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1. Recall Taylor's theorem from Calculus: "Assume a function $f(x)$ that has $k+1$ derivatives in an interval $[a,b]$, or simply, $f \in C^{k+1}[a,b]$ and $x_0 \in [a,b]$. Then, for every $x \in [a,b]$, $\exists \xi$ between x_0 and x such that

$$f(x) = \underbrace{\sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_k(x)} + \underbrace{\frac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}}_{R_k(x)}, \tag{1}$$

where $P_k(x)$ is called the kth Taylor polynomial for f around x_0 and $R_k(x)$ is called the remainder, or truncation error". Note that

$$\lim_{k\to\infty} P_k(x)$$

gives the Taylor series for the same function f about $x = x_0$ and also a function f is analytic in (a,b) if the Taylor series equals f $\forall x \in (a,b)$. Finally, the Taylor series around $x = x_0 \equiv 0$ are called Maclaurin ones.

   (a) Find $P_1(x)$, $P_2(x)$ and $P_3(x)$ around $x_0 = 0$ if $f(x) = x^2 - 4x + 3$. How $P_3(x)$ is related to $f(x)$?

   (b) Same as part (a) but consider $x_0 = 1$.

   (c) In general, given a polynomial $f(x)$ with degree m, what can you say about $f(x) - P_k(x)$ for $k \geq m$?

2. Given the function $f(x) = \cos x$, find both $P_2(x)$ and $P_3(x)$ about $x_0 = 0$, and use them to approximate $\cos(0.1)$. Show that in each case the remainder term provides an upper bound for the true (absolute) error.

3. If $f(x) = e^x$, then

   (a) derive the Maclaurin series of the function $f(x) = e^x$, i.e., the Taylor series about $x_0 = 0$ (write seperately $P_k(x)$ and $R_k(x)$),

   (b) find a minimum value of k necessarary for $P_k(x)$ to approximate $f(x)$ to within 10^{-6} on the interval $[0, 0.5]$ (here, you must use the remainder term).

4. Use Taylor series to show that the first derivative of a function $f(x)$ at x_0 can be approximated by:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2).$$

The above formula is called the centered difference formula for the first derivative. Then, if $f(x) = e^x$ and $x_0 = 1/2$ write a MATLAB script which does the following: approximates the first derivative of $f(x)$ given using the above formula and shows in a log-log plot the absolute error by halving h at each step, i.e., use $h = 0.1$, $h = 0.05$, $h = 0.025$ etc. Also, show that your approximation has $O(h^2)$ truncation error, i.e., the order of accuracy is 2. When your approximation starts getting worse and why does this happen?

Hints: Note that if error $\approx Ch^q$ *where* q *is the order of accuracy, then* $\log(\textit{error}) \approx \log C + q \log h$. *Therefore in the log-log plot, you must determine the slope of the straight line obtained therein which eventually is the value of* q. *Recall that for a straight line of the form of* $y = b + ax$, *two points can be selected on the* x*-axis, e.g.,* x_1 *and* x_2 *and we can find the corresponding ordinates* $y_1 = y(x_1)$ *and* $y_2 = y(x_2)$. *Then, we obtain the slope via*

$$a = \frac{y_2 - y_1}{x_2 - x_1}.$$

5. Consider the integral given by

$$y_n = \int_0^1 \frac{x^n}{x+\alpha} dx, \quad \alpha > 0, \tag{2}$$

where n is a non-negative integer, that is, $n \in \mathbb{Z}^*$.

(a) Show that $0 < y_n < 1$ for $n \geq 1$.

(b) Show that the recurrence relation

$$y_n = \frac{1}{n} - \alpha y_{n-1} \tag{3}$$

holds for any positive integer, that is, $n \in \mathbb{Z}^+$.

(c) Write a MATLAB script that computes the first 20 values of Eq. (3) with $\alpha = 10$. Attach your code and also a figure containing the graph of y_n as a function of n. Note that you have to initialize Eq. (3), that is, determine y_0. What is happening?