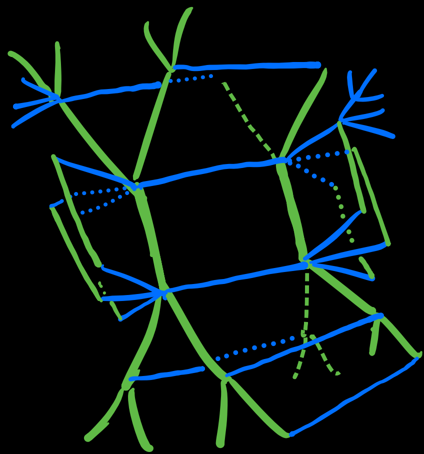
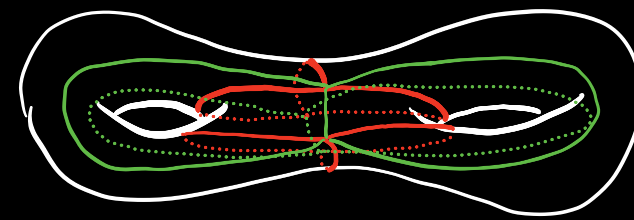
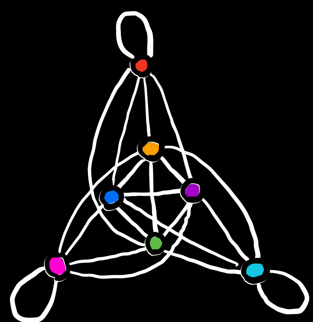
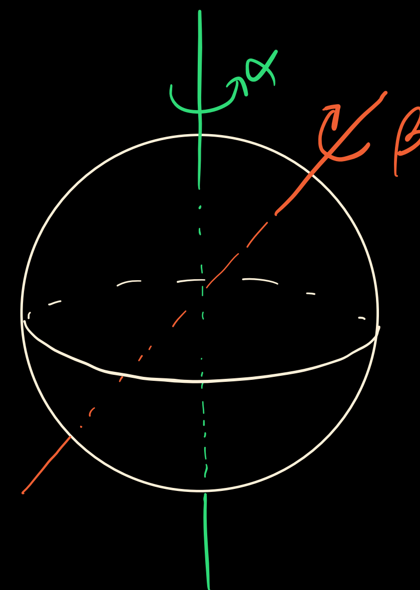
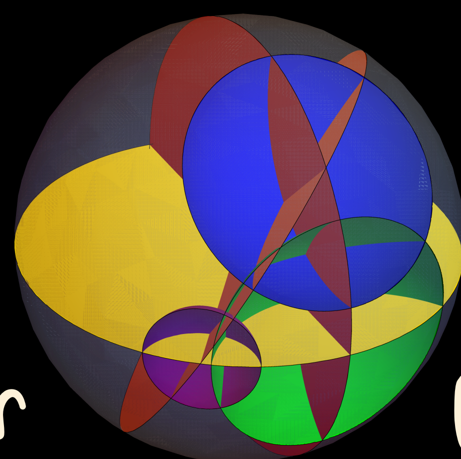




Subgroups of $SO_3(\mathbb{Q})$







Geometry & Topology Seminar

Yale University

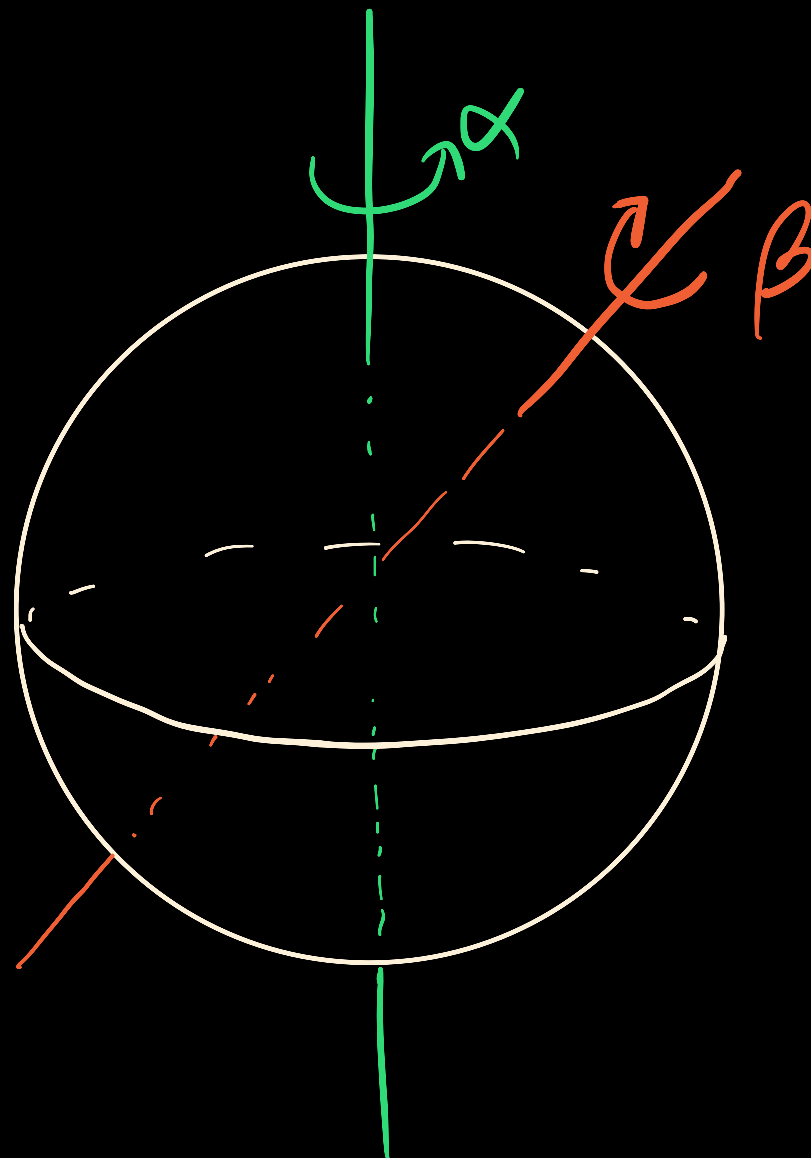
Nic Brody

UC Santa Cruz

April 25, 2023

Central question

Let A be a finite set of rotations of the 2-sphere. What is the group generated by A?



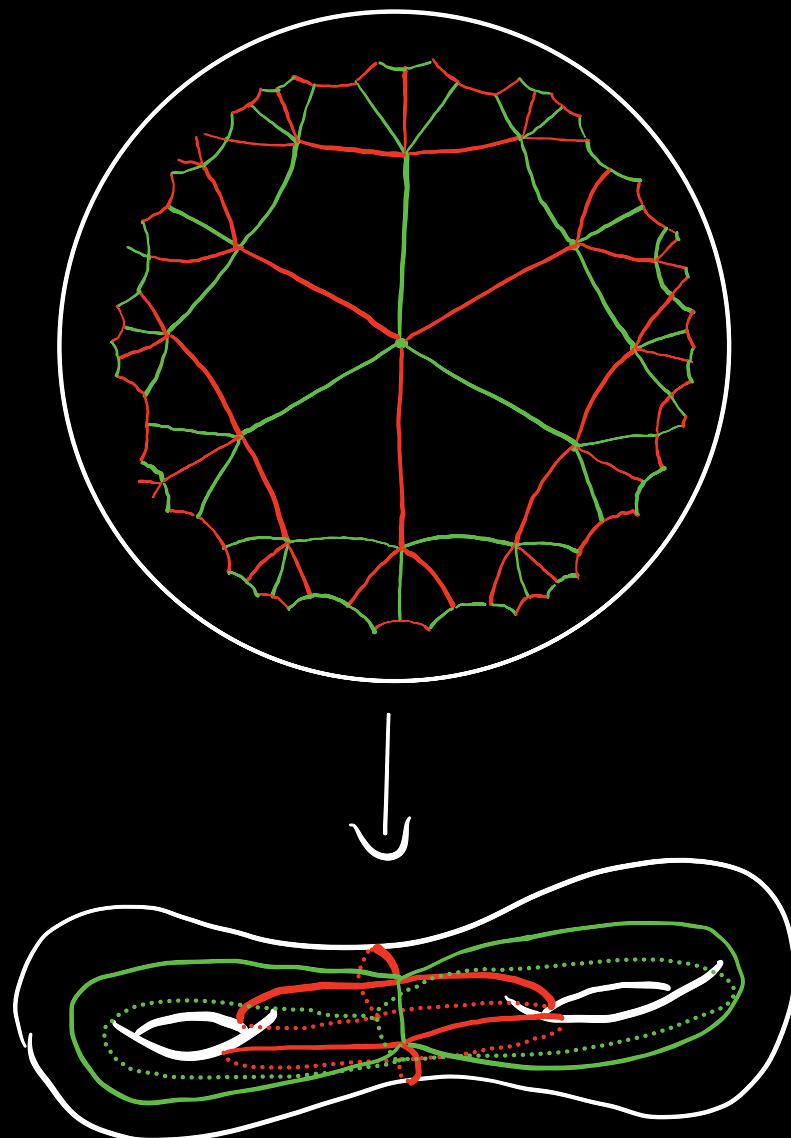
α
β

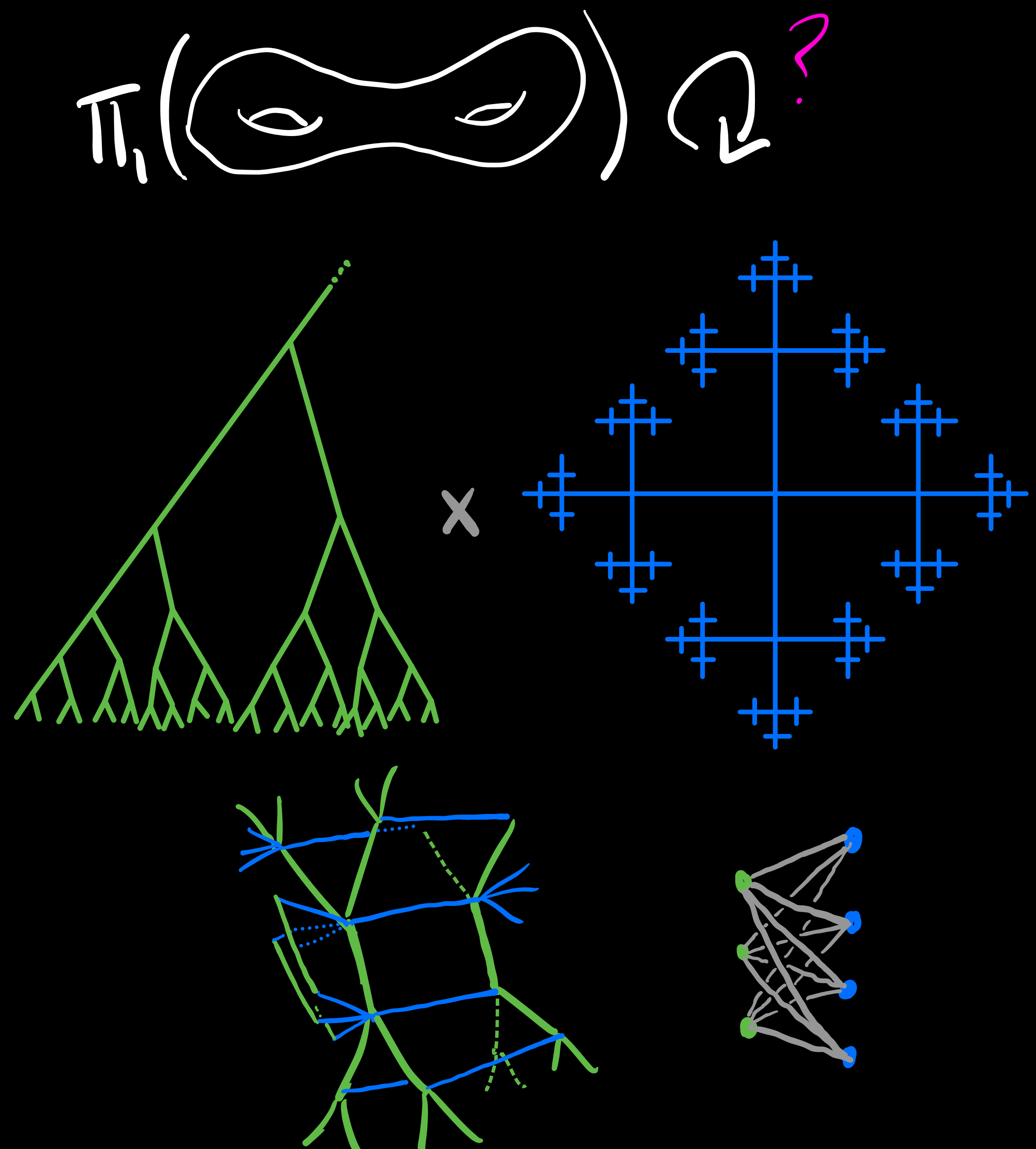
My goal for today's talk is to describe how thinking about one problem led me to another.

Q 1 Can a hyperbolic surface group act properly on a locally compact product of trees?

Def A *jungle* is a locally compact product of trees

A surface group




π_1
?
×

It is possible to obtain proper actions of surface groups on

1) finite products of locally infinite trees
filling multicurves

2) infinite products of locally finite trees.
fully residually free

(Fisher-Larsen-Spatzier-Stover)

but it is unclear if one can take a finite product of locally finite trees

There is a "folklore" action of a surface group on $T_3 \times T_4$, suggested by Long + Reid in '99, which is faithful but it is unknown if it has vertex stabilizers.

They consider the fundamental group of the hyperbolic orbifold

$\Sigma =$ 

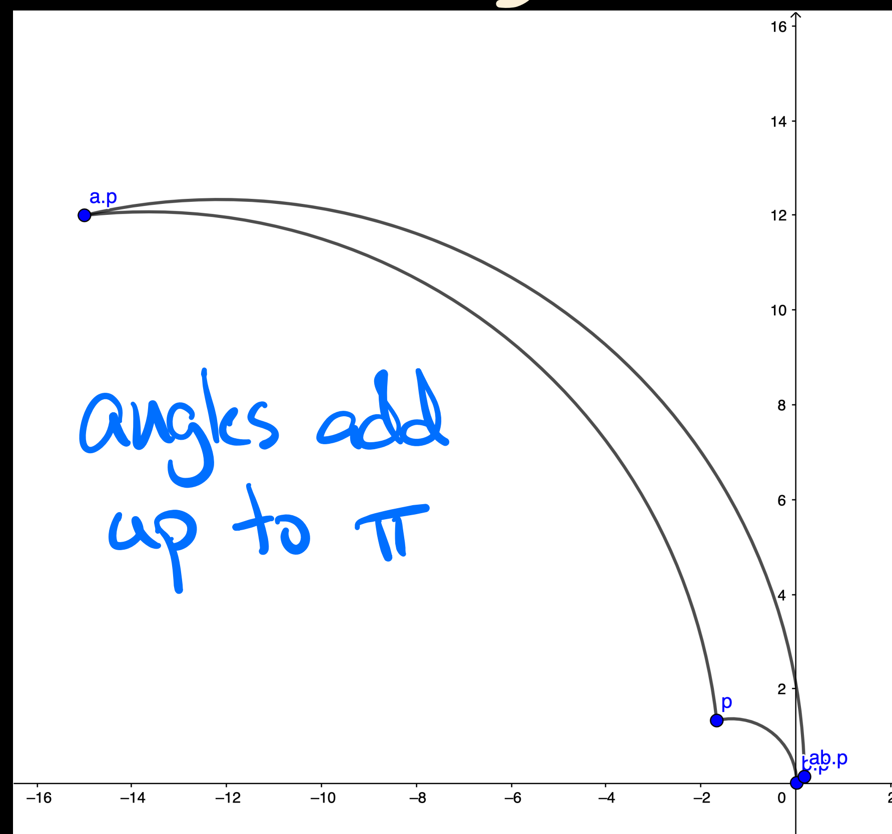
$$G = \pi_1^{orb}(\Sigma) = \langle a, b \mid [a,b]^2 = 1 \rangle$$

They find a particular representation

$G \to PSL_2 \mathbb{R}$ defined by

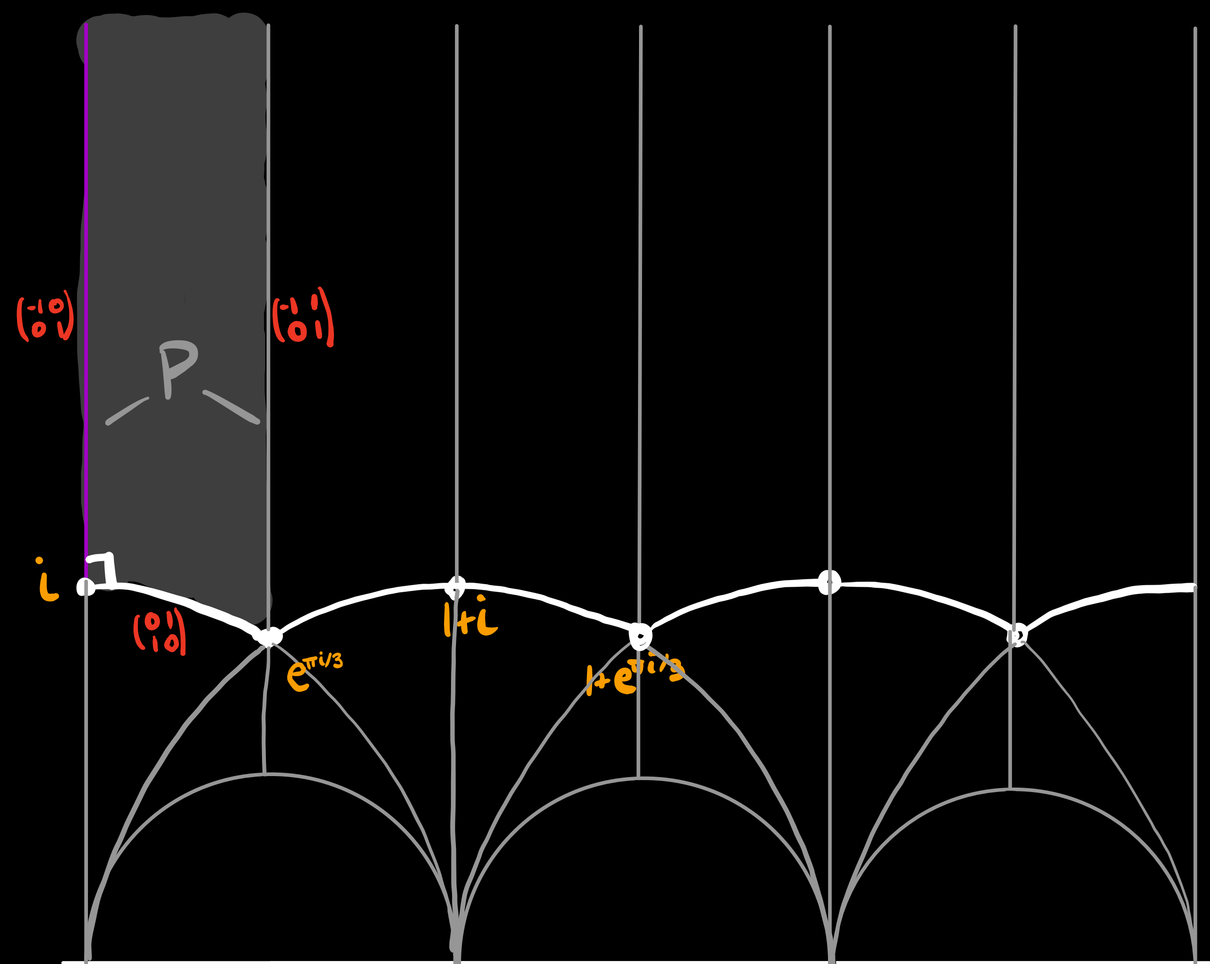
$$a \mapsto \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} 1/8 & 9 \\ 2/64 & 82/8 \end{pmatrix}$$

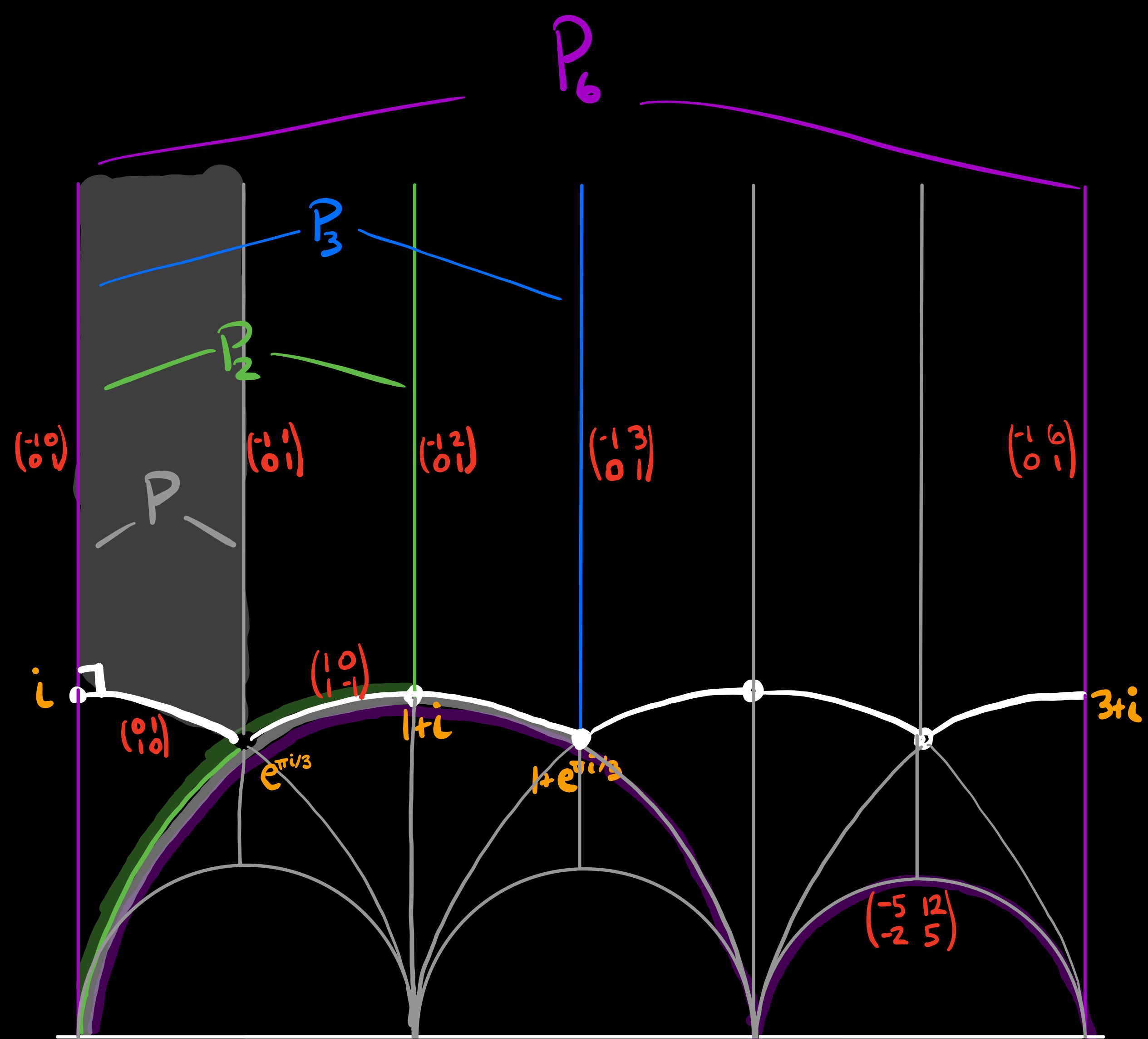


Since this representation corresponds to a hyperbolic structure on the orbifold, it is guaranteed to be a faithful rep into $PSL_2(\mathbb{Z}[\frac{1}{6}])$

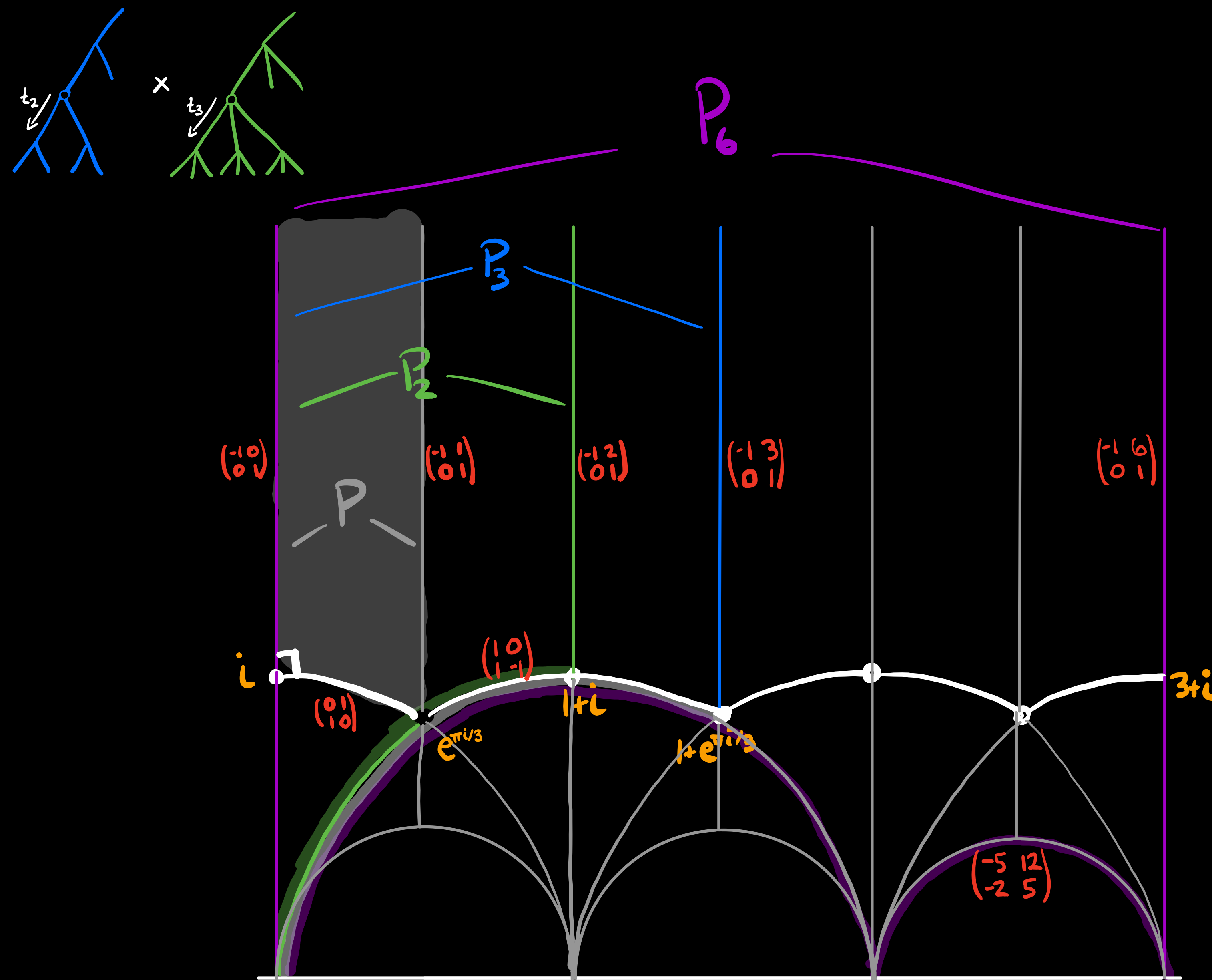
Recall $PSL_2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$



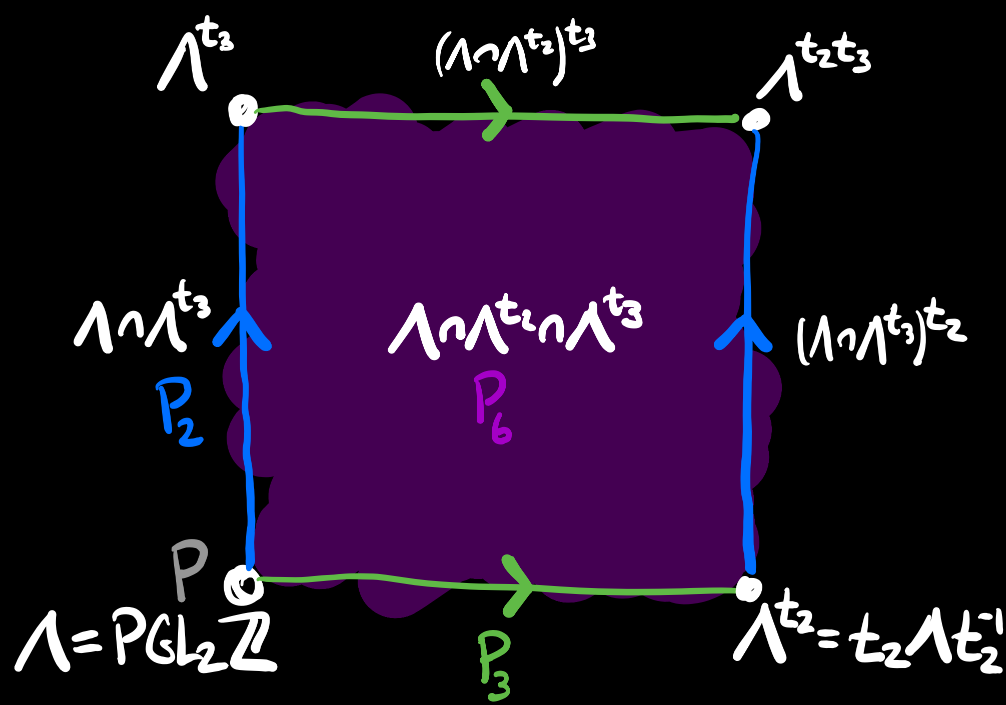
P
i
1+i
$e^{\pi i/3}$
$1+e^{\pi i/3}$
$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$
$\begin{pmatrix}-1&1\\0&1\end{pmatrix}$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$



P6
P3
P2
P
(-1 0 / 0 1)
(-1 1 / 0 1)
(-1 2 / 0 1)
(-1 3 / 0 1)
(-1 6 / 0 1)
(1 0 / 1 -1)
(0 1 / 1 0)
(-5 12 / -2 5)
i
e^πi/3
1+i
1+e^πi/3
3+i

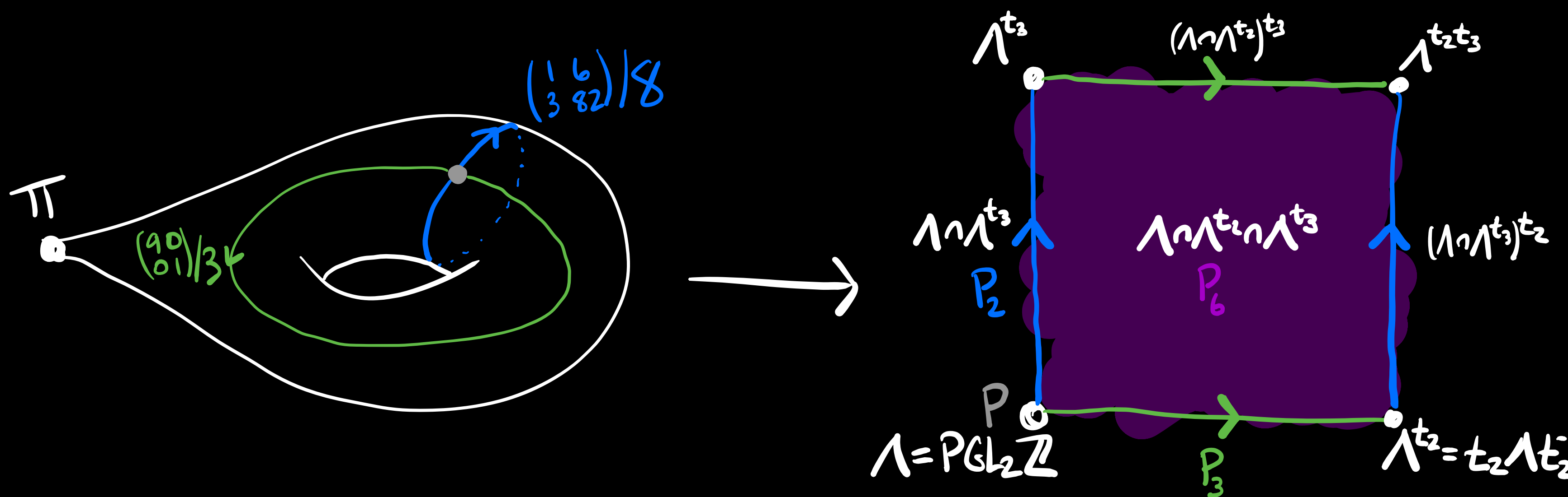


t_2
×
t_3
P_6
P_3
P_2
P
$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$
$\begin{pmatrix}-1&1\\0&1\end{pmatrix}$
$\begin{pmatrix}-1&2\\0&1\end{pmatrix}$
$\begin{pmatrix}-1&3\\0&1\end{pmatrix}$
$\begin{pmatrix}-1&6\\0&1\end{pmatrix}$
$\begin{pmatrix}1&0\\1&-1\end{pmatrix}$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$\begin{pmatrix}-5&12\\-2&5\end{pmatrix}$
i
$1+i$
$e^{\pi i/3}$
$1+e^{\pi i/3}$
$3+i$



$$PSL_2\mathbb{Z}[\tfrac{1}{6}] \leq PSL_2\mathbb{R} \times PSL_2\mathbb{Q}_2 \times PSL_2\mathbb{Q}_3$$

acts properly on $\mathbb{H}^2 \times T_3 \times T_4$



π
(9 0 / 0 1)/3
(1 6 / 3 82)/8
Λ^{t₃}
(Λ∩Λ^{t₂})^{t₃}
Λ^{t₂t₃}
Λ∩Λ^{t₃}
Λ∩Λ^{t₂}∩Λ^{t₃}
(Λ∩Λ^{t₃})^{t₂}
P₂
P₆
P
Λ=PGL₂ℤ
P₃
Λ^{t₂}=t₂Λt₂⁻¹

Now Long-Reid ask:

If $g \in G$ has infinite order, is $tr(g) \notin \mathbb{Z}$?

If this always holds, then this determines a proper action of G on $T_3 \times T_4$!

Computing traces of millions of group elements, everybody has non-integral trace!

To understand this, we now discuss $\underline{PGL_2\mathbb{Q}_p \circlearrowright T_{p+1}}$

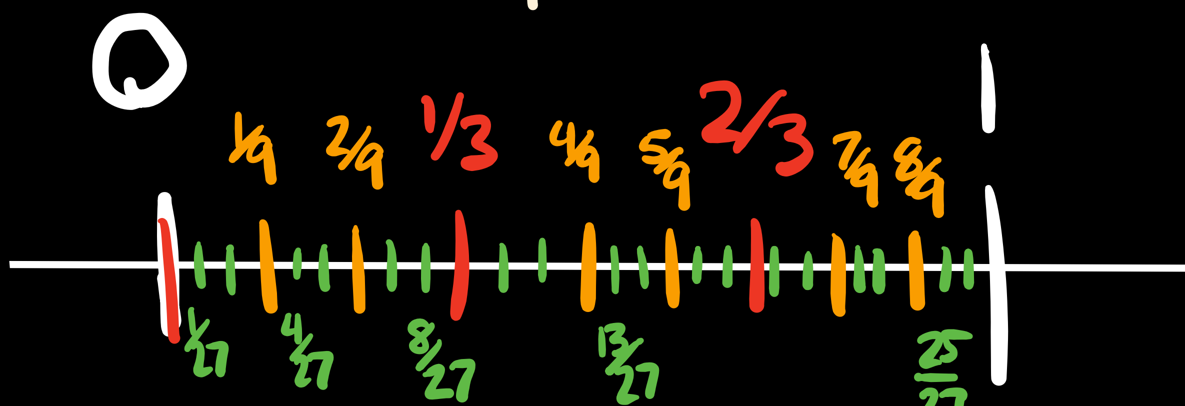
(p+1)-regular tree

We will only consider subgroups of $PSL_2\mathbb{Q}$ acting on the tree, and we can accomplish everything without explicit use of p-adic numbers

:25

First, consider the set

$$\mathbb{Z}[1/p]/\mathbb{Z}$$



$$\frac{25}{27} = \frac{2}{3} + \frac{2}{9} + \frac{1}{27}$$

We can write a general $q \in \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as $q = \sum_{i=m}^{-1} a_i p^i$, where $a_i \in \{0, \dots, p-1\}$, $m < 0$

Similarly, $\mathbb{Z}[\frac{1}{p}]/p^n\mathbb{Z}$ consists of "n-truncated p-rational numbers" $\lfloor q \rfloor_n$

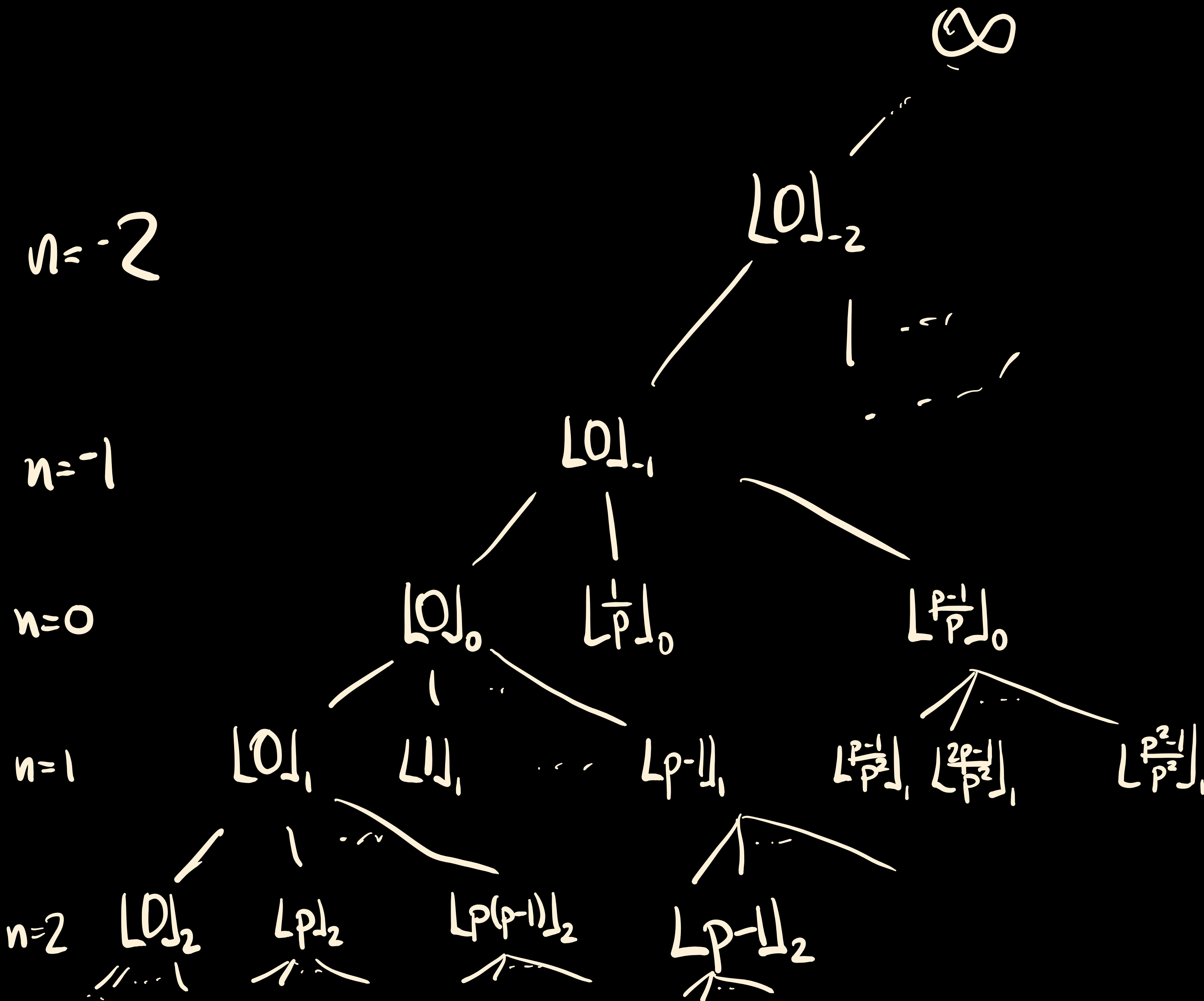
There is a natural (p-to-1) map

$$\mathbb{Z}[\tfrac{1}{p}]/p^n\mathbb{Z} \to \mathbb{Z}[\tfrac{1}{p}]/p^{n-1}\mathbb{Z}$$

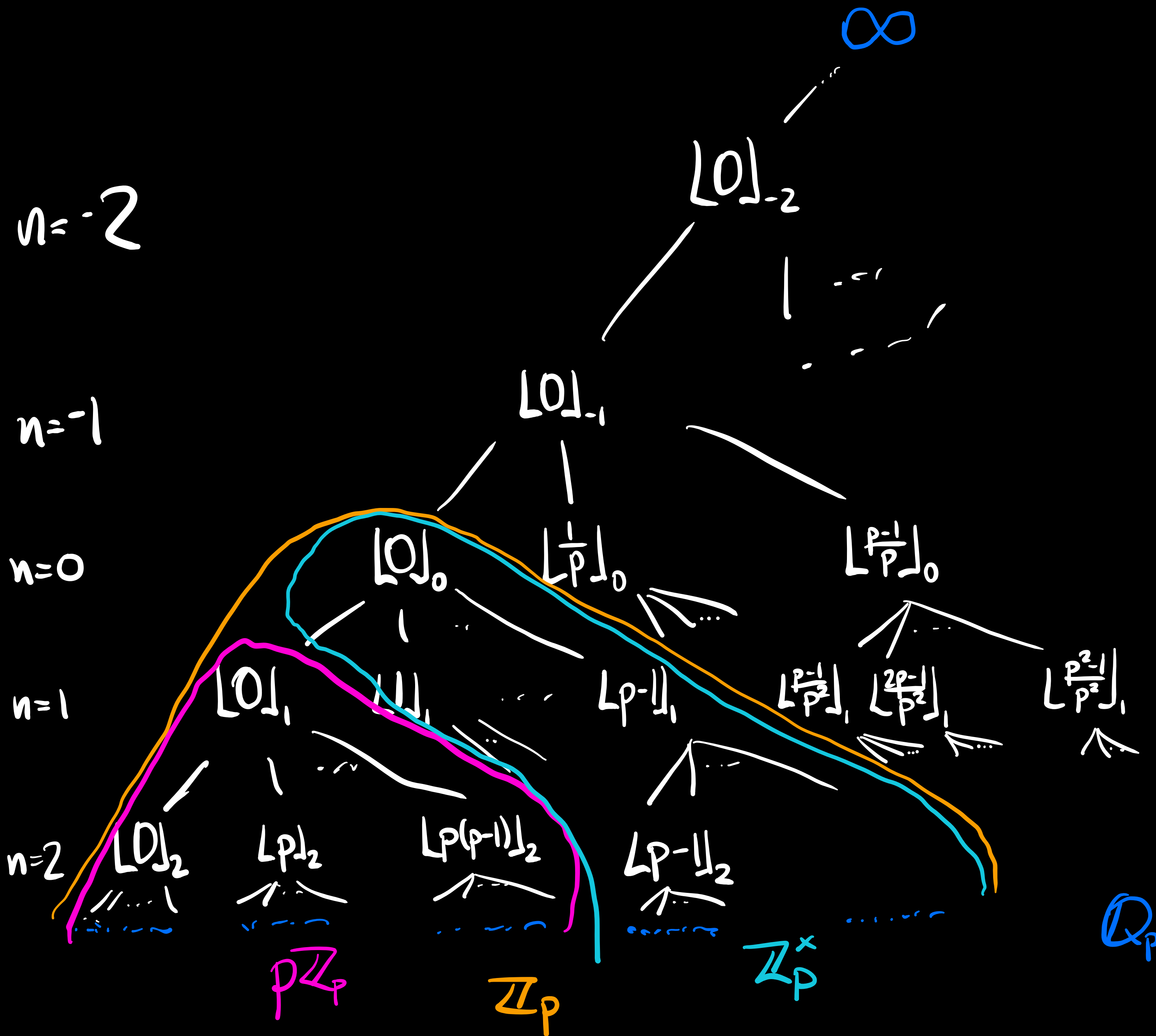
for each n, which just truncates one term earlier.

We obtain a $(p+1)$-regular tree from this data, whose vertices are labeled by $\coprod_{n\in\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]/p^n\mathbb{Z}$

$$\begin{tikzcd}[column sep=small]
 & & \lfloor q \rfloor_{n-1} & \\
 & \lfloor q \rfloor_n \arrow[ur, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\lfloor q \rfloor_{n+1} & \lfloor q+p^n \rfloor_{n+1} & \cdots & \lfloor q+(p-1)p^n \rfloor_{n+1}
\end{tikzcd}$$



∞
n = -2
$\lfloor 0 \rfloor_{-2}$
n = -1
$\lfloor 0 \rfloor_{-1}$
n = 0
$\lfloor 0 \rfloor_0$
$\lfloor \frac{1}{p} \rfloor_0$
$\lfloor \frac{p-1}{p} \rfloor_0$
n = 1
$\lfloor 0 \rfloor_1$
$\lfloor 1 \rfloor_1$
$\lfloor p-1 \rfloor_1$
$\lfloor \frac{p-1}{p^2} \rfloor_1$
$\lfloor \frac{2p-1}{p^2} \rfloor_1$
$\lfloor \frac{p^2-1}{p^2} \rfloor_1$
n = 2
$\lfloor 0 \rfloor_2$
$\lfloor p \rfloor_2$
$\lfloor p(p-1) \rfloor_2$
$\lfloor p-1 \rfloor_2$


∞
$\lfloor 0 \rfloor_{-2}$
$n=-2$
$n=-1$
$\lfloor 0 \rfloor_{-1}$
$n=0$
$\lfloor 0 \rfloor_0$
$\lfloor \frac{1}{p} \rfloor_0$
$\lfloor \frac{p-1}{p} \rfloor_0$
$n=1$
$\lfloor 0 \rfloor_1$
$\lfloor 1 \rfloor_1$
$\lfloor p-1 \rfloor_1$
$\lfloor \frac{p-1}{p^2} \rfloor_1$
$\lfloor \frac{2p-1}{p^2} \rfloor_1$
$\lfloor \frac{p^2-1}{p^2} \rfloor_1$
$n=2$
$\lfloor 0 \rfloor_2$
$\lfloor p \rfloor_2$
$\lfloor p(p-1) \rfloor_2$
$\lfloor p-1 \rfloor_2$
$p\mathbb{Z}_p$
$\mathbb{Z}_p$
$\mathbb{Z}_p^\times$
$\mathbb{Q}_p$

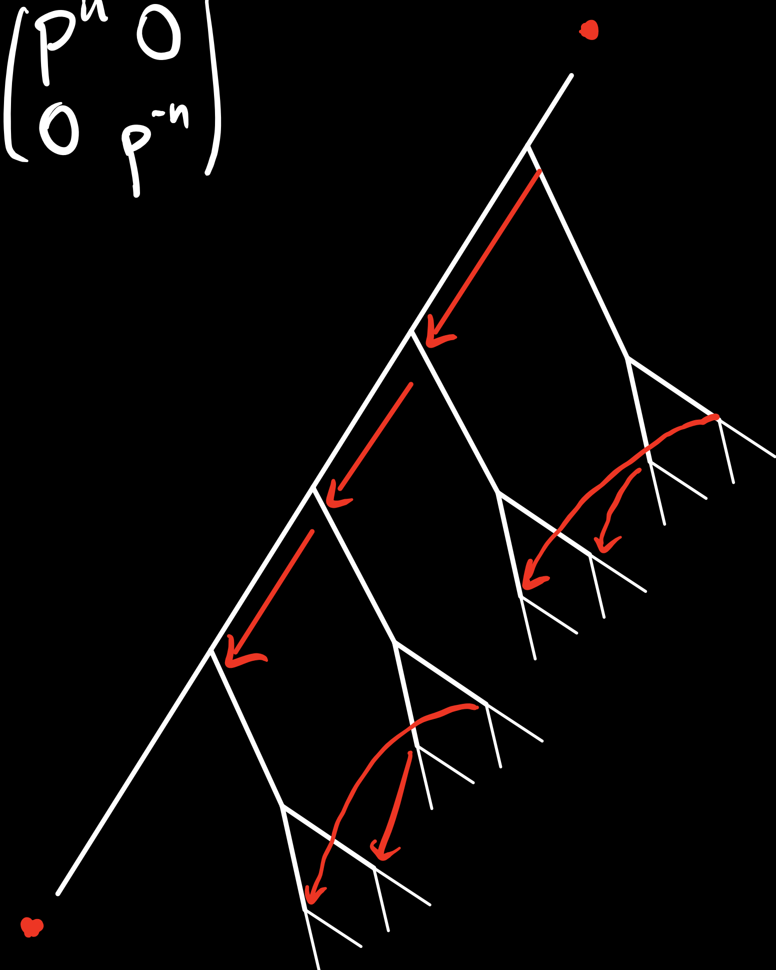
Now, much like $PSL_2\mathbb{R} \curvearrowright \mathbb{H}^2$, we have an action $PSL_2\mathbb{Q}_p \curvearrowright T_{p+1}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \lfloor q \rfloor_n = \begin{cases} \left\lfloor \frac{aq+b}{cq+d} \right\rfloor_{n-2v_p(cq+d)} & \text{if } v_p(cq+d) \leq v_p(cp^n) \\ \left\lfloor \frac{a}{c} \right\rfloor_{-n-2v_p(c)} & \text{if } v_p(cq+d) \geq v_p(cp^n) \end{cases}$$

Thm Classification of elements of $PSL_2 \mathbb{Q}_p$

1) Hyperbolic   Fixes no point in the tree

$$\sim \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix}$$

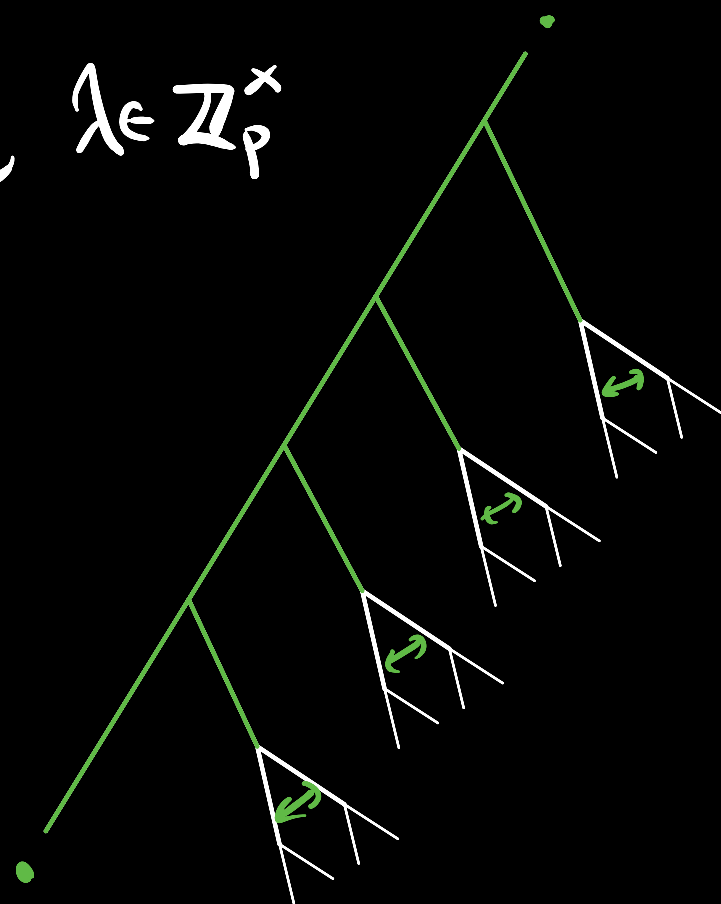


Thm Classification of elements of $PSL_2\,\mathbb{Q}_p$

1) Hyperbolic — Fixes no point in the tree

2) Loxodrom-ish elliptic — Fixes a point in the tree and 2 points in ∂T

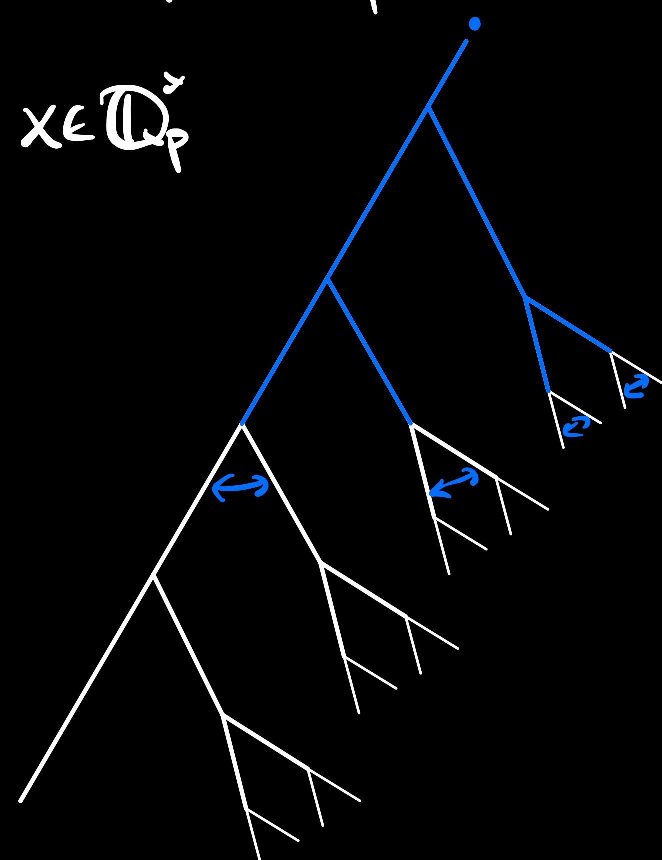
$\sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{Z}_p^\times$



Thm Classification of elements of $PSL_2 \mathbb{Q}_p$

1) Hyperbolic — Fixes no point in the tree
2) Loxodrom-ish elliptic — Fixes a point in the tree and 2 points in ∂T
3) Parabol-ike elliptic — Fixes a point in the tree and 1 point in ∂T

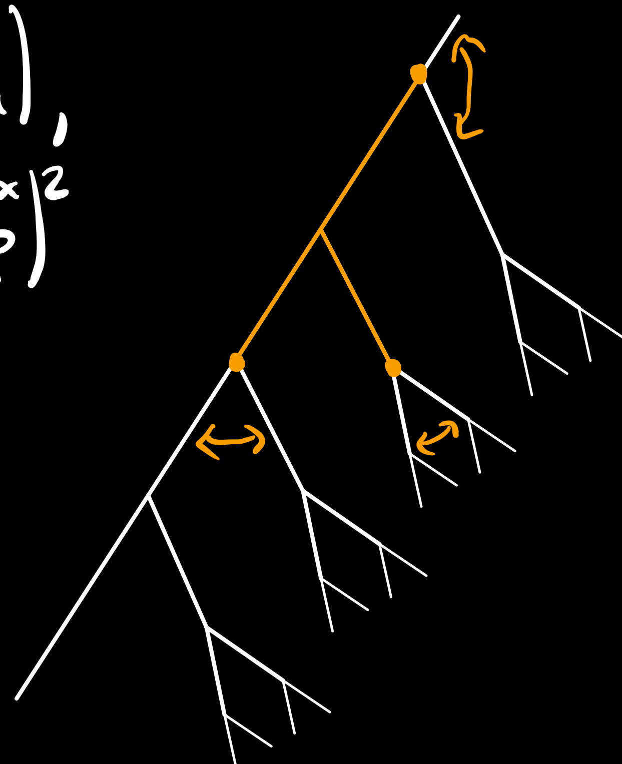
$\sim \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ x \in \mathbb{Q}_p^\times$



Thm (B) Classification of elements of $PSL_2\,\mathbb{Q}_p$

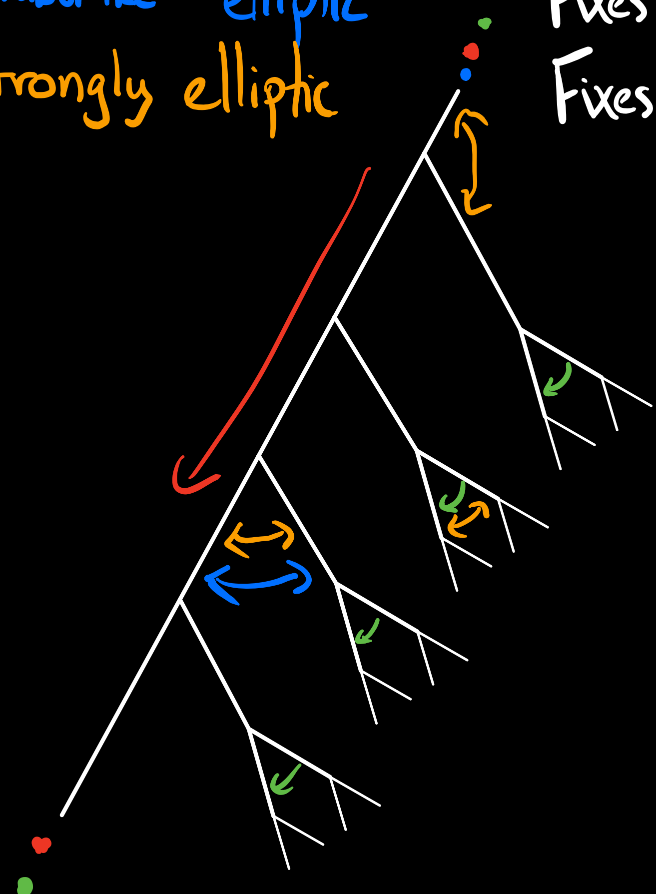
1) Hyperbolic — Fixes no point in the tree
2) Loxodrom-ish elliptic — Fixes a point in the tree and 2 points in ∂T
3) Parabol-ike elliptic — Fixes a point in the tree and 1 point in ∂T
4) Strongly elliptic — Fixes a point in the tree and 0 points in ∂T

$$\sim \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}, \quad a^2-4 \notin (\mathbb{Q}_p^\times)^2$$



Thm Classification of elements of $PSL_2 \mathbb{Q}_p$

1) Hyperbolic — Fixes no point in the tree
2) Loxodrom-ish elliptic — Fixes a point in the tree and 2 points in ∂T
3) Parabol-ike elliptic — Fixes a point in the tree and 1 point in ∂T
4) Strongly elliptic — Fixes a point in the tree and 0 points in ∂T



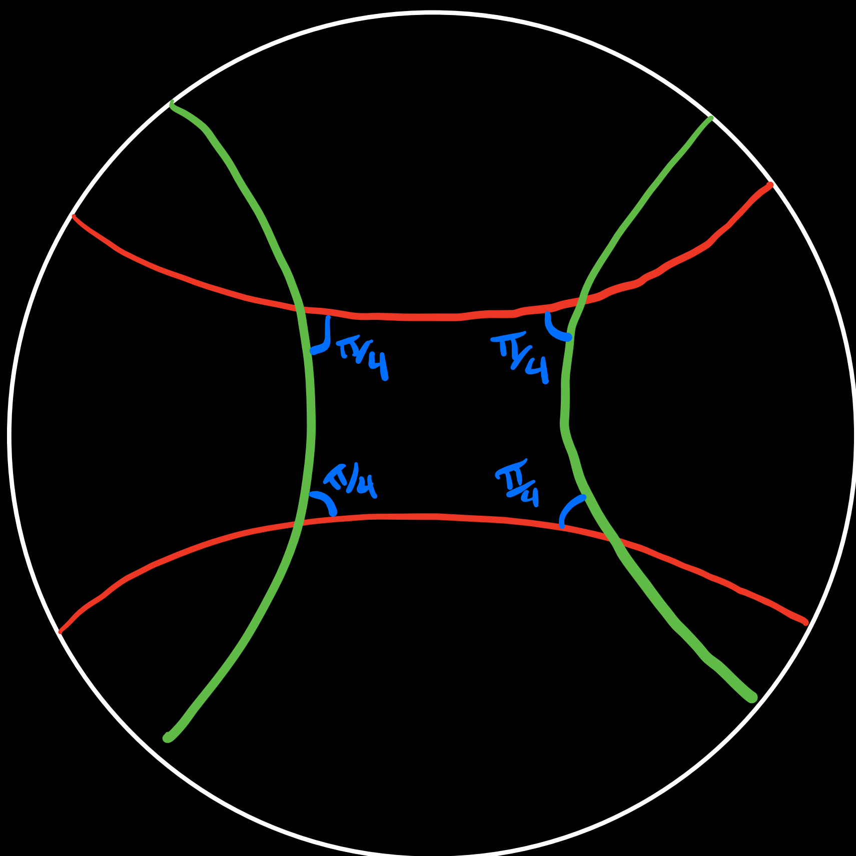
This action is easy to work with in these coordinates – one can quickly compute

$$g \cdot [q]_n$$

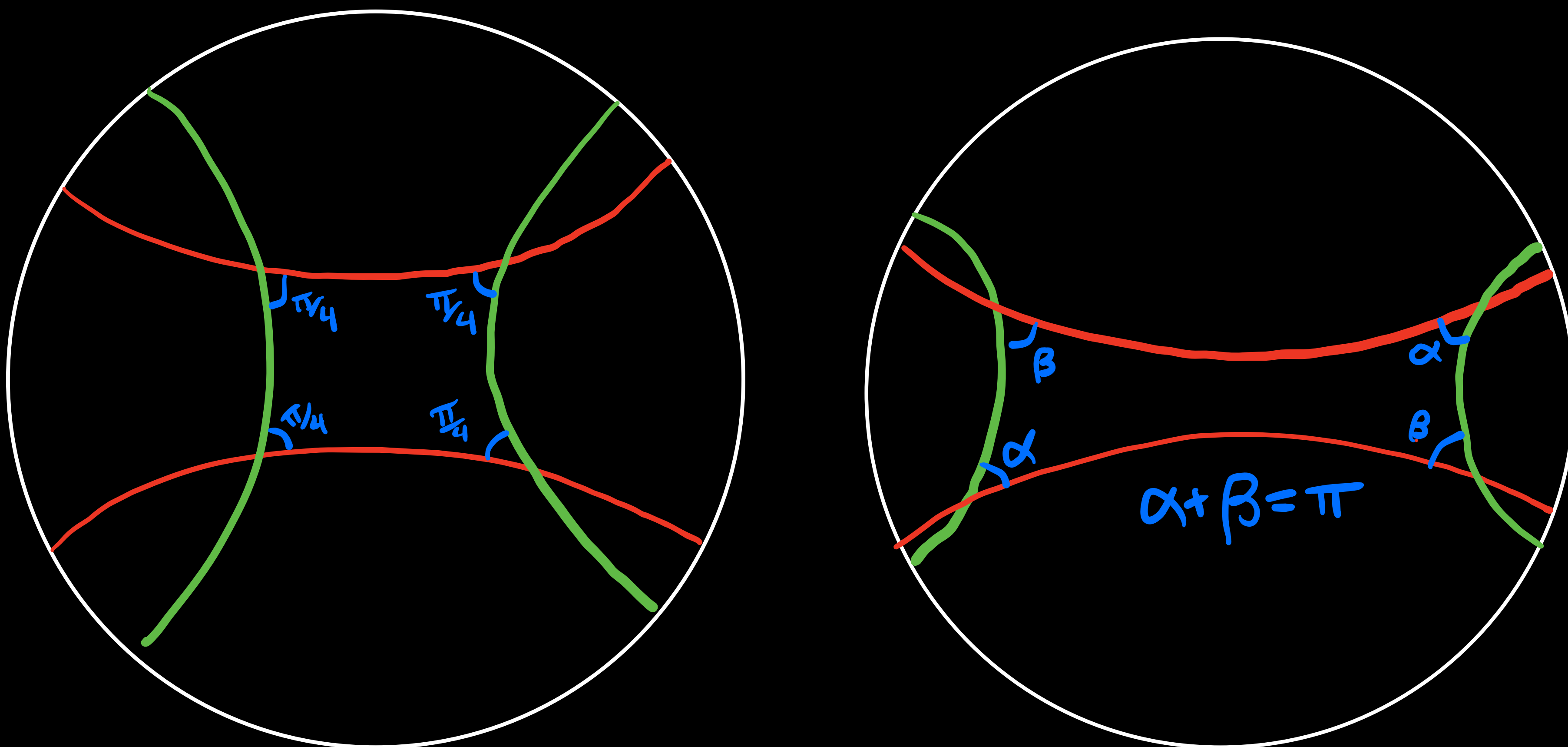
How can we find representations

$$\langle a, b \mid [a,b]^2 \rangle \longrightarrow PSL_2 \ ?$$

$\pi_1^{orb}\Big($ 2  $\Big) = \langle a, b \mid [a,b]^2 \rangle$



$\pi_1^{orb}\Big($ 2  $\Big) = \langle a, b \mid [a,b]^2 \rangle$



There is a rich deformation theory of hyperbolic surfaces!

:20

Note $g \in PSL_2$ has order 2 iff $tr(g) = 0$

Setting $x = tr(\rho(a))$

$y = tr(\rho(b))$ $\quad z = tr(\rho(ab))$,

a lengthy computation yields

that $\rho([a,b])$ has order 2 iff

$$\underline{x^2 + y^2 + z^2 = xyz + 2}.$$

$$F(X;Y;Z;W) = 2W^3 - W(X^2+Y^2+Z^2) + XYZ$$

$F^{-1}(0)$ is a smooth cubic surface.

Can find two skew lines defined over $k = \mathbb{Q}(\zeta)$, where $\zeta = \frac{1+i}{\sqrt{2}}$ satisfies $\zeta + \zeta^{-1} = \sqrt{2}$, $\zeta^2 = i$.

L_1: $X = \zeta Z \cap Y = \sqrt{2} W$

L_2: $X = 0 \cap W = 0$

A pair of points in $L_1 \times L_2$ determines a line, which by Bezout's theorem intersects $F^{-1}(0)$ in a unique 3^{rd} point. Obtain a k-rational map $(\mathbb{P}^1(k))^2 \to F^{-1}(0)$

$$g(r,s) = \left(s, \frac{r^2(s^2-1-i) - 2rs + 1+i}{\xi(r^2 - rs + 1)}, \frac{(r^2+i)s - 2(1+i)r}{\xi(r^2 - rs + 1)} \right)$$

$$g\left(\frac{31+20i}{1361}\left((5-4i)-(8+2i)\sqrt{2}\right),6\right)=\left(6,\tfrac{5}{2},\tfrac{7}{2}\right)$$

on unit circle

$$\rightsquigarrow \Gamma_1=\left\langle \begin{pmatrix}3&4\\2&3\end{pmatrix},\begin{pmatrix}2&-2\\0&\tfrac{1}{2}\end{pmatrix}\right\rangle \leq PSL_2\mathbb{Z}[\tfrac{1}{2}]$$

on unit circle

$$g\left(\frac{45-44i}{3961}\left(5+4i-14\sqrt{2}i\right),10\right)=\left(10,\tfrac{7}{2},\tfrac{7}{2}\right)$$

$$\rightsquigarrow \Gamma_2=\left\langle \begin{pmatrix}5&6\\4&5\end{pmatrix},\begin{pmatrix}\tfrac{3}{2}&-2\\-1&2\end{pmatrix}\right\rangle \leq PSL_2\mathbb{Z}[\tfrac{1}{2}]$$

Thm Let $G = \pi_1^{orb}\big(2$  $\big)$

$= \langle g, h \mid [g,h]^2 = 1 \rangle.$

Then there is no faithful representation $G \longrightarrow PSL_2(\mathbb{Z}_2)$.

contains all $\mathbb{Z}[\frac{1}{n}]$,
n odd

Pf 1 The only commutator in $PSL_2(\mathbb{F}_2)$ with order dividing 2 is the identity, so $\rho([g,h]) \equiv I \mod 2$. Since $ad-bc=1$ and $2|b$, $2|c$, we have $ad \equiv 1 \mod 4$. This means $a+d \equiv 2 \mod 4$, but $g \in PSL_2$ has order 2 iff $tr(g)=0$.

Pf 2 Very lengthy diophantine analysis on $x^2+y^2+z^2=xyz+2$, the Markoff surface associated to 2 

Q Fix g, and let Σ_g be a surface of genus g.

Is $\left|\{\text{primes } p \mid \pi_1(\Sigma_g) \leq PSL_2(\mathbb{Z}[\tfrac{1}{p}])\}\right| < \infty$?

(For 2 , this set is just $\{2\}$)

After noticing $PSL_2\mathbb{Z}$ contains no surface groups since it is virtually free, one might ask:

Does every surface group in $PSL_2\mathbb{Q}$ determine a proper action on its p-adic trees?

One can quickly find surface groups in $PSL_2\mathbb{Q}$ which have elements with integral trace.

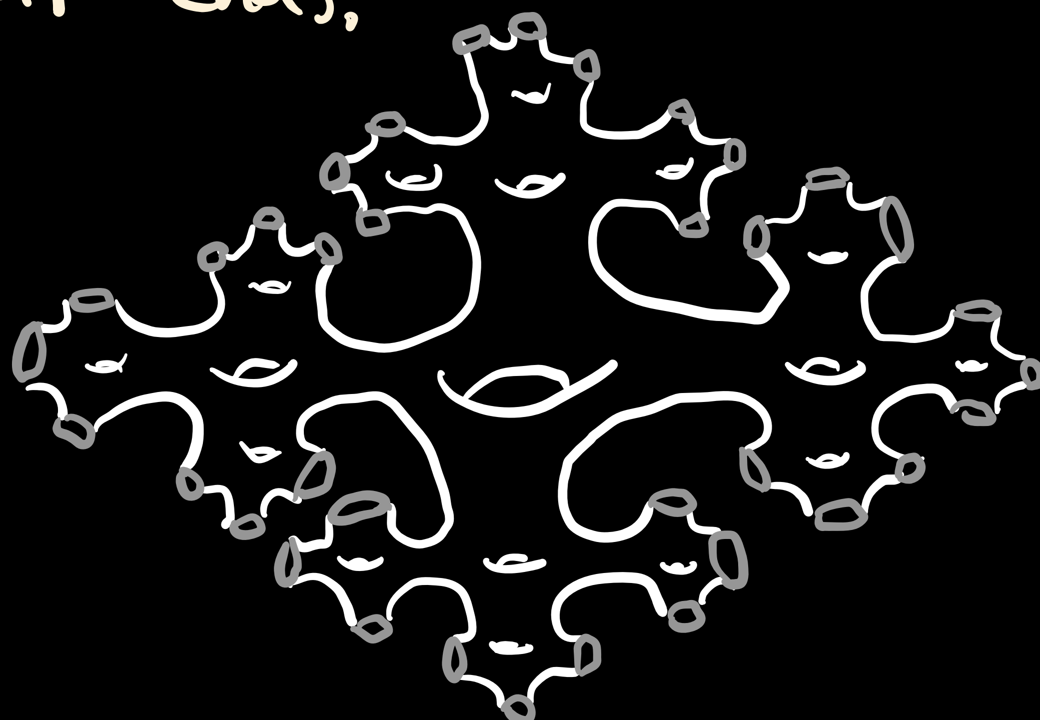
For example,

$$S = \left\langle \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 0 & 1/2 \end{pmatrix} \right\rangle$$

is another faithful rep of G

This comes from essentially the same construction as Long-Reid, but is clearly indiscrete (in fact dense) in $PSL_2\,\mathbb{Q}_2$

Thm $\mathbb{H}^2/(S \cap PSL_2\mathbb{Z})$ is a surface of infinite type with no planar ends.



Note it is <u>so</u> easy to check if $g \in PSL_2\mathbb{Q}$ is in the subgroup $PSL_2\mathbb{Z}$.

How to check if $g \in PSL_2\mathbb{Q}$ lies in $\left\langle \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 0 & 1/2 \end{pmatrix} \right\rangle$?

<u>Q</u> Are arithmetic groups characterized by the subgroup membership problem?

Also,

$$\mathbb{H}^2 \Big/ \left(\left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 1/8 & 9 \\ 2/64 & \frac{82}{8} \end{pmatrix} \right\rangle \cap PSL_2 \mathbb{Z} \right)$$

is either $\mathbb{H}^2$, or it is as above.

Discouraged by a failure to find a suitable method to guarantee $\left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 1/8 & a \\ 2/64 & 84/8 \end{pmatrix} \right\rangle$ has no integral elements, I tried another approach.

:38

Def Let k be a number field. Say $G \leq PSL_2 k$ is ***totally unitary*** if $\forall \sigma : k \to \mathbb{C}$, $\sigma(G)$ is contained in a compact subgroup of $PSL_2 \mathbb{C}$.

Thm If G is finitely generated and totally unitary, G acts properly on a jungle.

Goal Find a faithful, totally unitary rep $G \to PSL_2 k$!

With Fuchsian representations $\pi_1(\Sigma) \to PSL_2 \mathbb{Q}$, the challenge is to show $\rho(\pi_1(\Sigma)) \cap PSL_2 \mathbb{Z} = 1$

ρ is automatically faithful

– is it discrete?

For totally unitary reps, the action is automatically discrete

– is it faithful?

A couple totally unitary reps:

$$(a,b) \longmapsto \left(\frac{1}{\sqrt{13}} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}, \frac{1}{2\sqrt{3}} \begin{pmatrix} 1+2i & 2+3i \\ -2+3i & 1-2i \end{pmatrix} \right)$$

$$(a,b) \longmapsto \left(\frac{1}{85} \begin{pmatrix} 36 & 77 \\ -77 & 36 \end{pmatrix}, \frac{1}{154} \begin{pmatrix} 55+71i & 79+97i \\ -79+97i & 55-71i \end{pmatrix} \right)$$

land in $PU_2(\mathbb{Q}(i))$

$$\left(\left(\frac{4}{\sqrt{13}}, \frac{\sqrt{2}}{3}, \frac{-4\sqrt{2}}{3\sqrt{13}} \right) \text{ and } \left(\frac{72}{85}, \frac{5}{7}, \frac{-373}{595} \right) \right)$$

The first one is not faithful according to a computation.

(There are fewer words of length 7 in $\rho(G)$ than in G).

So we ask: What is the image? In other words, what are the possible subgroups of $SO_3 \mathbb{Q}$?

$\exists$ surface gps in $PSL_2\mathbb{Q}$
with elements of integral trace

$\exists$ irreps $\pi_1(\Sigma) \to SO_3\mathbb{Q}$
which are not faithful

Maybe all have these
properties?

The first rep above gives

$$\Gamma = \left\langle \frac{1}{13}\begin{pmatrix} -5 & 0 & 12 \\ 0 & 13 & 0 \\ -12 & 0 & -5 \end{pmatrix}, \frac{1}{9}\begin{pmatrix} -4 & 1 & 8 \\ 7 & -4 & 4 \\ 4 & 8 & 1 \end{pmatrix} \right\rangle$$

$$\leq SO_3\left(\mathbb{Z}\left[\tfrac{1}{3\cdot 13}\right]\right)$$

:41

Prop Nontrivial commensurated subgroups of $G \curvearrowright X$ have the same limit set as G.

$\Gamma' = \Gamma \cap SO_3(\mathbb{Z}[\frac{1}{3}])$ acts on T_4

with full limit set.

If this is because Γ' is

a lattice in T_4, then Γ

has finite index in $SO_3(\mathbb{Z}[\frac{1}{39}])$

A bit about arithmetic groups

Consider $\mathbb{Q}\left\{\begin{pmatrix}\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix}, \begin{pmatrix}0 & 1\\ -3 & 0\end{pmatrix}\right\}$.

This $\mathbb{Q}$-algebra is a rank 4 $\mathbb{Q}$-module, and can be written

$$\left\{a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + b\begin{pmatrix}\sqrt{2} & 0\\ 0 & -\sqrt{2}\end{pmatrix} + c\begin{pmatrix}0 & 1\\ -3 & 0\end{pmatrix} + d\begin{pmatrix}0 & \sqrt{2}\\ 3\sqrt{2} & 0\end{pmatrix} \,\middle|\, \begin{matrix}a,b,\\ c,d\end{matrix} \in \mathbb{Q}\right\}$$

The determinant of a matrix with coords (a,b,c,d) is $a^2-2b^2+3c^2-6d^2$.

It turns out the elements of determinant 1 form a finitely generated group, generated by

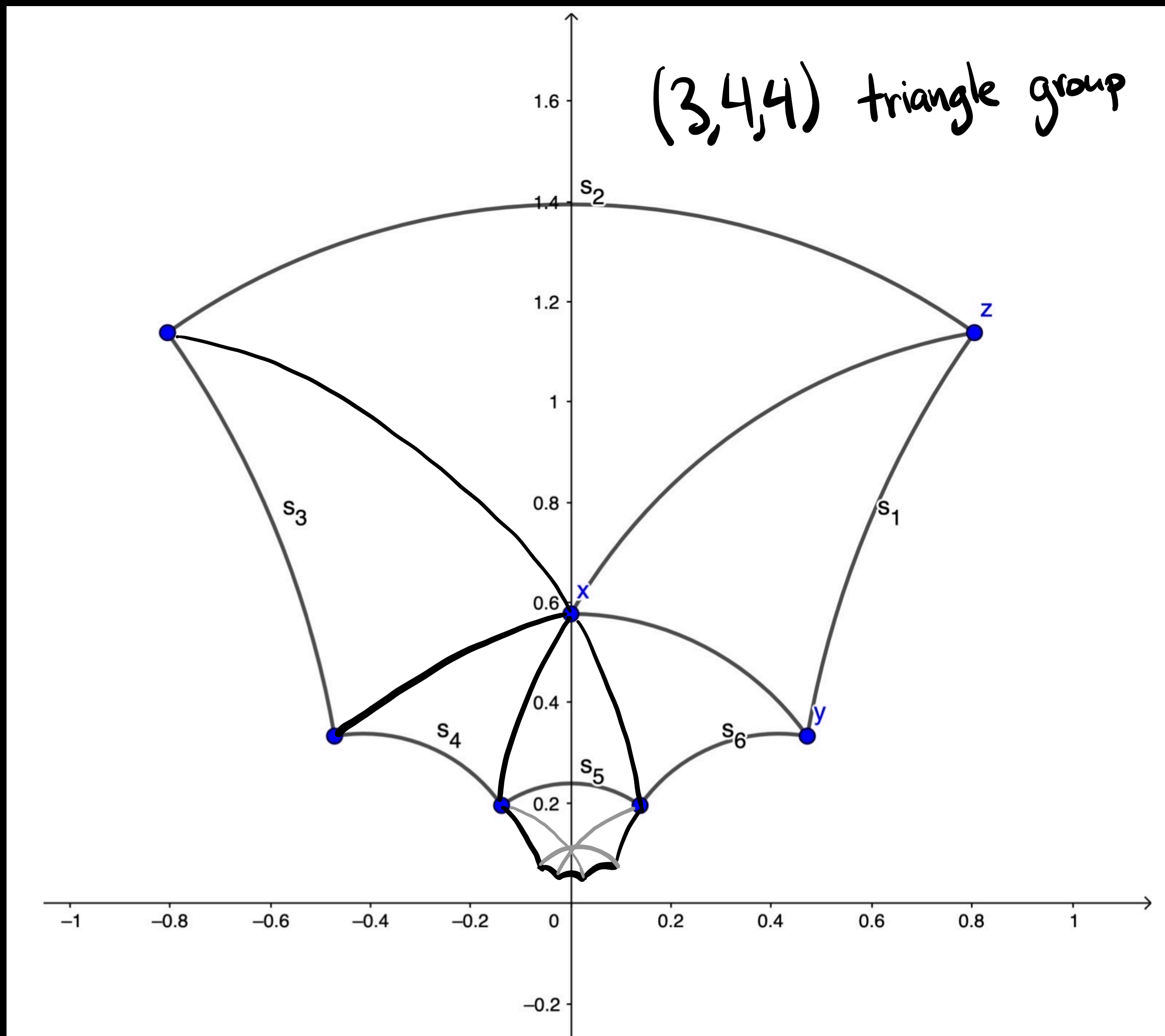
$$a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \text{ and } b = \frac{1}{\sqrt{2}}\begin{pmatrix} -1+\sqrt{2} & -1 \\ 3 & -1-\sqrt{2} \end{pmatrix}.$$

Then $ab = \frac{1}{2}\begin{pmatrix} 1+\sqrt{2} & -(1+\sqrt{2}) \\ 3(-1+\sqrt{2}) & -1+\sqrt{2} \end{pmatrix}$, and we have

$$\langle a, b \mid a^3 = b^4 = (ab)^4 = 1 \rangle$$

(Cheating a little bit)

Norm ± 1 elts in $\left(\frac{-2,3}{\mathbb{Z}}\right)$



An algebraic group is an algebraic variety endowed with a group structure so that the group operations are regular maps.

Vaguely, if G is defined over a subring $A \leq K$, and the group operations are defined locally by polynomials with coefficients in A, the A-points of the variety form a subgroup $G(A) \leq G(K)$, called an <u>arithmetic subgroup</u>.

Eg Let $f(x,y,z,w) = x^2+y^2+z^2-\sqrt{2}\,w^2$, defined over $\mathbb{Z}[\sqrt{2}]$. We let

$$\Gamma = O_f(\mathbb{Z}[\sqrt{2}]) = \left\{ g \in GL(\mathbb{Q}(\sqrt{2})^4) \;\middle|\; \begin{array}{c} f(gv) = f(v) \\ \forall v \in V \end{array} \right\}$$

Γ is a lattice in the Lie group $O_f(\mathbb{R}) \times O_{\sigma(f)}(\mathbb{R})$, where σ is the nontrivial Galois automorphism of $\mathbb{Q}(\sqrt{2})$.

The projection of this group onto its first factor has compact kernel, and so Γ is a lattice in $O_{3,1}(\mathbb{R})$, the isometry group of $\mathbb{H}^3$.

Γ is generated by reflections in hyperplanes orthogonal to

$$\Delta = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \right\}$$

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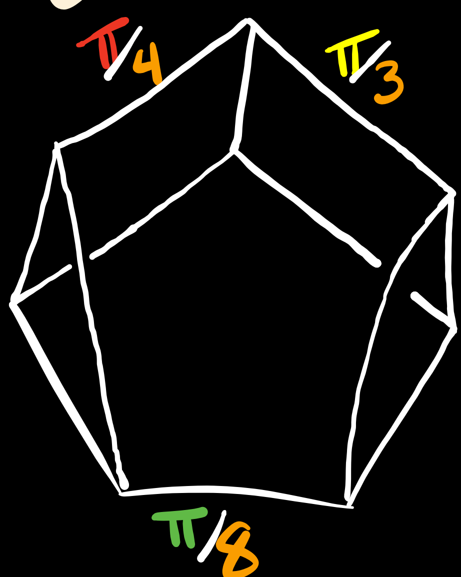
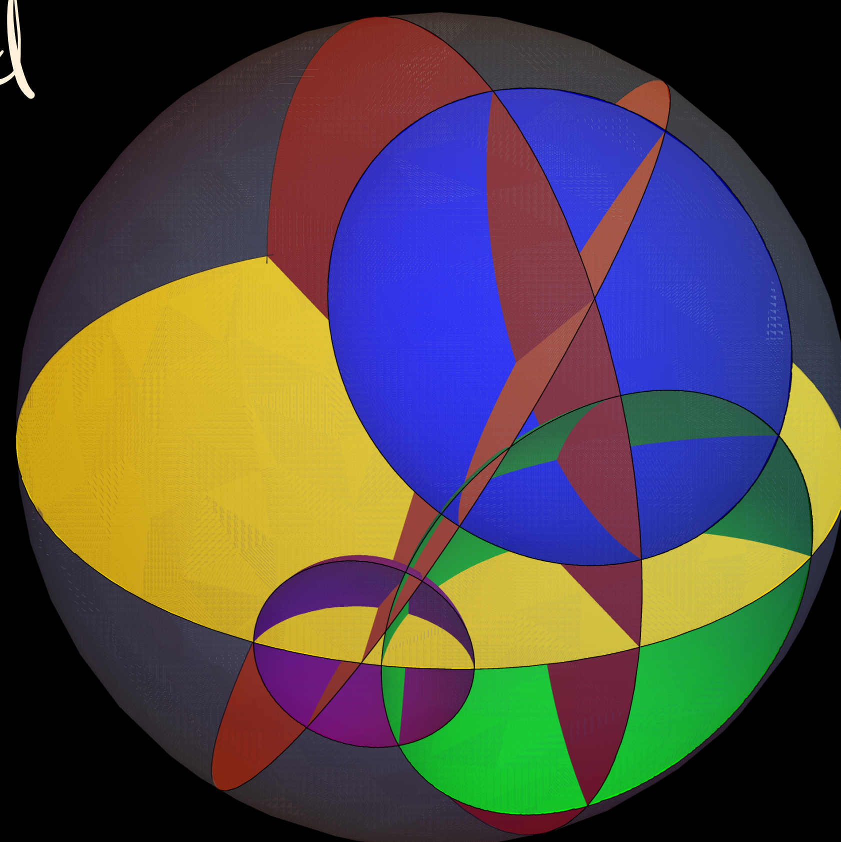
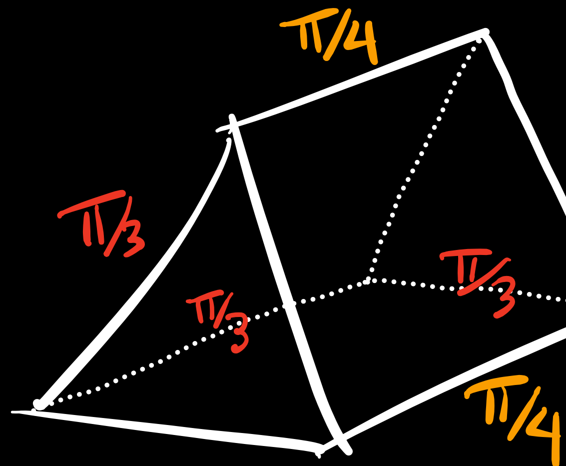
The fundamental domain is a pentagonal wedge, with all unlabeled edges right-angled



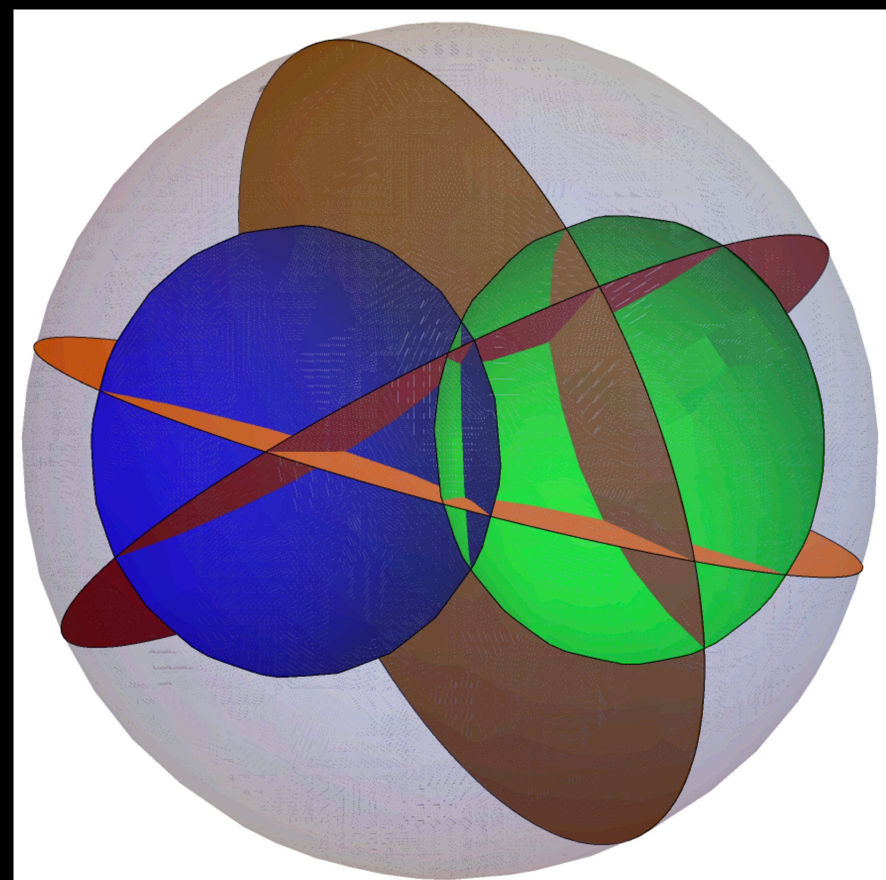
$O_{x^2+y^2+z^2-\sqrt{2}w^2}(\mathbb{Z}[\sqrt{2}])$



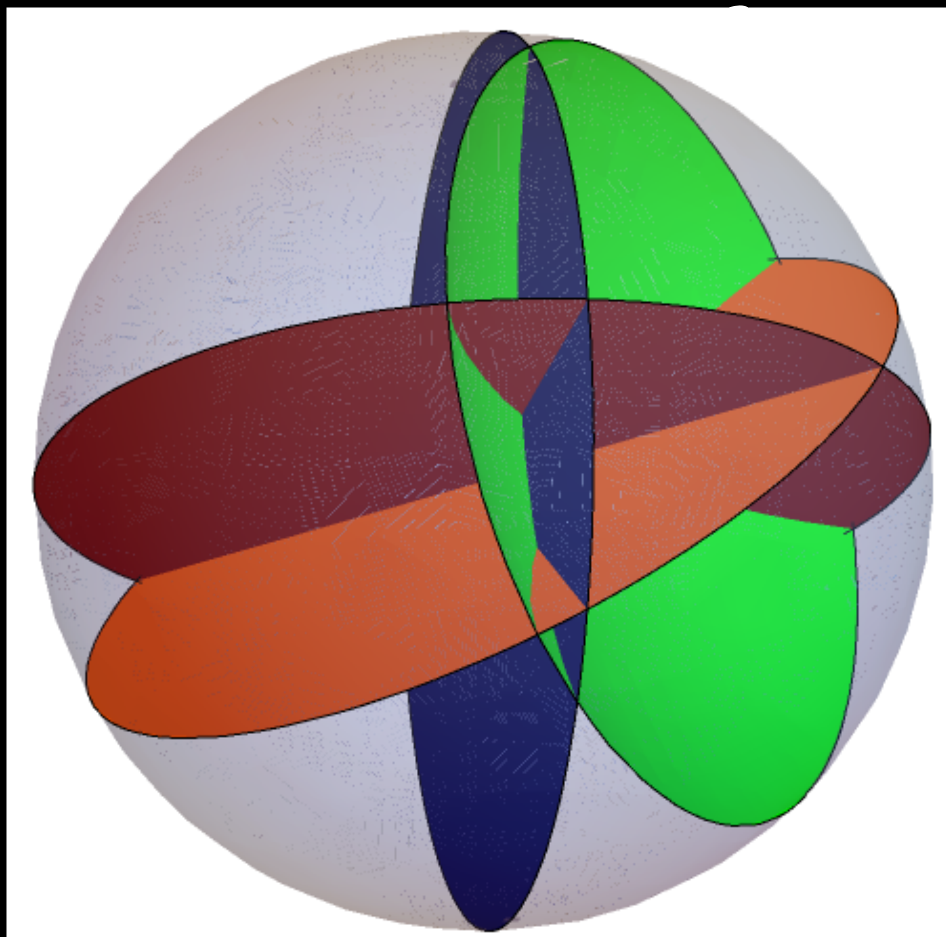
For fun: $O_{x^2+y^2+z^2-7w^2}(\mathbb{Z})$



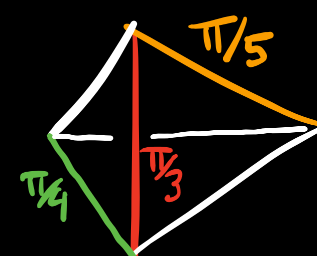
$$\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\\-2\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-3\\-1\end{pmatrix} \right\}$$



$$O_{x^2+y^2+z^2-\varphi w^2}(\mathbb{Z}[\varphi])$$



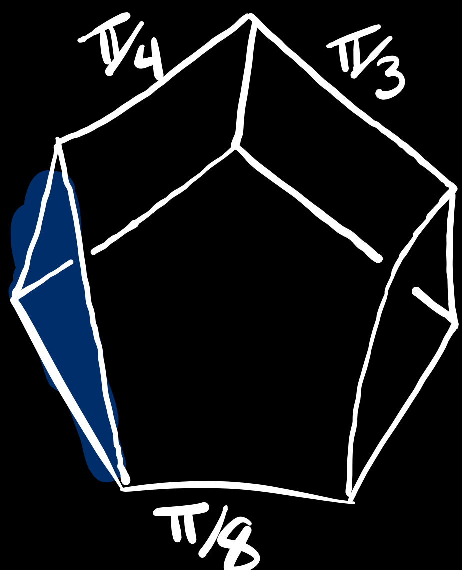
The principal 2-congruence subgp is refl. in right-angled dodecahedron (suggested by Sami Douba)



← index 60

Back to $O_{x^2+y^2+z^2-\sqrt{2}w^2}(\mathbb{Z}[\sqrt{2}])$

(A surface group in $SO_3(\mathbb{R})$)



The group preserving the blue face is the normalizer of $O_{x^2+y^2-\sqrt{2}w^2}(\mathbb{Z}[\sqrt{2}])$ and contains it with index 2.

$$f(x,y,z) = x^2 + y^2 - \sqrt{2}\, z^2 \quad \text{a quadratic form.}$$

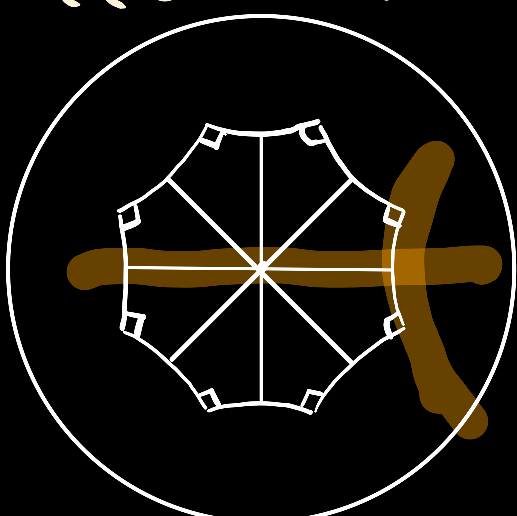
$$SO_f(\mathbb{Z}[\sqrt{2}]) = \left\{ A \in GL_3(\mathbb{Z}[\sqrt{2}]) \;\middle|\; \begin{array}{l} f(Av) = f(v) \\ \forall v \in \mathbb{Q}[\sqrt{2}]^3 \end{array} \right\}$$

$$= \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3+2\sqrt{2} & 0 & 4+2\sqrt{2} \\ 0 & 1 & 0 \\ 2+2\sqrt{2} & 0 & 3+2\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2+\sqrt{2} & 1+\sqrt{2} & 2+2\sqrt{2} \\ 1+\sqrt{2} & 2+\sqrt{2} & 2+2\sqrt{2} \\ 2+\sqrt{2} & 2+\sqrt{2} & 3+2\sqrt{2} \end{pmatrix} \right\rangle$$

$f(x,y,z) = x^2 + y^2 - \sqrt{2}\, z^2$ a quadratic form.

$$SO_f(\mathbb{Z}[\sqrt{2}]) = \left\{ A \in GL_3(\mathbb{Z}[\sqrt{2}]) \;\middle|\; \begin{array}{l} f(Av) = f(v) \\ \forall v \in \mathbb{Q}[\sqrt{2}]^3 \end{array} \right\}$$

$$= \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3+2\sqrt{2} & 0 & 4+2\sqrt{2} \\ 0 & 1 & 0 \\ 2+2\sqrt{2} & 0 & 3+2\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2+\sqrt{2} & 1+\sqrt{2} & 2+2\sqrt{2} \\ 1+\sqrt{2} & 2+\sqrt{2} & 2+2\sqrt{2} \\ 2+\sqrt{2} & 2+\sqrt{2} & 3+2\sqrt{2} \end{pmatrix} \right\rangle$$



$O_f(\mathbb{Z}[\sqrt{2}])$ contains a right-angled pentagon reflection group, and a regular right-angled octagon reflection group.

Let $\sigma: \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ the nontrivial Galois automorphism. Then

$$\sigma(SO_f(\mathbb{Z}[\sqrt{2}])) = SO_{x^2+y^2+\sqrt{2}z^2}(\mathbb{Z}[\sqrt{2}])$$

and so we obtain a surface group in $SO_3(\mathbb{Z}[\sqrt[4]{2}])$, and a faithful action of $\pi_1(\Sigma_g)$ on a sphere by rotations.

Similar arithmetic tricks can give rise to hyperbolic 3-manifold groups in $SO_3(\mathbb{R})$.

Hyperbolic 3-manifolds have lots of covers, hence lots of possible subgroups. Classifying subgroups of $SO_3\mathbb{R}$ appears quite daunting.

Frightened by the complexity of $SO_3(k)$ for $k \supsetneq \mathbb{Q}$ (and hopeful that understanding the simplest case might solve the general question) we focus on subgroups of $SO_3(\mathbb{Q})$.

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The geometry of $SO_3\mathbb{Q}$

$SO_3\mathbb{Q}$ is a lattice in $SO_3\mathbb{A}$, for $\mathbb{A}$ the <u>adeles</u>, a restricted product

$$\mathbb{R} \times \prod'_{p \text{ prime}} \mathbb{Q}_p.$$

$SO_3\mathbb{R}$ and $SO_3\mathbb{Q}_2$ are compact groups (by Hilbert reciprocity, this is a finite set w/ even cardinality), and for p odd,

$$PGL_2\mathbb{Q}_p \cong SO_3\mathbb{Q}_p.$$

(the adjoint action of $PGL_2\mathbb{Q}_p$ on its p-adic Lie algebra gives the iso).

So $SO_3\mathbb{Q}$ is a lattice in $\prod'_{p \text{ odd prime}} T_{p+1}$!

:50

$$\left(\text{And } SO_3\mathbb{Z}\left[\tfrac{1}{n}\right] \text{ is a lattice in } \prod_{p|n \text{ odd}} T_{p+1} \right)$$

Even better – we can obtain an (infinite) presentation for $SO_3\mathbb{Q}$!

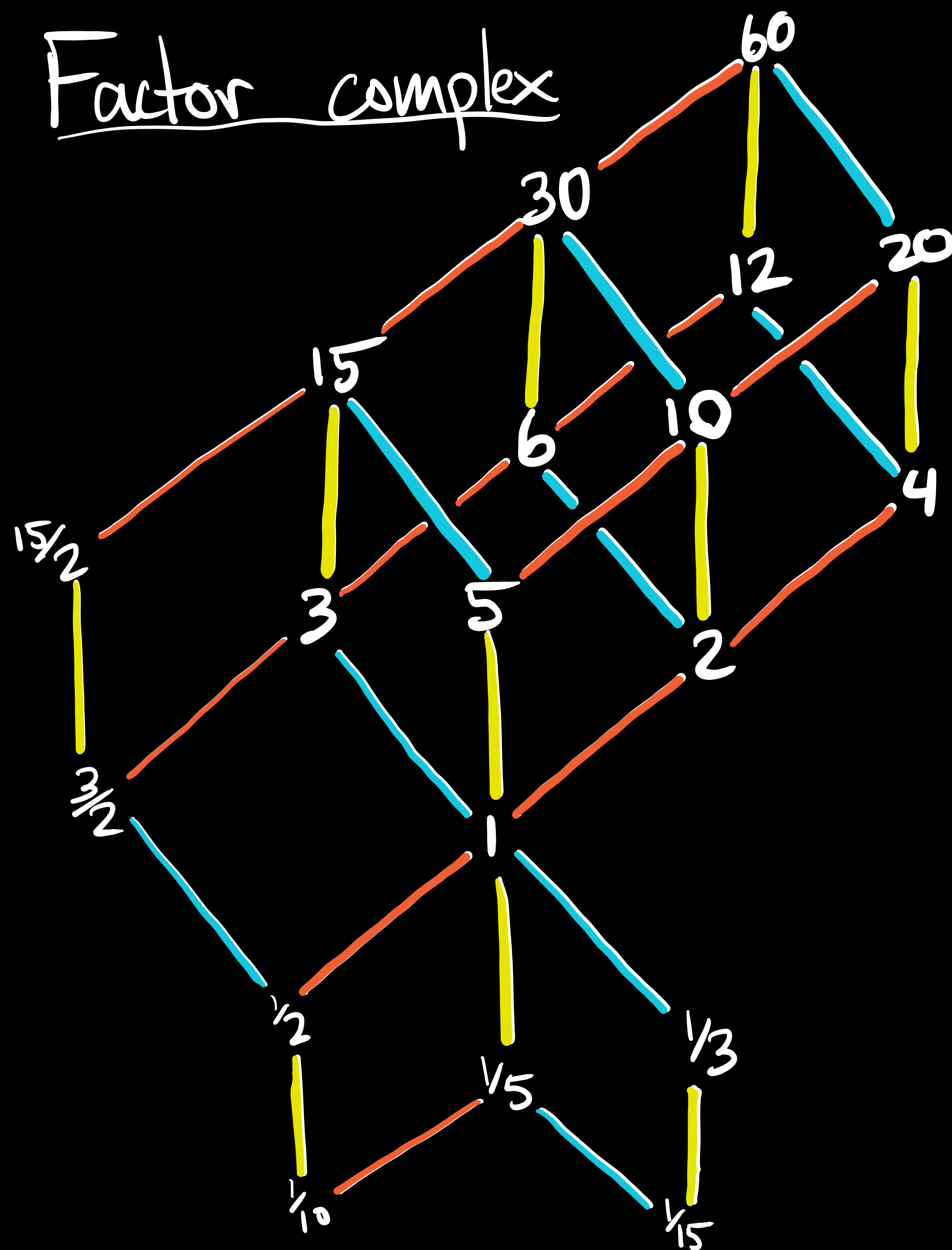
Fundamental Theorem of Arithmetic

Every positive integer can be expressed uniquely as a product of primes

GGT version

$\Longrightarrow$ $\mathbb{Q}^{\times}_{>0}$ acts properly + coboundedly on a restricted direct product of lines, one for each prime.

Factor complex



Fundamental theorem of quaternion arithmetic

(Hurwitz?)

Let q be a primitive integer quaternion. If p is an odd prime dividing $N(q)$, there is a unique quaternion of norm p (up to associates) which left-divides q.

Thm (Jacobi) The number of representations of an odd prime p as a sum of four squares is $8(p+1)$

If $N(q)=p$, then up to the action of $Q_8=\{\pm1,\pm i,\pm j,\pm k\}$, we can take

$$\mathrm{Re}(q)>0 \text{ and } \begin{cases} q\equiv 1 \pmod 2 & \text{if } p\equiv 1\ (4) \\ q\equiv 1+i+j \pmod 2 & \text{if } p\equiv 3\ (4) \end{cases}$$

$A_5=\{1\pm 2i, 1\pm 2j, 1\pm 2k\}$

$A_3=\{1\pm i\pm j\}$

$A_{13}=\{3\pm 2i/j/k,\ 1\pm 2i\pm 2j\pm 2k\}$

$A_{11}=\{3\pm i\pm j,\ 1\pm 3i\pm j,\ 1\pm i\pm 3j\}$

$A_{17}=\{1\pm 4i/j/k,\ 3\pm 2i\pm 2j,\ 3\pm 2i\pm 2k,\ 3\pm 2j\pm 2k\}$

Each A_p is a symmetric generating set for a free group of rank $\frac{p+1}{2}$.

A quaternion acts by conjugation on the pure rational quaternions, preserving a form.

$$\langle A_p \rangle \hookrightarrow SO_3(\mathbb{Z}[\tfrac{1}{p}]).$$

Note: $|A_p| = |\mathbb{P}^1 \mathbb{F}_p| =$ valence of p-adic tree...

There is a 2-congruence subgroup (of index 24) of $SO_3 \mathbb{Q}$ which acts simply transitively on the vertices of

$$X = \prod_{p \text{ odd}}{}' T_{p+1}$$

So $SO_3\mathbb{Q}$ has a presentation which consists of

gens: $\bigcup_{p \text{ odd}} A_p$

relations:

$$\begin{tikzcd} \bullet \arrow[r, "p_2"] & \bullet \\ \bullet \arrow[u, "q_2"] \arrow[r, "p_1"'] & \bullet \arrow[u, "q_1"'] \end{tikzcd}$$

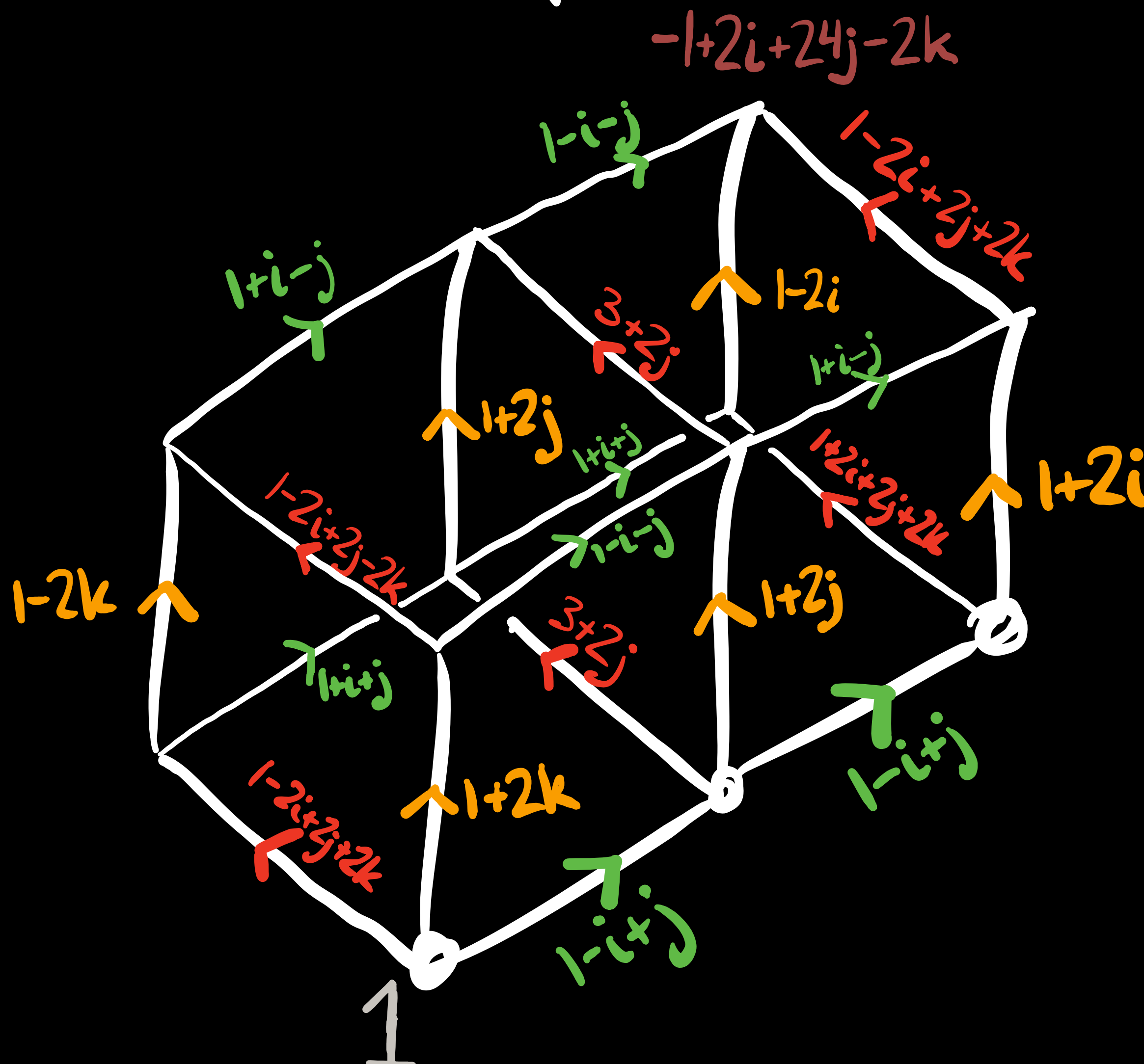
$$p_1 q_1 = q_2 p_2$$

To determine the relations, we can fix an isomorphism

$$A_p \leftrightarrow \mathbb{P}^1\mathbb{F}_p$$

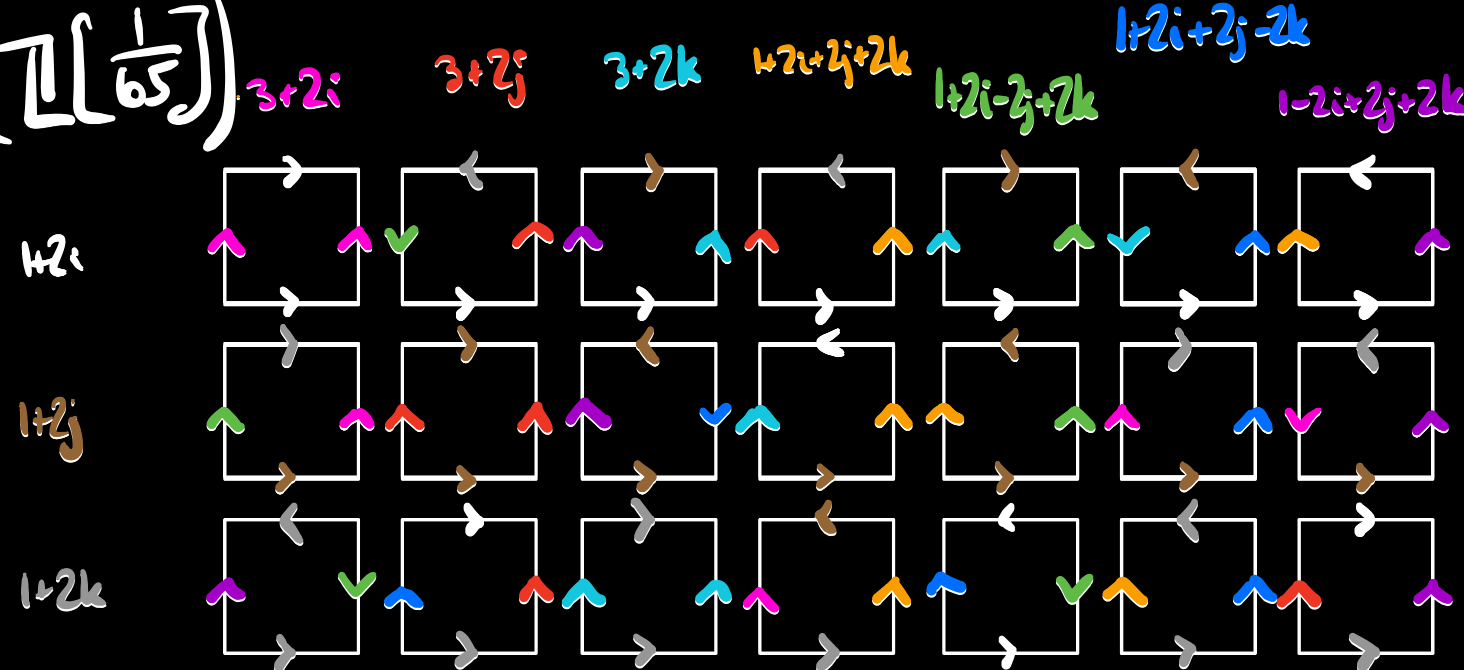
and work with the action of $A_q \circlearrowright \mathbb{P}^1\mathbb{F}_p$

For example...

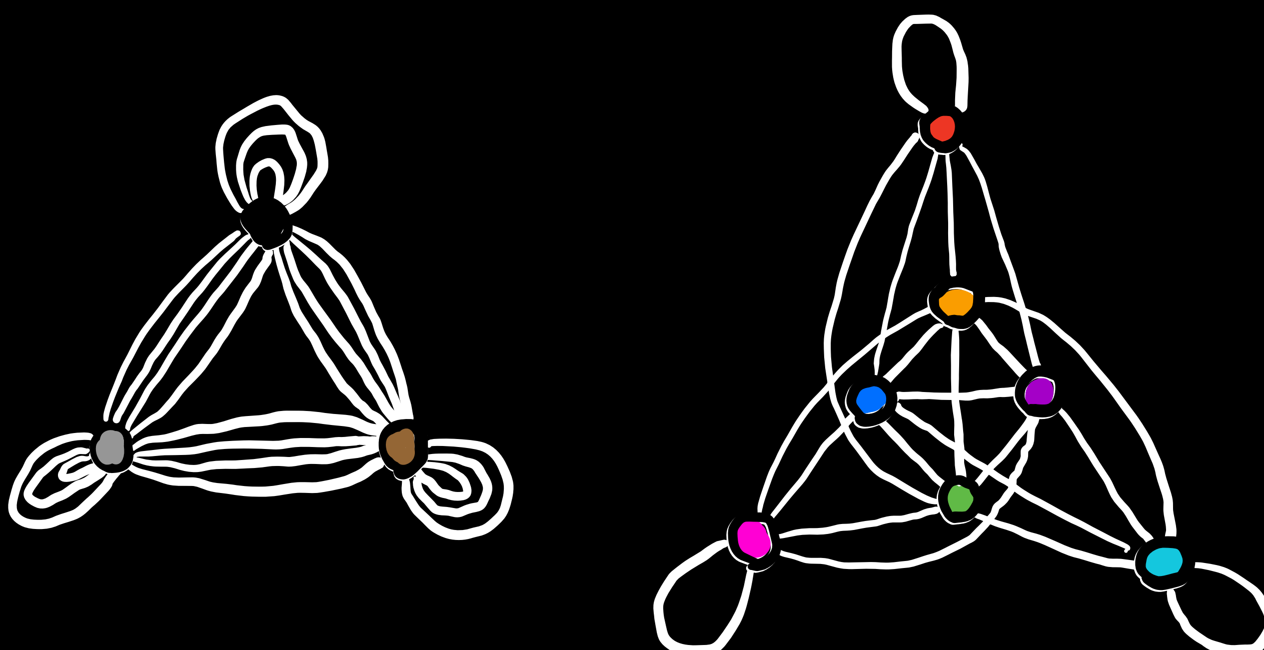


-1+2i+24j-2k
1-i-j
1+i-j
1-2i+2j+2k
1-2i
3+2j
1+2j
1+i-j
1+i+j
1+2i
1-2i+2j-2k
1+2i+2j+2k
1-i-j
1-2k
3+2j
1+2j
1+i+j
1-i+j
1-2i+2j+2k
1+2k
1-i+j
1

$SO_3\left(\mathbb{Z}\left[\frac{1}{65}\right]\right)$



Hyperplanes



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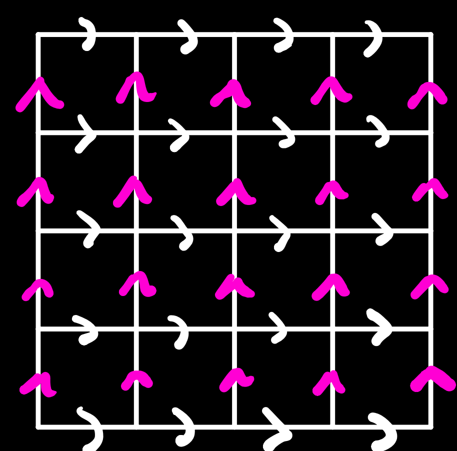
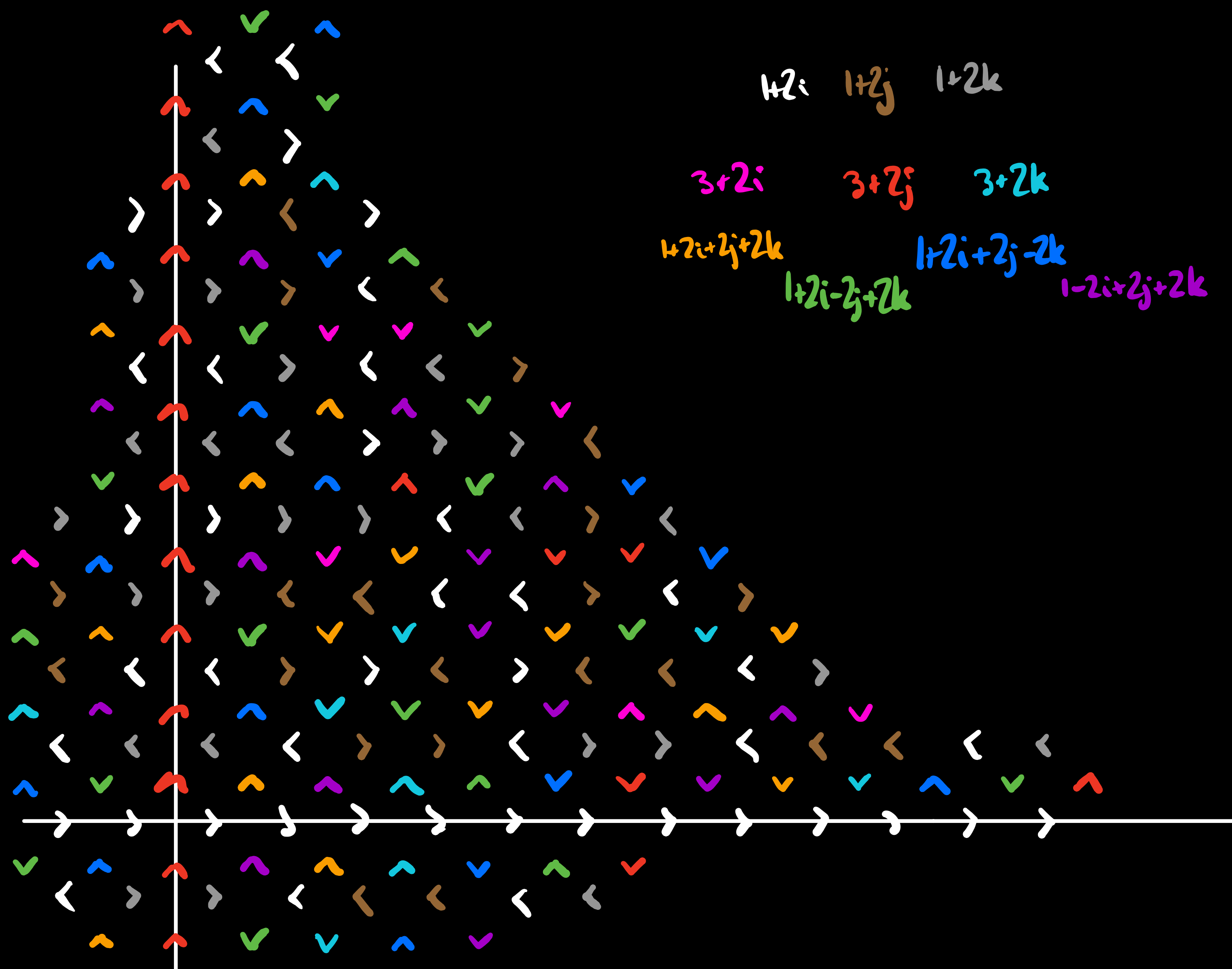
This is a square complex with one vertex, 3 vertical and 7 horizontal edges, and 21 squares. The universal cover is $T_6 \times T_{14}$.

There are some <u>periodic</u> flats in the universal cover



and some <u>aperiodic</u> flats-


1+2i
1+2j
1+2k
3+2i
3+2j
3+2k
1+2i+2j+2k
1+2i-2j+2k
1+2i+2j-2k
1-2i+2j+2k

Thm Let A_p be the set of quaternions of norm p, and $x \in A_q$. Then $\langle A_p, x \rangle = \langle A_p, A_q \rangle$

Thus, any subgroup of $SO_3\mathbb{Q}$ which contains an $SO_3(\mathbb{Z}[\frac{1}{p}])$ is arithmetic!

Now, let

$$\Gamma \leq SO_3 \mathbb{Q}.$$

It might happen that Γ is abelian, in which case Γ preserves some axis in $\mathbb{R}^3$. For example,

$$\left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a^2+b^2=1,\ a,b \in \mathbb{Q} \right\} \leq SO_3\mathbb{Q}$$

This group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{(P_1)}$ where P_1 is the set of primes equivalent to 1 modulo 4.

It may also happen that for some prime p, there are only finitely many $g \in \Gamma$ which have small powers of p in their denominators. We'll call such groups <u>primary</u>.

<u>Prop</u> If Γ is primary, there is a finite index subgroup $F \leq \Gamma$ which is free.

Conjecture Let $\Gamma \leq SO_3\mathbb{Q}$ be finitely generated. Then either

(i) Γ is abelian

(ii) Γ is primary

(iii) Γ is commensurable with $SO_3(\mathbb{Z}[\frac{1}{n}])$ for n odd composite.

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Let's call $\Gamma \leq SO_3\mathbb{Q}$ <u>full</u> if it is not abelian or primary. The conjecture is thus that Γ must be S-arithmetic.

A great example to have in mind is obtained by taking $\langle a^N, b^N \rangle$, where

$$a = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad b = \frac{1}{13}\begin{pmatrix} 13 & 0 & 0 \\ 0 & 5 & 12 \\ 0 & -12 & 5 \end{pmatrix}$$

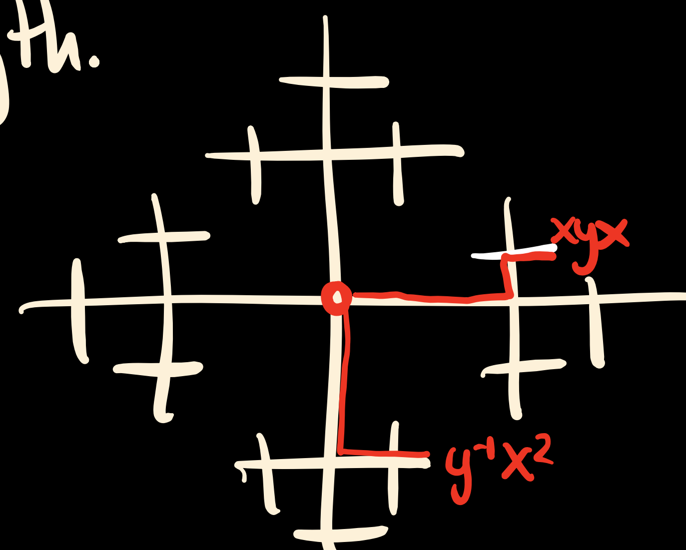
Geometric Rigidity

for full subgroups

Suppose $G \curvearrowright T$ a tree, and let $v \in T$.

We obtain a function $\tau_v : G \to \mathbb{N}$ defined by $\tau_v(g) = d_T(v, gv)$.

If $T = Cay(F_S, S)$, $\tau_{\{e\}}$ coincides with word length.



An action $G \curvearrowright T$ is <u>minimal</u> if there is no proper G-invariant subtree.

<u>Prop</u> (Chiswell) Suppose $G \curvearrowright T_1$ and $G \curvearrowright T_2$ are minimal actions and $v_i \in T_i$ such that $\tau_{v_1}(g) = \tau_{v_2}(g)$ $\forall g \in G$. Then there is a unique G-equivariant isometry $T_1 \to T_2$.

Let X be a product of trees, and suppose $\rho: G \to \mathrm{Aut}(X)$ has no convex G-invariant subcomplex. We say $\rho': G \to \mathrm{Aut}(X)$ is a geometric representation if $\forall\, T_i$, the translation length functions induced by ρ and ρ' agree.

Geometric Rigidity

Thm Let $\Gamma \leq SO_3\mathbb{Q}$ be a full group, so that $\Gamma \curvearrowright X = \prod_{p\in S} T_{p+1}$ discretely for some set S of odd primes p. Suppose $\rho: \Gamma \to \mathrm{Aut}(X)$ is a geometric representation. Then ρ has a unique extension to a representation of $\mathbb{G} = \prod_{p \in S} PGL_2\mathbb{Q}_p$.

$$\begin{tikzcd} \mathbb{G} \arrow[r, dashed, "\tilde{\rho}"] & \mathrm{Aut}(X) \\ \Gamma \arrow[u, hook] \arrow[ur, "\rho"'] & \end{tikzcd}$$

:02

<u>Lemma</u> $PGL_2 \mathbb{Q}_p \leq Aut(T_{p+1})$ is self-normalizing

<u>Pf</u> An automorphism of $PGL_2(k)$ is induced by an automorphism of k, but $\mathbb{Q}_p$ has no automorphisms. Since $PGL_2 \mathbb{Q}_p \curvearrowright T_{p+1}$ is minimal, it can't be centralized.

<u>Pf of thm</u> By Chiswell's theorem, ρ can be conjugated to the standard action by an element of $Aut(X)$. By strong approximation, we have that Γ acts densely on each factor, so $\tilde{\rho}$ unique.

Back to

$$\Gamma \leq SO_3(\mathbb{Q})$$

We might say $H \leq G$ is <u>coarsely maximal</u> if any intermediate subgroup has finite index in either side, and <u>roughly maximal</u> if any finite index subgroup of H is coarsely maximal.

<u>Question</u> For n odd composite,

is $SO_2(\mathbb{Z}[\frac{1}{n}]) \leq SO_3(\mathbb{Z}[\frac{1}{n}])$ roughly maximal?

(This translates into a number theoretic statement.)

:04

If so, we can prove:

Any subgroup of $SO_3\mathbb{Q}$ containing a $\mathbb{Z}^2$ is abelian or arithmetic.

Consequently, a counterexample to the conjecture would either yield

1) A hyperbolic group acting properly on a product of trees

2) A CAT(0) group which is not hyperbolic and has no "poison subgps"

Consider $A = \mathbb{Z}[\frac{1}{n}]$, n odd composite. Let

$$S^2(A) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in A^3 \,\middle|\, x^2+y^2+z^2 = 1 \right\},$$

and note $G = SO_3(A) \curvearrowright S^2(A)$ transitively, with stabilizers commensurable with $\mathbb{Z}^k$, $k = \#$ prime factors of n.

Note that G-invariant partitions of $S^2(A)$ correspond exactly to subgroups intermediate to

$$O_2(A) \leq SO_3(A).$$

This subgroup is coarsely maximal iff every G-inv partition is finite or cofinite.

Prop $O_2(\mathbb{R}) \leq SO_3(\mathbb{R})$ is maximal

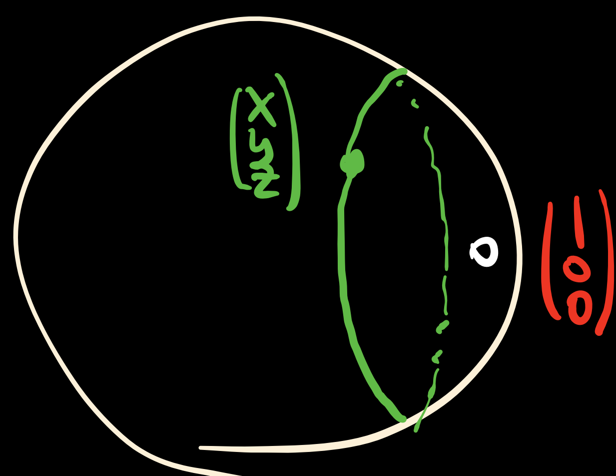
Pf Let $\mathcal{P} = \bigsqcup_{i \in I} P_i$ an SO_3-invt partition of S^2.

Set P_0 the partition containing $\begin{pmatrix}1\\0\\0\end{pmatrix}$. $P_0 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ or $\left\{\pm\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$, get SO_2, O_2.

Suppose $\begin{pmatrix}x\\y\\z\end{pmatrix} \in P_0$, $x \in (-1,1)$.

Since partition is G-invt and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & O_2 \end{pmatrix}$ fixes $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, any such matrix g has $g \cdot P_0 \cap P_0 \neq \emptyset$, so $g \cdot P_0 = P_0$.

Hence the whole green circle is in P_0.



Since also rotations fixing $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ must preserve P_0, we must have $P_0 \subseteq S^2$ open.

So $\lambda(P_0) > 0$ for λ the Lebesgue measure, and since S^2 has finite measure and SO_3 preserves λ, the partition has finitely many parts (in fact, must have $P_0 = S^2$)

This argument doesn't work for subrings of $\mathbb{R}$, and the conclusion doesn't hold for $\mathbb{Z}[\frac{1}{p}]$.

What about for $\mathbb{Z}[\frac{1}{pq}]$?

Note

Conj $\Longrightarrow$ every subgp of $SO_3\mathbb{Q}$ is a lattice in a jungle

$\Rightarrow$ Coherence is not a geometric property

Q Does $SO_3(\mathbb{Z}[\frac{1}{n}])$ have the congruence subgroup property?

Q Is thinness always detected in a simple factor?

Q Does every hyperbolic 3-manifold contain a nontrivial element of integral trace?

Q Can every hyperbolic n-mfld be built from arithmetic pieces, $n \geq 3$? Or surfaces with algebraic traces?