Geometry of Homogeneous Spaces

1) Symmetric Spaces
2) Subgroups of Semisimple Groups
3) Critical exponents
4) Unitary reps.

Dedicated to
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Lecture 1

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Sur une classe remarquable d'espaces de Riemann

1) Lie algebra $\mathfrak{g}$

$\mathfrak{g}$ is simple if $\dim(\mathfrak{g}) > 1$ and it has no nontrivial ideals.

semisimple if $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ simple ideals.

<u>Ex</u> $\mathfrak{g} = \mathfrak{sl}(n, K) = \{M \in M_n(K) \mid tr(M) = 0\}$, $K = \mathbb{R}$ or $\mathbb{C}$

$\mathfrak{g} = \mathfrak{so}(p,q) = \{M \in M(p+q, \mathbb{R}) \mid M I_{p,q} + I_{p,q} M^t = 0\}$

$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$

$p+q \geq 5$.

1890: Killing classifies complex simple Lie algs:

$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

(Proof not entirely rigorous)

1895: Cartan's thesis gave rigorous proof.

Killing constructed G_2, but not E or F.

1914: Cartan classifies real simple Lie algebras.

<u>Def</u> The Killing form B on $\mathfrak{g}$ is

$B(X,Y) = tr(ad_X \circ ad_Y) \qquad ad_X(Z) = [X,Z]$.

<u>Fact</u>
- $\mathfrak{g}$ semisimple iff $\mathfrak{g}$ has no solvable ideals iff B is nondegenerate
- $\exists$ a Cartan involution $\Theta \in Aut(\mathfrak{g})$, $\Theta^2 = 1$

$$B(\Theta X, X) \leq 0 \quad \forall X \in \mathfrak{g}$$

$$\mathfrak{g} = \underset{\Theta=1}{\mathfrak{k}} \oplus \underset{\Theta=-1}{\mathfrak{s}}$$

1.2 Lie groups

Let G be a connected Lie group

$\mathfrak{g} = \mathrm{Lie}(G)$, dg the left Haar measure

$\mathrm{Ad}: G \to GL(\mathfrak{g})$ Adjoint representation

$\Delta: G \to \mathbb{R}_{>0}$ modular character

$$g \mapsto \det(\mathrm{Ad}_g)$$

$$\forall \varphi \in C_c(G) \qquad \int_G \varphi(gg_0)\,dg = \Delta(g_0)\int_G \varphi(g)\,dg$$

compactly supported function

G is unimodular if $\Delta \equiv 1$.

(*) Assume G semisimple + $Z(G) = 1$. (Consequently G unimodular)

Let $K \subseteq G$ the connected subgroup s.t. $\mathrm{Lie}(K) = \mathfrak{k}$.

Fact K is a maximal compact subgroup

$X = G/K$ has a G-inv't Riemannian metric given by the Killing form B on $\mathfrak{s} \cong T_K(G/K)$

Eg $G = SL_n\mathbb{R}$, $\Theta(g) = {}^t g^{-1}$.

1.3 Symmetric spaces

$$\mathcal{E} = \left\{ \begin{matrix} M \text{ complete simply connected} \\ \text{Riemannian mfld with } \nabla R = 0 \end{matrix} \right\}$$

$$\mathcal{E}_- = \left\{ M \in \mathcal{E} \;\middle|\; \begin{matrix} \text{nonpositive curvature} \\ \text{and no Euclidean factor} \end{matrix} \right\}$$

Thm (1926) $\left\{ \begin{matrix} G\ (*) \text{ with no} \\ \text{compact factor} \end{matrix} \right\} \xleftrightarrow{\text{bijection}} \mathcal{E}_-$

$$G \longleftrightarrow G/K$$

Cor $G = SK$, $S = \exp(\mathfrak{s})$
Cartan decomposition

Ex $G = SL_n\mathbb{R}$, $K = SO_n$, $S = \left\{ g \in G \;\middle|\; g \begin{matrix} \text{symmetric} \\ + \text{positive definite} \end{matrix} \right\}$

1.4 Cartan decomposition

Def A Cartan subspace $\mathfrak{a} \subseteq \mathfrak{s}$ is a maximal abelian Lie subalgebra

Fact All such $\mathfrak{a}$ are conjugate by K
$\operatorname{rank}_{\mathbb{R}}(G) = r = \dim(\mathfrak{a})$

For $\alpha \in \mathfrak{a}^*$, $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a}\ [H, X] = \alpha(H) X\}$

- $\Sigma = \{\alpha \neq 0 \mid \mathfrak{g}_\alpha \neq 0\}$ "restricted roots"
- $\mathring{\mathfrak{a}}_+$ a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker(\alpha)$
- $\mathfrak{a}_+ = \overline{\mathring{\mathfrak{a}}_+}$ a Weyl chamber

<u>Thm</u> Every $g \in G$ can be written $g = k_1 e^X k_2$ for $k_1, k_2 \in K$, $X \in \mathfrak{a}_+$

X is unique, so we get $\kappa : G \to \mathfrak{a}_+$
Cartan projection.

<u>Eg</u> $G = SL_n \mathbb{R}$

$$\kappa(g) = \operatorname{diag}\left(\log\|g\|, \log\frac{\|\Lambda^2 g\|}{\|g\|}, \log\frac{\|\Lambda^3 g\|}{\|\Lambda^2 g\|}, \dots\right)$$

$\in \operatorname{End}(\Lambda^k \mathbb{R}^n)$

Formula for dg: $\int_G \varphi(g)\, dg$

$$= \int_{K\mathfrak{a}_+K} \varphi(k_1 e^X k_2) \prod_{\alpha \in \Sigma_+} \sinh(\alpha(X))^{m_\alpha}\, dk_1\, dX\, dk_2$$