CS 601 Spring 2023: Problem Set 4.

Problem 1. (25 points)

a) If Δ is the maximum degree of a vertex in an undirected graph G, describe an efficient (i.e., polynomial time) algorithm to color the graph using no more than $\Delta + 1$ colors.

b) Give an efficient algorithm to color a 3-colorable graph with $O(\sqrt{n})$ colors, where n is the number of vertices. Prove that your algorithm uses at most $O(\sqrt{n})$ colors and gives a legal coloring.
Hint: If the maximum degree of any vertex is less than $\sqrt{n}$, use the result of part a. If any vertex x has degree greater than $\sqrt{n}$, consider its immediate neighbors $N(x)$ – can any 3 of them be connected in a triangle? Find a 3-coloring of the vertices $N(x) \cup \{x\}$. Then what?

Problem 2. (15 points) Consider the Clustering (CLSTR) decision problem:
Instance: An n × n symmetric distance matrix D with non-negative entries, and two non-negative integers b and k.
Question: Does D allow a (b,k)-clustering? That is, does there exist a partition of {1, . . . , n} into k disjoint subsets (or clusters) $X_1, \ldots, X_k$ such that distances within each cluster are bounded by b? More formally, $\forall h \in \{1, \ldots, k\}$: $(\forall i, j \in X_h)[D[i,j] \leq b]$?

a) Show that CLSTR is in NP.
b) Show that 3COLORING $\leq_P$ CLSTR

Problem 3. (20 points) Define the language:

$$ODD3SAT = \{\phi : \phi \text{ is a } 3CNF \text{ formula over } n \text{ variables and has a satisfying assignment in which every clause has an odd number of } TRUE \text{ literals.}\}$$

Prove that $ODD3SAT \in P$.

To get started, instead of a boolean variable that is either TRUE or FALSE, think of x_i as either 1 or 0.

A. How would you replace the logical expressions $\overline{x_i}$ and $x_i \vee x_j$ by equivalent arithmetic expressions modulo 2?
B. Describe how to convert each clause of ϕ into a linear equation modulo 2, so that a truth assignment satisfies the clause if and only if the corresponding numerical values of the literals satisfy the equation.
C. Describe how to convert the conjunction of clauses of ϕ into a system of linear equations such that a truth assignment satisfies ϕ if and only if the corresponding numerical values simultaneously satisfy all the linear equations.
D. Describe a polynomial time algorithm to solve the system of linear equations.