Name and section: ____________________

ID number: ____________________

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1. (40 points) Determine the interpolating polynomial $p(x)$ that interpolates u at $\bar{x}$, $\bar{x}-h$ and $\bar{x}-2h$. Then, show that the one-sided approximation of $u'(\bar{x})$ is given by

$$D_2 u(\bar{x}) = \frac{3u(\bar{x}) - 4u(\bar{x}-h) + u(\bar{x}-2h)}{2h}. \tag{1}$$

Next, test this finite difference formula [cf. Eq. (1)] to approximate $u'(1.2)$ for $u(x) = \cos(x)\sin(x)$. Specifically, write a MATLAB script (or a code in **any** programming language) for this purpose and show the absolute error as a function of h in a log-log scale by halving h at each step, i.e., use $h = 0.1$, $h = 0.05$, $h = 0.025$ etc. Perform the halvation 10 times and present your results in a table with the following format:

- column 1: h (step-size)
- column 2: $E_2 u(\bar{x})$ (absolute error, i.e., $E_2 u(\bar{x}) = |D_2 u(\bar{x}) - u'(\bar{x})|$).

Subsequently, show that the approximation is indeed **second-order accurate**, i.e., $\mathcal{O}(h^2)$ (you might want to compute the slope of the straight line in the log-log scale). You are free to modify the script `dr_chapter_1_ex_fd.m` for this purpose.

Attach your code, the figure and MATLAB output.

2. (60 points) Consider the **fourth-order accurate** finite difference approximation

$$u''(\bar{x}) = c_1 u(\bar{x}-2h) + c_2 u(\bar{x}-h) + c_3 u(\bar{x}) + c_4 u(\bar{x}+h) + c_5 u(\bar{x}+2h) + \mathcal{O}\left(h^4\right), \tag{6}$$

which is based on 5 equally-spaced points, i.e., $x_1 = \bar{x}-2h$, $x_2 = \bar{x}-h$, $x_3 = \bar{x}$, $x_4 = \bar{x}+h$ and $x_5 = \bar{x}+2h$. In class we discussed about how to obtain the coefficients c_i, for $i = 1, 2, 3, 4, 5$ via solving the *Vandermonde* system given by

$$\frac{1}{(i-1)!} \sum_{j=1}^{n} c_j \left(x_j - \bar{x}\right)^{(i-1)} = \begin{cases} 1 & \text{if} \quad i-1 = k, \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

(a) (20 points) Set up the underlying 5×5 Vandermonde system.

(b) (10 points) Compute the coefficients c_i using the MATLAB function `fdstencil.m` that we discussed in class and verify that the c_i's satisfy the system you determined in part (a).

(c) (30 points) Test this finite difference formula [cf. Eq. (6)] to approximate $u''(1)$ for $u(x) = \sin(2x)$. Similarly, as you did in Question 1, write a MATLAB script (or a code in **any** programming language) for this purpose and show the absolute error as a function of h in a log-log scale. Present again your results in a table with the following format:

- column 1: h (step-size)
- column 2: $E_4 u(\bar{x})$ (absolute error).

Subsequently, show that the approximation is indeed **fourth-order accurate**, i.e., $\mathcal{O}(h^4)$. Compare the error against the predicted one from the leading term of the expression printed by `fdstencil.m`.

Attach your code, the figure and MATLAB output.