Monte Carlo parameter estimation in the Lorenz 63 system

Noah D. Brenowitz

May 12, 2015

1 Introduction

Operational and theoretical climate and weather prediction use a host of approximations known as *parameterizations* to account for sub-grid scale dynamics. Frequently these parameters are physically well motivated, but just as frequently these parameters are chosen an a *ad hoc* manner.

As a proof of concept, the point of this final project is to learn parameters in a simple chaotic dynamical system using Markov Chain Monte Carlo (MCMC) techniques. The approach used is to minimize the sum of squares of the *dynamic residuals* as proposed by Prof. Goodman in a conversation. As a simple prototype chaotic simple, we use the 3 dimensional ODE introduced in (Lorenz, 1963).

This system is one of the earliest examples of what would later become known as *chaotic* dynamical systems. It is obtained by projection of the Rayleigh-Benard equations for convection of a thin layer onto three orthogonal modes. The system is given by

$$\dot{X} = \sigma(Y - X) \tag{1}$$

$$\dot{Y} = -XZ + rX - Y \tag{2}$$

$$\dot{Z} = XY - bZ. \tag{3}$$

The parameters used by Lorenz are $\sigma = 10$, $b = 8/3$ and $r = 28$. The objective of this project is estimate these three parameters using Monte Carlo. This problem is the simplest prototype of parameter fitting in chaotic dynamical systems.

2 Parameter fitting procedure

The general outline here is to define a dynamics based loss function, and then samples from the associated Gibbs distribution. The loss function was proposed by Prof. Goodman in a meeting is sum of squares of the so-called *dynamic residuals*.

Let U be given by the dynamics

$$\dot{U} = F(U; \theta) + \epsilon \dot{W}$$

where θ is some set of parameters.

Let $T > 0$, N, and define $\tau = T/(N+1)$. Then, a temporal grid $t_j = j\tau$ can be defined for $j = 0 \ldots N$. Let Ψ_t^θ be the pushforward operator for deterministic part of the dynamics. Given a set of point estimates U_j and parameters, define the dynamic residual for a time t_j to be

$$e_j(\theta) := e(U_{j-1}, U_j, \theta) := \Psi_\tau^\theta U_{j-1} - U_j.$$

Then, a sensible way to choose the parameters is so that they minimize

$$H(\theta) = \frac{1}{N}\sum_{j=1}^{N} |e_j(\theta)|^2.$$

The corresponding Gibbs distribution is $f(\theta) \propto \exp\left(-\beta H(\theta)\right)$.

This Gibbs distribution is sampled using a Metropolis-Hastings algorithm with a proposal distribution that is independent in each direction. In other words,

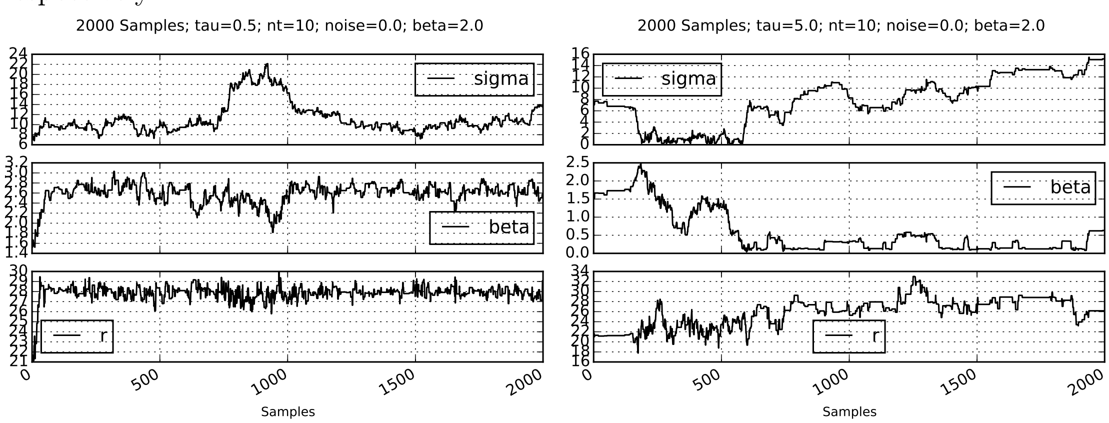
$$P(\theta_0, \theta_1) \propto \exp\left(-\frac{1}{2}(\theta_1 - \theta_0)^T C^{-1}(\theta_1 - \theta_0)\right),$$

where the covariance matrix is given by $C_{11} = .5^2$, $C_{22} = .1^2$, and $C_{33} = 1.0^2$, $C_{ij} = 0$ if $i \neq j$.

A reference value of $\beta = 2.0$ is used.

3 Results and Discussion

The traces of the sampler are shown here for short and long observation intervales of $\tau = .5$ and 5.0, respectively:



The performance of the method is strongly dependent on the observation time interval τ. Using the dynamic residuals method to find parameters works well for short observations times $\tau \leq .5$, but MCMC performance is very poor for larger τ regardless of the β parameter used. This makes sense if τ is approaching the predicatiblity limit of the system. Making a model error by introducing gaussian noise $\epsilon > 0$, can regularize the fit somewhat, but does not work miracles. Of all the parameters in Lorenz63, the exhange parameter σ is the most difficult to estimate and shows very long autocorrelation times compared to r and b.

Future directions include using the dynamics residuals approach for the state esimation problem, which might be more tractable given the simpler dependence of the $H(\theta)$ on the state estimates U_j. Since the parameters determine the shape of the Lorenz attractor, it probably makes more sense to pose the parameter-finding problem in terms of equilibirum statistics. The *dynamic residuals* approach is subject to the inherent predictability limit of the system and essentially misses the forest through the trees.

References

[1] Edward N. Lorenz. “Deterministic Nonperiodic Flow”. In: *Journal of the Atmospheric Sciences* 20.2 (Mar. 1963), pp. 130–141. DOI: 10.1175/1520-0469(1963)020<0130:dnf>2.0.co;2. URL: http://dx.doi.org/10.1175/1520-0469(1963)020%3C0130:DNF%3E2.0.CO;2.