Monte Carlo parameter estimation in the Lorenz 63 system

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Outline

Introduction

- The point of this final project is to learn parameters in Lorenz63 using monte carlo methods. The approach taken is to formulate a Hamiltonian using the dynamic residuals, and then sample from the corresponding Gibbs distribution.
- Another similar idea is to estimate the state of the system given noisy observations (e.g. data assimilation).

Lorenz63 system

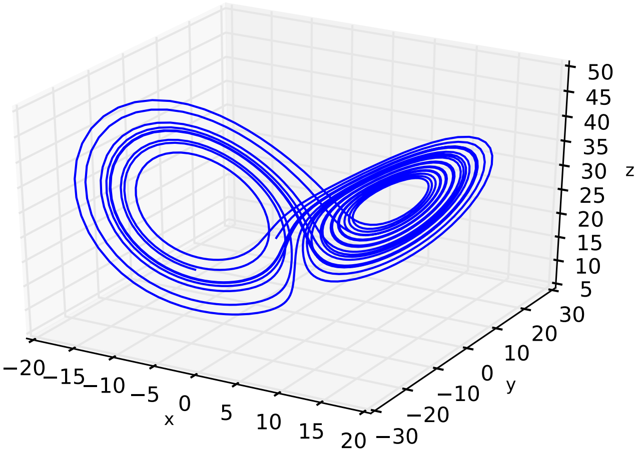
This system is one of the earliest examples of what would later become known as *chaotic* dynamical systems. It is obtained by projection of the Rayleigh-Benard equations for convection of a thin layer onto three orthogonal modes. The system is given by

$$\dot{X} = \sigma(Y - X) \tag{1}$$

$$\dot{Y} = -XZ + rX - Y \tag{2}$$

$$\dot{Z} = XY - bZ. \tag{3}$$

The parameters used by Lorenz are $\sigma = 10$, $b = 8/3$ and $r = 28$.



−20
−15
−10
−5
0
5
10
15
20
x
−30
−20
−10
0
10
20
30
y
5
10
15
20
25
30
35
40
45
50
z

Notation

- Let U be given by the dynamics

$$\dot{U} = F(U; \theta) + \epsilon \dot{W}$$

  where θ is some set of parameters.
- Let Ψ_t^θ be the *propagator*, defined by

$$\Psi_t^\theta U(s) = U(t+s).$$

- *Temporal grid*: Let $T > 0$, N, and define $\tau = T/(N+1)$. Then, a temporal grid $t_j = j\tau$ can be defined for $j = 0...N$.

Dynamic Residuals

Given a set of point estimates U_j and parameters, define the dynamic residual for a time t_j to be

$$e_j(\theta) := e(U_{j-1}, U_j, \theta) := \Psi_\tau^\theta U_{j-1} - U_j.$$

Then, a sensible way to choose the parameters is so that they minimize

$$\sum_{j=1}^{N} |e_j|^2.$$

Because θ is a potentially high dimensional object, Monte Carlo techniques can be used to perform this optimization.

Equilibrium distribution

- Vector of parameters $\theta = [\sigma, r, \beta]$
- Generate samples θ_k from

$$f(\theta) \propto \exp\left(\frac{\beta}{N}\sum_{j=1}^{N}|e_j(\theta)|^2\right).$$

- The mode of $f(\theta)$ is the value that minimizes the dynamic residuals

Data assimilation

Note that U_j is only in e_j and e_{j-1}, this can be used to simplify the optimization problem, and suggests some sort of resampling technique.

Proposal Distribution

- Independent gaussian proposal function:

$$P(\theta_0, \theta_1) \propto \exp\left(\frac{1}{2}(\theta_1 - \theta_0)^T C^{-1}(\theta_1 - \theta_0)\right)$$

- Covariance matrix: $C_{11} = .5^2$, $C_{22} = .1^2$, $C_{33} = 1.0^2$, $C_{ij} = 0$ if $i \neq j$.

Acceptance Probability

Metropolis-Hastings acceptance probability:

$$A(\theta_0, \theta_1) = \min\left\{1, \frac{f(\theta_1)P(\theta_1, \theta_0)}{f(\theta_0)P(\theta_0, \theta_1)}\right\}$$

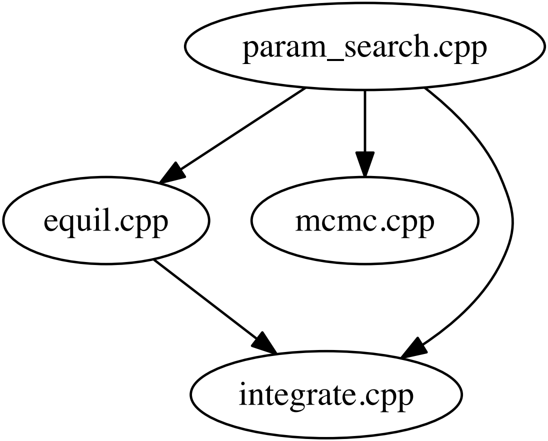
Languages and Libraries

- Language: C++ for sampler, Python for plotting/analysis
- C++ Libraries:
  - Armadillo++ for convenient vector arithmetic
  - GNU Scientific Library for random number generation

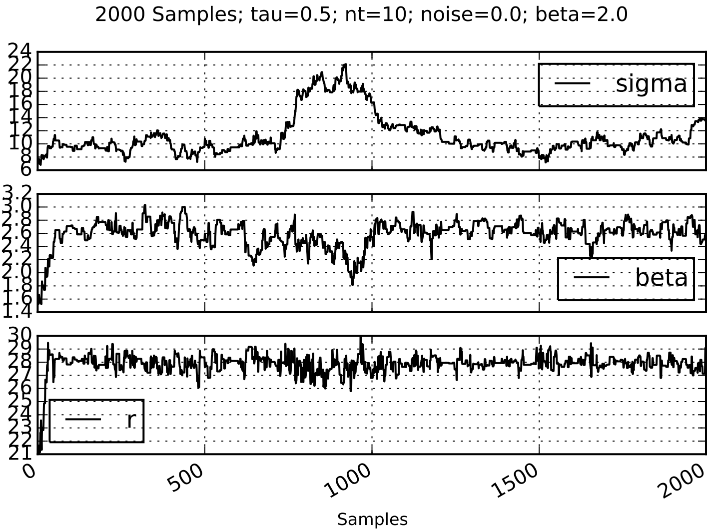
Code Architecture

- The integrator for the dynamical system is defined in `src/integrate.cpp`.
  - Second order explicit predictor/corrector method for deterministic dynamics
  - Forward Euler for the stochastic terms
- `src/equil.cpp` contains the equilibrium distribution
- The basic proposal distribution and metropolis acceptance function are defined in `src/mcmc.cpp`.
- `src/param_search.cpp` contains the parameter searching method.

Dependency Graph

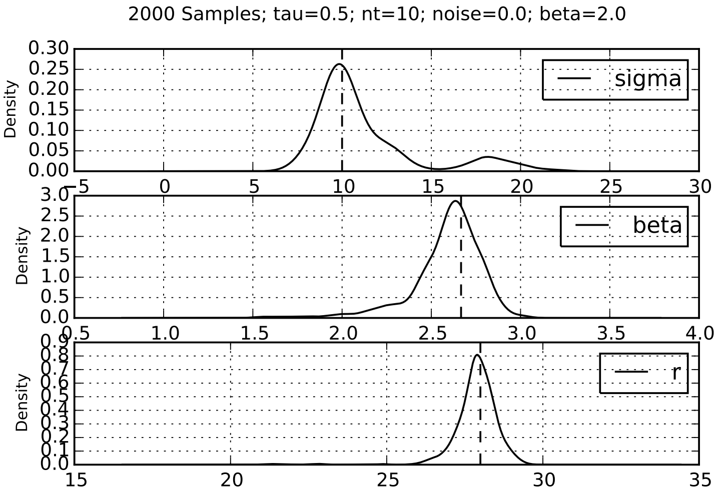


$\tau = .5$ Traces

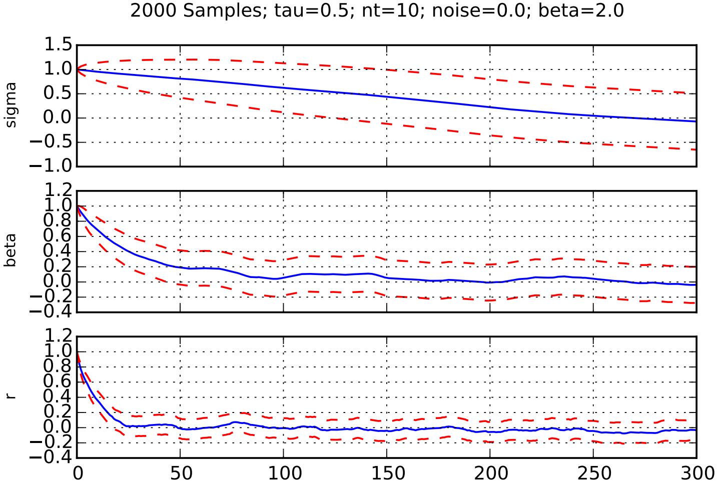


2000 Samples; tau=0.5; nt=10; noise=0.0; beta=2.0
sigma
beta
r
Samples
0
500
1000
1500
2000

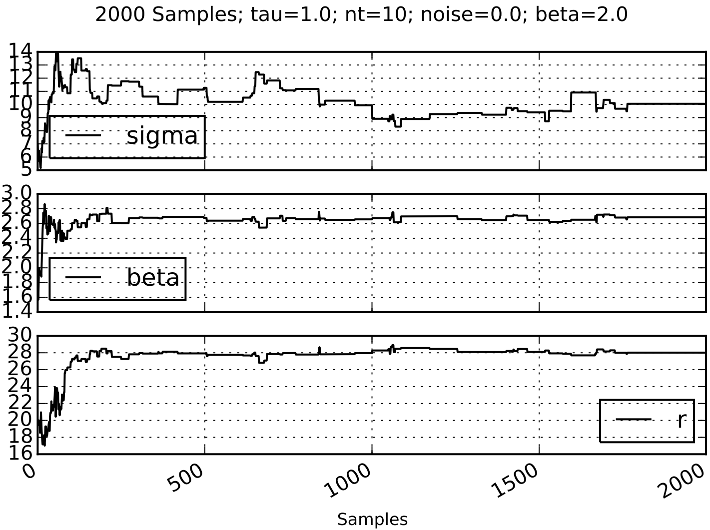
$\tau = .5$ Distributions



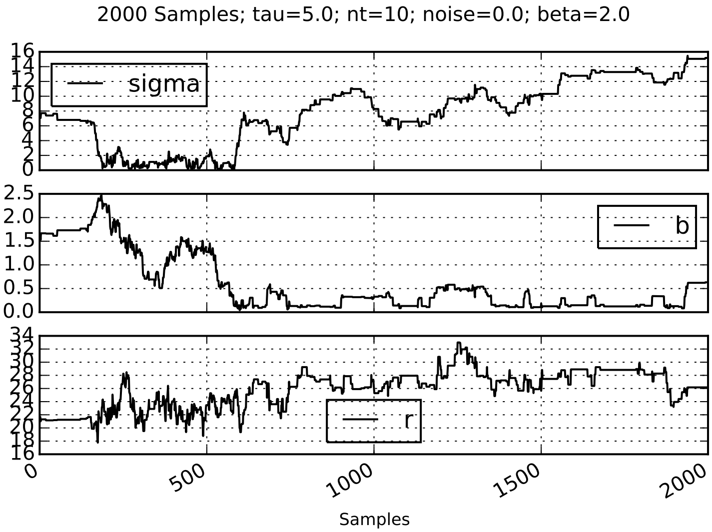
$\tau = .5$ Autocorrelation functions



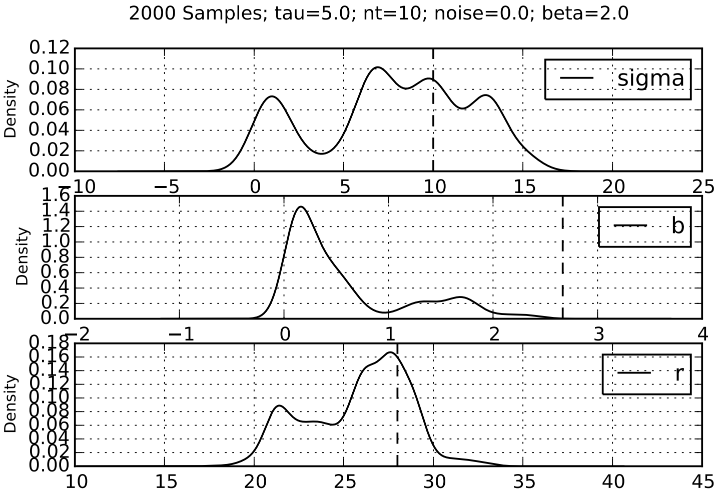
$\tau = 1.0$ Traces



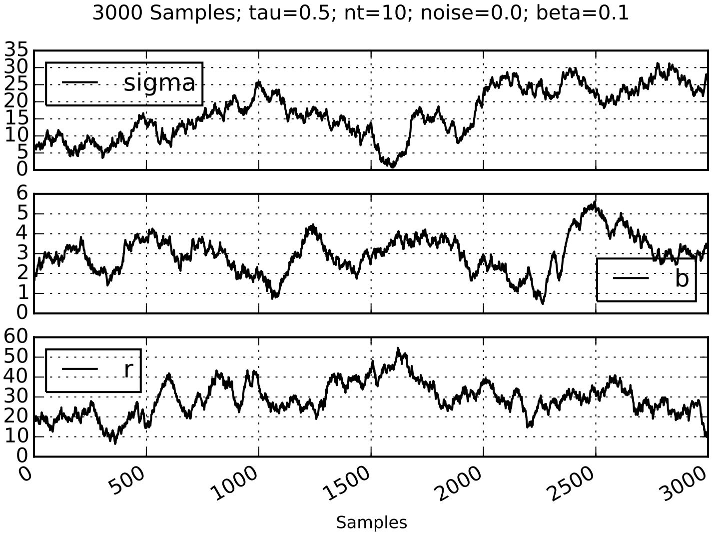
$\tau = 5.0$ Traces



$\tau = 5.0$ Distributions

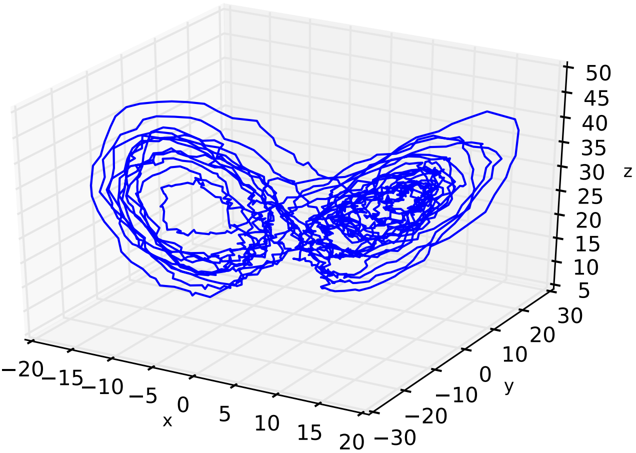


Reducing β gives large drift and autocorrelations

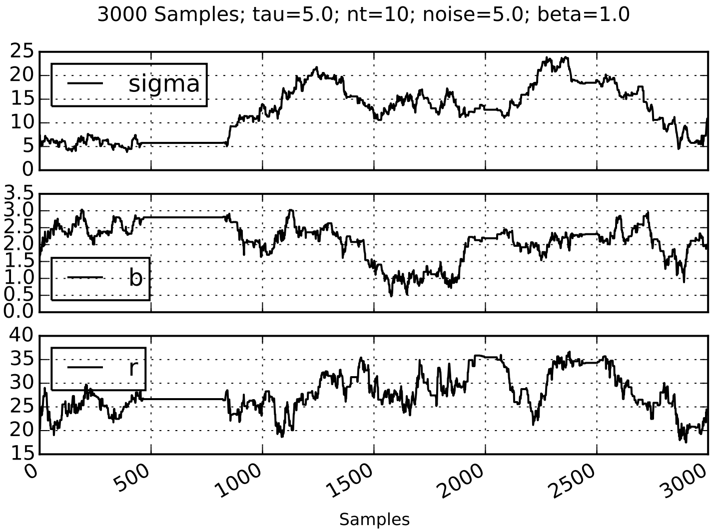


Model Error can help

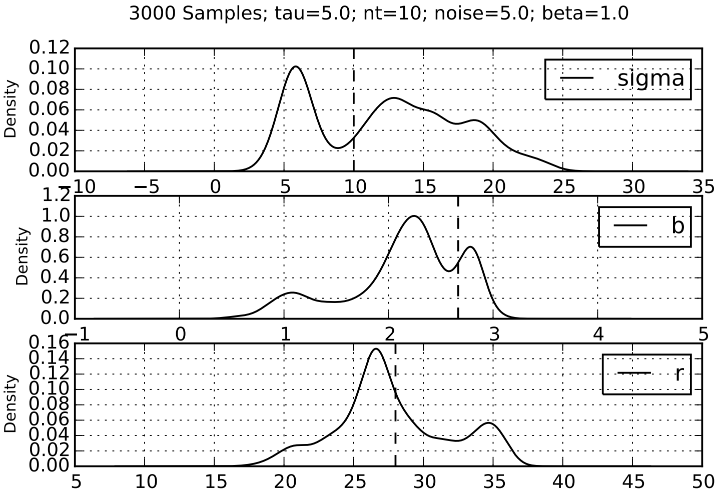
- Let $\epsilon = 5.0$



Model error can help a little



Model error can help a little



Conclusions

- Using the dynamic residuals method to find parameters works well for short observations times $\tau \leq .5$.
- MCMC performance is very poor for larger τ.
- Making a model error can help regularize the fit
- σ is the most difficult parameter to estimate

Future Directions

- Use dynamic residuals might work better for the data assimilation problem
- Parameter fitting might be better done by matching equilbrium *statistics* of the attractor rather than exact paths.