XII: 7.6.; XIII: 7,3; 9.3; 9.8; 9.9; 9.17.	Гуллин М. 432 гр.

XV: 7.1; 7; 4 7.4;

XVI: 7,1; 7; 7.17.

№7.6.

Схема Кранка-Николсона для ур. теплопроводности

$$\frac{u_j^{n+1}-u_j^n}{\tau}=\frac{D}{2h^2}\left(u_{j+1}^{n+1}+u_{j+1}^n-2(u_j^{n+1}+u_j^n)+u_{j-1}^{n+1}+u_{j-1}^n\right),$$ где $u_j^n=u(jh,n\tau)$

Усл-е монотонности =?

Схема двухслойная.

Теорема: $y_m^{n+1}=\sum_\ell \beta_\ell y_{m+\ell}^n$ монотонна $\Leftrightarrow$ тогда $\forall \ell\ \beta_\ell\geq 0$

Решение.

$$\frac{u_i^{n+1}-u_i^n}{\tau}=\frac{D}{2h^2}\left(u_{i+1}^{n+1}+u_{i+1}^n-2(u_i^{n+1}+u_i^n)+u_{i-1}^{n+1}+u_{i-1}^n\right).$$

Заменим $D=a$.

$$u_i^{n+1} = \sum_l \beta_l \, u_{i+l}^n.$$

$$u_i^{n+1} = \frac{\tau a}{2h^2} u_{i+1}^{n+1} + \frac{\tau a}{2h^2} u_{i+1}^n - \frac{\tau a}{h^2} u_i^{n+1} - \frac{\tau a}{h^2} u_i^n +$$

$$+ \frac{\tau a}{2h^2} u_{i-1}^{n+1} - \frac{\tau a}{2h^2} u_{i-1}^n + u_i^n.$$

$$u_{i+1}^{n+1} = u_i^n + u_t' \tau + u_x' h + O(\ldots)$$

$\cancel{u_i^{n+1}}$

$$u_{i-1}^{n+1} = u_i^n - u_x' h + u_t' \tau + O(\ldots)$$

$$u_i^{n+1}\left(1 + \frac{\tau a}{h^2}\right) = u_i^{n+1}\left(\frac{h^2 + \tau a}{h^2}\right) =$$

$$= \frac{\tau a}{2h^2}\left(u_i^n + u_t' \tau + u_x' h\right) + \frac{\tau a}{2h^2} u_{i+1}^n -$$

$$- \frac{\tau a}{h^2} u_i^n + \frac{\tau a}{2h^2}\left(u_i^n + u_t' \tau - u_x' h\right) - \frac{\tau a}{2h^2} u_{i-1}^n + u_i^n =$$

$$= \frac{\tau a}{2h^2} u_t' \tau + \frac{\tau a}{2h^2} u_{i+1}^n + \frac{\tau a}{2h^2} u_i^n + \frac{\tau a}{2h^2} u_t' \tau -$$

$$- \frac{\tau a}{2h^2} u_{i-1}^n + u_i^n + \frac{\tau a}{2h^2} u_i^n - \frac{\tau a}{h^2} u_i^n =$$

$$= -\frac{\tau a}{2h^2} u_{i-1}^n + u_i^n + \frac{\tau a}{2h^2} u_{i+1}^n + \frac{\tau a}{h^2} u_t' \tau \Rightarrow$$

$$u_i^{n+1} = -\frac{\tau a}{2(h^2 + \tau a)} u_{i-1}^n + \frac{h^2}{h^2 + \tau a} u_i^n + \frac{\tau a \, u_{i+1}^n}{2(h^2 + \tau a)} + \frac{\tau a}{h^2 + \tau a} u_t' \tau$$

По теореме: $\forall \ell \ \beta_\ell \geq 0$

$$\beta_{-1} = \frac{-\tau a}{2(h^2+\tau a)} \geq 0$$

чтобы схема была монотонна:

$$\begin{cases} \dfrac{-\tau a}{2(h^2+\tau a)} \geq 0 \\ \dfrac{h^2}{h^2+\tau a} \geq 0 \\ \dfrac{\tau a}{2(h^2+\tau a)} \geq 0 \end{cases}$$

$$\frac{\tau a}{2(h^2+\tau a)} = 0.$$

$\cancel{\tau a \neq 0}$

$\tau a = 0.$

При $\tau a = 0$ верно $\dfrac{h^2}{h^2+\tau a} \geq 0.$

<u>Ответ:</u> чтобы схема была монотонна: $\tau a = 0.$

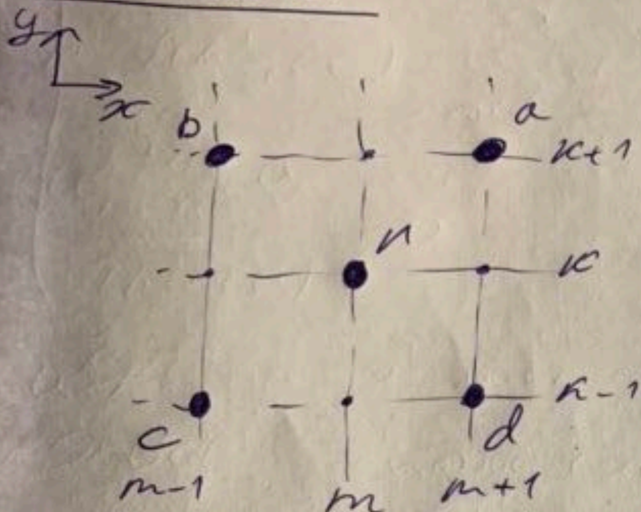
№ 7.8 из XII

РИК на шаблоне:

$Ш = \{ (x_{m-1}, y_{k-1}), (x_{m-1}, y_{k+1}), (x_{m+1}, y_{k+1}),$

$(x_{m+1}, y_{k+1}), (x_m, y_k) \}$ Апроксим. ур-я Лапласа.

Решение.



a, b, c, d, n - веса.

$$a u_{m+1}^{k+1} + b u_{m-1}^{k+1} + c u_{m-1}^{k-1} + d u_{m+1}^{k-1} + n u_m^k = 0. \quad (*)$$

$$\cancel{u_{m+1}^{k+1} = u_m^k + u'_t \tau + u'_x h}$$

$$u_{m+1}^{k+1} = u_m^k + h(u'_x + u'_y) + \frac{h^2}{2}(u''_{xx} + u''_{yy}) +$$
$$+ \frac{h^3}{6}(u'''_x + u'''_y) + O(\ldots). + h^2 u''_{xy}$$

$$u_{m-1}^{k+1} = u_m^k + h(-u'_x + u'_y) + \frac{h^2}{2}(u''_x + u''_y) +$$
$$+ \frac{h^3}{6}(-u'''_x + u'''_y) + O(\ldots). - h^2 u''_{xy}$$

$$u_{m-1}^{k-1} = u_m^k - h(u'_x + u'_y) + \frac{h^2}{2}(u''_x + u''_y) -$$
$$- \frac{h^3}{6}(u'''_x + u'''_y). + O(\ldots) + h^2 u''_{xy}$$

$$u_{m+1}^{k-1} = u_m^k + h(u'_x - u'_y) + \frac{h^2}{2}(u''_x + u''_y) + \frac{h^3}{6}(u'''_x - u'''_y) +$$
$$- h^2 u''_{xy}$$

$u_m^k = u_m^k$. Подставляем в (*)

$$a\Big[u^k_m + h(u'_x + u'_y) + \frac{h^2}{2}(u''_x + u''_y) + \frac{h^3}{6}(u'''_x + u'''_y) + h^2 u''_{xy}\Big] + b\Big[u^k_m + h(-u'_x + u'_y) + \frac{h^2}{2}(u''_x + u''_y) - h^2 u''_{xy} + \frac{h^3}{6}(-u'''_x + u'''_y)\Big] + c\Big[u^k_m - h(u'_x + u'_y) + \frac{h^2}{2}(u''_x + u''_y) + h^2 u''_{xy} - \frac{h^3}{6}(u'''_x + u'''_y)\Big] + d\Big[u^k_m + h(u'_x - u'_y) + \frac{h^2}{2}(u''_x + u''_y) - h^2 u''_{xy} + \frac{h^3}{6}(u'''_x - u'''_y)\Big] \cancel{=0} + n u^k_m = \cancel{0}(u''_x + u''_y)$$

u^k_m: $a + b + c + d + n = 0$ (1)

5 переменных. нужно 5 независ. ур-ний.

hu'_x: $a - b - c + d = 0$ (2)

hu'_y: $a + b - c - d = 0$ (3)

$\frac{h^2}{2}u''_x$: $a + b + c + d = \cancel{0}1$ (4)

$\frac{h^2}{2}u''_y$: —//—

$h^2u''_{xy}$: $a - b + c - d = 0$ (5)

Дальше не будем выписывать, ур-ний достаточно.

$$\begin{cases} a + b + c + d + n = 0 \\ a - b - c + d = 0 \\ a + b - c - d = 0 \\ a + b + c + d = \frac{2}{h^2} \\ a - b + c - d = 0 \end{cases} \qquad \begin{cases} n = -\frac{2}{h^2} \\ a = d \\ a = +b \\ c = +d \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = +b = +c = d = \frac{1}{2h^2} \\ n = -\frac{2}{h^2} \end{cases} \Rightarrow$$

$$\frac{1}{2h^2}u_{m+1}^{k+1}+\frac{1}{2h^2}u_{m-1}^{k+1}+\frac{1}{2h^2}u_{m-1}^{k-1}+\frac{1}{2h^2}u_{m+1}^{k-1}-\frac{2}{h^2}u_m^k=0$$

$$u_{m+1}^{k+1}+u_{m-1}^{k+1}-4u_m^k+u_{m-1}^{k-1}+u_{m+1}^{k-1}=0.$$

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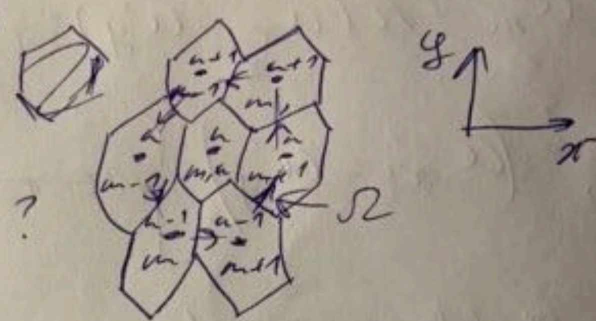
$$\begin{cases} \operatorname{div}\vec{W}+\alpha_0\varphi=Q(x,y) & (1)\\ \frac{1}{3}\operatorname{grad}\varphi+\alpha_1\vec{W}=0 & (2)\end{cases}$$



Схему аппроксимации?

Решение.

Ф-ла Остроградского-Гаусса

$$\iiint\limits_V \operatorname{div}\vec{F}dv=\iint\limits_S(\vec{F},\vec{n})\,ds \qquad S=\partial V$$

Пусть a — сторона шестиугольника,

$h=a\sqrt{3}$ — расстояние от центра до стороны.

Площадь такого шестиугольника: $S=a^2\frac{3\sqrt{3}}{2}$.

Применим О.Г. на (1):

$$\iiint\limits_V \operatorname{div}\vec{W}dv=\iiint\limits_V\left(Q(x,y)-\alpha_0\varphi\right)dv=$$

$$=\left(Q(x_m,y_n)-\alpha_0\varphi_{m,n}\right)S\tau.$$

$$\iiint\limits_V \operatorname{div}\vec{W}dv=\iint\limits_{\partial V}(\vec{W},\vec{n})\,ds=\tau\int\limits_{\partial\Omega}(\vec{W},\vec{n})\,d\ell.$$

Рассмотрим (2):

$\frac{1}{3}\operatorname{grad}\varphi + \lambda_1 \vec{W} = 0 \quad | \cdot \vec{n}$ скалярно $\Rightarrow$

$$(\vec{W}, \vec{n}) = -\frac{1}{3\lambda_1}(\operatorname{grad}\varphi\, \vec{n}) = -\frac{1}{3\lambda_1}\frac{\partial \varphi}{\partial n}$$

$$\oint_{\partial\Omega} (\vec{W}, \vec{n})\, dl = \oint \vec{W}\, d\vec{l} = -\frac{1}{3\lambda_1}\oint_{\partial\Omega} \operatorname{grad}\varphi\, d\vec{l} =$$

$$= -\frac{1}{3\lambda_1} a \left(\frac{\varphi_{m+1,n} - \varphi_{m,n}}{2h} + \frac{\varphi_{m,n+1} - \varphi_{m,n}}{2h} + \right.$$

$$+ \frac{\varphi_{m-1,n+1} - \varphi_{m,n}}{2h} + \frac{\varphi_{m-1,n} - \varphi_{m,n}}{2h} + \frac{\varphi_{m,n-1} - \varphi_{m,n}}{2h} +$$

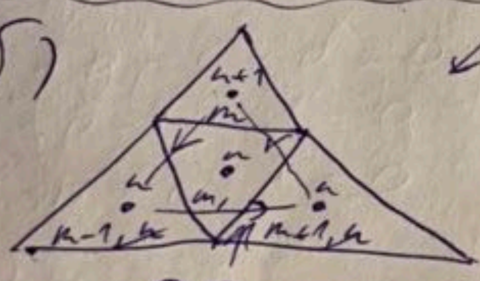
$$\left. + \frac{\varphi_{m+1,n-1} - \varphi_{m,n}}{2h} \right) = \left| h = a\sqrt{3} \right| =$$

$$= -\frac{1}{6\sqrt{3}\lambda_1} \left(\varphi_{m+1,n} + \varphi_{m,n+1} + \varphi_{m-1,n+1} + \right.$$

$$\left. + \varphi_{m-1,n} + \varphi_{m,n-1} + \varphi_{m+1,n-1} - 6\varphi_{m,n} \right) =$$

$$= \frac{3\sqrt{3}}{2} a^2 \left(Q(x_m, y_n) - \lambda_0 \varphi_{m,n} \right)$$

б) другая схема.



$\partial\Omega$.

Суть аналогична ~~но другая площадь~~

$$\iiint\limits_V (Q(x,y) - \lambda_0 \varphi)\, dv =$$

$$= \frac{\sqrt{3}}{4} \left(Q(x_m, y_n) - \lambda_0 \varphi_{m,n} \right) a^2 \tau$$

$$\iiint\limits_V \operatorname{div} \vec{\omega}\, dv = \iint\limits_{\partial V} (\vec{\omega}, \vec{n})\, ds = \tau \int\limits_{\partial \Omega} (\vec{\omega}, \vec{n})\, dl =$$

$$= \tau \int\limits_{\partial\Omega} \vec{\omega}\, d\vec{l} = -\frac{1}{3\lambda_1} \tau \int\limits_{\partial\Omega} \operatorname{grad} \varphi\, d\vec{l} =$$

$$= -\frac{1}{3\lambda_1} \tau \left(\frac{\varphi_{m,n+1} - \varphi_{m,n}}{h} + \frac{\varphi_{m-1,n} - \varphi_{m,n}}{h} + \right.$$

$$\left. + \frac{\varphi_{m+1,n} - \varphi_{m,n}}{h} \right) = -\frac{\sqrt{3}}{3\lambda_1} \tau \left(\varphi_{m,n+1} + \varphi_{m-1,n} + \right.$$

$$\left. + \varphi_{m+1,n} - 3\varphi_{m,n} \right).$$

<u>По итогу получим:</u>

$$a^2 \frac{\sqrt{3}}{4} \left[Q(x_m, y_n) - \lambda_0 \varphi_{m,n} \right] =$$

$$= -\frac{\sqrt{3}}{3\lambda_1} \left(\varphi_{m,n+1} + \varphi_{m-1,n} + \varphi_{m+1,n} - 3\varphi_{m,n} \right).$$

№4.3 из ФХ III.

Схема Алена-Чена:

$$\frac{y_m^{n+1}-y_m^n}{\tau}=D\,\frac{y_{m-1}^n-2y_m^{n+1}+y_{m+1}^n}{h^2} \qquad (1)$$

эквивал. явной схеме? при $\tau_1<\tau$.

$$\frac{y_m^{n+1}-y_m^n}{\tau_1}=D\,\frac{y_{m+1}^n-2y_m^n+y_{m-1}^n}{h^2} \qquad (2)$$

Решение:

Вычитаем из (2) − (1).

$$\frac{y_m^{n+1}-y_m^n}{\tau_1}-\frac{y_m^{n+1}-y_m^n}{\tau}=\frac{D}{h^2}\left(y_{m+1}^n-2y_m^n+y_{m-1}^n-\right.$$

$$\left.-y_{m-1}^n+2y_m^{n+1}-y_{m+1}^n\right)$$

$$\left(y_m^{n+1}-y_m^n\right)\left(\frac{1}{\tau_1}-\frac{1}{\tau}\right)=\frac{2D}{h^2}\left(y_m^{n+1}-y_m^n\right)$$

$$\left(y_m^{n+1}-y_m^n\right)\left(\frac{1}{\tau_1}-\frac{1}{\tau}-\frac{2D}{h^2}\right)=0.$$

$$\frac{1}{\tau_1}-\frac{1}{\tau}=\frac{2D}{h^2}$$

$$\frac{\tau-\tau_1}{\tau\tau_1}=\frac{2D}{h^2}$$

У нас $\frac{2D}{h^2}>0.$ $\Rightarrow$ $\tau-\tau_1>0$ $\Rightarrow$

$$\underline{\tau>\tau_1}$$

ч.т.д.

№9.3 из XIII

$$\frac{y_m^{n+1}-y_m^n}{\tau}=\frac{y_{m-1}^n-2y_m^n+y_{m+1}^n}{h^2}.$$

При каком $\frac{\tau}{h}$ порядок $O(\tau^2,h^4)$.

Решение.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_m^{n+1}-u_m^n}{\tau}=\frac{u_{m-1}^n-2u_m^n+u_{m+1}^n}{h^2}$$

$$u_m^{n+1}=u_m^n+u'_t\tau+\frac{\tau^2}{2}u''_{tt}+\frac{\tau^3}{6}u'''_t+O(\tau^4)$$

$$u_{m-1}^n=u_m^n-u'_xh+\frac{h^2}{2}u''_{xx}-\frac{h^3}{6}u'''_x+\frac{h^4}{24}u^{IV}_x-\frac{h^5}{120}u^V_x+\frac{h^6}{360}u^{VI}_x+O(h^7).$$

$$u_{m+1}^n=u_m^n+u'_xh+\frac{h^2}{2}u''_{xx}+\frac{h^3}{6}u'''_x+\frac{h^4}{24}u^{IV}_x+\frac{h^5}{120}u^V_x+\frac{h^6}{360}u^{VI}_x+O(h^7)$$

$$\frac{u'_t\tau+\frac{\tau^2}{2}u''_{tt}+\frac{\tau^3}{6}u'''_t+O(\tau^4)}{\tau}=$$

$$=\frac{h^2u''_{xx}+\frac{h^4}{12}u^{IV}_x+\frac{h^6}{360}u^{VI}_x+O(h^7)}{h^2}$$

$$\frac{\tau}{2}u''_{tt}+\frac{\tau^2}{6}u'''_t+O(\tau^3)=\frac{h^2}{12}u^{IV}_x+\frac{h^4}{360}u^{VI}_x+O(h^5)$$

Найдем нек.-ую зав-ть:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\Big| \Rightarrow \frac{\partial^2 u}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)=\frac{\partial^4 u}{\partial x^4}\Rightarrow$$

$$\frac{\tau}{2}=\frac{h^2}{12}\Rightarrow\boxed{\frac{\tau}{h^2}=\frac{1}{6}}\qquad \underline{u''_{tt}=u^{IV}_x}\Rightarrow.$$

$\boxed{1}$

№ 9.8 из XIII

$u_t = u_{xx}$ аппрокс:

$$\frac{u_m^{n+1} - u_m^{n-1}}{2\tau} = \frac{u_{m+1}^{n} - u_m^{n+1} - u_m^{n-1} + u_{m-1}^{n}}{h^2}$$

на устойчивость:

Решение.

Попробуем по спектр. признаку:

Подставим $u_m^n = \lambda^n e^{i\alpha m}$ в схему.

$$\frac{\lambda^{n+1} e^{i\alpha m} - \lambda^{n-1} e^{i\alpha m}}{2\tau} = \cancel{\lambda^n e^{i\alpha(m+1)}}$$

$$= \frac{\lambda^n e^{i\alpha(m+1)} - \lambda^{n+1} e^{i\alpha m} - \lambda^{n-1} e^{i\alpha m} + \lambda^n e^{i\alpha(m-1)}}{h^2}$$

сокращаем на $\lambda^n e^{i\alpha m} \Rightarrow .$

$$\cos x = \frac{e^{ix} + \frac{1}{e^{ix}}}{2}.$$

$$\frac{\lambda - \frac{1}{\lambda}}{2\tau} = \frac{e^{i\alpha} - \lambda - \frac{1}{\lambda} + \frac{1}{e^{i\alpha}}}{h^2}$$

$$\left(\lambda - \frac{1}{\lambda}\right) = \frac{2\tau}{h^2}\left(e^{i\alpha} + \frac{1}{e^{i\alpha}} - \lambda - \frac{1}{\lambda}\right)$$

$$(\lambda^2 - 1) = \frac{2\tau}{h^2}\left(\lambda e^{i\alpha} + \frac{\lambda}{e^{i\alpha}} - \lambda^2 - 1\right)$$

$$(\lambda^2 - 1) = \frac{2\tau}{h^2}\left(\lambda e^{i\alpha} + \frac{\lambda}{e^{i\alpha}}\right) - \frac{2\tau}{h^2}(\lambda^2 + 1)$$

$$\lambda^2 - 1 = \frac{2\tau}{h^2}\lambda\, 2\cos\alpha - \frac{2\tau}{h^2}\lambda^2 - \frac{2\tau}{h^2}$$

$$-\lambda^2\left(1 + \frac{2\tau}{h^2}\right) + \frac{4\tau}{h^2}\cos\alpha \cdot \lambda + 1 - \frac{2\tau}{h^2} = 0$$

$$\lambda^2\left(1 + \frac{2\tau}{h^2}\right) - \frac{4\tau}{h^2}\cos\alpha \cdot \lambda - 1 + \frac{2\tau}{h^2} = 0.$$

$$D = \left(\frac{4\tau}{h^2}\cos\alpha\right)^2 - 4\left[\frac{2\tau}{h^2}+1\right]\cdot\left[\frac{2\tau}{h^2}-1\right] =$$

$$= \frac{16\tau^2}{h^4}\cos^2\alpha - 4\left(\frac{4\tau^2}{h^4}-1\right) =$$

$$= \frac{16\tau^2}{h^4}\cos^2\alpha - \frac{16\tau^2}{h^4} + 4 =$$

$$= \frac{16\tau^2}{h^4}(\cos^2\alpha - 1) + 4. =$$

$$= 4 - \frac{16\tau^2}{h^4}\sin^2\alpha. = 4\left(1-\left(\frac{2\tau}{h^2}\sin\alpha\right)^2\right).$$

$$\lambda = \frac{\frac{4\tau}{h^2}\cos\alpha \pm 2\sqrt{1-\left(\frac{2\tau}{h^2}\sin\alpha\right)^2}}{2\left(1+\frac{2\tau}{h^2}\right)} =$$

$$= \frac{\frac{2\tau}{h^2}\cos\alpha \pm \sqrt{1-\left(\frac{2\tau}{h^2}\sin\alpha\right)^2}}{1+\frac{2\tau}{h^2}}.$$

По т. о спектр. устойчивости:

$$|\lambda| \le 1 + c_0\tau \,\vartheta^2$$

$$|\lambda|^2 \le 1 + 2c_0\tau + c_0^2\tau^2. \Rightarrow$$

$$|\lambda|^2 = \left(\frac{\frac{2\tau}{h^2}\cos\alpha + \sqrt{1-\left(\frac{2\tau}{h^2}\sin\alpha\right)^2}}{1+\frac{2\tau}{h^2}}\right)^2 =$$

$$= \frac{\frac{4\tau^2}{h^4}\cos^2\alpha + \frac{4\tau}{h^2}\cos\alpha\sqrt{1-\left(\frac{2\tau}{h^2}\sin\alpha\right)^2} + 1 - \left(\frac{2\tau}{h^2}\sin\alpha\right)^2}{\cancel{1+\ldots}\ \frac{(h^2+2\tau)^2}{h^4}} =$$

$$\cancel{= \frac{4\tau^2}{h^2}\cos^2\alpha + \frac{4\tau}{h^2}\cos\alpha}$$

$$\cancel{= \frac{4\tau^2}{h^2}\cos^2\alpha + 2\,\frac{4\tau^2}{h^4}\cos^2\alpha -}$$

$$|\lambda|^2 \le \frac{\frac{4\tau^2}{h^2}\left(\cos^2\alpha - \sin^2\alpha\right) + 1 + \frac{4\tau}{h^2}\cos\alpha\sqrt{1-\frac{4\tau^2}{h^4}}}{\frac{(h^2+2\tau)^2}{h^4}} \le$$

$$\le \frac{\frac{4\tau^2}{h^2} + 1 + \frac{4\tau}{h^2}\sqrt{1-\frac{4\tau^2}{h^4}}}{\frac{h^4+4h^2\tau+4\tau^2}{h^4}} =$$

$$= \frac{\left(\frac{4\tau^2}{h^2} + 1 + \frac{4\tau}{h^2}\cdot\sqrt{\frac{h^4-4\tau^2}{h^4}}\right)h^4}{h^4+4h^2\tau+4\tau^2} =$$

$$= \frac{4\tau^2h^2 + h^4 + 4\tau h^2\sqrt{h^4-4\tau^2}}{(h^2+2\tau)^2} \le$$

$$\le \frac{4\tau^2h^2 + h^4 + 4\tau}{h^4+4h^2\tau+4\tau^2} = \frac{h^4+4h^2\tau^2+4\tau}{h^4+4h^2\tau+4\tau^2} =$$

$$= 1 + \frac{4(\tau^2-\tau)}{2\tau+h^2} - \frac{4(2\tau^3-\tau^2-\tau)}{(2\tau+h^2)^2} \le 1 \Rightarrow$$

схема ус-ва по спектр. признаку

2) Замечание: $\tau \to 0, h \to 0$.

$$c = \frac{\tau}{h} = const$$

$$u_t + c^2 u_{tt} = u_{xx}.$$

$$u_m^{n\pm1} = u_m^n \pm u_t' \tau + u_t'' \frac{\tau^2}{2} \pm u_t''' \frac{\tau^3}{6} + o(\tau^4)$$

$$u_{m\pm1}^n = u_m^n \pm u_x' h + u_x'' \frac{h^2}{2} \pm u_x''' \frac{h^3}{6} + o(h^4).$$

$$\frac{2u_t'\tau + 2u_t''' \frac{\tau^3}{6} + o(\tau^4)}{2\tau} =$$

$$= \frac{2u_x'' \frac{h^2}{2} - 2u_t'' \frac{\tau^2}{2}}{h^2}$$

$$u_t' + u_t''' \frac{\tau^2}{6} + o(\tau^3) = u_x'' - c^2 u_t'' + o\left(h^2; \frac{\tau^4}{h^2}\right)$$

$$u_t + c^2 u_{tt} = u_{xx}.$$

~ 9.9. ug XIII

$$\frac{1}{12}\frac{u_{m+1}^{n+1}-u_{m+1}^{n}}{\tau}+\frac{5}{6}\frac{u_m^{n+1}-u_m^n}{\tau}+$$

$$+\frac{1}{12}\frac{u_{m+1}^{n+1}-u_{m-1}^{n}}{\tau}=\frac{1}{2}\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}+$$

$$+\frac{1}{2}\frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{h^2}$$

$$\cancel{u_{m+1}^{n+1}=u_m^n+u_t'\tau+u_x'h+u}$$

$$\frac{1}{12\tau}\Big[u_m^n+\tau u_t+hu_x+\frac{\tau^2}{2}u_t''+\frac{h^2}{2}u_x''+\frac{\tau^3}{6}u_t'''+$$

$$+\frac{h^3}{6}u_x'''-u_m^n-hu_x-\frac{h^2}{2}u_x''-\frac{h^3}{6}u_x'''+O(\tau^4,h^4)\Big]+$$

$$+\frac{5}{6\tau}\Big[\tau u_t'+\frac{\tau^2}{2}u_{tt}''+\frac{\tau^3}{6}u_t'''+O(\tau^4)\Big]+$$

$$+\frac{1}{12\tau}\Big[u_m^n+\tau u_t-hu_x+\frac{\tau^2}{2}u_t''+\frac{h^2}{2}u_x''+$$

$$+\frac{\tau^3}{6}u_t'''-\frac{h^3}{6}u_x'''-u_m^n+hu_x'-\frac{h^2}{2}u_x''+$$

$$+\frac{h^3}{6}u_x'''+O(\tau^4,h^4)\Big]=$$

$$=\frac{1}{2h^2}\Big[u_m^n+hu_x'+\frac{h^2}{2}u_x''+\frac{h^3}{6}u_x'''-2u_m^n+$$

$$+u_m^n-hu_x+\frac{h^2}{2}u_x''-\frac{h^3}{6}u_x'''+O(h^4)\Big]+$$

$$+\frac{1}{2h^2}\Big[u_m^n+\tau u_t'+hu_x'+\frac{\tau^2}{2}u_t''+\frac{h^2}{2}u_x''+\frac{\tau^3}{6}u_t'''+$$

$$+\frac{h^3}{6}u_x'''-2u_m^n-2\tau u_t'-2\frac{\tau^2}{2}u_t''-2\frac{\tau^3}{6}u_t'''+$$

$$+u_m^n+\tau u_t'-hu_x'+\frac{\tau^2}{2}u_t''+\frac{h^2}{2}u_x''+\frac{\tau^3}{6}u_t'''-\frac{h^3}{6}u_x'''+O(\tau^4,h^4)$$

$$2u'_t + \tau u''_t + \frac{\tau^2}{3} u'''_t + O(\tau^3, h^4) =$$
$$= u''_x + O(h^2) \Rightarrow$$

в) порядок аппроксимации: $O(\tau^3, h^4)$

г) На уст. исследуем по спектр. признаку.

$$u_m^n = \lambda^n e^{i\alpha m}$$

$$\frac{1}{12} \frac{\lambda^{n+1} e^{i\alpha(m+1)} - \lambda^n e^{i\alpha(m+1)}}{\tau} +$$
$$+ \frac{5}{6} \frac{\lambda^{n+1} e^{i\alpha m} - \lambda^n e^{i\alpha m}}{\tau} + \frac{1}{12} \frac{\lambda^{n+1} e^{i\alpha(m-1)} - \lambda^n e^{i\alpha(m-1)}}{\tau} =$$
$$= \frac{1}{2} \frac{\lambda^n e^{i\alpha(m+1)} - 2\lambda^n e^{i\alpha m} + \lambda^n e^{i\alpha(m-1)}}{h^2} +$$
$$+ \frac{1}{2} \frac{\lambda^{n+1} e^{i\alpha(m+1)} - 2\lambda^{n+1} e^{i\alpha m} + \lambda^{n+1} e^{i\alpha(m-1)}}{h^2}$$

$$\frac{1}{12} \frac{\lambda e^{i\alpha} - e^{i\alpha}}{\tau} + \frac{5}{6} \frac{\lambda - 1}{\lambda \tau} + \frac{1}{12} \frac{e^{-i\alpha}(\lambda - 1)}{\tau} =$$
$$= \frac{1}{2} \frac{e^{i\alpha} + e^{-i\alpha} - 2}{h^2} + \frac{1}{2} \frac{\lambda(e^{i\alpha} + e^{-i\alpha} - 2)}{h^2}$$

$$\frac{(\lambda - 1)}{\tau} \left(\frac{1}{12} e^{i\alpha} + \frac{5}{6} + \frac{1}{12} e^{-i\alpha} \right) = \frac{\cos\alpha - 1}{h^2} (\lambda + 1)$$

$$\frac{\lambda - 1}{6\tau} (\cos\alpha + 5) = \frac{\cos\alpha - 1}{h^2} (1 + \lambda) \Rightarrow$$

$$|\lambda| = \frac{\cos\alpha + 5 + 6\frac{\tau}{h^2}(\cos\alpha - 1)}{\cos\alpha + 5 - 6\frac{\tau}{h^2}(\cos\alpha - 1)} = 1 + \frac{12\frac{\tau}{h^2}(\cos\alpha - 1)}{\cos\alpha + 5 - 6\frac{\tau}{h^2}(\cos\alpha - 1)} \le 1 \Rightarrow$$

уст-ва по спектр. признаку

$\boxed{16}$