Low density limit

- Chemical potential μ controls density. Consider limit $e^{\mu/kT} \ll 1$

$$\overline{n_i} = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{kT}\right) \pm 1} \approx e^{-\frac{\epsilon_i - \mu}{kT}}$$

for both F-D and B-E distributions.

- Now

$$N = \sum_i \overline{n_i}$$
$$\approx e^{\mu/kT} \sum_i e^{-\epsilon_i/kT}$$
$$= e^{\mu/kT} Z(1)$$

'Canonical' partition function for a single particle at temperature T.

- Thus in low density limit

$$e^{-\mu/kT} \gg 1 \Rightarrow \frac{Z(1)}{N} \gg 1$$

we recover results for a low density gas treated semi-classically (chapter 10) where we didn't consider whether particles were Fermions or Boson

Ideal Fermi gas

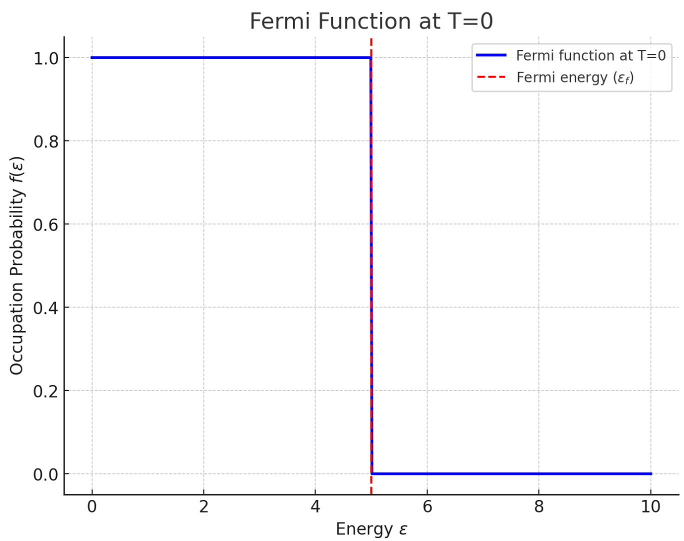
- The function f_+ is called the **Fermi function**
- Consider the limit $T \to 0$:

$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1} \to \begin{cases} 1 & \text{if } \epsilon < \epsilon_f \\ 0 & \text{if } \epsilon > \epsilon_f \end{cases}$$

where ϵ_f is the **Fermi energy**

- Key point 18: $\epsilon_f = \lim_{T \to 0} \mu(T)$

(Because $f_+(\epsilon)$ is discontinuous when $\epsilon = \mu$)



- All states up to ϵ_f are filled with probability 1.
- All states above ϵ_f are empty.
- Contrast with a classical gas where $T = 0$, all gas molecules would have zero energy.
- - It is a direct result of the **exclusion principle**, which leads to an "effective repulsion" between fermions.

Ideal Fermi gas

- Now calculate ϵ_f :

$$N = \sum_i f_+(\epsilon_i) \approx \int_0^\infty g(\epsilon) f_+(\epsilon)\, d\epsilon$$

- Recall (chapter 10.2) for spinless particles in a box, the density of states:

$$g(\epsilon) = DV\epsilon^{1/2}, \quad D = \left(\frac{2M}{h^2}\right)^{3/2} \frac{1}{4\pi^2}$$

- Incorporating the effects of spin increases the number of states by factor $2s + 1$:

$$g(\epsilon) = \tilde{D}V\epsilon^{1/2}, \quad \tilde{D} = (2s+1)D$$

Ideal Fermi gas

- Hence

$$N = \int_0^{\epsilon_f} \tilde{D} V \epsilon^{1/2} d\epsilon = \frac{2}{3} \tilde{D} V \epsilon_f^{3/2}$$

$$\Rightarrow \epsilon_f = \left(\frac{3N}{2\tilde{D}V} \right)^{2/3} = \frac{h^2}{2m} \left(\frac{6\pi^2 N}{(2s+1)V} \right)^{2/3}$$

- Find (Q4.6) that $E = \int_0^{\epsilon_f} g(\epsilon)\epsilon d\epsilon = \frac{3}{5} N \epsilon_f$

- Important points

  - ϵ_f decreases with the mass M of the fermion
  - ϵ_f increases with the number density N/V
  - ϵ_f defines a characteristic temperature through $\epsilon_f = kT_f$
  - At $T = 0$ there is a finite energy per particle $\epsilon = (3/5)\epsilon_f$

Low temperature behaviour

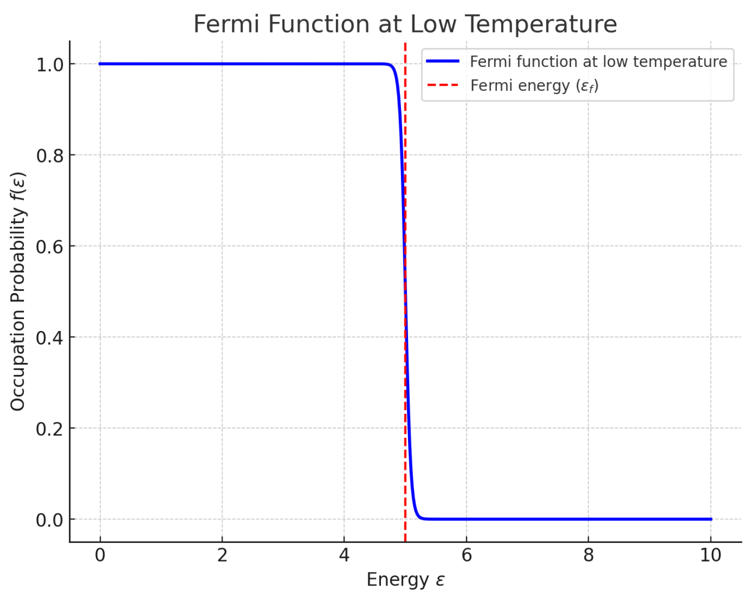
$$f_+(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1}$$

$f_+(\epsilon) \to 1$ if $(\epsilon - \mu)/kT \ll -1$

$f_+(\epsilon) \to 0$ if $(\epsilon - \mu)/kT \gg 1$

$f_+(\epsilon) = 1/2$ when $\epsilon = \mu$

$f_+(\epsilon)$ has a sigmoidal shape overall



- Differs from the $T = 0$ step function only when $|\epsilon - \mu| \sim O(kT)$

Low temperature behaviour

- At low T general scenario is similar to that at $T = 0$
- Difference is that some states within energy $O(kT)$ below ϵ_f are vacated and previously empty states within $O(kT)$ above ϵ_f are filled.
- That is some fermions are thermally excited above the Fermi energy.

- Rough calculation of $E(T)$:

$$E(T) - E(0) \sim N \cdot \frac{kT}{\epsilon_f} \cdot kT$$

$$C_V \sim \frac{E(T) - E(0)}{T} \sim \frac{Nk^2T}{\epsilon_f}$$

- Change in energy is N times the fraction of fermions excited (roughly kT/ϵ_f) times the typical excitation (roughly kT).

- The important point is that this is **linear in** T. This contrasts with the classical gas, where C_V is a constant (equal to $3Nk/2$).