y$ satisfies

$$(c_x + c_\Sigma)^2 + c_y^2 = (b_\Sigma E_{\text{size}})^2,$$

where c_Σ is a real shift.

Finding the envelope of these circles

Define

$$F(c_x, c_y, z) = (c_x + c_\Sigma)^2 + c_y^2 - (b_\Sigma z)^2.$$

The envelope of the family is obtained from

$$F = 0, \qquad \frac{\partial F}{\partial z} = 0.$$

We have

$$\frac{\partial F}{\partial z} = -2b_\Sigma^2\,z,$$

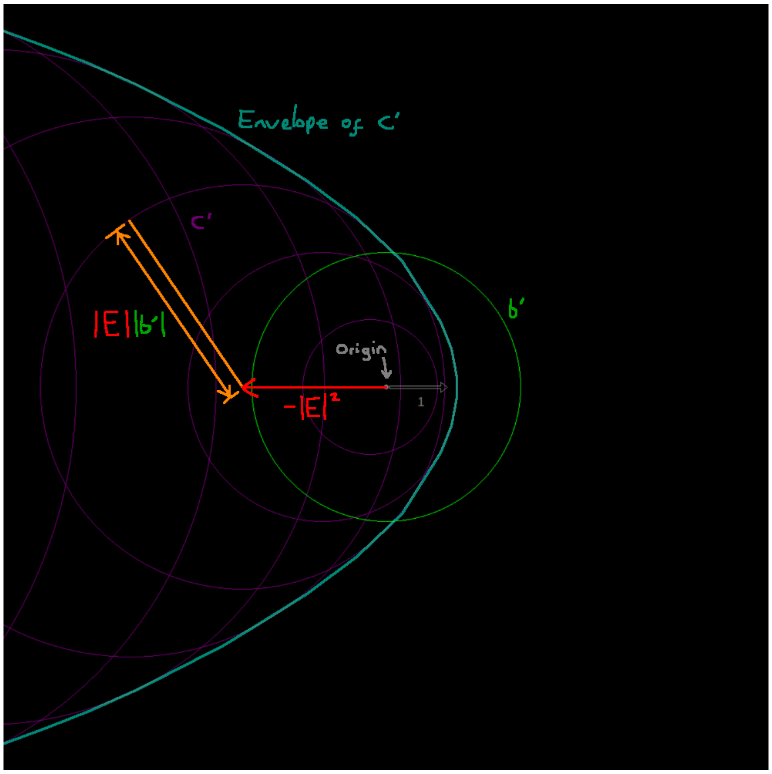
so the envelope occurs where $z = E_{\text{size}} = 0$.

Substituting $z = 0$ into $F = 0$ gives

$$(c_x + c_\Sigma)^2 + c_y^2 = 0.$$

Undoing the coordinate shifts and simplifications yields the envelope curve

$$c_y^2 = -\,c_x\left(c_x^2 + \frac{b_\Sigma^2}{4}\right), \qquad c_x \le 0.$$



The set of c' for which a solution E exists is the interior of this envelope.

Translating back to $c = ac'$ determines all triples (a, b, c) for which a solution exists.