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ABSTRACT
This article takes a fresh look at reswitching. When two production techniques are compared, reswitching occurs when one technique is more viable than the other at a high interest rate, switches to being less viable at a lower rate, and reswitches to being more viable again at even lower rates. For some, reswitching undermines the foundations of neoclassical economics because it belies the idea of a monotonic relationship between relative capital values and factor price. The reswitching equation is an nth degree polynomial having n roots, implying the existence of n interest rates. Conventional analysis uses one interest rate but ignores the others. We argue that the others should not be ignored because all rates are determined simultaneously, and when one rate shifts, all rates shift. We demonstrate that the Samuelson reswitching model possesses a 'dual' expression containing every interest rate, the rates being compressed into a composite, interest-rate variable, thereby establishing a role for interest rates previously thought lacking in use and meaning. The relationship between this composite interest rate and capital value does not exhibit reswitching. The notion of a composite interest rate has implications for economics beyond reswitching.

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1. Introduction

The calculation and analysis of interest rates in discrete time have to date failed to use the full properties of the underlying polynomials on which they are based. In particular, where the solution involves the calculation of interest rates by evaluating the roots of a polynomial, both complex and real, negative roots are arbitrarily discarded in favor of real, positive roots. This is perverse because there are problems where the complex roots provide valuable information, enabling accurate calculations where conventional analysis provides only approximations, for example the calculation of bond duration (Osborne 2014, pp. 82–91). It is well known that a polynomial can be characterized by the full set of its roots (Courant, Robbins, and Stewart 1996, p. 101). Less well known is that these roots, real and complex, provide a toolkit facilitating analysis of a problem, just as duality provides a complementary framework for addressing a range of economic problems.

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This insight is employed here to provide a novel perspective on a long-running issue in economics—reswitching—that forms the basis of the so-called 'Cambridge controversies' in capital theory. These controversies began in the 1930s, intensified into the 'Cambridge controversies' during the 1960s (see the 1966 symposium in the *Quarterly Journal of Economics*), simmered during the late twentieth century, and are stirring again because of the treatment of capital in Piketty (2014). See Galbraith (2014) for one view of Piketty (2014) and Solow (2014) for another. Harcourt (1972) provides an early survey of the Cambridge controversies; Cohen and Harcourt (2003) and Han and Schefold (2006) contain more recent accounts.

When two techniques of production are compared, reswitching occurs when one technique is most viable at a high interest rate, switches to being less viable at a lower rate, and reswitches to being most viable again at even lower rates. For some, the phenomenon undermines the foundations of neoclassical economic theory. Samuelson (1966, p. 568) expresses concern about reswitching as follows:

> The phenomenon of switching back at a very low interest rate to a set of techniques that had seemed viable only at a very high interest rate involves more than esoteric technicalities. It shows the simple tale told by Jevons, Bohm-Bawerk, Wicksell and other neoclassical writers—alleging that, as the interest rate falls in consequence of abstention from present consumption in favor of future, technology must become in some sense more 'roundabout', more 'mechanized' and more 'productive'––cannot be universally valid.

Decades later, Cohen and Harcourt (2003, p. 211) continue to express concern, as they believe that the questions at issue in the recurring capital controversies are 'very deep indeed'.

Samuelson (1966) contains a small model capturing the phenomenon of reswitching described in Sraffa (1960). In this article, we apply multiple-interest-rate analysis from Osborne (2014) to the Samuelson model. To contemporary eyes, the model is dated because it uses the labor theory of value. Nevertheless, we employ it because it is simple, and it enables focus on the most important aspect of the phenomenon: the reswitching equation is an nth degree polynomial having n roots, with every root containing an interest rate. In most economic analyses, including reswitching, it is normal to employ only one root, the 'orthodox root' yielding a positive, real interest rate. The remaining, $(n-1)$, 'unorthodox roots'—and their implied interest rates—are mostly complex numbers or real, negative numbers, and they are usually ignored. We argue that this neglect is misguided.

We demonstrate that the Samuelson equation possesses a dual expression that has four notable features. First, it contains every interest rate solving its conventional counterpart, the rates being combined into a meaningful, composite, interest rate variable. Second, the relationship between this composite interest rate variable and capital cost does not exhibit reswitching. Third, the relationship is positive, confirming the neoclassical 'tale' about lower rates promoting more 'roundabout' technology. Fourth, the analysis exposes a long-standing anomaly in conventional theory: its focus on one interest rate to the exclusion of all others, despite the others having use and meaning.

2. The Samuelson reswitching model

Samuelson (1966) assumes that capital cost today is the accumulated value of past labor inputs, meaning capital is stored (or dated) labor. Equation 1 is a polynomial containing past labor inputs (L_i from $i = 1$ to 3) compounded at an interest or profit rate (r) to a capital cost today (C):[1]

$$C = L_1(1+r) + L_2(1+r)^2 + L_3(1+r)^3 \tag{1}$$

Two production techniques, A and B, have labor inputs {0, 7, 0} and {6, 0, 2}, respectively. These numbers applied to Equation 1 result in the following equations:

$$C_A = 7(1+r)^2 \tag{2}$$

$$C_B = 6(1+r) + 2(1+r)^3 \tag{3}$$

Table 1 contains capital costs C_A and C_B as the rate of interest (r) varies between 150 percent and zero percent. As the rate of interest falls from 150 percent, technique A begins by being cheaper than technique B, switches to being more expensive at interest rates between 100 percent and 50 percent, and reswitches to being cheaper again at rates below 50 percent.[2]

Levhari (1965, p. 99) attempts to downgrade the importance of reswitching by claiming that it 'may indeed be observed in the production of a single good … [but] … it is impossible with the whole basis of production'. This was refuted by contributions to the 1966 symposium, notably Pasinetti (1966), but also Bruno, Burmeister, and Sheshinski (1966), Garegnani (1966) and Morishima (1966). These contributions persuaded neoclassical participants (Samuelson 1966; Levhari and Samuelson 1966) to admit the general possibility of a 'perverse relationship' between the relative viability of techniques and the interest rate. More recently, Han and Schefold (2006) employ OECD input–output data from industrial sectors in nine countries to demonstrate empirically the existence of reswitching (albeit in only a small percentage of the cases they examine).

This article does not argue with the theoretical and empirical evidence for reswitching as currently perceived. Instead, it argues that the relationship between capital cost and interest rate is subtler than it appears because of the functional form of Equation 1. The equation is a polynomial of degree three, and therefore, for each capital cost, there are three values of $(1 + r)$ solving the equation, not one. These three roots are determined simultaneously, and therefore all roots (and their implied interest rates) shift when capital cost shifts, and vice versa. Thus, the focus of this article is the use and meaning of all interest rates solving Equation 1.

3. Multiple interest rates

Courant, Robbins, and Stewart (1996, p. 101) summarize a well-known result about polynomials:

[1] A dilemma exists concerning terminology. Schools of thought employ the phrase 'rate of interest' or 'rate of profit' depending on their stance in the controversies, their definition of capital and their view of reswitching's significance. It is tedious and confusing to use both phrases, and partisan to use one. We do not intend to be partisan; we hope the new perspective is interesting to all schools, and therefore, *for simplicity only*, we follow Samuelson (1966) and employ the phrase 'interest or profit rate' initially and the word 'interest' thereafter.

[2] John Woods reminds us that this analysis applies to the phenomenon of 'capital reversing' that takes place inside, as well as on, the capital-cost/interest-rate envelope for multiple techniques, reswitching applying to the occasions when capital reversing bites on the envelope itself. Nevertheless, we stay with the better-known term 'reswitching', the essence of the argument remaining the same.

Table 1. C_A and C_B at interest rates from 150 to 0 percent

Interest rate (%)	Capital cost: technique A	Capital cost: technique B
150	43.75	46.25
145	42.02	44.11
140	40.32	42.05
135	38.66	40.06
130	37.03	38.13
125	35.44	36.28
120	33.88	34.50
115	32.36	32.78
110	30.87	31.12
105	29.42	29.53
100	28.00	28.00
95	26.62	26.53
90	25.27	25.12
85	23.96	23.76
80	22.68	22.46
75	21.44	21.22
70	20.23	20.03
65	19.06	18.88
60	17.92	17.79
55	16.82	16.75
50	15.75	15.75
45	14.72	14.80
40	13.72	13.89
35	12.76	13.02
30	11.83	12.19
25	10.94	11.41
20	10.08	10.66
15	9.26	9.94
10	8.47	9.26
5	7.72	8.62
0	7.00	8.00

The shaded areas indicate where $C_A < C_B$, the two switch points occurring at interest rates of 100 percent and 50 percent. The boxed entries highlight the values analyzed in the text.

Every polynomial of degree n, $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, can be factored into the product of exactly n factors, $f(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$, where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are complex numbers, the roots of the equation $f(x) = 0$.

It follows that capital cost Equation 1 rearranges and factorizes uniquely into Equation 4, in which $[(1 + r) - (1 + r_j)]$ is the jth factor, $(1 + r_j)$ is the jth root and r_j is the jth interest rate:

$$\begin{aligned}&(1 + r)^3 + (L_2/L_3)(1 + r)^2 + (L_1/L_3)(1 + r) - (C/L_3) \\ &\quad = [(1 + r) - (1 + r_1)][(1 + r) - (1 + r_2)][(1 + r) - (1 + r_3)]\end{aligned} \tag{4}$$

Equations like (4) can be interpreted by visualizing a three-dimensional surface. This visualization is achieved by means of the modulus of the equation (Aleksandrov, Kolmogorov, and Lavrent'ev 1969, p. 286). The coefficients (labor inputs and capital cost) are real numbers, and therefore the analysis begins in the single dimension of the real number line. Given values for these coefficients, the roots $(1 + r_j)$ from $j = 1$ to 3 are uniquely determined: they can be real numbers (positive and negative) and complex numbers of the form $a + bi$ where 'a' and 'b' are real and i is the imaginary number, $\sqrt{-1}$, thereby moving the analysis into the two-dimensional complex plane. The variable $(1 + r)$ can also be complex-valued. Given values for the coefficients and their associated roots, and given a value for the variable $(1 + r)$, we can determine the absolute value (modulus) of Equation 4. This modulus measures the height (h) above the point $(1 + r)$ in the complex plane, as in Equation 5, making the analysis three-dimensional.

$$\begin{aligned} &|(1+r)^3 + (L_2/L_3)(1+r)^2 + (L_1/L_3)(1+r) - (C/L_3)| = |h| \\ &\quad = |(1+r) - (1+r_1)||(1+r) - (1+r_2)||(1+r) - (1+r_3)| \end{aligned} \tag{5}$$

As the location of $(1 + r)$ moves around the plane, the height of Equation 5 varies, mapping a surface. Some locations of $(1 + r)$ and their associated heights are more interesting than others because of the relationships they reveal between coefficients (labor inputs, capital cost) and roots (interest rates).

An initial exploration of Equation 5 assumes that $(1 + r)$ takes the value of one of the roots, for example when $(1 + r)$ takes the value of root $(1 + r_1)$, the associated factor is zero, and therefore the equation's height is zero (the surface touches the complex plane). The left-hand side of (5) is also zero, and it rearranges into Equation 1a, a version of capital cost Equation 1 in which $r = r_1$:

$$C = L_1(1+r_1) + L_2(1+r_1)^2 + L_3(1+r_1)^3 \tag{1a}$$

Since there are three roots, it follows that there are three ways in which the height of Equation 5 can be zero, meaning there are two additional locations for $(1 + r)$ at which the surface touches the plane, and therefore two additional versions of capital cost Equation 1 coexist with Equation 1a:

$$C = L_1(1+r_2) + L_2(1+r_2)^2 + L_3(1+r_2)^3 \tag{1b}$$

$$C = L_1(1+r_3) + L_2(1+r_3)^2 + L_3(1+r_3)^3 \tag{1c}$$

In this analysis, the root $(1 + r_1)$ is designated the orthodox root, meaning r_1 is the real interest rate produced by a spreadsheet or financial calculator given values for the coefficients L_i and C. The remaining two roots are labeled 'unorthodox' because they are negative, real numbers or complex numbers. For example, technique B has three equations (6–8), employing rates r_{B1}, r_{B2} and r_{B3}:

$$C_B = 6(1+r_{B1}) + 2(1+r_{B1})^3 \tag{6}$$

$$C_B = 6(1+r_{B2}) + 2(1+r_{B2})^3 \tag{7}$$

$$C_B = 6(1+r_{B3}) + 2(1+r_{B3})^3 \tag{8}$$

When capital cost C_B takes the value 9.262, the orthodox root $(1 + r_{B1})$ is 1.1 and the two unorthodox roots are $(1 + r_{B2}) = -0.55 + 1.9767i$ and $(1 + r_{B3}) = -0.55 - 1.9767i$. The interest rates implied by these roots are $r_{B1} = 0.1$ and $|r_{B2}| = |r_{B3}| = |-1.55 \pm 1.9767i| = 2.512$. These roots and rates are illustrated in Figure 1.

Mathematically, each solution for the interest rate is equally valid. Conventional opinion in economics, however, does not reflect this. Economists have ignored the unorthodox solutions to equations of this type for centuries, probably because the negative and complex solutions are not obvious candidates for inclusion in a practical economic

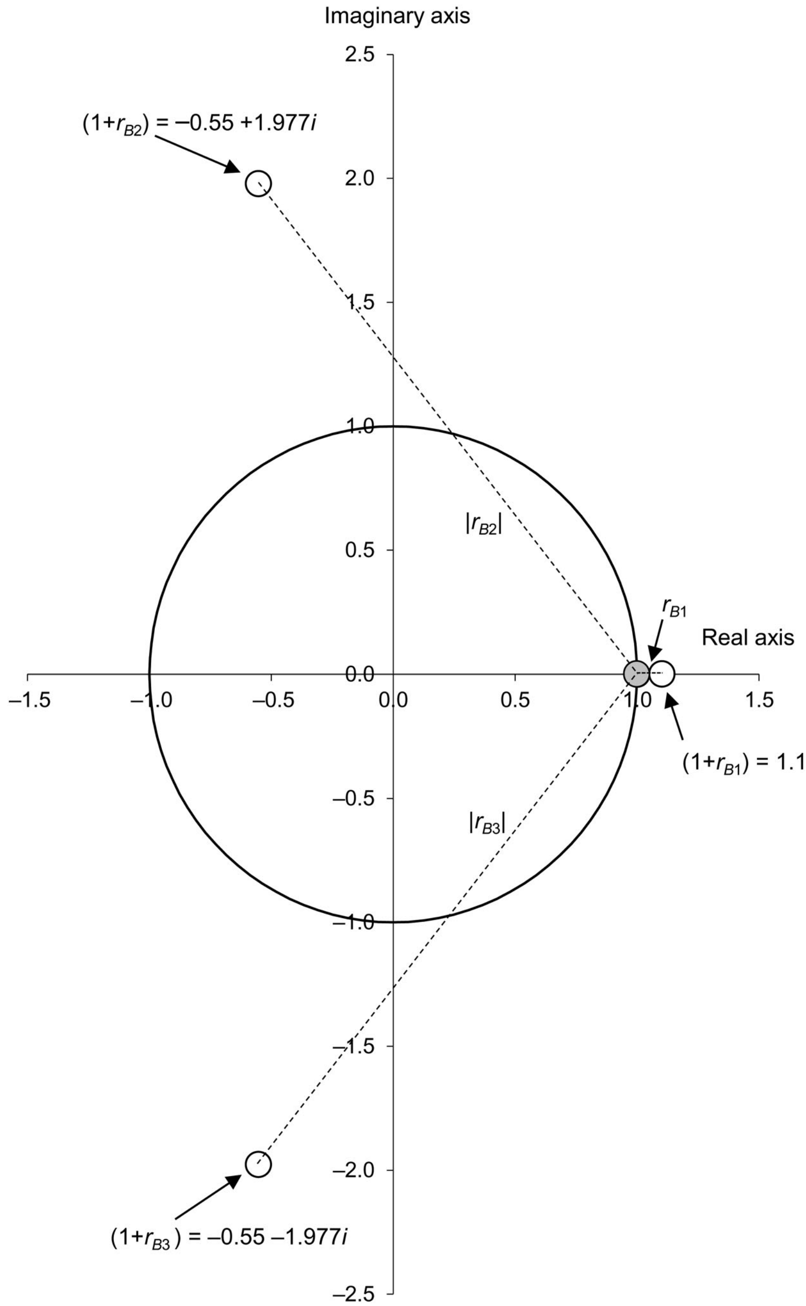

Figure 1. The roots (open dots) and interest rates (dashed lines) associated with Equations 6–8 for technique B when $C_B = 9.262$. The unit circle provides scale. The grey dot is unity, i.e., the point (1,0). The interest rates are the distances between the roots and unity, i.e., $r_{B1} = (1 + r_{B1}) - 1 = 0.1$ and $|r_{Bj}| = |(1 + r_{Bj}) - 1|$ for $j = 2$ and 3 (both being equal to 2.512). The product of the unorthodox interest rates is $2.512^2 = 6.31$.

theory. Some influential economists have stated explicitly that such solutions have neither use nor meaning.

> Now it is true that an equation of the nth degree has n roots of one sort or another ….Nevertheless, in the type of payment series with which we are most likely to be concerned, it is extremely probable that all but one of these roots will be either negative or imaginary, in which case they will have no economic significance (Boulding 1936, p. 440).

This conventional view is well entrenched; for example, when financial calculators and spreadsheets are given a value for capital cost C_B and asked to calculate the interest rate in Equation 3, they report only the orthodox interest rate, r_1. The conventional view is written into the software.[3]

It follows that conventional analysis focuses on Equation 1a. Given values for the coefficients L_i, the analyst chooses a value for the orthodox root $(1 + r_1)$ and solves for the associated capital cost (C), or vice versa. Conventional graphs plot pairs of observations (r_1, C), with the interest rate on the x-axis and capital cost on the y-axis.

Conventional analysis is partial analysis, however, because it ignores the unorthodox roots, and therefore neglects meaningful interest rate activity 'behind the scenes'. An account of this behind-the-scenes activity is as follows. Given values for the coefficients (capital cost and labor inputs), Equation 1a produces a value for the orthodox root $(1 + r_1)$. Given the same coefficients, Equations 1b and 1c produce values for the unorthodox roots $(1 + r_2)$ and $(1 + r_3)$. Thus, the given coefficients are uniquely associated with three, simultaneously determined interest rates, r_1, r_2 and r_3.

We analyze what happens when this situation changes by recalling an earlier observation that the height (h) of Equation 5 can be measured at any location of the variable $(1 + r)$. For example, when $(1 + r)$ is located at $(1 + r^*)$ on the real number line—$(1 + r^*)$ not equal to root $(1 + r_1)$—the equation's height (h^*) is given by the following expression abstracted from Equation 5:

$$|(1 + r^*)^3 + (L_2/L_3)(1 + r^*)^2 + (L_1/L_3)(1 + r^*) - (C/L_3)| = |h^*|$$

This expression is rearranged and interpreted to mean that the height at this location is the difference between the original, scaled capital cost (C/L_3) and a new, scaled capital cost (C^*/L_3):

$$|(1 + r^*)^3 + (L_2/L_3)(1 + r^*)^2 + (L_1/L_3)(1 + r^*)| = (C/L_3) + |h^*| = (C^*/L_3)$$

The last equation is a version of Equation 1a in which the coefficients L_i remain the same, and the new capital cost C^* is associated with a new root and its implied interest rate, meaning $r^* = r_1{}^*$, and the following version of Equation 1a holds:

$$L_3(1 + r_1^*)^3 + L_2(1 + r_1^*)^2 + L_1(1 + r_1^*) = C^*$$

Simultaneously, given L_i and C^*, Equations 1b and 1c produce new values for the unorthodox interest rates, r_2^* and r_3^*. Thus, a new configuration of coefficients, containing the original labor inputs and new capital cost, C^*, is uniquely associated with three new interest rates: r_1^*, r_2^* and r_3^*.

This procedure can be repeated for many different capital costs, each value being uniquely associated with a triplet of interest rates. To demonstrate this, vary the value of C_B in the root-finding formula in footnote 3 while holding the labor coefficients

[3]Software written for scientists and engineers is not so prejudiced; for example, when the formula solve (9.262 = 6x + 2x^3, x) is entered into www.wolframalpha.com, it yields all three values of $x = (1 + r)$ solving Equation 3.

constant, and observe the resulting triplet of roots. When capital cost changes, every root (interest rate) changes.

The result can be visualized using Figure 1, in which capital cost C_B is 9.262. When capital cost C_B changes value, every open dot (root) shifts position, and every dashed line (interest rate) changes length.

This observation prompts questions. Is the relationship between capital cost and interest rate best represented by the conventional Equation 1a involving the (x, y) pairs(r_1, C), (r_1^*, C^*), (r_1^{**}, C^{**}), and so on? Or is the relationship between capital cost and interest rate best represented by the (x, y) pairs $(\{r_1, r_2, r_3\}, C)$, $(\{r_1^*, r_2^*, r_3^*\}, C^*)$, $(\{r_1^{**}, r_2^{**}, r_3^{**}\}, C^{**})$, and so on, in which each element in curly brackets is some combination (yet to be determined) of the triplet of simultaneously determined interest rates? In other words, does another kind of equation exist—a dual expression to Equation 1—in which all simultaneously determined interest rates are visible, functional and somehow combined into a meaningful, composite, interest rate variable? A positive answer to this question follows.

4. The derivation of the dual expression to the capital cost equation

The previous section explores the factorized capital cost Equation 4 by setting the variable $(1 + r)$ to the value of the roots $(1 + r_j)$, thereby determining locations where the equation's height (h) is zero. The results are Equations 1a, 1b and 1c, conventional analysis voluntarily restricting itself to Equation 1a.

The current analysis removes this restriction. The objective of this section is to determine a single equation linking capital cost with *every* simultaneously determined interest rate. This is achieved by analyzing Equation 4 at a location for $(1 + r)$ other than one of the roots. For current purposes, we set $(1 + r)$ to unity, meaning the rate (r) is zero, and Equation 4 simplifies to Equation 9:

$$\sum_{i=1}^{3} L_i - C = L_3 \prod_{j=1}^{3} (-r_j) \tag{9}$$

Given constant labor inputs (L_i), the key relationship in Equation 9 is between capital cost and every simultaneously determined interest rate. Equation 9 rearranges into Equation 10 and we analyze its signs (+/−):

$$C = \sum_{i=1}^{3} L_i - L_3(-1)^3 \prod_{j=1}^{3} r_j \tag{10}$$

The visible minus signs on the right-hand side of Equation 10 cancel. The intrinsic signs of the interest rates r_j are determined by means of Descartes' rule of signs.[4] There is one change of sign in the sequence of coefficients on the left-hand side of Equation 4, and therefore the capital cost Equation 1 has one positive root $(1 + r_1)$, implying r_1 can be

[4]Descartes' rule of signs is as follows:

> If the coefficients of an equation are real and all its roots are also known to be real, then the number of its positive roots, with account taken of multiplicities, is equal to the number of changes of sign in the sequence of its coefficients. If it also has complex roots, then this number is equal to, or an even number less than, the number of these changes in sign. (Aleksandrov, Kolmogorov, and Lavrent'ev 1969, p. 295)

negative or positive, that is, $(1 + r_1) > 0$ implies $r_1 > -1$. The other two roots are unorthodox: they comprise either a complex conjugate pair[5] or a negative pair. Either way, their product is real and positive, and Equation 10 is written as dual Equation 11 containing the absolute values of the unorthodox interest rates:

$$C = \sum_{i=1}^{3} L_i + L_3 r_1 |r_2||r_3| \tag{11}$$

Given values for the labor inputs, dual Equation 11 reveals a relationship between capital cost and interest rate that differs in several respects from the conventional relationship embodied in capital cost Equation 1. These differences are discussed next.

5. Observations on the dual equation

The first observation on Equation 11 is that it employs every interest rate solving Equation 1, meaning every item of information from the three equations 1a–1c is compressed into Equation 11. This contrasts with conventional Equation 1a containing only one rate.

Second, the fundamental theorem of algebra implies that once one interest rate is known the other two rates are simultaneously and uniquely determined. Thus, the mapping from capital cost to interest rate, or vice versa, is not between C and r_1; instead, the mapping is between C and the composite variable $r_1|r_2||r_3|$. A mathematician's definition of a variable 'is "something" or, more accurately, "anything" that may take on various numerical values' (Aleksandrov, Kolmogorov, and Lavrent'ev 1969, p. 43). Interpreting the product of all simultaneously-determined interest rates as a composite, interest rate variable is a legitimate choice.

A third observation on Equation 11 concerns the economic meaning of the composite variable, $r_1|r_2||r_3|$. The meaning of the orthodox component (r_1) is known already: it is the mark-up employed during the accumulation of labor value using capital cost Equation 1a; but the unorthodox component requires interpretation. Our interpretation draws on analysis in Osborne (2014). We state (and prove in the Appendix) what observation of Equation 11 suggests: the product of the two, unorthodox rates ($|r_2||r_3|$) equals the number of applications of the orthodox rate (r_1) to an initially invested unit of labor (L_3) during the accumulation of labor value in Equation 1a. Furthermore, we state, and demonstrate in the Appendix, that this number depends on the structure of the adjusted labor inputs. Specifically, the number is a summary statistic—duration (Macaulay 1938)—describing the present-value weighted-average timing of the labor inputs (after a small prior adjustment). Thus, contrary to conventional wisdom, the unorthodox interest rates have meaning as well as use.

A final observation on Equation 11 is that the relationship between the composite interest rate variable ($r_1|r_2||r_3|$) and capital cost is linear. In other words, when r_1 shifts, affecting capital cost, the product of the unorthodox rates (the duration of the adjusted labor inputs) also shifts such that the overall interest-rate–cost relationship is linear. This linearity implies that, in the context of this model at least, switching between techniques can happen but reswitching cannot because two straight lines cross only once. Moreover, the

[5]When the coefficients of a polynomial are real, as they are in the reswitching equation, the complex roots always occur in conjugate pairs (Erdos and Turan 1950), each pair multiplying to a positive, real number.

relationship between capital cost and the composite interest rate is positive, implying that the neoclassical 'simple tale', that lower rates promote more roundabout technology, is valid when the interest rate variable is broadly defined.

The combined effect of these observations on dual Equation 11 is a new perspective on the phenomenon of reswitching. Multiple-interest-rate analysis leads to a reappraisal of the meaning of the term 'interest rate' by observing that there always are n interest rates solving an nth degree polynomial, the n interest rates combining (as a product) into a meaningful, composite, interest rate variable. The analysis reveals that a single interest rate, by itself, is not as informative as the full array of rates. The impact of the single, orthodox interest rate on capital cost depends on the context, on the pattern of labor inputs to which the rate is applied, and the influence of this pattern is transmitted via the unorthodox interest rates.

6. Numerical examples of the dual expressions to the Samuelson model

When the factorization procedure is applied to techniques A and B in the Samuelson model, the results are dual expressions (12) and (13) corresponding to conventional Equations 2 and 3:

$$C_A = 7 + 7r_{A1}|r_{A2}| \tag{12}$$

$$C_B = 8 + 2r_{B1}|r_{B2}||r_{B3}| \tag{13}$$

First, we analyze expressions (12) and (13) at one value of the orthodox interest rate. Table 2 contains values of the interest rates r_{Aj} and r_{Bj} when the two orthodox rates are 25 percent, meaning $(1 + r_{A1}) = (1 + r_{B1}) = 1.25$. The associated capital costs C_A and C_B are 10.9375 and 11.4063, respectively.

The interest rates in Table 2 are inserted into dual expressions (12) and (13), resulting in Equations 14 and 15:

$$10.9375 = 7 + 7(0.25)(2.25) \tag{14}$$

$$11.4063 = 8 + 2(0.25)(2.61008)(2.61008) \tag{15}$$

Equation 14 shows that the interest rate $r_{A1} = 0.25$ is applied 2.25 times to an initially invested unit of labor during the accumulation of labor value for technique A. Similarly, Equation 15 shows that the interest rate $r_{B1} = 0.25$ is applied $(2.61008)^2 = 6.8125$ times to

Table 2. The interest rates solving capital cost Equation 1: (a) for technique A when the orthodox interest rate is 25 percent and capital cost is 10.9375, and (b) for technique B when the orthodox interest rate is 25 percent and capital cost is 11.4063.

(a) Two capital cost equations with identical labor inputs, each equation having its own interest rate	The values of $(1 + r_{Aj})$	The implied values of r_{Aj}
$10.9375 = 7(1 + r_{A1})^2$	$(1 + r_{A1}) = 1.25$	$r_{A1} = 0.25$
$10.9375 = 7(1 + r_{A2})^2$	$(1 + r_{A2}) = -1.25$	$\|r_{A2}\| = 2.25$
(b) Three capital cost equations with identical labor inputs, each equation having its own interest rate	**The values of $(1 + r_{Bj})$**	**The implied values of r_{Bj}**
$11.4063 = 6(1 + r_{B1}) + 2(1 + r_{B1})^3$	$(1 + r_{B1}) = 1.25$	$r_{B1} = 0.25$
$11.4063 = 6(1 + r_{B2}) + 2(1 + r_{B2})^3$	$(1 + r_{B2}) = -0.625 + 2.0425i$	$\|r_{B2}\| = 2.61008$
$11.4063 = 6(1 + r_{B3}) + 2(1 + r_{B3})^3$	$(1 + r_{B3}) = -0.625–2.0425i$	$\|r_{B3}\| = 2.61008$

an initially invested unit of labor during the accumulation process for technique B. These numbers are confirmed by the accumulation schedules in Tables 3 and 4.

Second, we analyze expressions (12) and (13) at another value of the orthodox interest rate. Table 5 contains values of the interest rates r_{Aj} and r_{Bj} when the two orthodox rates are 75 percent, meaning $(1 + r_{A1}) = (1 + r_{B1}) = 1.75$. The associated capital costs C_A and C_B are 21.4375 and 21.2188, respectively.

The interest rates in Table 5 are inserted into dual expressions (12) and (13), resulting in Equations 16 and 17:

$$21.4375 = 7 + 7(0.75)(2.75) \tag{16}$$

$$21.2188 = 8 + 2(0.75)(2.9686)(2.9686) \tag{17}$$

Equation 16 shows that the interest rate $r_{A1} = 0.75$ is applied 2.75 times to an initially invested unit of labor during the accumulation of labor value for technique A. Similarly, Equation 17 shows that the interest rate $r_{B1} = 0.75$ is applied $(2.9686)^2 = 8.8125$ times to an initially invested unit of labor during the accumulation process for technique B. These numbers are confirmed by the accumulation schedules in Tables 6 and 7.

Table 8 combines the key data from the last two situations with similar data resulting from a third situation (not elaborated here) in which $r_{A1} = r_{B1} = 125$ percent, implying $C_A = 35.4375$ and $C_B = 36.2813$. Thus, Table 8 contains an alternative view of the boxed values in Table 1, exposing the role played by the previously ignored unorthodox interest rates in the determination of capital cost.

Table 3. The accumulation schedule for technique A when the interest rate is 25 percent.

Col.1 Time	Col. 2 Labor inputs	Col. 3 Cumulative units of labor	Col. 4 (extracted from Col. 3) Number of labor units marked up at each time by $(1 + r_{A1}) = 1.25$
2	7	7	
1	0	$7(1.25) + 0$	7
0	−10.9375	$7(1.25)^2 + 0(1.25) - 10.9375$	$7(1.25) + 0$
		The last entry above is equal to zero	Total number of labor units marked up during accumulation is the sum of cells above, i.e., $S = 15.75$

The number of times an initially invested labor unit is marked up at the orthodox rate $r_{A1} = 0.25$ is equal to $S/L_2 = 15.75/7 = 2.25$. This number is equal to the unorthodox rate, $|r_{A2}| = 2.25$, which is also equal to the present-value, weighted-average time (duration) of the adjusted labor inputs.
See the Appendix for an explanation of this table and Tables 4, 6 and 7.

Table 4. The accumulation schedule for technique B when the interest rate is 25 percent.

Col.1 Time	Col. 2 Labor inputs	Col. 3 Cumulative units of labor	Col. 4 (extracted from Col. 3) Number of labor units marked up at each time by $(1 + r_{B1}) = 1.25$
3	2	2	
2	0	$2(1.25) + 0$	2
1	6	$2(1.25)^2 + 0(1.25) + 6$	$2(1.25) + 0$
0	−11.40625	$2(1.25)^3 + 0(1.25)^2 + 6(1.25) - 11.40625$	$2(1.25)^2 + 0(1.25) + 6$
		The last entry above is equal to zero	Total number of labor units marked up during accumulation is the sum of cells above, i.e., $S = 13.625$

The number of times an initially invested labor unit is marked up at the orthodox rate $r_{B1} = 0.25$ is equal to $S/L_3 = 13.625/2 = 6.8125$. This number is equal to the product of the two unorthodox rates, $|r_{B2}|$ and $|r_{B3}|$, i.e., $2.61008 \times 2.61008 = 6.8125$, which is also equal to the present-value, weighted-average time (duration) of the adjusted labor inputs.

Table 5. The interest rates solving capital cost Equation 1: (a) for technique A when the orthodox interest rate is 75 percent and capital cost is 21.4375, and (b) for technique B when the orthodox interest rate is 75 percent and capital cost is 21.2188.

(a) Two capital cost equations with identical labor inputs, each equation having its own interest rate	The values of $(1+r_{Aj})$	The implied values of r_{Aj}
$21.4375 = 7(1+r_{A1})^2$	$(1+r_{A1}) = 1.75$	$r_{A1} = 0.75$
$21.4375 = 7(1+r_{A2})^2$	$(1+r_{A2}) = -1.75$	$\lvert r_{A2}\rvert = 2.75$
(b) Three capital cost equations with identical labor inputs, each equation having its own interest rate	**The values of $(1+r_{Bj})$**	**The implied values of r_{Bj}**
$21.2188 = 6(1+r_{B1}) + 2(1+r_{B1})^3$	$(1+r_{B1}) = 1.75$	$r_{B1} = 0.75$
$21.2188 = 6(1+r_{B2}) + 2(1+r_{B2})^3$	$(1+r_{B2}) = -0.875 + 2.3015i$	$\lvert r_{B2}\rvert = 2.9686$
$21.2188 = 6(1+r_{B3}) + 2(1+r_{B3})^3$	$(1+r_{B3}) = -0.875 - 2.3015i$	$\lvert r_{B3}\rvert = 2.9686$

Table 6. The accumulation schedule for technique A when the interest rate is 75 percent.

Col.1	Col. 2	Col. 3	Col. 4 (extracted from Col. 3)
Time	Labor inputs	Cumulative units of labor	Number of labor units marked up at each time by $(1+r_{A1}) = 1.25$
2	7	7	
1	0	$7(1.75) + 0$	7
0	−21.4375	$7(1.75)^2 + 0(1.75) - 21.4375$	$7(1.75) + 0$
		The last entry above is equal to zero	Total number of labor units marked up during accumulation is the sum of cells above, i.e., $S = 19.25$

The number of times an initially invested labor unit is marked up at the orthodox rate $r_{A1} = 0.25$ is equal to $S/L_2 = 19.25/7 = 2.75$. This number is equal to the unorthodox rate, $\lvert r_{A2}\rvert = 2.75$, which is also equal to the present-value, weighted-average time (duration) of the adjusted labor inputs.

Table 7. The accumulation schedule for technique B when the interest rate is 75 percent.

Col.1	Col. 2	Col. 3	Col. 4 (extracted from Col. 3)
Time	Labor inputs	Cumulative units of labor	Number of labor units marked up at each time by $(1+r_{B1}) = 1.75$
3	2	2	
2	0	$2(1.75) + 0$	2
1	6	$2(1.75)^2 + 0(1.75) + 6$	$2(1.75) + 0$
0	−21.2188	$2(1.75)^3 + 0(1.75)^2 + 6(1.75) - 21.2188$	$2(1.75)^2 + 0(1.75) + 6$
		The last entry above is equal to zero	Total number of labor units marked up during accumulation is the sum of cells above, i.e., $S = 17.625$

The number of times an initially invested labor unit is marked up at the orthodox rate $r_{B1} = 0.75$ is equal to $S/L_3 = 17.625/2 = 8.8125$. This number is equal to the product of the two unorthodox rates, $\lvert r_{B2}\rvert$ and $\lvert r_{B3}\rvert$, i.e., $2.9686 \times 2.9686 = 8.8125$, which is also equal to the present-value, weighted-average time (duration) of the adjusted labor inputs.

Table 8. Capital costs for techniques A and B at various interest rates and 'products-of-interest-rates'.

Orthodox rate of interest	Product of the unorthodox rates	Product of all interest rates	Dual equation for capital cost
0.25	**2.25**	**0.5625**	$\boldsymbol{C_A = 7 + 7(0.5625) = 10.9375}$
0.25	6.8125	1.7031	$C_B = 8 + 2(1.7031) = 11.4063$
0.75	**2.75**	**2.0625**	$\boldsymbol{C_A = 7 + 7(2.0625) = 21.4375}$
0.75	8.8125	6.6094	$C_B = 8 + 2(6.6094) = 21.2188$
1.25	**3.25**	**4.0625**	$\boldsymbol{C_A = 7 + 7(4.0625) = 35.4375}$
1.25	11.3125	14.1406	$C_B = 8 + 2(14.1406) = 36.2813$

These three sets of data correspond to the boxed entries in Table 1. The product of the unorthodox rates in column two is equal to duration calculated from capital cost and the *adjusted* labor inputs.

7. Discussion and concluding remarks

In this article the reswitching phenomenon is re-examined in the context of the Samuelson model, the analysis focusing on the dual capital cost equation rather than the conventional one. The dual expression contains every possible interest rate solving its conventional counterpart. The interest rates are considered together, as a product, the orthodox interest rate acting as a unit of value, and the product of its companion rates measuring the number of units, this number being a summary statistic (duration), reflecting pattern in the labor inputs. Thus, the reswitching phenomenon is seen in a new light by redefining the interest rate variable to be a composite entity containing rates previously thought lacking in use and meaning.

At its heart, this analysis is about information. Equation 4 involves a third-degree polynomial, with three distinct items of information on the left-hand side (three coefficients), and three distinct items of information on the right-hand side (three interest rates). The characteristics of the coefficients on the left-hand side (their size and timing) can be reconstructed from the combination of every interest rate on the right-hand side, that is, by expanding all right-hand-side factors. In contrast, knowledge of one item of information on the right-hand side of Equation 4—one factor, root, rate—is insufficient to reconstruct the coefficients on the left-hand side, meaning a single interest rate cannot characterize fully the relationships between capital cost and labor inputs.

A small, multiple-interest-rate literature exists in the context of the Cambridge capital controversies. Bharadwaj (1970) mentions all roots but explicitly excludes the complex solutions, asserting, like Boulding (1936), that they are not relevant. Bruno, Burmeister, and Sheshinski (1966) mention multiple roots, while Hagemann and Kurz (1976, p. 704) discuss multiple roots at length, concluding that 'a close connection between the multiplicity of the internal rate of return and the reswitching of techniques … does not exist'. However, these two works restrict their analyses to the multiple *real-valued* roots; they do not consider all roots together, including the complex.

To our knowledge, Dorfman (1981) is the only twentieth-century work employing all solutions for the interest rate as we do here—not singly but in combination, as components in the formula for another economic concept—making that work truly seminal.

Issues remain. First, the analysis involves comparative statics; time is not passing, as Harcourt (1972, p. 122) notes:

> Following Joan Robinson's strictures that it is most important not to apply theorems obtained from the analysis of differences to situations of change … modern writers usually have been most careful to stress that their analysis is essentially the comparisons of different equilibrium situations one with another.

The behavior described by Harcourt applies here: different capital costs are determined alongside their associated sets of interest rates, and these different values are compared with each other without reference to the passage of time. Incorporating time into the analysis remains a challenge.

Second, we have merely pointed out the existence of a new perspective on the reswitching phenomenon. We do not pursue its implications for the capital controversies in particular, and for theories of capital and growth in general. The analysis is offered to economists better versed than we are in these subjects in the hope they find it interesting and fruitful.

Finally, and most importantly, we observe that the Samuelson model of reswitching is only one example of economic theory's inordinate focus on the single, orthodox interest rate solving the time–value polynomial. The equation is probably the most common in financial economics—variants of it are solved millions of time every day in the computers of financial institutions. In most applications, the unorthodox solutions for the interest rate are discarded. This profligacy is puzzling because complex numbers are employed in the analyses of many other subjects, for example computing, mobile communication, satellite navigation and medical imaging. Applications of multiple interest rate analysis to economics outside of reswitching exist already (Dorfman 1981; Osborne 2014), some having implications for economic policy and practice; we predict more applications will be found.

Pursuit of these issues is left for future research. In the meantime, this article provides a different perspective on a famous debate, a perspective from the complex plane. A toy-like model, introduced by Samuelson to illustrate reswitching in a simple way, not only sheds light on the phenomenon but also raises profound and provocative questions about the use and meaning of an 'interest rate'.

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Appendix

On the meaning of unorthodox interest rates

Conventional analysis focuses on the orthodox rate of interest (r_1) solving capital cost Equation 1 and analyzes what happens to capital cost as r_1 varies, or vice versa, as in Equation 1a. In contrast, this article derives dual Equation 11, in which the right-hand-side variable is a composite of every simultaneously-determined interest rate solving Equation 1, and analyzes what happens to capital cost as the composite interest rate varies, or vice versa:

$$C = L_1(1+r) + L_2(1+r)^2 + L_3(1+r)^3 \tag{1}$$

$$C = L_1(1+r_1) + L_2(1+r_1)^2 + L_3(1+r_1)^3 \tag{1a}$$

$$C = \sum_{i=1}^{3} L_i + L_3 r_1 |r_2||r_3| \tag{11}$$

The text asserts that the product of the unorthodox interest rates possesses two meanings. The first meaning is suggested by observation of Equation 11: the product $|r_2||r_3|$ measures the number of times the orthodox interest rate (r_1) is applied to an initially invested unit of labor (L_3) during the accumulation of labor value in (1a). The following proof (based on analysis in Osborne 2014) goes beyond mere observation.

Inspection of the accumulation schedule for Equation 1a in Table A1 reveals that the assertion is true if the following equality is true: $|r_2||r_3| = S/L_3$.

Equation 1a is rearranged and written down four times in matrix M1. The first line of M1 is Equation 1a divided throughout by $(1 + r_1)$; the second line is Equation 1a divided throughout by $(1 + r_1)^2$, and so on:

$$\begin{aligned}
0 &= -\frac{C}{(1+r_1)^1} + L_1(1+r_1)^0 + L_2(1+r_1)^1 + L_3(1+r_1)^2 \\
0 &= -\frac{C}{(1+r_1)^2} + \frac{L_1}{(1+r_1)^1} + L_2(1+r_1)^0 + L_3(1+r_1)^1 \\
0 &= -\frac{C}{(1+r_1)^3} + \frac{L_1}{(1+r_1)^2} + \frac{L_2}{(1+r_1)^1} + L_3(1+r_1)^0 \\
0 &= -\frac{C}{(1+r_1)^4} + \frac{L_1}{(1+r_1)^3} + \frac{L_2}{(1+r_1)^2} + \frac{L_3}{(1+r_1)^1}
\end{aligned} \tag{M1}$$

Every line in M1 sums to zero, and therefore the entire matrix sums to zero. The six elements in the triangle with its right angle at the top right-hand corner of M1 comprise S in Table A1. It follows that the ten elements in the opposing triangle in M1 comprise minus S, giving Equation A1:

$$\begin{aligned}
-S = &-C\left[\sum_{i=1}^{4} \frac{1}{(1+r_1)^i}\right] + L_1\left[\sum_{i=1}^{3} \frac{1}{(1+r_1)^i}\right] + L_2\left[\sum_{i=1}^{2} \frac{1}{(1+r_1)^i}\right] \\
&+ L_3\left[\frac{1}{(1+r_1)}\right]
\end{aligned} \tag{A1}$$

The elements in square brackets simplify and (A1) rearranges to (A2):

$$\begin{aligned}
S = &\frac{C}{r_1}\left[1 - \frac{1}{(1+r_1)^4}\right] - \frac{L_1}{r_1}\left[1 - \frac{1}{(1+r_1)^3}\right] - \frac{L_2}{r_1}\left[1 - \frac{1}{(1+r_1)^2}\right] \\
&- \frac{L_3}{r_1}\left[1 - \frac{1}{(1+r_1)}\right]
\end{aligned} \tag{A2}$$

Equation A2 expands and rearranges to A3:

$$r_1 S = C - \sum_{i=1}^{3} L_i + \left[-\frac{C}{(1+r_1)^4} + \frac{L_1}{(1+r_1)^3} + \frac{L_2}{(1+r_1)^2} + \frac{L_3}{(1+r_1)}\right] \tag{A3}$$

The element in square brackets on the right-hand side of (A3) is the bottom line of M1, which is zero. Equation A4 follows:

$$C = \sum_{i=1}^{3} L_i + r_1 S \tag{A4}$$

Equation A4 is set alongside dual Equation 11:

$$C = \sum_{i=1}^{3} L_i + L_3 r_1 |r_2||r_3| \tag{11}$$

Comparing terms in (A4) and (11) proves the equality $|r_2||r_3| = S/L_3$. This proof generalizes to equations of nth degree.

The text asserts a second way in which the product of the unorthodox interest rates possesses meaning: the product is equal to the present-value, weighted-average time to maturity (duration) of the adjusted labor inputs. The reasoning is as follows.

Equation 1, with its given coefficients L_i and C, determines every root $(1 + r_j)$, meaning the three coefficients on the left-hand side of Equation 4 determine the three roots on the right-hand side. On the right-hand side, we replace the orthodox root $(1 + r_1)$ by $(1 + \text{zero})$, meaning there is a *ceteris paribus* shift in the orthodox interest rate. Then we return from the right-hand side of Equation 4 to the left-hand side, expanding the amended set of factors to a new set of coefficients containing *adjusted* labor inputs and unchanged capital cost. The adjusted set of labor inputs is similar to, but not the same as, the original set, because two of the three roots are unchanged, and the third root is smaller by only r_1. We assert (following analysis in Osborne 2014) that the duration of these values—capital cost and adjusted labor inputs—is equal to the product of the unorthodox interest rates.

The last point is illustrated by putting numbers into Equation 8. When $(1 + r_{B1})$ has value 1.1, capital cost (C_B) is 9.262, as in (A5):

$$9.262 = 6(1.1) + 0(1.1)^2 + 2(1.1)^3 \tag{A5}$$

These coefficients are contained in the following version of Equation 4:

$$\begin{aligned}(1 + r)^3 &+ \left(\frac{0}{2}\right)(1 + r)^2 + \left(\frac{6}{2}\right)(1 + r) - \left(\frac{9.262}{2}\right)\\ &= [(1 + r) - (1.1)][(1 + r) - (- 0.55 + 1.9767i)][(1 + r) - (- 0.55 - 1.9767i)]\end{aligned}$$

The orthodox root on the right-hand side of the last equation is shifted from (1.1) to $(1 + \text{zero})$, and this amended right-hand side is expanded, producing a new set of labor coefficients to accompany unchanged capital cost on the left-hand side, as in the following expression:

$$\begin{aligned}(1 + r)^3 &+ \left(\frac{0.22}{2.2}\right)(1 + r)^2 + \left(\frac{6.842}{2.2}\right)(1 + r) - \left(\frac{9.262}{2.2}\right)\\ &= [(1 + r) - (1 + \text{zero})][(1 + r) - (- 0.55 + 1.9761i)][(1 + r) - (- 0.55 + 1.9761i)]\end{aligned}$$

The variable $(1 + r)$ is then located at the new, orthodox root, $(1 + \text{zero})$, and the equation simplifies and rearranges to give (A6) in which capital cost is the sum of the adjusted labor inputs:

$$9.262 = 6.842(1 + \text{zero}) + 0.22(1 + \text{zero})^2 + 2.2(1 + \text{zero})^3 \tag{A6}$$

Thus, the *ceteris paribus* shift in the orthodox interest rate from 10 to 0 percent transforms (A5) into (A6), and the duration formula (Macaulay 1938) is applied to the coefficients in (A6).

Figure 1 provides a visual argument why the labor inputs are adjusted before applying the duration formula. Our focus is the product of the unorthodox rates, that is, the product of distances between the two unorthodox roots (open dots) and the root (1 + zero) (grey dot). Expanding the factors containing these three roots produces adjusted coefficients, whose duration is equal to the product of the unorthodox rates.

Before applying the duration formula to the adjusted coefficients in (A6), however, one further rearrangement of the equation is necessary. Duration was developed in the context of bond analysis where future cash flows are discounted to the present value of a bond, in contrast to the current analysis in which past labor inputs are compounded into the present value of capital cost. Thus, before applying duration, the 'compounding format' of Equation A6 is transformed into the 'discounting format' of Equation A7, meaning (A6) is divided throughout by $(1 + \text{zero})^3$ and rearranged:

$$2.2 = -\frac{0.22}{(1 + \text{zero})} - \frac{6.842}{(1 + \text{zero})^2} + \frac{9.262}{(1 + \text{zero})^3} \tag{A7}$$

Duration is the present-value, weighted-average time to maturity of the coefficients in Equation A7, as given by (A8), the asterisk indicating duration is calculated from the adjusted labor inputs, not the original ones. It is a summary statistic, capturing pattern in the sequence of coefficients:

$$D^* = \frac{1}{2.2}\left[-\left(\frac{0.22}{(1 + \text{zero})}\right)1 - \left(\frac{6.842}{(1 + \text{zero})^2}\right)2 + \left(\frac{9.262}{(1 + \text{zero})^3}\right)3\right] = 6.31 \tag{A8}$$

This value of duration is equal to the product of the unorthodox interest rates, meaning, $D^* = |r_2||r_3| = 2.51197^2 = 6.31$, as in Figure 1. Like the first result, this result generalizes.

Hence, dual Equation 11 is restated as Equation A9, in which the excess of capital cost over the sum of labor inputs is equal to the initial input of labor (L_3) marked up by an interest rate per period (r_1) for a number of periods (duration, D^*):

$$C = \sum_{i=1}^{3} L_i + L_3 r_1 D^* \tag{A9}$$

The entity r_1D^* is the composite interest rate variable, both r_1 and D^* changing simultaneously with any shift in capital value. Thus, the unorthodox interest rates do possess 'economic significance' when viewed as a cluster.[6]

[6]We thank Roger Garrison for pointing us to Cachanosky and Lewin (2014) connecting duration with the Austrian capital-theory concept of roundaboutness. One comment is offered here. They apply the duration formula to their original data, whereas we apply the same formula to our adjusted data, that is, the original data *after* the influence of the orthodox interest rate has been removed from it. It is the latter result that is equal to the product of the unorthodox interest rates. The connection between these works is a topic for further research.

Table A1. The accumulation schedule for the three-period capital cost Equation 1a.

Col.1	Col.2	Col.3	Col.4 (extracted from Col. 3)
Time	Labor inputs	Cumulative units of labor	The number of labor units marked up at each time by $(1 + r_1)$
3	L_3	L_3	
2	L_2	$L_3(1 + r_1) + L_2$	L_3
1	L_1	$L_3(1 + r_1)^2 + L_2(1 + r_1) + L_1$	$L_3(1 + r_1) + L_2$
0	$-C$	$L_3(1 + r_1)^3 + L_2(1 + r_1)^2 + L_1(1 + r_1) - C$	$L_3(1 + r_1)^2 + L_2(1 + r_1) + L_1$
		The last equation above is equal to zero	The total number of labor units marked up during the accumulation process is the sum of cells above $= S$

The number of times an initially invested labor unit is marked up by $(1 + r_1)$, and therefore by r_1, is S/L_3.