12. Quantum gases

N as a function of μ

- Grand canonical distribution applies to an “open” system in equilibrium with a reservoir of energy and particles
- Microstates of all energies and particle numbers are possible

$$\overline{N} = \sum_r N_r P_r = \frac{1}{Z} \sum_r N_r \exp\left(\beta \left[N_r \mu - E_r\right]\right)$$

$$\overline{N} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu}$$

- Find $\dfrac{\left(\Delta \overline{N^2}\right)^{1/2}}{\overline{N}} \sim \dfrac{1}{N^{1/2}}$

- Key point 16: choose μ to fix $\overline{N} = N(\mu)$

- ie. N is sharp about $\overline{N}$

Indistinguishable particles

Recall:

- **Distinguishable particles**: A microstate is specified by $i_1, i_2, \ldots, i_N$, i.e., the quantum state of each particle.
- **Indistinguishable particles**: A microstate is specified by $n_1, n_2, \ldots,$ where n_i is the occupation number, i.e., the number of particles in single-particle state i.

- Thus, for indistinguishable particles in microstate r specified by the set $\{n_i\}$:

$$N_r = \sum_i n_i \quad \text{and} \quad E_r = \sum_i n_i \epsilon_i$$

where ϵ_i is the energy of single-particle state i.

- Hence $\quad \beta(N_r\mu - E_r) = \beta \sum_i n_i(\mu - \epsilon_i)$

Indistinguishable particles

$$
\begin{aligned}
\mathscr{Z} &= \sum_r \exp\left(\beta\left[N_r\mu - E_r\right]\right) \\
&= \left[\sum_{n_1}\sum_{n_2}\cdots\right]\exp\left(\beta\sum_i n_i(\mu-\epsilon_i)\right) \\
&= \left[\sum_{n_1}\exp\left(\beta n_1(\mu-\epsilon_1)\right)\right]\times\left[\sum_{n_2}\exp\left(\beta n_2(\mu-\epsilon_2)\right)\right]\times\cdots \\
&= \mathscr{Z}_1\times\mathscr{Z}_2\times\cdots = \prod_i \mathscr{Z}_i
\end{aligned}
$$

where $\mathscr{Z}_i$ is the partition function for quantum state i

- Thus, a factorization into single-state partition functions occurs.

Fermions and Bosons

- **Fermions**: spin is a half-integral multiple of $\hbar$; there are $2s+1$ spin states. Eg. electrons, neutrons, protons, and composite particles etc,
- **Bosons**: spin is an integral multiple of $\hbar$$ (including spin zero). Eg. photons and composite particles made up of an even number of fermions, e.g., He^4

- **Pauli Exclusion Principle**

There can be at most one fermion in any quantum state.

- Single state partition function for **Fermions**

$$\mathcal{Z}_i = \sum_{\substack{n_i=0,1 \\ \uparrow \\ \text{Pauli}}} \exp\left(\beta n_i(\mu - \epsilon_i)\right) = 1 + \exp\left(\beta(\mu - \epsilon_i)\right)$$

Fermions and Bosons

- Single state partition function for **Bosons**

$$\mathscr{Z}_i = \sum_{n_i=0}^{\infty} \exp\left(\beta n_i(\mu - \epsilon_i)\right) = \frac{1}{1 - \exp\left(\beta(\mu - \epsilon_i)\right)}$$

For $\exp\left(\beta(\mu - \epsilon_i)\right) < 1$

No restriction

Fermions and Bosons

- Now calculate $\bar{n}_i$, the average number of particles in quantum state i for Bosons and Fermions
- Write: $\mathcal{Z}_i = \left[1 \pm \exp\left(\beta(\mu - \epsilon_i)\right)\right]^{\pm 1}$ where + refers to bosons, and − refers to fermions.

$$
\begin{aligned}
\bar{n}_i = \sum_{n_i} n_i P(n_i) &= \frac{1}{\beta}\frac{\partial \ln Z_i}{\partial \mu} \\
&= \pm\frac{1}{\beta}\frac{\partial}{\partial \mu} \ln\left[1 \pm \exp\left(\beta(\mu - \epsilon_i)\right)\right] \\
&= \pm\frac{1}{\beta}(\pm\beta)\frac{\exp\left(\beta(\mu - \epsilon_i)\right)}{1 \pm \exp\left(\beta(\mu - \epsilon_i)\right)} \\
&= \frac{\exp\left(\beta(\mu - \epsilon_i)\right)}{1 \pm \exp\left(\beta(\mu - \epsilon_i)\right)} \\
&= \frac{1}{\exp\left(\beta(\epsilon_i - \mu)\right) \pm 1}
\end{aligned}
$$

Fermions and Bosons

- Key point 17: $\bar{n}_i = f_{\pm}(\epsilon_i) = \dfrac{1}{\exp\left(\beta(\epsilon_i - \mu)\right) \pm 1}$

  where + refers to bosons, and – refers to fermions.

- Thus $N(\mu)$ is determined via

$$N = \sum_i \bar{n}_i = \sum_i f_{\pm}(\epsilon_i)$$