Name and section: ______________________________

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1. (30 points) Consider the **second-order** accurate finite difference formula for u''

$$D^2 u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}, \quad j = 0, 1, \ldots, m+1, \tag{1}$$

where $u_j \doteq u(x_j)$, $x_j = jh$ and $h = 1/(m+1)$. Furthermore, consider **periodic boundary conditions**

$$u(0) = u(1),$$

i.e., the values of the solution at the endpoints of the interval **are the same**. Note that at the discrete level $u_0 = u_{m+1}$ as well as $u_{-n} = u_{m+1-n}$ and $u_{m+1+n} = u_n$ hold $\forall n$ integer.

Then:

(a) (10 points) Write the **matrix representation** of Eq. (1) with periodic boundary conditions and denote your matrix by A. What about the dimension of A?

(b) (20 points) If $u_j^p = e^{2\pi i p j h}$ $(i = \sqrt{-1})$ stands for the jth component of the pth eigenvector of $A\mathbf{u}^p = \lambda_p \mathbf{u}^p$, show that the eigenvalues of your matrix obtained in part (a) are given by

$$\lambda_p = \frac{2}{h^2}\left[\cos\left(2\pi p h\right) - 1\right]. \tag{2}$$

2. (70 points) Write a MATLAB script (or a code in **any** programming language) to solve the **nonlinear** BVP:

$$\begin{aligned} u''(x) &= \frac{3}{2} u(x)^2, \quad x \in (0,1), \\ u(0) &= 4, \quad u(1) = 1, \end{aligned} \tag{5}$$

using finite differences with $m = 124$ interior points and **Newton's method**. Use as stopping criteria

$$||\mathbf{U}^{(k+1)} - \mathbf{U}^{(k)}||_2 < \epsilon \left(1 + ||\mathbf{U}^{(k)}||_2\right), \tag{6}$$

where $\epsilon = 10^{-10}$ and $\mathbf{U}^{(k)}$ stands for the approximate solution vector obtained at the kth Newton step. Make a plot of your initial guess and approximate solution (upon successful convergence) **on the same graph**. Please, state **which is which** by including a legend (for instance, you can use a **dashed-dotted black line** for the initial guess and **solid red line** for the approximate one).

Attach your code, the figure and MATLAB output. As per the latter, present a table with the following format:

- column 1: k (Newton's step)
- column 2: $||\mathbf{F}(\mathbf{U}^{(k)})||_2$ (ℓ2-norm of the residual)
- column 3: $||\mathbf{U}^{(k+1)} - \mathbf{U}^{(k)}||_\infty$ (infinity norm of the difference between successive iterates).

3. (100 points) (**Extra credit**) Consider the problem of fluid injection through one side of a long vertical channel. The Navier-Stokes and heat transfer equations can be reduced to the following nonlinear system:

$$
\begin{aligned}
& u''''(x) = R\left[u'(x)u''(x) - u(x)u'''(x)\right], \quad x \in (0,1), && (10)\\
& w''(x) + R\,u(x)w'(x) + 1 = 0, && (11)\\
& \theta''(x) + P\,u(x)\theta'(x) = 0, && (12)\\
& u(0) = u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 0, && \\
& w(0) = w(1) = 0, && \\
& \theta(0) = 0, \quad \theta(1) = 1, && (13)
\end{aligned}
$$

where $u(x)$ and $w(x)$ are two potential functions and $\theta(x)$ is the temperature distribution function. The parameters R and P correspond to the *Reynolds number* and *Peclet number*, respectively. Specifically, consider the values of $P = 0.7R$ and $R = 1000$. Subsequently, consider the **second-order** accurate finite difference formulas

$$
\begin{aligned}
D^4 u_j &= \frac{u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}}{h^4},\\
D^3 u_j &= \frac{-u_{j-2} + 2u_{j-1} - 2u_{j+1} + u_{j+2}}{2h^3},\\
D_0 u_j &= \frac{u_{j+1} - u_{j-1}}{2h}, \qquad (14)
\end{aligned}
$$

for u'''', u''' and u' at the jth grid point (use Eq. (1) for u'').

Eq. (10) is a **nonlinear problem** for u and can be solved again via **Newton's method** (same as Question 2). Recall that Newton's method requires the Jacobian at each iteration. However, and due to the complicated nature of the underlying **difference** equations, the Jacobian matrix can be **approximated** numerically using **finite differences**. To this end, utilize the MATLAB function `dfdjac.m` for this purpose with the following syntax:

```
Jx = dfdjac( fcn_name, u, epsilon );
```

where `fcn_name` is the function containing the nonlinear equations, `u` is the input vector corresponding to the **current iterate**, `epsilon` is the "width" of the finite difference representation employed therein (use `epsilon=1e-6`) and `Jx` is the (output) Jacobian matrix.

Then:

(a) Write a MATLAB script (or a code in **any** programming language) in order to solve the nonlinear BVP, i.e., Eq. (10) by considering $m = 124$ interior points and using as stopping criteria those of Question 2. Also present your results using exactly the same format as you did in Question 2.

(b) Upon successful convergence in part (a), write a MATLAB script (or a code in **any** programming language) to solve subsequently the **linear** BVPs, i.e., Eqs. (11) and (12) by considering $m = 124$ interior points again **and** the computed u.

(c) Make **three** separate graphs:

- The derivative of the potential function, i.e., $u'(x)$ as a function of x.
- The approximate solutions $w(x)$ and $\theta(x)$ both as functions of x.

Attach your codes, the figures and MATLAB output.