13. The Ideal Fermi gas

Blundell and Blundell chapter 30

Low density limit

The imposed value of μ for a system in the grand canonical ensemble sets the average particle number $\overline{N}$. Consider the limit $e^{\mu/kT} \ll 1$, i.e., μ large and negative. Then:

$$\overline{n_i} = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{kT}\right) \pm 1} \approx e^{-\frac{\epsilon_i - \mu}{kT}}$$

for both Fermi-Dirac (F-D) and Bose-Einstein (B-E) distributions.

Now fix $\mu(N, T)$ through the constraint

$$N = \sum_i \overline{n_i} \approx e^{\mu/kT} \sum_i e^{-\epsilon_i/kT} = e^{\mu/kT} Z(1)$$

where we have identified the canonical single particle partition function $Z(1)$. Thus we see that in the limit $e^{\mu/kT} \ll 1$:

$$\frac{Z(1)}{N} \gg 1$$

which, referring to chapter 10.1, 10.4, is the low density/high temperature limit where a semi-classical treatment is adequate.

Ideal Fermi gas

We return now to the Fermi-Dirac distribution from key point 17 (chapter 12). The function f_+ is often referred to simply as the **Fermi function**.

Consider the limit $T \to 0$ ($\beta \to \infty$):

$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1} \to \begin{cases} 1 & \text{if } \epsilon < \epsilon_f \\ 0 & \text{if } \epsilon > \epsilon_f \end{cases}$$

where ϵ_f is the **Fermi energy** defined by:

Key Point 18

$$\epsilon_f = \lim_{T \to 0} \mu(T)$$

You should convince yourself that because the occupation numbers n_i are restricted to $0, 1$ for fermions, the Fermi-Dirac distribution is equivalent to the probability that a quantum state of energy ϵ is occupied.

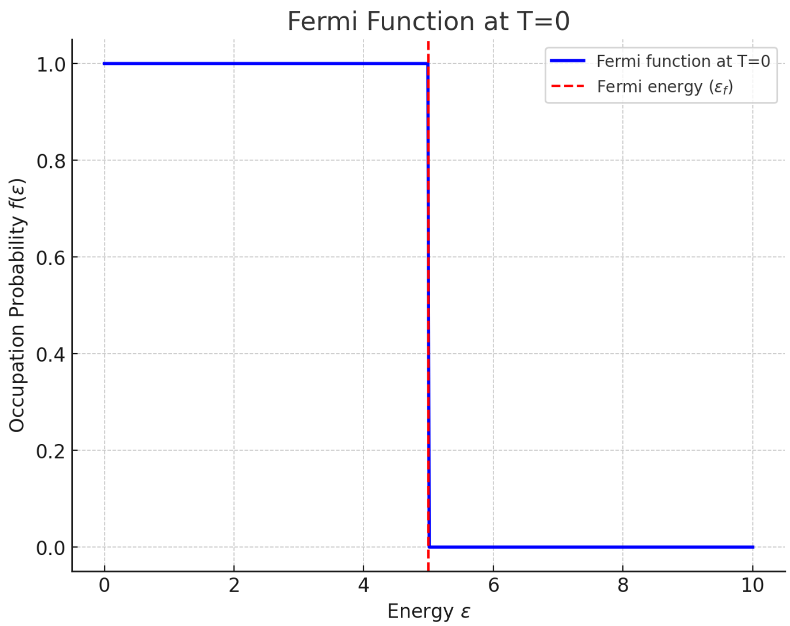

Figure 1: Fermi function at zero temperature

In Figure 1, we observe the following features:

- All states up to ϵ_f are filled with probability 1.
- All states above ϵ_f are empty.
- This is very different from a classical gas where at zero temperature, all gas molecules would have zero energy.
- It is a direct result of the **exclusion principle**, which leads to an "effective repulsion" between fermions.

We now calculate ϵ_f. First, turn the sum over quantum states into an integral:

$$N = \sum_i f_+(\epsilon_i) \approx \int_0^\infty d\epsilon\, g(\epsilon) f_+(\epsilon)$$

where $g(\epsilon)$ is the density of states, a concept first introduced in **chapter 10**. As a reminder:

$$g(\epsilon)d\epsilon = \text{number of states in energy range } [\epsilon, \epsilon + d\epsilon]$$

We saw in **chapter 10** that for spinless particles in a box:

$$g(\epsilon) = DV\epsilon^{1/2}, \quad D = \left(\frac{2M}{h^2}\right)^{3/2} \frac{1}{4\pi^2}$$

Incorporating the idea of spin, each translational ("standing wave") state corresponds to $2s+1$ states since the particle has $2s+1$ possible spin states. Therefore:

$$g(\epsilon) = \tilde{DV}\epsilon^{1/2}, \quad \tilde{D} = (2s+1)D$$

Thus:

$$N = \int_0^{\epsilon_f} \tilde{D}V\epsilon^{1/2}d\epsilon = \frac{2}{3}\tilde{D}V\epsilon_f^{3/2}$$

which implies:

$$\epsilon_f = \left(\frac{3N}{2\tilde{D}V}\right)^{2/3}$$

or equivalently:

$$\epsilon_f = \frac{h^2}{2M}\left(\frac{6\pi^2 N}{(2s+1)V}\right)^{2/3}$$

In **Question 4.6**, it is shown that:

$$E = \int_0^{\epsilon_f} g(\epsilon)\epsilon d\epsilon = \frac{3}{5}N\epsilon_f$$

Important Points:

- ϵ_f decreases with the mass M of the fermion.
- ϵ_f increases with the number density N/V.
- ϵ_f defines a characteristic 'Fermi temperature' through $\epsilon_f = kT_f$.
- At zero temperature, there is a finite energy per particle $\epsilon = \frac{3}{5}\epsilon_f$.

Low temperature behaviour

Now consider the Fermi function at low but finite T. The meaning of "low" will be specified shortly. Note that:

$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon-\mu}{kT}\right)+1}$$

approaches 1 if $(\epsilon-\mu)/kT \ll -1$ and 0 if $(\epsilon-\mu)/kT \gg 1$, with $f_+(\epsilon) = 1/2$ when $\epsilon = \mu$. The Fermi function is a sigmoid shape illustrated in Figure 2. It differs from the zero-temperature step-function only when $|\epsilon-\mu| \sim O(kT)$.

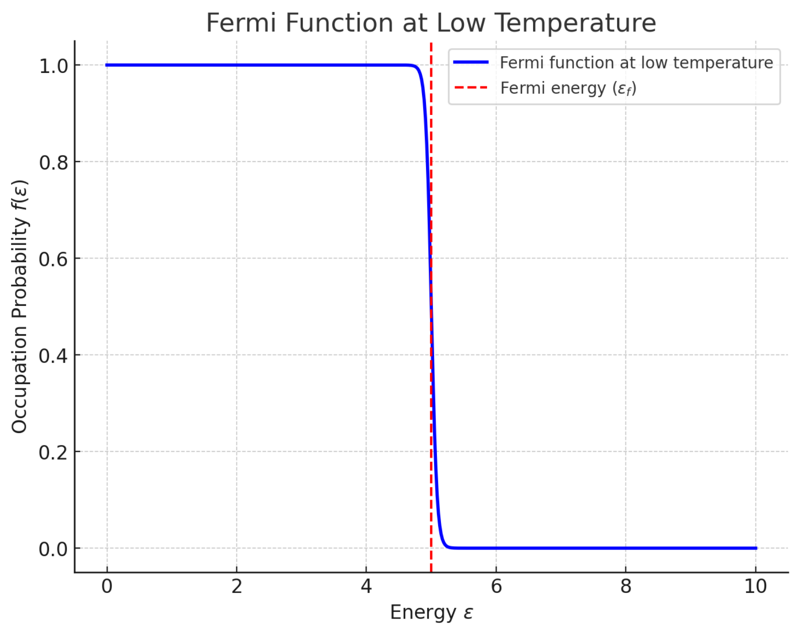

Figure 2: Fermi function at zero temperature

For the Fermi function to retain its characteristic shape, we must have $kT \ll \mu(T) \approx \epsilon_f$, implying $T \ll T_f$.

An intuitive interpretation of Figure 2 is that at low temperatures, the general scenario is similar to that at zero temperature, where all states up to energy ϵ_f are filled. The difference is that states within energy $O(kT)$ below ϵ_f are vacated with some probability, and previously empty states within $O(kT)$ above ϵ_f are filled with some probability. In other words, some fermions are thermally excited above the Fermi energy.

We investigate the result of this thermal excitation by calculating the heat capacity. To avoid a complicated calculation required to get the exact result, we instead make a rough estimate (for a more careful argument, see Baierlein 9.1). We expect:

$$E(T) - E(0) \sim N \cdot \frac{kT}{\epsilon_f} \cdot kT$$

i.e the change in energy is N times the fraction of fermions excited (roughly kT/ϵ_f) times the typical excitation (roughly kT).

Therefore, the heat capacity is approximately:

$$C_V \sim \frac{E(T) - E(0)}{T} \sim \frac{Nk^2T}{\epsilon_f}$$

The important point is that this is **linear in** T. This contrasts with the classical gas, where C_V is a constant (equal to $3Nk/2$).