# 13. Ideal Fermi gas

# Low density limit



- Chemical potential $\mu$ controls density. Consider limit $e^{\mu/kT} \ll 1$

$$\overline{n_i} = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{kT}\right) \pm 1} \approx e^{-\frac{\epsilon_i - \mu}{kT}}$$

for both F-D and B-E distributions.

- Now

$$N = \sum_i \overline{n_i}$$
$$\approx e^{\mu/kT} \sum_i e^{-\epsilon_i/kT}$$
$$= e^{\mu/kT} Z(1)$$

- Thus in low density limit

$$\frac{Z(1)}{N} \gg 1$$

# Ideal Fermi gas



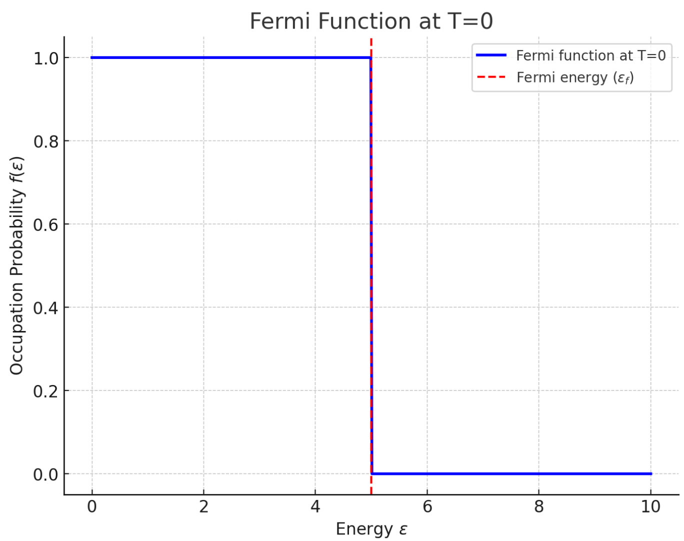
- The function $f_+$ is called the **Fermi function**
- Consider the limit $T \to 0$:

$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1} \to \begin{cases} 1 & \text{if } \epsilon < \epsilon_f \\ 0 & \text{if } \epsilon > \epsilon_f \end{cases}$$

where $\epsilon_f$ is the **Fermi energy**

- Key point 18: $\epsilon_f = \lim_{T \to 0} \mu(T)$

(Because $f_+(\epsilon)$ is discontinuous when $\epsilon = \mu$)




- All states up to $\epsilon_f$ are filled with probability 1.
- All states above $\epsilon_f$ are empty.
- Contrast with a classical gas where $T = 0$, all gas molecules would have zero energy.
- - It is a direct result of the **exclusion principle**, which leads to an "effective repulsion" between fermions.

# Ideal Fermi gas



- Now calculate $\epsilon_f$

$$N = \sum_i f_+(\epsilon_i) \approx \int_0^\infty d\epsilon\, g(\epsilon) f_+(\epsilon)$$

- Recall (chapter 10.2) for spinless particles in a box, the density of states:

$$g(\epsilon) = D\epsilon^{1/2}, \quad D = \left(\frac{2m}{h^2}\right)^{3/2} \frac{1}{4\pi^2}$$

- Incorporating the effects of spin increases the number of states by factor $2s+1$:

$$g(\epsilon) = \tilde{D}\epsilon^{1/2}, \quad \tilde{D} = (2s+1)D$$

- Hence

$$\epsilon_f = \left(\frac{3N}{2\tilde{D}V}\right)^{2/3} = \frac{h^2}{2m}\left(\frac{6\pi^2 N}{(2s+1)V}\right)^{2/3}$$

# Ideal Fermi gas

- Important points
  - $\epsilon_f$ decreases with the mass $M$ of the fermion
  - $\epsilon_f$ increases with the number density $N/V$
  - $\epsilon_f$ defines a characteristic temperature through $\epsilon_f = kT_f$
  - At $T = 0$ there is a finite energy per particle $\epsilon = (3/5)e_f$

# Low temperature behaviour



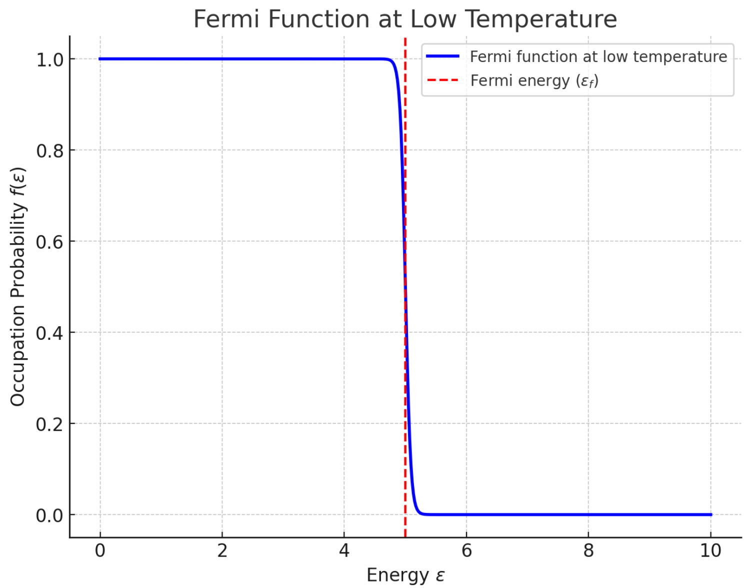
$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1}$$

$f_+(\epsilon) \to 1$ if $(\epsilon - \mu)/kT \ll -1$

$f_+(\epsilon) \to 0$ if $(\epsilon - \mu)/kT \gg 1$

$f_+(\epsilon) = 1/2$ when $\epsilon = \mu$

$f_+(\epsilon)$ has a sigmoidal shape overall




- Differs from the $T = 0$ step function only when $|\epsilon - \mu| \sim O(kT)$

# Low temperature behaviour



- At low $T$ general scenario is similar to that at $T = 0$
- Difference is that some states within energy $O(kT)$ below $\epsilon_f$ are vacated and previously empty states within $O(kT)$ above $\epsilon_f$ are filled.
- That is some fermions are thermally excited above the Fermi energy.

- Rough calculation of $E(T)$:

$$E(T) - E(0) \sim N \cdot \frac{kT}{\epsilon_f} \cdot kT$$

$$C_V \sim \frac{E(T) - E(0)}{T} \sim \frac{Nk^2T}{\epsilon_f}$$

- Change in energy is $N$ times the fraction of fermions excited (roughly $kT/\epsilon_f$) times the typical excitation (roughly $kT$).

- The important point is that this is **linear in** $T$. This contrasts with the classical gas, where $C_V$ is a constant (equal to $3Nk/2$).