# 13. The Ideal Fermi gas

*Blundell and Blundell chapter 30*

## Low density limit

Consider the limit $e^{\mu/kT} \ll 1$, i.e., $\mu$ large and negative. Then:

$$\overline{n_i} = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{kT}\right) \pm 1} \approx e^{-\frac{\epsilon_i - \mu}{kT}}$$

for both Fermi-Dirac (F-D) and Bose-Einstein (B-E) distributions.

Now fix $\mu(N, T)$ through the constraint

$$N = \sum_i \overline{n_i} \approx e^{\mu/kT} \sum_i e^{-\epsilon_i/kT} = e^{\mu/kT} Z(1)$$

where we have identified the canonical single particle partition function $Z(1)$. Thus we see that in the limit $e^{\mu/kT} \ll 1$:

$$\frac{Z(1)}{N} \gg 1$$

which, referring to chapter 10.1, 10.4, is the low density/high temperature limit where a semi-classical treatment is adequate.

## Ideal Fermi gas

We return now to the Fermi-Dirac distribution from key point 17 (chapter 12). The function $f_+$ is often referred to simply as the **Fermi function**.

Consider the limit $T \to 0$ ($\beta \to \infty$):

$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1} \to \begin{cases} 1 & \text{if } \epsilon < \epsilon_f \\ 0 & \text{if } \epsilon > \epsilon_f \end{cases}$$

where $\epsilon_f$ is the **Fermi energy** defined by:

Key Point 18

$$\epsilon_f = \lim_{T \to 0} \mu(T)$$

You should convince yourself that the Fermi-Dirac distribution is equivalent to the probability that a quantum state of energy $\epsilon$ is occupied.

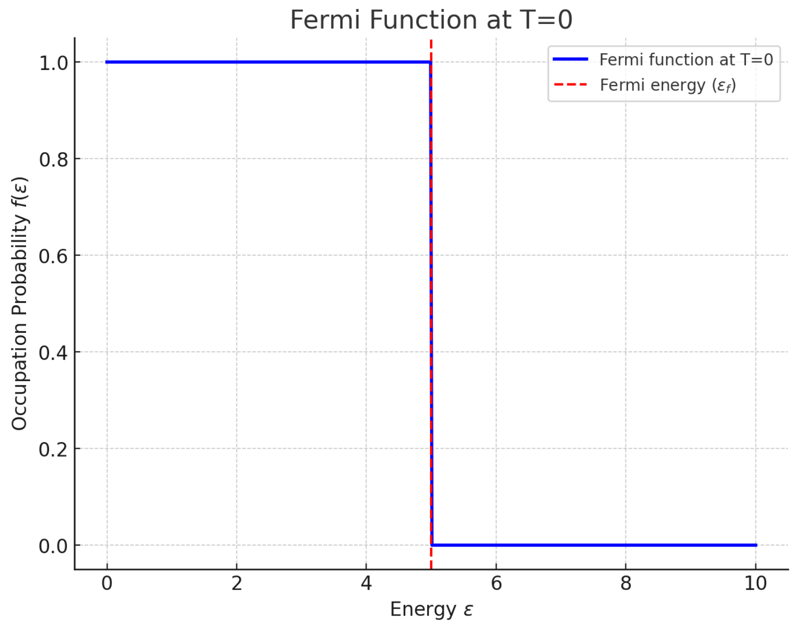


Figure 1: Fermi function at zero temperature

In Figure 1, we observe the following features:

- All states up to $\epsilon_f$ are filled with probability 1.
- All states above $\epsilon_f$ are empty.
- This is very different from a classical gas where at zero temperature, all gas molecules would have zero energy.
- It is a direct result of the **exclusion principle**, which leads to an "effective repulsion" between fermions.

We now calculate $\epsilon_f$. First, turn the sum over quantum states into an integral:

$$N = \sum_i f_+(\epsilon_i) \approx \int_0^\infty d\epsilon\, g(\epsilon) f_+(\epsilon)$$

where $g(\epsilon)$ is the density of states, a concept first introduced in **chapter 10**. As a reminder:

$$g(\epsilon)d\epsilon = \text{number of states in energy range } [\epsilon, \epsilon + d\epsilon]$$

We saw in **chapter 10** that for spinless particles in a box:

$$g(\epsilon) = D\epsilon^{1/2}, \quad D = \left(\frac{2m}{h^2}\right)^{3/2} \frac{1}{4\pi^2}$$

Incorporating the idea of spin, each translational (“standing wave”) state corresponds to $2s+1$ states since the particle has $2s+1$ possible spin states. Therefore:

$$g(\epsilon) = \tilde{D}\epsilon^{1/2}, \quad \tilde{D} = (2s+1)D$$

Thus:

$$N = \int_0^{\epsilon_f} \tilde{D}\epsilon^{1/2} d\epsilon = \frac{2}{3}\tilde{D}\epsilon_f^{3/2}$$

which implies:

$$\epsilon_f = \left(\frac{3N}{2\tilde{D}V}\right)^{2/3}$$

or equivalently:

$$\epsilon_f = \frac{h^2}{2m}\left(\frac{6\pi^2 N}{(2s+1)V}\right)^{2/3}$$

In **Question 4.5**, it is shown that:

$$E = \int_0^{\epsilon_f} g(\epsilon)\epsilon d\epsilon = \frac{3}{5}N\epsilon_f$$

**Important Points:**

- $\epsilon_f$ decreases with the mass $M$ of the fermion.
- $\epsilon_f$ increases with the number density $N/V$.
- $\epsilon_f$ defines a characteristic temperature through $\epsilon_f = kT_f$.
- At zero temperature, there is a finite energy per particle $\epsilon = \frac{3}{5}\epsilon_f$.

### Low temperature behaviour

Now consider the Fermi function at low but finite $T$. The meaning of "low" will be specified shortly. Note that:

$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon-\mu}{kT}\right)+1}$$

approaches 1 if $(\epsilon-\mu)/kT \ll -1$ and 0 if $(\epsilon-\mu)/kT \gg 1$, with $f_+(\epsilon) = 1/2$ when $\epsilon = \mu$. The Fermi function is a sigmoid shape illustrated in Figure 2. It differs from the zero-temperature step-function only when $|\epsilon-\mu| \sim O(kT)$.

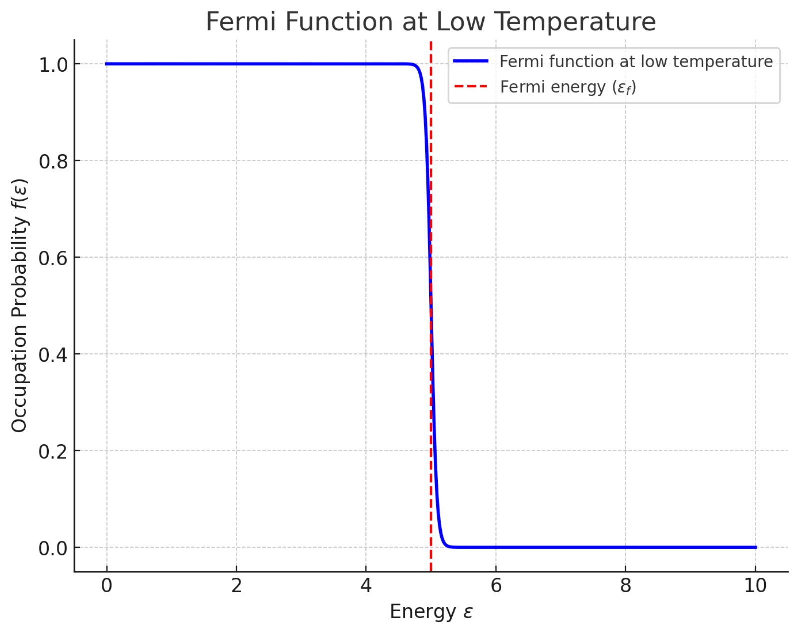


Figure 2: Fermi function at zero temperature

For the Fermi function to retain its characteristic shape, we must have $kT \ll \mu(T) \approx \epsilon_f$, implying $T \ll T_f$. We assume that for low $T$:

$$\mu(T) \approx \epsilon_f \left(1 - O\left(\frac{T^2}{T_f^2}\right)\right)$$

Thus, the change in $\mu$ is second-order in $T/T_f$ and can safely be ignored in the regime $T \ll T_f$.

An intuitive interpretation of Figure 2 is that at low temperatures, the general scenario is similar to that at zero temperature, where all states up to energy $\epsilon_f$ are filled. The difference is that states within energy $O(kT)$ below $\epsilon_f$ are vacated with some probability, and previously empty states within $O(kT)$ above $\epsilon_f$ are filled with some probability. In other words, some fermions are thermally excited above the Fermi energy.

We investigate the result of this thermal excitation by calculating the heat capacity. To avoid a complicated calculation required to get the exact result, we instead make a rough estimate (for a more careful argument, see Baierlein 9.1). We expect:

$$E(T) - E(0) \sim N \cdot \frac{kT}{\epsilon_f} \cdot kT$$

Therefore, the heat capacity is approximately:

$$C_V \sim \frac{E(T) - E(0)}{T} \sim \frac{Nk^2T}{\epsilon_f}$$

The important point is that this is **linear in** $T$. This contrasts with the classical gas, where $C_V$ is a constant (equal to $3Nk/2$).