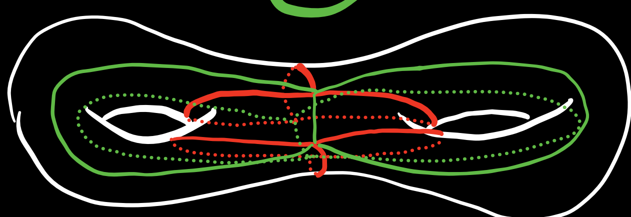
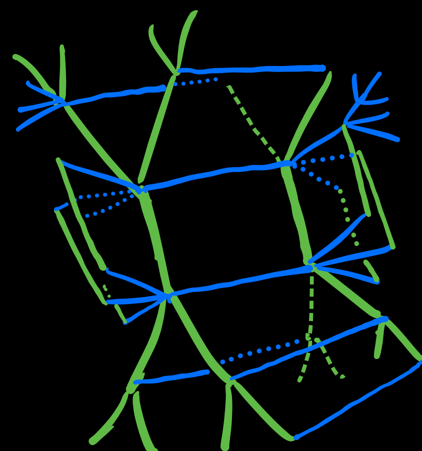
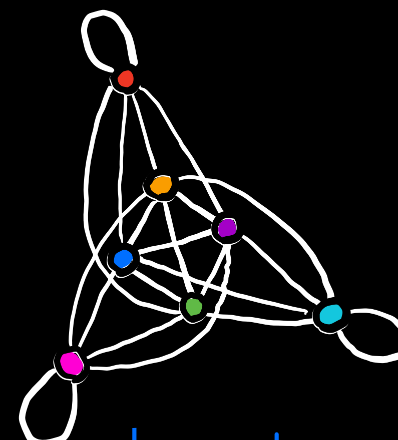
10/4/23

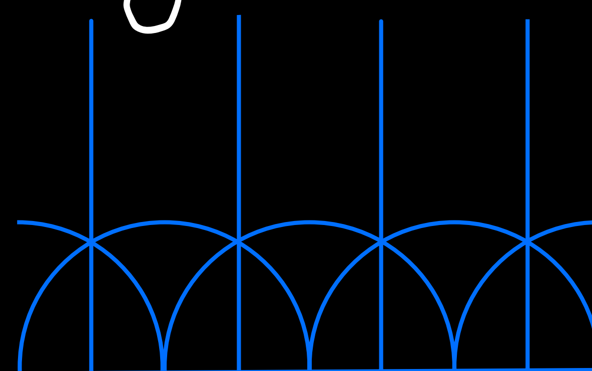
Two-by-Two Matrices

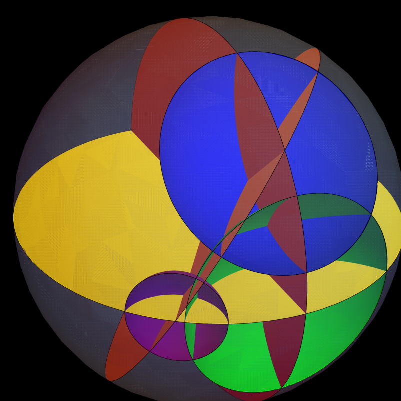
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$











University of San Francisco Colloquium

Nic Brody
UC Santa Cruz

Motivation (for experts)

Low-dimensional topologists are quite interested in understanding "hyperbolic 3-manifolds" - which are essentially the same as "discrete subgroups of $PSL_2\mathbb{C}$"

Largely, this talk is concerned with — What happens if we drop the discreteness assumption?

<u>A</u> We can construct a higher-dimensional space, and use discreteness there to understand the group!

Part I Rotations of the sphere

Part II Continued Fractions

Part III Matrices + Quaternions

Part IV Classification of linear (matrix) groups (??)

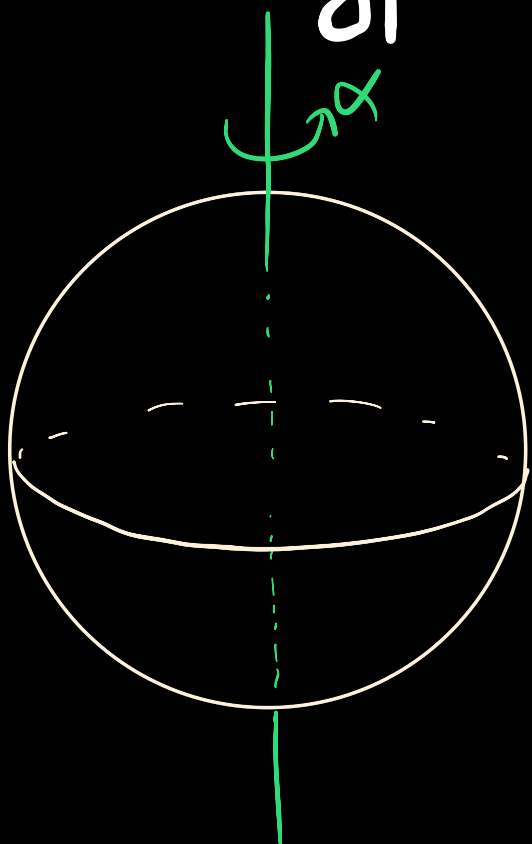
We will begin by studying two seemingly unrelated dynamical systems, and then relate them to questions about 2-by-2 matrices

Part I

Spinning in Circles

Suppose you have a globe, and you are allowed to make four types of <u>moves</u>:

Suppose you have a globe, and you are allowed to make four types of moves:

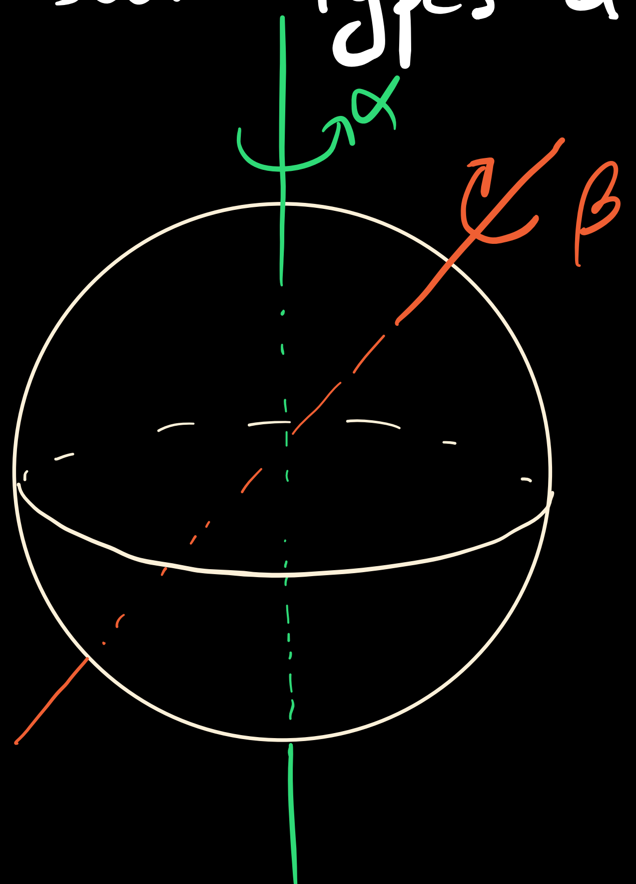


You can rotate by a fixed angle α about the green axis in either direction

Call these rotations

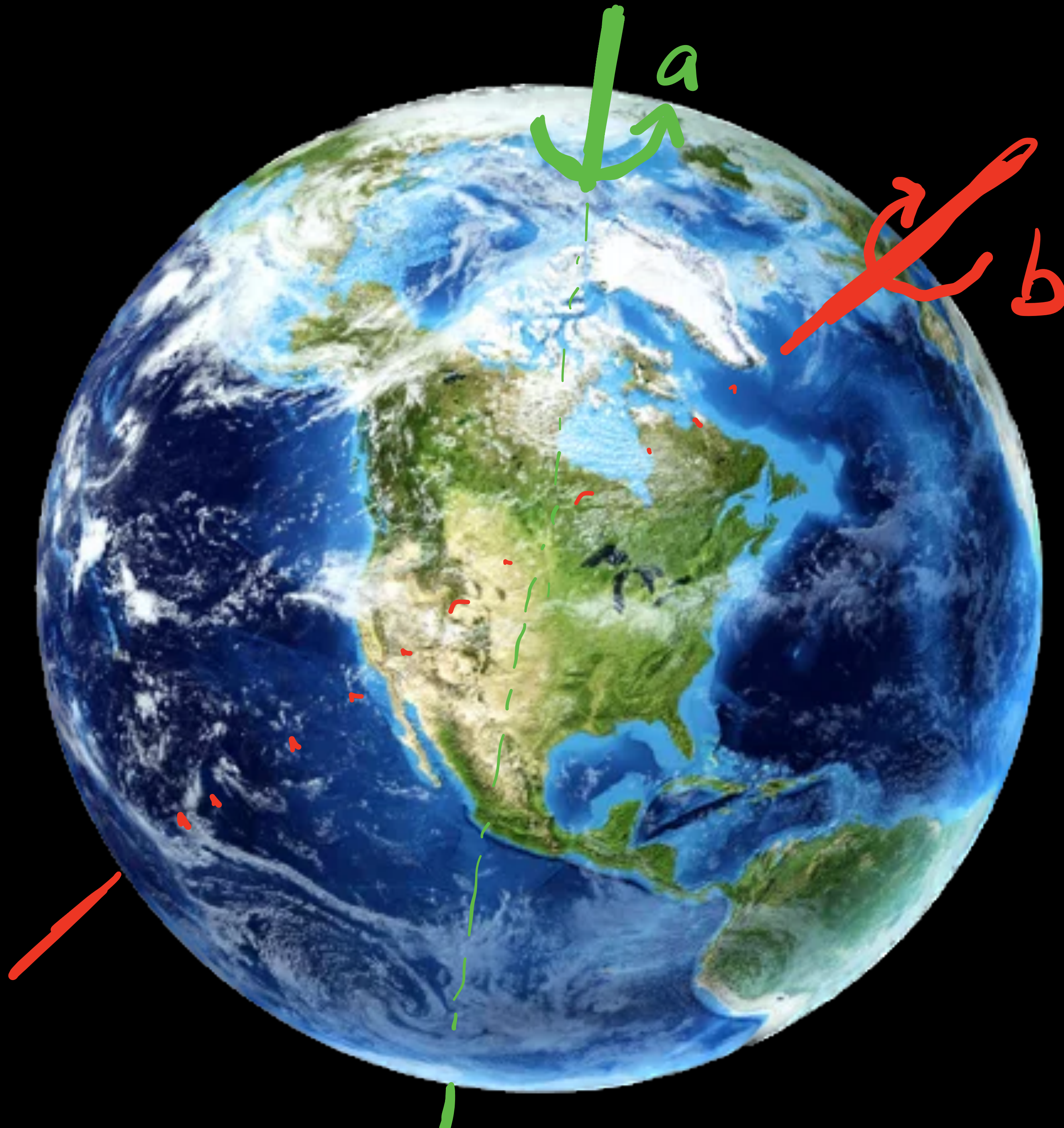
a, a^{-1}

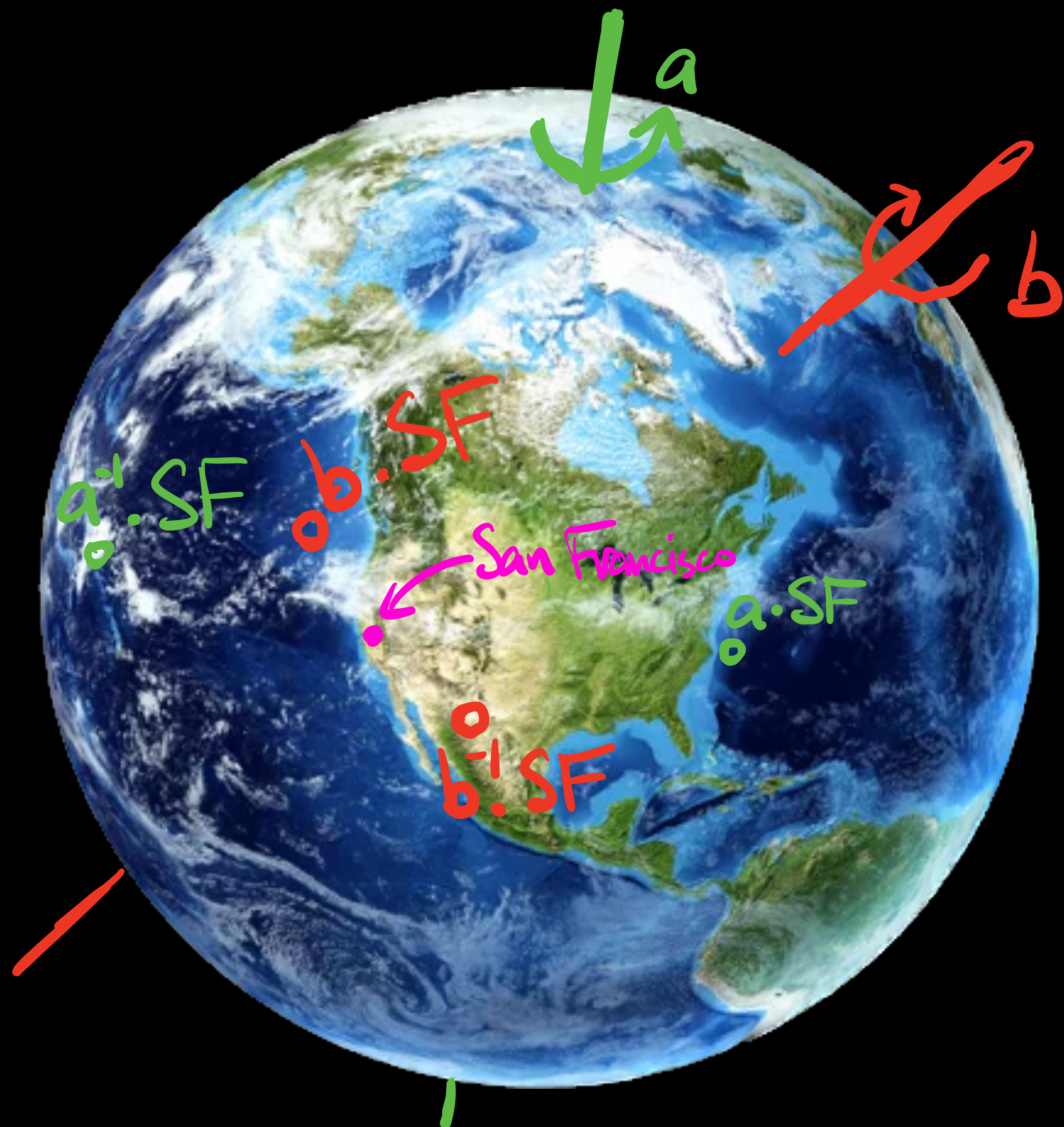
Suppose you have a globe, and you are allowed to make four types of <u>moves</u>:



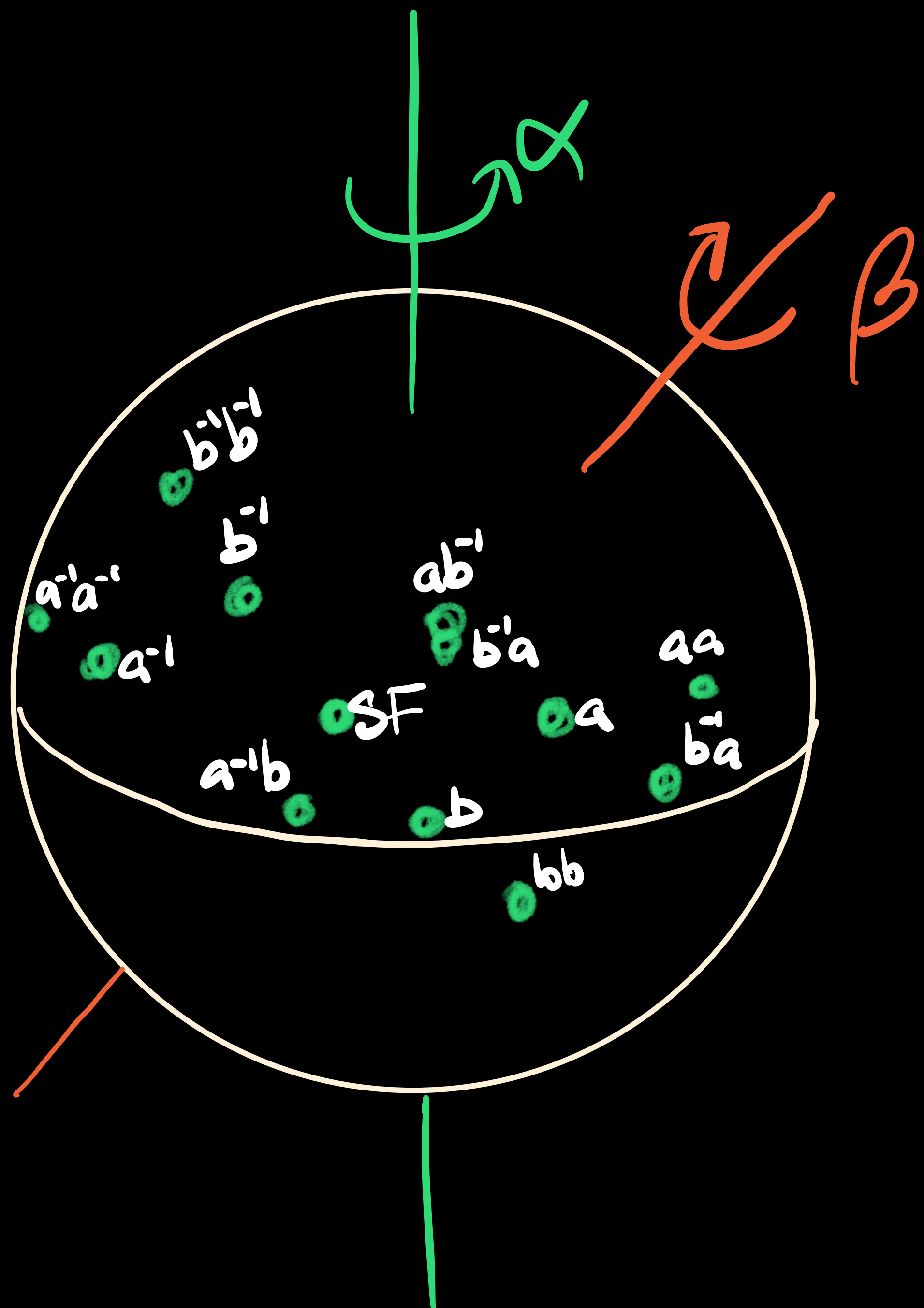
You can rotate by a fixed angle β about the red axis in either direction

b, b^{-1}


a
b



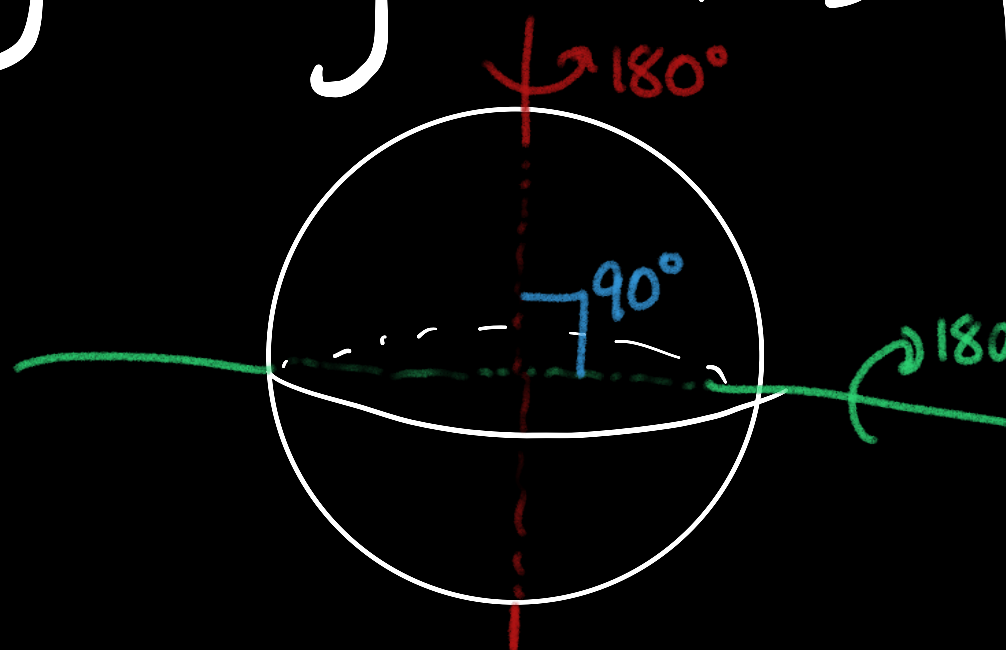
a
b
b·SF
a·SF
San Francisco
a·SF
b·SF

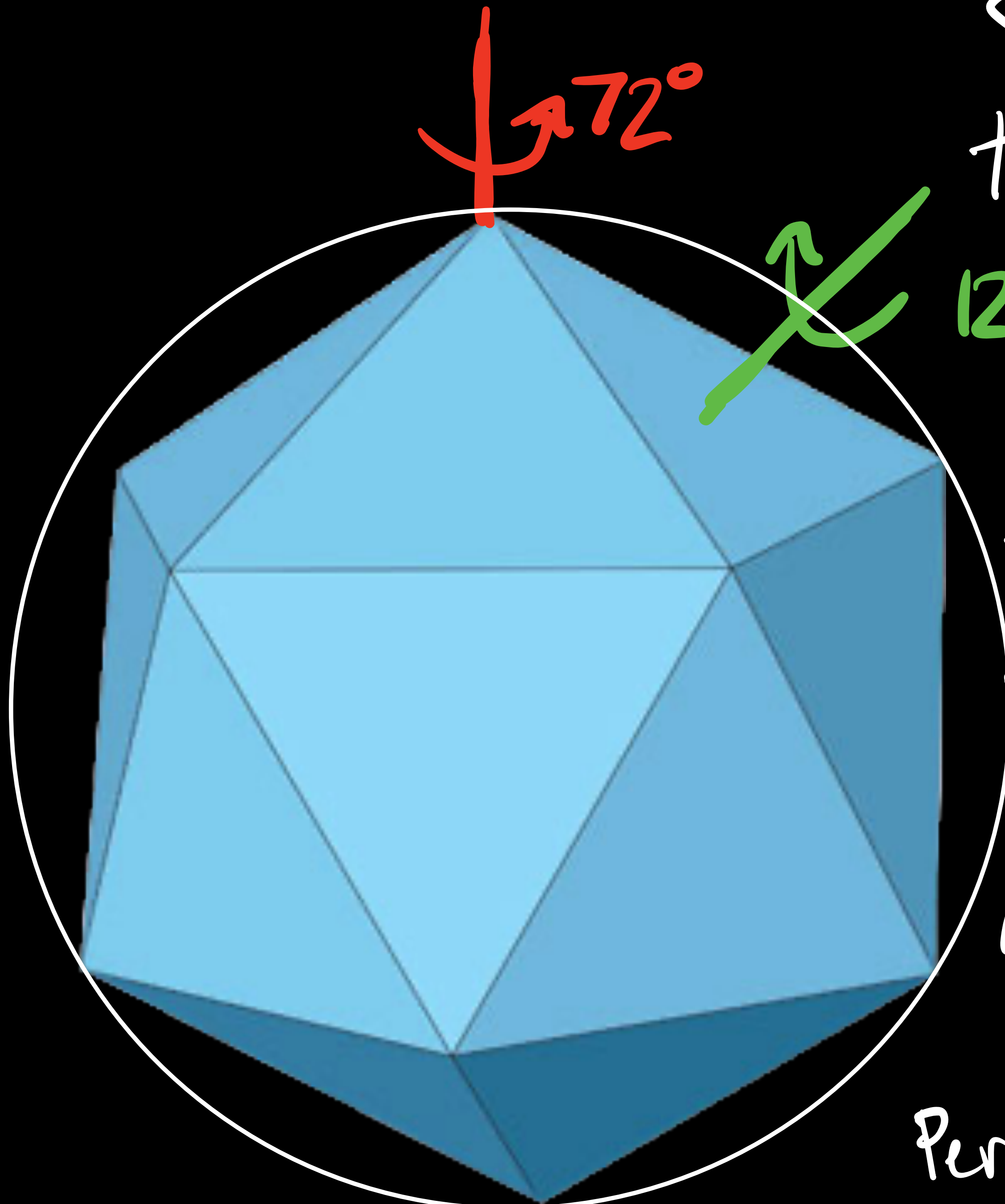

α
β
$b^{-1}b^{-1}$
b^{-1}
$a^{-1}a^{-1}$
a^{-1}
ab^{-1}
$b^{-1}a$
aa
SF
a
$b^{-1}a$
$a^{-1}b$
b
bb

We now ask:
which points on the globe can be reached via a sequence of these moves?

For example - is it possible to move San Francisco onto Santa Cruz?

If you are incredibly precise with the choice of axes and angles, you may have only finitely many rotations possible:

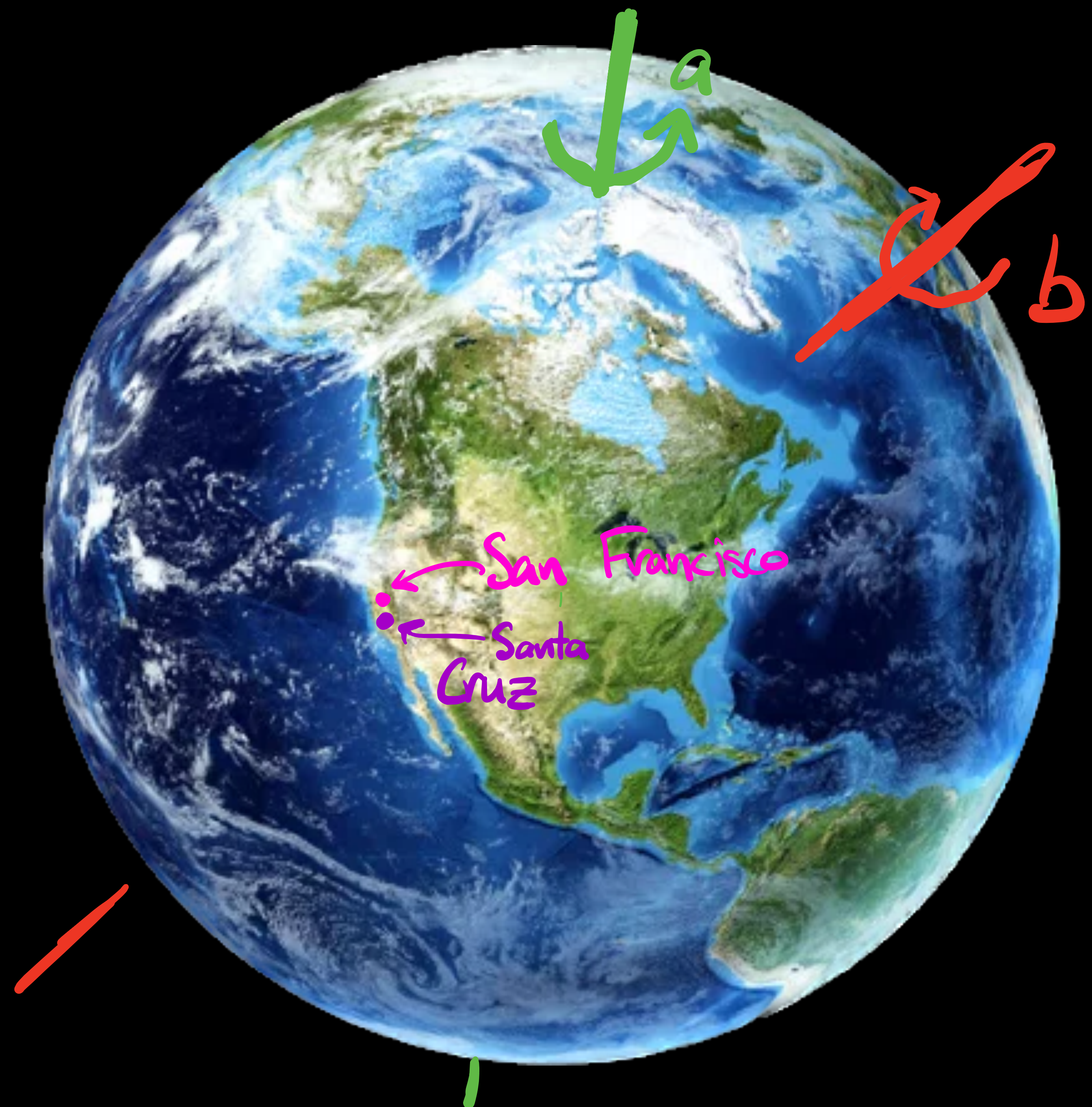




Since these two rotations send vertices to vertices, there are only finitely many rotations attainable!

Permutes the 12 vertices

However, if the parameters are chosen "randomly," there is a 100% chance that there will be infinitely many rotations that can be achieved by composing sequences of our four moves


a
b
San Francisco
Santa Cruz

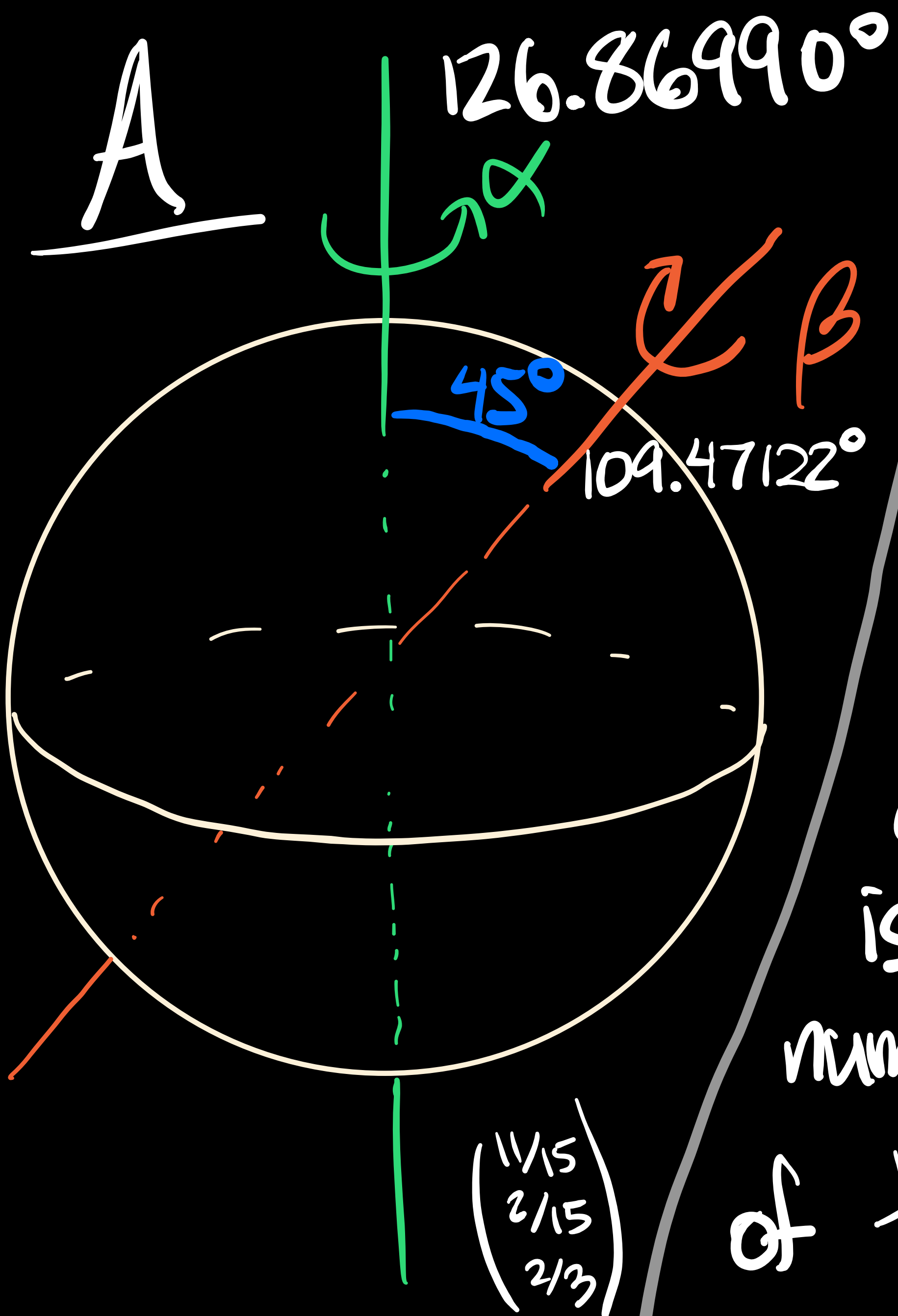
<u>A</u> With 100% probability in the choice of moves, it is <u>always</u> possible to move San Francisco on top of Santa Cruz.

In fact, for any choice of two atoms on the surface of the earth, it will be possible to move one to overlap the other via a finite sequence of moves!

(we will ignore important quantum mechanical questions raised here)

However, this is a math talk, and we are pickier than that... We would like to know exactly which points can be moved to which points!

Note the set of points on the sphere is <u>uncountable</u>, while the set of points we can reach from our game is <u>countable</u>. Is it possible to describe the set of reachable points?

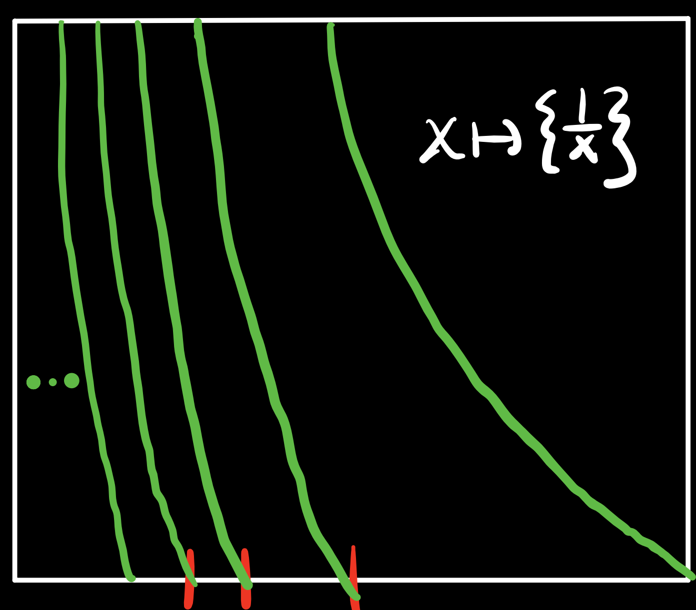


For these parameters, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ can be moved to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ exactly when $x^2+y^2+z^2=1$ and each coordinate is a rational number with denominator of the form $3^m \cdot 5^n$.

Before attempting to explain how we are able to answer this question, we now look at another type of problem...

Part II

Continued Fractions



Define $\frac{1}{0}=\infty$, $\frac{1}{\infty}=0$.

$\infty \pm 1 = \infty$.

<u>Q</u> Which numbers can be obtained from ∞ via a sequence of the operations

$x \mapsto \frac{1}{x}$ $\quad$ $x \mapsto x+1$ $\quad$?

$\quad\quad\quad\quad\;\,$ $x \mapsto x-1$

Step 1 For any real number x, there is a unique integer a_1 so that $0 \leq x - a_1 < 1$.

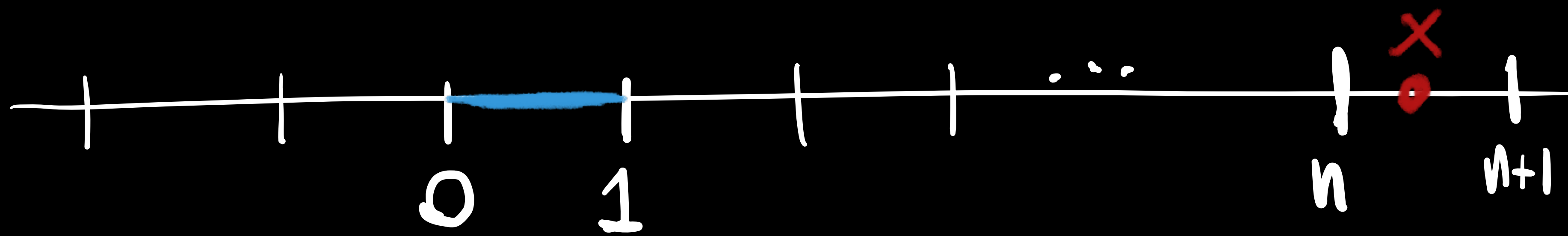
Eg

$$0 \leq \pi - 3 < 1$$

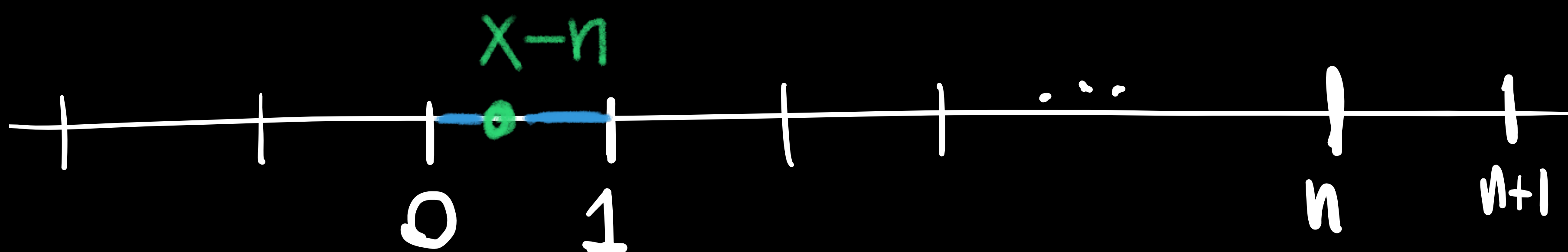
$$0 \leq \frac{11}{2} - 5 < 1$$

$$0 \leq \sqrt{2} - 1 < 1$$

$$0 \leq -5 + 5 < 1$$



x
0
1
n
n+1



x−n
0
1
n
n+1

Step 2 If $x - a_1 \neq 0$, invert it.

Apply Step 1 to $\frac{1}{x - a_1}$ (if $\underset{\text{done}}{x - a_1 = 0,}$)

$\ast \qquad \ast$

If $0<x<1$ is rational, $\mathrm{Numerator}(x) < \mathrm{Denominator}(x)$.

Hence, the denominator of $\frac{1}{x}$ is smaller than that of x.

Since shifting by an integer does not change the denominator, this process must eventually terminate!

$$\frac{40}{31} \xrightarrow{-1} \frac{9}{31} \xrightarrow{\text{Invert}} \frac{31}{9}$$

$$\xrightarrow{-3} \frac{4}{9} \xrightarrow{\text{Invert}} \frac{9}{4}$$

$$\xrightarrow{-2} \frac{1}{4} \xrightarrow{\text{Invert}} 4$$

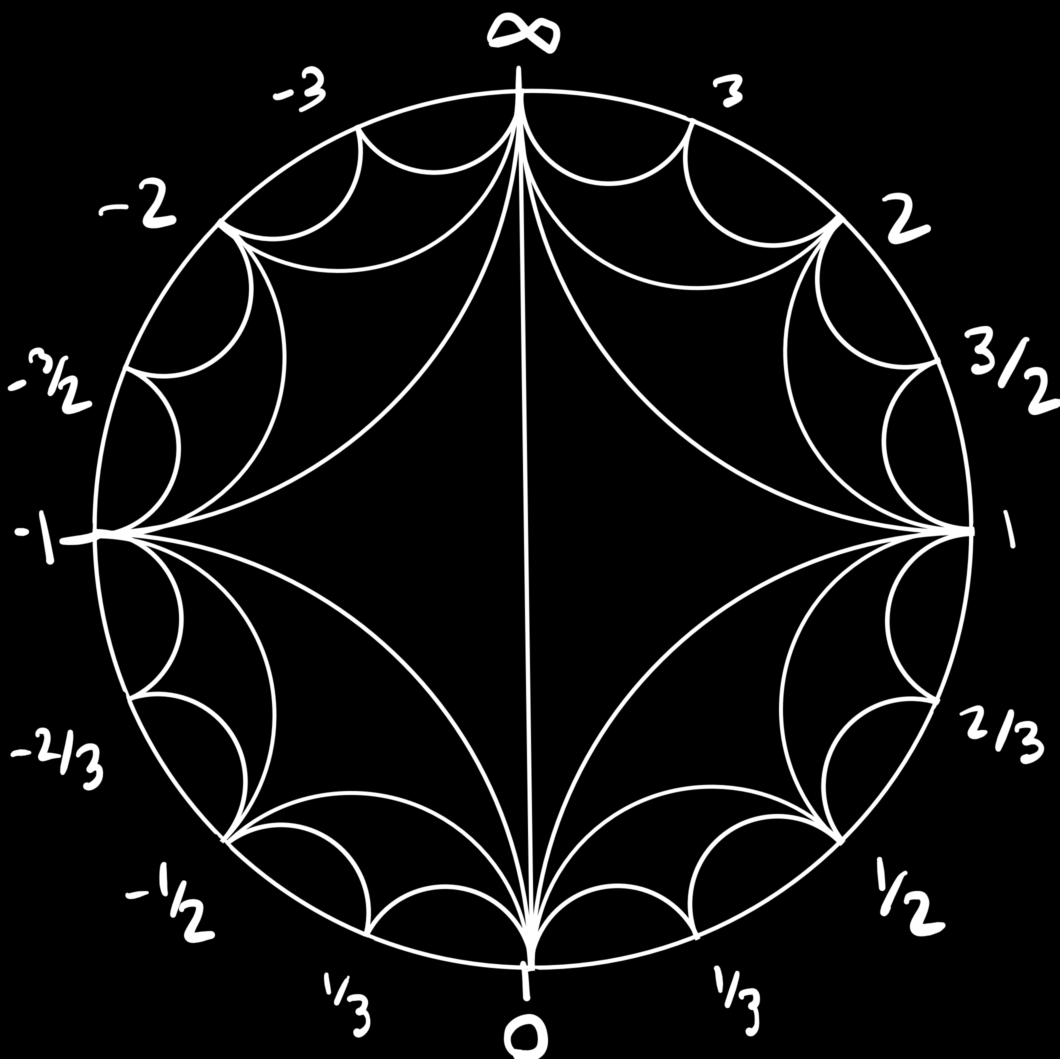
$$\xrightarrow{-4} 0 \xrightarrow{\text{Invert}} \infty$$

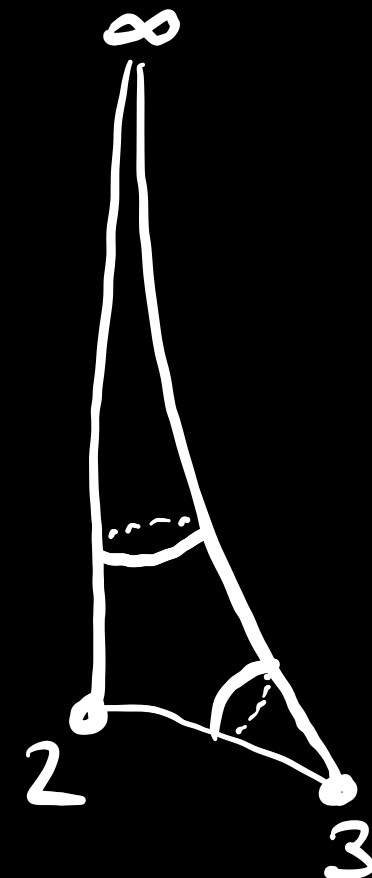
Reversing this process, we see

$$1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{4}}} = \frac{40}{31}$$

Cons Every rational number can be written $a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\dots+\cfrac{1}{a_k}}}$

$a_0 \in \mathbb{Z}$, $a_i \in \mathbb{Z}_{\geq 1}$.


∞
-3
3
-2
2
-3/2
3/2
-1
1
-2/3
2/3
-1/2
1/2
1/3
0
1/3


∞
2
3

This has been understood for centuries. However, what if we replace the operations with

$$x \overset{A}{\longmapsto} 2x \qquad x \underset{A^{-1}}{\longmapsto} \frac{x}{2}$$

$$x \overset{R}{\longmapsto} \frac{x-1}{x+1} \;?$$

We will now prove that choosing different operations can sometimes produce a different continued fraction algorithm!

We will say a fraction $\frac{p}{q}$ is <u>simpler</u> than $\frac{p'}{q'}$ if $\max\{|p|,|q|\} < \max\{|p'|,|q'|\}$ when written in reduced form (and <u>no more complicated</u> if $\leq$)

<u>Eg</u> $\frac{99}{100}$ is simpler than $\frac{1}{101}$,

$\frac{48}{96}$ is simpler than $\frac{1}{3}$.

$\frac{4}{5}$ is no more complicated than $1/5$.

Important Observation

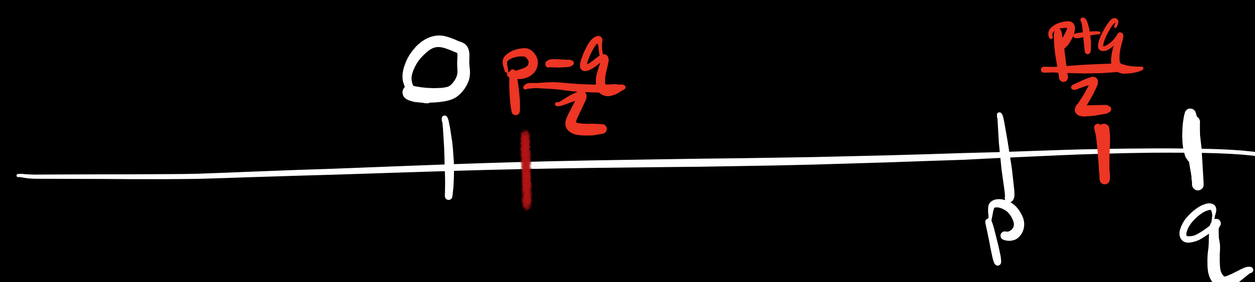
If p, q are odd and $|p| \neq |q|$

$\frac{p-q}{p+q}$ is simpler than $\frac{p}{q}$.

Pf Since p, q are both odd, $p+q$ and $p-q$ are both even.

Therefore, $\frac{p-q}{p+q} = R\left(\frac{p}{q}\right)$ simplifies to $\frac{\left(\frac{p-q}{2}\right)}{\left(\frac{p+q}{2}\right)}$, and

$$\max\left\{\left|\frac{p-q}{2}\right|, \left|\frac{p+q}{2}\right|\right\} < \max\{|p|, |q|\}$$



We also note: If the denominator is even,

$$A\left(\frac{p}{q}\right) = 2\cdot\frac{p}{q} = \frac{p}{(q/2)}$$ is

no more complicated than $\frac{p}{q}$

(similarly, if p even, $A^{-1}\left(\frac{p}{q}\right)$ is no more complicated)

Algorithm

	p even	p odd
q even	Not reduced	$2\frac{p}{q}$ is nmc
q odd	$\frac{1}{2}\frac{p}{q}$ nmc	$\frac{p-q}{p+q}$ is simpler

Apply until we reach $|p|=|q|=1$.

$$\frac{13}{28} \overset{A^2}{\rightsquigarrow} \frac{13}{7} \overset{R}{\rightsquigarrow} \frac{6}{20} = \frac{3}{10}$$

$$\overset{A}{\rightsquigarrow} \frac{3}{5} \overset{R}{\rightsquigarrow} \frac{-2}{8} = -\frac{1}{4}$$

$$\overset{A^2}{\rightsquigarrow} -1 \overset{R}{\rightsquigarrow} \frac{-2}{0} = \infty$$

Thus, every rational number can be reached from ∞ via the operations!

This is supposed to be a talk about 2-by-2 matrices, and so far none have shown up!

Part III

Matrices + Quaternions

Note that 2-by-2 matrices can operate on vectors $\begin{pmatrix} u \\ v \end{pmatrix}$

via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au+bv \\ cu+dv \end{pmatrix}$.

This preserves "scaling" $g.(\lambda v) = \lambda(g.v)$

They can hence operate on "slopes" $\frac{u}{v}$ of vectors $\begin{pmatrix} u \\ v \end{pmatrix}$

We'll focus on the case of matrices with rational entries, so that the slope is either a rational number or ∞.

Under $\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax+b}{cx+d}$,

we have $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x = x+1$

$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} x = x-1$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x = \frac{1}{x}$.

Multiplying matrices corresponds to composing functions.

We can now translate

$$\text{“}\; 1+\cfrac{1}{3+\cfrac{1}{2+1/4}} = \frac{40}{31} \;\text{”}$$

to

$$\text{“}\; \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&4\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}40&9\\31&7\end{pmatrix} \;\text{”}$$

The operations

$$x \mapsto 2x \ , \ x \mapsto \frac{x-1}{x+1}$$

are then represented by

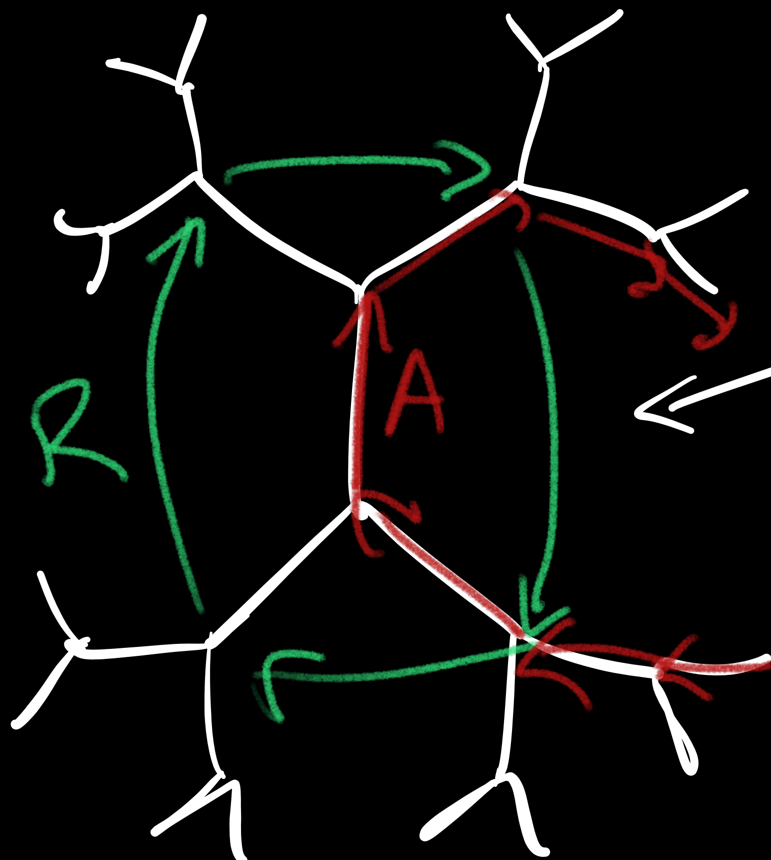
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$\frac{13}{7} \overset{R}{\rightsquigarrow} \frac{13-7}{13+7} = \frac{6}{20} = \frac{3}{10} \overset{A}{\rightsquigarrow}$$

$$\frac{3}{5} \overset{R}{\rightsquigarrow} \frac{-2}{8} = \frac{-1}{4} \overset{A^2}{\rightsquigarrow} \frac{-1}{1} \overset{R}{\rightsquigarrow} \frac{-2}{0} = \infty$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{-1}{8} \begin{pmatrix} 13 & -11 \\ 7 & -1 \end{pmatrix}$$

$$\frac{1 + \dfrac{1 + \frac{1 - 1/4}{1 + 1/4}}{2\left(1 - \frac{1 - 1/4}{1 + 1/4}\right)}}{1 - \dfrac{1 + \frac{1 - 1/4}{1 + 1/4}}{2\left(1 - \frac{1 - 1/4}{1 + 1/4}\right)}} = \frac{13}{7} \quad \rightsquigarrow \quad \begin{pmatrix} 13 & -11/64 \\ 7 & -1/64 \end{pmatrix} \in G.$$



gives another continued fraction calculator!

<u>Conj</u> Suppose $\Gamma \leq PGL_2\mathbb{Q}$ is a lattice in a completion of $\mathbb{Q}$.

Then $\mathbb{Q}\mathbb{P}^1/\Gamma$ is finite

$$\left(\text{True for } \begin{array}{l} \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \underset{L}{\leq} PGL_2\mathbb{R} \\ \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \underset{L}{\leq} PGL_2\mathbb{Q}_2 \end{array} \right)$$

Prop If a, b, c, d are real numbers with $a^2+b^2+c^2+d^2=1$, the matrix $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ determines a rotation of the sphere, and multiplying such matrices corresponds to concatenating rotations

* It is somewhat miraculous that we can take two solutions to $a^2+b^2+c^2+d^2=1$ and produce another! *

Similarly, if $a^2+b^2=x^2+y^2=1$,

$$(ax-by)^2+(ay+bx)^2=1$$

We will let the expression "$a+bi+cj+dk$" be shorthand for the matrix

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

We have a dictionary

$$
\begin{tikzcd}
 & \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
a+bi+cj+dk \arrow[rr, leftrightarrow] & & \text{Rotation of a sphere about the axis } \begin{pmatrix} b \\ c \\ d \end{pmatrix} \text{ by angle } 2\cos^{-1}(a).
\end{tikzcd}
$$

Defn If S is a finite set of 2-by-2 matrices, $\langle S \rangle$ will denote the set of all matrices one can obtain by multiplying matrices in S and their inverses.

We will call this set of matrices "the group generated by S"

Operations

If $g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$,

their product is

$$g_1 g_2 = \begin{pmatrix} a_1 d_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & a_1 b_2 + d_1 d_2 \end{pmatrix}$$

In particular — if A is a set of numbers closed under addition and multiplication, then $g_1 g_2$ has entries in A

Moreover, if $\frac{1}{ad-bc}$ is in A,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Fundamental Theorem of Arithmetic

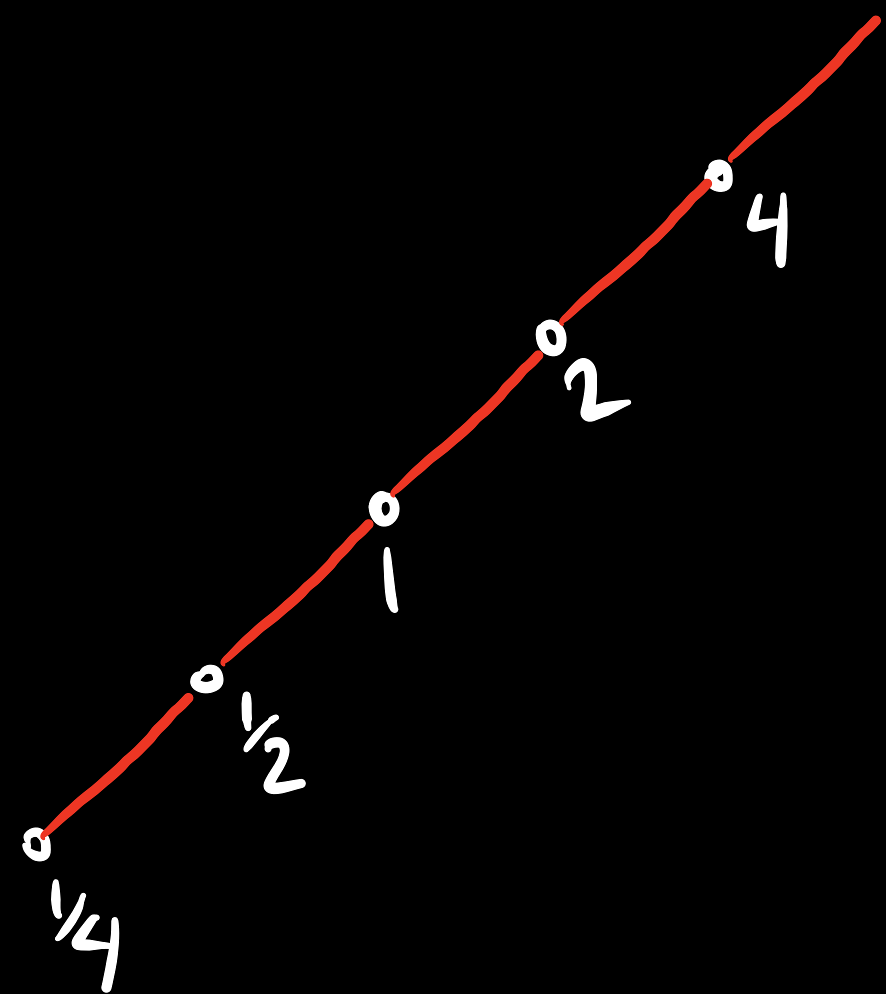
Every positive integer can be expressed uniquely as a product of primes

Geometric Interpretation

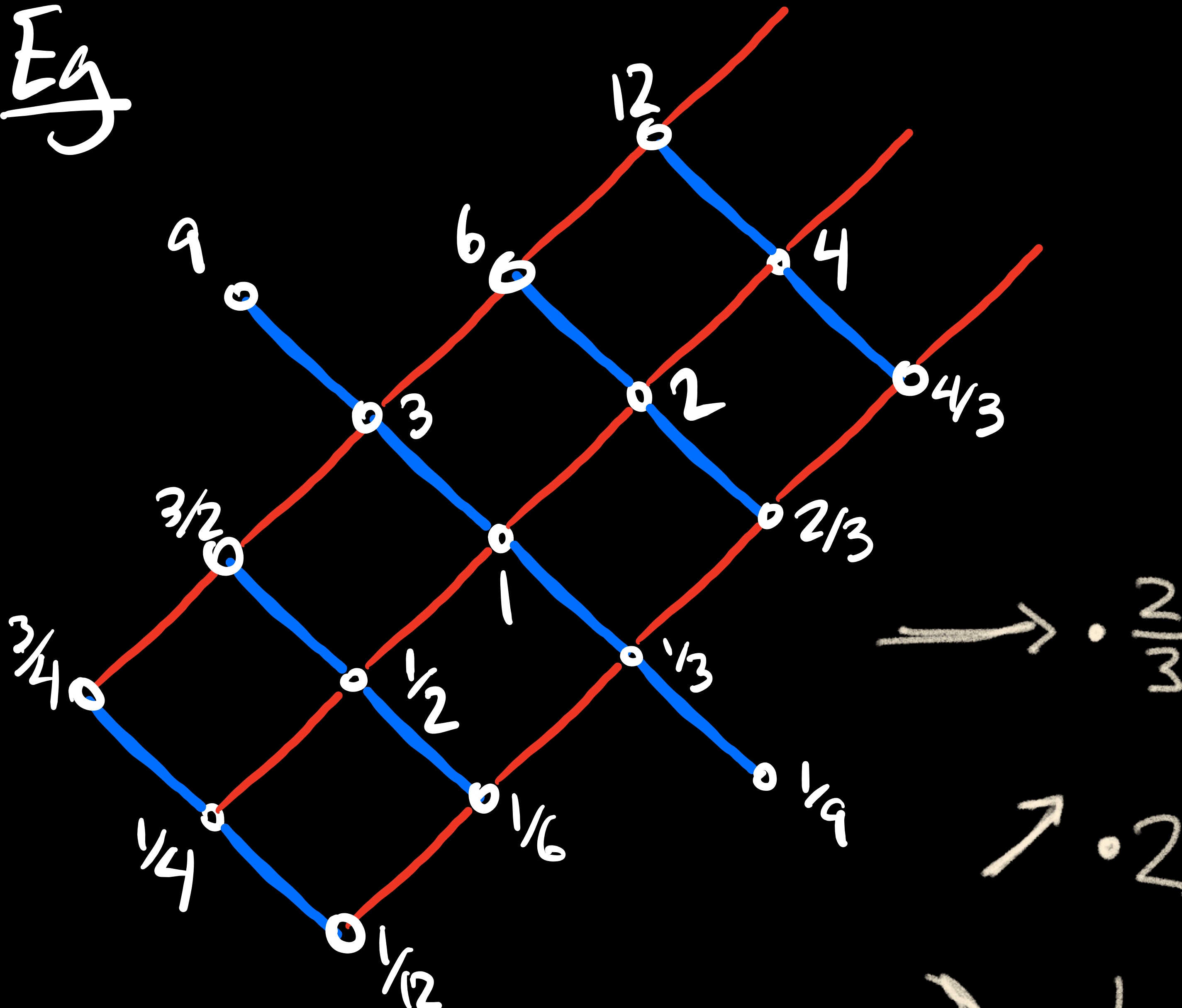
We can arrange the positive rationals in an infinite-dimensional grid.

Multiplying by a given rational $\frac{p}{q}$ will act as a translation on this grid.

Eg

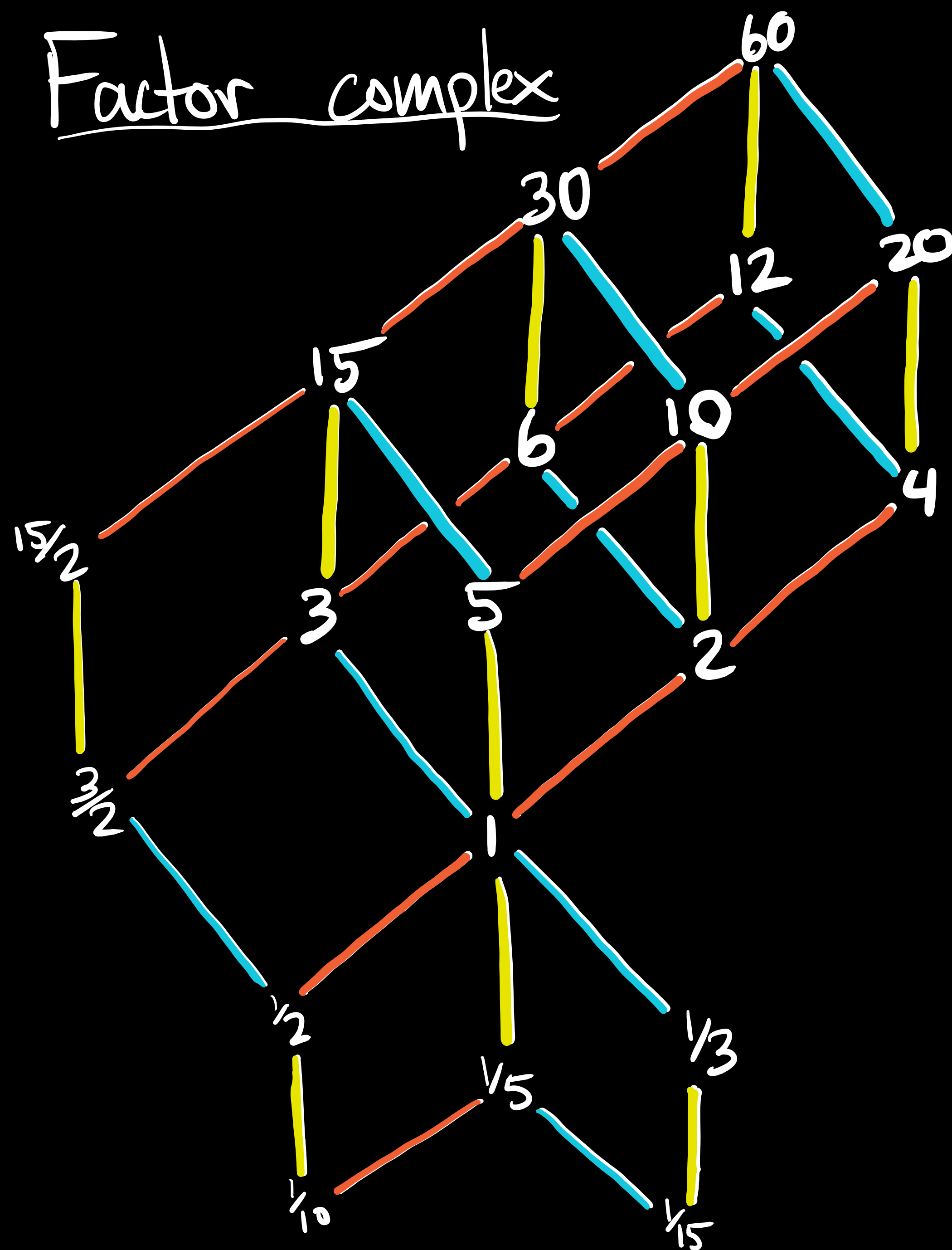


Eg



12
9
6
4
3
2
4/3
3/2
1
2/3
3/4
1/2
1/3
1/4
1/6
1/9
1/12
$\cdot \frac{2}{3}$
$\cdot 2$
$\cdot \frac{1}{3}$

Factor complex



Fundamental theorem of quaternion arithmetic

Let q be a primitive integer quaternion. ahh!

If p is an odd prime dividing $N(q)$,

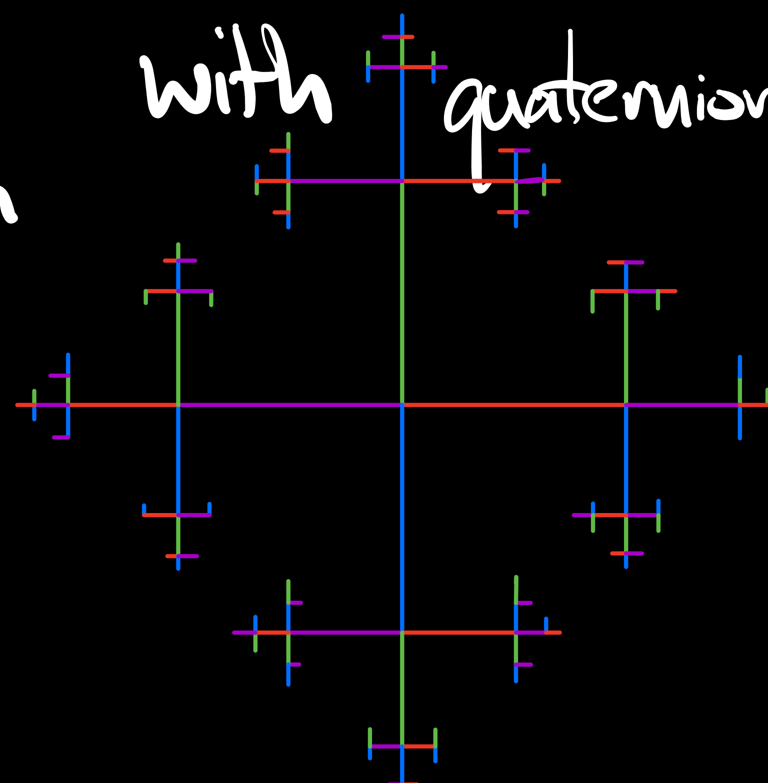
there is a unique quaternion of norm p Scary!

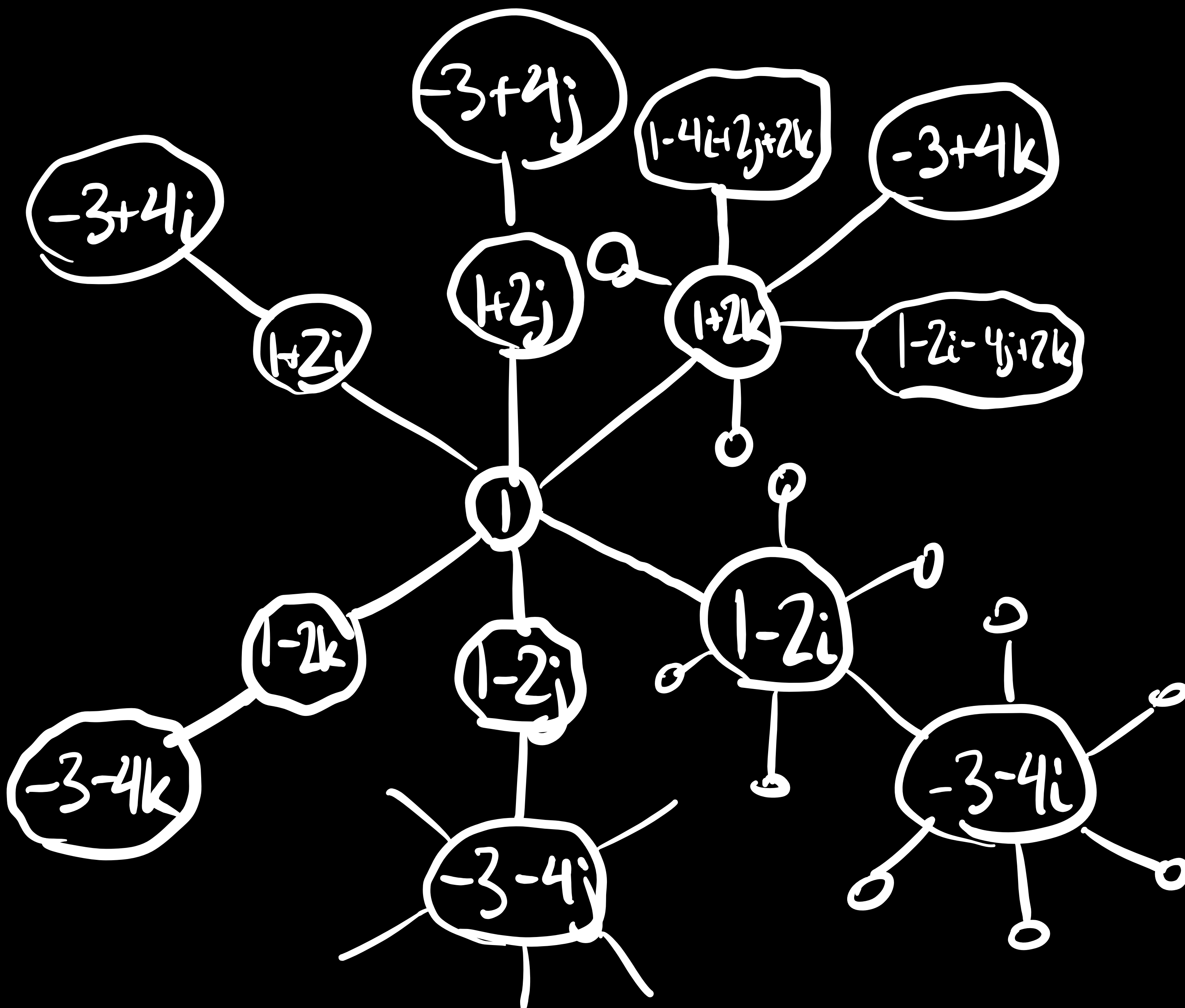
(up to associates) which left-divides q.

Thm (Jacobi) The number of representations of an odd prime p as a sum of four squares is $8(p+1)$

Consequence

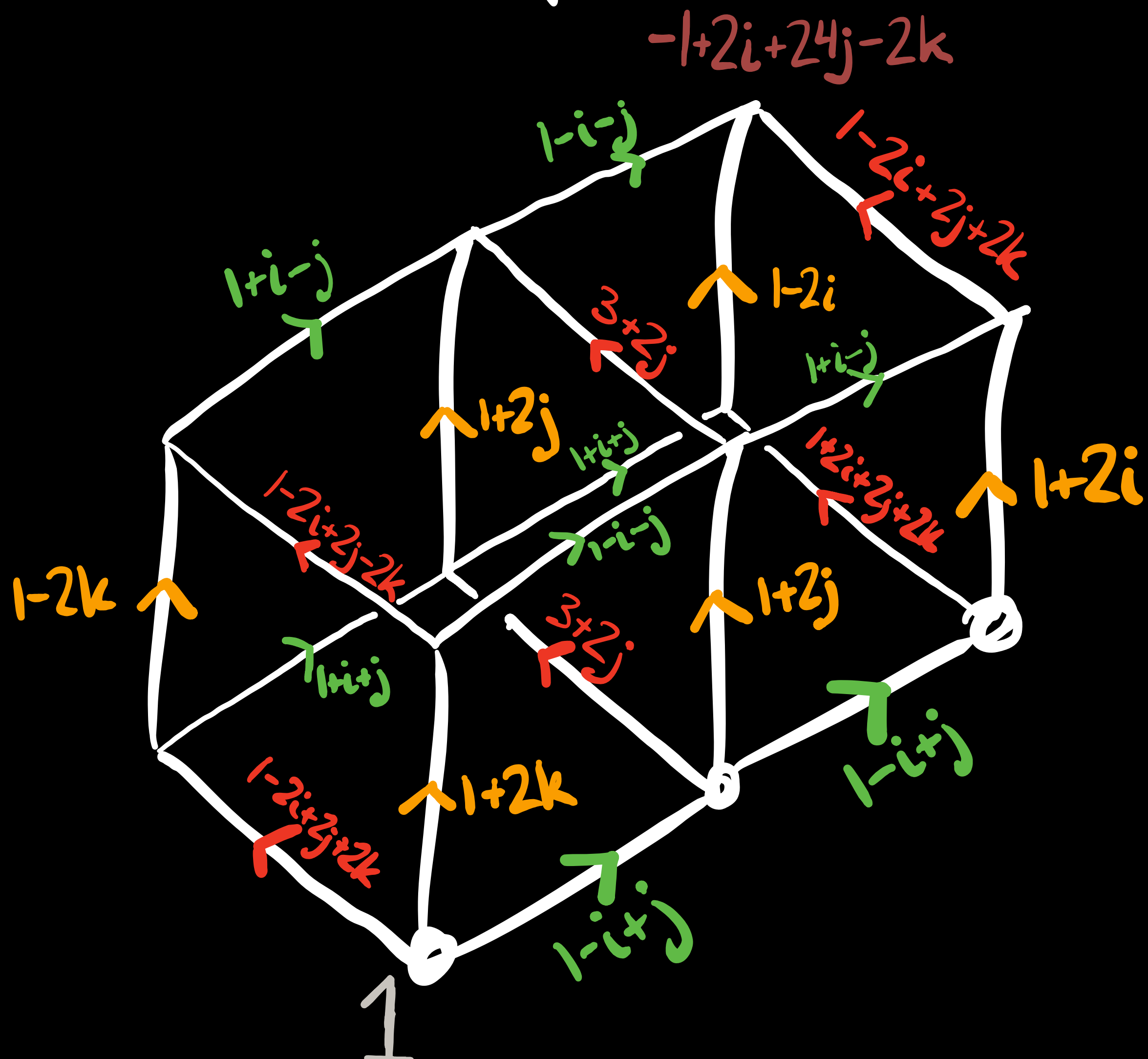
For an odd prime p, we can label the vertices of an infinite tree with quaternions of norm p^n





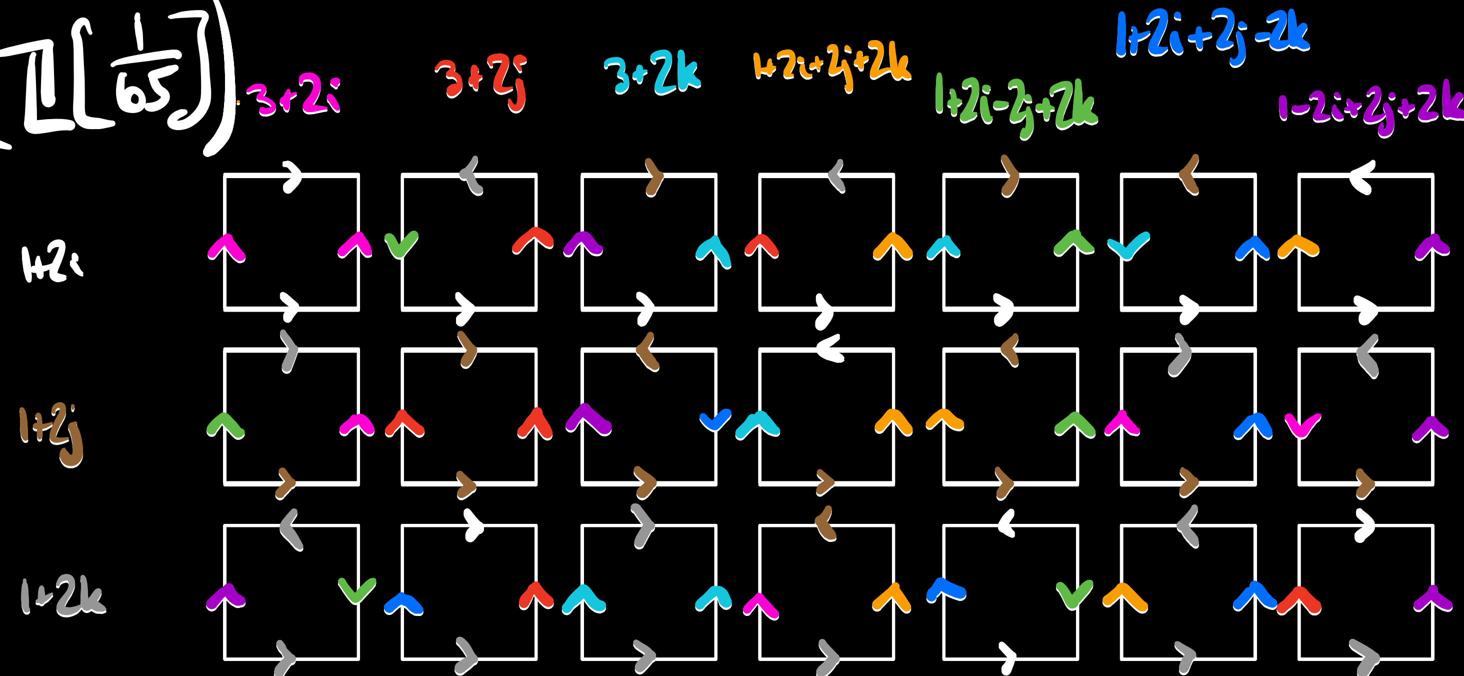
-3+4j
1-4i+2j+2k
-3+4k
-3+4i
1+2j
1+2k
1+2i
1-2i-4j+2k
1
1-2k
1-2j
1-2i
-3-4k
-3-4j
-3-4i

For example...

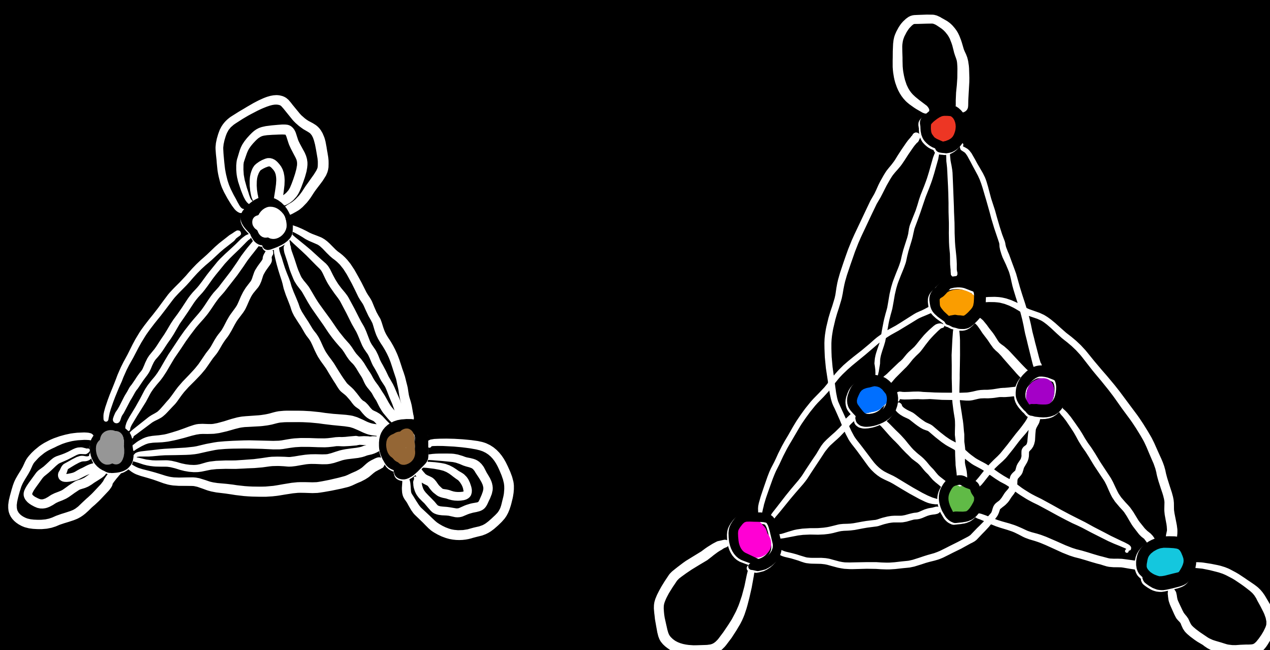


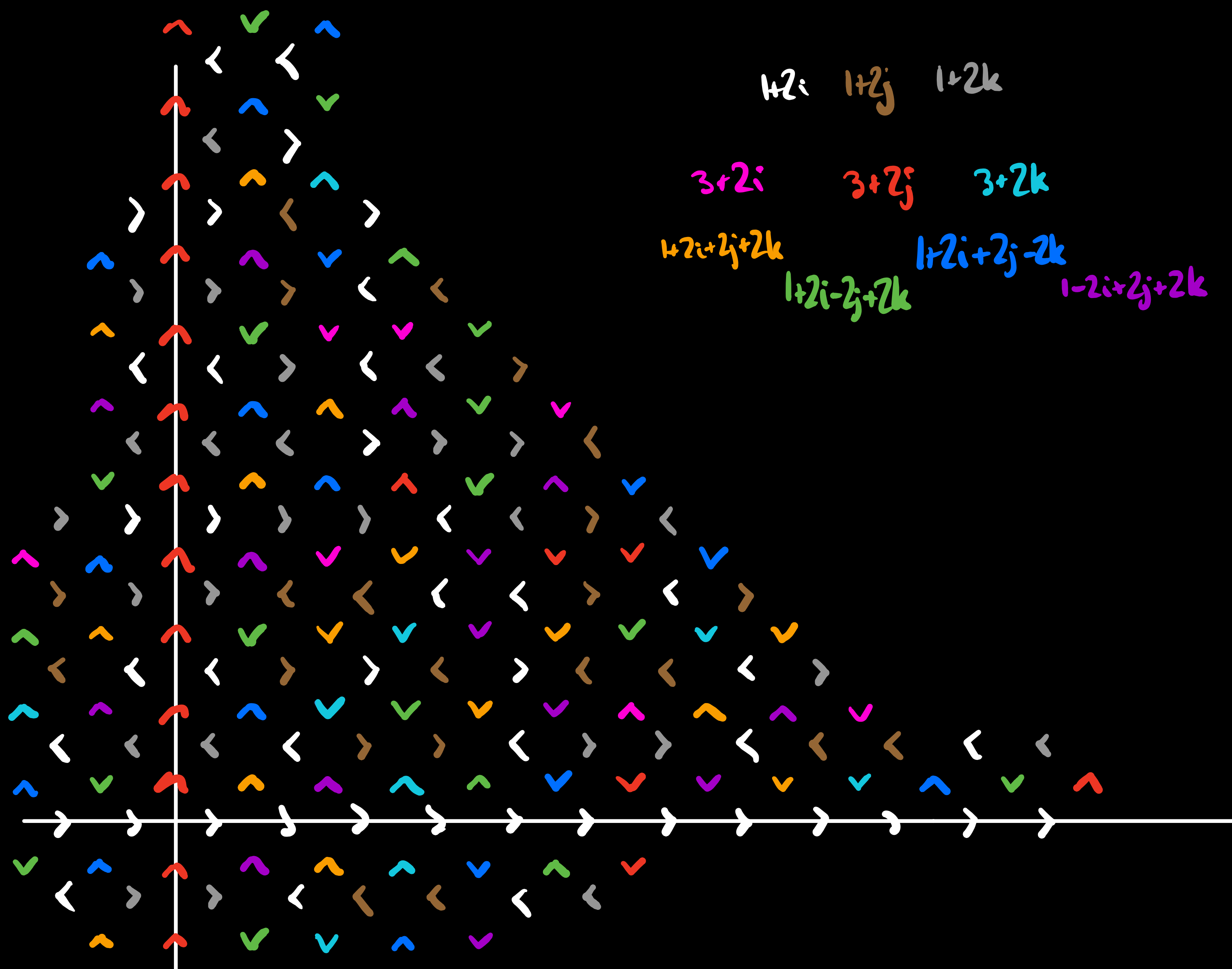
-1+2i+24j-2k
1-i-j
1+i-j
1-2i+2j+2k
3+2j
1-2i
1+2j
1+i-j
1+i+j
1-2i+2j-2k
1+2i+2j+2k
1+2i
1-i-j
1-2k
1+2j
1+i+j
3+2j
1+2k
1-2i+2j+2k
1-i+j
1-i+j
1

$SO_3(\mathbb{Z}[\frac{1}{65}])$



Hyperplanes

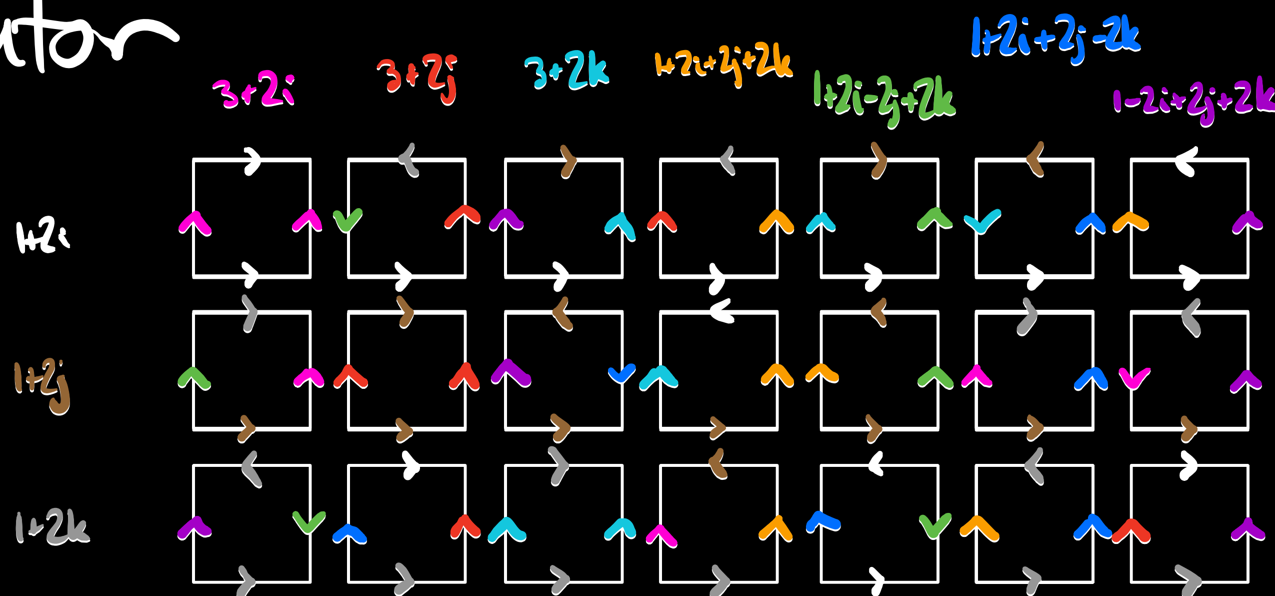




1+2i
1+2j
1+2k
3+2i
3+2j
3+2k
1+2i+2j+2k
1+2i-2j+2k
1+2i+2j-2k
1-2i+2j+2k

Takeaway

A set of quaternions determines a square-like calculator

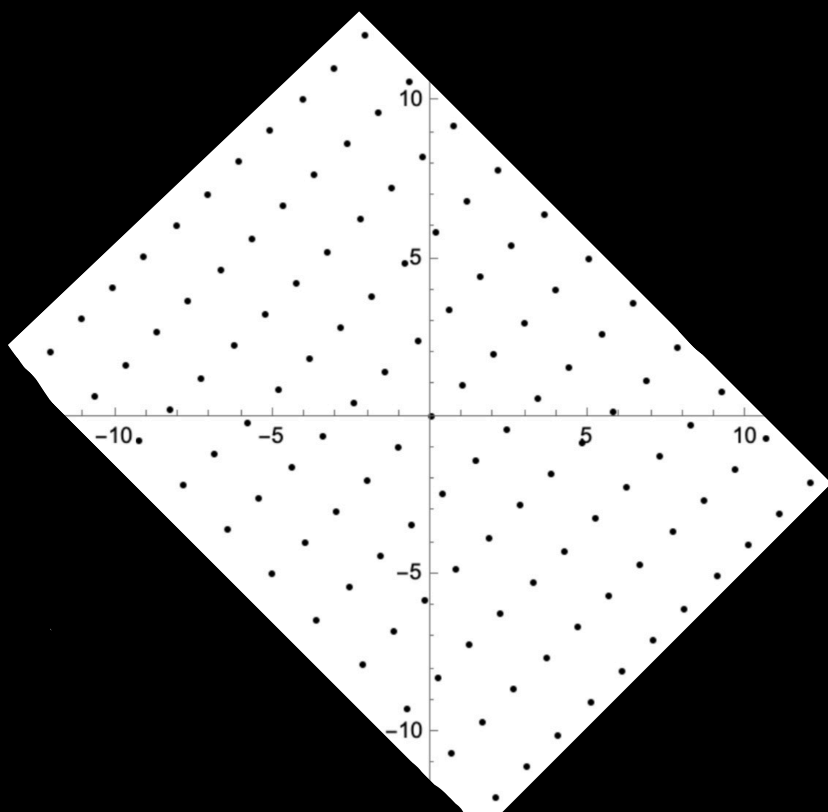


This calculator not only allows us to say which rotations are in $\langle S \rangle$, but also tells us how to perform a given rotation in terms of our basic moves in a maximally efficient way

Part IV

Towards a Classification of Linear Groups

Note $\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{R}$ is dense ..



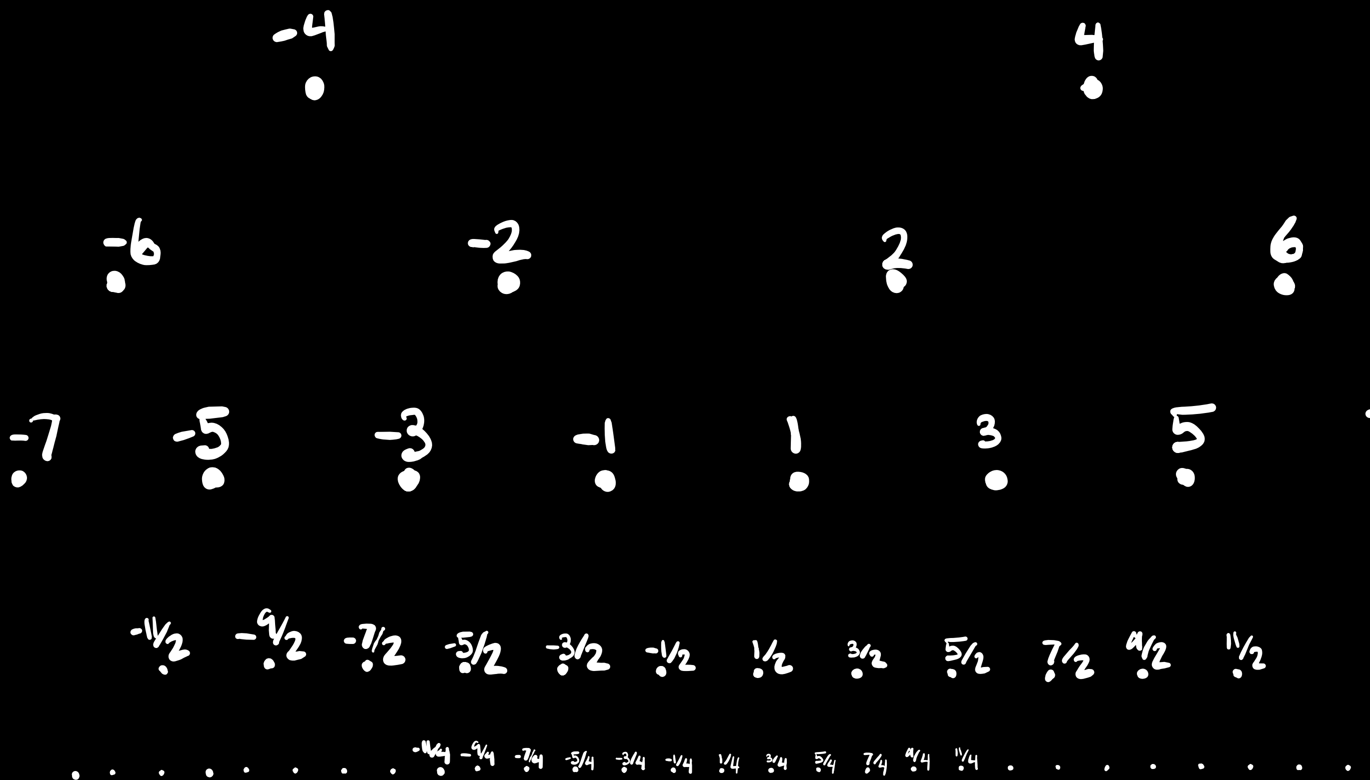
Idea Send

$$a+b\sqrt{2} \mapsto (a+b\sqrt{2},\ a-b\sqrt{2}).$$

Discrete!

(a lattice, even!)

Similarly, $\mathbb{Z}[\frac{1}{2}] \leq \mathbb{R}$ is dense.



$\mathbb{Z}[\frac{1}{2}] \leq \mathbb{R} \times \mathbb{Q}_2$ discrete! (again a lattice)

Trick Any f.g. $\Gamma \leq G(\overline{\mathbb{Q}})$ is contained in $G(A)$ for A finitely generated, and

$$G(A) \underset{\text{lattice}}{\leq} \prod_{i=1}^{n} G(k_{v_i})$$ for

finitely many completions of k

For $G = PGL_2$, get a finite product of

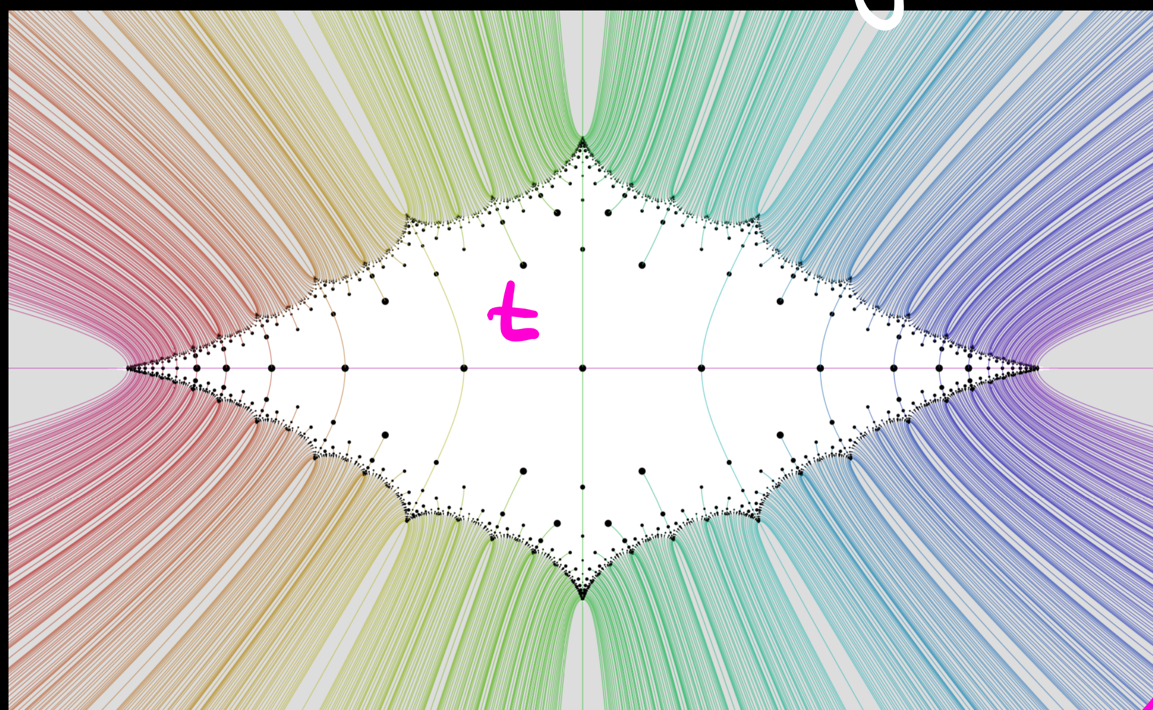
(i) trees
(ii) $\mathbb{H}^2$
(iii) $\mathbb{H}^3$

Our observation is that, surprisingly often, $\Gamma \leq PGL_2(A)$ has finite index!

Let $\mathcal{S}$ denote the set of irr. factors of polynomials q which can be obtained from $\begin{pmatrix}1\\0\end{pmatrix}$ via $\begin{pmatrix}p\\q\end{pmatrix} \mapsto \begin{pmatrix}p \pm q\\q\end{pmatrix}$

$$\begin{pmatrix}p\\q\end{pmatrix} \mapsto \begin{pmatrix}p\\q \pm pt\end{pmatrix}$$

Let $\mathcal{R}$ denote the set of irr. polynomials all of whose roots lie in the white region



$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\rangle$

Prop $\mathcal{R} = \mathcal{S}$

Conj 1 $R = S$

Let k be a number field, and G a simple k-algebraic group.

A subgroup $\Gamma \leq \mathbb{G}(k)$ is

① <u>arithmetic</u> if it is commensurable with $\mathbb{G}(A)$ for a subring $A \leq k$

② <u>algebraic</u> if it is contained in some $\mathbb{H}(k) \lneq \mathbb{G}(k)$

③ <u>geometric</u> if it is discrete with respect to some valuation topology on K.

<u>Conj 1</u> $\Rightarrow$ These are all the possibilities!

In fact, if one can show that a singly degenerate Kleinian group cannot lie in $PSL_2\overline{\mathbb{Q}}$, we obtain an algorithm

$$\text{Finite set } S \subseteq PGL_2\overline{\mathbb{Q}} \rightsquigarrow \text{Finite}^* \text{ presentation of } \langle S \rangle$$

$*$ or, $\langle S \rangle = \mathbb{Z}^n \rtimes A$, $A \leq \mathbb{C}$.

Algorithm Input finite $S \subseteq GL_2 \overline{\mathbb{Q}}$.

Step 1 Is $\langle S \rangle$ algebraic?

Compute Zariski tangent space of Z. closure

If $\langle S \rangle$ Zariski dense, it must be geometric or arithmetic.

Step 2 Let A be the ring generated by all of the entries of matrices in S. Since this ring is f.g., there is a finite set of valuations so that

$A \leq k_v$ is unbounded

For each valuation, check if $\langle S \rangle$ is discrete by computing a Dirichlet domain in the symmetric space

$$\left(\begin{array}{c} \text{either Bruhat-Tits building} \\ \text{or } \mathbb{H}^2/\mathbb{H}^3 \text{ for } PGL_2 \end{array} \right)$$

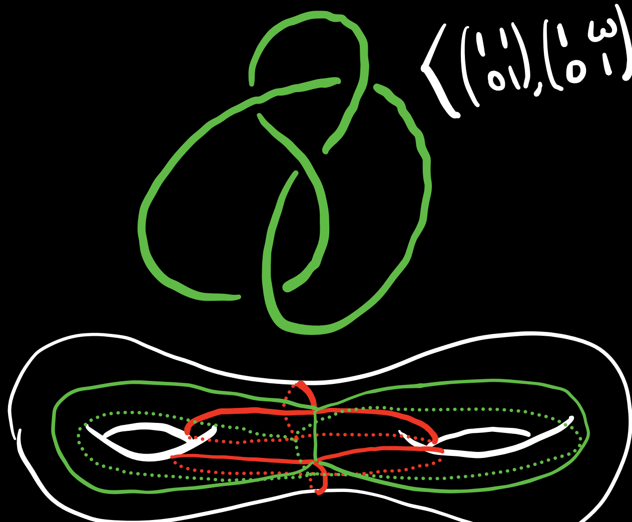
If none are discrete, $\langle S \rangle$ must be arithmetic!

Look at numerators of entries to determine congruence subgroup

<u>Output</u> Presentation

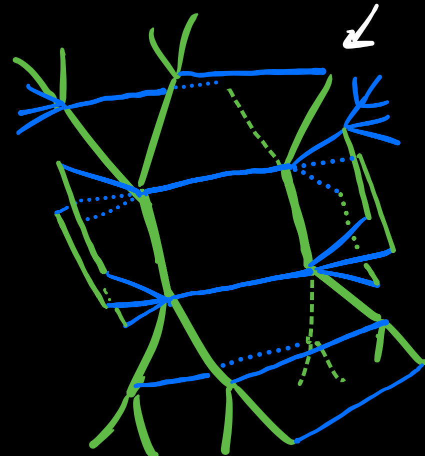
Thank you!

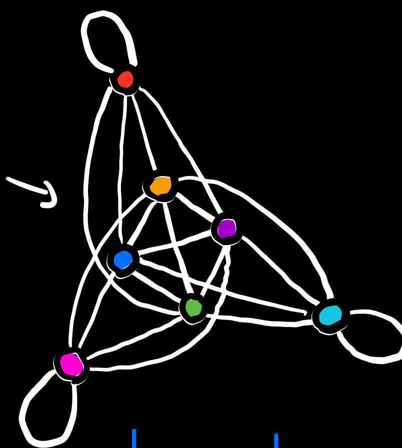
$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$

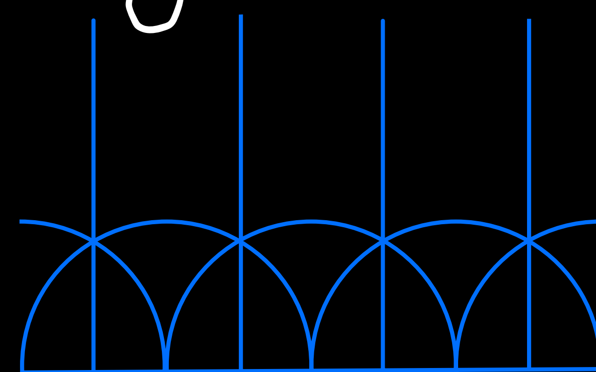


f.i. in $\begin{pmatrix} a+b\sqrt{2} & -3(c-d\sqrt{2}) \\ c+d\sqrt{2} & a-b\sqrt{2} \end{pmatrix}$

$SU_2(\mathbb{Z}[\frac{1}{65}])$

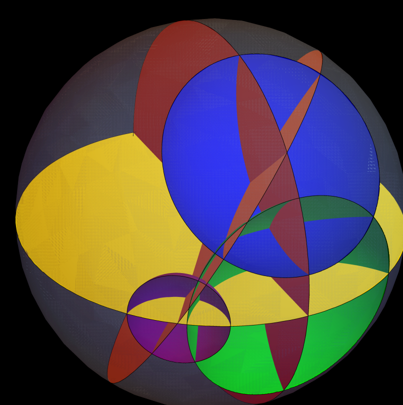






$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \frac{1}{8}\begin{pmatrix} 82 & 9 \\ 2 & 1 \end{pmatrix} \right\rangle$



$\begin{pmatrix} a+b\varphi & c+d\varphi \\ -c+d\varphi & a-b\varphi \end{pmatrix}$

Some consequences of Conj

$PSL_2\mathbb{Z}[1/p]$ is coherent (Serre '74)

$SO_3\mathbb{Q}$ is coherent, and all its hyp subgps are virt free.

$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \right\rangle$ is never free, for $|q|<4$ rational (Lyndon-Ullman '69)

$\left\langle \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}^n, \begin{pmatrix} 3+2i & 0 \\ 0 & 3-2i \end{pmatrix}^n \right\rangle$ is never free ($n \geq 1$). (Wise '02)

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\rangle$$ is arithmetic if it is not geometric

Further Qs

Can we classify geometric subgroups further? Must they all have integral elements?

Can surface gps act properly on a product of trees?

$$\left\langle \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 0 & 1/2 \end{pmatrix} \right\rangle$$

$$\left\langle \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \frac{1}{8}\begin{pmatrix} 82 & 9 \\ 2 & 1 \end{pmatrix} \right\rangle$$

Hyp 3-mflds - classification of nonarithmetic?

When does $PSL_2(\mathbb{Z}[\frac{1}{p}])$ contain a surface group?

Does a fixed surface group only embed in finitely many $PSL_2(\mathbb{Z}[\frac{1}{p}])$?

Is every surface gp in $PSL_2\mathbb{Q}$ Fuchsian?

Our goal today is to understand the group

$$\mathrm{PGL}_2(\overline{\mathbb{Q}})$$

$\overline{\mathbb{Q}}$ denotes the complex numbers which are roots of a polynomial with integer coefficients:

$$i \in \overline{\mathbb{Q}}, \text{ because } i^2+1=0$$

$$\sqrt[3]{2} \in \overline{\mathbb{Q}}, \text{ because } (\sqrt[3]{2})^3-2=0$$

$$\frac{1}{5} \in \overline{\mathbb{Q}}, \text{ because } 5\left(\frac{1}{5}\right)-1=0$$

The remainder of the talk will aim to describe these possibilities, and try to show evidence for the conjecture.

Algebraic Subgroups:

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a=d \\ b=-c \end{array} \right\}$$

$SO_3\,\mathbb{Q}$, $BS(1,n)$

Arithmetic $PGL_2\,\mathcal{O}_K$

Geometric Hyperbolic 3-mflds