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Compact 3-manifolds of constant negative curvature fibering over the circle

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1. Introduction

Let **H** denote the 3-dimensional hyperbolic space. It is well known that **H** is a Riemannian manifold of constant negative curvature. For any discontinuous group G of isometries of **H**, it is possible to view the coset space $\mathbf{H}/G$ as a manifold of constant negative curvature, allowing for branch lines or more complicated singularities when G has torsion (i.e., non-trivial finite subgroups).

In the following, certain special examples of groups and quotient manifolds will be described for the purpose of justifying:

THEOREM 1. *There exist compact 3-manifolds of constant negative curvature, which fiber over the circle.*

In topology it has been rather well understood which compact 3-manifolds fiber over the circle since the appearance of Stallings' paper [19]. Other contributions to this subject are presented in [6], [13] and [17]. It is the negative curvature which is the crucial ingredient of Theorem 1. In this connection the papers [11], [15] and [16] may be mentioned.

THEOREM 2. *There exist torsion-free, discrete, co-compact groups of hyperbolic isometries which map onto the integers with finitely generated kernel.*

Stallings' theorem makes Theorem 1 an immediate consequence of Theorem 2. Thus, the problem is to construct a finitely generated group for which some automorphism merely amounts to conjugation by a transformation extending the group infinitely to a discontinuous, co-compact group.

We start with a group G generated by two elements X and Y whose commutator has finite order. The automorphism will be the one determined by $(X, YX) \mapsto (XY^{-1}, Y)$. It is in a sense the simplest among infinitely many essentially different automorphisms, each of which could have been used equally well [7]. The chosen limitation makes an elementary and very explicit treatment possible.

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The quotient manifold $\mathbf{H}/G$ is homeomorphic to the product of the real line and a torus with one branch point. For the extended group G^*, the quotient manifold is a twisted product of the circle and a torus with one branch point.

Instead of the group G, one may start with its extension by the Lie product of a pair of generators, thereby obtaining a manifold whose fibers are spheres with four branch points, three of which have order 2.

It is more important to observe that whenever N^* is a normal subgroup of finite index in G^*, then the intersection of N^* and G is a normal subgroup of finite index in G, which again admits an infinite cyclic extension. With use of a lemma of Selberg, this permits us to obtain from the original constructions torsion-free groups and hence smooth quotient manifolds as examples for Theorem 2 and Theorem 1.

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2. The Poincaré model

For convenience we identify the hyperbolic space $\mathbf{H}$ with the upper half-space in $\mathbf{R}^3$ (bounded by the extended complex plane, $\mathbf{C} \cup \{\infty\}$). The points of $\mathbf{H}$ may be written in quaternion form $h = z + tj$, where z is the orthogonal projection of h onto $\mathbf{C}$ and t is the (Euclidean) distance from h to $\mathbf{C}$. Thus, as a set $\mathbf{H} = \mathbf{C} + \mathbf{R}_+ j$. With respect to the hyperbolic metric $ds^2 = t^{-2}(|dz|^2 + dt^2)$, the Euclidean hemispheres (including half-planes) perpendicular to $\mathbf{C}$ are the planes. Hence, the hyperbolic straight lines are semicircles perpendicular to $\mathbf{C}$. The isometries of $\mathbf{H}$ are complex Möbius transformations $h \mapsto A(h) = (\alpha h + \beta)(\gamma h + \delta)^{-1}$, where $\alpha\delta - \beta\gamma = 1$ (and the computation is carried out within the algebra of quaternions). The group of isometries may be identified with the group of pairs of opposite, complex, unimodular 2-by-2 matrices, $\mathrm{SL}(2, \mathbf{C})/\{\pm \mathrm{id}\}$, discontinuous subgroups of the former thereby becoming identical with discrete subgroups of the latter.

3. Isometric hemispheres

With respect to the Euclidean metric $|dz|^2 + dt^2$ on $\mathbf{H}$, each Möbius transformation $h \mapsto A(h)$ is a conformal mapping with infinitesimal scale ratio $|dA(h)|/|dh|$ equal to $|\gamma h + \delta|^{-2}$. If $\gamma \neq 0$, the locus where the scale ratio is equal to $+1$ forms a Euclidean hemisphere $|\gamma h + \delta| = 1$, with centre $-\gamma^{-1}\delta$ and radius $|\gamma|^{-1}$. This locus is called the *isometric hemisphere* of A (in the complex plane, the circle $|\gamma z + \delta| = 1$ is called the isometric circle of A).

We will be particularly interested in the exterior E_A of this isometric hemisphere, defined by the inequality $|\gamma h+\delta|>1$, or in other words $|dA(h)|/|dh|<1$. By A, it is mapped onto the interior of the isometric hemisphere of A^{-1}. In terms of hyperbolic geometry, E_A is of course a half-space bounded by a plane.

The virtue of these half-spaces is the following. If G is a discrete group of Möbius transformations, then intersecting the half-spaces E_A, $A \in G$, we obtain a hyperbolic convex polyhedron which can sometimes be taken as a fundamental domain for the action of G on **H**, or from which it is often a simple matter to obtain such a domain.

Certain general facts about isometric hemispheres follow from the chain rule for differentiating composite functions. We shall make use of the following two.

If the isometric hemispheres of A and B intersect, then also the isometric hemispheres of A^{-1} and BA^{-1} and of B^{-1} and AB^{-1} intersect, and the three edges of intersection are mapped onto each other by A, B and AB^{-1}.

If f is a Euclidean similarity transformation of **H**, then the isometric hemisphere of fAf^{-1} is the image by f of the isometric hemisphere of A.

4. Lie products

Let A and B denote two complex, unimodular 2-by-2 matrices as well as the Möbius transformations (acting on $\mathbf{H} \cup \mathbf{C} \cup \{\infty\}$) which they represent. Suppose that the trace of their commutator is different from 2 (or, what can be shown to be equivalent, that A and B have no common fixed points in the extended complex plane). Then, since $AB - BA$ has determinant equal to $2 -$ trace $ABA^{-1}B^{-1}$, the Lie product of A and B defines a Möbius transformation φ, which is elliptic of order 2 (its trace being equal to 0). It is clear that

$$(AB - BA)A^{-1} = A(BA^{-1} - A^{-1}B) \text{ and } (AB - BA)B^{-1} = B(B^{-1}A - AB^{-1}) .$$

Using the fact that the sum of any two mutually inverse elements of SL(2, C) is a multiple of the unit element, we obtain

$$AB - BA = BA^{-1} - A^{-1}B \text{ and } AB - BA = B^{-1}A - AB^{-1} .$$

Together, these four identities show that

$$A^{-1} = \varphi A \varphi^{-1} \text{ and } B^{-1} = \varphi B \varphi^{-1} .$$

Geometrically, this means that the axis of φ is the common perpendicular to the axis of A and the axis of B, suitably interpreted in case of A or B being parabolic.

It may be observed that the mapping $\varphi A^{-1}B^{-1}$ is a square root of $ABA^{-1}B^{-1}$. In fact, we have $\varphi = \varphi^{-1}$ and hence

$$(\varphi A^{-1}B^{-1})^2 = \varphi A^{-1}\varphi\varphi B^{-1}\varphi A^{-1}B^{-1} = ABA^{-1}B^{-1} ,$$

since φ transforms A^{-1} into A and B^{-1} into B.

5. Certain numbers

Below, square roots of real numbers represent positive real and "positive imaginary" numbers:

$$(\mathbf{R}_+)^{1/2} = \mathbf{R}_+ \text{ and } (\mathbf{R}_-)^{1/2} = \mathbf{R}_+ i .$$

Let n be a natural number greater than or equal to 2. The number

$$2\psi = 1 + \left(17 - 8\cos\frac{\pi}{n}\right)^{1/2}$$

then belongs to the open interval between 4 and 6. For

$$\lambda = \exp\frac{\pi i}{2n} ,$$

one has the two identities

$$\lambda + \lambda^{-1} = (\psi + 2)^{1/2}(-\psi + 3)^{1/2} ,$$
$$\lambda - \lambda^{-1} = (\psi - 2)^{1/2}(-\psi - 1)^{1/2} .$$

Furthermore, consider the numbers ρ (and ρ^{-1}), x and y defined by

$$2\rho = (\psi + 2)^{1/2} + (\psi - 2)^{1/2} ,$$
$$2\rho^{-1} = (\psi + 2)^{1/2} - (\psi - 2)^{1/2} ,$$
$$2x(\psi - 2)^{1/2} = (-\psi + 3)^{1/2} + (-\psi - 1)^{1/2} ,$$
$$2y(\psi - 2)^{1/2} = -(-\psi + 3)^{1/2} + (-\psi - 1)^{1/2} .$$

Notice that x and $-y$ are conjugate complex numbers. Their absolute value is $(\rho - \rho^{-1})^{-1}$, which is greater than 1.

It is not difficult to show that

$$(\lambda - \lambda^{-1})xy + x + y = 0 \tag{1}$$

and

$$1 + x^2 + y^2 = xy . \tag{2}$$

Further elementary computations give

$$\begin{aligned} -x^2 + \rho^2(1 + x^2) &= \lambda^{-1}y , \\ -x^2 + \rho^{-2}(1 + x^2) &= -\lambda y \end{aligned} \tag{3}$$

and

$$\begin{aligned}\lambda^{-1}(\rho^2-1)x+(\lambda-\lambda^{-1})y+\rho^{-2}&=0\ ,\\ \lambda(1-\rho^2)x+(\lambda-\lambda^{-1})y+\rho^2&=0\ .\end{aligned}\tag{4}$$

6. The matrices

For given $n \geqq 2$ as before, the group to be considered is generated by

$$T=\begin{pmatrix}\rho & 0\\ 0 & \rho^{-1}\end{pmatrix} \quad\text{and}\quad X=\begin{pmatrix}-\lambda x & -(1+x^2)\\ 1 & \lambda^{-1}x\end{pmatrix}.$$

Writing Y instead of $TX^{-1}T^{-1}X$ and applying (3), we obtain

$$Y=\begin{pmatrix}\lambda^{-1}y & \lambda^{-1}x(1+x^2)(\rho^2-1)\\ \lambda x(\rho^{-2}-1) & -\lambda y\end{pmatrix},$$

$$YX=\begin{pmatrix}-\lambda^{-1}[x+(\lambda-\lambda^{-1})xy] & -\lambda^{-2}(1+x^2)[\rho^2+(\lambda-\lambda^{-1})y]\\ \lambda^2[\rho^{-2}+(\lambda-\lambda^{-1})y] & \lambda[x+(\lambda-\lambda^{-1})xy]\end{pmatrix},$$

$$XY=\begin{pmatrix}\lambda[x+(\lambda-\lambda^{-1})xy] & (1+x^2)[\rho^2+(\lambda-\lambda^{-1})y]\\ -[\rho^{-2}+(\lambda-\lambda^{-1})y] & -\lambda^{-1}[x+(\lambda-\lambda^{-1})xy]\end{pmatrix}.$$

It is convenient to rewrite the expressions for YX and XY. Applying (1) in and (4) outside the main diagonals, we see that

$$YX=\begin{pmatrix}\lambda^{-1}y & \lambda^{-1}x(1+x^2)(1-\rho^{-2})\\ \lambda x(1-\rho^2) & -\lambda y\end{pmatrix},$$

$$XY=\begin{pmatrix}-\lambda y & -\lambda x(1+x^2)(1-\rho^{-2})\\ -\lambda^{-1}x(1-\rho^2) & \lambda^{-1}y\end{pmatrix}.$$

Observe that $YX = T^{-1}YT$. By the definition of Y, the element TXT^{-1} is equal to

$$XY^{-1}=\begin{pmatrix}-\lambda x & -\rho^2(1+x^2)\\ \rho^{-2} & \lambda^{-1}x\end{pmatrix}.$$

Finally, using (3) and (2), we obtain

$$XYX^{-1}Y^{-1}=\begin{pmatrix}-\lambda^2 & 0\\ 0 & -\lambda^{-2}\end{pmatrix}.$$

7. The Möbius transformations

From now on T and X are thought of as Möbius transformations acting on the compactified upper half-space, $\mathbf{H}\cup\mathbf{C}\cup\{\infty\}$. The group generated by T and X is denoted by G^*. The subgroup generated by X and $Y=TX^{-1}T^{-1}X$ is denoted by G. It is the commutator subgroup of G^*.

The group F generated by T and $K = XYX^{-1}Y^{-1}$ is abelian and consists of Euclidean similarity transformations.

Consider the set $I = \{fXf^{-1}, fYf^{-1}, fX^{-1}f^{-1}, fY^{-1}f^{-1} \mid f \in F\}$ and denote by P the set of points in $\mathbf{H}$ lying exterior to all isometric hemispheres of the elements of I, that is

$$P = \bigcap_{i \in I} E_i \,.$$

As we shall see, P is an infinite hyperbolic polyhedron whose faces are hyperbolic congruent hexagons. One gets a fairly good impression of what P looks like from considering the orthogonal projection of its boundary onto the complex plane. Locally the picture is as sketched in Figure 4. There A and B denote some pair of generators for G belonging to I and having commutator $ABA^{-1}B^{-1}$ equal to K, and the hexagonal polygons are projections of faces of P situated on the isometric hemispheres of A, B, AB^{-1}, BA and their inverses. In the total picture, such hexagons fill up $\mathbf{C}\backslash\{0\}$ with the group F of similarity transformations acting as a group of automorphisms. Figure 1 shows the qualitative configuration of the isometric circles.

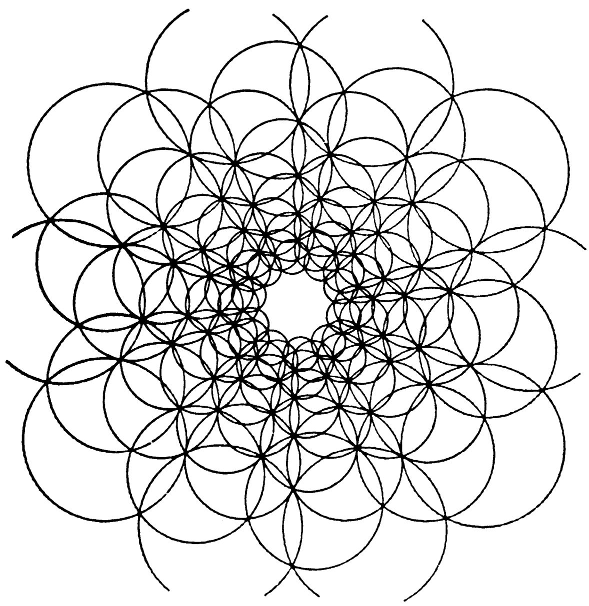

FIGURE 1

One checks that the relative positions of the poles of YX, X, Y and XY^{-1} and their inverses are as described in Figure 2, and that the radii of the isometric circles of these four pairs of mutually inverse elements are ρ^{-1}, 1, ρ and ρ^2, respectively. Thereafter, calculations show that the points of the isometric hemisphere of X or of Y lying exterior to the surrounding hemispheres make up a hyperbolic hexagon whose projection has the proportions given in Figure 3. Easy considerations show that the isometric hemispheres of X^{-1} and Y^{-1} carry faces which are images of those found for

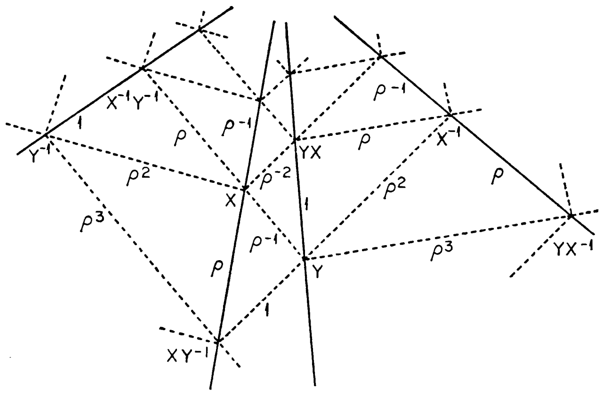

FIGURE 2

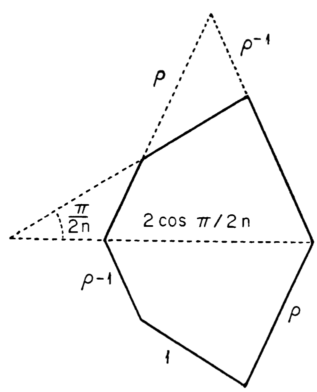

FIGURE 3

X and Y under the smaller square roots of K and K^{-1}, respectively. All together, we obtain a description modulo F of the boundary of P; it follows that P looks as asserted.

A fundamental polyhedron for G^* can now be constructed as follows. First choose a plane through the line of fixed points of K (the j-axis) and intersect P with the infinite $2\pi/n$-angular sector between this plane and its image under K. Any set Q, say, obtained in this manner will be a fundamental polyhedron for G (as we will see later). Intersecting such a polyhedron with the domain between a hyperbolic plane perpendicular to the j-axis and its image under T, we obtain a finite hyperbolic polyhedron R. It is easily seen to possess the characteristic properties of a fundamental polyhedron. In fact, locally we know the boundary identifications (Figure 4 shows us some), and it follows from chain-rule considerations that the angles at equivalent edges add up correctly. Hence, by a theorem of Poincaré (see Maskit [12]) R is a fundamental polyhedron for the group generated by the transformations pairing its faces, that is for G^*. It follows at once that $Q = \bigcup_{n \in Z} T^n R$ is a fundamental polyhedron for G.

The (geometrical) presentations

$$G = \langle X, Y \mid (XYX^{-1}Y^{-1})^n = 1 \rangle$$

and

$$G^* = \langle X, Y, T \mid (XYX^{-1}Y^{-1})^n = X^{-1}TXT^{-1}Y = TYXT^{-1}Y^{-1} = 1 \rangle$$

can be read off from the boundary identifications of suitably chosen polyhedra Q and R (with the angle at the j-axis known). Of course, it is possible to define G^* abstractly using only two generators and two relations.

8. The manifolds

Imagine such a portion of the boundary of P which by orthogonal projection could have given rise to Figure 4. The Lie product φ of A and B

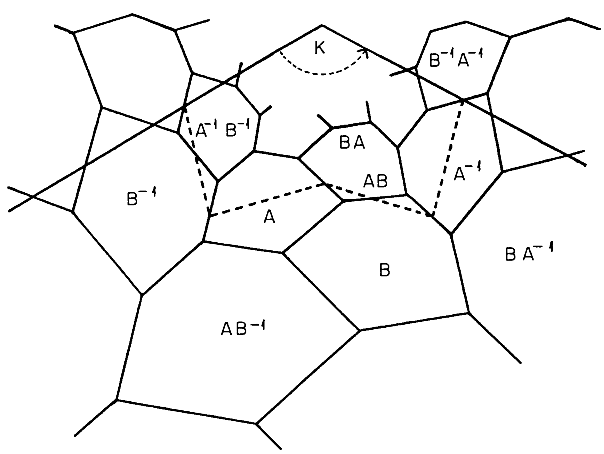

FIGURE 4

has the same isometric hemisphere as AB (and BA) and is easily seen to map the face of P situated on this hemisphere onto itself. When φ is followed by the (smaller) square root $K^{1/2}$ of K, the total mapping amounts to the same as AB. Similarly, the face of P belonging to the isometric hemisphere of A is mapped onto itself by $K^{-1/2}A$, that is, by the Lie product of B and AB^{-1}. Each face of P possesses such a symmetry.

Suppose that a fundamental polyhedron for G has been constructed as suggested previously. When the natural boundary identifications have been made, it represents $\mathbf{H}/G$. Among the faces of this polyhedron, consider those which also belong to the boundary of P as made up by hyperbolic line segments perpendicular to the respective axes of symmetry. Together, these segments yield a fibration by polygonal arcs of the considered part of the boundary of P. In Figure 4, the dotted line segments are supposed to show the projection of one such arc. It is not difficult to extend this fibration of the entire polyhedron with fibers which look as sketched in Figure 5,

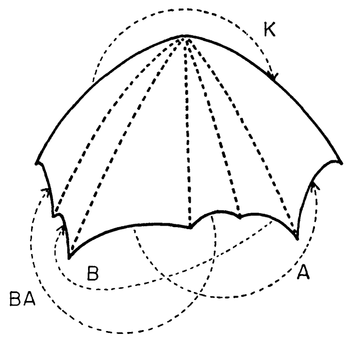

FIGURE 5

each representing a torus with one branch point. In fact, each fiber could be made up by Euclidean line segments (of sufficiently high slopes relatively to the complex plane) joining one fixed point of K and the points of one boundary fiber. We see that $\mathbf{H}/G$ topologically is the product of the real line and the torus with one branch point of order n.

The fibration of $\mathbf{H}/G$ can be made periodic with respect to T and one period will then give us (a model of) $\mathbf{H}/G^*$ as a fiber space with the circle as base and the torus with one branch point of order n as fiber.

9. Smooth examples

Every finitely generated group of n-by-n matrices over a field of characteristic zero contains a normal subgroup of finite index in which no element different from the identity has finite order [18]. It is a consequence that G^* possesses torsion-free, normal subgroups of finite index. If N^* is such a subgroup, then the group $N = G \cap N^*$ is a normal subgroup of finite index in G. Since G is finitely presentable, so is N. Let S denote some power of T belonging to $N^* \backslash \{\mathrm{id}\}$. The group $N(S)$ obtained by adjoining S to N then contains N as a non-trivial, finitely generated normal subgroup. The quotient group $N(S)/N$ is infinite cyclic. Since $N(S)$ is torsion-free, the fundamental groups of the quotient manifolds $\mathbf{H}/N(S)$ and $\mathbf{H}/N$ are isomorphic to $N(S)$ and N, respectively. Furthermore, the manifold $\mathbf{H}/N(S)$ is smooth and compact. Stallings' fibration theorem [19] tells us that N is isomorphic to the fundamental group of a compact surface and that $\mathbf{H}/N(S)$ fibers over the circle with this surface as fiber. Thus, Theorems 1 and 2 are true.

10. Other examples

Denote by L the group obtained by adjoining to G the Lie product of X and Y. The index of G in L is 2 and L acts discontinuously on $\mathbf{H}$. In a sense, the manifold $\mathbf{H}/L$ is half as much as $\mathbf{H}/G$. It is the product of the real line and a sphere with three branch points of order 2 and one branch point of order $2n$. This is seen as before by constructing a fundamental polyhedron for L. This time the polyhedron P must be "divided" by the cyclic group generated by $K^{1/2}$ and the Lie products come into play, pairing faces of the boundary. A presentation is

$$L = \langle l_1, l_2, l_3 \,|\, l_1^2 = l_2^2 = l_3^2 = (l_1l_2l_3)^{2n} = 1\rangle ,$$

where the elements l_1, l_2 and l_3 can be thought of as the Lie products of certain pairs of generators for G and their product $l_1l_2l_3$ as a square root of K.

Let us for a moment go back to Section 5 and there extend the domain of definition for ψ and λ by allowing n to be half an integer ($\geqq 3/2$). For $2n = 2m - 1$ and thus

$$2\psi = 1 + \left(17 - 8\cos\frac{2\pi}{2m-1}\right)^{1/2} \quad \text{and} \quad \lambda = \exp\frac{\pi}{2m-1} ,$$

this gives rise to an infinite family of new groups M. In these examples, the mapping determined by $K = XYX^{-1}Y^{-1}$ is the square root of the mapping determined by K^m. Thus, the group M generated by X and Y contains the Lie product of X and Y. It is easy to see that M has a fundamental polyhedron which looks like the polyhedron for L. The geometrical presentation is

$$M = \langle l_1, l_2, l_3 \,|\, l_1^2 = l_2^2 = l_3^2 = (l_1l_2l_3)^{2m-1} = 1\rangle ,$$

but, as we know, the rank of M is only 2 (a fact which surprisingly enough has been observed only recently). Of course, the manifold $\mathbf{H}/M$ is the product of the real line and a sphere with branch points of orders 2, 2, 2 and $2m-1$. The manifolds $\mathbf{H}/L^*$ and $\mathbf{H}/M^*$ of the extended groups fiber over the circle.

11. Remarks

The idea behind the work which has been presented comes from my study of certain spaces of finitely generated discrete groups [7]. Presumably, the groups which can occur as kernels in Theorem 2 are all boundary points of spaces of quasi-Fuchsian groups and may be thought of as fixed points for automorphisms acting on these spaces in a rather natural manner.

The groups G we have studied were fixed points for the automorphism given by $(X, YX) \mapsto (XY^{-1}, Y)$. For each of these, the polyhedron P was bounded by hexagonal faces. If the automorphism $(X, YX^2) \mapsto (XY^{-2}, Y)$ had been used instead, we would have met polyhedra with both quadrangular and octagonal faces. In general, there is a simple connection between the structure of the automorphisms and the geometry associated with their fixed points.

It is interesting to observe that G coincides with the commutator subgroup of G^*. In other examples I have examined, the kernel is still only a finite extension of the commutant of the full group. It might be worthwhile to pursue this observation.

Another special feature of the above examples becomes apparent when we compare to Kleinian groups. The work of Ahlfors (see Greenberg [5]) implies that finitely generated (non-elementary) Kleinian groups, in contrast to the groups G, must have finite index whenever they occur as normal subgroups of other discontinuous groups of rigid motions of $\mathbf{H}$.

In conclusion, I would like briefly to explain how the numbers and matrices of Sections 5 and 6 were found, starting from the assumption that X should be conjugate to XY^{-1} and YX to Y under a Möbius transformation T and, furthermore, that the commutator of X and Y be elliptic of order n with trace equal to $-2\cos \pi/n$. In general the conjugacy class of a group generated by two transformations is determined by the traces of a pair of generators and the trace of their product. Under the present conditions, the traces must satisfy certain equations, the solutions to which are few. Among these solutions, if I remember correctly, there is essentially only one which gives rise to a discrete group. However, there remain some rather unpleasant (elementary) computations to be done in order to bring the matrices to a suitable form. For that reason, and also because I am unable here to explain completely my preliminary insight into this subject, I have presented the material in its final form, as facts merely to be verified.

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