**Name and section:** ______________________________

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1. Use the known values of the function $f(x) = \sin(x)$ (with $y = f(x)$) at $x = 0, \pi/6, \pi/4, \pi/3$ and $\pi/2$ in order to derive an interpolating polynomial $p(x)$ using a **monomial basis**. What is the degree of your polynomial? What is the interpolation error magnitude $|p(1.2) - \sin(1.2)|$? Plot your data points and the underlying interpolating polynomial for $x \in [0, \pi/2]$ in the same figure. Attach any codes/scripts producing your figure and the figure itself.

2. Assume the data pairs $\{(x_i, y_i)\}_{i=0}^n$ together with the functions:

$$\begin{aligned}\rho_j &= \prod_{i \neq j} (x_j - x_i), \quad j = 0, 1, \ldots, n, \\ \psi(x) &= \prod_{i=0}^{n} (x - x_i).\end{aligned}$$

   (a) Show that:
   $$\rho_j = \psi'(x_j).$$

   (b) Show that the interpolating polynomial of degree at most $n$ is given by
   $$p_n(x) = \psi(x) \sum_{j=0}^{n} \frac{y_j}{(x - x_j)\,\psi'(x_j)}.$$

3. Let $f(x) = 1/x$ and data points $x_0 = 2$, $x_1 = 3$ and $x_2 = 4$. Note that you can use the abscissae to find the corresponding ordinates.

   (a) Find **by hand** the **Lagrange form**, the **standard form**, and the **Newton form** of the interpolating polynomial $p_2(x)$ of $f(x)$ at the given points. **State which is which!** Then, expand out the Newton and Lagrange form to verify that they agree with the standard form of $p_2$ that you obtained (this is true due to the uniqueness of polynomial interpolation). Also, verify that $p_2(x_i) = f(x_i)$ for $i = 0, 1, 2$.

   (b) Use the Polynomial Interpolation Error theorem to find an upper bound for the error
   $$||f - p_2||_\infty = \max_{2 \leq x \leq 4} |f(x) - p_2(x)|.$$

(c) Find the exact value of $||f - p_2||_\infty$ to at least 5 decimal places of accuracy. Of course, the answer should be less than or equal to the upper bound you found in part (b).

4. Consider the data set $\{x_i\}_{i=0}^n$ containing $(n+1)$ **distinct points** and the corresponding Lagrange basis functions $\{L_i(x)\}_{i=0}^n$. Then, prove that

$$\sum_{j=0}^{n} L_j(x) = 1.$$

*Hint: Consider interpolating the function $f(x) = 1$ at the points given!*

5. For some function $f$, the divided difference table is given:

| $i$ | $x_i$ | $f[\cdot]$ | $f[\cdot,\cdot]$ | $f[\cdot,\cdot,\cdot]$ | $f[\cdot,\cdot,\cdot,\cdot]$ |
|---|---|---|---|---|---|
| 0 | 1 | $f[x_0]$ | – | – | – |
| 1 | 5 | $f[x_1]$ | $f[x_0,x_1]$ | – | – |
| 2 | 6 | 4 | 0 | $-1/4$ | – |
| 3 | 4 | 2 | $f[x_2,x_3]$ | $f[x_1,x_2,x_3]$ | $f[x_0,x_1,x_2,x_3]$ |

Fill in the unknown entries in the table.