

RESOLVING THE ST. PETERSBURG PARADOX: A TRIUMPH FOR AUSTRIAN ECONOMICS

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1. Introduction

The St Petersburg paradox has been of academic interest for more than 300 years. It should not have been since in reality there is no paradox. This article demonstrates if two fundamental precepts of Austrian economics are applied this becomes clear. The first is utility, but not cardinal utility, and the second is placing the game in a real-world setting, evoking Austrian Economics' dislike of economic theory in a vacuum. By doing this then it becomes clear that the application of Austrian economics leads to the resolution of the so-called paradox. It is worth emphasising that this article is merely about solving a specific problem and not an attempt to restate any economic discipline or school of economics.

2. Dislike of cardinal utility (mathematical) functions

Austrian Economics has its critics. Bryan Caplan is one such critic (Caplan 1997). He set-out his views, in an unpublished document, as to why he no-longer was an Austrian Economist and later published an article expressing his views (Caplan 1999). These attracted some comments (Block 1999; Hülsmann 1999), to which he responded (Caplan 2001). One of Caplan's central points of disagreement with Austrian Economics is its unhappiness with cardinal utility, which nowadays plays a central role in neo-classical economics. He wrote (Caplan 1997, para. 2.1):

Modern neoclassical economists habitually use "utility functions" to describe individuals' preferences. For example, they may posit that an individual's [expected] utility $U=a*\ln(\text{quantity of apples})+(1-a)*\ln(\text{quantity of oranges})$. Rothbard instead preferred to discuss the "value scales" of individuals.

Caplan's above equation, the Expected Utility Value (EUV) equation is a well-known feature of cardinal utility and can be written in its usual form:

$$E\{U\} = \sum \pi_i U(C_i)$$

Where π_i is the probability that the outcome C_i is returned with $U(C_i)$ being the utility of C_i .

According to Caplan, Austrian economists, as represented by Von Mises (1881-1973) and Rothbard (1926-95) reject cardinal utility, preferring to remain with ordinal utility:

According to Rothbard, the mainstream approach credulously accepted the use of cardinal utility, when only the use of ordinal utility is defensible.

So, to Caplan his sticking point with Austrian Economics is cardinal utility with its mathematical representation of utility functions. This allows the application of mathematics to aspects of economics which is not possible if utility can only be represented as an ordinal. Rothbard had earlier published his views reconstructing utility and welfare economics (Rothbard 1956), which is not formally cited *per se* by Caplan 1997 but is a useful document to be consulted to obtain an understanding of Austrian Economists' views on the

points raised by Caplan. It is perhaps unfortunate that Rothbard did not write this piece a bit later since at the time a number of developments were taking place, and possibly these matters would better have been considered at a later date. These developments include the seminal work on the use of cardinal utility by Von Neumann and Morgenstern 1944 and the endorsement thereof by Friedman and Savage 1948 (Von Neumann and Morgenstern 1944; Friedman and Savage 1948, 1952) and other developments which were causing a stir when Rothbard wrote his rejoinder. Daniel Bernoulli's famous article, from which the origin of cardinal utility is derived, had then only recently been translated from Latin into English (Bernoulli 1954 [1738]). Samuelson's history of the St Petersburg Paradox appeared later (Samuelson 1977). A further development was the introduction of Prospect Theory by two psychologists (Kahneman and Tversky 1979) which has given impetus to behavioural economics. It also can be said that Stigler's 1950 systematic history of the development of both forms of utility was premature by a few years (Stigler 1950a, 1950b).

3. Origin of cardinal utility: St Petersburg Paradox

The origin of cardinal utility is Daniel Bernoulli's 1738 solution of the Petersburg Paradox (Bernoulli 1954) set-out in 1713 by Nicolas Bernoulli, Daniel Bernoulli's cousin. He devised several games which he thought demonstrated that, that which today is referred to as probability theory, did not provide answers to all games of chance. The St Petersburg game is one of those games. Nicolas circulated his games to interested parties in Europe soliciting assistance. It was accepted that the St Petersburg paradox could not be resolved in terms of the usual application of probability theory; the very point Nicolas was trying to make. In an article published in 1738 Daniel Bernoulli came-up with his cardinal utility solution to the paradox. This solution to what appeared to be an otherwise insoluble problem ensured cardinal utility's place in history. To understand the solution, the St Petersburg game must be considered. The game is simple enough. A coin is flipped until a head appears at which point the game stops. If it appears after the n^{th} flip the payout is 2^{n-1} ducats (or dollars). As with all games of chance, further games can be played, notionally over time an infinite number of games can be played. The first step is to determine the mathematical expected value of playing the game. Daniel Bernoulli did not provide a detailed derivation but merely stated "... the standard calculation shows that the value of Paul's [gambler, consumer] expectation is infinitely great ..." (Bernoulli 1954, para. 17). Bernoulli presumably thought the solution was so obvious that it did not need to be explained in detail. The original article was translated into English in 1954. Karl Menger, the son of Carl Menger one of the founders of Austrian Economics, was the technical adviser on the translation and explained in a footnote how this solution can be arrived at (Bernoulli 1954, n. 9):

The probability of heads turning up on the 1st throw is $\frac{1}{2}$. Since in this case Paul receives one ducat [or dollar], this probability contributes $\frac{1}{2} \cdot 1 = \frac{1}{2}$ ducat [dollar] to his expectation. The probability of heads turning up on the 2nd throw is $1/4$. Since in this case Paul receives 2 ducats [dollars], this possibility contributes $1/4 \cdot 2 = \frac{1}{2}$ to his expectation. Similarly, for every integer n , the possibility of heads turning up on the n -th throw contributes $1/2^n \cdot 2^{n-1} = \frac{1}{2}$ ducats [dollars] to his expectation. Paul's total expectation is therefore $\frac{1}{2} + \frac{1}{2} + , \dots, + \frac{1}{2} + \dots$ that is, infinite.

Karl Menger himself had previously published a work on the St Petersburg game (Menger 1934; Peters 2011). So according to the mathematicians of the time the mathematical expected value is infinite. But on the other hand as Bernoulli pointed out, "it has ... to be admitted that any fairly reasonable man would sell his chance, with great pleasure, for twenty ducats." (Bernoulli 1954, para. 17)

According to Bernoulli the reasonable gambler would only be prepared to pay a modest amount, in the order of \$20 or less to play the game; not the very large sum as determined by the purported application of standard probability theory; hence the paradox. The application of mathematics produces one answer and the imagined behaviour of consumers produced a different answer: a moderate sum. Bernoulli resolved the paradox by evoking what is now known as the expected cardinal utility solution, something he called the moral expectation. Accepting the gambler's utility function can be represented by the natural log then:

Expected Utility = $\sum 1/2^n \ln[(W_0 + 2^{n-1})/W_0]$ where n tends to infinity and W_0 is the gambler's initial wealth.

According to Bernoulli's cardinal utility solution, the stake which would be gambled depends on the gambler's initial wealth. Bernoulli lists some values. If $W_0 = 0$, the stake would be \$2, if W_0 is \$10 the stake would be \$3, \$4 if W_0 was \$100 and \$6 of W_0 was \$1 000 (Bernoulli 1954, para. 19). According to this cardinal utility solution, the wealthier the individual, the more the individual would stake to play the game. Bernoulli settled on the value of \$13. Todhunter, Stigler and others redid Bernoulli's calculations confirming these results (Todhunter 1865; Stigler 1950b). For our purposes, by way of example, a figure of \$10 to play a game will do. This modest amount confirmed the view that gamblers would only stake modest amounts.

3.1. Adam Smith (1776) and the rise of ordinal utility

Thirty-eight years after Daniel Bernoulli published his cardinal utility solution Adam Smith published his *Wealth of Nations* in which he drew attention to the two values; the value in use and the value in exchange. The distinction is between the price paid (value in exchange) and the consumer's utility (value in use). This launched the quest to define utility from which three authors Jevons, Menger and Walrus, working independently, produced the theory of ordinal utility. By the early 1900s doubts existed as to the measurability of utility and it became the settled view of many economists that ordinal utility was defensible but not cardinal utility. The possible measurability of utility was not dismissed by all economist (Stigler 1950a, 1950b).

3.2. Von Neumann and Morgenstern 1944

Then in 1944 Von Neumann and Morgenstern in their widely acclaimed work revived Daniel Bernoulli's cardinal utility (Von Neumann and Morgenstern 1944). Austrian Economists remained sceptical. Von Neumann and Morgenstern's work ushered in the modern interest in cardinal utility, what Rothbard called neo-cardinal utility. Although Stigler's articles on the development of utility included the St Petersburg Paradox it did not take into consideration the then unfolding developments flowing from the work of Von Neumann and Morgenstern although he does cite Friedman and Savage 1948 (Stigler 1950b, n. 172). In any event Rothbard dismissed Von Neumann and Morgenstern's contribution with, "The errors of this theory are numerous and grave" (Rothbard 1956, 17). A large number of other academics did not dismiss their work that easily. This work elevated the expected utility solution, with its mathematical utility functions as a central part of modern mainstream economics. Cardinal Utility as part of mainstream economics became assured, hence Caplan's understandable criticism of Austrian Economics rejection of cardinal utility.

4. Austrian economics' preference for real world setting

The second factor which is important to resolve the St Petersburg Paradox is Austrian Economics preference for economics in a real-world setting, not in an abstract setting. Rothbard expressed dislike for analysis *in vacuo* (Rothbard 1956, 6):

One of the most absurd procedures based on a constancy assumption has been the attempt to arrive at a consumer's preference scale not through observed real action, but through quizzing him by questionnaires. *In vacuo*, a few consumers are questioned at length on which abstract bundle of commodities they would prefer to another abstract bundle, and so on.

To resolve the St Petersburg game, it is necessary to place it in a real-world setting. This is difficult since the St Petersburg game unlike Lotto or Powerball and so on, has, as far as is known, never been offered by any gambling institution, such as a casino. It is not a real economic problem. It is one of those hypothetical problems where answers can be obtained by resorting to quizzing students to provide solutions (Neugebauer 2010). This is the methodology Rothbard rejected. Student quizzing has nowadays become common currency (Kahneman and Tversky 1979). Despite the passing of 300 years "... only a few experiments on the St Petersburg gamble have been documented." (Neugebauer 2010)

In addition to Rothbard's criticism of this methodology another can be raised. The game as usually presented is not placed in a real-world setting. This is now done.

In a real economic setting, there would be a supplier of games, say a casino, and demand for games from gamblers; consumers. In this setting two pieces of insight from neo-classical economics are of use. The first is

that the price is determined by suppliers seeking to make a profit and secondly the consumer's decision to participate in the gamble is as a consequence of the consumer's [ordinal] utility, as Roy put it, "the assumption [is] that entrepreneurs maximise their profits and consumers their utility ..." (Roy 1952, 447; Weintraub 2002).

4.1. First: the casino, as the supplier of games

The casino would be a firm as the supplier of games. As standard neo-classical economics points out firms maximise profits. Suppliers set their prices to earn a profit. Casinos would enter into the market based on the mathematical expectation of the game. This price, according to the traditional solution of the St Petersburg game would have to be very large; an infinite sum. But this solution, explained above, contains an error. Karl Menger, who was the technical adviser to the English translation noted an error in Bernoulli's solution to the paradox, but oddly never appreciated the full significance of that error, nor the implications of that error to the solution which he himself provided as a footnote! In deriving the utility solution Bernoulli wrote (Bernoulli 1954, para. 18):

18. The number of cases to be considered here is infinite: in one half of the cases the game will end after the first throw, in one quarter of the cases it will conclude at the second, in the eighth part of the cases with the third, in a sixteenth with the fourth, and so on. If we designate the number of cases through [to] infinity by N ...

Menger noted the error in this approach (which is equally applicable to the solution he provided in the earlier footnote):

Since the number of cases is infinite, it is impossible to speak about one half of the cases, one quarter of the cases, etc., and the letter N in Bernoulli's argument is meaningless.

So, if instead of $N = \infty$ which as Menger points out is meaningless let $N = 2^k$. In this case a possible and realistic mathematical outcome is arrived at (Vivian 2003, 2004, 2013). The number of games which would terminate at each position in the series, using Bernoulli methodology would be:

$$2^{k-1} + 2^{k-2} + 2^{k-3} + 2^{k-4} + \dots + 2^{k-k}$$

This is a simple geometric series and if summed produces a total of $2^k - 1$ games.

If 2^k games are played all the games, except one, are expected to terminate within a series which is k in length. Thus, only one game is expected to end beyond a series k in length. Thus, for any finite number of games played the series is finite. As Menger correctly points out each term in the series, k in length, contributes $\frac{1}{2}$ to the total expectation.

Thus, the expected value of the series is $k/2 + \lambda$.

$k/2$ accounts for the games which terminate within the series k in length and λ is added to account for the game which terminates outside of the range of k games. By way of example λ is taken to be 1 for 50 % confidence level of the game terminating at the first term after the series k in length.

If the game is played say 2^{18} (ie. 262,144) times then mathematically the expected value is \$10 at a 50% confidence level. So, the mathematical expectation is not infinite. The casino can accept bets priced at \$10 providing no more than 262,144 games are permitted if the casino is profit motivated. The outcome of playing 262,144 games can be observed by getting a computer to play 262,144 games and the average pay-out determined. So, the game can be placed in a real-world setting.

4.2. Second: Gambler's utility explanation

Again as pointed out in neo classical economics the consumer enters into exchange transactions to maximise his or her utility (Weintraub 2002; Roy 1952). Austrian economists would agree with this. Where Austrian economist disagree is that individual's utility can be determined using a mathematical utility function. What utility explains is why consumers enter into exchange transactions. The explanation is that utility is the driving force behind the consumer entering into any exchange transaction or as Irving Fisher pointed out "An individual acts as he desires" (Fisher 1926; Rothbard 1956). In a competitive market the price at which the consumer contracts is not determined by his or her utility but by the market price set by suppliers maximising profits. So, in a real-world setting the price would be the price set by the casino based on profit maximisation and not on the consumer's utility. This realisation produces Alfred Marshall's consumer's surplus and leads to Pigou's Economic Welfare theory (Pigou 1932). What the utility theory does is give an indication of the amount the gambler will be willing to bet, not the amount they actually bet. In the example used in this article the casino sets the price at \$10 per game and if the gambler accepts this figure the gambler demonstrates his or her utility leads to the acceptance of the gamble. The gambler receives greater utility by playing the game than retaining the \$10. If the gambler accepts that price the gambler demonstrates his preference.

$$U(G) > U(\$10)$$

To the gambler the utility of playing the game exceeds his or her utility of retaining the stake of \$10.

There is no need to consider utility beyond this point to resolve the St Petersburg Game. Cardinal Utility is not required. Irving Fisher was correct in this regard when he wrote "[t]o fix the idea of utility the economist should go no further than is serviceable in explaining *economic* facts (Fisher 1926, 11). The emphasis is in the original. The economic fact is the consumer exchanged \$10 to play the game. If the consumer sought the advice of a mathematical expert, based on the traditional solution, the expert would advise that the consumer should accept the modest amount to play the game.

The consumer does not enter into gambling transactions to maximise his or her profits but because of the possibility of winning a large sum of money. If the head appears say after 64 flips, the number of squares on chess board, the consumer wins $\$1.8446741 \times 10^{19}$, all this for a mere \$10. It is the supplier which takes the risk not the consumer.

5. Results

The results of the computer playing the game a number of times are indicated below.

k	Number of Games	Income	Expenditure	Profit	Profit as % of income	Expenditure per game	
						Expected	Actual
0	1	\$10	\$1	\$9	90.00	\$1.00	\$1.00
5	32	\$320	\$102	\$218	68.13	\$3.50	\$3.19
10	1 024	\$10 240	\$4 973	\$5 267	51.44	\$6.00	\$4.86
15	32 768	\$327 680	\$277 285	\$50 395	15.38	\$8.50	\$8.46
18	262 144	\$2 621 440	\$2 545 781	\$75 659	2.89	\$10.00	\$9.71
20	1 048 576	\$10 485 760	\$10 865 471	(\$379 711)	(3.62)	\$11.00	\$10.36

When $k=0$ only one game was played, and the head appeared on the first flip of the coin resulting in a payment of \$1. The casino received an income of \$10 earning a profit of \$9. The expected value per game = $0/2 + 1 = \$1$ and the measured actual value was \$1.

When the game was played 262 144 times which is the breakeven number of games with an expected value of $k/2 + \$1 = \$18/2 + \$1 = \$9 + \$1 = \10 . The income was \$2 621 440 with an expenditure of \$2 545 781 giving a small profit of \$75 659. The actual expenditure per game was \$9.71, instead of the expected \$10. The casino operator is expected to make a loss where more than 262 144 games were played. This was realised when 1 048 576 games were played; 2^{20} games. When the game was played 1 048 576 times the expected expenditure per game is \$11 but since the income is still only \$10 per game the casino operator can expect to make a loss, which is what happened. The casino operator made a loss of \$379 711

The large expected value as calculated by Nicolas and Daniel Bernoulli never materialised. The expenditure is in line with the predicted expected expenditure when the expected value is correctly calculated. There is no paradox requiring an explanation and thus no need to evoke cardinal utility to resolve the St Petersburg game. The game and its outcome when placed in a real world setting is consistent with Austrian Economics. There is nothing out of the ordinary calling for an explanation.

As indicated above the St Petersburg Game is not a game which is known to be offered by casinos even although as indicated above it is possible to determine the expected costs (and hence profits). It is suggested the work of Frank Knight provides an explanation for this omission. Knight explained the relationship between risk, uncertainty and profit (Knight 1921). As Kenneth Arrow pointed out institutions exist which manage risk, usually converting risk into a determinable fixed cost (Arrow 1951, 1971). In gambling the risk is reduced to a determinable fixed cost per game via the law of large numbers. In the St Petersburg Game because the expected value is a function of the number of games played and uncertainty exists in the form of λ the expected cost is not reduced to a fixed cost. Thus the expected profit cannot satisfactorily be determined. Thus, despite the ability to determine the expected costs firms which have the objective of maximising profits will not find the St Petersburg Game an attractive proposition.

6. Conclusion

Austrian Economics does not favour individual's utility being represented by mathematical functions preferring instead the concept of a demonstrated preference (Rothbard 1956, 2). Cardinal Utility was introduced to resolve the St Petersburg Paradox and thrust into mainstream economics by Von Neumann and Morgenstern. Cardinal Utility is however not required to resolve the St Petersburg Paradox. If the game is offered at a mathematically determined price and accepted by gamblers, utility of the gamblers demonstrate their preference by electing to play the game. Their actual utility function becomes largely irrelevant. The price of the game is determined by the casino operator with the goal of maximising its profits. Utility is not necessary to determine the price. It turns out that the historical view that the mathematical expectancy of the St Petersburg is infinite, is incorrect. It is a function of the number of games played and is quite modest. Today, knowing this, it is clear that there is no paradox and thus cardinal utility is not necessary to resolve the paradox. The correct price can be obtained from the correct application of probability theory.

If two precepts of Austrian economics are applied, the use of utility as the explanation as to why gamblers gamble and placing the game in a real word setting, then that there is no paradox becomes clear. The application of these precepts leads to the resolution of the paradox. The rejection of the usefulness of cardinal utility is not a weakness of Austrian economics but a strength as is placing the game in a real-world setting.

7. References

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