Name and section: ______________________________

ID number: ______________________________

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1. (20 points) Consider the boundary value problem (BVP):

$$\begin{aligned} &u''(x) = f(x), \quad x \in (0,1), \\ &u(0) = u(1) = 0. \end{aligned} \tag{1}$$

   In class, we divided $[0,1]$ into $m+1$ equal subintervals such that $x_j = jh$, $j = 0, 1, \ldots, m+1$ and $h = 1/(m+1)$. This way, $U_j \doteq U(x_j)$ denotes the approximation to $u_j \doteq u(x_j)$. If $h = 1/4$, then write the underlying system of equations in the form of $A\,\mathbf{U} = \mathbf{F}$ by considering the **second-order** accurate finite difference formula for $u''(\bar{x})$:

$$D^2 u(\bar{x}) = \frac{u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)}{h^2}. \tag{2}$$

   How many equations and how many unknowns are there in this problem?

2. (30 points) Consider the following yet more general BVP

$$\begin{aligned} &u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad x \in (a,b), \\ &u(a) = \alpha, \quad u(b) = \beta, \end{aligned} \tag{5}$$

   where $p(x), q(x), f(x) \in C[a,b]$. Similarly as in Question 1, we are seeking approximate values $U_j \doteq U(x_j)$ for the exact values $u_j \doteq u(x_j)$ where $j = 0, 1, \ldots, m+1$ and $x_j = a + j\,h$ with $h = (b-a)/(m+1)$. Next, consider the finite difference formulas

$$\begin{aligned} D_0 u(\bar{x}) &= \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}, \\ D^2 u(\bar{x}) &= \frac{u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)}{h^2}, \end{aligned} \tag{6}$$

   for approximating $u'(\bar{x})$ and $u''(\bar{x})$, respectively.

   Then, write the underlying system of equations in the form of $A\,\mathbf{U} = \mathbf{F}$ by explicitly presenting the form of A, $\mathbf{U}$ and $\mathbf{F}$. How many unknowns the vector $\mathbf{U}$ contains? Subsequently, how the form of the matrix A and $\mathbf{F}$ will change if we consider the **Neumann boundary condition** $u'(a) = \sigma$ at the left-end point? How many unknowns the vector $\mathbf{U}$ contains in this case?

3. (50 points) Consider the BVP

$$\begin{aligned} &u''(x) - 400u(x) = 400\cos^2(\pi x) + 2\pi^2\cos(2\pi x), \quad x \in (0,1), \\ &u(0) = u(1) = 0, \end{aligned} \tag{10}$$

whose exact solution is given by

$$u(x) = \frac{e^{-20}}{1+e^{-20}}e^{20x} + \frac{1}{1+e^{-20}}e^{-20x} - \cos^2(\pi x). \tag{11}$$

Write a MATLAB script (or a code in **any** programming language) to solve the above BVP using finite differences with $m = 124$ interior points. Furthermore, make a plot of the exact and approximate solutions **on the same graph** and state **which is which** by including a legend (for instance, you can use a **solid black line** for the exact solution and **red open circles** for the approximate one). Calculate the infinity norm of the **global error**, i.e.,

$$||\mathbf{E}||_\infty = \max_{1\le j\le m} |U_j - u(x_j)|,$$

where U_j is the jth component of the approximate solution vector $\mathbf{U}$.

Attach your code, the figure and MATLAB output.